ISAR TARGET RECONSTRUCTION VIA DIPOLE MODELING

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ISAR TARGET RECONSTRUCTION VIA DIPOLE MODELING

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ABSTRACT

ISAR TARGET RECONSTRUCTION VIA DIPOLE MODELING

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This thesis addresses a unique method of ISAR dipole model image by employing vector dyadic contrast function technique. The current ISAR imaging algorithms rely upon the assumption that the area under investigation consists of a superposition of infinitesimally small isotropic scatterers (i.e., the point scatterer model). This approximation fails to capture the true real-world scattering mechanisms occurring within the targets of interest. Therefore, this thesis proposes a better imaging technique which is based upon the assumption that targets can be modeled as a collection of infinitesimally small dipoles. And the reason is that it is the simplest antenna with well-known radiation pattern, theory and it is more realistic approximation model. The word “small” implies that it is relative to the wavelength of operation. So that the absolute size of the dipole does not matter, only the size of the wire relative to the wavelength of the frequency matters. Also the field of small dipole antenna is function of the polar angle and this tells us the orientation of the target relative to the incident field. The orientation of each dipole is accounted in a dyadic contrast function. The image reconstruction, i.e., retrieval of the
dyadic reflectivity function from measured data, will not only provide information regarding the shape but also the direction of predominant edges of the target.
Dedicated to my wife and my children,

for all the patience and support that made this work possible.
ACKNOWLEDGEMENTS

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I would like to thank all of the students, faculty, and staff for their help and support here at the University of Dayton. I would also like thank all of my friends in the Mumma Radar Lab for their invaluable friendship and support for such tough and long period.

Finally, thanks to my family who deserves special thanks, since they have supported me such long days of this hard process. My parents, my children and my wife were always there for me. To my wife, I express my deepest appreciation. Your unconditional love, constant encouragement, and unwavering devotion sustained me during the many long days and without you this work wouldn’t be possible.
PREFACE

The problem of inverse synthetic aperture radar imaging has been studied for decades. Researchers have continued developing new ways and algorithms to improve the current method of imaging which is point scattering model. This method assumes that every target can be approximated by the superposition of infinitesimal point scatterers then sums up to form the image. Realistically speaking there is no such a target that radiates equally all direction no matter what aspect angle is illuminated.

Therefore, we developed a more sensitive technique to model and image objects of interest known as dipole model (DM). The dipole model is based on the vector dyadic reflectivity function which permits degrees of freedom not otherwise available with traditional point scattering model based. Vector dyadic is second order tensor of reflectivity function which comprised nine elements rather than one and that provides more details of the target such as orientation, more features and so on.

This thesis is divided into five chapters. The first three chapter’s covers introduction and literature review which is only relevant to the thesis. The fourth chapter covers the current model which is known as point source and the fifth chapter talks about the proposed method known as dipole model and the results as shown below.

- Chapter 1: The introduction of ISAR and its basic mathematical theory.
o Chapter 2: Here is presented the literature overview of electromagnetics that is only relevant to this thesis. Also the problem of electromagnetic scattering and its general phenomena is covered.

o Chapter 3: Brief overview of Green’s function, dyadic Green’s function in electric field and a detailed description of field quantity in terms of dyadic.

o Chapter 4: This chapter describes the classical implementation of ISAR and examples of point scattering model. It describes in more detail point source model of ISAR imaging technique: The forward model and which is used to reconstruct a scalar contrast function by employing back-projection inversion methods is described.

o Chapter 5: This chapter discusses the numerical results and the proposed method of ISAR imaging known as dipole model. This dipole model is based upon vector dyadic. Vector dyadic is second order tensor of reflectivity function which comprised nine elements rather than one which point source model (PSM) is based upon. This technique allows not only detecting the target but also determines the orientation of the target and gives more features of the target.
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<table>
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<th>Abbreviation</th>
<th>Description</th>
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<tr>
<td>ISAR</td>
<td>Inverse Synthetic Aperture Radar</td>
</tr>
<tr>
<td>Tx</td>
<td>Transmitter</td>
</tr>
<tr>
<td>Rx</td>
<td>Receiver</td>
</tr>
<tr>
<td>RCS</td>
<td>Radar Cross Section</td>
</tr>
<tr>
<td>RF</td>
<td>Radio frequency</td>
</tr>
<tr>
<td>PEC</td>
<td>Perfect electric conductor</td>
</tr>
<tr>
<td>PMC</td>
<td>Perfect magnetic conductor</td>
</tr>
<tr>
<td>TI</td>
<td>Target of interest</td>
</tr>
<tr>
<td>EFIE</td>
<td>Electric field integral equation</td>
</tr>
<tr>
<td>MFIE</td>
<td>Magnetic field integral equation</td>
</tr>
<tr>
<td>RAM</td>
<td>Radar Absorbing Material</td>
</tr>
<tr>
<td>TE</td>
<td>Transverse Electric</td>
</tr>
<tr>
<td>TM</td>
<td>Transverse Magnetic</td>
</tr>
<tr>
<td>FFT</td>
<td>Fast Fourier Transfer</td>
</tr>
<tr>
<td>IFFT</td>
<td>Inverse Fast Fourier Transfer</td>
</tr>
<tr>
<td>EM</td>
<td>Electromagnetic</td>
</tr>
<tr>
<td>SNR</td>
<td>Signal to Noise Ratio</td>
</tr>
<tr>
<td>AC</td>
<td>Alternating Current</td>
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</tbody>
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xxii
1D  One dimensional
2D  Two-dimensional
3D  Three-dimensional
CT  Computed Tomography
$hh$  Transmitting horizontally and receiving horizontally
$hv$  Transmitting horizontally and receiving vertically
$vh$  Transmitting vertically and receiving horizontally
$vv$  Transmitting vertically and receiving vertically
MoM  Method of Moments
FEM  Finite Element Method
FDTD  Finite Difference Time Domain
GTD  Geometrical Theory of Diffraction
PO  Physical Optics
PTD  Physical Theory of Diffraction
SBR  Shooting and Bouncing Rays
CGM  Conjugate Gradient Method
PSM  Point Source Model
DM  Dipole Model
$G_t$  Gain of the transmitter
$G_r$  Gain of the receiver
$B_n$  Noise bandwidth
$T_0$  Temperature
$K$  Boltzmann constant
# LIST OF SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$E^i$</td>
<td>Incident electric field</td>
</tr>
<tr>
<td>$E^s$</td>
<td>Scattered electric field</td>
</tr>
<tr>
<td>$S$</td>
<td>Poynting vector</td>
</tr>
<tr>
<td>$i$</td>
<td>$\sqrt{-1}$</td>
</tr>
<tr>
<td>$\ddot{G}$</td>
<td>Dyadic Green’s function</td>
</tr>
<tr>
<td>$g$</td>
<td>Scalar Green’s function</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>Reflection coefficient</td>
</tr>
<tr>
<td>$k_0$</td>
<td>Wave number of free space</td>
</tr>
<tr>
<td>$\nabla \times$</td>
<td>Curl operation</td>
</tr>
<tr>
<td>$\nabla \cdot$</td>
<td>Divergence operation</td>
</tr>
<tr>
<td>$n$</td>
<td>Index of refraction</td>
</tr>
<tr>
<td>$A$</td>
<td>Magnetic vector potential</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Electric scalar potential</td>
</tr>
<tr>
<td>$x$</td>
<td>X-coordinate</td>
</tr>
<tr>
<td>$y$</td>
<td>Y-coordinate</td>
</tr>
<tr>
<td>$z$</td>
<td>Z-coordinate</td>
</tr>
<tr>
<td>$\mu_0$</td>
<td>Permeability of free space</td>
</tr>
</tbody>
</table>
\( \varepsilon_0 \)  
Permittivity of free space

\( \sigma \)  
Conductivity

\( \eta_0 \)  
Free space impedance

\( \omega \)  
Angular frequency

\( \tau \)  
Pulse width

\( \lambda_0 \)  
Free space wavelength

\( \delta \)  
Delta function

\( \rho \)  
Reflectivity function

\( \Delta r_r \)  
Range resolution

\( \Delta r_c \)  
Cross range resolution

\( K \)  
Chirp rate

\( f_d \)  
Doppler frequency

\( t_d \)  
Delay time

\( F \)  
Lorentz force

\( E \)  
Electric field

\( B \)  
Magnetic field

\( H \)  
Auxiliary field

\( D \)  
Displacement field

\( J \)  
Volume density of electric current

\( \rho_{\text{free}} \)  
Volume density of electric charge

\( \varepsilon_r \)  
Relative permittivity

\( \mu_r \)  
Relative permeability
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
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<tbody>
<tr>
<td>$exp$</td>
<td>Exponential operator</td>
</tr>
<tr>
<td>$Im$</td>
<td>Imaginary operator</td>
</tr>
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CHAPTER 1

INVERSE SYNTHETIC APERTURE RADAR (ISAR)

1.1 Introduction

Inverse synthetic aperture radar (ISAR) imaging is a radar technique to generate a two-dimensional high resolution image of a target by using the information of both down range direction and cross range directions. Range is defined as the axis parallel to the direction of EM wave propagation from the radar towards the target of interest (TI) and the cross range is defined as the axis orthogonal to the range direction. Range resolution is obtained either using short pulse or pulse compression where the cross-range resolution is obtained from Doppler history of different parts of the target.

Figure 1.1 Inverse SAR imaging attempts to reconstruct an image of moving target.
ISAR technology utilizes the movement of the target rather than the transmitter. For small angles, an ISAR image is the two dimensional Fourier transform of the received signal as a function of frequency and target aspect angles. The change in aspect due to the motion between the radar and target provides the cross range dimension needed to form an image. Individual scatterers are resolved in range because of the fine time sampling capabilities of the radar.

For ISAR, the surface character of the target is complex but it usually can be seen as made of many scatterers points in different position and with different reflection coefficient. The purpose of ISAR imaging is to show correctly the relative position and the field intensity of the echo of every scattering point.

ISAR processing techniques are developed based upon the fundamental assumption that the targets to be imaged are composed of a superposition of infinitesimally small isotropic sources. Spotlight ISAR (or SAR) cross range resolution is limited by the extent of the platform (or target) motion, e.g., rotation and translation, within the antenna beam. Extended dwell time processing allows for improved resolution. However, as the platform (or target) rotates, the scattered electric field changes both in phase and amplitude, invalidating the fundamental assumption in SAR/ISAR of isotropic point target scattering. As target returns decorrelate over increasingly wide observation angles, the respective coherently-formed SAR/ISAR images will degrade. Typically, this problem is mitigated by arbitrarily reducing the coherent dwell to a value that will likely preserve the target’s electromagnetic signature.
1.2 The Mathematical Theory of ISAR

Figure 1 & 2 show the concept of (ISAR) geometry. Figure 1 shows a stationary radar illuminating a moving plane. The radar waveform has pulse width $\tau$ and pulse repetition interval (PRI) or $T$. The instantaneous frequency is as follows:

$$f = f_c + K(t - nT) \quad (1.1)$$

where $f_c$ is the carrier frequency, $t$ corresponds to the center of the pulse, $K$ is the chirp rate, and $B = K\tau$ is the bandwidth of the pulse. If two scatterers are separated by $\Delta r$ then they will appear in the same range bin. Since there are $N$ range bins obtained from an $N$-point FFT over the returns from the radar, the spatial resolution is achievable by pulse compression as shown below:

$$\Delta r = \frac{c_0}{2K\tau} \quad (1.2)$$

The angular interval is the angle through which the target is viewed during coherent processing over an aperture. If we take two separate scatterers, point A and point B located on a target that rotates with an angular rate of $\omega$, these points will have a separable Doppler rate. This Doppler frequency ($f_d$) and the phase of the returned signal produced by the scattering points are shown below:
\[ g = \omega_c \frac{2 \left| R - \int V_R(t) dt \right|}{c_0} \]  \hspace{1cm} (1.3)

\[ f_d = \frac{2 \Delta r \omega_c \cos(\theta)}{\lambda_0} \]  \hspace{1cm} (1.4)

The cross range resolution \( \Delta r_{cr} \) depends on how well the frequency difference can be resolved. The frequency resolution \( \Delta f_d \) is the difference between Doppler frequencies from two scatterers, but separated in cross range distance \( \Delta r_{cr} \). The cross-range resolution is obtained as

\[ \Delta r_{cr} = \frac{\lambda_0}{2 \Delta \theta} \]  \hspace{1cm} (1.5)

where, \( \lambda_0 = \frac{c_0}{f_d} \).

For a continuous wave radar that transmits a signal in the form of a series of \( N \) chirps, the received signal is delayed with respect to the transmitted signal by the amount shown below:

\[ t_d = \frac{2 * R(t)}{c_0} \]  \hspace{1cm} (1.6)
\[ Tx = A(t)e^{j\theta_{tx}(t)} \]  \hspace{1cm} (1.7)

where,

\[ \theta_{tx}(t) = \frac{\omega R(t)}{c_0}, \]

\( A(t) \) is the amplitude modulation, and \( R(t) \) is the target distance from the radar. The transmitted signal in the complex exponential form is,

\[ Rx = \int Tx \cdot \rho(r')e^{j\omega(t-\frac{2R(t)}{c_0})} \, dr' \]  \hspace{1cm} (1.8)

In equation (1.8), \( \rho(r') \) is the reflectivity of the target, \( Tx \) and \( Rx \) are the transmitted and received signals. The received signal is nothing more than the scaled and delayed version of the transmitted signal.

Figure 1.2 Mono-static ISAR configuration.
One method of achieving good a quality image is frequency stepping waveforms. High speed analog to digital converters are needed to achieve a high range resolution. The transmitted radar frequency (RF) and receiver local oscillator reference frequency are stepped from pulse to pulse as shown in figure 1.3 below. Once some of the radar returns are stored in the signal processor, the discrete Fourier transform (FFT) is executed to get the range profile which is one dimensional (1D). As shown in equation (1.2) the range resolution $\Delta r_r$ is the reciprocal of the total span of the frequency stepping bandwidth.

\[ f_n = f_0 + n\Delta f \]

\( f_0 \) is starting carrier frequency
\( \Delta f \) is step size
\( n \) is number of pulses

ISAR is coherent radar which sends out a set of pulses that are digitally sampled and coherently integrated to increase the signal to noise ratio as can be seen in equation (1.9) because the signal returns are almost the same from pulse to pulse where the noise is different, in other words independent from pulse to pulse.
Signal to noise ratio

\[
\frac{S}{N} = \frac{P_i G_t^2 \lambda^2 \sigma}{(4\pi)^3 R^4 k T_o BF_n L}
\]  

(1.9)

1.3 Simple ISAR Imaging

ISAR imaging is a process that takes a three dimensional (3D) target and maps it into two dimensional (2D) space. This 2D image space is range and cross range. Range is defined as the axis parallel to the direction of EM wave propagation from the radar towards the target of interest (TI) and the cross range is defined as the axis orthogonal to the range direction.

1.3.1 Range profile (1D)

Range profile is the returned waveform shape from the target that is illuminated by the radar. If the returned signal collected by the receiver is a time domain pulse we will have 1D characteristic in terms of intensity vs time, or range, by converting the time to range by implying equation (1.10) shown below. But if the received signal is the stepped frequency waveform, we take the inverse Fourier transform (IFT) to get the time and then plot the range profile of the target as shown below:

\[
r = \frac{C_0 t}{2}
\]

(1.10)

where \(r\) is the distance between the radar and the target, \(t\) is the round trip time and \(C_0\) is the speed of light in free space.
1.3.2 Cross range profile (1D)

Cross range profile is obtained by collecting the signal from different angles as shown in figure 1.5. The aspect of the incident angles is used to resolve the required cross range point to form a 1D cross range profile. The cross range profile is obtained by processing the returned signal at one frequency, but different incident angles, whereas the range profile is acquired at one fixed angle but different frequencies. In other words, range profile is a function of frequency where cross range profile is a function of incident angle.
1.4 Antenna Theory

An antenna is defined by the IEEE as “transmitting or receiving system that is designed to radiate or capture electromagnetic waves”. In other words an antenna is a transducer which converts radio frequency (RF) power at its feed point into electromagnetic radiation (EMR) in transmitting mode and intercepts energy from passing electromagnetic radiation (EMR), which appears as RF voltage across the antenna’s feed point, in receiving mode. An antenna can be any size or shape. The intensity of the radiation launched by an antenna is generally not the same in all directions whether it is transmitting or receiving. Antennas are passive devices, meaning...
the power radiated cannot be greater than the power entering from the transmitter. The gain of the antenna doesn’t mean the signal is amplified but it means that the antenna radiates in certain directions better than others. Antennas are reciprocal in nature meaning the same antenna can be used as a transmitting antenna or receiving antenna. Lists of some of the common types of antennas are as follows:

- Wire antenna such as a dipole
- Aperture antenna such as a horn
- Micro-strip antenna
- Reflector antenna such as a dish
- Arrays such as linear or planner antennas
- And so on

1.4.1 Dipole antenna

In this section we will talk about the dipole antenna briefly. The reason is that it is a simple antenna with a well-known radiation pattern which we will be using in chapter 5 of this thesis. A dipole is a type of wire antenna defined as a pair of equal and opposite electric charges separated by a small distance. At the feed of a center fed dipole antenna the current is at the peak, and is the lowest at the two ends of the conductors. The radiation pattern of a dipole looks like a donut. A dipole is an omnidirectional antenna, meaning it radiates uniformly in all directions in one plane where the radiated power decreases with elevation angle above or below the plane, dropping to zero on the antenna axis as discussed below.
A linear wire is positioned symmetrically at the origin and oriented along z-direction as shown figure 1.6 below. The wire is considered to be a perfect electric conductor (PEC). To calculate its far field radiation pattern, we need only to find the vector potential $A$: 

$$A(r) = \int J(r')G_0(\vec{r}, \vec{r'})dr'$$  \hspace{1cm} (1.11)$$

where $\vec{r}$ is the observation point, $\vec{r'}$ is the source location on the wire, and $J(r')$ is the current vector.

Figure 1.6 Small wire dipole antenna.

$$E_{\text{ff}} = -j\omega A(r)$$  \hspace{1cm} (1.12)$$

Now let us look at an infinitesimal wire element of length $l$ lying along the z-axis and centered at $(0, 0, 0)$. The wire is excited by an incident electric field $E^i_z$ and we are interested to compute the current generated on the wire due to the excitation. Normally a constant current distribution on the wire element is not considered unless the wire length is
less than $\lambda/50$. But a more accurate and common current distribution is the triangular shape current which has the following equation:

$$I(z') = \begin{cases} 
I_0 \left(1 - \frac{2\Delta z}{l}\right), & 0 \leq \Delta z \leq \frac{l}{2} \\
I_0 \left(1 + \frac{2\Delta z}{l}\right), & -\frac{l}{2} \leq \Delta z \leq 0 
\end{cases} \quad (1.13)$$

Figure 1.7 Infinitesimal wire dipole model.

When a conductor’s dimensions are much less than wavelength $l << \lambda$, it can be supposed that the current is uniform across the conductor which leads to the conclusion shown below.

$$I = \int I d\tilde{z} = \int J dz = Jz \quad (1.14)$$
In a thin wire, current flow is mainly confined along its axis in such a way that $\hat{z}$ and $\hat{z}'$ have practically the same direction. The dipole antenna shown in Figure 1.8 is modeled through a 3D computational solver known as FEKO.

The current distribution along a thin wire can be found by deriving the Pocklington integro-differential equation from the time harmonic electric field through the magnetic vector potential and the electric scalar potential. The wave equation for the $z$-axis thin wire is as

$$\nabla^2 A_z + k^2 A_z = -\mu J_z \quad (1.15)$$

Away from the wire $J_z = 0$ and then equation (1.15) reduces to

$$\left[ \nabla^2 + k^2 \right] A_z = 0 \quad (1.16)$$
The above equation is called the Helmholtz equation which has the following solution known as a Green’s function:

\[ G_0(\vec{r}, \vec{r}') = \frac{e^{-jk|\vec{r} - \vec{r}'|}}{4\pi |\vec{r} - \vec{r}'|} \]

where:
- \( \vec{r} = \) observation point
- \( \vec{r}' = \) source point

The total magnetic vector potential is found by integrating the Green’s function everywhere as.

\[ A_z = \mu \oint_S I_z(\vec{r}')G_0(\vec{r}, \vec{r}')dz' \]

where \( A_z \) relates the induced current with the field and it is covered in section 3.2 and the Green’s function is also covered in section 3.1.

To solve the surface current on the wire, we must enforce the boundary condition by demanding that the total tangential electric field disappears on the surface of the perfectly conducting wire.

\[ E^{tot} = E^i_z + E^s_z = 0 \]
where $E_z^i$ is the source or impressed field and $E_z^s$ can be computed from the current density induced on the cylindrical wire antenna due to the incident. For radiated electric field in $z$-direction, it only looks like as:

\[
E_z^s = -E_z^i \\
E_z^i = \frac{j}{\omega \mu \varepsilon} \left[ k^2 + \frac{\partial^2}{\partial z^2} \right] A_z
\]

Substituting equation (1.18) into equation (1.20) leads to

\[
Pocklington integro–differential equation \\
E_z^s = -\frac{j}{\omega \mu \varepsilon} \int_{-z/2}^{z/2} \left[ k^2 + \frac{\partial^2}{\partial z^2} \right] I_z(\vec{r'})G(\vec{r},\vec{r'})dz'
\]

Figure 1.9 Far field radiation pattern.
Figure 1.10 3D radiation pattern of dipole antenna.

