DIGITAL PHASE CORRECTION OF A PARTIALLY COHERENT SPARSE APERTURE SYSTEM

Thesis
Submitted to
The School of Engineering of the
UNIVERSITY OF DAYTON

In Partial Fulfillment of the Requirements for
The Degree of
Master of Science in Electro-Optics

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UNIVERSITY OF DAYTON
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August 2015
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ABSTRACT

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Sparse aperture image synthesis requires proper phasing between sub-apertures. Phasing can be difficult due to hardware misalignments, atmospheric turbulence, and many other causes of optical path differences (OPD). Common synthesis techniques include incoherent and coherent methods. Incoherent methods utilize passive illumination and adaptive optics while coherent methods rely on active illumination and phase reconstruction approaches such as phase retrieval or spatial heterodyne. In this thesis, we present a partially coherent technique with the capability to use either active or passive illumination to digitally correct for piston phase errors. This technique requires an anamorphic pupil relay system and a piston correction algorithm. The anamorphic pupil relay causes two closely spaced sub-apertures in the entrance pupil to appear to be shifted further apart in the exit pupil. Analytic and numerical wave optics models demonstrate the effectiveness of this relay system, matching with experimental results. An analytic model shows that the higher frequency terms are equivalent to scaled cross-
correlations of the two sub-apertures, which are shifted due to the anamorphic separation. The constant shifts due to the separation are found experimentally using a registration algorithm with a calibration target. The cross-correlations are dependent on the piston phase errors between sub-apertures. We show that a piston correction algorithm can be used to shift the cross-correlations to their original positions dictated by the entrance pupil, multiply a cross-correlation with the complex conjugate of the auto-correlation, use the summation of this product to calculate the piston, and correct the phase error in each cross-correlation before recombining them with the auto-correlation. Examples show diffraction limited results for both simulated and experimental images that are supported by analytical, numerical, and experimental analysis of the system’s modulation transfer function (MTF). In addition, analysis of the piston for multiple wavelengths reveals two effects of bandwidth on the system. First, the impact of a constant spatial separation resulting in different spatial frequency shifts for different wavelengths, referred to as field dependent contrast (FDC), is addressed. Second, the inverse relationship between the bandwidth of the system and OPD tolerances is shown. In future research, diffraction gratings could eliminate the FDC so that partially coherent illumination with a small enough system bandwidth could relax OPD requirements for the sparse aperture array.
ACKNOWLEDGEMENTS

I would like to thank Brian Ewert, Chief of the LADAR Technologies branch of the Air Force Research Lab, AFRL/RYMM for the opportunity, laboratory space, and access to equipment needed to complete this thesis. I highly appreciate the guidance of Dr. David Rabb, my thesis advisor, and owe him a thank you for his time and patience. I would also like to thank Dr. Matthew Dierking, Larry Barnes, Dr. John McCalmont, and Dr. Edward Watson for their advice while working on this project. Gratitude is given to Dr. Michael Eismann for his initial conceptualization of this thesis topic, Jason Stafford for his help in the laboratory, and Doug Moore in the machine shop for modifying equipment needed for the experiment.

This effort was supported in part by the U.S. Air Force through contract number FA8650-12-D-1377, and the University of Dayton Electro-Optics Program (EOP). The views expressed in this article are those of the authors and do not reflect on the official policy of the Air Force, Department of Defense, or the U.S. Government.
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CHAPTER 1

INTRODUCTION

Aperture synthesis is a useful technique for imaging at higher resolutions while reducing the size, weight, and cost of the imaging system. However, combining the images from multiple sub-apertures calls for precise alignment, which can be difficult to achieve. One of the most common issues in aperture synthesis of sparse aperture systems is piston phase errors caused by path differences between sub-apertures that reduce the resolution of a system. A partially coherent aperture synthesis method is proposed that can digitally correct piston errors caused by hardware misalignments and atmospheric turbulence. Past aperture synthesis methods generally used fully incoherent or coherent light. A typical incoherent sparse aperture system uses broadband, passive illumination and hardware piston corrections. Conversely, coherent systems require narrowband laser illumination and allow for digital corrections. Partially coherent systems can be used for digital piston phasing of passive and active illumination while easing alignment requirements. Before discussing digital piston correction a brief overview of sparse aperture systems, piston phase errors, and incoherent and coherent phase correction techniques is included.
1.1 Sparse Aperture Imaging [1]

The diffraction limited resolution of a single monolithic aperture is proportional to the aperture’s diameter, allowing higher resolutions to be imaged by using a larger aperture. However, the difficulties involved in manufacturing and handling large mirrors and lenses can make it more convenient to use sparse aperture systems. Traditionally, sparse aperture systems utilize multiple small sub-apertures in order to minimize the light collecting area while maintaining a large aperture diameter. However, in cases where the goal is to maintain the collecting area while minimizing the total volume of the system, the sub-apertures may be close together which can improve the ability to digitally correct for piston errors where such techniques are possible. The diagram below illustrates a sparse aperture system with two sub-apertures.

Each sub-aperture in the system has a cutoff frequency from diffraction in object space defined by the diameter $2w$,

$$ f_{sub} = \frac{2w}{\lambda z} \quad (1) $$
Where $\lambda$ is the wavelength, $z$ is the distance from the object to the sub-aperture, and $f_{\text{sub}}$ is the incoherent cutoff frequency of the sub-aperture in cycles per meter. With aperture synthesis, the array in Figure 1 has a resolution along the $y$-axis equivalent to the cutoff frequency of a larger aperture,

$$f_{\text{total}} = \frac{D}{\lambda z}$$  \hspace{1cm} (2)

Where $f_{\text{total}}$ is the incoherent cutoff frequency of the sparse aperture along the synthesized dimension. Without aperture synthesis, the resolution of the array will be the same as that of a single sub-aperture.

One of the earliest uses of aperture synthesis techniques was the construction of radio interferometers in the 1940s. For example, the antenna array at the Very Large Array (VLA) collects both amplitude and phase information from astronomical events. The complex data from each antenna is used to construct a radio image of much higher resolution than any single antenna could create. Similar sparse aperture techniques have been used to create higher resolution images in infrared and optical wavelengths. However, image synthesis at shorter wavelengths faces additional difficulties with phasing aperture data than compared with radio interferometry.

1.2 Detecting Piston Phase Errors

Aperture synthesis is only possible if the inter-aperture phase relationships between all the sub-apertures of an array are correct. Optical path differences between sub-apertures lead to piston phase differences and can be caused by hardware misalignment or atmospheric turbulence. Since radio antennae are capable of collecting both amplitude
and phase information, post-collection piston corrections are relatively simple. Data from a single radio event can be isolated at each antenna [2]. The amplitudes and phases from each antenna are compared to find the phase relationships, allowing piston corrections to be applied before the final images are synthesized.

Phasing apertures in the infrared and visible range is much more difficult. Compared to gigahertz radio frequencies, near-infrared and visible frequencies have wavelengths orders of magnitude smaller. At these wavelengths most sensors only record intensity data. The lack of complex data restricts the ability to directly calculate inter-aperture phase relationships and correct for piston errors. Common methods for calculating and correcting phase errors use either incoherent or coherent light. Incoherent techniques use passive illumination and interfere images onto a single focal array. Piston is physically corrected with adaptive optics hardware prior to image interference. Coherent techniques utilize active illumination and record the irradiance fields from each sub-aperture on separate focal plane arrays. Digital reconstruction methods such as phase retrieval or spatial heterodyne can be used to find and correct the piston phase errors [3], [4]. Examples of current passive and active sparse aperture piston correction techniques are discussed in section 1.3. Furthermore, the micrometer wavelength scale may lead to phase differences within a single sub-aperture such as image jitter from “tip” and “tilt” as well as blurring from higher order aberrations. However, for the work presented here these additional intra-aperture phase errors are negligible due to well aligned and diffraction limited sub-apertures, leaving only piston to affect image synthesis.
1.3  Current Sparse Aperture Techniques

Both incoherent and coherent piston correction methods have been developed. Incoherent, or passive, synthesis is discussed in section 1.3.1 and coherent, or active, aperture synthesis is covered in section 1.3.2.

1.3.1  Passive Aperture Synthesis

Incoherent aperture synthesis can only occur in optical systems when images from multiple sub-apertures are combined optically on a single focal plane array. The interference of the images on the focal plane array preserves the phase relationships between all sub-apertures. Piston, tip, and tilt errors can be physically corrected before the interfering beams reach the image plane using adaptive optics. Adaptive optics systems use wavefront sensors and fast response mirrors to correct for wavefront errors by reducing the optical path difference between the sub-apertures to a fraction of a wavelength. They are passively illuminated and the only coherence requirement is related to the spatial coherence diameter of the atmosphere. The coherence diameter is a measure of atmospheric turbulence that is affected by many factors including wind, temperature, and dust, which work to create fluctuations in the atmosphere’s index of refraction. As long as the sub-apertures are smaller than the atmospheric coherence area, the only correction required is the piston phasing of the sub-apertures. When they are larger, a more complex adaptive optics system is required to effectively correct phase errors, typically using a deformable mirror. However, the effectiveness of such a setup is tempered by the complexity of the equipment and the amount of loop bandwidth needed to control the adaptive optic mirrors.
Adaptive optics can be applied to single or sparse aperture systems. The Large Binocular Telescope (LBT) is an interesting example of an adaptive optics system. This telescope consists of two 8.4m primary mirrors spaced 6m apart. Since these two mirrors are much larger than the coherence area of the atmosphere, phase corrections are applied to each mirror separately before the images from the two mirrors are interferometrically phased and interfered at the image plane. Light collected from a laser guide star is measured by a set of wavefront sensors that control two deformable secondary mirrors [5]. This setup is shown in Figure 2.

The secondary mirrors are 0.91m in diameter and each contains 672 actuators that correct for tip, tilt, and piston errors [7]. Each actuator controls a small portion of the mirror that can be thought of as representing a single sub-aperture in an array with a fill factor of unity. The apparent size of an actuator at the conjugate plane to the secondary mirror should be smaller than the coherence area of the atmosphere. If the system only corrects for piston between the two primary mirrors, without correcting for intra-aperture phase
errors, the resolution of each sub-aperture will correspond to the coherence area at that conjugate plane.

Infrared interferometry between the two primary mirrors can be done with the Large Binocular Telescope Interferometer (LBTI) after the corrections for each individual mirror are complete. The LBTI utilizes a second set of wavefront sensors and a single fast response mirror for further compensation of piston error, which co-phases the corrected beams before they interfere on the detector [6].

Figure 3: Optical design of LPTI beam combiner [6]

A laser guide star is used to find the required atmospheric turbulence corrections. Since turbulence is relatively constant over small angles, any star within the isoplanatic angle of the laser guide star can be observed with phase correction [8]. Theoretically, placement of adaptive optics at multiple points in the system will lead to a corrected imaging area greater than the isoplanatic patch.

A simpler adaptive optics setup is demonstrated with Star 9. Star 9 is a nine telescope phased sparse array. Each telescope has two parts: a light collector and a set of
adaptive relay optics to correct tip, tilt, and piston. The relay optics lead to a combiner that interferes all nine images on one focal plane array [9].

![Figure 4: Optical design for one aperture of Star 9](image)

The relay optics consist of five mirrors. The first two mirrors adjust for tip and tilt while the second two correct for piston. The last mirror directs the beam into the combiner. The phase diversity method is used to calculate the wavefront information using a focused and an unfocussed image in order to direct the mirrors [9]. Aperture synthesis is achieved by the adaptive optics systems of both the LBT and Star 9 and results in diffraction limited images with resolutions corresponding to the diameter of each telescope’s sparse aperture array.

### 1.3.2 Coherent Aperture Synthesis

Coherent aperture synthesis techniques address the need for imaging at high resolutions while retaining a short system depth. In a coherent system a laser is used to illuminate the field of view, light reflects off the target, and the field is recorded and reconstructed independently for each aperture in the sparse array. The coherence length...
of the illumination source must be greater than depth variation over the scene in order to properly reconstruct complex pupil fields. The complex fields can be reconstructed using digital holography techniques based on interference with a local oscillator at each aperture, but phase retrieval and self-referencing interferometry could also be used to reconstruct the field without a local oscillator. The sub-aperture images are not interfered physically, but digitally in processing. An example of a coherent aperture synthesis system is shown in Figure 5 [4].

![Diagram of a coherent aperture synthesis technique](image)

**Figure 5: Diagram of a coherent aperture synthesis technique [4]**

An algorithm can be written to correct for the phase differences between sub-apertures in the array. In some cases, this is done using the sharpness metric of the interfered image to find the optimal phase relationships between sub-apertures [4]. Coherent illumination and sharpness metrics make it possible to digitally phase arrays and correct for piston as well as reduce the size and cost of an optical system. However, they come with a new set of problems. For instance, coherent light causes speckle which must be averaged over multiple images.
1.4 Proposed Digital Piston Correction

The proposed aperture synthesis method physically interferes the images on a single focal plane array while reducing the size, cost, and complexity of the hardware by digitally correcting the piston errors. The long coherence length of a coherent system is not required, but a coherence length longer than the dynamic path difference of the sub-apertures is still needed. This relaxed partial coherence requirement permits the use of passive as well as active illumination and reduces the need for speckle averaging. The phasing technique used for partially coherent digital piston correction is similar to some coherent aperture synthesis methods. It also provides the ability to phase apertures independently for different areas within the field of view, which has the potential to extend the system’s effective field of view beyond the isoplanatic angle. This phase correction technique is possible through an anamorphic pupil relay system.

The entrance pupil of this imaging system resembles the diagram in Figure 1. The sub-apertures are close enough together to give a filled frequency response. This provides complete spatial frequency information in the image within the frequency cutoff of the optical system. The frequency response of the entrance pupil has two parts: the auto- and cross-correlations of the individual sub-apertures. The cross-correlations contain the inter-aperture phase information while the auto-correlations do not. The phase differences between the sub-apertures can be digitally found and corrected by isolating the cross-correlations from the auto-correlations. This isolation is done by physically increasing the relative separation of the sub-apertures between the entrance and exit pupil planes. The separated cross-correlations can be digitally registered with the auto-correlations in order to calculate the phase difference before being recombined.
with the auto-correlations to resemble the spatial frequency of the system without the relative increase in pupil separations. This will result in a diffraction limited image.

More rigorous analytical and simulated treatments of this pupil separation are covered in CHAPTER 2 and CHAPTER 3. CHAPTER 4 illustrates the experimental setup and data collection while CHAPTER 5 discusses the algorithm written to digitally correct for piston errors from the data collection. The experimental results are displayed and compared with analytical and numerical models in CHAPTER 6. CHAPTER 7 concludes this discussion of partially coherent digital phase corrections and mentions some potential plans for future research.
CHAPTER 2

PARTIALLY COHERENT APERTURE SYNTHESIS THEORY

Partially coherent digital piston correction can be obtained by using an anamorphic pupil relay system and a piston correction algorithm. Knowledge of the effects of the anamorphic pupil relay on the frequency response of the system can be used in the piston correction algorithm. Section 2.1 explains the terminology and describes the association between the pupil configuration and spatial frequency of an imaging system. This is important for the quantitative analysis of pupil separation in section 2.2. The equations in section 2.2 support a brief description of the piston correction algorithm presented in section 2.3. Further analytical models of the MTF of the system as well as in an in-depth discussion of the effects of partially coherent, “broadband” illumination on the system’s performance are found in sections 2.4 and 2.5.

2.1 Simple Imaging System [10]

The theory behind digital piston correction relies heavily upon the relationship between the spatial and spatial frequency domains of the imaging system. Final imagery is given in spatial irradiance while the digital corrections must be done using the spatial frequency domain. It is helpful to understand the mathematical connection between these two domains in order to correctly model and code the piston correction. This relationship
is explained through a simple imaging system before more complicated anamorphic sparse aperture systems are introduced.

