DESIGNING SHAPE CHANGING MECHANISMS FOR PLANAR AND SPATIAL APPLICATIONS

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# DESIGNING SHAPE CHANGING MECHANISMS FOR PLANAR AND SPATIAL APPLICATIONS

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ABSTRACT

DESIGNING SHAPE CHANGING MECHANISMS FOR PLANAR AND SPATIAL APPLICATIONS

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Rigid-body shape changing mechanisms are a growing area of research due to their numerous practical uses. Rigid-body shape-change describes mechanisms comprised of rigid links connected with revolute and prismatic joints that are able to approximate a set of prescribed "morphing" curves. Planar rigid-body shape-changing mechanisms are synthesized to achieve different positions within a common plane. Designing for spatial tasks, however, is an area of emerging research. This thesis addresses topics in both planar and spatial shape-changing devices. First, a practical application for planar shape-changing devices is developed through the design and testing of variable geometry dies for polymer extrusion. Second, a synthesis methodology for spatial shape-changing is developed for serial chains of spherical four-bar mechanisms that can achieve specified helices.

Variable geometry dies enable the extrusion of plastic parts with a varying cross-section. Extrusion accounts for 40% of all manufactured plastic parts due to its relatively low-cost and high-production-rate. Conventional polymer extrusion technology, however, is limited to fixed dies that produce continuous plastic products of constant cross section defined by the die exit profile. A shape-changing die allows the cross-section of the extruded part to change over its length, thereby
introducing the capacity to manufacture plastic faster and with lower tooling costs than injection molding. This thesis discusses design guidelines that were developed for movable die features including revolute and prismatic joint details, land length, and the management of die leakage. To assess these guidelines, multiple dies have been designed and constructed to include an arbitrary four-sided exit profile where changes were made to the internal angles and length of sides as the extruder was operating. Experimental studies were conducted by using different extruder line settings and time between die movements. Test results are presented that include shape repeatability and the relationship between extrudate profile and die exit geometry.

A spatial shape-change application is then introduced with serial chains of spherical four-bar mechanisms. The chains are comprised of identical copies of the same four-bar mechanism by connecting the coupler of the prior spherical mechanism to the base link of the subsequent spherical mechanism. Although having a degree of freedom per mechanism, the design methodology is based upon identically actuating each mechanism. With these conditions, the kinematic synthesis task of matching periodically spaced points on up to five arbitrary helices may be achieved. Due to the constraints realized via the spherical equivalent of planar Burmester Theory, spherical mechanisms produce at most five prescribed orientations resulting in this maximum. The methodology introduces a companion helix to each design helix along which the intersection locations of each spherical mechanisms axes must lie. As the mechanisms are connected by rigid links, the distance between the intersection locations along the companion helices is a constant. An extension to the coupler matches the points along the design helices. An approach to mechanically reducing the chain of mechanisms to a single degree of freedom is also presented. Finally, an example shows the methodology applied to three design helices.
To The Giaier Family

Specifically Mom, Dad and Kellie
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CHAPTER I

INTRODUCTION

Multi-actuator systems with high levels of electronic integration are becoming commonplace in engineering design and manufacturing due to their reduced cost and the increased availability of components. High degree of freedom (DOF) mechanical systems such as robots and articulated servo driven devices offer large range of motion and flexibility in uses. However, these devices are still more costly, lack accuracy, and require more maintenance than low DOF mechanisms designed for specified tasks. Designing mechanisms to achieve prescribed motion, or place specific components in chosen positions, is termed rigid-body guidance or motion generation. Applications of rigid-body guidance mechanisms include pick and place devices for manufacturing [3], vehicle suspensions and steering mechanisms [4, 5], and bucket guidance mechanisms for a front loader [6].

1.1 Shape Change

A novel application of rigid-body guidance is shape-changing mechanisms [7, 8]. These mechanisms are comprised of rigid planar links connected with revolute or prismatic joints so they can be actuated to approximate a series of “morphing” curves. The morphing curves are defined as design profiles and can either be open or closed. Shape change allows for precise and predictable geometric control over body profiles, and can be designed for high strength applications with minimal DOF.
mechanization. Until recently, shape change was only synthesized for profiles of similar arc length, however the addition of the prismatic joint allows for curves with varying arc length [1].