Figure 1.11 Radiation pattern of vertically oriented dipole antenna.
1.5 Radar Cross Section

Radar cross section measures a target’s ability to scatter signals in all directions including the direction of the radar receiver. The spatial distribution of energy is known as scattering and the object itself is called a scatterer. The energy scattered back to the receiver is called backscattering. How much energy is returned back to the source is determined by different factors such as:

- Material composition
- Polarization of the Tx and Rx
- Frequency of the transmitter
- Angle of illumination
- Size and shape

![Monostatic RCS](image-url)
The intensity of the return compared to the incident describes the radar cross section of the object as shown below [1].

\[
\sigma = \lim_{R \to \infty} 4\pi R^2 \frac{|E_s|^2}{|E_i|^2} \quad (1.22)
\]

where:

- \( R \) is the distance from the target to the receiver or vice versa.
- \( E_i \) is the electric field incident wave impinging on the target
- \( E_s \) is the electric field scattered wave at the receiver.

By the way, the above formula assumes a target extracts power from an incident wave and then scatters that power isotopically.

### 1.5.1 Monostatic RCS of PEC sphere

When a target has very small physical dimensions with respect to the wavelength of the incident wave, the analysis is done in the so called Rayleigh region [2]. In this region the shape of the target doesn’t influence in determining the target’s RCS. For targets that are comparable in size to the wavelength of the incident wave the RCS varies depending on the frequency and this region is called the resonant region (or Mie region).

When the dimensions of the target are large compared with the wavelength of the incident wave, the RCS of the target is determined by using the method known as geometrical optics (GO) and this region is known optical region [3]. Below we have shown the RCS of a PEC sphere by using the exact formula, an approximated formula,
and finally we used computational electromagnetic solver known as FEKO by plotting the scattered electric field of the sphere. The first figure is the RCS curve of PEC sphere as a function of the ratio of the target by employing the exact formula. The second figure is the same thing as the first one but using the approximated formula. And the final figure is the scattered electric field of the sphere where we used FEKO.

Figure 1.13 Normalized monostatic RCS of PEC sphere by using the exact formula.

\[ \sigma = \lim_{R \to \infty} 4\pi R^2 \left| \frac{\mathbf{E}}{E'} \right|^2 \]  

(1.23)
Figure 1.14 Normalized monostatic RCS of PEC sphere by using the approximated formula.

\[ \sigma = \frac{\lambda}{4\pi} \left[ \sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)}{\hat{H}_{n}^{(2)}(\beta a)\hat{H}_{n}^{(2)}(\beta a)} \right]^2 \]  

(1.24)

The example shown below demonstrates the accuracy of (FEKO) for scattering from a conducting sphere. The monostatic scattering pattern from an incident plane wave is computed at a frequency of $5 \times 10^7$ to $5 \times 10^9$ Hz.

The sphere is considered perfectly conducting and measures 0.1m in diameter. The sphere is centered at the origin and an incident plane wave arrives from the z-direction as shown in figure 1.15 below. The near field monostatic RCS pattern is
computed in points along the negative z-direction of the sphere in the x-y plane as shown in the Figure 1.16 below. The plane wave is shown as the blue arrow at the bottom of the figure and the electric field is oriented along the x-axis.

Figure 1.15 Perfect electric conductor (PEC) sphere.

Figure 1.16 The magnitude and the phase of the scattered electric field $E_x$ of (PEC) sphere.
Figure 1.17 The magnitude of the scattered electric field \((E_x, E_y, E_z)\) of (PEC) sphere.

Figure 1.18 The phases of the scattered electric field \((E_x, E_y, E_z)\) of (PEC) sphere.
CHAPTER 2

LITERATURE REVIEW OF ELECTROMAGNETICS

2.1 Background

In the modern world electromagnetic fields are used in many different devices for different reasons. Common examples are wireless communication such as antenna cell phones and satellites, heating and cooking foods in microwave ovens and medical equipment such as x-ray and tomography imaging. Therefore, it has become very vital to have good understanding the theory of electromagnetics and how it interacts with matter in general.

In this chapter only the parts of electromagnetic waves and field theory that are relevant to this thesis are summarized. The relations that describe the variations of the electric and magnetic fields, charges, and currents connected with electromagnetic waves are governed by physical laws that are known as Maxwell’s equations. These laws were put together by James Clerk Maxwell, a Scottish physicist and mathematician. These relations and basics are used in all sections of this thesis to formulate and solve in both direct and inverse scattering problems using electric field integral equations (EFIE).
2.1.1 Maxwell's equation

In this section, we will formulate the Maxwell equations that govern the behavior of the electromagnetic fields. Electromagnetism is concerned with the interaction between charged particles. Stationary charges cause an electric field, while moving charges known as currents cause both an electric and a magnetic field. Below are given the differential and integral forms of Maxwell’s equations. The differential form of Maxwell’s equations describe and relate the field vectors, current densities, and charge densities at any point in space at any time, provided that the field vectors are single-valued, bounded, continuous functions of position and time and have continuous derivatives. The integral form of Maxwell’s equations describe the relations of the field vectors, charge densities, and current associated with electromagnetic waves and are solved through electromagnetic boundary problem [4], [5].

\[
\nabla \cdot \mathbf{D} = \rho_{\text{free}} \quad (2.1)
\]

\[
\nabla \cdot \mathbf{B} = 0 \quad (2.2)
\]

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (2.3)
\]

\[
\nabla \times \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \quad (2.4)
\]

\[
\int \int \int_A \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int \int \int_V \rho_{\text{free}} dV \quad (2.5)
\]
Gauss’s Law for Magnetism

\[ \iint_A \vec{B} \cdot d\vec{A} = 0 \]  \hspace{1cm} (2.6)

Faraday’s Law

\[ \oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \oint_A \vec{B} \cdot d\vec{A} \]  \hspace{1cm} (2.7)

Ampere’s Law

\[ \oint_C \vec{B} \cdot d\vec{l} = \mu_0 \int_A \left( \vec{J} + \varepsilon_o \frac{d\vec{E}}{dt} \right) \cdot d\vec{A} \]  \hspace{1cm} (2.8)

where:

\( \vec{E} \) = Electric field [V/m]

\( \vec{B} \) = Magnetic field [Wb/m^2]

\( \vec{H} \) = Auxiliary field [A/m],

\( \vec{D} \) = Displacement field [C/m^2]

\( \vec{J} \) = volume density of electric current [A/m^2]

\( \rho_{\text{free}} \) = volume density of electric charge [C/m^3]

The force acting on a particle of electric charge \( q \) with instantaneous velocity \( \vec{v} \), due to an external electric and magnetic field, is given by the Coulomb and Lorentz force as shown below.

Lorentz force

\[ F = q \left( \vec{E} + (\vec{v} \times \vec{B}) \right) \]  \hspace{1cm} (2.9)
2.1.2 Constitutive relations

The electric and magnetic flux densities \( \mathbf{D} \) and \( \mathbf{B} \) are related to the field intensities \( \mathbf{E} \) and \( \mathbf{H} \) via the constitutive relations, whose precise form depends on the material in which the fields exist. For homogeneous, isotropic, dielectric and magnetic materials these relations are:

\[
\mathbf{D} = \varepsilon \mathbf{E} \quad (2.10)
\]
\[
\mathbf{B} = \mu \mathbf{H} \quad (2.11)
\]

where \( \varepsilon \) and \( \mu \) are respectively the electric permittivity and magnetic permeability of the medium. Permittivity is the physical quantity for the ability of a material to store electrical energy in electric field. Permeability is a material property which measures the ability of a material to support the formation of a magnetic field within itself. Their numerical values in vacuum is given below.

\[
\varepsilon_0 = 8.85 \times 10^{-12} \, \text{C}^2 \text{Nm}^{-1} \quad (2.12)
\]
\[
\mu_0 = 4\pi \times 10^{-7} \, \text{NAm}^{-1} \quad (2.13)
\]

The relative permittivity \( \varepsilon_r \) and relative permeability \( \mu_r \) are defined as the material properties relative to vacuum as such

\[
\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} \quad (2.14)
\]
\[ \mu_r = \frac{\mu}{\mu_0} \quad \text{(2.15)} \]

From the two quantities \( \varepsilon \) and \( \mu \) we can define two other physical constants such as the speed of light \( c_0 \) and the characteristic impedance \( \eta_0 \), which leads to the following [6], [7].

\[ c_0 = \frac{1}{\sqrt{\mu \varepsilon}} = 3 \times 10^8 \left[ \frac{m}{s} \right] \quad \text{(2.16)} \]

\[ \eta_0 = \sqrt{\frac{\mu}{\varepsilon}} = 377 \Omega \quad \text{(2.17)} \]

2.1.3 Boundary conditions

Reflection and refraction at an interface where the media of different electromagnetic properties show a discontinuities, the field vectors are discontinuous and their behavior across the boundaries are governed by boundary conditions. Boundary conditions are derived by applying the integral form of Maxwell’s equations (2.18-2.21) to a small region at an interface of the two media, e.g. the application of the integral form a curl equation to a flat closed path at an interface with top and bottom sides in the two touching media gives the boundary conditions for the tangential components; and the application of the integral form of a divergence equation to a shallow pillbox at an interface with the top and bottom faces in the two contiguous media gives the boundary conditions...
conditions for the normal components [8], [6], [9]. The boundary conditions for the electromagnetic fields across material boundaries are given shown below.

Figure 2.1 The surface of discontinuity and a thin slab of thickness h and area ΔS.

Normal component of \( \mathbf{D} \) is discontinuous by the free surface charge density

\[
\left( \mathbf{D}_2 - \mathbf{D}_1 \right) \cdot \mathbf{u}_n = \rho_e
\]  

(2.18)

Tangential components of \( \mathbf{E} \) are continuous

\[
\mathbf{u}_n \times \left( \mathbf{E}_2 - \mathbf{E}_1 \right) = \mathbf{j}_m
\]  

(2.19)

Normal components of \( \mathbf{B} \) are continuous

\[
\left( \mathbf{B}_2 - \mathbf{B}_1 \right) \cdot \mathbf{u}_n = \rho_m
\]  

(2.20)

Tangential components of \( \mathbf{H} \) are discontinuous by the free surface current density

\[
\mathbf{u}_n \times \left( \mathbf{H}_2 - \mathbf{H}_1 \right) = \mathbf{j}_e
\]  

(2.21)

Energy conservation follows from Maxwell’s equations. The vector identity is as:
\[ \nabla \cdot (a \times b) = b \cdot (\nabla \times a) - a \cdot (\nabla \times b) \quad (2.22) \]
yields the Poynting theorem in the time domain:

\[ \nabla \cdot \left( E \times H \right) + H \cdot \frac{\partial B}{\partial t} + E \cdot \frac{\partial D}{\partial t} = -E \cdot J \quad (2.23) \]

and the Poynting vector defines as

\[ S = E \times H \quad (2.24) \]

### 2.1.4 Phasor form Maxwell’s equations

Since we are often interested in electromagnetic waves, it helps to simplify Maxwell’s equations for the special case of time-harmonic fields. These fields, whose only time dependence is sinusoidal, may be represented in the phasor domain just like scalar time-harmonic functions. For example, we may write the relationship between time harmonic electric field, \( \vec{E}(\vec{r}, t) \) and its phasor field \( \vec{E}(\vec{r}) \) as shown below

\[ \vec{E}(\vec{r}, t) = \text{Re} \left\{ \vec{E}(\vec{r}) e^{j\omega t} \right\} \quad (2.25) \]

where \( \omega \) is the angular frequency. The phasor representation simplifies the analysis of electromagnetic waves by removing the time dependency from all the field components. So now we wrote all of the point-form Maxwell’s equations in phasor form by taking advantage of the fact that time-differentiation becomes a simple \( j \omega \) multiplication in the phasor domain as shown in equations (2.26-2.29) below.
\[ \nabla \times \vec{E}(r) = -i\omega \vec{B}(r) \quad (2.26) \]

\[ \nabla \cdot \vec{D}(\vec{r}) = q(\vec{r}) \quad (2.27) \]

\[ \nabla \times \vec{H}(\vec{r}) = i\omega \vec{D}(\vec{r}) + \vec{J}(\vec{r}) \quad (2.28) \]

\[ \nabla \cdot \vec{B}(\vec{r}) = 0 \quad (2.29) \]

The space and material through which a wave travels constitute the medium of propagation. There are countless types of propagation media such as isotropic, anisotropic, and dispersive and many more but we are only covering here a simple media. A simple medium is said to be linear, isotropic, and homogenous and source free regions. Each one of them will be discussed briefly and separately in section 2.5, [6], [9], [8].

2.2 Wave Equation

Even-though there are many techniques for solving Maxwell’s equation, in this thesis we are primarily interested in solving the differential Maxwell’s equation in the temporal transform domain. Even with the time derivatives eliminated, Maxwell’s equation still shows complicated set of two first order, vector, and coupled partial
differential equation. These equations can be decoupled in several ways as considered in
the next sections.

### 2.2.1 Vector wave equation

In this section we will develop the basic vector wave equations used in all
electromagnetics and describe how waves propagate in free space. We will drive the
vector wave equation from the phasor form of Maxwell’s equation in a linear, isotropic
and homogenous medium known as simple medium [6], [4].

If we take the curl of equation (12) each side we get as shown below

\[
\nabla \times \nabla \times \vec{H}(\vec{r}) = i\omega \varepsilon \nabla \times \vec{E}(\vec{r})
\]

\[
= i\omega \varepsilon \left[ -i\omega \mu \vec{H}(r) \right] 
\]

\[
= \omega^2 \varepsilon \mu \vec{H}(r)
\]

\[
= k^2 \vec{H}(r)
\]

\[
\nabla \times \nabla \times \vec{H}(\vec{r}) - k^2 \vec{H}(r) = 0
\]

where \( k^2 = \omega^2 \varepsilon \mu \) \hspace{1cm} (2.31)

Also if we take the curl of equation (2.36) each side and do the same manipulations we
get the same vector wave equation but this time in terms of electric field as shown below
\[ \nabla \times \nabla \times \vec{E}(r) = -i \omega \mu \nabla \times \vec{H}(r) \]
\[ = -i \omega \mu \left[ i \omega \varepsilon \vec{E}(\vec{r}) \right] \]
\[ = \omega^2 \varepsilon \mu \vec{E}(\vec{r}) \]
\[ = k^2 \vec{E}(\vec{r}) \]

\( \nabla \times \nabla \times \vec{E}(r) - k^2 \vec{E}(\vec{r}) = 0 \)

where \( k^2 = \omega^2 \varepsilon \mu \)

Equation (2.41) and equation (2.43) can be written in compact form as follows

\[ \left( \nabla \times \nabla \times -k^2 \right) \left[ \begin{array}{c} \vec{E} \\ \vec{H} \end{array} \right] = 0 \]  \( (2.34) \)

The vector wave equation can be thought of as one big differential operator: a double-curl operation with a constant subtracted from it. This operates on both \( \vec{E} \) and \( \vec{H} \) and for all time-harmonic fields in a simple medium.

**2.2.2 Scalar wave equation**

In this section we will drive the scalar wave equation by simplifying the vector wave equation. First, we will apply the following vector calculus identity:
The above identity shows that any double-curl operation on a vector may be rewritten as a gradient-divergence and a Laplacian operation. Therefore, we will use this relationship to fields in a simple medium which states that the divergence of $\vec{E}$ and $\vec{H}$ are going to evaluate to be zero and that simplifies the complexity that comes with it [6].

\[
\nabla \cdot \left[ \vec{E} \right] = 0 \quad (2.36)
\]

This leads to the following

\[
\nabla \times \nabla \times \vec{E} = \nabla \left( \nabla \cdot \vec{E} \right) - \nabla^2 \vec{E} = \nabla^2 \vec{E} \quad (2.37)
\]

where $\left( \nabla \cdot \vec{E} \right) = 0$

So if we rewrite equation (2.44) we get the following scalar wave equation known as also Helmholtz equation.

\[
\left( \nabla^2 + k^2 \right) \left[ \vec{E} \right] = 0 \quad (2.38)
\]
2.3 Electromagnetic Waves in Free Space

In free space meaning no charges and currents Maxwell’s equations have a trivial solution in which all the fields vanish. However, there are also non-trivial solutions with considerable practical importance. In general, it is difficult to write down solutions to Maxwell’s equations, because two of the equations involve both the electric and magnetic fields. However, by taking additional derivatives, it is possible to write equations for the fields that involve only either the electric or the magnetic field. This makes it easier to write down solutions [6], [10].

\[ \nabla \cdot \vec{E} = 0 \]  \hspace{1cm} (2.39)

\[ \nabla \cdot \vec{B} = 0 \]  \hspace{1cm} (2.40)

\[ \nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \]  \hspace{1cm} (2.41)

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]  \hspace{1cm} (2.42)

Our aim is to obtain a form of the equations in which the fields E and B are separated. To do so we take the curl of both sides of equation (2.52), and interchange the
order of differentiation on the right hand side which we are allowed to do, since the space
and time coordinates are independent. We found as shown below:

\[ \nabla \times \nabla \times \vec{E} = -\frac{\partial}{\partial t} \nabla \times \vec{B} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \] (2.43)

\[ \Delta \vec{E} = \varepsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2} \] (2.44)

\[ \Delta \vec{B} = \varepsilon \mu \frac{\partial^2 \vec{B}}{\partial t^2} \] (2.45)

Where:

\[ \nabla \times \nabla \times \vec{E} \equiv \nabla \left( \nabla \cdot \vec{E} \right) - \Delta \vec{E} \] (2.46)

And \( \Delta \equiv \nabla^2 \)

Equation (2.44-45) is the wave equation in three spatial dimensions. Note that each
component of the electric field is independently satisfied the wave equation. Below one
dimensional wave equation which is a plane wave is shown.

\[ E(r) = Ae^{i\vec{k} \cdot \vec{r}} + Be^{-i\vec{k} \cdot \vec{r}} \] (2.47)
\[ H(r) = Ce^{i\mathbf{k} \cdot \mathbf{r}} + De^{-i\mathbf{k} \cdot \mathbf{r}} \]  \hspace{1cm} (2.48)

where:

\[ k = \omega \cdot c_0 = \omega \sqrt{\varepsilon \mu} \]

### 2.3.1 Plane wave in free space

As we discussed earlier, taking the second derivatives of Maxwell’s equations allows us to decouple the equations for the electric and magnetic fields. To do so we substituted the expressions for the fields of equations (2.57 and 2.58) into equation (2.49 and 2.50) and we obtain as shown below.

\[ \mathbf{k} \cdot \mathbf{E}_0 = 0 \]  \hspace{1cm} (2.49)

\[ \mathbf{k} \cdot \mathbf{B}_0 = 0 \]  \hspace{1cm} (2.50)

Since \( \mathbf{k} \) represents the direction of propagation of the wave, we see that the electric and magnetic fields must at all times and all places be perpendicular to the direction in which the wave is travelling and we see that the quantities \( \mathbf{k} \) appearing in equation (2.49 and 2.50) must be the same in each case. Also, we have the following relations between the magnitudes and directions of the field [4], [10].
\[ \vec{k} \times \vec{E}_0 = \omega \vec{B}_0 \] (2.51)

\[ \vec{k} \times \vec{B}_0 = -\omega \vec{E}_0 \] (2.52)

By choosing a coordinate system so \( \vec{E}_0 \) is parallel to the x axis and \( \vec{B}_0 \) is parallel to the y axis, then \( \vec{k} \) is forced to be parallel to the z axis since they have be orthogonal to each other at all time which satisfies this vector product \( \vec{E}_0 \times \vec{B}_0 \). \( \vec{k} \) is the propagation vector which shows the direction that the wave is propagating as shown in figure 2.2 below. The magnitudes of the electric and magnetic fields are related by the speed of light as shown in equation (2.53) below.

![Figure 2.2 Transfer electric and magnetic field (TEM) propagation in x-direction.](image-url)
\[ \frac{|\vec{E}|}{|\vec{B}|} = c \]  

(2.53)

2.3.2 Propagation of electromagnetic waves in free space

There are two basic ways of transmitting and receiving electromagnetic waves through either a guided or unguided medium. Guided mediums such as coaxial cables and fiber optics is much safer when it comes to conveying information carrying EM signal compared with the unguided medium. As EM wave travels through free space, it goes through different kinds of propagation effects such as reflection, diffraction and scattering, due to the presence obstructions such as buildings, mountains and so on which makes the signal to fade. Reflection occurs when the EM waves is incident on objects which greater than the wavelength of the traveling EM wave. Scattering occurs when the medium through the wave is traveling contains objects which are much smaller than the wavelength of the EM wave. These different phenomena’s due to the different frequency leads to propagation loses [6], [11].

2.3.2.1 Reflection

Reflection occurs when an electromagnetic wave falls on an object, which has very large dimensions as compared to the wavelength of the propagating wave. For example, such objects can be large like the earth, buildings and walls. Whenever EM wave incident upon another medium with different electromagnetic properties, part of it is transmitted into it, while the other part is reflected back to where it come from. Let’s
say the EM wave is incident on dielectric medium, then some energy is reflected back and some energy is transmitted. But if the medium is a perfect electric conductor known as (PEC), all energy is reflected back to the first medium meaning the reflection coefficient is $|\Gamma| = -1$. Therefore, the amount of energy that is reflected back always depends on medium which wave strikes, the polarization of the EM wave and incident angle as shown below [11], [12].

$$\theta_r = \theta_i$$ (2.54)

$$\Gamma = \frac{E_r}{E_i} = \frac{\eta - \eta_0}{\eta + \eta_0}$$ (2.55)

Figure 2.3 Reflection affects polarization.