A simple imaging system contains an object plane, pupil plane, and image plane in which \((x, y)\) are the spatial coordinates for the object and image planes while \((\xi, \eta)\) are the spatial coordinates in the pupil plane. If a complex target at the object plane is defined as \(u_{\text{obj}}(x, y)\), the Fresnel diffraction integral can be used to propagate the field a distance \(d_0\),

\[
e^{\frac{ikd_0}{i\lambda d_0}} e^{\frac{i\pi}{\lambda d_0}(\xi^2+\eta^2)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_{\text{obj}}(x, y) e^{\frac{i\pi}{\lambda d_0}(x^2+y^2)} e^{-\frac{2\pi}{\lambda d_0}(x\xi+y\eta)} dx dy
\]

For the purposes of this analysis, \(\frac{e^{ikd_0}}{i\lambda d_0}\) is a propagation constant that can be ignored.

Equation (3) represents the complex field located just before the pupil plane. Without the outer curvature term, \(e^{\frac{i\pi}{\lambda d_0}(\xi^2+\eta^2)}\), the complex field is defined relative to a curved reference surface focused on the object plane,

\[
U_{\text{obj}}(\xi, \eta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_{\text{obj}}(x, y) e^{\frac{i\pi}{\lambda d_0}(x^2+y^2)} e^{-\frac{2\pi}{\lambda d_0}(x\xi+y\eta)} dx dy
\]

If the pupil is a lens with a clear aperture described by \(P(\xi, \eta)\), which for this work is assumed to be purely real, and a focal length \(f\), the complex field directly after the lens is defined as,

\[
e^{\frac{i\pi}{\lambda d_0}(\xi^2+\eta^2)} e^{-\frac{i\pi}{\lambda f}(\xi^2+\eta^2)} U_{\text{obj}}(\xi, \eta) P(\xi, \eta)
\]

Which can be rewritten as,
\[ e^{\frac{i\pi}{\lambda} (\xi^2 + \eta^2)} e^{-\frac{i\pi}{\lambda} (\xi^2 + \eta^2)} U(\xi, \eta) \]

Where \( U(\xi, \eta) \) is the product of \( U_{obj}(\xi, \eta) \) and the pupil function,

\[ U(\xi, \eta) = P(\xi, \eta) \int \int u_{obj}(x, y) e^{\frac{i\pi}{\lambda d_0} (x^2 + y^2)} e^{-\frac{i\pi}{\lambda d_0} (\xi x + \eta y)} \, dx \, dy \]

From here, the field can be propagated a distance \( d_i \) to the image plane with a second Fresnel diffraction integral,

\[ e^{\frac{i\pi}{\lambda d_i} (x^2 + y^2)} \int \int U(\xi, \eta) e^{\frac{i\pi}{\lambda d_i} (\xi^2 + \eta^2)} e^{\frac{i\pi}{\lambda d_i} (\xi x + \eta y)} e^{-\frac{i2\pi}{\lambda d_i} (\xi x + \eta y)} \, d\xi \, d\eta \]

The phase curvatures inside the integral can be rewritten so that,

\[ e^{\frac{i\pi}{\lambda d_i} (x^2 + y^2)} \int \int U(\xi, \eta) e^{\frac{i\pi}{\lambda d_i} (\xi^2 + \eta^2)} \left( \frac{1}{d_0} + \frac{1}{d_i} \right) \frac{1}{f} e^{-\frac{i2\pi}{\lambda d_i} (\xi x + \eta y)} \, d\xi \, d\eta \]

The phase curvature term inside the integral of equation (9) is unity when the thin lens equation is taken into account,

\[ \frac{1}{d_0} + \frac{1}{d_i} = \frac{1}{f} \]

Therefore, the final complex field incident on a focal plane array located at the image plane is,

\[ e^{\frac{i\pi}{\lambda d_i} (x^2 + y^2)} \int \int U(\xi, \eta) e^{-\frac{i2\pi}{\lambda d_i} (\xi x + \eta y)} \, d\xi \, d\eta \]

Equation (11) can be described as the complex image multiplied by an additional curvature term if the complex image is defined relative to a curved surface focused on the pupil plane.
\[
    u_{img}(x, y) = \int_{-\infty}^{\infty} U(\xi, \eta) e^{-i \frac{2\pi}{\lambda d_i} (x\xi + y\eta)} d\xi d\eta
\]

So equation (11) can be re-written as,

\[
e^{i \frac{\pi}{\lambda d_i} (x^2 + y^2)} u_{img}(x, y)
\]

The \(\frac{1}{\lambda d_i}\) term in the exponent of equation (12) acts as a scaling factor that shows the relationship between spatial and spatial frequency coordinate systems,

\[
f_x = \frac{\xi}{\lambda d_i}
\]

\[
f_y = \frac{\eta}{\lambda d_i}
\]

If these expressions for the spatial frequencies are applied to equation (12) for a change of variables and the resulting constant, \(\lambda d_i\), is again ignored, the complex image is simply the Fourier transform of the complex pupil,

\[
u_{img}(x, y) = \mathcal{F}\{U(f_x \lambda d_i, f_y \lambda d_i)\}
\]

And equation (11) can be g as,

\[
e^{i \frac{\pi}{\lambda d_i} (x^2 + y^2)} \mathcal{F}\{U(f_x \lambda d_i, f_y \lambda d_i)\}
\]

The progression of a point source target at the object plane through a circular pupil to the image plane is displayed in Figure 6.
Figure 6: Progression of field through a simple lens system. From left to right: The magnitude of the target, complex pupil field, and complex image. The complex pupil and complex image can be defined by equations (7) and (16) respectively.

A focal plane array records the intensity of the complex field at the image plane, which is the squared modulus of equation (13). The curvature term cancels and the intensity simplifies to the squared modulus of the complex image.

\[
I(x, y) = \left| e^{\frac{i\pi}{\lambda d_i} (x^2 + y^2)} u_{img}(x, y) \right|^2 = \left| u_{img}(x, y) \right|^2
\]

The spatial frequency response of a system is commonly derived through a Fourier transform of the image intensity for the point object,

\[
G(f_x, f_y) = \mathcal{F}\{I(x, y)\}
\]

However, a second definition for the frequency response is found by substituting equations (16) and (18) into equation (19) and applying the autocorrelation theorem. This results in the frequency response being equal to a scaled auto-correlation of the complex pupil field,

\[
G(f_x, f_y) = \left[ U(f_x, f_y, \lambda d_i) \ast U(f_x, f_y, \lambda d_i) \right]
\]

The relationships between the complex image, complex pupil, intensity, and frequency response described in equations (16) through (20) are shown in the figure below.
The relationships shown above for a simple imaging system will be useful in describing the more complex system needed for digital piston correction. This system must contain a pupil relay that distorts the exit pupil.

### 2.2 Pupil Relay Sparse Aperture System

The sparse aperture system used for digital piston correction should have a continuous frequency response which provides all of the frequency information below the system’s cutoff frequency. Continuous frequency content is important for a complete, accurate image. A simple form for a sparse aperture entrance pupil contains two circular sub-apertures as shown in the top left diagram of Figure 8. The auto-correlation of the entrance pupil is shown to its right. The central circle corresponds to the auto-
correlations of the single sub-apertures and the circles on either side correspond to the
cross-correlations of the two sub-apertures. The cross-correlations contain the piston
relationship between the two sub-apertures. Since the cross-correlations overlap with the
auto-correlations, the frequency information for the system is continuous.

Most sparse aperture imaging systems require the exit pupil to be an exact, scaled
 replica of the entrance pupil in order to maintain proper phase relationships between the
target and image. In a Fizeau interferometer, this is particularly important since the
images from each optical path are recombined to reveal the interference between sub-
apertures due to higher frequency target content. If the exit pupil is shifted, the cross-
correlations of the spatial frequency response are also shifted and the resulting
interference fringes will be biased. Additionally, fringes will be inaccurate if the sub-
apertures are not phased. A successful Fizeau interferometer matches the phases between
two optical paths while maintaining a constant pupil function [11]. One good example of
this is the LBTI. The secondary deformable mirrors in Figure 2 and the adjustable
mirrors from the beam combiner in Figure 3 correct for piston while the beam combiner
itself creates a corrected but exactly scaled exit pupil that matches the entrance pupil
defined by the primary mirrors [5], [6].
Since the phase information between sub-apertures is contained in the cross-correlations of the frequency response function, it can easily be digitally corrected if they are isolated from the auto-correlations. Unlike the exact scaling required of a Fizeau interferometer, this is achieved by physically separating the sub-apertures at the exit pupil. The resulting lens setup could be described as an anamorphic pupil relay that produces a disproportionately scaled version of the system’s frequency response. A diagram of the separation can be seen in the lower images of Figure 8. The anamorphic magnification between the sub-apertures and their separation distance allows phase corrections to be applied digitally without the reference beam required by active illumination techniques.

Figure 9 demonstrates the separation and scaling between the entrance and exit pupil. The upper diagram of Figure 9 represents an entrance pupil where the sub-apertures have a radius of \( w \) and an original separation of \( \eta_0 \), where \( \eta_0 \) is defined as the distance between the center of a sub-aperture and the optical axis. To provide a continuous frequency response \( \eta_0 \) should be shorter than \( 2w \). The center diagram
introduces an additional separation of $a$ to each sub-aperture. Full separation of the cross- and auto-correlations occurs if $a$ is at least $2w - \eta_0$. That is, the total distance between the optical axis and each shifted pupil, $\eta_0 + a$, should be greater than or equal to $2w$. At the exit pupil, these separated sub-apertures will be magnified by a factor of $m$.

The final radius of each sub-aperture in the exit pupil will be $mw$ while the separation distance from the optical axis will be $m(\eta_0 + a)$, as shown in Figure 9. If the separation distance, $a$, is known, the separated sub-apertures in the exit pupil can be digitally shifted back to the original positions seen in the entrance pupil. Both the complete separation of the cross- and auto-correlations of the sub-apertures and the ability to shift them back to their original positions are needed to digitally find and correct the piston phase errors while avoiding misplacement or loss of spatial frequency content captured by the system.

Figure 9: Entrance pupil (left), separated sub-apertures (center), Exit pupil (right)
Section 2.2.1 and 2.2.2 contain a more rigorous discussion of this pupil relay system. Section 2.2.1 covers the equations describing a simple multi-aperture system without a change between the entrance and exit pupils while section 2.2.2 will compare this to a relay system with sub-aperture separation. The pupils can be described mathematically using the same terminology introduced for the simple lens system. The calculations for the additional sub-aperture separation will be used for the piston correction algorithm discussed in CHAPTER 5.

### 2.2.1 Pupil Relay for a Simple Sparse Aperture System

The simplest sparse aperture system contains two sub-apertures. The equation representing the complex field at the exit pupil corresponds to the field defined in equation (7) and is defined as:

\[
U_{array}(\xi, \eta) = P_{sub}(\xi, \eta + \eta_0)U_{obj}(\xi, \eta)e^{j\varphi_1} + P_{sub}(\xi, \eta - \eta_0)U_{obj}(\xi, \eta)e^{j\varphi_2}
\]

Where \(P_{sub}\) is the pupil function of a sub-aperture, \(U_{obj}\) is the field directly before the entrance pupil defined relative to a curved reference surface focused on the object plane, and \(\varphi_1\) and \(\varphi_2\) are the piston errors for each of the two optical paths. This can be rewritten as:

\[
U_{array}(\xi, \eta) = U_1(\xi, \eta + \eta_0)e^{j\varphi_1} + U_2(\xi, \eta - \eta_0) e^{j\varphi_2}
\]

Where \(U_1\) and \(U_2\) represent the complex fields after the lower and upper sub-apertures. For this system, the expression for the complex image could be described using the scaled Fourier transform of \(U_{array}(\xi, \eta)\).
\[
u_{\text{array}}(x, y) = u_1(x, y) e^{i \frac{2 \pi y \eta_0}{\lambda d_i}} e^{i \phi_1} + u_2(x, y) e^{i \frac{2 \pi y \eta_0}{\lambda d_i}} e^{i \phi_2}
\]

The new exponents are the linear phase tilts caused by the displacement of the sub-apertures from the origin while \(u_1\) and \(u_2\) are the complex images from each optical path.

An additional curvature term is needed to define the complex field incident on the focal plane array, as seen in equation (13). The focal plane array records the irradiance, or modulus squared, of this field.

\[
I(x, y) = \left| e^{i \frac{\pi}{\lambda d_i} (x^2 + y^2)} \ u_{\text{array}}(x, y) \right|^2
\]

\[
= u_1(x, y) u_1^*(x, y) + u_2(x, y) u_2^*(x, y)
\]

\[
+ u_1(x, y) u_2^*(x, y) e^{i \frac{4 \pi y \eta_0}{\lambda d_i}} e^{-j(\phi_2 - \phi_1)}
\]

\[
+ u_1^*(x, y) u_2(x, y) e^{-i \frac{4 \pi y \eta_0}{\lambda d_i}} e^{j(\phi_2 - \phi_1)}
\]

Where the additional curvature term cancels immediately in the irradiance. The spatial frequency spectrum of a simple multi-aperture system can then be defined using the relationships in equations (19) and (20),

\[
G(f_x, f_y) = U_1(f_x \lambda d_i, f_y \lambda d_i) \otimes U_1^*(-f_x \lambda d_i, -f_y \lambda d_i)
\]

\[
+ U_2(f_x \lambda d_i, f_y \lambda d_i) \otimes U_2^*(-f_x \lambda d_i, -f_y \lambda d_i)
\]

\[
+ [U_1(f_x \lambda d_i, f_y \lambda d_i) \otimes U_2^*(-f_x \lambda d_i, -f_y \lambda d_i)] \otimes \delta(f_x \lambda d_i, f_y \lambda d_i + 2 \eta_0) e^{-j(\phi_2 - \phi_1)}
\]

\[
+ [U_1^*(-f_x \lambda d_i, -f_y \lambda d_i) \otimes U_2(f_x \lambda d_i, f_y \lambda d_i)] \otimes \delta(f_x \lambda d_i, f_y \lambda d_i - 2 \eta_0) e^{j(\phi_2 - \phi_1)}
\]

The first two terms in equation (25) represent the auto-correlation of each sub-aperture while the second two terms are the cross-correlations that contain the piston difference between the apertures in the forms \(e^{-j(\phi_2 - \phi_1)}\) and \(e^{j(\phi_2 - \phi_1)}\). For this system the auto-
and cross-correlations overlap. In the next section an anamorphic pupil relay will be used to separate the cross-correlations of the frequency response from the auto-correlations by adding a sub-aperture separation.