2.3.2.2 Scattering
Scattering generally occurs when electromagnetic (EM) waves encounters an object with different electromagnetic properties such conductivity (σ), permittivity (ε) and permeability (μ) and bounces off in different directions. The amount of scattering always depends on the frequency, polarization, aspect angle, the size and the structure of the object as we stated earlier. As shown below there are three regions of scattering [13].

- Rayleigh Region ($2\pi a/\lambda < 1$): The RCS is inversely proportional to the 4th power of the wavelength as indicated by the slope of the curve in that region.
- Mie Region ($1 < 2\pi a/\lambda < 10$): In this resonance region, a creeping wave travels around the sphere and back towards the receiver where it interferes constructively or destructively with the specular backscatter to produce a cyclical variation in the RCS.
- Optical Region ($2\pi a/\lambda > 10$): The RCS of the sphere approaches its geometric projected area $2\pi a$

Figure 2.4 Sphere scattering energy all directions
2.3.2.3 Diffraction

Diffraction is a phenomena which occurs when the wave interacts with a surface having sharp irregularities or it incident upon an obstacle and they tend to travel around them [11]. This means the signal is still received from the transmitter though it is blocked by large object between transmitter (Tx) and receiver (Rx). This mostly happens at the lower frequency meaning longer wavelength. To understanding it better we have to look at Huygens’s principle which says each point of spherical wave front can be considered as a source of secondary wave front. And this is the reason why signals with longer wavelengths are able to provide coverage houses on the other side of the hills or mountains even though there is a shadow zone.

2.4 Uniqueness Theorem

Whenever we solve a problem it is good to know the problem has unique solution. Then if that is case we would love to know under what condition is this possible. Normally electromagnetic field (E, H) in a lossy medium is uniquely quantified by its sources \((J_s, M_s)\) within the medium plus (1) the tangential component of the electric field over the boundary, or (2) the tangential component of the magnetic field over the boundary, or (3) the former over part of the boundary and the latter over the rest of boundary. Here M is the magnetic current density which its existence is derived from
\[ M_s = -\hat{n} \times E_2 \], where \( E \) is the electric field on a surface and \( \hat{n} \) is the normal vector on that surface as shown in figure 2.5 below [6].

For the actual problem on the left-hand side, if we are interested only to find the fields \((E_2, H_2)\) outside \( S \) \((V_2)\), we can replace region \((V_1)\) with a perfect conductor as on the right-hand side which makes the fields inside it to be zero. We also need to place a magnetic current density \( M_s = -\hat{n} \times E_2 \) on the surface of the perfect conductor in order to satisfy the boundary condition on \( S \). Now for both the actual problem and the equivalent problem, there are no sources inside \((V_2)\). In the actual problem, the tangential component of the electric field at the outside of \( S \) is \(-\hat{n} \times E_2\). In the equivalent problem, the tangential component of the electric field at the outside of \( S \) is also \(-\hat{n} \times E_2\) as a magnetic current density \( M_s = -\hat{n} \times E_2 \) has been specified on \( S \) already.
By using the uniqueness theorem, the fields \((E_2, H_2)\) in \((V_2)\) in the equivalent problem will be the same as those in the actual problem [6], [7], [10].

Note that the requirement for the fields inside \((V_1)\) is to be zero is to meet the boundary condition specified on the tangential component of the electric field across \(S\). Because now in the equivalent problem just outside \(S\), the electric field is \(E_1\) while there is also a magnetic current density \(M_s\) [6]. But just inside \(S\), the electric field is zero. Therefore on \(S\) we have as follows,

\[
M_s = -\hat{n} \times E_2 = -\hat{n} \times (E_2 - 0)
\]
\[
J_s = -\hat{n} \times H_2 = \hat{n} \times (H_2 - 0)
\]  

\[2.56\]

2.5 Simple Medium

In this section we are going to describe a linear, isotropic, and homogeneous and source free medium known as simple medium. A medium is called linear when the flux density vectors are proportional to their corresponding field components. Electric flux density, \(\vec{D}\) and electric field, \(\vec{E}\), must be proportional to one another in order to be linear medium as shown in equation (2.69). Also, magnetic flux density, \(\vec{B}\) and magnetic field \(\vec{H}\), must also be proportional to one another as shown in equation (2.68)

A medium is said to be isotropic if its material properties do not vary as a function of field orientation otherwise it is called anisotropic. The linearity and isotropy of a
material is characterized by two scalar constants known as permittivity (\( \varepsilon \)) and permeability (\( \mu \)) and mathematically is written as follows [6].

\[
\vec{D} = \varepsilon \vec{E} \quad \text{(2.57)}
\]

\[
\vec{B} = \mu \vec{H} \quad \text{(2.58)}
\]

A medium is called homogeneous when it is material properties is not function of space otherwise it is inhomogeneous. This simple media is an ideal one, since we are assuming that the material has electromagnetic properties across the space.

A medium is known source-less when there is no electromagnetic charges or currents and mathematically written as shown below [5], [6].

\[
\vec{J} = 0 \quad \text{(2.59)}
\]

\[
q = 0 \quad \text{(2.60)}
\]
2.6 Long and Thin Metallic Object (2D problem)

In this section we will discuss scattering from thin metallic object. Therefore let’s consider a plane wave incident upon a cylinder with negligible radius \( r \ll \lambda \) with infinite length along \( z \) axis as shown in figure 2.7. The incident plane wave is linearly polarized in the \( y-z \) plane with an angle \( \theta \) with respect to the \( z \) axis. Since the long dimension of the target is along the \( z \)-axis, the three dimensional problem is reduced to two dimensional problem. And if the cross sectional area of the target and source is independent in the \( z \)-direction then the field solution is also independent of \( z \)-direction. Therefore, the problem can be decomposed as linear superposition of two orthogonal components. First one corresponds to the part of the incident field parallel to the cylinder and the second one corresponds to the orthogonal component. In another wards this problem is decomposed into two cases: \( \text{TM}_z \) and \( \text{TE}_z \). In \( \text{TM}_z \), the magnetic field intensity is confined to the \( x-y \) plane and it is transverse to the \( z \)-axis. In \( \text{TE}_z \) is transverse electric field where the current source flows in the \( x-y \) plane as shown in figure 2.6. Both \( \text{TM}_z \) and \( \text{TE}_z \) are discussed below [14].

When the cylinder is made up of Perfect Electric Conductor, the electromagnetic field inside the cylinder is zero according to the boundary condition and a surface current density \( \mathbf{J}_S \) is present on the scatterer surface \( S \). Moreover when the object is illuminated by incident field \( \mathbf{E}' \) the total electric field outside the cylinder is given by equation (2.72).

Since the presence of the cylinder doesn’t affect the field radiated by \( \mathbf{J}_S \) outside the scatterer, the total electric field \( \mathbf{E}' \) can be written as:
\[ E' = E' + E^s = E' + j \omega \mu_b \int_S J_S(r') \cdot \mathbf{G}(r', r')dr' \] (2.61)

Which in the inverse scattering problem represents an integral equation having \( S \) and \( J \) as the unknown to be determined from through the inversion technique.

Figure 2.6 (a) TM polarization incidence. (b) TE polarization incidence.

In figure 2.6 two cases are shown: (a) parallel polarization where the electric field is parallel to the plane of incidence and (b) perpendicular polarization where the electric field is orthogonal to the plane of incidence. These two waves are independent. In fact,
for two dimensional problems the electromagnetic field can separated into two independent waves such as TM and TE as follows.

Let’s take a look a specific example where the incident wave is striking the cylinder with an angle relative to z-axis and the cylinder is oriented along the z-axis as the figure 2.7 is shown. From this we can take a look both TM and TE mode and compare their magnitude.

Figure 2.7 Describes what the scattering problem looks like.

2.6.1 TM excitation mode

Consider a perfect conducting cylinder (PEC) excited by impressed electric field as shown in figure 2.6a. TM mode has only one component which corresponds to the part that of the incident field that is oriented along the z axis as shown in figure 2.6a and the
wave is travelling in x-direction with a wave number \( k = \frac{2\pi}{\lambda} \). The impressed electric field induces surface currents due to the electric field parallel to the axis of the cylinder exerting a force which in turn produces scattered electric field. In this mode the magnetic field H lies on the x-y plane and the incident field as follows:

\[
\mathbf{E}_i = \mathbf{\hat{z}} E_0 e^{-j k x} \quad (2.62)
\]

The incident wave can be expanded in cylindrical coordinate in order to employ the imposed boundary condition as follows:

\[
\mathbf{E}_i = \mathbf{\hat{z}} E_0 \sum_{n=-\infty}^{\infty} j^n J_n(k \rho) e^{j n \phi} \quad (2.63)
\]

where \( J_n \) denotes a Bessel function of first kind of order n. This incident wave generates a scattered wave and we assume the scattered wave has the same polarization with the incident field which travels outward and then it looks like as follows [5], [6], [14], [15].

\[
\mathbf{E}^s = \mathbf{\hat{z}} E_0 \sum_{n=-\infty}^{\infty} c_n H_n^{(2)}(k \rho) \quad (2.64)
\]
Here the H with the superscript (2) is the Hankel function of the second kind and has the following form

\[ H_a^{(2)}(x) = J_a(x) - iY_a(x) \]  \hspace{1cm} (2.65)

Imposing a boundary condition on the surface of the PEC cylinder would allow us to determine the value of the coefficient \( c_n \). And the total tangential electric field on the surface of the PEC cylinder with radius \( a \) must be equal to zero:

\[ \hat{E}' = \hat{E}^i + \hat{E}^e = 0 \bigg|_{\rho = a} \]

\[ \Rightarrow \hat{z}E_0 \sum_{n=-\infty}^{\infty} \left[ j^{-n} J_n(k\rho)e^{jn\phi} + c_n H_n^{(2)}(k\rho) \right] = 0 \]  \hspace{1cm} (2.66)

Where:

\[ c_n = -j^{-n} \frac{J_n(ka)e^{jn\phi}}{H_n^{(2)}(ka)} \]  \hspace{1cm} (2.67)
Physically, the Bessel function $J_n(k\rho)$ represents standing waves whereas the Hankel function of second kind $H_n^{(2)}(k\rho)$ represents the outgoing wave and propagates $+\rho$ direction.

The above equation leads to the scattered field to become as follows.

$$\mathbf{E}^s = \hat{z}\mathbf{E}_0 \sum_{n=-\infty}^{\infty} -j^n J_n(ka)e^{jn\phi} H_n^{(2)}(ka) - H_n^{(2)}(k\rho) (2.68)$$

The total magnetic field on the surface of the PEC cylinder is:

$$\mathbf{H}^t = -\frac{1}{j\omega\mu_0} \nabla \times \mathbf{E}^s (2.69)$$

Since the total electric field only has components in the $z$ direction, the total magnetic field will have two components in the $\rho$ and $\phi$ directions. And only the tangential $\phi$ component will contribute to the most of the induced current density [14], [15].
\[ \mathbf{J}_{TM} = \hat{n} \times \mathbf{H}' \]
\[ = \hat{\rho} \times (\hat{\rho} H'_\rho + \hat{\phi} H'_\phi) \]

*when \( \rho = a \)*
\[ = \hat{\rho} \times \hat{\phi} H'_\phi = \hat{\mathbf{z}} H'_\phi \]  

(2.70)

The induced current is parallel to both the incident and the scattered field as equation (2.71) shows. After little manipulation we get the flowing form of induced current:

\[ \mathbf{J}_{TM} = \hat{\mathbf{z}} \frac{2E_0}{a \pi \omega \mu_0} \sum_{n=-\infty}^{\infty} -j^{-n} \frac{e^{jn\phi}}{H_n^{(2)}(ka)} \]  

(2.71)

For given frequency and radius of the cylinder, the magnitude of the induced current depends on only \( \phi \) which is the position around the cylinder. And if the radius of the cylinder is so small the summation of equation (2.72) drops and the current is equal to:

\[ \mathbf{J}_{TM} \approx \hat{\mathbf{z}} \frac{2E_0}{a \pi \omega \mu_0 H_0^2(ka)} \]  

(2.72)

2.6.2 TE excitation mode
At this time let the (PEC) cylinder is excited by impressed electric field as shown figure 2.6b. The TE mode, the magnetic field $H$ has only one component which corresponds to the part that of the incident field that is oriented along the $z$ axis as shown in figure 2.6b and the wave is travelling in $x$-direction with a wave number $k = \frac{2\pi}{\lambda}$ and the electric field $E$ lies on the $x$-$y$ plane and the incident magnetic field is [14], [15].

\[
H^i = \hat{z} H_0 e^{-j k x} = \hat{z} H_0 \sum_{n=-\infty}^{\infty} j^{-n} J_n(k \rho) e^{j n \phi}
\]  

(2.73)

This results a scattered magnetic field of:

\[
H^s = \hat{z} H_0 \sum_{n=-\infty}^{\infty} c_n H_n^{(2)}(k \rho)
\]  

(2.74)

Where

\[
c_n = -j^{-n} \frac{J_n'(ka) e^{j n \phi}}{H_n^{(2)}(ka)}
\]  

(2.75)
The tangential electric field has only $\phi$ component:

$$E^s = -\frac{1}{j\omega \mu_0} \nabla \times H^s \quad (2.76)$$

Thus, the total magnetic field on the surface of the cylinder is

$$H' = H^i + H^s = 0 \bigg|_{\rho=a}$$

$$\Rightarrow -\hat{\mathbf{\hat{\mathbf{j}}}} \frac{2H_0}{\pi k a} \sum_{n=-\infty}^{\infty} j^{n} e^{in\phi} \left[ \frac{e^{in\phi}}{H_n^{(2)}(ka)} \right] \quad (2.77)$$

The induced current on the surface of the cylinder is:
\[ \mathbf{J}_{TE} = \hat{n} \times \mathbf{H}' \]
\[ = \hat{\rho} \times \hat{z} \mathbf{H}'_z \]
\[ when \ \rho = a \]
\[ = -\hat{\phi} \mathbf{H}'_{\phi} \]
\[ = \hat{\phi} \frac{2\mathbf{H}_0}{\pi k a} \sum_{n=-\infty}^{\infty} \left[ j^{-n} \frac{e^{in\phi}}{H_n^{(2)}(ka)} \right] \]

Figure 2.8 Comparison between TM and TE impressed currents.

Comparing between the two allows us to see the significant difference between the two impressed current. In the TM case is larger than the magnitude of the impressed current in the TE case as shown in figure 2.8 which makes the TM excitation dominant.
This means that when the plane wave hits on the thin cylinder at an angle with respect to the axis of the cylinder, the impressed current will be mostly aligned with the cylinder itself. Therefore, the scattered field also becomes aligned with the cylinder due to a depolarization effect [14], [15].

### 2.7 General Properties of Near Field and Far Field

Since it always depends on the distance between the observer and the source, now let’s see the common cases such as the near field \( (k |\vec{r} - \vec{r}'| << 1) \) and the far field \( (k |\vec{r} - \vec{r}'| >> 1) \). In the far field, we have \(|\vec{r} - \vec{r}'| \approx \vec{r} - \hat{r} \cdot \vec{r}'\). The electric field becomes as [5], [6], [8], [16]

\[
\vec{E}(\vec{r}) = j \omega \mu \left[ \vec{I} + \frac{\nabla \nabla}{k^2} \right] \cdot \frac{e^{jkr}}{4\pi r} \int_{S} J(\vec{r}') e^{jk \vec{r}' \cdot \vec{r}} d\vec{r}'
\] (2.79)

The integral in equation (2.80) results in a function that depends only on \( \theta \) and \( \phi \). We can define a vector current moment as:

\[
\vec{f} (\theta, \phi) = \int_{S} J(\vec{r}') e^{jk \vec{r}' \cdot \vec{r}} d\vec{r}'
\] (2.80)
In the far field region, we only keep terms on the order of \( \frac{1}{kr} \) and neglect all the higher order terms. Using:

\[
\nabla = \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\varphi} \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \varphi}
\]

(2.81)

Then equation (2.80) becomes as follows:

\[
\tilde{E}(\vec{r}) = j \omega \mu \left[ \hat{I} + \hat{\varphi} \right] \cdot \frac{e^{jkr}}{4\pi r} \tilde{f}(\theta, \phi)
\]

(2.82)

This is an outgoing wave with a spherical wave front. The electric field is perpendicular to the propagation direction. At large distance, the wave can be approximated by a plane wave [16].

In the near field, we can approximate as \( e^{jk|\vec{r} - \vec{r}'|} = 1 + jk|\vec{r} - \vec{r}'| + \ldots \approx 1 \) and when the distance between observer and the source is smaller than the wavelength of the operating frequency we found as shown below [5], [6], [8], [16].
This is a quasi-static field because of the absence of the oscillating exponential term. Note the field is not truly static since there is an implicit time harmonic factor $e^{j\omega t}$. Similarly, the magnetic field is \cite{6}, \cite{16}:

\[
\vec{H}(\vec{r}) = \nabla \times \int_{S} \frac{J(\vec{r}')}{4\pi |\vec{r} - \vec{r}'|} d\vec{r}'
\] (2.84)

Because in near field region, we have $\left( k |\vec{r} - \vec{r}'| << 1 \right)$, the contribution from the second term in the parenthesis of equation (2.84) dominates and the magnetic field can usually be neglected because magnetic field gets differentiation once while the electric field gets differentiation twice. When magnetic field is neglected, the solution of fields satisfies the electrostatic equation or the Poisson equation as shown below \cite{4}, \cite{16}.

\[
\nabla \cdot E_0 = 0
\] (2.85)
\[ \nabla^2 \phi = 0 \]  \hspace{1cm} (2.86)

2.8 Derivation of Magnetic (MFIE) and Electric Field Integral Equation (EFIE)

When electromagnetic radiation is incident upon a dielectric or metallic object, it induces a current density in the material due to the force it exerts. This current density generates electromagnetic radiation, which is observed as the scattered field by the object and it can be determined via solving on of the two integral equations known as electric field integral equation (EFIE) and magnetic field integral equation (MFIE) [17]. Inside the medium, the radiation from the current density adds to the incident field, giving the refracted field. One approach for solving several such scattering problems is the boundary condition method, which is also referred to as the differential equation method. For the material, a constitutive equation relates the current density to the local electric field [5], [6], [18].

Figure 2.9 PEC object illuminated
In figure 2.9 when electromagnetic field hits perfect electric conductor (PEC), it induces a surface current density which in turn produces a radiated field. The unit normal vector \( \hat{n} \) at point \( r \) is directed from the medium towards free space, and the origin of coordinate’s can be chosen arbitrarily. The Magnetic Field Integral Equation relates the surface current density \( J(r') \) at all other points \( r \). The magnetic component of this field will be written as

\[
B(r,t) = \text{Re}\left[ B(r')e^{-j\omega t} \right] \tag{2.87}
\]

With incident magnetic field \( B_i(r) \) the complex amplitude, and all time dependent fields will have the same time dependence, in particular \( J(r,t) \). The incident field illuminates the material, as illustrated in Figure 1, and a current density \( J(r) \) is induced in the surface at the point \( r \). For this situation, an integral equation for \( J(r) \) can be derived, which has incident magnetic field \( B_i(r) \) as inhomogeneous term [6], [18].

Let an electromagnetic field with angular frequency \( \omega \) be incident upon an object with a surface \( S \) of arbitrary shape, as in Figure 1. For perfectly-conducting material, all induced current is surface current \( J(r) \). The magnetic field \( B(r) \) at point \( r \) can then be written as
Magnetic field integral equation (MFIE)

\[
B(r,t) = B^i(r) - \frac{\mu_0}{4\pi} \int j(r) \times \nabla g(r-r')ds'
\] (2.88)

where the incident magnetic field \(B^i(r)\) is given. The second term on the right-hand side equals the magnetic field generated by the current, with the free-space Green's function for the Helmholtz equation shown below [6], [18].

\[
g(r-r') = \frac{e^{jk|r-r'|}}{|r-r'|} \] (2.89)

\[
B(r) = \mu_0 J(r) \times \hat{n}(r) \] (2.90)

Which is the usual boundary condition for an interface carrying a surface current density \(J(r)\). When the material is a perfect conductor, the electromagnetic field inside vanishes and equation (2.91) can be written as

\[
\hat{n} \times B^i(r) = J(r) + \frac{\mu_0}{4\pi} \times \hat{n} - \frac{\mu_0}{4\pi} \int J(r) \times \nabla g(r-r')ds'
\] (2.91)
After taking the cross product with \( \hat{n} \). This integral equation for the unknown current density \( I(r) \) is the Magnetic Field Integral Equation. MFEI is the Fredholm integral equation of the second kind which the unknown is present both inside and outside the kernel as shown in equation (2.91) and this is good for structures with large smooth surface [18].

The integral equation that couples the incident electric field to induced surface current is known the electric field integral equation (EFIE) and it is good for specific structures such as thin wire structures of small conductor volume. The EFIE and MFIE are both derived from the Statton Chu formula which states that for points on the surface [19] [18].

Knowing magnetic field integral equation (MFIE), one can derive the electric field integral equation or vise verse by employing the relationship between them.

Therefor:

\[
\begin{align*}
E(r_+) = \frac{\omega \mu_0}{k} \hat{n} \times H(r_+) = \eta_0 \hat{n} \times H(r_+) \quad (2.92)
\end{align*}
\]

Where:

\[
\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}
\]
2.9 Scattering Problems

In general, electromagnetic scattering problems can be characterized into two groups: (A) direct scattering problems which the material properties and source are given and the goal is to find the field radiated and (B) inverse scattering problems which involve finding the material properties and the shape of unknown scatterer from measurement of fields scattered by scatterer. Antenna and radar wave bouncing off a target is considered as direct scattering problem. Medical imaging of biological tissues using microwave or ultrasound is examples of inverse scattering problems. This thesis is based on inverse problem and quite challenging to retrieve the unknown reflectivity function through inverse technique since there are many similar scatterers that scatter an incident field almost the same way [14], [20].

2.9.1 Electromagnetic scattering and material characterization

The scattering of the electromagnetic field by material objects is usually used for the determination of the material electromagnetic parameters [20]. Exposing a material sample to an incident electromagnetic field leads to the excitation of time-varying electric and magnetic polarizations within the material as well as a conduction current if free charges exist. Time-varying polarizations result in what is called polarization currents. The polarization currents and conduction current radiate back an additional electromagnetic field into both the material sample and its surrounding space, which is known as the scattered field. Being solely generated by the material polarization and possibly conduction currents, the scattered field carries information about the target’s
electromagnetic properties and their representative parameters. Such information can be used for the determination of those parameters if the scattered field is properly measured
[6], [8], [20], [20].

2.9.2 Inverse scattering problems

The inverse scattering problem is that of determining the nature of the target from the knowledge of the scattered field as we stated earlier. Figure 4.3 shows a target illuminated by a number of transmitters. The scattered field is measured by the receivers both in bistatic and monostatic modes. The task is to determine the shape and type of the target under test [14], [21].

The methods used for the inverse problem depend on the electrical size of the inhomogeneity. If D is the characteristic dimension of the scatterer and k is the wavenumber, the quantity k*D gives a measure of electrical length of the target. When k*a << 1, scattering is weak, and we may apply low frequency methods such as Rayleigh and Born approximations. On the other hand, when k*D >>1, we may use high frequency asymptotic techniques such as geometrical or physical optics methods [21].
CHAPTER 3

DYADIC GREEN’S FUNCTION IN TERMS OF ELECTRIC FIELD

3.1 Green’s Function

Green’s function is one of the very useful tools for solving problems in electromagnetics. It has the same meaning and importance for electromagnetics as the impulse response for circuits. Green’s function describes the response of an electromagnetic system to a delta function source. Once the Green’s function is found, the response to an arbitrary excitation is obtained through the superposition principle. At the bottom we will derive the 3D free space Green’s function and we will see why it is important shortly [22].

We begin when the source is assumed to be at origin of coordinates and the Green’s function is expressed as follows.

\[
\left[ \nabla^2 + k^2 \right] G_0(r) = -\delta(r) \quad (3.1)
\]

Since the free space is assumed, \( G(r, r') \) is only function of \( r \) due to the symmetry. By using the Laplacian in the spherical coordinates
\[ \nabla^2 \phi = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d \phi}{dr} \right) + \frac{1}{r^2 \sin(\theta)} \left( \sin(\theta) \frac{d \phi}{d \theta} \right) + \frac{1}{r^2 \sin^2(\theta) d \theta} \left( \frac{d^2 \phi}{d \phi} \right) \]  

(3.2)

So if we rewrite the equation (3.2) in spherical form we get as follows:

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d G_0(r)}{dr} \right) + k^2 G_0(r) = -\delta(r) \]  

(3.3)

Since the right side of equation (3.3) is zero except the origin both sides of the equation are multiplied by \( r \) and we found as shown below

\[ \frac{d}{dr} \left( r G_0(r) \right) + k^2 \left( r G_0(r) \right) = 0 \]  

(3.4)

The above equation has the following solution

\[ G_0(r) = C \frac{e^{-jk r}}{r} \]  

(3.5)

where \( C \) is an arbitrary constant and the \(+r\) is chosen for outgoing wave from the origin.
To solve the constant $C$ we plug the equation (3.5) into equation (3.3) back and take the integral on both sides with small sphere as $[6], [8]:$

\[
\int_{V} \nabla^2 G dV = \int_{dV} \nabla G \cdot dS
\]

\[
= 4\pi^2 \hat{r} \cdot \nabla G
\]

\[
= -4\pi C r^2 \left( \frac{e^{-jkr}}{r^2} - \frac{jkr e^{-jkr}}{r} \right)
\]

(3.6)

\[
\int_{V} k^2 G dV = 4\pi k^2 C \left[ -\frac{e^{-jkr}}{jk} + \frac{1}{k^2} \left( e^{-jkr} - 1 \right) \right]
\]

(3.7)

By taking the limit as $r \to 0$ we obtain the arbitrary constant to be as

\[
C = \frac{1}{4\pi i}
\]

And when the source is located somewhere other than the origin we found the 3D Green’s function to be as:
\[ G_0(\vec{r},\vec{r}') = \frac{e^{-jk|\vec{r} - \vec{r}'|}}{4\pi |\vec{r} - \vec{r}'|} \]  

(3.8)

### 3.2 Vector Potentials

In this section we will talk about vector potentials briefly. Solving radiation integral directly is quite challenging in most cases. Therefore, we will derive set of auxiliary vector potentials which we will be using to solve for the radiated fields in this thesis especially the next section which will covered the radiation integral. As we will see later on, these potentials are acquired through integrals of currents which then the radiated field is found from the potentials as shown in figure 3.1 below. There are two types of vector potentials such as magnetic vector potential and electric vector potential which cover them below [6].

#### 3.2.1 Magnetic vector potential

In here we will derive the magnetic vector potential for a homogeneous and source free region as follows:

There is no isolated magnetic charges
\[ \nabla \cdot \hat{\vec{B}} = 0 \]  

(3.9)

\[ \vec{B} = \nabla \times \vec{A} \]  

(3.10)
\[ \nabla \cdot \left( \nabla \times \vec{A} \right) = 0 \quad (3.11) \]

We know from Maxwell’s equation that:

\[ \nabla \times \vec{E} = -j \omega \vec{B} \quad (3.12) \]

\[ \nabla \times \vec{B} = \mu \vec{J} + j \omega \mu \varepsilon \vec{E} \quad (3.13) \]

Substituting the equation (3.10) into equation (3.12) we get

\[ \nabla \times \vec{E} = -j \omega (\nabla \times \vec{A}) \quad (3.14) \]

The above equation can be written as:

\[ \nabla \times (\vec{E} + j \omega \vec{A}) = 0 \quad (3.15) \]

By using the following identity as

\[ \nabla \times (-\nabla \Phi_e) = 0 \quad (3.16) \]

Writing equation (3.15) by using equation (3.16) we get as follows
\[
\vec{E} = -j\omega\vec{A} - \nabla\Phi_e \quad (3.17)
\]

where \( \Phi \) is electric scalar potential. Now we will use another identity as

\[
\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \quad (3.18)
\]

Taking the curl of (3.10) on both sides and we get as follows:

\[
\nabla \times \vec{B} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \quad (3.19)
\]

So now equating the right side of equation (3.13) with the right side of equation (3.19)
leads to as shown below

\[
\mu\vec{J} + j\omega\mu\varepsilon\vec{E} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \quad (3.20)
\]

Plugging equation (3.17) into equation (3.20) and then rearranging it leads as follows

\[
\nabla^2 \vec{A} + k^2 \vec{A} = -\mu\vec{J} + \nabla(\nabla \cdot \vec{A} + j\omega\mu\varepsilon\Phi_e) \quad (3.21)
\]

By using the well-known Lorentz Gauge the magnetic vector potential and electric scalar
potential for time harmonic are coupled:
By plugging equation (3.22) into equation (3.21) will lead to as

\[ \nabla^2 \vec{A} + k^2 \vec{A} = -\mu J \]  

(3.23)

Equation (3.23) is not only an inhomogeneous vector Helmholtz equation of \( \vec{A} \) but it also relates the source to the magnetic vector potential. So now we can find the electric field everywhere in the free space through the following equation.

\[ \vec{E} = -j \omega A - \frac{j}{\omega \mu \varepsilon} \nabla (\nabla \cdot \vec{A}) \]

where

\[ \vec{A} = \mu \int_{s} \vec{J}'(\vec{r}') G(\vec{r}, \vec{r}') ds' \]  

(3.24)

3.2.2 Electric vector potential

We can derive the electric vector potential \( \vec{F} \) similar way by applying Maxwell’s equation as follows:
\[ \vec{E} = \frac{1}{\varepsilon} \nabla \times \vec{F} \quad (3.25) \]

\[ \Phi_m = -\frac{1}{j \omega \mu \varepsilon} \nabla \cdot \vec{F} \quad (3.26) \]

\[ \nabla^2 \vec{F} + k^2 \vec{F} = -\varepsilon \vec{M} \quad (3.27) \]

\[ H = -j \omega \vec{F} - \frac{j}{\omega \mu \varepsilon} \nabla (\nabla \cdot \vec{F}) \]

where

\[ \vec{F}(\vec{r}) = \varepsilon \int_S \vec{M}_s(\vec{r}') G_0(\vec{r}, \vec{r}') d\vec{s}' \quad (3.28) \]

### 3.3 Radiation Integral

Radiation integral also known as the Stratton-Chu integrals give the electric and magnetic field intensities at the observation point (P) for a known distribution of electric and magnetic currents \((J_s, M_s)\) [23]. Figure 3.1 shows the geometry current distribution. These currents can be volume currents \((J_v, M_v)\) or surface currents \((J_s, M_s)\). As usual the primed quantities are associated with the source \(J_s(\vec{r}')\) and unprimed quantities are associated with the observation point \((P)\) \([E(\vec{r}), H(\vec{r})]\).
Figure 3.1 The geometry of current distribution.

The electric vector potential and magnetic vector potential at the observation point is as follows:

\[ \vec{A}(\vec{r}) = \mu \int_S \vec{J}_s(\vec{r'}) G_0(\vec{r},\vec{r'}) d\vec{s}' \quad (3.29) \]

\[ \vec{F}(\vec{r}) = \varepsilon \int_S \vec{M}_s(\vec{r'}) G_0(\vec{r},\vec{r'}) d\vec{s}' \quad (3.30) \]

where:

\( G_0(\vec{r},\vec{r'}) \) is the Green’s function which covered or derived in the previous section

\( \vec{R} = \vec{r} - \vec{r}' \)

\( R = |\vec{R}| \)
The fields at the observation point (P) for the volume integrals are:

\[
\hat{E}(\vec{r}) = -j \omega \vec{A}(\vec{r}) - \frac{j}{\omega \mu \varepsilon} \nabla \left( \nabla \cdot \vec{A}(\vec{r}) \right) - \frac{1}{\varepsilon} \nabla \times \vec{F}(\vec{r}) \quad (3.31)
\]

\[
\hat{H}(\vec{r}) = -j \omega \vec{F}(\vec{r}) - \frac{j}{\omega \mu \varepsilon} \nabla \left( \nabla \cdot \vec{F}(\vec{r}) \right) - \frac{1}{\mu} \nabla \times \vec{A}(\vec{r}) \quad (3.32)
\]

### 3.4 Dyadic Green’s Function

In this section we describe the dyadic Green’s function and the reason is that most electromagnetic problems are vectorial in nature. Therefore, the need for extending to one-dimensional scalar Green’s function to multi-dimensional vectors is necessary. Vectors and dyadic are often used to describe linear transformations within a given orthogonal system and to simplify the mathematical manipulations [24], [25]

Consider an arbitrary time harmonic electric current distribution \( \vec{J}(\vec{r}) \) in an unbounded homogeneous medium with permittivity (\( \varepsilon \)) and permeability (\( \mu \)). Starting from Maxwell’s equation the vector wave equation for the electric field can be obtained as given below by

\[
\hat{E}(\vec{r}) = -j \omega \vec{A}(\vec{r}) - \frac{j}{\omega \mu \varepsilon} \nabla \left( \nabla \cdot \vec{A}(\vec{r}) \right) - \frac{1}{\varepsilon} \nabla \times \vec{F}(\vec{r}) \quad (3.31)
\]

\[
\hat{H}(\vec{r}) = -j \omega \vec{F}(\vec{r}) - \frac{j}{\omega \mu \varepsilon} \nabla \left( \nabla \cdot \vec{F}(\vec{r}) \right) - \frac{1}{\mu} \nabla \times \vec{A}(\vec{r}) \quad (3.32)
\]
\[ \nabla \times \nabla \times \mathbf{E}(r) - k^2 \mathbf{E}(r) = i\omega \mu \mathbf{J}(r) \] (3.33)

The dyadic Green’s function \( \mathbf{G} \) is essentially defined by the electric field \( \mathbf{E}(r) \) at the field point \( r \) generated by a radiating electric dipole \( \mathbf{p} \) located at the source point \( r' \). In mathematical terms this reads as [6]:

\[ \mathbf{E}(r) = k^2 \mathbf{G}(r, r') \] (3.34)

Let us consider the following general inhomogeneous equation

\[ Lx(r) = b(r) \] (3.35)

where:

\( L \) is a vector differential operator which acting on the vector field \( x(r) \) representing the unknown response of the system. The vector field \( b(r) \) is known source function and makes the differential equation inhomogeneous equation [26]. The general linear differential equation is equal to the sum of the complete homogeneous solution \( b(r) = 0 \) and the particular inhomogeneous solution. It is difficulty to solve equation.
(3.35) but it’s easier to consider the special inhomogeneity \( \delta(r - r') \) which is zero everywhere except in the point \( r = r' \) then the linear equation becomes as

\[
L \vec{G}(r,r') = n_i \delta(r - r')
\]

(3.36)

where:

\( (i = x, y, z) \) and \( n_i \) denotes an arbitrary constant unit vector.

In order to account for all orientation we write the general definition of the dyadic Green’s function for the electric field as [26].

\[
\nabla \times \nabla \times \vec{G}(r,r') - k^2 \vec{G}(r,r') = \vec{I} \delta(r - r')
\]

(3.37)

\( \vec{I} \) is the unit dyad (unit tensor). Thus if we know the dyadic greens function \( \vec{G} \), we can find the electric field which the sum of particular and homogeneous solution as:

\[
\mathbf{E}(r) = \mathbf{E}_0(r) + i \omega \mu_0 \mu \int \vec{G}(r:r') \mathbf{J}(r') dV' \]

(3.38)
The corresponding magnetic field is:

\[
H(\mathbf{r}) = H_0(\mathbf{r}) + \int_V \left( \nabla \times \mathbf{G}(\mathbf{r}, \mathbf{r'}) \right) J(\mathbf{r'}) dV' \quad (3.39)
\]

Above figure describes the dyadic green’s function \( \mathbf{G}(\mathbf{r}, \mathbf{r'}) \). This function renders the electric field at the field point \( \mathbf{r} \) due to single point source \( J(\mathbf{r'}) \) at the source point \( \mathbf{r}' \). Since the field at \( \mathbf{r} \) depends on the orientation of \( J(\mathbf{r'}) \) the green’s function must account all possible orientations in the form of tensor. With \( G_0 \left( r, r' \right) \) as a scalar Green’s function would imply that a component of the source \( J(\mathbf{r'}) \) parallel to a given axis produces a field parallel to the same axis [6], [26].
\[
\bar{G}(r,r') = \left[ \mathbf{I} + \frac{\nabla \nabla}{k^2} \right] G_0(r,r')
\]

\[
= \begin{bmatrix}
G_{xx} & G_{xy} & G_{xz} \\
G_{yx} & G_{yy} & G_{yz} \\
G_{zx} & G_{zy} & G_{zz}
\end{bmatrix}
\]

(3.40)

\[
G_0(r,r') = \frac{e^{-jk|r-r'|}}{4\pi |r-r'|}
\]

(3.41)

\(G_0(r,r')\) is the scalar green’s function which the dyadic green’s function \(\bar{G}(r,r')\) can be calculated from. The first column of the tensor \(\bar{G}(r,r')\) represents the field due to a point source in x-direction, the second column is to the field due to the point source in y-direction and the third column is the field due to the a point source in z-direction. Therefore, the dyadic green’s function is nothing more than the compact form of three vectorial green’s function. Since the field at point (r) depends on the orientation of the induced current (J), the green’s function must account for all possible orientation in the form of tensor or in other words dyadic green’s function. Therefore the dyadic green’s function correctly represents the depolarization effects [6], [26].
The scattered field depends on the orientation of induced current where the induced current depends on the polarization of incident field. To account depolarization effect dyadic green’s function must be used to account all possible orientation.

3.5 Derivation of Field Quantities from the Dyadic Green’s Function

Consider a homogeneous medium bounded by a closed surface $S$ which includes an arbitrary electric current distribution $J(r)$ [27]. Using the vector wave equation (3.33 and 3.37) in conjunction with the vector-dyadic Green’s theorem given by

$$
\iiint_{V} \left[ \hat{P} \cdot \nabla \times \nabla \times \hat{Q} - (\nabla \times \nabla \times \hat{P}) \cdot \hat{Q} \right] \, dv
$$

$$
= -\iiint_{S} \left[ (\hat{n} \times \nabla \times \hat{P}) \cdot \hat{Q} + (\hat{n} \times \hat{P}) \cdot \nabla \times \hat{Q} \right] \, ds
$$

(3.42)

explicit expression for the electric field due to the impressed electric current can be obtained. By letting $\hat{P} = \vec{E}(\vec{r})$ and $\hat{Q} = \vec{G}(\vec{r}, \vec{r}')$ it can easily be shown the following [27].

$$
\vec{E}(\vec{r}') = ik\eta \iiint_{V} J(\vec{r}) \cdot \vec{Q} \, dv
$$

$$
- \iiint_{S} \left[ (\hat{n} \times \nabla \times \vec{E}(\vec{r})) \cdot \vec{G}(\vec{r}, \vec{r}') + (\hat{n} \times \vec{E}(\vec{r})) \cdot \nabla \times \vec{G}(\vec{r}, \vec{r}') \right] \, ds
$$

(3.43)

Knowing that $\nabla \times \vec{E}(\vec{r}) = ik\eta \vec{H}(\vec{r})$ equation (3.43) can be written as follows:
Similarly we can find the expression for the magnetic field by starting with vector wave equation as follows

\[ \nabla \times \nabla \times \vec{H}(\vec{r}) - k^2 \vec{H}(\vec{r}) = \nabla \times \vec{J}(\vec{r}) \quad (3.45) \]

We let \( \vec{P} = \vec{H}(\vec{r}) \) and \( \vec{Q} = \vec{G}(r, r') \) as we did earlier in equation (3.42) and we get as follows:

\[
\vec{H}(\vec{r}') = \iiint_{V} \left[ \nabla \times \vec{J}(\vec{r}) \right] \cdot \vec{Q} dV
\]
\[
- \iiint_{S} \left[ (\hat{n} \times \nabla \times \vec{H}(\vec{r})) \cdot \vec{G}(r, r') + (\hat{n} \times \vec{H}(\vec{r})) \cdot \nabla \times \vec{G}(r, r') \right] ds
\]

(3.46)

By employing the dyadic identity such as

\[
\nabla \cdot (\vec{a} \times \vec{b}) = \nabla \times \vec{a} \cdot \vec{b} - \vec{a} \cdot \nabla \times \vec{b}
\]

(3.47)
Equation (14) can be written as

\[
\iiint_v \left[ \nabla \times \bar{J}(\bar{r}) \right] \cdot \bar{G}(r,r') \, dv = \iiint_v \left\{ \nabla \cdot \left[ \bar{J}(\bar{r}) \times \bar{G}(r,r') \right] + \bar{J}(\bar{r}) \cdot \nabla \times \bar{G}(r,r') \right\} \, dv
\]

(3.48)

Again using the divergence theorem we can rewrite it as

\[
\iiint_v \nabla \cdot \left[ \bar{J}(\bar{r}) \times \bar{G}(r,r') \right] \, dv = \iint_s \hat{n} \cdot \left[ \bar{J}(\bar{r}) \times \bar{G}(r,r') \right] \, ds = \iint_s \left[ \hat{n} \times \bar{J}(\bar{r}) \right] \cdot \bar{G}(r,r') \, ds
\]

(3.49)

From Maxwell’s equation we have as follows

\[
\nabla \times \vec{H}(\bar{r}) = \frac{ik}{\eta} \vec{E}(\bar{r}) + \vec{J}(\bar{r})
\]

(3.50)

Plugging equation (3.50) into equation (3.46) we obtain as follows

\[
\vec{H}(\bar{r}') = \iiint_v \bar{J}(\bar{r}) \cdot \nabla \times \vec{Q} \, dv + \iint_s \left\{ \frac{ik}{\eta} \left[ \hat{n} \times \vec{E}(\bar{r}) \right] \cdot \bar{G}(r,r') - (\hat{n} \times \vec{H}(\bar{r})) \cdot \nabla \times \bar{G}(r,r') \right\} \, ds
\]

(3.51)
3.6 Frequency Dependent

The derivation of the Green function for the electric field is most conveniently accomplished by considering the time-harmonic vector potential $A$ and the scalar potential $\phi$ in an infinite and homogeneous space which is characterized by the constants $\varepsilon$ and $\mu$. In this case, $A$ and $\phi$ are defined by the relationships [26].