2.2.2 Pupil Relay with Added Sub-Aperture Separation

The calculations done for the simple sparse aperture system assume the entrance and exit pupil are co-planar at the two sub-apertures that comprise the aperture stop. However, the anamorphic lens setup needed for digital piston correction is not a simple system. The entrance pupil is relayed to an exit pupil with a magnification of $m$ and a sub-aperture shift of $a$, as illustrated in Figure 9. A conventional sparse aperture system would have a shift of $a = 0$. The equation representing the field at the exit pupil is defined as:

$$U_{array}(\xi, \eta) = U_1 \left( \frac{1}{m} \xi, \frac{1}{m} (\eta + m(\eta_0 + a)) \right) e^{j\phi_1}$$

$$+ U_2 \left( \frac{1}{m} \xi, \frac{1}{m} (\eta - m(\eta_0 + a)) \right) e^{j\phi_2}$$

This corresponds to equation (22) of the simple sparse aperture system for the exit pupil. The complex image incident on the focal plane array is,

$$u_{array}(x, y) = u_1 \left( \frac{1}{m} x, \frac{1}{m} y \right) e^{j\frac{2\pi my(\eta_0 + a)}{\lambda d_i}} e^{j\phi_1}$$

$$+ u_2 \left( \frac{1}{m} x, \frac{1}{m} y \right) e^{j\frac{2\pi my(\eta_0 + a)}{\lambda d_i}} e^{j\phi_2}$$

Again, an additional curvature term is needed to define the complex field relative to the plane of the focal plane array. It follows that the irradiance recorded by the detector at the image plane is,
\[ I(x, y) = \left| e^{\frac{i \pi}{\lambda d_i}(x^2 + y^2)} u_{array}(x, y) \right|^2 \]

\[ = u_1 \left( \frac{1}{m} x, \frac{1}{m} y \right) u_1^* \left( \frac{1}{m} x, \frac{1}{m} y \right) + u_2 \left( \frac{1}{m} x, \frac{1}{m} y \right) u_2^* \left( \frac{1}{m} x, \frac{1}{m} y \right) \\
+ u_1 \left( \frac{1}{m} x, \frac{1}{m} y \right) u_2^* \left( \frac{1}{m} x, \frac{1}{m} y \right) e^{\frac{j4 \pi m y (\eta_0 + a)}{\lambda d_i}} e^{-j(\varphi_2 - \varphi_1)} \\
+ u_2 \left( \frac{1}{m} x, \frac{1}{m} y \right) u_1^* \left( \frac{1}{m} x, \frac{1}{m} y \right) e^{\frac{j4 \pi m y (\eta_0 + a)}{\lambda d_i}} e^{j(\varphi_2 - \varphi_1)} \]

Once more the curvature term drops out when calculating the irradiance. The spatial frequency can be defined as,

\[ G(f_x, f_y) = G_1 \left( \frac{1}{m} f_x, \frac{1}{m} f_y \right) \otimes G_1^* \left( -\frac{1}{m} f_x, -\frac{1}{m} f_y \right) + G_2 \left( \frac{1}{m} f_x, \frac{1}{m} f_y \right) \otimes G_2^* \left( -\frac{1}{m} f_x, -\frac{1}{m} f_y \right) \\
+ \left[ G_1 \left( \frac{1}{m} f_x, \frac{1}{m} f_y \right) \otimes G_2^* \left( -\frac{1}{m} f_x, -\frac{1}{m} f_y \right) \right] \otimes \delta(f_x d_i, f_y d_i + 2m(\eta_0 + a)) e^{-j(\varphi_2 - \varphi_1)} \\
+ \left[ G_2 \left( \frac{1}{m} f_x, \frac{1}{m} f_y \right) \otimes G_1^* \left( -\frac{1}{m} f_x, -\frac{1}{m} f_y \right) \right] \otimes \delta(f_x d_i, f_y d_i - 2m(\eta_0 + a)) e^{j(\varphi_2 - \varphi_1)} \]

This equation separates and magnifies the spatial frequencies, as shown in Figure 9, while retaining the common spatial frequency information in the regions where the auto- and cross-correlations overlapped before separation. Therefore, the equations (25) and (29) are equal if \( m = 1 \) and \( a = 0 \). Since the cross-correlations’ components in equation (29) contain the piston differences between sub-apertures as well as the complex information common with the auto-correlations, the isolation of the cross- and auto-correlations of the sub-apertures in this analytical model provide the information needed to return the spatial frequency profile to its proper aspect ratio and correct for piston error.
2.3 Digital Registration and Piston Correction

The frequency responses of the images recorded by the laboratory setup will be defined by $G(f_x, f_y)$ in equation (29). The frequency content in the cross-correlations is shifted from its original positions due to the sub-aperture separation, separating the cross-correlations from the auto-correlations. To help calibrate the system, a registration algorithm was used to find the exact shifts of the cross-correlations with regard to the autocorrelation. This registration algorithm is discussed in section 2.3.1. Notice that the piston difference, given as $e^{-j(\phi_2-\phi_1)}$ in equation (29), can be found in one cross-correlation and its complex conjugate in the other. Digitally calculating and correcting this piston difference can be completed using the calibrated shift, which is covered in section 2.3.2.

2.3.1 Registration

Registration is primarily a numerical process, but the general theory is discussed here. The technique depends on the common frequency content found in the separated auto- and cross-correlations. Therefore, a target containing strong spatial frequencies in both the auto- and cross-correlations should be used, such as a Ronchi ruling with a fundamental frequency which is near the center of the overlap region. To find the shift of the cross-correlations, each cross- and auto-correlation should be cropped into separate spectrums. Then, a registration of one of the cross-correlations and the auto-correlation can be performed. This registration can be accomplished through a cross-correlation of the two spectrums.

When a cross- and auto-correlation are registered, a peak occurs in their cross-correlation. The position of this peak corresponds to the distance each of the sub-
aperture cross-correlations should shift in order to overlay the auto-correlation so that the common spatial frequencies are properly aligned. As calculated from the Dirac delta functions in the cross-correlation components of equation (29), the values that can be used to digitally correct the shifts in the frequency response caused by the anamorphic pupil relay are \( \pm \frac{2ma}{\lambda d_i} \). The registration peak should be found at one of these distances depending on which cross-correlation was used for the registration. The shift corresponding to the anamorphic pupil relay in the laboratory will be constant and should only need to be calculated once.

### 2.3.2 Piston Correction

Once the constant shift from the sub-aperture separation is known, it can be used to calculate the piston error for any image taken by the system for which the target has spatial frequency information common to the auto- and cross-correlations. First, the cross-correlations of the frequency response can be shifted back to their original positions before sub-aperture separation. Equation (30) shows the shifted version of equation (29).

\[
G(f_x, f_y) = U_1 \left( \frac{1}{m} f_x \lambda d_v \frac{1}{m} f_y \lambda d_l \right) \ast U_1^\dagger \left( -\frac{1}{m} f_x \lambda d_v - \frac{1}{m} f_y \lambda d_l \right) + U_1 \left( \frac{1}{m} f_x \lambda d_v \frac{1}{m} f_y \lambda d_l \right) \ast U_2^\dagger \left( -\frac{1}{m} f_x \lambda d_v - \frac{1}{m} f_y \lambda d_l \right)
\]

\[
+ \left[ U_1 \left( \frac{1}{m} f_x \lambda d_v \frac{1}{m} f_y \lambda d_l \right) \ast U_2 \left( -\frac{1}{m} f_x \lambda d_v - \frac{1}{m} f_y \lambda d_l \right) \ast \delta \left( f_x \lambda d_v f_y \lambda d_l + 2m\eta_0 \right) e^{i(\varphi_2 - \varphi_1)} \right]
\]

\[
+ \left[ U_1^\dagger \left( -\frac{1}{m} f_x \lambda d_v - \frac{1}{m} f_y \lambda d_l \right) \ast U_2 \left( \frac{1}{m} f_x \lambda d_v \frac{1}{m} f_y \lambda d_l \right) \ast \delta \left( f_x \lambda d_v f_y \lambda d_l - 2m\eta_0 \right) e^{i(\varphi_2 - \varphi_1)} \right]
\]

Aside from the piston phase errors, the phase of each term is a function of the object spatial frequency content. This content is common in the overlap of the auto-correlation and cross-correlation terms. By multiplying the elements of the cross-correlation by the complex conjugate of the auto-correlation the phase from the object
content cancels. With the common phase removed, only the piston phase error will remain if higher order wavefront errors are negligible. The angle of the sum of the pixels in this complex product gives the piston difference between the sub-apertures.

\[ \varphi_2 - \varphi_1 = \angle \left( \sum_{f_x} \sum_{f_y} \text{cross} \otimes \text{auto}^* \right) \]  

(31)

Where \text{cross} can refer to line 3 or 4 of equation (30) while \text{auto} refers to the first two lines. A similar value for the piston difference can be calculated by taking the weighted average of the angle for each element in the complex product. Since there is a complex symmetry in the frequency response, the piston difference for one cross-correlation is the complex conjugate of the piston difference for the other, so the piston only needs to be calculated for one cross-correlation.

Piston difference can be corrected by multiplying each shifted cross-correlation with the complex conjugate of its phase so the phases cancel. Once piston correction is complete, the shifted cross-correlations can be recombined with the auto-correlation matrix and equation (30) becomes,

\[
\mathcal{G}(f_x, f_y) = U_1(f_x\lambda d_i, f_y\lambda d_i) \otimes U_2^*(-f_x\lambda d_i, -f_y\lambda d_i) \\
+ U_2(f_x\lambda d_i, f_y\lambda d_i) \otimes U_2^*(-f_x\lambda d_i, -f_y\lambda d_i) \\
+ \left[ U_1(f_x\lambda d_i, f_y\lambda d_i) \otimes U_2^*(-f_x\lambda d_i, -f_y\lambda d_i) \otimes \delta(f_x\lambda d_i, f_y\lambda d_i + 2m\eta_0) \right] \\
+ \left[ U_1^*(-f_x\lambda d_i, -f_y\lambda d_i) \otimes U_2(f_x\lambda d_i, f_y\lambda d_i) \otimes \delta(f_x\lambda d_i, f_y\lambda d_i - 2m\eta_0) \right]
\]  

(32)

This resembles a scaled, piston corrected version of the original frequency response first described with equation (25) before sub-aperture separations were applied. A more in-depth, quantitative discussion of registration and piston correction is provided in CHAPTER 5. An IFFT of the equation above will provide the diffraction limited images.
shown in the results of CHAPTER 6. To further analyze these results, the modulation transfer function (MTF) and an effect of non-monochromatic illumination called field dependent contrast (FDC) can be studied. The theoretical background for MTF and FDC is discussed below.

2.4 Modulation Transfer Function (MTF)

After experimental data is collected, registered, and piston corrected, the effectiveness of the digital phase correction should be tested. An image’s frequency response can be useful for analyzing the improved resolution of the optical system. The frequency response of a point source target for a diffraction limited system is the optical transfer function (OTF). The MTF is the modulus of OTF. Under diffraction limited conditions the OTF is real and positive so it is equal to the MTF. The MTF can be calculated analytically by taking a normalized, scaled auto-correlation of the pupil function where the point source target results in a plane wave at the complex field of the pupil, defined by $U_{obj}$ in equation (4).

2.4.1 MTF of a Circular Aperture

A typical, simple imaging system contains a circular aperture at the pupil plane. Goodman defines the MTF for a diffraction limited single circular aperture as [10],

$$
\mathcal{H}(f_x, f_y) = \begin{cases} 
\frac{2}{\pi} \left[ \arccos \left( \frac{\rho}{2\rho_0} \right) - \frac{\rho}{2\rho_0} \sqrt{1 - \left( \frac{\rho}{2\rho_0} \right)^2} \right] & \rho \leq 2\rho_0 \\
0 & \text{otherwise} 
\end{cases} 
$$

(33)

Where the frequencies are defined in polar coordinates,
\[ \rho = \sqrt{f_x^2 + f_y^2} \]  

(34)

And the cutoff frequency of the coherent system is,

\[ \rho_0 = \frac{w}{\lambda d_i} \]  

(35)

As before, \( w \) is the radius of the aperture and \( d_i \) is the distance from the pupil to the image plane.

### 2.4.2 MTF of a Simple Sparse Aperture System

A simple system with multiple circular sub-apertures at the pupil field contains equation (33) but also has similar profiles at higher frequencies. The MTF of a two aperture system is calculated as,

\[
\mathcal{H}_i(f_x, f_y) = \frac{2}{\pi} \left[ \cos^{-1} \left( \frac{\rho}{2\rho_0} \right) - \frac{\rho}{2w} \sqrt{1 - \left( \frac{\rho}{2\rho_0} \right)^2} \right] \\
+ \frac{1}{\pi} \left[ \cos^{-1} \left( \frac{\rho_1}{2\rho_0} \right) - \frac{\rho_1}{2\rho_0} \sqrt{1 - \left( \frac{\rho_1}{2\rho_0} \right)^2} \right] \\
+ \frac{1}{\pi} \left[ \cos^{-1} \left( \frac{\rho_2}{2\rho_0} \right) - \frac{\rho_2}{2\rho_0} \sqrt{1 - \left( \frac{\rho_2}{2\rho_0} \right)^2} \right]
\]  

(36)

The additional components of this MTF are defined in terms of shifted radial frequencies shown in equations (37) and (38) respectively. The shifted components of the equation above are the cross-correlations of a point source frequency response and the un-shifted component is the auto-correlation.
\[ \rho_1 = \sqrt{f_x^2 + \left( f_y - \frac{2\eta_0}{\lambda d_i} \right)^2} \]  

(37)

\[ \rho_2 = \sqrt{f_x^2 + \left( f_y + \frac{2\eta_0}{\lambda d_i} \right)^2} \]  

(38)

While equation (36) represents the MTF of the sparse-aperture system, it does not take into account the MTF of the camera. When the camera MTF is largely a function of the area integration over the pixel area, it can be found through the Fourier transform of the pixel response. The focal plane array used in the lab had square pixels with a side length of \( d \).

\[ \text{PIXEL}(x, y) = \text{rect} \left( \frac{x}{d}, \frac{y}{d} \right) \]  

(39)

A Fourier transform gives,

\[ \mathcal{H}_{\text{camera}}(f_x, f_y) = \text{sinc}(f_xd, f_yd) \]  

(40)

The product of the system and camera MTFs is the MTF recorded by the focal plane array. The camera MTF is close to unity near the spatial frequencies affected by the system, but the small change in values is large enough to be noticeable when comparing analytical and experimental results.

\[ \mathcal{H}_{\text{simple}}(f_x, f_y) = \mathcal{H}_1(f_x, f_y)\mathcal{H}_{\text{camera}}(f_x, f_y) \]  

(41)

However, the anamorphic pupil relay required for digital piston correction results in a slightly different MTF due to the added sub-aperture separation.
2.4.3 Altered Frequency Response from Added Sub-Aperture Separation

The spatial frequency response from a point target for an anamorphic pupil relay system strongly resembles the MTF of a simple, two aperture system as given in equation (36),

\[
\mathcal{H}_2(f_x, f_y) = \frac{2}{\pi} \left[ \cos^{-1}\left(\frac{\rho}{2m\rho_0}\right) - \frac{\rho}{2m\rho_0} \sqrt{1 - \left(\frac{\rho}{2m\rho_0}\right)^2} \right]
\]

\[
+ \frac{1}{\pi} \left[ \cos^{-1}\left(\frac{\rho_3}{2m\rho_0}\right) - \frac{\rho_3}{2m\rho_0} \sqrt{1 - \left(\frac{\rho_3}{2m\rho_0}\right)^2} \right]
\]

\[
+ \frac{1}{\pi} \left[ \cos^{-1}\left(\frac{\rho_4}{2m\rho_0}\right) - \frac{\rho_4}{2m\rho_0} \sqrt{1 - \left(\frac{\rho_4}{2m\rho_0}\right)^2} \right]
\]

(42)

The only differences between the two equations is the magnification, \( m \), and the additional separations seen in the shifted radial spatial frequency components of equations (43) and (44).

\[
\rho_3 = \sqrt{f_x^2 + \left(f_y - \frac{2m(\eta_0 + a)}{\lambda d_i}\right)^2}
\]

(43)

\[
\rho_4 = \sqrt{f_x^2 + \left(f_y + \frac{2m(\eta_0 + a)}{\lambda d_i}\right)^2}
\]

(44)

Unlike equation (36), equation (42) is not the MTF for a system with sub-aperture separation between the entrance and exit pupils created by an anamorphic pupil relay. The spatial frequency components controlled by \( \rho_3 \) and \( \rho_4 \) are not in their proper locations. This is, however, the frequency response that is collected at the image plane. Again, the product of the separated pupil relay system and camera MTF is the frequency response recorded by the focal plane array.
\[ \mathcal{H}_{\text{shift}}(f_x, f_y) = \mathcal{H}_2(f_x, f_y) \mathcal{H}_{\text{camera}}(f_x, f_y) \]  

(45)

In order to find the MTF of the anamorphic system using this shifted frequency response, the displaced spatial frequencies in the cross-correlations must be moved back to their original locations from before sub-aperture separation. This involves a shift of \( \pm \frac{2m\eta}{\lambda d_i} \), as described in section 2.3.1. Analytically, this MTF will resemble a magnified version of equations (36), (37), and (38).

\[ \mathcal{H}_3(f_x, f_y) = 2 \frac{2}{\pi} \left[ \cos^{-1} \left( \frac{\rho}{2m\rho_0} \right) - \frac{\rho}{2m\rho_0} \sqrt{1 - \left( \frac{\rho}{2m\rho_0} \right)^2} \right] \]

\[ + \frac{2}{\pi} \left[ \cos^{-1} \left( \frac{\rho_5}{2m\rho_0} \right) - \frac{\rho_5}{2m\rho_0} \sqrt{1 - \left( \frac{\rho_5}{2m\rho_0} \right)^2} \right] \]

\[ + \frac{2}{\pi} \left[ \cos^{-1} \left( \frac{\rho_6}{2m\rho_0} \right) - \frac{\rho_6}{2m\rho_0} \sqrt{1 - \left( \frac{\rho_6}{2m\rho_0} \right)^2} \right] \]

(46)

Where,

\[ \rho_5 = \sqrt{\frac{f_x^2}{\lambda d_i} + \left( f_y - \frac{2m\eta_0}{\lambda d_i} \right)^2} \]

\[ \rho_6 = \sqrt{\frac{f_x^2}{\lambda d_i} + \left( f_y + \frac{2m\eta_0}{\lambda d_i} \right)^2} \]

(47)

(48)

However, the camera MTF cannot be directly applied to \( \mathcal{H}_3(f_x, f_y) \) since it is not the frequency response that is collected at the image plane. Instead, the separated cross-correlations in equation (42) must be shifted back to their correct positions after the camera MTF is realized in equation (45). Analytically, the \( \mathcal{H}_{\text{camera}}(f_x, f_y) \) must be applied separately to the auto-correlation and each of the cross-correlations in the scaled
MTF given in equation (46) before recombining the spectrums. The auto-correlation needs no shift of the camera MTF. The two cross-correlations require camera MTF shifts of $\pm \frac{2ma}{\lambda d_i}$. The MTF of the piston corrected system is the sum of the corrected auto- and cross-correlations.

$$MTF_{\text{Final}} = \mathcal{H}_4(f_x, f_y)\mathcal{H}_{\text{camera}}(f_x, f_y)$$

$$+ \mathcal{H}_5(f_x, f_y)\mathcal{H}_{\text{camera}}(f_x, f_y + \frac{2ma}{\lambda d_i})$$

$$+ \mathcal{H}_6(f_x, f_y)\mathcal{H}_{\text{camera}}(f_x, f_y - \frac{2ma}{\lambda d_i})$$

(49)

In which $\mathcal{H}_4$, $\mathcal{H}_5$, and $\mathcal{H}_6$ represent the first, second, and third lines of equation (46). This analytical model will be useful when analyzing the simulated and experimental MTF results.