$$E(\mathbf{r}; \omega) = i\omega A(\mathbf{r}; \omega) - \nabla \phi(\mathbf{r}; \omega)$$  \hspace{1cm} (3.52)$$

$$H(\mathbf{r}; \omega) = \frac{1}{\mu_0 \mu} \nabla \times A(\mathbf{r}; \omega)$$  \hspace{1cm} (3.53)$$

We can insert these into the Maxwell equation

$$\nabla \times H(\mathbf{r}; \omega) = -i\omega D(\mathbf{r}; \omega) + J(\mathbf{r}; \omega)$$  \hspace{1cm} (3.54)$$

And obtain

$$\nabla \times \nabla \times A(\mathbf{r}; \omega) = \mu_0 \mu J(\mathbf{r}; \omega)$$

$$-i\omega \mu_0 \mu \varepsilon_0 \varepsilon [i\omega A(\mathbf{r}; \omega) - \nabla \phi(\mathbf{r}; \omega)]$$  \hspace{1cm} (3.55)$$
where we used $D = \varepsilon_0 \varepsilon E$. The potentials $A$ and $\varphi$ are not uniquely defined by equations (3.52-3.53). We are still free to define the value of $\nabla \cdot A$ which we choose as the Lorentz gauge [4], [21], [26].

$$\nabla \cdot A \left( r; \omega \right) = i \omega \mu_0 \mu \varepsilon_0 \varepsilon \varphi \left( r; \omega \right) \quad (3.56)$$

Using the identity, $\nabla \times \nabla \times = -\nabla^2 + \nabla \cdot$ along with the Lorentz gauge we can rewrite the equation (3.55) as

$$\left[ \nabla^2 + k^2 \right] A \left( r; \omega \right) = -\mu_0 \mu J \left( r; \omega \right) \quad (3.57)$$

Which is the inhomogeneous Helmholtz equation. It holds independently for each component $A_i$ of $A$. A similar equation can be derived for the scalar potential $\varphi$

$$\left[ \nabla^2 + k^2 \right] \varphi \left( r; \omega \right) = -\frac{\rho \left( r; \omega \right)}{\varepsilon_0 \varepsilon} \quad (3.58)$$

Thus, we obtain four scalar Helmholtz equations of the form

$$\left[ \nabla^2 + k^2 \right] f \left( r; \omega \right) = -g \left( r; \omega \right) \quad (3.59)$$
To derive the scalar Green’s function $G_0 (r, r'; \omega)$ for the Helmholtz operator we replace the source term $g (r)$ by a single point source $\delta (r - r')$ and obtain

$$[\nabla^2 + k^2] G_0 (r, r'; \omega) = -\delta (r - r') \quad (3.60)$$

The coordinate $r$ denotes the location of the field point, i.e. the point in which the fields are to be evaluated, whereas the coordinate $r'$ designates the location of the point source. Once we have determined $G_0$ we can state the particular solution for the vector potential in equation (3.60) as

$$A (r; \omega) = \mu_0 \mu \int G_0 (r, r'; \omega) J (r'; \omega) dV' \quad (3.61)$$

A similar equation holds for the scalar potential. Both solutions require the knowledge of the Green function defined through equation (3.59). In free space, the only physical solution of this equation is $[4], [21], [26]$

$$G_0 (r, r'; \omega) = \frac{e^{i k |r - r'|}}{4\pi |r - r'|} \quad (3.62)$$

The solution with the plus sign denotes a spherical wave that propagates out of the origin whereas the solution with the minus sign is a wave that converges towards the origin. In
the following we only retain the outwards propagating wave. The scalar Green function can be introduced and the vector potential can be calculated by integrating over the source volume $V$ to get equation (3.61). Thus, we are in a position to calculate the vector potential and scalar potential for any given current distribution $j$ and charge distribution $\rho$. Notice, that the Green function in equation (3.62) applies only to a homogeneous three-dimensional space. The Green function of a two-dimensional space or a half-space will have a different form.

So far we reduced the treatment of Green functions to the potentials $A$ and $\phi$ because it allows us to work with scalar equations. The formalism becomes more involved when we consider the electric and magnetic fields. The reason for this is that a source current in $x$-direction leads to an electric and magnetic field with $x$, $y$, and $z$-components. This is different for the vector potential: a source current in $x$ gives only rise to a vector potential with $x$ component. Thus, in the case of the electric and magnetic fields we need a Green function which relates all components of the source with all components of the fields, or, in other words, the Green function must be a tensor. This type of Green function is denoted as dyadic Green function and has been introduced in the previous section. To determine the dyadic Green function we start with the wave equation for the electric field $[4], [21], [26]$

$$\nabla \times \mu^{-1} \nabla \times E (r; \omega) - k_0^2 \varepsilon E (r; \omega) = i\omega\mu_0 j (r; \omega) \quad (3.63)$$
We can define for each component of $j$ a corresponding Green function which can all be brought together to obtain the general definition of the dyadic Green function for the electric field,

$$
\nabla \times \nabla \times \vec{G}(\mathbf{r}, \mathbf{r}'; \omega) - k^2 \vec{G}(\mathbf{r}, \mathbf{r}'; \omega) = \vec{I} \delta(\mathbf{r}-\mathbf{r}') \quad (3.64)
$$

The first column of the tensor $\vec{G}$ corresponds to the field due to a point source in $x$-direction, the second column to the field due to a point source in $y$-direction, and the third column is the field due to a point source in $z$-direction [26]. Thus a dyadic Green function is just a compact notation for three vectorial Green functions. As before, we can view the source current in equation (3.64) as a superposition of point currents. Thus, if we know the Green function $\vec{G}$ we can state a particular solution of equation (3.64) as

$$
\mathbf{E}(\mathbf{r}; \omega) = j\omega \mu_0 \mu \int_{V} \vec{G}(\mathbf{r}, \mathbf{r}'; \omega) \mathbf{J}(\mathbf{r}'; \omega) dV' \quad (3.65)
$$

However, this is a particular solution and we need to add any homogeneous solutions $\mathbf{E}_0$. Thus, the general solution turns out to be

$$
\mathbf{E}(\mathbf{r}; \omega) = \mathbf{E}_0(\mathbf{r}; \omega) + j\omega \mu_0 \mu \int_{V} \vec{G}(\mathbf{r}, \mathbf{r}'; \omega) \mathbf{J}(\mathbf{r}'; \omega) dV' \quad (3.66)
$$

The corresponding magnetic field reads as
\[ \mathbf{H}(\mathbf{r}, \omega) = \mathbf{H}_0(\mathbf{r}, \omega) + \int \nabla \times \mathbf{G}(\mathbf{r}, \mathbf{r}'; \omega) \mathbf{J}(\mathbf{r}', \omega) \, dV' \] (3.67)

These equations are denoted as volume integral equations. They are very important since they form the basis for various formalisms such as the method of moments, the Lippmann-Schwinger equation, or the coupled dipole method. We have limited the validity of the volume integral equations to the space outside the source volume \( V \) in order to avoid the apparent singularity of \( \mathbf{G} \) at \( r = r' \). This limitation will be relaxed in later \([4],[26]\).

Note that the Green functions used above are electric field Green functions that describe the propagation of electric fields. A similar magnetic Green function exists from solving equations involving magnetic field sources

\[ \nabla \times \nabla \times \mathbf{H}(\mathbf{r}, \omega) - k^2 \mathbf{H}(\mathbf{r}, \omega) = i\nu \mu_0 \mathbf{F} \times \mathbf{J}(\mathbf{r}, \omega) \] (3.68)

Which gives the magnetic field Green function via

\[ \nabla \times \nabla \times \mathbf{G}_m(\mathbf{r}, \mathbf{r}'; \omega) - k^2 \mathbf{G}_m(\mathbf{r}, \mathbf{r}'; \omega) = \mathbf{1} \delta(\mathbf{r}-\mathbf{r}') \] (3.69)
This function looks exactly the same as equation (3.64) magnetic sources are dealt with differently from electric sources and the magnetic field boundary conditions are different from the electric field boundary conditions. However, it is possible to calculate the magnetic field Green function in the exact same manner as the electric field Green function by interchanging $\mu$ with $\varepsilon$ and $E$ with $H$ [4], [21], [26]. Some common relations used in derivations involving Green Functions are [26].

\[ G_{ij}^* (r, r'; \omega) = G_{ij}^* (r', r; \omega) \]  \hspace{1cm} (3.70)

\[ G_{ji} (r, r'; \omega) = G_{ij} (r', r; \omega) \]  \hspace{1cm} (3.71)

\[ Im\vec{G} \ (r, r'; \omega) = \int \left[ \vec{G} \ (r, s; \omega) \times \vec{\nabla}_s Im\mu^{-1}(s, \omega) [\vec{\nabla}_s \times \vec{G}^* (s, r'; \omega)] \right] dV \right. \\
\left. + \vec{G} \ (r, s; \omega) Im\varepsilon(s, \omega) \vec{G}^* (s, r'; \omega) \right] \hspace{1cm} (3.72)

Where the following relation has been used,

\[ \vec{G} (r, r'; \omega) \times \vec{\nabla}' = \left[ \vec{\nabla}' \times \vec{G}^T (r, r'; \omega) \right]^T \]  \hspace{1cm} (3.73)

The solution form Maxwell’s equation in the frequency domain in terms of Dyadic Green’s function is as follows:
\[ E(r) = \int_{V} G_{ee}(r, r') \times J_{e}(r')dV + \int_{V} G_{em}(r, r') \times J_{m}(r')dV' \quad (3.74) \]

\[ H(r) = \int_{V} G_{me}(r, r') \times J_{e}(r')dV + \int_{V} G_{mm}(r, r') \times J_{m}(r')dV' \quad (3.75) \]

\[
\begin{pmatrix}
E \\
H
\end{pmatrix}
= \int_{V}
\begin{pmatrix}
G_{ee} & G_{em} \\
G_{me} & G_{mm}
\end{pmatrix}
\times
\begin{pmatrix}
J_{e} \\
J_{m}
\end{pmatrix}
\text{d}V' \quad (3.76)
\]

Where:

\[ G_{ee} = -jw\mu(I + \frac{\nabla \nabla}{k^2}) \frac{e^{-jk|r-r'|}}{4\pi |r - r'|} \]

\[ G_{em} = -\nabla \times I \frac{e^{-jk|r-r'|}}{4\pi |r - r'|} \quad (3.77) \]

\[ G_{mm} = -jw\varepsilon(I + \frac{\nabla \nabla}{k^2}) \frac{e^{-jk|r-r'|}}{4\pi |r - r'|} \]

\[ G_{me} = \nabla \times I \frac{e^{-jk|r-r'|}}{4\pi |r - r'|} \]

Where also:

\[
G(r_x, r_y) = \begin{pmatrix}
G_{xx} & G_{xy} & G_{xx} \\
G_{yx} & G_{yy} & G_{yx} \\
G_{zx} & G_{zy} & G_{zz}
\end{pmatrix} = (I + \frac{\nabla \nabla}{k^2}) \frac{e^{-jk|r_n - r_p|}}{4\pi |r_n - r_p|} \quad (3.78)
\]
\[ \nabla = \left( \begin{array}{c}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{array} \right) \]

\[ \nabla \nabla = \left( \begin{array}{ccc}
\frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial x \partial z} \\
\frac{\partial^2}{\partial y \partial x} & \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial y \partial z} \\
\frac{\partial^2}{\partial z \partial x} & \frac{\partial^2}{\partial z \partial y} & \frac{\partial^2}{\partial z^2}
\end{array} \right) \]

\[ (I + \frac{\nabla \nabla}{k^2}) = \left( \begin{array}{ccc}
1 + \frac{\partial^2}{k^2 \partial x \partial x} & \frac{\partial^2}{k^2 \partial x \partial y} & \frac{\partial^2}{k^2 \partial x \partial z} \\
\frac{\partial^2}{k^2 \partial y \partial x} & 1 + \frac{\partial^2}{k^2 \partial y \partial y} & \frac{\partial^2}{k^2 \partial y \partial z} \\
\frac{\partial^2}{k^2 \partial z \partial x} & \frac{\partial^2}{k^2 \partial z \partial y} & 1 + \frac{\partial^2}{k^2 \partial z \partial z}
\end{array} \right) \quad (3.79) \]
\[ r = r_n - r_p \]

\[ G_{xx} = \frac{e^{-ikr}}{4\pi kr} \left( \frac{3x^2}{r^5} + \frac{3jx^2}{r^4} - \frac{x^2k^2 + 1}{r^3} - \frac{jk}{r^2} + \frac{k^2}{r} \right) \]

\[ G_{xy} = \frac{e^{-ikr}}{4\pi kr} \left( \frac{3xy}{r^5} + \frac{3jxyk}{r^4} - \frac{xy^2k^2}{r^3} \right) \]

\[ G_{xz} = \frac{e^{-ikr}}{4\pi kr} \left( \frac{3xz}{r^5} + \frac{3jxz}{r^4} - \frac{k^2xz}{r^3} \right) \]

\[ G_{yx} = \frac{e^{-ikr}}{4\pi kr} \left( \frac{3xy}{r^5} + \frac{3jxyk}{r^4} - \frac{xy^2k^2}{r^3} \right) \]

\[ G_{yy} = \frac{e^{-ikr}}{4\pi kr} \left( \frac{3y^2}{r^5} + \frac{3jy^2}{r^4} - \frac{y^2k^2 + 1}{r^3} - \frac{jk}{r^2} + \frac{k^2}{r} \right) \]

\[ G_{yz} = \frac{e^{-ikr}}{4\pi kr} \left( \frac{3yz}{r^5} + \frac{3jyz}{r^4} - \frac{yz^2k^2}{r^3} \right) \]

\[ G_{zx} = \frac{e^{-ikr}}{4\pi kr} \left( \frac{3xz}{r^5} + \frac{3jxz}{r^4} - \frac{xzk^2}{r^3} \right) \]

\[ G_{zy} = \frac{e^{-ikr}}{4\pi kr} \left( \frac{3y^2}{r^5} + \frac{3jy^2}{r^4} - \frac{y^2k^2 + 1}{r^3} - \frac{jk}{r^2} + \frac{k^2}{r} \right) \]

\[ G_{zz} = \frac{e^{-ikr}}{4\pi kr} \left( \frac{3z^2}{r^5} + \frac{3jz^2}{r^4} - \frac{z^2k^2 + 1}{r^3} - \frac{jk}{r^2} + \frac{k^2}{r} \right). \]
3.7 Time Dependent

It is also useful to consider the derivation of the time dependent Green function, especially in the context of ultrafast phenomena. We can rewrite equations (3.52 and 3.53) in the time domain as [4], [21], [26].

\[
E (r; t) = - \frac{\partial A (r; t)}{\partial t} - \nabla \varphi (r; t) \tag{3.81}
\]

\[
H (r; t) = \frac{1}{\mu_0 \mu} \nabla \times A (r; t) \tag{3.82}
\]

From which we find the time-dependent Helmholtz equation in the Lorentz gauge by assuming a none dispersive medium.

\[
\left[ \nabla^2 - \frac{n^2}{c^2} \frac{\partial}{\partial t} \right] A (r; t) = -\mu_0 \mu J (r; t) \tag{3.83}
\]

With a similar equation for the scalar potential \( \varphi \). The definition of the scalar Green function is now generalized to,

\[
\left[ \nabla^2 - \frac{n^2}{c^2} \frac{\partial}{\partial t} \right] G_0 (r, r'; t, t') = -\delta (r - r') \delta (t - t') \tag{3.84}
\]
The point source is now defined with respect to space and time. The solution for $G_0$ is

\[ G_0 \left( r, r'; t, t' \right) = \frac{\delta \left( t' - [t \mp n c |r - r'|] \right)}{4\pi |r - r'|} \] (3.85)
CHAPTER 4

CLASSICAL ISAR IMAGING

4.1 Introduction

Classical inverse synthetic aperture radar (ISAR) processing techniques are developed based upon the fundamental assumption that the targets can be thought of as a collection of point scattering centers [28], [29], [30] see figure 4.1 and the two dimensional images of the targets can be considered as spatial maps of these scattering centers. The cross section of the target is characterized by a two dimensional distribution of scattering centers, which is called the reflectivity function. The target is illuminated by a plane wave, and the scattered field is collected by receivers located around the target of interest. ISAR cross-range resolution is limited by the extent of the target’s motion, e.g., rotation and translation, within the antenna beam. ISAR technology utilizes the movement (rotation) of the target rather than movement of the sensor. For small angles, an ISAR image is the two dimensional Fourier transform of the received signal as a function of frequency and target aspect angles. Even though, the target scatters the incident field in all directions, only the portion that is reflected back to the radar is of interest [31], [32], [33].
4.2 Classical ISAR Image Reconstruction

To construct an ISAR image we need to better understand electromagnetic scattering. Imaging in this context is simply the reconstruction of the spatial distribution of the conductivity of the imaged target, \( \sigma(x, y, z) \). According to the geometric theory of diffraction (GTD), if the wavelength at the operating frequency is small relative to the target size, then the scattered field consists of contributions from a number of electrically isolated scattering centers. ISAR imaging algorithms rely upon an assumption that the area under observation consists of a collection of infinitesimally small isotropic scatterers (i.e., the point scatterer model). This approximation is based upon the application of a scalar contrast function which ignores directional dependency, since point scatterers radiate isotropically in all directions [32], [33], [34], [35], [36], [37], [38].

From a purely electromagnetic perspective, the scattering process occurring from a target is properly described using the electric field integral equation (EFIE) and the magnetic field integral equation (MFIE), both derived from Maxwell’s equations. See [39].

When electromagnetic fields scatter from thin wires, we solve an integral equation of first kind, with complex kernels. These integrals are inherently ill-posed, meaning that the solutions are generally unstable and small changes may cause very large changes in the results. The scattering model of the rigorous EFIE/MFIE interpretation is based on a point-scattering model assumption that [8], [33], [40]:

1) The target consists of a collection of point scatterers scatters (i.e., equal radiation in all direction)
2) Internal multipath is ignored, meaning the first order Born approximation holds.

In other words, the successes of these approaches or approximations are dependent on the target being a weak scatterer. At this stage, metallic surfaces are ignored because their inclusion in the model would dramatically complicate the mathematical formulation [35], [41], [42].

To summarize, classical ISAR imaging assumes that the target consists of a collection of point scatterers as shown in figure 4.1 below:

![Figure 4.1 Superposition of point sources.](image)

Note: small lowercase letters represent scalars, bold, capital letters represent vectors. The next sections describe a mathematical formulation of the forward and inverse problem, as well as the simulations procedure [33].
4.3 Scattering by an Electrically Large Object

Consider a transmitter located at \( r' \) emitting a vertically polarized wave along the vector \( \mathbf{a}' \), a receiver located at \( r'' \) receiving this vertically polarized wave along the vector \( \mathbf{a}'' \) and a point under illumination at location \( r \). The time-harmonic incident electric field \( \mathbf{e} \) at \( r \) is [8], [33], [43].
\[ \mathbf{E}^i (\mathbf{r}) = G_0 (\mathbf{r}, \mathbf{r}') \cdot \mathbf{a}' \]

*Where:*

\[ G_0 (\mathbf{r}, \mathbf{r}') \equiv \frac{e^{-j \mu |r-r'|}}{4\pi r} \]

\( G_0 (\mathbf{r}, \mathbf{r}') \) is the Green’s function that is the solution of the field equation for a point source. Using the principle of linear superposition, the solution of the field due to the general source is just the convolution of the Green’s function with the source [44], [45].

Let us assume that a small cylinder is oriented along \( \mathbf{z} \) is present at point \( \mathbf{r} \). When the electric field \( \mathbf{E}^i \) impinging on \( \mathbf{r} \) is parallel to \( \mathbf{z} \), then a current \( \mathbf{J}(\mathbf{r}) \) is induced on the cylinder. This current will generate a scattered field propagating in all directions, and towards the receiver antenna. The measured electric field will be [33], [34], [46], [47].

\[ \mathbf{E}^s = \mathbf{a}' \cdot G_0 (\mathbf{r}', \mathbf{r}) \cdot \mathbf{J}(\mathbf{r}) \]  

(4.2)

Clearly, if the small cylinder is not oriented along the incident field, only a fraction of the current is excited. Mathematically, the current induced in the cylinder can be expressed as:
\[
\mathbf{J}(\mathbf{r}) = \mathbf{v}(\mathbf{r}) \cdot \mathbf{E}'(\mathbf{r})
\]  
(4.3)

Here, the dyadic \( \mathbf{v}(\mathbf{r}) \) has the same meaning as the reflectivity function in SAR/ISAR.

By using equation (4.2), the received electromagnetic signal in the frequency domain for a bistatic ISAR configuration is

\[
\mathbf{H}' = \frac{jk\mathbf{H}_0 e^{j(2\mathbf{k}r - \omega t)}}{(2\pi)^3 r} \iiint_D \mathbf{v}(\mathbf{r}) e^{j(2\mathbf{k}r')} \cdot d\mathbf{r}'
\]  
(4.4)

Since \( \mathbf{E}^s \) is orthogonal both the magnetic field and the direction of propagation in the far field zone

\[
\mathbf{E}^s = j\mathbf{k} \times \mathbf{H}^s
\]
\[
= \frac{k^2 \mathbf{H}_0 e^{j(2\mathbf{k}r - \omega t)}}{(2\pi)^3 r} \hat{\mathbf{k}} \times \iiint_D \mathbf{v}(\mathbf{r}) e^{j(2\mathbf{k}r')} \cdot d\mathbf{r}'
\]  
(4.5)

where

\( \overline{\mathbf{k}} = k\hat{\mathbf{k}} \)

To write the above equation as a discrete form of point scatterer model which is the standard model used in ISAR processing is shown below.
where $D$ is the domain of interest, $\Gamma_n$ is the location of the $n$th pixel, $P$ is the total number of pixels, and $\nu(r)$ is the unknown contrast function.

The scattered field is acquired from FEKO, both in amplitude and phase, on a square investigation domain around the target, which is sufficiently extent to be within the radiating near-field region. The incident field oriented along the cylindrical target axis and a probe of the same kind is used to collect on the acquisition domain for the different positions of illumination. The measured scattered field data are subsequently processed to solve the inverse scattering problem and successfully retrieve the reflectivity function profile of the cylinder [7], [35].

### 4.4 Inverse Scattering Technique

Mathematically, the problem of finding the solution of the reflectivity function profile is determined by computing the inverse linear operator of equation (4.6). This equation is the forward scattering model which relates the unknown reflectivity dyadic function $V$ to the simulated scattered electric field. Therefore, we can write an equation (4.6) in a simplified form as shown below [48], [49]:

$$E^s = \sum_{n=1}^{P} a^r \cdot G_0\left(r^r, r_n\right) \cdot \nu\left(r\right) \cdot G_0\left(r_n, r^t\right) \cdot a^t \quad (4.6)$$
\[ E^s = L \cdot v \quad (4.7) \]

where \( E^s \) is the known vector containing the scattered field collected at different positions and directions with a single frequency, \( L \) is a large matrix whose value is computed theoretically from (4.6), and \( v(r) \) is the vector representing the reflectivity function of the target, which is zero outside the investigation domain. The reflectivity function can be easily recovered from equation (4.7) by using the inversion technique, [35], [50], [51]:

\[ v \cong L^H E^s \quad (4.8) \]

Figure 4.3 Configuration setup for imaging two cylinders.
The target is illuminated with monochromatic plane waves, and the scattered waves, after the interactions of the incident plane waves with the scattering target, are measured by receivers placed in the near field around the target.

4.5 Scalar Simulation

In this section two types of simulation is performed by employing an accurate EM simulation software tool known as FEKO and one experiment. The first simulation is imaging a 2D thin cylinder which is vertically oriented in z-direction and compared with experiment. The second simulation is imaging the 3D of two thin cylinders with three different separations. Both simulations and the experiment are based point scattering model which assumes the target radiates equally in all direction.

4.5.1 2D setup and results

The 2D image was reconstructed through both experimentally and using accurate EM simulation software tool known as FEKO. In this case, a thin cylinder as shown in (Figure 4.4) representing a wire is oriented vertically at the center and the transmitting and receiving antennas are located in the x-y plane. In both the simulation and the experiment we used 72 transmitters and 72 receivers. In the simulation both the transmitters and receivers transmit and receive in one of the three orthogonal polarizations such as \( \hat{x}, \hat{y}, \text{ and } \hat{z} \), where in the experiment they only transmit and receive vertically polarized wave and in this case z direction only. The transmitting antennas are placed along a radius of \( 13\lambda \) and the receiving antennas are also placed \( 10\lambda \) with respect
to the target. The operating frequency is 10 GHz \( (\lambda = 3\,\text{cm}) \) and the total measurement collected from this is \( (N_t \times N_r \times 9 = 72 \times 72 \times 9 = 46656) \); where \( N_t \) and \( N_r \) are number of transmitters and number of receivers and the nine is the nine pixels of reflectivity function. The area of investigation is 0.4 m by 0.4 m which divided into pixels with size of \( 0.05\lambda \) with the total of \( (N_t \times N_r \times 9 = 72 \times 72 \times 9 = 46656) \) unknowns. Data is collected using a monostatic platform encircling the cylinder (360 degrees, see Figure 4.3). The objective is to determine 1) the location, and 2) make 2D image of the target. (Figures 4.8-4.10) show the simulation results and the target is indeed present at (0,0) at all figures due to the point source model (PSM). PSM is based on the assumption that a the returned signal from the target can be expressed as a sum of scattering points responses [52], as shown in figure 4.7. Therefore, the polarization of the illuminator doesn’t matter since the object will radiate in all directions equally. The figure 4.19 shows also the results of experimental data which is almost the same thing like the simulation. If we would have used the dipole model which is covered in chapter 5 the target is only detected when plotting the image corresponding to a \( z \) directed dipoles. Therefore, using point source model one cannot determine the orientation of the target since all three figures 4.8-4.10 appear equal.

![Figure 4.4 FEKO model of cylinder vertically oriented.](image-url)
Figure 4.5 Near field pattern of two cylinders oriented vertically.

Figure 4.6 Radiation pattern of cylinder vertically oriented.
Figure 4.7 Dipole antenna illuminating an object that is radiating equally in all directions. This object is known as isotropic radiator.

Figure 4.8 The cylinder is oriented along $z$-axis while the antenna is along the $x$-axis and detection is declared due to the assumption of isotropic radiator.
Figure 4.9 The cylinder is oriented along $z$-axis while the antenna is along the $y$-axis but still the target is detected because of the point source model.

Figure 4.10 The cylinder and the antenna are both oriented along $z$-axis. The target is detected due to the same alignment with incident field even though orientation doesn’t matter due to the point source model.
4.5.2 3D setup and results

The reflectivity function was found by using FEKO, an accurate EM simulation software tool. In this simulation, we performed three cases: 0.5λ, λ and 2λ separation between two thin cylinders in order to see how the separation between them effects the construction of the image. In the first case, the first cylinder is placed at (2.5, 2.5, 0 cm) and the second one is at (-2.5, -2.5, 0 cm), in the second case the first cylinder is placed at (5, 5, 0 cm) and the second one is at (-5, -5, 0 cm) and in third case the first cylinder is placed at (10, 10, 0 cm) and the second one is at (-10, -10, 0 cm). In all three cases the two thin cylinders are oriented vertically. The transmitting and receiving antennas are located in the x-y plane. In the simulation we used 30 transmitters and 180 receivers, both of which transmit and receive vertically polarized waves in the z direction only. The transmitting antennas are placed along a radius of 4λ, and the receiving antennas are placed at 2λ with respect to the target. The operating frequency is 3 GHz (λ = 10cm), and the total measurement collected from this is (Nt × Nr × 1 = 30 × 180 = 5400); where Nt and Nr are number of transmitters and number of receivers and the one is one of the nine pixels of reflectivity function and this case it’s the \( V_{zz} \). The area of investigation is 0.25 m by 0.25 m, which was divided into pixels of size 0.05λ with the total of (Nt × Nr × 1 = 30 × 180 = 5400) unknowns. Data is acquired using a monostatic platform encircling the two cylinders, as shown in figure 4.3. The objective is to: 1) determine the location, and 2) reconstruct the 3D image of the two cylinders.
Figure 4.11 Matlab setup for imaging. The inner circle represents number of receivers (Rx) and the outer circle represents number of transmitters (Tx).

Figure 4.12 FEKO setup. Two thin cylinders illuminated with incident plane wave.
Figure 4.13 Radiation pattern of two cylinders oriented vertically.

Figure 4.14 Two thin cylinders vertically oriented and separated by half of wavelength. The cylinders are not clearly resolved due to the mutual coupling caused by the small separation.
Figure 4.15 Two thin cylinders vertically oriented. The two cylinders are resolved as the separation between the two is increased to one wavelength.

Figure 4.16 Two thin cylinders vertically oriented. The two cylinders are clearly resolved as the separation between them is increased to two wavelengths.
4.6 Scalar 2D Experimental Result

The scalar version of this theory (i.e., scalar reflectivity function) has been already demonstrated experimentally by our team [53] and other scientists worldwide. To this aim, the robotic tomographic chamber at the University of Dayton MUMMA radar laboratory was used for this experiment. The MUMMA tomographic chamber is equipped with four robotic arms, each one having one dual polarized horn antenna as shown in figure 5.30 and operating up to 12 GHz. In this experiment, the target of interest (TI) is a thin cylinder which was placed on the turn table and the target is oriented vertically as shown in figure 4.18. The transmitter is placed 40 cm away from the target, where the receiver is placed 30 cm away with respect to the target. The transmitter and receiver are separated by 60 cm with 60° bistatic angle. Inverse synthetic aperture (ISAR) measurement is performed in this experiment where both transmitter and receiver were fixed and the target is rotated from 0° to 355° with 5° increment total of 72 points. As the
turn table rotates, data is collected through an 8-port vector network analyzer. The frequency domain data is stored and used to fill the measurement vector $\mathbf{E}^* \text{in}(4.7)$. The image was reconstructed using the procedure described in sections 3-4 and the experimental data is show in figure 4.19.

Figure 4.18 The cylinder is oriented vertically at the center and both transmitter (Tx) and receiver (Rx) are vertically polarized.
Figure 4.19 Experimental image of vertically oriented cylinder with vertically polarized Tx and Rx.
CHAPTER 5
DIPOLE MODELING ISAR IMAGING

5.1 Introduction

The inverse scattering problem is a one that determines the nature of a target of interest from the knowledge of the scattered electromagnetic fields. Figure 5.1 shows a cylinder oriented vertically and illuminated with a monochromatic plane wave, after the interaction of the incident plane wave with the target, the scattered fields are measured by receivers placed in the near field of the target. The methods used for solving inverse problems depend upon the electrical size of the target in a medium. If D is the characteristic dimension of the scatterer and k is the wavenumber, the quantity k*D gives a measure of the electrical length of the target. When k*D << 1, scattering is weak, and we may apply low frequency methods as well as the Rayleigh and Born approximations. On the other hand, when k*D >>1, we may use high frequency asymptotic techniques such as geometrical or physical optics methods [21]. The main contribution of this research is enhanced imaging of targets under observation using a novel sensing approach in which the orientation of the current dipole is a critical variable for interference suppression and feature exploitation. Using vector dyadic reflectivity function, small features are better observed, even when in the presence of dominant scatterers [5], [54].
5.2 Inverse Synthetic Aperture Radar (ISAR)

ISAR, provides a powerful sensing and signal processing technique for imaging moving targets in the range and cross-range domains, as shown in figure 5.2. This technique is developed based upon the fundamental assumption that the target to be imaged is a collection of infinitesimally small isotropic sources. In ISAR, cross-range resolution is limited by the extent of the platform (or target) motion, e.g., translation and rotation within the antenna main beam over an observation interval. Extended dwell time processing allows for improved resolution. However, as the target rotates (and translates), the scattered electric fields changes both in phase and amplitude, invalidating the fundamental assumption of isotropic point target scattering. As target returns decorrelate over time and observation angle, a coherently-formed ISAR image will degrade.
Typically, this problem is mitigated by reducing the coherent dwell to a value that will “likely” preserve the target’s exploitable electromagnetic signature [32], [34], [33].

Many researchers have addressed the problem of wide angle ISAR imaging, with very successful results [32]. Some approaches resort to computationally-intense signal processing to correct for phase errors. Other approaches simply compute a set of coherently-formed images, and then attempt to “merge” such images to form an improved representation of the target. These methods are all useful, but a substantial increase in performance could be achieved if the underlying model captured more realistically the scattering phenomena and mechanisms occurring at the target. From a pure electromagnetic perspective, the scattering process occurring at a target is properly described using the electric field integral equation (EFIE) and the magnetic field integral equation (MFIE), both derived in section 2.8. In principle, if ISAR data is processed using the EFIE and MFIE as the scattering model, then the entire data set could be
processed coherently, leading to an extremely accurate representation of the target [4], [5], [8], [32].

$$E(r) = -\frac{i\omega\mu}{4\pi} \int \left[ \mathbf{I} + \frac{\nabla \nabla}{k^2} \right] G_0(r,r') \cdot J(r') \, dr'$$  \hspace{2cm} (5.1)$$

$$H(r) = -\frac{1}{i\omega\mu} \nabla \times E(r)$$ \hspace{2cm} (5.2)$$

Practically, inversion of the EFIE of equation (5.1) or MFIE equation (5.2) is extremely difficult because when electromagnetic fields scatter from thin wires, we must solve an integral equation of first kind, with complex kernels. These integrals are inherently non-linear and ill-posed, meaning that the solutions are generally unstable and small physical changes may cause very large changes in the results, plus they are computationally intensive [55]. Therefore, we propose a scattering model that lies in between the rigorous EFIE/MFIE interpretation and the point-scattering assumption. Specifically:

1) The target consists of a collection of infinitesimally small dipoles with arbitrary orientation,

2) The target’s internal multipath is negligible, meaning the first order Born approximation holds, and the linearized equation as shown below is applicable.
\[ E_{\text{tot}}(r) = E'(r) + k_0^2 \int_{\nu} G_0(r,r') \cdot E_{\text{tot}}^s(r') \, dr' \]

\[ = E'(r) + E^s(r) = \Gamma \Psi E_{\text{tot}}^s(r) + E^i(r) \]

\[ = \frac{I - \Gamma \Psi^{-1} E^i(r)}{1 - \Gamma \Psi^{-1}} = \left[ I + \Gamma \Psi + \Gamma \Psi \Gamma \Psi + \cdots \right] E^i(r) \quad (5.3) \]

where:

Geometrical series expansion

\[ I - \Gamma \Psi^{-1} = 1 + \Gamma \Psi + \Gamma \Psi \Gamma \Psi + \cdots \]

In the above equation, \( E_{\text{tot}}^s(r,r') \) is the scattered field resulting from the interaction of the incident wave \( E^i(r,r') \) with the scattering/contrast function \( \rho(r') \), and \( G_0(r,r') \) is the free space Green’s function. Using the far field approximation for the radiated spherical wave \( G_0(r,r') \) we obtain

\[ E^s(r,r') = k_0^2 \int_{\nu} G_0(r,r')(r,r') \cdot E^i(r,r') \rho(r') \, dr' \quad (5.4) \]

When adopting the first order Born approximation, the total field \( E_{\text{tot}}(r,r') \) in equation (5.4) is replaced with the incident field \( E^i(r,r') \) which leads to linear equation. The Born approximation is based upon weak scattering phenomena which occur when the incident wave is only scattered once and this incident wave basically
undergoes very little perturbation as it interacts with the target of interest. This is beneficial in that the wave can generally be approximated as the incident field, allowing the problem to be linearized in order to find a solution. By linearizing the problem one can establish a Fourier relationship between the measured scattered field data and the object/contrast function [7], [21], [49], [56], [57].

The problem of determining the image of an object from scattered fields has received a great deal of attentions due to the myriad of applications. For objects that are sufficiently weakly scattering, it's well-known that the inverse scattering problem can be linearized as equation (5.3) shows and image reconstruction can be achieved by extracting the Fourier data describing the target from scattered fields. In other words, the success of this approximation is dependent upon the target being modeled as a collection of weak scatterers. The target is assumed to be a wire-frame representation of the original, and the goal is to reconstruct an image of the target of interest. At this stage, small metallic surfaces are often represented by their thin-wire boundary approximation. Inclusion of a patch model would dramatically complicate the mathematical formulation. Irregular metallic patches may reflect the energy diffusely in a variety of different directions, thus providing a small contribution to the reflected energy. If the receiver’s direction is pointing near the specular region of the metallic patch under illumination, then the resulting scattering process can be equally described as a superposition of many infinitesimally small dipoles oriented along the same polarization as the incident waveform and distributed along the entire surface of the metallic patch. In essence, the scattering processes due to flat metallic patches are accounted for in the dipole model.
The idea of extending the point-scattering model to a dipole model is not new. Several researchers, especially B. Yazici and M. Gustaffson [47], have realized the importance of such developments and have proposed imaging schemes based upon exploitation of a dipole model of a target.

What we explore in this thesis is the incorporation of a target’s dipole locations and direction under a simple 3x3 matrix function (i.e., a dyadic function). This can be estimated using classical back-projection algorithms. In other words, extending the scattering model from a point target to a dipole target would only require invoking the existing ISAR algorithms multiple (3 by 3 matrix) times.

As shown in the result section 5.8 below, the reconstructed “dyadic” image (which can either be interpreted as nine different but interrelated images, or a single image with each pixel having 9 parameters) is unintelligible, but the structure of each 3x3 dyadic “pixel” incorporates the direction of the dipole scatterer included within. Another important advantage obtained using a dipole model is the capability to provide additional information about each reconstructed pixel - information that can be used for better target recognition. Many researchers have developed feature extraction algorithms that attempt to retrieve shapes including plates, corner reflectors, edges, cylinders, conical sections, or spheres, from the target’s scattered waveform [31]. This additional information provides extra dimensions in the feature space for better classification and identification. A dipole-based representation of the target will provide additional dimensions to the feature space.
The next sections describe a mathematical formulation of the forward and inverse problem, as well as simulations and forthcoming experiments [35], [42].

5.3 Near to Far Field Transformation

It is not feasible most of the time to simulate a large object in most of the anechoic chambers and still get the far field radiation of the scatterer accurately. Because as equation (5.6) shows below, the larger the object in size and the higher the frequency of operation is the larger the distance that is needed to meet the conditions of the far field. The boundary between the near-field and far-field regions is not sharp because the near-field gradually becomes less dominant as the distance from the source increases. And the reason why meeting the conditions of far-field is important is as follows:

- On the far-field the source and the target are separated by enough distance with which when the target is illuminated by a source antenna, the phase across the target doesn’t very more than 22.5°
- In the far field the radiation pattern doesn’t change its shape with distance although the field decays \( E = \frac{1}{R} \)
- The electric and (E) and magnetic (H) field vector are orthogonal to each other and to the direction of propagation as with plane waves.
- E and H propagate in phase, therefore the ratio of their magnitude is constant with the intrinsic impedance of free space as shown below
where $D$ is the largest dimension of the target and $\lambda$ is the wavelength of the operation.

But we can compute the far fields from the near fields post-processing by employing transformation technique which based on equivalence theorem, Green’s function and radiation integral. To do this we have to surround the target and the sources in the simulation domain with closed surface and find the equivalent sources on the surface. This allows that the target to be replaced by current source in free space. This fictitious sources are said to be equivalent within the region because they produce within
that region the same fields as the actual sources. The surface equivalence theorem is based on the uniqueness theorem which is covered in section 2.4 and Green’s function is covered in 3.1 as well. As stated earlier the far fields due to the source are acquired via the computation of radiation integral. According to the Huygens, it is known that the field values can be computed outside a closed volume, if the tangential fields on the test surface (A) are known. The Huygens’s principle which states that each point on a primary waveform can be considered as the envelope of these secondary spherical waves is used to determine the equivalent sources on the surface and find the fields radiated by the source outside the surface [6].

5.4 Analysis of Scattering from a Thin PEC Cylinder

The physical optics approximation (POA) is widely used to predict scattering from a large target. An irregularly shaped object is decomposed into many flat elementary patches, which have a simple geometry such as rectangular or triangular shape [6], [58].

Consider an infinite length PEC (perfectly electrical conductor) cylinder lying along the z-axis. The electrical properties are taken as homogenous and isotropic with electric and magnetic properties \( \sigma, \varepsilon, \mu \), and having a small radius as compared with the free space wavelength. The surrounding media is assumed to be free space where \( (\sigma = 0, \varepsilon_0, \mu_0) \) as shown in figure 5.4. The cylinder is illuminated by a uniform incident plane wave \( E_z^i \). The incident electric field is parallel to the axis of the cylinder and makes the problem entirely two dimensional.
\[ E^i_z = e_0 \exp(-j k x) \]

where: \( k = (\varepsilon \mu_0^{1/2}) \omega \) \hspace{1cm} (5.7)

\( k \) is the background wave number which specifies the direction of an incident plane wave that illuminates the target.

The incident field in scattering problem is mostly a plane wave as equation (5.7) shows. Because the source in the scattering problem is located far from the target of interest (TI) and the illuminated field due to the source can be approximated near the target as a plane wave. The importance of illuminating an object with plane wave is covered in section 5.4. Equation (5.7) is a single incident plane wave propagating along \( x \)-direction but the combination of many plane wave coming from different direction is shown below as phasor form.

\[ \tilde{E}^i = \int \tilde{E}^i_0(\hat{k}) e^{-j \hat{k} \cdot \hat{r}} d\hat{k} \] \hspace{1cm} (5.8)

It is clear that when the incident wave impinging upon on the thin cylinder a current will be induced and if it is sufficiently thin, we conclude that only the axially induced current needs to be considered. The scattered field \( E^i_z \) will be azimuthally
symmetric and will have the form of the radiated electric field from an infinite line source [4], [6], [5], [8], [10].

In general the scattered field $E_z^s$ is described to be the perturbation due to the incident field by the target. As shown below in equation (5.10), the measurable field around the target is the total field which is the combination of the incident plus the scattered field.

Figure 5.4 Electromagnetic wave impinging on a thin wire.

From Maxwell’s equations, one finds the electromagnetic field equations

$$\nabla \times \nabla \times E - k^2 E = 0$$
$$\nabla \times \nabla \times H - k^2 H = 0 \quad (5.9)$$
The fields consist of incident \((\mathbf{E}^i, \mathbf{H}^i)\) and scattered field as well \((\mathbf{E}^s, \mathbf{H}^s)\).