2.5 Field Dependent Contrast (FDC)

Sections 2.2 and 2.4 cover the basic equations needed to create the algorithms for digital piston correction and to analyze the results of the system. This section assumes the success of this correction technique for a single wavelength of light and discusses the effects of using partially coherent illumination. In the case where the added sub-aperture separation is a fixed physical distance across the bandwidth of illumination, the spatial frequencies vary as a function of wavelength. Understanding this effect identifies a potential limitation for passive systems since field dependent contrast (FDC) can restrict the allowable system bandwidth. With additional research, this limitation may be eliminated by creating a variable sub-aperture separation as a function of wavelength so
that the shift is constant in the spatial frequency domain. Diffraction gratings or other dispersive optics can be used to create this variable separation [12].

A target with a single spatial frequency is useful when studying the effects of a system bandwidth. When the target is propagated through the pupil relay system with additional sub-aperture separation defined in section 2.2.2, the fundamental spatial frequencies at the image plane will be ±$f_t$, which can be written as,

$$\delta(f_y - f_t) + \delta(f_y + f_t)$$

(50)

Where $f_t$ is only in the $f_y$ direction. However, the additional separation of the sub-apertures divides this frequency into two separate values in the auto- and cross-correlations. This separation is $\frac{2ma}{\lambda z}$, as given in equation (29) and discussed in section 2.3.1. With monochromatic light, the spatial frequencies in the cross-correlations will be found at,

$$\delta\left(f_y - \left(f_t + \frac{2ma}{\lambda d_t}\right)\right) + \delta\left(f_y + \left(f_t + \frac{2ma}{\lambda d_t}\right)\right)$$

(51)

The dependence of the separation’s spatial frequency on wavelength leads to a different spatial frequency profile for non-monochromatic light. If over some bandwidth, $\Delta v$, a sensor has a flat response, its temporal frequency spectrum can be defined as rectangular.

$$rect\left(\frac{v - \bar{v}}{\Delta v}\right)$$

(52)

In which $\bar{v}$ is the mean frequency and can be defined as,
Where $\bar{\lambda}$ is the mean wavelength and the bandwidth is,

$$\Delta v = -\frac{c}{\bar{\lambda}^2} \Delta \lambda$$

(54)

For a rectangular frequency spectrum, only the magnitude of the bandwidth is needed and these expressions allow equation (52) to be rewritten as,

$$\text{rect}\left(\frac{v - \frac{c}{\bar{\lambda}}}{\frac{c\Delta \lambda}{\bar{\lambda}^2}}\right)$$

(55)

The additional sub-aperture separation needed for digital piston correction leads to a similar rectangular spectrum in the spatial frequency domain. The relationship between distance and spatial frequency is,

$$f_y = \frac{y}{\lambda d_i}$$

(56)

Through which it can be seen that spatial frequency is also affected by wavelength, similar to the relationship between temporal frequency and wavelength shown in equation (53). The spatial frequency bandwidth is,

$$\Delta f = -\frac{y}{d_i \lambda^2} \Delta \lambda$$

(57)

Equations (54) and (57) reveal that the magnitudes of $\Delta v$, $\Delta \lambda$, and $\Delta f$ are proportional to one another. Therefore, a temporal spectrum represented by a rect function will lead to a spatial frequency spectrum which is also rectangular.


\[
rect\left(\frac{f_y - (f_t + f_a)}{\Delta f}\right) + rect\left(\frac{f_y + (f_t + f_a)}{\Delta f}\right)
\]

The center frequencies of the rectangles in the cross-correlations are $\pm (f_t + f_a)$, where $f_a$ is set as the spatial frequency shift at the mean wavelength,

\[
f_a = \frac{2ma}{\lambda d_i}
\]

And the distance of the sub-aperture separation is $2ma$. Equation (59) and the magnitude of equation (57) allow equation (58) to be rewritten as,

\[
rect\left(\frac{f_y - (f_t + \frac{2ma}{\lambda d_i})}{2ma\Delta \lambda/\lambda^2}\right) + rect\left(\frac{f_y + (f_t + \frac{2ma}{\lambda d_i})}{2ma\Delta \lambda/\lambda^2}\right)
\]

When the cross-correlations are registered and shifted back to their positions relative to the auto-correlation, the function above will become,

\[
rect\left(\frac{f_y - f_t}{2ma\Delta \lambda/\lambda^2}\right) + rect\left(\frac{f_y + f_t}{2ma\Delta \lambda/\lambda^2}\right)
\]

The widths of these rectangular spectra change as a function of $\Delta \lambda$. Applying a Fourier transform to the function above gives a restored image without the contribution of the DC content,

\[
e^{j2\pi f_t y}\text{sinc}\left(\frac{y2ma\Delta \lambda}{\lambda^2}\right) + e^{-j2\pi f_t y}\text{sinc}\left(\frac{y2ma\Delta \lambda}{\lambda^2}\right)
\]

\[
= 2\text{sinc}\left(\frac{y2ma\Delta \lambda}{\lambda^2}\right)\cos(2\pi f_t y)
\]

Where the cosine term represents the original single frequency modulation of the target. Interestingly, the sinc function traces out an envelope that restricts the spatial frequency amplitude within the imaging field. This is referred to as field dependent contrast (FDC).
As the bandwidth increases, the envelope becomes narrower. Conversely, at the limit of $\Delta \lambda = 0$ the envelope goes to 1 at all points and normalizing equation (62) gives,

$$\cos(2\pi f_t y)$$ (63)

Which perfectly defines the single spatial frequency modulation of the target. In a simple pupil relay system without additional separation, the FDC would always be unity.

Equation (62) above only represents the shape of the FDC. The visibility of the envelope depends on the ratio of the frequency responses of the system in the auto- and cross-correlations at the spatial frequency of the target. The sinc envelope was derived from only the cross-correlation. Therefore, its amplitude can be defined as the frequency response at $f_t + f_a$, while the amplitude not affected by the envelope in the auto-correlation is defined by the frequency response at $f_t$. The final FDC for a target with a single frequency of $f_t$ is defined as,

$$FDC(f_t) = 2\mathcal{H}_{shift}(0, f_t) \cos(2\pi f_t y)$$

+ $$2\mathcal{H}_{shift}(0, f_t + f_a) \text{sinc} \left(\frac{y^2 ma \Delta \lambda}{\lambda^2}\right) \cos(2\pi f_t y)$$ (64)

This can be rewritten as,

$$FDC(f_t) = 2 \cos(2\pi f_t y) \left[\mathcal{H}_{shift}(0, f_t) + \mathcal{H}_{shift}(0, f_t + f_a) \text{sinc} \left(\frac{y^2 ma \Delta \lambda}{\lambda^2}\right)\right]$$ (65)

Both $\mathcal{H}_{shift}(0, f_t)$ and $\mathcal{H}_{shift}(0, f_t + f_a)$ are constants for a specific target frequency. Often, it is convenient to view only the envelope while ignoring the target frequency. This simplifies the FDC to,
\[ FDC(f_t) = \mathcal{H}_{\text{shift}}(0, f_t) + \mathcal{H}_{\text{shift}}(0, f_t + f_a) \text{sinc} \left( \frac{\gamma 2ma\Delta \lambda}{\lambda^2} \right) \]  

(66)

The values of \( \mathcal{H}_{\text{shift}}(0, f_t) \) and \( \mathcal{H}_{\text{shift}}(0, f_t + f_a) \) change for different target frequencies. If \( f_t \) is too low to be affected by the separation, \( \mathcal{H}_{\text{shift}}(0, f_t + f_a) \to 0 \) and the normalized FDC is unity. If \( f_t \) is so high that it only appears in the cross-correlation of the MTF, \( \mathcal{H}_{\text{shift}}(0, f_t) \to 0 \) and the image goes to 0 with the sinc function. Intermediate target frequencies found in the overlapping auto- and cross-correlations of the system’s MTF are partially affected by the envelope, but also maintain a limited visibility at the zeroes of the sinc function.

Knowledge of FDC is useful when considering a passive digital phase correction system. It allows us to visualize the largest viewable area available at different bandwidths, and it may restrict system bandwidth in future research or indicate specifications for the diffraction grating to be used. In the next chapter, the computational simulations of the models for lens setup are discussed.
CHAPTER 3

SIMULATIONS

There were two main goals for the digital piston correction simulations. The first goal was to identify a suitable optical setup. The general lens configuration is presented in section 3.1. Initial calculations were done with thin lens equations assuming paraxial rays, which is discussed in section 3.2. Based on these calculations, specific lenses were identified and the setup was modeled using OSLO lens design software to calculate more accurate placements of optics and check for potential aberrations. This is discussed in section 3.3. The second goal was to show, through wave optics propagation, a model of the field propagating through the system at key points such as the target, pupils, and image plane. This is covered in section 3.4.

3.1 General Lens Configuration

Section 2.2 describes the anamorphic pupil relay that can be used to demonstrate partially coherent digital phase corrections. The experimental setup should have an entrance pupil with a continuous spatial frequency response as demonstrated in section 2.2.1 and an anamorphic pupil relay that separates the cross-correlations as seen in section 2.2.2. To meet these criteria, a Galilean imaging system was designed and a diagram is shown in Figure 10.
Light from a target travels a distance $d_1$ to a set of lenses with focal length $F_A$. These convex lenses make up the two sub-apertures of the entrance pupil. They both have a radius $w$ and the distance from the center of a lens to the optical axis is $\eta_0$. The light propagates from these lenses a distance $d_2$ until the two beams reach a set of concave lenses. These lenses must be placed off-center from the two optical beams in order to act as optical wedges that focus the beams back down to a single image at the image plane. If the optical beams have a height $\eta_b$, then the two concave lenses should have a height $h_b$, to align the images and a width $w_b$ in order to avoid vignetting.

The two concave lenses create an exit pupil located between the two sets of lenses along the back projection of $d_3$. At the exit pupil, the sub-aperture radii will be magnified by $m$ while the distance of each beam center from the optical axis will be $m(\eta_0 + a)$, where $a$ is the additional beam separation as seen in the plane of the exit pupil. This description of the entrance and exit pupils matches the requirements given in the diagram from Figure 9. The simulations used to calculate the values for each of these variables are covered in the following two sections.
3.2 YNU Paraxial Ray Tracing

The general lens configuration from Figure 10 was finalized by finding the correct distances, lens sizes, and focal lengths. An ideal technique for modelling a system with multiple sets of lenses is the geometrical optics YNU paraxial ray tracing method [13]. This method uses two equations to trace multiple rays through an optical system. The first equation is used to calculate the height of the rays,

\[ h_{i+1} = y_i + d_i u'_i \]  \hspace{1cm} (67)

And the second equation finds the ray angles at each component of the system,

\[ n'_i u'_i = n_i u_i - h_i K_i \]  \hspace{1cm} (68)

Where \( h \) is the height, \( n' \) is the index of refraction, \( u \) is the angle of the ray before each component, \( i \), of the system, and \( u' \) is the angle of the ray after passing through each component. The distance between components is \( d \) and the power of each component is \( K \). The power is the reciprocal of the effective focal length for each component.

These equations were input into Excel to allow rays to be traced along the object plane, image plane, entrance pupil, exit pupil, and each of the lenses along the system. Chief rays were traced along the optical path for a single sub-aperture and the marginal ray was found for the system as a whole. The system constraints are shown in Table 1.
### Table 1: Paraxial ray tracing constraints

<table>
<thead>
<tr>
<th>Constraint Type</th>
<th>Cause</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total distance</td>
<td>Table space</td>
<td>6705.6 mm (22 ft)</td>
</tr>
<tr>
<td>Target to lens distance</td>
<td>Table separation</td>
<td>4470.4 mm (14.67 ft)</td>
</tr>
<tr>
<td>Front lens to image distance</td>
<td>Table length</td>
<td>2235.2 mm (7.33 ft)</td>
</tr>
<tr>
<td>Sub-aperture separation</td>
<td>Minimum mount thickness</td>
<td>3.4 mm</td>
</tr>
<tr>
<td>Image height</td>
<td>Pixel array size</td>
<td>12.8 mm</td>
</tr>
<tr>
<td>Nyquist frequency</td>
<td>Pixel size</td>
<td>40 cycles/mm</td>
</tr>
</tbody>
</table>

Besides these specific constraints, vignetting and pupil separations of the anamorphic system had to be watched closely. It was found that the best lenses for the system were 25.4 mm (1 in.) in diameter with two 300 mm focal length convex lenses and two -200 mm focal length concave lenses. The image magnification was slightly over 1 and the maximum target diameter was 13.5 mm. Lens thicknesses were assumed to be 0 for these initial calculations.

The YNU paraxial ray tracing method worked well as an initial set of calculations for the laboratory setup. However, there are some calculations that cannot be completed with the paraxial approximation. More exact wave tracing, wavefront analysis, and MTF calculations were done using OSLO.

### 3.3 OSLO Ray Tracing

After the initial distance and focus values were found with YNU ray tracing, OSLO was used to verify the paraxial results, add in the lens thicknesses, and fine tune the approximate distance calculations. First, the paraxial measurements were input into the program along with the thicknesses of the convex, 3.521 mm, and concave, 2.5 mm,
lenses. The aperture stop was placed directly before the first two convex lenses (Figure 11).

Figure 11: First set of lenses and aperture stop as defined in OSLO

The components were placed non-sequentially in order to provide for an anamorphic pupil relay with two separate ray paths (Figure 12).

Figure 12: Non-sequential lenses. The final plane seen in this diagram after the last lens surface is an artifact of the non-sequential system and does not affect the experimental setup.

After inputting the paraxial measurements, OSLO’s wavefront analysis tool was used to find the point along the optical axis with the smallest peak to valley optical path distance (P-V OPD). All beams from the second set of lenses should focus at this location (Figure 13).
Figure 13: Focus at point of minimized P-V OPD

Placing a focal plane array at this point would lead to the best image possible for an on axis point source. For simplicity, only one optical path was analyzed at a time. The minimum P-V OPD for the given system was .06344 of a wavelength and is shown in Figure 14.

While minimizing the P-V OPD, it was necessary to iteratively adjust the positions of the concave lenses in order to ensure that a ray beginning in the center of the object plane would be focused to the center of the image plane for both ray paths. The beam footprint of this centered spot is shown below. OSLO created this footprint by tracing rays through the center of the object to multiple points on the entrance pupil.
At the final position with both a centered beam footprint and a minimized P-V OPD, the image plane is shifted nearly 96 mm further from the pupil plane than in the paraxial calculations. The concave lenses shifted inward by .18 mm, from \( h_b = 17.897 \text{mm} \) to \( h_b = 17.717 \text{mm} \), in order to adjust for the angle needed to interfere both optical paths at the image plane. Vignetting was analyzed again with these new measurements using OSLO’s beam footprint application. The ray bundles were observed at the concave lenses to assure no vignetting and at the focal plane array to ensure the target size was within the field of view. The diameter of the focal plane array was 12.8 mm.

Figure 16: Footprint at concave lens with no vignetting (left). Footprint at image plane (right). Part of the field of view is lost if the small green dot representing the target height moves outside the area of the focal plane array.
The maximum diameter of the object remaining within the focal plane array’s field of view was 12.46 mm, providing a magnification for the system that was slightly smaller than unity. The system’s depth of focus was large and had very little effect on the MTF of each sub-aperture. Also, less than a quarter wave P-V wavefront error can be maintained if the concave lenses are moved ± .55 mm along the optical axis. The final, scaled lens diagram given by OSLO is shown in Figure 17. Close-ups of the various components of this diagram are provided in Figure 11, Figure 12, and Figure 13.

![Figure 17: Final OSLO diagram with correct scaling](image)

The distances, lens sizes, and focal lengths of the system are shown in the table below.

<table>
<thead>
<tr>
<th>Variable Set</th>
<th>Variable</th>
<th>Value (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lens set A:</td>
<td>W</td>
<td>12.7 (CLAP:11.9)</td>
</tr>
<tr>
<td></td>
<td>η₀</td>
<td>13.6</td>
</tr>
<tr>
<td></td>
<td>Fₐ</td>
<td>300</td>
</tr>
<tr>
<td>Lens set B:</td>
<td>wₜ</td>
<td>12.7 (CLAP:11.9)</td>
</tr>
<tr>
<td></td>
<td>ηₜ</td>
<td>14.47</td>
</tr>
<tr>
<td></td>
<td>hₜ</td>
<td>17.717</td>
</tr>
<tr>
<td></td>
<td>Fₜ</td>
<td>-200</td>
</tr>
<tr>
<td>Distances:</td>
<td>d₁</td>
<td>2645</td>
</tr>
<tr>
<td></td>
<td>d₂</td>
<td>167</td>
</tr>
<tr>
<td></td>
<td>d₃</td>
<td>1368</td>
</tr>
</tbody>
</table>
These values should be the proper specifications of an anamorphic pupil relay that will allow digital piston corrections and the MTF for each of the sub-apertures shows that the final, corrected image should be diffraction limited. These measurements will be used in a wave optics simulation of the system as well as in the experimental setup.

3.4 Wave Optics Simulation Using Matlab

The simulations created in Matlab had the general purpose of verifying the analytical model and helping to understand the experimental setup in the lab by providing visualizations of the optical field at key points throughout the system. In order to create simulations that can be readily compared to the experimental data, it is desirable to have targets in the simulation that are similar to those used in the lab. These targets will be discussed in section 3.4.1. Section 3.4.2 covers the propagation of the beams through the system and section 3.4.3 addresses the speckle caused by simulating monochromatic illumination.

3.4.1 Targets

Three types of targets were used to prepare and verify the digital piston correction code. These were Ronchi rulings, a star target, and a USAF 1951 resolution target. The Ronchi rulings were the closest to single spatial frequency targets that were available in the laboratory. In the wave optics simulation single frequency targets were created using the equation,

\[
\frac{1}{2} (1 + \cos(2\pi f_t y)) e^{i2\pi \theta(x, y)}
\]

(69)

Where the cosine portion creates a single frequency pattern with a DC offset to give non-negative amplitude values. \( \theta(x, y) \) is a uniform randomly distributed variable that
generates a random phase for the target to simulate its roughness. This will give rise to speckle in the imagery after propagation of monochromatic light. An example of a single frequency target is shown in the left image of Figure 18.

The star target was needed as a target containing a continuous progression of spatial frequencies and angles. A digital copy of a star target was found online [14] and is shown in the center image of Figure 18. The USAF 1951 target was the most difficult to simulate because the spatial frequency elements of the target had to be matched to the size of the pixels at the object plane of the simulation to avoid aliasing. This was done by finding a scalable vector graphics (SVG) file of the target [15]. The SVG file was saved at a resolution of 4050 x 4050 pixels, a fast Fourier transform (FFT) was used to transform the image into the spatial frequency domain where the information was cropped to the correct size, and an inverse FFT (IFFT) was performed on the cropped image in order to change it back to the spatial domain. Since the object plane of the simulation was defined as 2048 x 2048 pixels with .0124 mm/pixel, the sizes of USAF 1951 targets of various resolutions are shown in Table 3.
Table 3: Sizes of different resolutions for USAF 1951 targets

<table>
<thead>
<tr>
<th>Groups</th>
<th>Image Size (in Pixels)</th>
<th>Image Size (in mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1, 0, 1, 2</td>
<td>3645 x 3645</td>
<td>45.2 x 45.2</td>
</tr>
<tr>
<td>1, 2, 3, 4</td>
<td>900 x 900</td>
<td>11.16</td>
</tr>
<tr>
<td>2, 3, 4, 5</td>
<td>459 x 459</td>
<td>5.69 x 5.69</td>
</tr>
</tbody>
</table>

The last target in the table is the closest in size and resolution to the one available in the lab. It takes up about half of the length of the object plane. Next, each of these targets should be propagated through the simulated system.

3.4.2 Light Propagation

The wave optics simulation presented here follows the general guidelines introduced for a simple imaging system in section 2.1. Propagation between planes is achieved through a series of Fresnel diffraction integrals as defined in equation (3). Phase curvatures at the lenses are accounted for by multiplying a field by the field curvature expression, \( e^{-\frac{i\pi}{\lambda} \sqrt{\xi^2 + \eta^2}} \).

In order to properly display the numerically calculated fields, the pixel scales should be traced through the system as well as the fields. These pixel values are used to create a grid at each plane that physically represents the size of the field at that plane. Pixel scales can be calculated through the formula,

\[
d\xi = \frac{\lambda z}{Nd\xi}
\]  

(70)

Where \( d\xi \) is the pixel size at the first plane, \( d\xi \) is the pixel size at the second plane, \( z \) is the general distance between planes, and \( N \) is the number of pixels in one dimension of the pixel array.
As with the analytical model, the wave optics simulations begin with a complex target at the object plane, \( u_{obj}(x, y) \). However, to best illustrate this simulation a single point source target will be used. A zoomed in image of the point target is shown below. The phase at this point is a uniformly distributed random variable between \([-\pi, \pi]\).

The simulations reflect the analytical models and the target is propagated through two different lens systems. The first is the simple sparse aperture system discussed in section 2.2.1. The second is the pupil relay with the added sub-aperture separation presented in section 2.2.2.

### 3.4.2.1 Pupil Relay for a Simple Sparse Aperture System

The first simulation calculates the image and frequency response of a simple sparse aperture imaging system, as illustrated in Figure 20. Since the target used here is a point source, the intensity and frequency response correspond to the PSF and MTF of the system.
Conventionally, this simple system could be modelled by propagating the field to the pupil using the Fresnel diffraction integral, multiplying by the pupil function, and propagating to the image plane as was discussed analytically in section 2.1. However, this technique leads to long processing times and large memory requirements when simulating the anamorphic pupil relay. To reduce the memory requirements, two separate tilts are applied to the target to direct the optical axes separately toward the two sub-apertures of the entrance pupil. This constrains sample sizes by simulating each optical path individually before combining them at the image plane without the need to include the space between the beam paths. These tilts can be defined as,

$$u_{obj}(x, y)e^{\pm iky \frac{n_0}{d_1}}$$

(71)

In which $k$ is the wavenumber and $\frac{n_0}{d_1}$ is the slope from the object plane to the center of each sub-aperture at the entrance pupil. Each tilted field is propagated to the entrance pupil using a Fresnel diffraction integral. For a point source the amplitude and phase at this plane are unity and the pixel size at the entrance pupil can be calculated using equation (70). The tilts are removed after propagation and the fields at the entrance pupil
are each multiplied by a sub-aperture mask. The combined fields at the entrance pupil are shown in the left image of Figure 21. This is analogous to equations (21) and (22) in section 2.2.1.

After applying the aperture masks, the phase transformations for the two convex lenses are multiplied with the pupil fields as well. The optical axes for the two fields are now directed toward the effective image plane of the lens system using new tilts that replace \( d_1 \) in equation (71) with \( d_{\text{virt}} \), the virtual distance from the pupil to its effective image plane. This distance is described as virtual since the thesis did not require the simple sparse aperture setup to be physically realized in the lab. The pupil fields with the curvatures and new tilts are shown in the equation below,

\[
U(\xi, \eta \pm \eta_0) e^{-i \frac{\pi}{\lambda F_{\text{A}}} [\xi^2 + (\eta \pm \eta_0)^2]} e^{\pm i k y \frac{\eta_0}{d_{\text{virt}}}}
\]

Where the “\( \pm \)” refers to the top and bottom sub-aperture. These fields are Fresnel propagated a distance \( d_{\text{virt}} \) to a virtual image plane. The pixel scaling between the pupil and image plane can be found through,

\[
x_{\text{virt}} = \frac{\lambda d_{\text{virt}}}{N d^2}
\]

When the tilts are removed and the two image fields are interfered as described in equation (23), the intensity of the combined complex image is the PSF for the system. This defines the intensity from equation (24) for a point source target and is displayed in the center image of Figure 21. An FFT of the PSF provides the MTF for the simple sparse aperture system, as shown in the rightmost image of Figure 21. This exactly matches the analytical MTF given by equation (36).
3.4.2.2 Pupil Relay with Added Sub-Aperture Separation

The anamorphic pupil relay system that results in separated sub-apertures in the exit pupil is the experimental lens setup that is used in this thesis. To review, the diagram for this setup can be seen in Figure 10 and the wave optics simulation of this setup is based on the discussion from section 2.2.2. The complex fields of interest in this system are found at the object, entrance pupil, concave lens, image, and exit pupil planes. The system will be numerically solved in this order. A few extra steps will be necessary in order to scale the pixel sizes for the grid at each plane properly. A detailed explanation of this propagation through the anamorphic system follows.

The propagation from the target to the entrance pupil is the same as described in section 3.4.2.1 above. The amplitude at the entrance pupil is shown below and matches with equations (21) and (22).
In this case, however, the tilts are not removed at the entrance pupil plane. Instead, the sub-aperture masks and convex phase bowls were applied immediately to give the complex field at each of the sub-apertures defined by,

\[
U(\xi, \eta \pm \eta_0) e^{-\frac{i \pi}{\lambda R_A} [\xi^2 + \eta^2]} e^{\pm i k y_0} \frac{d_1}{d_2}
\]  

(74)

Typically, these fields could simply be Fresnel propagated a distance of \(d_2\), as given in Figure 10, to the concave lenses. However, the pixel scaling from this particular propagation results in a pixel size that is far too small to display both sub-aperture fields on one grid in Matlab. Two-step propagation was used to avoid this problem [15]. In two-step propagation, a field is propagated once to a plane past the one of interest, and a second time back to that plane. In this lens setup, the first propagation travels \(d_{\text{virt}}\) past the concave lens plane to the virtual image plane as seen in Figure 20. The fields from the bottom and top sub-apertures create the images \(u_1(x, y)\) and \(u_2(x, y)\) from equation (23) consecutively. The second Fresnel propagation travels a distance \(d_2 - d_{\text{virt}}\) back to the concave lens plane. The fields at this plane could be defined as,
$U_B(\xi_2, \eta_2 \pm \eta_b)$

(75)

Where $\eta_b$ is the height of the field at the second set of lenses, as seen in Figure 10. This allows for a much more manageable pixel size with which to display the field. The pixel size for the first propagation is the same as the $dx$ calculated in equation (73). The pixel size for the second propagation is,

$$d\xi_2 = \frac{\lambda(d_2 - d_{virt})}{Nd_{vrt}}$$

(76)

The amplitude of the field directly before the concave lenses is shown in Figure 23. Since this plane is not a pupil, the amplitude is not quite unity.

![Figure 23: Amplitude of field directly before concave lenses](image)

It is at this plane that the initial tilts, $e^{\pm ik\eta_0/d_1}$, are removed. The phase transformations from the two concave lenses are multiplied by the field, and a new tilt is added to the field propagating from each sub-aperture in order to direct the optical axes back toward the center of the image plane. The field directly after the concave lenses is,
\[ U_B(\xi_2, \eta_2 \pm \eta_b) e^{-\frac{\pi}{\lambda f b} [\xi^2 + \eta^2]} e^{\pm i k y \frac{\eta_b}{d_3}} \] (77)

Where the tilt has the slope, \( \frac{\eta_b}{d_3} \) and \( d_3 \) is the distance between the second set of lenses and the image plane. This tilt is due to the concave lenses acting as a wedge. After using another Fresnel diffraction integral to propagate the field to the image plane, the tilt is removed so that the fields again have a common optical axis and the images can be interfered to form the complex field defined by equation (27). The pixel size is calculated by,

\[ dx_{img} = \frac{\lambda d_3}{N d_\xi_2} \] (78)

The intensity of the combined image, from the analytical model in equation (28), defines the PSF of the anamorphic system when the target is a point source. An FFT of the PSF gives the frequency response of the system, which matches equation (42).

Figure 24: PSF (left) and frequency response (right) of anamorphic relay system
This frequency response, however, is not the MTF of the system due to the shift in the cross-correlations. To find the MTF, the cross-correlations should be digitally shifted back to their proper locations. This will be covered in CHAPTER 5.

The frequency response shown here is the frequency response as seen from the shifted fields at the exit pupil. The complex field at the exit pupil can be back-calculated from the image plane. If the distance from the image plane to the exit pupil is defined as $d_{exit}$, then the tilt from equation (77) can be re-applied and a Fresnel propagation of $d_{exit}$ will result in the field at the exit pupil. This shifted field is defined by equations (26) and (27). The pixel size at this field is found through,

$$ d\xi_{exit} = \frac{\lambda d_{exit}}{Nd_{img}} \quad (79) $$

The amplitude at the exit pupil is shown in Figure 25.

![Figure 25: Amplitude (left) and phase (right) at the exit pupil](image)

The amplitude of the complex field at the exit pupil is unity, which supports the fact that exit pupil plane is $d_{exit}$ behind the image plane.
The images in this section demonstrate the path that light follows through the system that will be used for digital piston correction. However, this path is only illustrated above for a point source. Images created using more complex targets are discussed in the next section.

3.4.3 Complex Targets with Speckle Effects

The PSF found using a point source target and shown in section 3.4.2.2 shows no speckle impacts. However, the image of a star target, which is comprised of a large number of point sources that interfere, shows fully realized speckle in the imagery.

![Figure 26: Single realization of simulated image of star target](image)

It is the monochromatic illumination used in the simulation that leads to the speckle structure. In order to obtain a clearer simulated image it is necessary to speckle average over multiple images. About 100 images are needed to be able to resolve the components of the image and speckle averaging over 1000 images almost completely restores the image to the resolution of the target. Therefore, 1000 images were speckle averaged.
before the simulated images of each target were analyzed alongside the experimental data.

Figure 27: Speckle averaging over 100 images (left) and 1000 images (right)

The speckle noise visible in the progression of 1, 100, and 1000 star target images helps to describe the signal to noise ratio (SNR) found in speckle averaging. The SNR of speckle is based on the number, \( n \), of summed realizations.

\[
SNR = \sqrt{n}
\]  

This relationship results in a speckle noise SNR of 1 for 1 realization, 10 for 100 realizations, and 32 for 1000 realizations.

Simulations with multiple wavelengths have similar speckle averaging requirements. The simulations discussed until this point used only one wavelength, which is convenient for the initial setup and analysis of digital piston correction based on a coherent laser source illumination. However, multiple wavelengths are required for simulations of partially coherent light and the resulting field dependent contrast of the system. To simulate multiple wavelengths in Matlab, the propagation described in
section 3.4.2.2 above is looped with a small change in $\Delta \lambda$ for each iteration. A full bandwidth is defined over 100 iterations and 10 of these multi-wavelength images are averaged to provide speckle averaging over 1000 realizations. For example, an image with a wavelength bandwidth of 50nm can be simulated as a series of images with illuminations ranging from 1475nm to 1525nm. In a set of 100 images, the wavelength of each image would differ by .5nm. In this case, speckle averaging occurs over both multiple realizations and wavelengths. Incoherent summation of the intensities for speckle averaging is possible assuming each realization represents an integration time much longer than the coherence time.