The total fields in the surrounding media are equal to the sum of the incident and the scattered fields. In the far zone \((\text{i.e. } r \to \infty)\) the scattered field expression simplifies as follows

\[
E^{\text{tot}} = E^i(r) + E^s(r)
\]

where:

\[
E^s(r) = -j\omega A - j \frac{1}{\omega \mu \varepsilon} \nabla(\nabla \cdot A)
\quad (5.10)
\]

\[
H^{\text{tot}} = H^i(r) + H^s(r)
\]

\[
E^s(r) = -j\omega A
\]

\[
H^i(r) = \frac{E^s(r)}{\eta_0}
\]

where:

\[
A = \frac{\mu_0}{4\pi} \int_{s'} \mathbf{J}(r') G_0(r,r') \, ds'
\quad (5.11)
\]

\(\mathbf{J}(r')\) is the induced surface current and \(G_0(r,r')\) is the scalar free space far field Green’s function for outgoing waves [8], [10]:

\[
G_0(r,r') = \frac{e^{jkr}}{4\pi r} e^{-jk\rho r'}
\quad (5.12)
\]
Assuming that \( \mathbf{J}(\mathbf{r}') \) is arbitrary, one can show that the homogeneous medium dyadic Green’s function is the solution to the following equation [4], [7], [8], [58].

\[
\nabla \times \nabla \times \mathbf{G}(\mathbf{r}, \mathbf{r}') - k^2 \mathbf{G}(\mathbf{r}, \mathbf{r}') = \mathbf{I} \delta(\mathbf{r}, \mathbf{r}') \quad (5.13)
\]

Its solution is given by

\[
\mathbf{G}(\mathbf{r}/\mathbf{r}') = \left[ \mathbf{I} + \frac{\nabla \nabla}{k^2} \right] G_0(\mathbf{r}, \mathbf{r}')
= \left[ \mathbf{I} - \hat{\mathbf{r}}\hat{\mathbf{r}} \right] \frac{e^{jk\mathbf{r} \cdot \mathbf{r}'}}{4\pi r}
\]

Because this dyadic Green’s function generates an electric field from electric current, it is also known as the electric dyadic Green’s function. An electric field integral equation (EFIE) is obtained by enforcing the boundary condition for the electric field at the perfectly conducting surface. The total electric field satisfies boundary condition on a PEC thin cylinder [4], [5], [8], [10].

\[
n \times E_{\text{tot}}^{\text{tan}} \bigg|_{x=R} = 0 \quad (5.15)
\]

Upon noting that the total electric field on the surface of the PEC is the sum of the incident and scattered fields,
\[ E^{tot} = E^i + E^s \quad (5.16) \]

Then

\[ n \times \left[ \int_V jk \eta_0 \mathbf{j}(\mathbf{r}') \cdot \mathbf{G}(\mathbf{r} / \mathbf{r}') \cdot d\mathbf{r}' + E^i(\mathbf{r}) \right] = 0 \quad (5.17) \]

For \( \mathbf{r} \in V \) the above equation is written as

\[-n \times E^i(\mathbf{r}) = jk \eta_0 n \times \int_D \mathbf{j}(\mathbf{r}') \cdot \mathbf{G}(\mathbf{r}', \mathbf{r}) \cdot d\mathbf{r}' \quad (5.18)\]

Also the scattered field \( E^s \) and the incident field \( E^i \) both satisfy the boundary condition

\[ E^s \big|_{x=R} = -E^i \big|_{x=R} \]

where:

\[ E^i(\mathbf{r}) = -jk \eta_0 \int_D \mathbf{j}(\mathbf{r}') \cdot \mathbf{G}(\mathbf{r}', \mathbf{r}) \cdot d\mathbf{r}' \quad (5.19) \]

As stated earlier, the incident field is that which would exist in the computational domain in which no target is present [4], [5], [8].

5.5 Imaging with a Dipole Model

To summarize, classical ISAR imaging assumes that the target consists of a collection of point scatterers see figure 5.5, while the proposed model assumes that the target consists of a collection of dipole scatterers see figure 5.6. Up to now we have only
discussed the inverse scattering infinitesimal dipole model, where the target is located in a homogeneous background medium [42].

Figure 5.5 Target model is based upon a superposition of point sources.

Figure 5.6 Target model is based upon a vector summation of small dipoles.
Hereafter, small letters represent scalars, bold small letters represent vectors, and bold capital letters are 3x3 dyadics.

Consider a transmitter located at $\mathbf{r}^t$ emitting a waveform polarized along the vector $\mathbf{a}^t$, a receiver located at $\mathbf{r}^r$ polarized along the vector $\mathbf{a}^r$, and a point under illumination at location $\mathbf{r}$. The time-harmonic incident electric field at location $\mathbf{r}$ is shown below [44], [59], [42]:

$$E^i = \mathbf{G} \cdot \mathbf{a}^i$$

$$
\begin{bmatrix}
E^i_x \\
E^i_y \\
E^i_z
\end{bmatrix}
= 
\begin{bmatrix}
a^i_x G_{xx} + a^i_y G_{xy} + a^i_z G_{xz} \\
a^i_x G_{yx} + a^i_y G_{yy} + a^i_z G_{yz} \\
a^i_x G_{zx} + a^i_y G_{zy} + a^i_z G_{zz}
\end{bmatrix}
$$

(5.20)
and

\[ H^i(r, r', a') = \frac{1}{\omega \mu} k^i \times E^i(r, r', a') \] (5.21)

where, \( \tilde{G} \) is the free space far fields dyadic Green’s function of the background medium which relates the vector electromagnetic fields to vector current source and \( G_0(r, r') \) is the green’s function which is the solution of the field equation for a point source. Using the principle of linear superposition, the solution of the field due to a general source is the convolution of the green’s function with the source [44], [60], [22], [37], [61], [46].

Let us assume that an infinitesimally small dipole oriented along \( t \) is present at point \( r \). When the electric field \( e \) impinging on \( r \) is parallel to \( t \), then a current \( j \) is induced on the dipole. This current will generate a scattered field propagating in all directions, and towards the receiver antenna. The measured electric field will be [46].

\[
E^s = a^r \cdot \tilde{G} \cdot J
\] (5.22)
We assume that the axial current along the infinitesimal dipole is uniform. With a $\ll \lambda$, we may assume that any circumferential currents are negligible. The infinitesimal dipole with a constant current along its length is a nonphysical antenna. However, it approximates several physically realizable sources. Clearly, if the small dipole is not oriented along the incident field, only a fraction of the current is excited. Mathematically, the current induced in the dipole $t$ can be expressed as [44], [42].

$$\mathbf{j}(\mathbf{r}) = \frac{\left[ \mathbf{E}'(\mathbf{r}) \cdot \mathbf{t}(\mathbf{r}) \right] \mathbf{t}(\mathbf{r})}{\|\mathbf{t}(\mathbf{r})\|}$$

(5.23)

After some mathematical manipulation, equation (5.23) can be rewritten in dyadic form as shown below [59], [42].

$$\mathbf{j}(\mathbf{r}) = \begin{bmatrix} t_x^2(\mathbf{r}) & t_x(\mathbf{r})t_y(\mathbf{r}) & t_x(\mathbf{r})t_z(\mathbf{r}) \\ t_x(\mathbf{r})t_y(\mathbf{r}) & t_y^2(\mathbf{r}) & t_y(\mathbf{r})t_z(\mathbf{r}) \\ t_x(\mathbf{r})t_z(\mathbf{r}) & t_y(\mathbf{r})t_z(\mathbf{r}) & t_z^2(\mathbf{r}) \end{bmatrix} \cdot \frac{\mathbf{E}'(\mathbf{r})}{\|\mathbf{t}(\mathbf{r})\|}$$

$$= \frac{1}{\|\mathbf{t}(\mathbf{r})\|} \mathbf{\rho}(\mathbf{r}) \cdot \mathbf{E}'(\mathbf{r})$$

(5.24)
Figure 5.8 Dipole antenna oriented along z-direction.

where, \( \hat{n}(r') \) is the outward unit vector of the illuminated targets surface. The vector \( r' \) is defined from the origin to any point on the illuminated surface \( S \), and this is known as the physical optics approximation. The scattered field at the far field region along the observation vector \( r \) is given by

\[
E^s(r) = \int \left( a^r \right)^T \cdot \vec{G}(r', r) \cdot j(r) \cdot dr
\]  

(5.25)

Plugging equation (5.23) into equation (5.25) will yield the scattered field in terms of the incident electric field as

\[
E^s(r) = \int_{s} \left( a^r \right)^T \cdot \vec{G}(r', r) \cdot \left[ \frac{E'(r) \cdot t(r)}{\|t(r)\|} \right] \cdot t(r) \cdot dr
\]  

(5.26)
where, $\mathbf{a}^r$ is the polarization vector of scattered electric field and $\mathbf{a}'$ is the polarization vector of the incident wave. Now let us assume that the receiving antenna has a particular polarization such that it collects the scattered field in the $\mathbf{a}^r$ direction [44], [59], [42]. Equation (5.26) is modified as shown below:

$$
E^s(r) = \int \left( \mathbf{a}^r \right)^T \mathbf{G}(\mathbf{r}^r, \mathbf{r}) \cdot \rho(\mathbf{r}_n) \cdot \mathbf{G}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{a}' \cdot d\mathbf{r}
$$

(5.27)

where, $\rho(\mathbf{r}_n)$ is the contrast function and has the same meaning as the classical reflectivity function in ISAR, with the difference that for each point $\mathbf{r}$ there exists nine different parameters to be estimated.

Using the dyadic notation, equation (5.27) is transformed into a discrete form as:

$$
E^s = \sum_{n=1}^{P} \left( \mathbf{a}^r \right)^T \mathbf{G}(\mathbf{r}^r, \mathbf{r}_n) \cdot \rho(\mathbf{r}_n) \cdot \mathbf{G}(\mathbf{r}_n, \mathbf{r}') \cdot \mathbf{a}'
$$

(5.28)

where, $D$ is the domain of interest $\mathbf{r}_n$ is the location of the $n$th pixel, $P$ is the total number of pixels, and $\rho(\mathbf{r}_n)$ is the unknown reflectivity/contrast function which looks like as follows [44], [42]:
where, the contrast function $\rho(\mathbf{r}_n)$ is composed of the sum of the main diagonal and off-diagonal elements. The first term which is the main diagonal matrix represents the effects of the target when there is no depolarization. While the off-diagonal is non zero when there is a depolarization. The second term is obtained through dyadic product.

### 5.6 Formation of a Matrix Equation

Mathematically, the problem of finding the reflectivity function requires that we compute the inverse linear operator of equation (5.28). This is the dipole-based forward scattering model, and relates the unknown reflectivity dyadic function $\rho(\mathbf{r}_n)$ to the measured signal. However, equation (5.28) returns the value corresponding to a single measurement, i.e., a specific transmitter location and orientation, a specific receiver location and orientation, and a specific frequency. If any of these parameters vary, a new measurement must be acquired. Let us collect a set of $(m = 1, M)$ measurements. For a specific measurement $m$ and a specific pixel $p$, equation (5.28) can be rewritten as shown below [42], [62], [59], [63]:
\[ E_{mp} = l_{xx}^{mp} \rho_{mp}^{xx} + l_{yy}^{mp} \rho_{mp}^{yy} + l_{xz}^{mp} \rho_{mp}^{xz} + \ldots \\
+ l_{yx}^{mp} \rho_{mp}^{yx} + l_{yy}^{mp} \rho_{mp}^{yy} + l_{yz}^{mp} \rho_{mp}^{yz} + \ldots \\
+ l_{zx}^{mp} \rho_{mp}^{zx} + l_{zy}^{mp} \rho_{mp}^{zy} + l_{zz}^{mp} \rho_{mp}^{zz} \\
= l_{mp}^{T} \cdot \rho_{mp} \]

(5.30)

where, each value of \( l \) can be determined by properly re-arranging and recasting the terms in of the result in (5.28). By extending the above equation to all pixels in the region \( D \) and all possible measurement configurations \( M \), one obtains [64], [65], [45]:

\[
\begin{bmatrix}
  e_{1} \\
  \vdots \\
  e_{M}
\end{bmatrix}
= \begin{bmatrix}
  l_{11}^{T} & \cdots & l_{1P}^{T} \\
  \vdots & \ddots & \vdots \\
  l_{M1}^{T} & \cdots & l_{MP}^{T}
\end{bmatrix}
\begin{bmatrix}
  \rho_{1} \\
  \vdots \\
  \rho_{MP}
\end{bmatrix}
\]

(5.31)

\[
E' \quad = \quad L \cdot \rho
\]

Simply speaking, \( E' \) is a (known) vector containing all measurements collected at different positions / directions / frequencies, \( L \) is a (large) matrix whose entries can be calculated theoretically from equation (5.28), and \( \rho(r_{n}) \) is a vector representing the (dyadic) reflectivity function of the target. After the creation of the \( L \) matrix from
equation (5.28), equation (5.31) known as forward model must be inverted in order to recover the dyadic reflectivity function. The easiest way to invert equation (5.31) is to employ any regularized method either direct such as Truncated Singular Value Decomposition (TSVD) or iterative method such as Conjugate Gradient (CGM). This thesis conjugate gradient method is used for inversion [33], [45] [42]. Conjugate gradient method is an iterative method which covered in more details in section 5.7.1.1. The reflectivity function can be easily estimated as

$$\rho \equiv L^H E^s$$  \hspace{1cm} (5.32)

However, $\rho(r_n)$ by itself is not intelligible. To retrieve the information concerning each elementary dipole present at the $p$-th pixel, one needs to:

- Estimate the nine reflectivity values (from measured data) at pixel $p$
- Recast them in the dyadic form $\rho(r_n)$
- Solve for $t_x, t_y, t_z$
- Form the vector $t_p = t_x \hat{x} + t_y \hat{y} + t_z \hat{z}$, which represents the direction of the infinitesimal dipole at pixel $p$. 

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5.7 Computational Electromagnetics (CEM)

As the power of computer has increased significantly over the last half century, computational electromagnetics has become very important for analyzing any electromagnetic problems. The numerical approximations of Maxwell’s equation are known as CEM. Maxwell’s theory can predict the design performance or incremental outcome Maxwell’s equations are solved correctly. Maxwell’s theory is valid over broad frequencies from DC to optics and over large dynamic range of length scales, from subatomic to intergalactic. The applications of CEM include but not limited biological EM effects, medical imaging and treatment, electronic packing, superconductivity, microwave devices, antenna, radar and remote sensing, optics and many more [66], [67], [68].

There are many CEM algorithms that have developed over years to solve Maxwell’s equation each with its own strength and weakness. These algorithms are divided into two groups: group (i) is exact solution or low frequency and group (ii) approximate solution or high frequency. Shortly, we will only talk about briefly the method that is relevant to this thesis [19], [69], [70], [71], [72].

Low frequency or exact method is used to solve Maxwell’s equation with no approximations and typically limited to problems of small electrical sizes due to the limited space and computation time. By the way, electrically small in size refers that the
wavelength of operation longer compared to the length of the device. The most common low frequency or exact method are:

- Method of Moments (MoM)
- Finite Element Method (FEM)
- Finite Difference Time Domain Method (FDTD)

High frequency method known as approximation method is used to solve electrically large objects. Common examples of large problems are the computation of radar cross section (RCS), and antenna radiation pattern specially when mounted on large structure. When the target is large in terms of wavelength, the high frequency methods which approximate the fields don’t produce results that are accurate enough. Also the most commonly use ones are:

- Geometrical Theory of Diffraction (GTD)
- Physical Optics (PO)
- Physical Theory of Diffraction (PTD)
- Shooting and Bouncing Rays (SBR)

5.7.1 Method of moment

In this section we will talk about one of low frequency method known as MoM and the reason is that we employed a 3D computational solver known as FEKO in this thesis and FEKO is mainly based on MoM.
The method of moments is a powerful numerical method capable of applying weighted residual techniques to convert integral equation to matrix equation [73], [74], [75]. Then the matrix equation is executed through inversion, elimination or iterative techniques such conjugate gradient method (CGM) which we will use in this thesis to solve some of the equations covered in chapter 5. This method is the most commonly used numerical technique in electrically small objects. It is applied to the problems involving currents on a metallic and dielectric objects and radiation in free space.

In MoM the radiating or scattering objects is replaced by equivalent currents. Derivation of equivalent currents is covered in section 2.4. This surface current is discretized into wire segments or surface patches. A matrix equation is derived representing the effect of every segment and then computed using Green’s function. The theory and derivation Green’s function is covered in section 3.1.

Since MoM is low frequency method, the objects are electrically small with respect to the wavelength of operation and they are typically made of good conductors such as metals. As we stated earlier, MoM is a full wave solution of Maxwell’s integral equations in the frequency domain. The advantage of MoM over the other low frequency method is that only the object of interest is investigated but not the free space as with field method such as FEM and FDTD [70], [73]. The MoM is used to solve equation of the form shown below:
\[ Ax = b \quad (5.33) \]

where \( b \) is a known excitation, \( A \) is linear operator and \( x \) is unknown response.

MoM is applied to integral equations of the following form:

\[ E_z^i = -\int I(z')K(z,z')dz' \]

where \( K(z,z') \) is the kernel \( (5.34) \)

Equation (5.34) is the form of Pocklington’s integral equation which is covered in section 1.4.1. The general form of Pocklington’s integral equation by using the thin wire approximation is shown in equation (1.21).

5.7.1.1 Conjugate gradient method (CGM)

In (chapter 2) of this thesis, we derived set of integral equations in order to solve radiation and scattering problems which is basis of this thesis. These equations cannot be solved analytically. Therefore we have to employ some sort of computational method such as Conjugate Gradient (CGM) in order to get a solution.

The conjugate gradient is a method for solving a matrix system where \( (A) \) is symmetric and positive definite. It is related to the method of steepest descent by minimizing a quadratic function in terms of generating a sequence of conjugate direction
search vectors that are \((A)\) orthogonal at each step \([76], [77]\). The conjugate gradient algorithm can be applied directly to any discrete equation as equation (5.37). The first step is to choose a discretization scheme and then apply to the integral equation. If the conjugate gradient is stopped after several iterations, it is better than the Gaussian elimination in terms of efficiency. We can apply CGM to the following equations shown below \([75], [76], [77], [78]\).

\[
A^\dagger Ax = A^\dagger b
\]

where \(A^\dagger\) is the conjugate gradient \hspace{1cm} (5.35)
and \(A^\dagger A\) is symmetric and positive definite.

An algorithm for the preconditioned (CGM) for the normal equations is summarized below \([77], [78]\):
Initial steps:

Guess \( x_0 \)

\( r_0 = b - Ax_0 \)

\( \bar{r}_0 = A^t r_0 \)

\( z_0 = M^{-1} \bar{r}_0 \)

\( p_0 = z_0 \)

For \( i = 1, 2, \ldots \) until it converges

\( w_{i-1} = Ap_{i-1} \)

\( a_{i-1} = \frac{(z^*_{i-1} \cdot \bar{r}_{i-1})}{\|w_{i-1}\|^2} \)

\( x_i = x_{i-1} + a_{i-1} p_{i-1} \)

\( r_i = r_{i-1} - a_{i-1} w_{i-1} \)

if \( \|r_i\|_2 < \varepsilon \|b\| \), stop the iteration

\( \bar{r}_i = A^t r_i \)

\( z_i = M^{-1} \bar{r}_i \)

\( \beta_{i-1} = \frac{(z^*_{i} \cdot \bar{r}_{i})}{(z^*_{i-1} \cdot \bar{r}_{i-1})} \)

\( p_i = z_i + \beta_{i-1} p_{i-1} \)

end

Where \( M \) is the preconditioner matrix. The algorithm shown above requires two matrix vector products per iteration, one with \( A \) and one with \( A^t \).

CGM is the most prominent iterative method for solving linear equation. Iterative methods of solving \( \mathbf{E}^s = \mathbf{L} \cdot \mathbf{p} \), such as the conjugate gradient method, create a
sequence of approximations that converge in theory to the exact solution [72], [68]. These methods require forming products $L \cdot \rho$ and updating $E^s$ as a result. This method is useful for my thesis for the following reasons.

- You only have to form products of a sparse matrix and a vector.
- Iterative method is self-correcting, meaning it rounds off error.
- And if very accurate solution is not needed, it can be stopped very early and we chose the maximum iteration to be 10 for this thesis.

5.8 Simulation Results

Below, we have done 2D and 3D simulation analysis. The first simulation is based on constructing 2D image and the results and procedures are shown in section 5.8.1 below. The second simulation is reconstructin 3D image and the result and how its done is shown section 5.8.2 as well.

5.8.1 2D simulation results

The proposed algorithm was tested using an accurate EM simulation software tool known as FEKO and Matlab to reconstruct a 2D image of the target (thin cylinder) and 2D image of two thin cylinders with three different orientations. In the first example we studied and constructed a 2D image which shows three versions of the target such as $\rho_{xy}$
, $\rho_{yy}, \rho_{zz}$. This yields more target information. In the second example we have done three cases. In the first case we both placed two thin cylinders and source along $x$-direction as shown in figure 5.17. The second case we placed the two thin cylinders and again source along $y$-direction as shown in figure 5.19. The third case we placed two cylinders along $x$-direction with source and another two cylinders along $y$-direction with another separate source which forms a ring comprising four cylinders shaped as a square as shown in figure 5.21.