In sections 3.1 through 3.3, the design of the laboratory setup for digital piston correction was introduced and the measurements for the system were finalized through paraxial ray tracing and other geometrical optics techniques. Section 3.4 tested these physical designs through wave optics simulations to ensure that the hardware and beam paths were correct.
CHAPTER 4

EXPERIMENT

There are three points of emphasis associated with the physical system setup for an anamorphic pupil relay system. First, the general setup of the hardware is discussed in section 4.1, which utilizes the hardware parameters calculated with paraxial ray tracing and geometrical optics analysis. Second, the ability to shift between viewing spatial and spatial frequency domains and its importance for system alignment is described in section 4.2. Third, images collected from the aligned, anamorphic system are presented and compared with simulated results in section 4.3.

4.1 Physical Setup

There are two parts to the physical setup used for pupil separation. The first part is the setup of the lenses, camera, and target. The second part is the setup for the system’s illumination.

4.1.1 Imaging System

The hardware used to separate the sub-apertures of the sparse aperture system has four key components. As with most imaging systems, a target is placed in the object plane and a focal plane array is placed in the image plane. The sparse array discussed in this thesis will contain two convex lenses that serve as the aperture stop and entrance
pupil of the system. The separation of the sub-aperture fields at the exit pupil come from the two separate convex lenses placed at the entrance pupil. A set of two concave lenses is placed a distance $d_2$ behind the entrance pupil and act as wedges to direct both sub-aperture complex fields towards interference at the image plane. The diagram for the experimental setup shown in section 3.1 is redisplayed in Figure 28.

![Figure 28: Diagram of experimental setup](image)

A photograph of the setup is shown below.

![Figure 29: Image of laboratory setup for anamorphic pupil relay. The target is located approximately 2.6m before the front set of lenses.](image)
The targets used in this experiment were a star target, a USAF 1951 target, and nine Ronchi rulings with the spatial frequencies of 1, 2, 3, 4, 5, 6, 8, 10, and 25 line pairs per mm. All targets were chrome on glass and back illuminated. The star target was used for alignment while the USAF target was used to quantify the image system’s resolution. The Ronchi rulings helped to verify the MTF and FDC calculations. Images of a few of the targets are shown below.

![Figure 30: Star target (left), USAF 1951 target (center), 2 line pairs/mm Ronchi ruling (right)]

Both the concave and complex lenses were stock lenses from Newport Corporation. They were 25.4mm (1 in.) in diameter with BK7 glass and a broadband anti-reflective (BBAR) coating for wavelengths between 1.0μm and 1.55μm. The lenses at the entrance pupil were plano-convex with focal lengths of 309.3mm at $\lambda = 1560$nm and a central thickness of 3.521mm. These convex lenses were the two sub-apertures of the entrance pupil. It was important that the sub-apertures were spaced as closely together as possible to allow for the highest amount of overlap between the auto- and cross-correlations. In order to do this, the lens mounts were filed to a minimum thickness of 1.9mm along one side so the lenses could be placed next to each other edgewise. These were the mounts for the front two lenses in Figure 29. The second two lenses were
plano-concave with focal lengths of -206.2mm at \( \lambda = 1560\text{nm} \) and central thicknesses of 2.5mm. This second set of lenses had to be placed precisely in order to align and focus the images. These two lenses were each placed on three translation mounts to allow adjustments in all three dimensions. A close-up of the anamorphic lens setup is shown in Figure 31.

![Figure 31: Overhead view of anamorphic pupil relay system](image)

The camera was a Sensors Unlimited GA1280JSX with a 1280 x 1024 pixel array and a pixel pitch of 12.5\(\mu\)m. It had a maximum rate of 60fps, exposure times between 30\(\mu\)s and 16.5ms, and NIR/SWIR sensitivity between 0.7 and 1.9\(\mu\)m [17]. The focal plane array was windowed to a 1024 x 1024 pixel portion of the original array size for data analysis.

![Figure 32: Sensors Unlimited GA1280JSX [17]](image)
The distances for each component in the setup were all within a few millimeters of the simulated OSLO measurements in Table 2. The heights of each of the lenses are within 1 mm of the simulations.

4.1.2 Light Source

Two different light sources were used to back illuminate the targets in the above system. An incoherent, white light lamp was placed directly behind the target for preliminary alignments. However, final alignments and all data collects were completed with an expanded laser beam and a spinning diffuser. The laser used was a model 6300-LN (TLB-6328) New Focus Velocity Tunable Diode Laser. This laser had a tuning range of 1520 to 1570nm and a course tuning resolution of .02nm [18]. The laser illumination setup is shown in Figure 33.

![Figure 33: Beam expander and diffuser diagram](image)

The distances, lens widths, and focal lengths are as follows.
Table 4: Laser illumination setup measurements

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1$</td>
<td>8.9</td>
</tr>
<tr>
<td>$F_2$</td>
<td>150</td>
</tr>
<tr>
<td>$w_1$</td>
<td>9.0</td>
</tr>
<tr>
<td>$w_2$</td>
<td>50.8</td>
</tr>
<tr>
<td>$d_1$</td>
<td>340</td>
</tr>
<tr>
<td>$d_2$</td>
<td>60</td>
</tr>
<tr>
<td>$d_3$</td>
<td>245</td>
</tr>
</tbody>
</table>

Directly after the laser free space output, the beam passed through a concave lens with a diameter of 9mm and a focal length of -9.42mm at the wavelength 1560nm. The beam expanded until it reached a convex lens with a focal length of 153.41mm at 1560nm and a 50.8mm (2 in.) diameter. The second lens of the system focused the beam on the object plane of the pupil relay for maximum intensity at the transmissive target. A holographic diffuser with a 15 degree beam divergence was placed between the focusing lens and the target. The diffuser served the joint purpose of creating multiple speckle realizations of the target when rotated and providing a beam large enough to fill the system’s field of view. The total distance of this laser alignment from the concave lens to the target was 645mm (25.4 in.) and the length of the entire system from the laser expansion to the detector was 4735mm (15.53 ft.). A photograph of the laser illumination setup is shown below.
4.2 Alignment

Before any alignments were done, the right and left optical paths were brought into preliminary focus by individually adjusting the back set of concave lenses along the optical path. Once each image was visible on the focal plane array at the image plane, a nominal alignment was done with a white light lamp to get a working understanding of the system. Image alignment was done by eye in the spatial domain by adjusting the concave lenses up, down, right, and left. Images from the right and left optical paths, as well as the interfered image, are shown in Figure 35. These first images were taken with National Instruments Measurement & Acquisition Explorer (NI MAX).
Due to the directionality of the lamp and the lack of a diffuser, unequal intensities between the left and right beam paths were evident. The broad spectrum of this lamp was greater than what was desired for the partially coherent illumination, so it was not used for further research purposes.

After the laser was expanded, diffused, and transmitted through the target, it was found that the system focus needed to be readjusted. This shift in the focus between the white light and coherent laser source at 1560nm demonstrated that the lamp did not emit much light around 1560nm and was more likely dominated by light near the camera’s lower cutoff of 0.9 microns. This also explained why the original distance between the back lenses and the image plane had been approximately 450mm (17.7 in.) too short while the distance for the laser alignment at 1560nm was very close to the OSLO calculations.

Using the laser as a light source, the two images were aligned by eye again. An initial image of a Ronchi ruling is shown below. The diffuser was present for both images, stationary in the left image where speckle is visible, and spinning in the right. The center of the spinning diffuser, as well as the focal point of the laser beam, can be
seen in the second image. The beam path was shifted for future images to avoid these two artifacts.

Further imaging and alignment acquisition software was developed using Labview. This software allows the user to control frame rate, exposure time, and whether to view in the spatial or spatial frequency domain of the camera data. The frame rate was set at 15Hz and the exposure time was set at 10ms. After setting the frame rate and exposure time, Labview’s IMAQ toolbox was used to grab (continuous acquisition) camera frames as well as to snap (single shot) camera frames and convert them to data arrays. Additional code allowed data to be Fourier transformed, displayed, and saved. A simple preprogrammed grab loop was used to continuously stream images from the camera. An option was added to stream the Fourier transforms of the images instead of the images themselves. The code was also given the ability to snap one image and display it in another window. This snapped image could either be dropped or saved to a file while the grab loop continued to stream. The option to display the Fourier transforms
was used for additional alignments while the snap option was used to collect images to use for digital piston correction.

Figure 37: Labview GUI imaging a Ronchi ruling of 3 line pairs/mm

An image of a Ronchi ruling is shown below along with its Fourier transform. This was collected by Labview and displayed with Matlab. The camera field is 12.8mm x 12.8mm.
A more exact alignment was attempted by viewing the Fourier transform of the Ronchi ruling and adjusting the concave lenses to find the highest intensity of the shifted fundamental frequency peak in the cross-correlation. This shifted fundamental frequency is specified in Figure 38. Unfortunately, maximizing the intensity weakened the interference between the two images and reduced the contrast of the image. This occurred for multiple trials and Ronchi rulings. It was determined that a local maximum in the MTF of the system caused by some aberration (likely defocus) was affecting this alignment approach. Under this assumption, each Ronchi frequency may be affected differently and the ideal alignment cannot be found with a single frequency target.

Instead, a star target was used. A star target covers a continuous frequency spectrum radially for a number of angles. When the target is in focus, these frequencies are clearly visible in the auto-correlation of the spatial frequency domain. When the two images of the target are completely aligned, the frequencies are also clearly visible in the
cross-correlations. An image of a star target illuminated at 1560nm and its Fourier transform are displayed in Figure 39.

![Figure 39: Star target (left) and logarithmically scaled FFT (right)](image)

The position of the lenses at which the frequency information in both the auto- and cross-correlations had maximum visibility was found to be the best possible alignment.

### 4.3 Data Collected

Data was collected for eleven targets. These targets were a USAF 1951 resolution target, a star target, and nine Ronchi rulings at the frequencies 1, 2, 3, 4, 5, 6, 8, 10, and 25 line pairs per mm. Each target was imaged at least 50 times while varying the wavelength between 1521 and 1571nm with one image taken for each wavelength. A few of the targets were imaged additional times at 1560nm to obtain repeatable monochromatic results. A few examples of these images and their Fourier transforms are shown in this section and are compared with the images and spatial frequencies generated by the wave optics simulations.
The experimental and simulated star target images are shown side by side below. There are uncharacteristically high spatial frequencies near the center of each star target. These spatial frequencies have a bias due to the additional pupil separation. When the spatial frequency content of the cross correlations is digitally shifted back together, these offset spatial frequencies will be corrected.

![Figure 40: Experimental (left) and simulated (right) star targets. The offset frequencies are visible as slightly darker regions near the center of each image.](image)

The Fourier transforms of each image are shown in Figure 41. These frequency responses are very similar, but call attention to a few of the differences between the experimental and simulated images. In the auto-correlations, notice that the experimental Fourier transform has a flatter response near DC. This is caused by the non-uniform background illumination. In the cross-correlations, the simulated Fourier transform has fewer spatial frequency rays due to fewer and thicker radial spokes in the spatial domain.
The next target displayed is the USAF 1951 resolution target. As with the star target, the experimental and simulated images are shown in Figure 42. The biased higher frequencies affected by the pupil separation can also be seen here. This bias is only found along the axis of separation. In the other dimension, the highest resolution that can be seen is Group 2 Element 4. The direction orthogonal to the aperture separation will not be affected by the digital corrections.
Figure 42: Experimental (left) and simulated (right) USAF 1951 resolution target. In the simulation, the group numbers -2 and -1 correspond to the actual groups 2 and 3.

The Fourier transforms for each image are as follows,

Figure 43: Experimental (left) and simulated (right) FFTs of USAF 1951 resolution target
Single frequency images also give some interesting insight into the system. Figure 44 shows experimental and simulated single frequency targets at 3 line pairs per mm.

When looking at the spatial frequency domain for these images, the DC offset, fundamental frequency, and multiple harmonic frequencies are visible. Below, notice that more harmonics are visible in the Fourier transform of the Ronchi ruling than that of the simulated cosine target. This is caused by the square wave nature of the Ronchi ruling. The first harmonic in each image is the fundamental frequency of the target. The ratio of the amplitude of this harmonic to the amplitude of the DC offset can be used to find the frequency response of the system. By collecting this ratio with multiple single frequency targets the MTF can be determined. It is also useful to notice that the fundamental frequency is visible in both the auto- and cross-correlations for the target frequency used in Figure 44.
A look at a single frequency target of 10 line pairs per mm gives slightly different results than the 3 line pairs per mm images shown in Figure 44. In that image, the original target frequency could be seen as well as a shifted copy due to the pupil separation. However, the only frequency visible in Figure 46 is the biased frequency content from the cross-correlation which has been shifted by the pupil separation.
The Fourier transforms are,

When observing the frequency content for a 25 line pairs per mm target, only the DC content would be visible since the fundamental frequency is too high for both the auto- and cross-correlations. In contrast, a frequency of 1 line pair per mm would be low
enough to only appear in the auto-correlation and would not have any shifted content due to the pupil separation.

Some final images of interest are those collected for Ronchi rulings at multiple wavelengths. The previous images shown in this section have been monochromatic. A single spatial frequency image illuminated by a small bandwidth will look similar to its monochromatic counterpart.

![Figure 48: Part of the image for the experimental (left) and simulated (right) 3 line pairs/mm targets with 50 nm bandwidths](image)

In the spatial frequency domain, however, there is a noticeable difference for a spatial frequency in the cross-correlation. In Figure 49, the fundamental frequency seen in the cross-correlation is elongated. While the pupil separation is constant for the wavelengths used, the resulting frequency shift is slightly different for each wavelength and spreads out the spatial frequency recorded on the focal plane array. This elongation was analytically modeled in section 2.5 and is the cause of field dependent contrast for frequencies in the cross-correlation.
The digital process used to shift the cross-correlation frequency content to match the unbiased frequency response and correct for piston will be discussed in CHAPTER 5.
CHAPTER 5

DIGITAL REGISTRATION AND PISTON CORRECTION

As discussed in section 2.3, there are two digital corrections that should be applied to the frequency response of the image recorded by the anamorphic pupil relay system in order to obtain a diffraction limited image. The first correction is to find the distances needed to shift the cross-correlations back to their original positions before the sub-apertures were separated. This is done once with a registration technique and is described in section 5.1. The second correction is to digitally calculate and correct for piston in each of the cross-correlations, which is discussed in section 5.2. Once the piston phase errors can be measured, a path difference between the system’s two optical paths can be calculated. The relationship between optical path difference (OPD) and the partial coherence of the illumination is explained in section 5.3.

5.1 Registration

Each of the simulated and experimental frequency responses in section 4.3 have shifted frequency content due to the sub-aperture separation caused by the anamorphic pupil relay. This shift causes a frequency bias in the images, which is best seen in Figure 42. The bias can be corrected by returning the cross-correlations to their original, non-shifted positions. In section 2.3, the simulated shift is defined by $\pm \frac{2ma}{\lambda d_i}$. The
experimental shift, however, will be slightly different. An exact value for this shift is most easily calculated for a target with a frequency response at only one spatial frequency. A Ronchi ruling with 3 line pairs/mm was chosen since its fundamental frequency is located in the center of the overlap between the auto- and cross-correlations. In order to find the shift, the fundamental frequency in the auto-correlation and its copy in one of the cross-correlations were registered. Therefore, the auto-correlation and each of the cross-correlations should be divided into separate matrices.

![Figure 50: Separated auto- and cross-correlations of a 3 lp/mm Ronchi ruling](image)

The fundamental frequency was isolated in the auto-correlation and one of the cross-correlations. Registration was performed by cross-correlating this frequency with its shifted copy. The result is shown below.
The distance of the peak in the image to the right of Figure 51 with regard to the center of the field corresponds to the number of pixels needed to shift the cross-correlations to their proper positions without the sub-aperture separation. The function, `dftregistration.m` was used to register the frequencies at a sub-pixel level [19]. This algorithm upsampled the data around the peak in order to give a more precise registration shift along both the x and y axes.