In both 2D simulations, the transmitting and receiving antennas are located in the $x$-$y$ plane. In the simulation we used 72 transmitters and 72 receivers. All 144 antennas are assumed to transmit and receive either one of the three orthogonal polarizations $(\hat{x}, \hat{y}$ and $\hat{z})$ or two of them combined. The transmitting antennas are placed along a radius of $13\lambda$ and the receiving antennas are also placed at $10\lambda$ with respect to the target. The operating frequency is $10 \text{ GHz} \ (\lambda = 3 \text{ cm})$ and the total measurement collected size is $(N_t \times N_r \times 9 = 72 \times 72 \times 9 = 46656)$; where $N_t$ and $N_r$ are number of transmitters and receivers and the 9 values of reflectivity function. The area under investigation is $0.4 \text{ m}$ by $0.4 \text{ m}$, which divided into pixels with a size of $0.05\lambda$ and comprised of $(N_t \times N_r \times 9 = 72 \times 72 \times 9 = 46656 \text{ unknowns})$. Data is collected using a sensor platform encircling the cylinder (360 degrees, see figure 5.14). After the scattered field is collected, we computed the $L$ matrix from equation (5.28) as stated earlier in section 5.5. $L$ matrix is nothing more than computing the dyadic Green’s function in homogeneous medium through Matlab. In equation (5.28) the dyadic Green’s function relates the
transmitter to the target of interest (TI) and from the target back to the receiver. After acquiring $\mathbf{E}_s$ through FEKO simulation and calculated the $L$ matrix, we generated the image of the target by employing technique known as conjugate gradient method (CGM). (CGM) is the most prominent iterative method for solving linear equation.

The objective of this simulation is to employ the vector dyadic reflectivity function which comprises nine elements rather than one scalar element. The nine elements not only allow us to determine: 1) the location, but also 2) the orientation of the electrically small dipole in which the scalar con not be determined due to the point source model. Figures 5.18, 5.20 and 5.22 are based on the nine elements of the vector dyadic reflectivity function which allows us to determine the orientation and gives us more details of the target. For instance, figure 5.18 represents when both the target and the dipole orientation are along the $x$ direction and indeed detection occurs, while figures 5.20 and 5.22 represents dipoles oriented along the $y$ and $z$ directions, respectively. The dipole is detected only when both the target and the sensor antenna have the same polarization. Therefore, using the dipole-based model in equation (5.28) one can determine the orientation of such dipoles, thus providing additional information concerning the target. Note that if a point-scattering model were used, all the nine elements of figures 5.18, 5.20 and 5.22 would have appeared equal like figure 5.15.
Figure 5.9 A cylinder rotated 45°. The target is illuminated with plane wave (θ = 0:360°) from elevation angle.

Figure 5.10 This is the wire grid of the figure 5.9 and it is reconstructed through (NEC). NEC is a numerical electromagnetic code which employs method of moment. Method of moment (MoM) is covered in section 5.7.1.
By using this MoM, we converted figure 5.9 into a series of wires creating wireframe model of the cylinder. The series of wires of figure 5.10 is then broken down into segments. Each segment is short with respect to the wavelength of interest. These segments carry current which then affects the current on each one of them. To obtain the currents on each segment a set of linear equations is created and solved through computer. Once the current on each segment is known, we can compute both near and far field through superposition principle.

Figure 5.11 In the left we have the far field radiation pattern of scattered electric field of a single wire from figure 5.10 oriented along z-axis and in the right we have the current distribution of the same wire. This is done through (NEC).
Figure 5.12 The illumination geometry of a dipole model from FEKO.

Figure 5.13 Radiation pattern of dipole from FEKO.
Figure 5.14 Matlab simulation configuration. A cylinder is placed at \((x = y = 0, z)\). The inner circle (blue) represents number of receivers where the outer circle (red) represents number of transmitters.

The first time we have used the classical model known as point source model (PSM), see figure 5.15 and the second time we will employ the proposed method known as dipole model (DP), see figure 5.16.
Figure 5.15  This image is the reconstruction of figure 5.14 in terms of point source model (PSM).

When used point source model as seen above, all nine elements have shown equal response no matter what angle that the target is illuminated from because this method is
based upon the assumption that the target of interest consists of a superposition of infinitesimally small isotropic scatterers. This method is inefficient in term of either determining the orientation of the target or showing the features of the target compared with dipole model as shown below.

\[
\rho(r) = \begin{bmatrix}
\rho_{xx} & \rho_{xy} & \rho_{xz} \\
\rho_{yx} & \rho_{yy} & \rho_{yz} \\
\rho_{zx} & \rho_{zy} & \rho_{zz}
\end{bmatrix}
\]

Figure 5.16 This image is the reconstruction of figure 5.14 in terms of dipole model (DM).
As can be seen in figure 5.16 we got three responses out of the nine elements of $\rho(r_n)$ such as $\rho_{xx}$, $\rho_{yy}$ and $\rho_{zz}$. $\rho_{xx}$ is in the upper left corner and that shows the edge of the cylinder along $x$-direction, where $\rho_{yy}$ is in the middle and that also shows the edge of the cylinder along $y$-direction and $\rho_{zz}$ is in the lower right corner which shows the full image of the cylinder since its oriented along $z$.

In figure 5.16 we could see that not only we get strong response when the incident field is aligned with dipole for detection but also we could see more details of the target such as features and orientation of the cylinder which point source model neglects to capture.
Figure 5.17 FEKO model. Two cylinders oriented along x-direction where $E^i$ is the incident plane wave and $k^i$ is the propagation vector.

The goal is to image the two cylinders by employing the proposed method known as dipole model (DM) and show not only we detect the target and image it but also discern the orientation of the target which is very important.

The electric field $E^i$ and the target are aligned in the same direction and this will mean only one of the nine elements of reflectivity function will have response as shown in figure 5.18.
Figure 5.18 Simulation result: Two thin cylinders along x-direction are imaged.
The above figure is based on the nine elements of vector of dyadic reflectivity function and the aim is to construct the image of figure 5.17. In Figure 5.18 the target was illuminated from along x-axis and we got strong response only when both the antenna and the target have the same orientation along x-axis as shown on $\rho_{xx}$. If this would be point source model (PSM), we would have gotten the same response in all nine elements since it assumes a scatterer radiates equally in all direction as shown in figures 4.8-4.10, 4.19 and 5.15.

![Diagram](image)

Figure 5.19 FEKO model. Two cylinders oriented along y-direction where $E^i$ is the incident plane wave and $k^i$ is the propagation vector. This time we only expect $\rho_{yy}$ response out of the nine elements of reflectiveity function as shown below.
Figure 5.20 Simulation result: Two thin cylinders are placed along y-direction.

$$\rho \left( \mathbf{r}_n \right) = \begin{bmatrix} \rho_{xx} & \rho_{xy} & \rho_{xz} \\ \rho_{yx} & \rho_{yy} & \rho_{yz} \\ \rho_{zx} & \rho_{zy} & \rho_{zz} \end{bmatrix}$$
Figure 5.20, the target and the source (dipole) are placed along y-axis and we got strong response from $\rho_{yy}$ out of the nine elements of reflectivity function $\rho(r_n)$. This shows again that when both the antenna and the target have the same orientation along y-axis $\rho_{yy}$ detection is declared only on a specific location of the reflectivity function and that allows to extract the orientation of the target.

Figure 5.21 The geometry of the problem from FEKO. This is four cylinders shaped as a square which is illuminated with incident plane wave $E^i$ both along x - axis and y - axis to create the full image of the four cylinders and $k^i$ is the propagation vector.
\[ \rho(r_n) = \begin{bmatrix} \rho_{xx} & \rho_{xy} & \rho_{xz} \\ \rho_{yx} & \rho_{yy} & \rho_{yz} \\ \rho_{zx} & \rho_{zy} & \rho_{zz} \end{bmatrix} \]

Figure 5.22 Simulation result: The reconstructed image of two thin cylinders are placed along x-direction, another two are placed along y-direction which formed square.
Figure 5.22, is based on reconstructing the full image of figure 5.21 which comprises four cylinders shaped as a square. Finally, we illuminated the target both from x-axis and y-axis as shown in figure 5.21 and we got $\rho_{xx}$, $\rho_{yy}$ and $\rho_{zz}$ responses as expected and shown in the figure 5.22. This shows that we only get response when the target and the source (dipole) are aligned because when the incident field $E^i$ strikes the target in parallel, a current is induced which in turn creates radiated field $E^s$. That radiated field is collected by the receiver (dipole), processed and detection is declared. Field alignments with the dipole not only detection occurs but also it provides more about the target such as features and orientation, otherwise we would have gotten an equal response in all nine elements no matter which angle we illuminated from the target as point source model assumes and shown in figure 5.15.

Figure 5.23 A cylinder is placed at 225° angle in the xy plane.
Figure 5.24 This figure is based on figure 5.23. A cylinder is placed at 225° in the x-y plane and we got a response only at $\rho_{xx}, \rho_{xy}, \rho_{yx}, \rho_{yy}$, and $\rho_{zz}$ for the image due to the dipole model.
Figure 5.25 A cylinder is placed at 45° angle in the xy plane.

\[ \begin{pmatrix} \rho_{xx} & \rho_{xy} & \rho_{xz} \\ \rho_{yx} & \rho_{yy} & \rho_{yz} \\ \rho_{zx} & \rho_{zy} & \rho_{zz} \end{pmatrix} \]

Figure 5.26 Notice that there are x and y component but not z component due to the orientation of the cylinder, which led to three responses out of the nine elements as shown in the above figure.
5.8.2 3D simulation results

In the first 3D simulations we used 20 transmitters and 150 receivers. All 170 antennas are transmitting and receiving only in one polarization either \((\hat{x}\text{or }\hat{y}\text{or }\hat{z})\) for the first three figures and they are located in the x-y plane and mixed polarization for last one. The transmitting antennas are placed along a radius of \(7\lambda\) and the receiving antennas are also placed at \(5\lambda\) with respect to the target. The operating frequency is \(10\text{ GHz } (\lambda = 3\text{cm})\) and the total measurement collected size is \((N_t \times N_r \times 1 = 20\times150\times1 = 3000)\); where \(N_t\) and \(N_r\) are number of transmitters and receivers and the one is one of the nine pixels of reflectivity function and this case it’s the \(\rho_{zz}\) which has the same orientation of the cylinder. The simulation area is 0.25 m by 0.25 m, which divided into pixels with a size of \(0.05\lambda\) and comprised of \((N_t \times N_r \times 1 = 20\times150\times1 = 3000 \text{ unknowns})\). We measured multiple cut along z-axis and finally summed up all the cuts for figure 5.27 and did the same thing figures 5.28-5.29. The images are generated the same procedure stated in section 5.6 and the theory come from sections 5.4-5.5. The result obtained from this simulation is shown in figures 5.27-5.31 below.
Figure 5.27 Simulation result: 3D image of vertically oriented cylinder. The cylinder and the antenna are both oriented along z - axis. The target is detected due to the same alignment with incident field.

Figure 5.28 3D image of a horizontally oriented cylinder. The cylinder and the antenna are both oriented along the x - axis. The target is detected due proper alignment with incident field.
Figure 5.29 3D image of a horizontally oriented cylinder. The cylinder and the antenna are both oriented along the y-axis. The target is detected due to the same alignment with incident field.

Figure 5.30 Both the antenna and the cylinder are rotated by 45° along x-axis and detection occurs.
In the second simulation we used the same transmitter and receivers used in the first simulation but this time all 170 antennas are assumed to transmit and receive in all three orthogonal polarizations ($\hat{x}$, $\hat{y}$ and $\hat{z}$). And the reason is that, we need the capture the orientation of the cylinders that are either oriented along x-axis or y-axis as shown in figure 5.18 and 5.20. The operating frequency is the same with all simulation and the total measurement collected size is $(N_t \times N_r \times 1 = 20 \times 150 \times 9 = 27000)$; where $N_t$ and $N_r$ are number of transmitters and receivers and the 9 values of reflectivity function and this time we added the result of 9 elements to generate the image shown in figure 5.27. The simulation area is 0.15 m by 0.15 m, which divided into pixels with a size $0.05\lambda$ and comprised of $(N_t \times N_r \times 1 = 20 \times 150 \times 9 = 27000 \text{ unknowns})$. If this simulation is not used the dipole model which based on dyadic reflectivity function we would not be able to tell the cylinders that are either oriented in x-direction or y-direction but now we see

![Field Strength](image)

Figure 5.31 3D quiver image for vertical cylinder
two cylinders are along x-direction while the other two is along y-direction as shown in the 2D model of figures 5.18, 5.20 and 5.22. The image is generated the same procedure stated in section 5.6 and the theory come from sections 5.4-5.5. Excellent results illustrate the power of the dyadic kernel based approach to image reconstruction. The result obtained from this simulation is shown in figure 5.32 below.

Figure 5.32 Simulation result: This is based on the 3D image of figure 5.21. The target is placed at XY plane and then taken a scan cut along z- direction and sum them up to generate the 3D version of $\rho_{zz}$ on the right and one slice cut is on the left.
5.9 Experimental

To this aim, the robotic tomographic chamber at the University of Dayton MUMMA radar laboratory was used to conduct an experiment. The tomographic chamber is equipped with four robotic arms, each one having a dual polarized antenna as shown in figures 5.33-5.35, operating up to 12 GHz.

5.9.1 Setup and instructions

- Thin wire: The radius of the wire is small relative to the wavelength of operation \( r << \lambda \) and the current is constant azimuthally.
- This produces both vertical and horizontal response of the wire.
- The wire is illuminated by the transmitter either orthogonal polarization (TE) or parallel polarization (TM).
- When the target is illuminated horizontally polarized wave, the backscattered wave can have contributions in both horizontal and vertical polarization due to depolarization effect. Therefore, the receiver receives both vertically and horizontally polarized wave as shown figure 5.33.
The electromagnetic wave vector interrogation with material is described by the scattering operator shown below $S(k_s, k_i)$, [64].

Where $k_s$, $k_i$ represents the wave vector of the scattered and incident fields.

$$E^i(r) = E_0^i e^{-jk_i r}$$

$$E^s(r) = E_0^s e^{-jk_s r}$$

Scattered field $E^s(r)$ is related to the incident field $E^i(r)$ as shown below

$$\begin{pmatrix} E^s_h \\ E^s_v \end{pmatrix} = e^{-jk_v r} \begin{bmatrix} S_{hh} & S_{hv} \\ S_{vh} & S_{vv} \end{bmatrix} \begin{pmatrix} E^i_h \\ E^i_v \end{pmatrix}$$

Total scattered power

$$|S_{hh}|^2 + 2|S_{hv}|^2 + |S_{vv}|^2$$
Figure 5.34 Flow chart measurement set up.

Measurement setup

Thin/Thick wire response

Co polarization S11 response

Yes

No

Cr- polarization S22 response

Yes

No

Plot range profile

Co polarization S11 linear scan

Yes

No

Cr- polarization S22 linear scan

Yes

No

Compute image/contrast function through inverse technique as

\[ \rho \equiv L^H E^z \]

to generate 2-D image of the thin/thick wire.

Where

- \( \rho \) is the unknown image function
- \( E^z \) is the measured scattered field of both polarizations
- \( L^H \) is large matrix whose value is computed theoretically.

Exit
A thin cylinder oriented horizontally

Figure 5.35 Experimental setup in Mumma Radar Lab.

- 8 Channel Network Analyzer 13GHz
- 4 Channel Oscilloscope 13GHz
- 4 Channel AWG 600MHz
- Two Signal Generators 20GHz
- Spectrum Analyzer 20GHz
- Channel Emulator GPS/LTE/SATCOM
- Vector Signal Generator and Analyzer
- Power Meter
- Soldering Rework Station
- Atomic Clock
- Five Workstations
- FEKO

Figure 5.36 Measurement equipment used for the experiment.
Specifications of Mumma Tomographic Chamber (MTC)

- **Sensing Capability**
  - Four Independent TX and RX channels
  - Simultaneous transmission and reception
  - Output Power 1W
  - Receiver Gain 20dB
  - Dynamic Range > 100dB

- **Antenna Placement Accuracy**: 150 μm

- **Frequency of Operation**
  - 3KHz to 13GHz Nominal
  - 2-12 GHz Antenna Limited
  - 8-12 GHz Quiet Zone Limited

- **Data Acquisition**
  - Network Analyzer S Parameters
  - Arbitrary Waveforms 600MHz Tuned to 2-12 GHz

- **Maximum speed**
  - Location dependent
  - Typically 25m/s
5.9.2 Results

In this experiment, the target of interest (TI) is a thin cylinder which was placed on the top of the absorbing material. The target is oriented horizontally as illustrated figure 5.35. The transmitter is placed 40 cm from the target, and the receiver is placed 30 cm from the target. The transmitter and receiver are separated by 30 cm with a 30° bistatic angle. Imaging measurements are performed in this experiment where both the transmitter and receiver are repositioned along the target linearly. As they move, data is collected using an 8-port vector network analyzer. The frequency domain data is stored...
and used to populate the measurement vector $E^s$ in (5.32) and then acquire the reflectivity function which corresponds the image function. The image was reconstructed using the procedure described in sections 5.4 and 5.5 and our experiments result in the image is shown in figures 5.38-5.40.

Figure 5.38 Experimental data: Range profile of the target.

Figure 5.39 Experimental data: Image of a 6 inch copper cylinder in slow time and fast time.
10 Conclusions

In this thesis we developed an efficient, practical and improved method for ISAR imaging, known as the dipole model based approach, and we rely upon a vector dyadic reflectivity function which comprises a nine elements rather than point source model. Point source model is based on scalar reflectivity function which assumes all targets scatter energy equally in all direction. Under the dipole model based approach, we
constructed a 2D ISAR images of two parallel cylinders with different orientation by computing (and plotting) the nine dyadic elements. Each time, detection declaration occurred when the antenna and the cylinders are co-polar, as shown in figures 5.18, 5.20 and 5.22. and see the 3D version of figure 5.22 which is corresponds figure 5.32. When the scalar method, which is based upon a point source model, is used, in chapter 4 detection occurred each time no matter which angle we illuminated from the target and we were not be able to discern the orientation of the target or have more information about it. Also, we constructed an image of a single cylinder along z-axis, as illustrated above, see figure 5.16 where in the upper left corner and in the upper right corner show the edges of the cylinder along (x and y-direction). Included herein is the image of a half inch diameter, 6 inch long copper cylinder, produced experimentally by exploiting co-pol and cross-pol see figures 5.38-5.40.

5.11 Future Work

In the course of this work, it has become clear that the principle of imaging an object to dipole based model rather than to a point can also be extended to dynamic imaging. We mean dynamic imaging an object changes its shape as it moves such telling what the time is in real time on a clock. As time goes by the hands of the of the clock moves by facing different direction and that leads to different shape.

So the future plan is take a clock mount it on one of our Mummm radar lab robot and tell what the time is by imaging the clock in real time as shown below.
Figure 5.41 The future goal is to image this clock in real time.

So far we have done two static examples on this issue by using 3D computational software known as FEKO see figures 5.42-5.44 below.

Figure 5.42 This clock shows the time is 9AM and the static image of this is shown below.
Figure 5.43 We reconstructed the static image of see figure 5.42 by employing the dipole model (DP). Even though three out of the nine of reflectivity function $\rho_{zz}$ have responses, looking at the right lower corner of this figure which corresponds $\rho_{zz}$ shows that the time on the clock is 9 AM.
Figure 5.44 Out of the nine reflectivity function, only three of them have responses because the clock is placed in the xy plane. Looking at the right lower corner of this figure which corresponds $\rho_{zz}$ shows that the time on the clock is 3 PM.
LIST OF MY PUBLICATIONS


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APPENDIX

The expansion of equation (5.28)

\[
L = \sum_{n=1}^{P} (\mathbf{a}^r)^T \cdot \mathbf{G}(\mathbf{r}^r, \mathbf{r}_n) \cdot \mathbf{G}(\mathbf{r}_n, \mathbf{r}^t) \cdot \mathbf{a}^t
\]

\[
= \begin{bmatrix}
L_{xx} & L_{yy} & L_{zz} & L_{xy} & L_{xz} & L_{yz} & L_{zx} & L_{zy}
\end{bmatrix}
\]

where:

\[
L_{xx} = \left( a_x^r G_{xx} + a_y^r G_{yx} + a_z^r G_{zx} \right) \cdot \left( G_{xx} a_x^t + G_{xy} a_y^t + G_{xz} a_z^t \right)
\]

\[
\mathbf{G}(\mathbf{r}_n, \mathbf{r}^t) = \begin{bmatrix}
G_{xx}^t & G_{xy}^t & G_{xz}^t \\
G_{yx}^t & G_{yy}^t & G_{yz}^t \\
G_{zx}^t & G_{zy}^t & G_{zz}^t
\end{bmatrix}
\]

\[
\mathbf{a}^t = \begin{bmatrix}
a_x^t \\
a_y^t \\
a_z^t
\end{bmatrix}
\]

Dyadic Green's function

Tx polarization
Dyadic Green's function

\[ \bar{G}(r', r) = \begin{bmatrix} G_{xx}^r & G_{xy}^r & G_{xz}^r \\
G_{yx}^r & G_{yy}^r & G_{yz}^r \\
G_{zx}^r & G_{zy}^r & G_{zz}^r \end{bmatrix} \]

Rx polarization

\[ a^r = \begin{bmatrix} a_x^r & a_y^r & a_z^r \end{bmatrix} \]

Reflectivity function

\[ \rho = \begin{bmatrix} \rho_{xx} & \rho_{xy} & \rho_{xz} \\
\rho_{yx} & \rho_{yy} & \rho_{yz} \\
0 & 0 & 0 \end{bmatrix} \]