To assess the accuracy of this registration approach, single frequency simulations with a known separation of 93.7714 pixels along the y-axis were tested. Speckle reduced the accuracy by introducing noise near the fundamental frequency, which increased the difficulty of registration. The averaged shifts calculated from 10 images each made from 100 speckle realizations are given in Table 5.
Table 5: Simulation registration values

<table>
<thead>
<tr>
<th>Output</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>y-axis</td>
<td>93.7719 ± .003 pixels</td>
</tr>
<tr>
<td>x-axis</td>
<td>0.0004 ± .002 pixels</td>
</tr>
</tbody>
</table>

The values for the y shift and x shift are both within one standard deviation of their known values, so the function was used to calculate the shift for the experimental images. It was applied to twenty images of the 3 line pairs/mm Ronchi ruling and the experimental shifts are shown below.

Table 6: Experimental registration values

<table>
<thead>
<tr>
<th>Output</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>y-axis</td>
<td>90.8917 ± .009 pixels</td>
</tr>
<tr>
<td>x-axis</td>
<td>-0.3569 ± .003 pixels</td>
</tr>
</tbody>
</table>

The y shift here is fairly similar to the known value of 93.7714 pixels for the simulations. The x shift given above is caused by a slight asymmetry in the two optical paths that create the anamorphic pupil relay. Once the experimental shift for the anamorphic pupil relay was found, it was set as a constant and used to digitally calculate piston phase errors.

5.2 Piston Correction

Once the constant shift is known for the laboratory setup, the piston error between the sub-apertures can be calculated for each image. To begin the piston error calculations, each cross-correlation should be shifted back to its original position with respect to the auto-correlation from before the sub-aperture separation.
Figure 52: Auto-correlation of a star target with its cross-correlations shift back to their original positions from before sub-aperture separation

After the cross-correlations have been shifted, each element of a cross-correlation and the complex conjugate of the auto-correlation should be multiplied. This process can be seen in Figure 53.
The complex product of the element-wise multiplication is only non-zero in the overlap between the cross- and auto-correlation. It gives the phase offset between the cross- and auto-correlation for each frequency along with a “confidence” in the offset. The phase offset and “confidence” correspond to the phase and amplitude respectively, shown in Figure 52. The angle of the sum of all the pixels in the complex product is the piston error. Quantitatively, this can be defined as,

$$\varphi_2 - \varphi_1 = \angle \left( \sum_{f_x} \sum_{f_y} \text{cross} \odot \text{auto}^* \right)$$

(81)

Which is a reiteration of equation (31), where $\varphi_1$ and $\varphi_2$ are the piston angles from the analytical equations used in CHAPTER 2 and $\varphi_2 - \varphi_1$ is referred to as the piston difference. A similar result can be found by taking the weighted average of all the phase
offsets from the complex product. The complex symmetry between the top and bottom
cross-correlations only makes it necessary to calculate the phase error for one cross-
correlation since the phase error for the other will be its complex conjugate.

An algorithm was written to complete this digital piston calculation and was
tested with multiple simulated images containing known piston phase errors. Twenty
simulated images of the USAF 1951 resolution target were created with different piston
errors between 0 and $2\pi$ radians with steps of $\pi/5$ on one of the two optical paths. The
standard deviation of the calculated piston from the actual values was $\pm0.02$ radians.
This error was small enough to allow piston phase corrections for the experimental
images. By multiplying the complex conjugate of the piston difference with each cross-
correlation in equation (30) the piston phase errors can be corrected.

$$Cross_{corrected} = Cross_{uncorrected}e^{\pi j(\varphi_2-\varphi_1)}$$

(82)

After digitally correcting the piston errors, the cross-correlations can be recomposed with
the auto-correlation to form the frequency response of a diffraction limited sparse
aperture system without sub-aperture separation. This frequency response corresponds to
equation (32) in the analytical model and the correction process is demonstrated in Figure
54 below.
Figure 54: Piston corrections and recombination of the frequency response of a star target.

The piston difference for the star target used in Figure 54 is -2.03 radians. An IFFT of the corrected frequency response results in a diffraction limited image of the star target.

Figure 55: Uncorrected (left) and corrected (right) image of a star target
Notice that the image to the right has higher frequency content in the direction of the aperture separation that is not visible in the uncorrected image to the left. This example helps to verify the effectiveness of digital piston correction for monochromatic light.

The ability to calculate piston error makes it possible to measure the fluctuations between the two optical paths of the system due to atmospheric turbulence (negligible in the lab), table vibrations, and other small changes in the system within calculated noise limits. To find the range of these fluctuations, images were taken of a 3 line pairs per mm Ronchi ruling once an hour for 5 hours. The wavelength, $\lambda$, was 1560 nm.

<table>
<thead>
<tr>
<th>Time</th>
<th>Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>1:00</td>
<td>-1.6604</td>
</tr>
<tr>
<td>2:00</td>
<td>-1.4987</td>
</tr>
<tr>
<td>3:00</td>
<td>-1.6818</td>
</tr>
<tr>
<td>4:00</td>
<td>-1.6326</td>
</tr>
<tr>
<td>5:00</td>
<td>-1.5394</td>
</tr>
</tbody>
</table>

Table 7: Phase variation with time

These results can be used to find small changes in the optical paths when the phase fluctuations are under $2\pi$ radians.

$$\Delta l = \frac{\varphi_{\text{max}} - \varphi_{\text{min}}}{2\pi} \lambda \tag{83}$$

In which $\varphi_{\text{max}}$ and $\varphi_{\text{min}}$ are the minimum and maximum piston errors from Table 7. This gives a pathlength change of 45nm over the course of the measurements. The next section explores the ability to use a range of wavelengths to calculate larger optical path differences (OPD) between the two sub-apertures and discusses the effects of partially coherent illumination on digital piston corrections.
5.3 Optical Path Difference (OPD)

For narrowband illumination the OPD between the two paths is expected to remain constant, but this will result in a piston phase offset which will vary with wavelength. If a series of monochromatic images between 1521 and 1571 nm at 1 nm intervals are analyzed, the resulting pistons have a slope related to the OPD. This slope can be seen in Figure 56.

![Figure 56: Piston varying over wavelength](image)

The slope can be used to find the OPD between the two sets of lenses through the equation,

\[
\Delta \varphi = \varphi_1 - \varphi_2 = 2\pi OPD \left( \frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right)
\]

Where \( \varphi_1 \) and \( \varphi_2 \) are the piston errors corresponding to the wavelengths \( \lambda_1 \) and \( \lambda_2 \).

Solving for the OPD while assuming \( \lambda_1 \approx \lambda_2 \approx \bar{\lambda} \) and defining \( \bar{\lambda} \) as the mean wavelength, 1546 nm, gives,

\[
OPD \approx \frac{\bar{\lambda}^2}{2\pi} \frac{\Delta \varphi}{\Delta \lambda}
\]
The OPD for the experimental setup was calculated at 66.57μm. This is a rather large path difference between the two sub-apertures and is caused by a variety of system misalignments.

When different wavelengths are collected in separate images so that each image is monochromatic, it is possible to digitally correct for the OPD by applying a separate piston phase correction for each individual image. Summing all of these images simulates the result of a system with partially coherent illumination and an OPD of 0. However, a true partially coherent system cannot digitally correct for each piston phase separately since the wavelengths are not collected separately. Instead, the OPD of the system can only be physically corrected so that the variation in piston across the wavelengths remains at an acceptably small level. Larger variations could lead to a more rapid decrease in the visibility of spatial frequencies in the cross-correlations, which will be discussed later in this section. This correction was accomplished by adjusting the concave lenses.

The beams from each sub-aperture pass through slightly different locations on each of the concave lenses. The curved nature of the lenses leads to different glass thicknesses for each beam. The positions of these secondary lenses can be fine-tuned to reduce the

![Figure 57: Illustration of the physical OPD correction method](image_url)
The system’s OPD. The final OPD of the system was 7.76μm. The reduced slope for this OPD is displayed in Figure 58.

![Figure 58: Piston slope after physically correcting the OPD](image)

The registration shift was recalculated after the OPD was reduced in the laboratory. Please note that all experimental images shown in section 4.3 were taken after this final pathlength correction to simplify the presentation of the results.

<table>
<thead>
<tr>
<th>Output</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>y-axis</td>
<td>90.7890 ± .009 pixels</td>
</tr>
<tr>
<td>x-axis</td>
<td>-0.3664 ± .003 pixels</td>
</tr>
</tbody>
</table>

The OPD is a concern since the higher spatial frequencies in the cross-correlations of the frequency response wash out when the piston difference between the highest and lowest wavelengths reaches $2\pi$ radians. The visibility, V, for these higher spatial frequencies can be defined as a function of OPD using the same argument as Goodman presented for fringe visibility with respect to time offset and illumination bandwidth in a Michelson interferometer [20]. Fringe visibility is written as,
\[ V_{fringe} = \text{sinc}(\Delta \nu \tau) \]  

(86)

Where \( \Delta \nu \) is the bandwidth of the illumination in frequency and \( \tau \) is the time offset. The bandwidth is related to coherence length, \( l \), through,

\[ \Delta \nu = \frac{c}{\lambda^2} \Delta \lambda = \frac{c}{l} \]  

(87)

And the time offset for the two paths used here is derived from the OPD.

\[ \tau = \frac{\text{OPD}}{c} \]  

(88)

Substituting equations (87) and (88) into equation (86) gives a function equivalent to the cross-correlation visibility for an anamorphic sparse aperture array with partially coherent piston correction.

\[ V = \text{sinc} \left( \frac{\text{OPD}}{l} \right) \]  

(89)

The equation above shows that there is an inverse relationship between the OPD and the system bandwidth. The alignment tolerance relaxes as the coherence length increases and the system bandwidth decreases. Conversely, alignment requirements are stricter when digitally correcting for piston using wider system bandwidths. When the OPD is 7.76\( \mu \)m, the visibility of the cross-correlation frequencies is 0.5 when \( l = 12.86 \mu \)m. Using equation (87), this corresponds to a wavelength bandwidth of \( \Delta \lambda = 186 \text{nm} \).
CHAPTER 6

RESULTS

Digital piston corrections for both monochromatic and multi-wavelength images are discussed in this chapter. Examples of diffraction limited images with monochromatic illumination and their simulated counterparts are shown in section 6.1. Section 6.2 discusses the MTF of the experimental and simulated systems and compares them with the analytical calculations. Partially coherent images are covered in section 6.3, including FDC and how these measurements were extracted from the data.

6.1 Registered and Piston Corrected Images

Before presenting the results of digital piston corrections for partially coherent illumination, it is helpful to demonstrate the effectiveness of the correction technique with the simpler condition of monochromatic illumination. This section presents several examples of diffraction limited, single wavelength images. To begin, Figure 59 shows three images of the USAF 1951 resolution target at different stages of phase correction. The first is an image taken directly from the anamorphic system, as seen in section 4.3. In the second image the spatial frequency shift is corrected, but the piston is not. This corresponds to the image that would have been collected for a simple sparse aperture system with no sub-aperture separation. The third image is piston corrected.
The piston error in the center image manifests as a small, downward blurring of each component in the resolution target. The phase difference between sub-apertures calculated and corrected in the rightmost image was -2.16 radians. The recombined spatial frequency of this target is shown in Figure 60.

Simulated versions of the above three images are shown below with a piston error of 1.88 radians. The images contain 100 speckle realizations.
Figure 61: Simulated uncorrected image (left), image corrected for shift only (center), and diffraction limited image (right)

An upward blur similar to the downward one discussed for the experimental results is visible in the center simulated image. This blur will affect different locations on the image depending on the value of the phase error. If the phase error cannot be calculated, then the exact effect of the blur will be unknown and the images from the two sub-apertures will not be synthesized. Therefore, the actual resolution of the image will be defined by the diameter of a single sub-aperture even though the center image appears to be diffraction limited with resolutions reaching into Group 3. This is difficult to see from just one piston error. In order to better visualize this frequency cutoff, simulations with multiple piston errors were combined into one image to display all possible blurs. This image and a fully corrected simulation are placed next to each other in Figure 62.
Figure 62: Piston blurred (left) and corrected (right) simulations

The blurred simulation on the left has a vertical and horizontal resolution of 5.66 lp/mm, which corresponds to Group 2 Element 4 of the USAF 1951 resolution target. In contrast, the corrected image has a vertical resolution of 11.31 lp/mm from Group 3 Element 4 while the horizontal resolution remains at 5.66 lp/mm. Thus, the corrected image has a resolution along the axis of separation that depends on the diameter of the sparse array. The resolution along the other axis only depends on the diameter of a single sub-aperture. The incoherent cutoff frequency, \( f_c \), along the axis of separation is,

\[
f_c = \frac{2m(\eta_0 + w)}{M\lambda d_i} = 12.36 \text{ lp/mm}
\]  

(90)

And along the axis of a single sub-aperture is,

\[
f_c = \frac{2mw}{M\lambda d_i} = 5.77 \text{ lp/mm}
\]  

(91)
Where \( M \) is the image magnification of the system defined by the pupil magnification, \( m \), the distance from the target to the entrance pupil, \( d_1 \), and the distance from the exit pupil to the image plane, \( d_i \).

\[
M = \frac{md_1}{d_i} = 0.979 \tag{92}
\]

The resolved elements in the USAF 1951 resolution targets of Figure 62 agree with these cutoff frequencies. Therefore, the corrected simulation on the right is diffraction limited. Experimental results displayed in Figure 63 match the simulations.

![Figure 63: Piston blurred (left) and corrected (right) experimental images](image)

A star target more readily displays the difference between images with only shift corrections and those with piston corrections. As in Figure 59, the figure below shows an uncorrected image, an image corresponding to a simple sparse array, and a piston corrected image. The biased frequencies from the sub-aperture separation are visible in the image to the left. The loss of contrast seen in the center image is due to the phase errors between the sub-apertures that are corrected in the rightmost image.
Figure 64: Experimental uncorrected image (left), image corrected for shift only (center), and diffraction limited image (right). These images are zoomed in to show the central region of the target.

The frequency response of the corrected star target image is shown below.

Figure 65: Final frequency response for star target

Both of the targets discussed so far have more than one spatial frequency. Digital phase corrections can only be accurately calculated using this technique when the target has frequency content which falls inside both the auto- and cross-correlations of the system’s frequency response. One such target is the 3 line pair/mm Ronchi ruling shown
in Figure 66. There is a definite increase in contrast between the center image, where only shift is corrected, and the right image with piston correction.

The FFT of this target has a response at the DC offset, fundamental frequency, and a second harmonic. The fundamental frequency for this target is near the center of the overlap of the auto- and cross-correlations.
For targets that do not have frequencies within this overlap region, however, piston correction is not possible. The fundamental frequency for a Ronchi ruling of 10 line pairs/mm is only in the cross-correlation.

![Figure 68: Final frequency response for a 10 lp/mm Ronchi ruling](image)

This means that the biased frequency from the sub-aperture separation can be corrected between the first and second images in Figure 69, but the phase offset cannot be accurately found and corrected between the second and third images. Instead, a piston shift is found based on the uncorrelated noise in the overlap regions, leading to an essentially random phase correction. In this example, the random phase is 1.1 radians.
A target with frequency content that falls only in the auto-correlation would show no impacts due to the corrections. A target that would meet this requirement for the pupil relay used in this experiment would have a frequency of less 0.82 line pairs/mm in the direction of separation with no higher harmonics. This target would not be affected by the separation between the sub-apertures and piston correction would not be necessary since it would only affect high frequency noise.

6.2 MTF

The MTF of a single sub-aperture and the full experimental system were measured and compared with the models in order to test the quality of the results. A discussion of the analytical model was given in section 2.4. The MTF of the simulation and the experimental system can be found using a collection of single frequency targets. The ratio of the amplitude at the target frequency to the amplitude of the DC offset in the spatial frequency domain gives a single point of the system’s MTF. However, a perfect single frequency target is difficult to create. Instead, the wave-optic simulations utilized a cosine target and the experiment used Ronchi rulings. The harmonics in both of these targets subtract from the intensity of the fundamental frequency and should be taken into
account when calculating the MTF. The MTFs for the simulation and laboratory setup are shown in sections 6.2.1 and 6.2.2 respectively.

### 6.2.1 Wave Optics Simulation MTF

The effect of harmonics on the frequency response of a cosine target is found by applying Euler’s formula to a positive cosine function. For simplicity, the targets used for this discussion are only defined along the axis of the sub-aperture separation.

\[
\frac{1}{2} + \frac{1}{2}\cos(x) = \frac{1}{2} + \frac{1}{4}e^{ix} + \frac{1}{4}e^{-ix}
\]

(93)

The expression for the target’s irradiance as recorded by the focal plane array can be found by normalizing the product of equation (93) and its complex conjugate.

\[
\frac{1}{6}e^{-2ix} + \frac{2}{3}e^{-ix} + 1 + \frac{2}{3}e^{ix} + \frac{1}{6}e^{2ix}
\]

(94)

The Fourier transform for this irradiance is,

\[
\frac{1}{6}\delta(f + 2f_x) + \frac{2}{3}\delta(f + f_x) + \delta(f) + \frac{2}{3}\delta(f - f_x) + \frac{1}{6}\delta(f - 2f_x)
\]

(95)

The DC offset is defined by the expression, \(\delta(f)\), while the coefficient for the fundamental frequency, \(f_x\), is 2/3. Therefore, all of the ratios collected by simulated cosine targets should be scaled by 2/3. The fundamental frequency utilized in equation (95) is the frequency in the image plane and is calculated by multiplying the frequency of the target with the system magnification. The magnification of the wave optics model is 0.979.

Table 9 presents the target and image frequencies, the analytical and simulated frequency responses for a single sub-aperture, and the analytical and simulated frequency
responses for the system as a whole. The analytical values for a single sub-aperture were calculated using the first component of equation (49) and the analytical values for the system were calculated using all three components of the same equation.

Table 9: Scaled frequency response data for wave optic simulation along with target frequencies, image frequencies, and the analytical values

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Sub-Aperture</th>
<th>System</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Target Frequency</td>
<td>Image Frequency</td>
</tr>
<tr>
<td>1</td>
<td>0.979</td>
<td>0.789</td>
</tr>
<tr>
<td>2</td>
<td>1.959</td>
<td>0.583</td>
</tr>
<tr>
<td>3</td>
<td>2.938</td>
<td>0.390</td>
</tr>
<tr>
<td>4</td>
<td>3.918</td>
<td>0.218</td>
</tr>
<tr>
<td>5</td>
<td>4.891</td>
<td>0.079</td>
</tr>
<tr>
<td>6</td>
<td>5.877</td>
<td>0.000</td>
</tr>
<tr>
<td>8</td>
<td>7.835</td>
<td>0.000</td>
</tr>
<tr>
<td>10</td>
<td>9.794</td>
<td>0.000</td>
</tr>
</tbody>
</table>

The data for a single sub-aperture is graphed in Figure 70.

Figure 70: Analytical and simulated MTF for a single sub-aperture
And the data for the full system is graphed in Figure 71.

![Figure 71: Analytical and simulated MTF for the full system](image)

The simulated and analytical results are in agreement.

6.2.2 Experimental MTF

The process for correcting the frequency response data of a Ronchi ruling is much the same as the one discussed above for the cosine target. A square wave can be written as,

\[
\frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \cos[(2n-1)x]
\]

(96)

\[
= \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} [e^{i(2n-1)x} + e^{-i(2n-1)x}]
\]

The normalized product of equation (96) and its complex conjugate gives the irradiance. The coefficient of the DC offset should be 1 and the coefficient of the fundamental...
frequency is $2/\pi$. In ideal conditions, the experimental ratios would be scaled by $2/\pi$. However, the background illumination also affects the scaling factor. The scaling factor is 0.43 for a single sub-aperture and 0.45 for the laboratory setup. Conveniently, the magnification for the experimental system was unity so the target and image frequencies are equal.

Table 10: Scaled frequency response data for experimental system along with target frequencies, image frequencies, and the analytical values

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Sub-Aperture</th>
<th>System</th>
</tr>
</thead>
<tbody>
<tr>
<td>Target Frequency</td>
<td>Image Frequency</td>
<td>Analytical</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.784</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.575</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0.379</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0.205</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>0.067</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>0.000</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>0.000</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>0.000</td>
</tr>
</tbody>
</table>

The data for a single sub-aperture is graphed in Figure 72.
The experimental data tends to agree with the analytical model with the exception of some small aberrations from the system. The data for the full system is graphed in Figure 73.

Figure 73: Analytical and experimental MTF for the full system
In this graph, the experimental data is very close to the analytical at 1 lp/mm and 6 lp/mm, but falls away from it elsewhere due to aberrations. The aberrations are seen in both the auto- and cross-correlations so the responses of the spatial frequencies within the overlap region are doubly affected. This concludes the analysis of monochromatic images. The next section discusses the effects of digital piston corrections on multi-wavelength systems.

6.3 FDC

The results in the last two sections demonstrate the ability to digitally correct phase errors using an anamorphic pupil relay and an image registration algorithm. This section discusses the effects of partially coherent illumination on the FDC of a corrected image. As defined in section 2.5, FDC occurs when the constant sub-aperture separation of the imaging system results in a varying shift in the spatial frequency content of the cross-correlations as a function of wavelength. The shifted cross-correlation frequencies cause a loss of contrast for higher frequencies across the area of the field of the image plane. A system with monochromatic illumination would have no FDC while larger wavelength spreads lead to greater contrast reduction. The visibility of the higher frequency content across an image has a sinc function envelope that becomes narrower with increasing system bandwidth. Ideally, FDC could be avoided by using hardware such as diffraction gratings to create a varying aperture separation as a function of wavelength so that the shift of the frequency content in the cross-correlations is constant. Analysis of the FDC in the anamorphic pupil relay used for this thesis could be helpful in the design of similar setups in the future for both active and passive illuminations.
6.3.1 FDC Images

Simulations of the USAF 1951 target were run for wavelength bandwidths of 10, 30, 50, 70, and 100nm. Digitally corrected images of the USAF 1951 resolution target at these five bandwidths demonstrate the effects of FDC. The first two bandwidths do not differ much from the monochromatic image.

The next three bandwidths, on the other hand, show a steady degradation of the outer spatial frequency elements of the images. The 100nm bandwidth example is almost completely blurred except for a small area in the center of the image field.
The standard deviation of the pistons calculated for each of these bandwidths also becomes less reliable as the bandwidth increases, as can be seen in Table 11.

**Table 11: Pistons calculated for USAF 1951 simulations of different bandwidths**

<table>
<thead>
<tr>
<th>Bandwidth (nm)</th>
<th>Piston (radians)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.006 ± 0.02</td>
</tr>
<tr>
<td>10</td>
<td>-0.0001 ± 0.01</td>
</tr>
<tr>
<td>30</td>
<td>-0.004 ± 0.02</td>
</tr>
<tr>
<td>50</td>
<td>0.004 ± 0.03</td>
</tr>
<tr>
<td>70</td>
<td>-0.001 ± 0.04</td>
</tr>
<tr>
<td>100</td>
<td>0.047 ± 0.07</td>
</tr>
</tbody>
</table>

The actual piston error in these simulations was 0 radians. The accuracy of the piston calculations begins to decrease and the standard deviation to increase for a wavelength spread of 100nm. The images in Figure 75 illustrate that a system bandwidth of 50nm or less should be used in order to retain visibility across an acceptable portion of the field. However, the data in Table 11 demonstrates that digital piston corrections are effective for bandwidths up to 100nm. The laser used in the laboratory had a maximum range of 50nm. The simulated analysis fits the experimental data collected for bandwidths of 27nm and 50nm. FDC is slightly visible in Group 2, Elements 1 and 2 of the 50nm bandwidth image below, but is not strong enough to be seen in the 27nm bandwidth image.
6.3.2 Isolating Target Frequency

The analysis in the last section demonstrated the general effects of FDC on an image. However, it is difficult to isolate the effects of FDC as a function of frequency using the image of a USAF 1951 resolution target. This relationship between FDC and frequency was discussed in section 2.5. Single frequency images are better for this analysis. Unfortunately non-uniform illumination in the laboratory, as well as a laser range of only 50nm, makes it difficult to see the effects of FDC on the images, as shown in Figure 77.
Therefore, it is necessary to isolate the fundamental frequency of the target in each image. This can be done by taking an FFT of the image and cropping the fundamental frequency in the frequency response. An example of a crop for a 4 lp/mm Ronchi ruling is shown in Figure 78.

An IFFT of the cropped frequency response can be used to recreate the image. The image below shows the normalized absolute value of the IFFT for a bandwidth of 50nm.
The FDC is visible in this image. Unfortunately, the high content density of the image leads to aliasing in the display. A more quantitative analysis of the amplitude envelope that defines the FDC can be done by collecting the local maxima for each column across the image in Figure 79 and plotting those values as a function of their height in the image.

![Figure 80: Plot of local maxima for an experimental 4 lp/mm Ronchi ruling. Each line shows the local maxima for a single column in Figure 79.](image)

Ideally, the Matlab function `polyfit` could be used to fit the local maxima to a 10\(^{th}\) degree polynomial to approximate the FDC envelope. However, the image is affected by non-uniform background illumination, which prevents `polyfit` from producing the correct FDC. To simplify the correction process for background illumination, only the final, fitted FDC envelopes are normalized. Since the experimental images were collected one wavelength at a time, the background illumination can be isolated from the FDC and cancelled. To isolate the background radiation, the fundamental frequencies are cropped for each single-wavelength image individually. The local maxima obtained from these monochromatic images are only affected by the
background illumination and not the FDC so a \textit{polyfit} of these points will characterize the non-uniform background.

![Figure 81: Plot of non-uniform background for each image of a 4 lp/mm Ronchi ruling between the wavelengths of 1521 and 1571 nm. Each line shows the local maxima for a single column.]

Dividing the fit for the non-uniform illumination in Figure 81 from the original FDC envelope in Figure 80 effectively cancels out the non-uniform background of the experimental FDC envelopes. Figure 82 shows the polynomial fits for the local maxima displayed in Figure 80 and Figure 81.
Figure 82: Polynomial curves fit to the original FDC data found in Figure 80 (left) and non-uniform background illumination data found in Figure 81 (right)

Examples of the normalized, experimental FDC without non-uniform background illumination are displayed below and compared with the analytical results from section 2.5.
Figure 83: FDC plots for 1, 2, 3, 4, 5, and 6 lp/mm. Blue is monochromatic, green is 27nm, and red is 50nm. Dashed lines are experimental Ronchi rulings while solid lines are analytical.
Noise and other artifacts cannot be completely cancelled, but the experimental results show a strong relationship with the analytical models in the central regions of the images. Weak background illumination causes the polynomial fits to be less accurate near the edges of the images. The curves of the monochromatic images were used to normalize the FDC envelopes for the images with higher system bandwidths. This served two purposes. First, it allowed the monochromatic curves to always be unity. Second, it helped to cancel out additional image artifacts. The 1 lp/mm target has a fundamental frequency almost completely in the auto-correlation so FDC was nearly non-existent. The fundamental frequencies 2 lp/mm through 5 lp/mm are in the overlap region of the cross- and auto-correlations and 5 lp/mm has a stronger FDC since its frequency response is stronger in the cross-correlations. At 6 lp/mm and above, the fundamental frequency is completely in the cross-correlation. The sharp change in amplitude at approximately 4mm in the analytical result in the bottom right image of Figure 83 is caused by plotting the absolute value of a contrast reversal. It could not be fully represented by a differentiable polynomial function, leading to smoother curves in the simulations.

Finding the FDC envelopes for simulated images is much simpler. Only speckle makes it difficult to see the effects of FDC in simulated images since the background illumination is uniform. After cropping the fundamental frequencies and graphing the local maxima, the speckle simply averages out when a polynomial is fit to the local maxima. Isolating the background illumination and dividing it from the local maximum values is unnecessary. Figure 84 shows the simulated equivalent to the experimental local maxima displayed in Figure 80. The simulated data contains fewer artifacts than the experimental results.
After applying the `polyfit` function to the simulated data displayed in plots like the one above, the curves of the monochromatic images were used to normalize the FDC envelopes of the images with higher system bandwidths, just as they were for the experimental images. Examples of the simulated FDC are displayed below and compared with the analytical results from section 2.5.
Figure 85: FDC plots for 1, 2, 3, 4, 5, and 6 lp/mm. Blue is monochromatic, green is 27nm, and red is 50nm. Dashed lines are cosine target simulations while solid lines are analytical.
The simulated results are very close to the analytical models. The polynomial fits are only slightly wider than the sinc function envelope given for the analytical model in section 2.5. The experimental and simulated FDCs shown in Figure 85 and Figure 83 demonstrate the relationship between the system bandwidth, spatial frequency, and the area of visibility in the images.
CHAPTER 7

CONCLUSION

7.1 Results

Aperture synthesis for a sparse aperture system relies on the ability to phase multiple sub-apertures. Common aperture synthesis techniques exist for incoherent and coherent illumination. Incoherent methods typically utilize adaptive hardware corrections to continuously equalize the OPD between sub-apertures to a fraction of the wavelength of the passive illumination. Coherent methods use active illumination to find and correct piston phase errors through phase retrieval, digital holography, self-referencing interferometry, or other approaches. This thesis demonstrated a partially coherent digital phase correction technique that can be used with active or passive illumination and relaxes the OPD tolerance of the system.

Partially coherent digital phase corrections are possible with the help of an anamorphic pupil relay system and an algorithm designed to correct for the anamorphic separation between sub-apertures while calculating and correcting the piston phase error between those sub-apertures. The anamorphic pupil relay for this experiment contains a simple sparse aperture array with two closely spaced sub-apertures in the entrance pupil. The fields from these sub-apertures are shifted in order to appear to have a further
separation in the exit pupil, which subsequently shifts the higher spatial frequencies in the cross-correlations of the frequency response to biased positions. This shift separates the cross-correlations from the auto-correlation, making it possible to compare the frequencies common to both the auto- and cross-correlations in order to digitally calculate the piston phase error between sub-apertures. This system was designed using paraxial and OSLO ray tracing techniques and the numerical wave optics simulations of the system matched the images collected in the laboratory.

The separated cross-correlations are digitally registered with the auto-correlations in order to find the distance needed to shift the higher frequencies back to their original positions. With this shift, the auto-correlation and a complex conjugate of the cross-correlation can be multiplied in order to identify the phase difference between the sub-apertures. Image synthesis is achieved by applying the piston phase to each of the cross-correlations before recombining them with the auto-correlations. An IFFT of the corrected frequency response provides a diffraction limited image based on the analytical cutoff frequencies calculated through the MTF. Both simulated and experimental images were successfully synthesized using this piston correction algorithm.

The inverse relationship between the OPD tolerance of the system and system bandwidth was found using the piston phases calculated for a series of regularly varying wavelengths. While the FDC also affects the system bandwidth, this could be corrected by using a diffraction grating to correct the wavelength shift in the frequency response. If the FDC is corrected, only the OPD will determine the partial coherence requirements of the system. For example, a higher OPD tolerance would call for a smaller system bandwidth and therefore a longer coherence length. In the laboratory setup discussed
here, the OPD was found to be 7.8μm. At this OPD, spatial frequencies in the cross-correlations of the frequency response have a visibility of 0.5 if the system bandwidth is $\Delta \lambda = 186\text{nm}$, which corresponds to a coherence length of $l = 12.86\mu\text{m}$.

### 7.2 Future Work

Future work could include a system analysis of this digital piston phase correction technique, which would test the robustness of the piston correction algorithm by studying the effects of varying aperture size, aperture separation, illumination brightness, speckle size, and many other variables. In addition, FDC correction could be demonstrated with the application of diffraction gratings. Work could also be done to show that digital piston corrections allow for digital anisoplanatic turbulence corrections over large fields of view without requiring a multi-conjugate adaptive optics solution. Research into FDC and digital anisoplanatic turbulence corrections would help with the creation of a sparse aperture system with increased resolution in both dimensions for observations at greater distances using active and passive illumination.

### 7.3 Original Contributions

- Developed an analytical model for an anamorphic pupil relay system
- Developed a wave optics simulation of an anamorphic pupil relay system
- Wrote an algorithm that can synthesize partially coherent sparse aperture images and return a diffraction limited image
- Confirmed the inverse relationship between OPD and system bandwidth
- Calculated the effects of system bandwidth on FDC
REFERENCES