There are numerous applications for shape changing mechanisms. Automotive convertible roofs [9], and portable performance stages [6] involve shape change in structural applications. Single DOF automotive seats to accommodate drivers ranging in height from the 1st percentile female to the 99th percentile male have been synthesized in Shamsudin et al. [1]. Some systems that could see benefit from using shape changing mechanisms include morphing airfoils [10], active aperture antennas [11], and deformable mirrors for optic systems [12]. This list only contains concepts that have been previously considered as potential shape changing applications and are being addressed by multi-actuator systems. Some less common applications include vehicle spoilers to enhance performance during corning and reduce drag during high speed sections, military aircraft turrets which are only used during combat flight conditions, kinetic artwork and variable geometry dies for polymer extrusion [13]. The variable geometry die concept will be explored in great detail within this thesis.

The synthesis process of rigid-body shape-changing mechanisms begins by dividing $p$ profiles into $E$ segments. The segments can either be constant length and shape segments (M-segments) or constant curvature variable length segments (C-segments). The segments are optimized such that one edge approximates a portion on all the design curves with a minimized error and are connected by revolute joints or fused. An example shape changing mechanism approximating 3 profiles with 6 segments (4 M-segments, 2 C-segments) is shown in Fig. 1.1. Using more segments to approximate design curves results in more accurate shape-changing mechanisms. Following segmentation, mechanization defines the full body shape of each segment and additional links and fixed pivots are added to the system to reduce the DOF of the mechanism. A balance is required however as the end goal in many applications of this processes is to create a single DOF device. Adding additional
Figure 1.1: A shape changing mechanisms capable of morphing between a circle, tear drop and oval shape (adapted from [1]).

segments makes the mechanization phase to reduce the number of DOF more complicated and can penalize the curve approximation accuracy.

With no fused components, a closed chain contains $K$ rigid links and $K$ joints connecting them (realizing that each C-segments contains 2 links). These rigid links can be constrained with the addition of $K$ links to create a single DOF device. For open chains, $K$ links are connected with $K - 1$ joints and can be constrained to a single DOF with $K + 1$ additional links. The example in Fig. 1.1 is shown mechanized and configured in several positions in Fig. 1.2. Equation

Figure 1.2: The shape changing mechanism has been constrained to a single degree of freedom through mechanization (adapted from [1]).
based methods and graphical methods have been developed for mechanization and can found in Shamsudin et al. and Myszka [1, 13]. Graphical synthesis has been updated to use geometric packages within CAD software to assist in mechanization [14]. This is the process most typically used for variable geometry die design and will be described in more detail in Chapter 2.

1.2 Motivation for Variable Geometry Dies

The commercial importance of polymer processing has substantially increased in the past 40 years, with 5.6% annual growth in plastic consumption [15]. This is more than ten times that of steel and more than twice that of aluminum. Among production processes, extrusion accounts for 40% of all manufactured plastic parts primarily because it offers relatively low cost and high production rates [16]. The output of a common single screw extruder can exceed four tons of plastic product per hour [17]. In contrast, the maximum production rate of a single injection molding machine is only about 500 pounds per hour because cycle times are dictated by the time required to solidify the part, which depends on the material and size [18]. Besides cost, other advantages of extrusion include the abilities to knit different materials together such that portions of the part have different properties and/or to apply coatings to select regions of the exterior surface. Structural components, such as steel spines, can also be imbedded within an extruded profile. Looking forward, demand for extruded plastics in the United States is projected to grow 2.4% yearly to nearly 40 billion pounds in 2013, valued at over $23 billion [19].

As depicted in Fig. 1.3, extrusion is a continuous process in which a rotating screw conveys polymer pellets inside a heated barrel [20, 21]. The pellets are softened by friction and heat to form a pressurized melt. This polymer melt flows from the barrel exit through a transition region known as the die channel and out of a properly shaped die orifice. A complete “die” consists of a die
channel, which may have a tapered cross section, and a die land, which is the final constant cross-sectional element. As the polymer exits the die land orifice, it is lightly pulled through a cooling trough where a calibrator may help maintain the profile shape during initial cooling. Finally, the extruded piece is cut to length as the final product. Any secondary processes are typically performed prior to cutting, but can follow it as well.

Figure 1.3: Schematic of a typical profile extrusion line for plastic parts.

The function of the die is to shape the molten polymer leaving the extruder barrel’s circular exit into the desired cross section exiting the die land orifice [22, 23]. Ideally, the die channel balances the melt flow to yield a uniform exit velocity across the entire profile at the orifice with a minimal drop in pressure from the inlet to the channel [18, 24]. In general, the exit velocity uniformity can be improved by increasing the die land length. However, any increase in land length causes a corresponding increase in the melt pressure, which requires greater power and will create larger forces on the die [25, 26].

Traditional profile extrusion die designs span the range from relatively simple plate dies to more complex streamlined dies. A typical plate die, as illustrated in Fig. 1.4a, consists of a flat plate die land with a constant cross sectional orifice and a relatively short, circular die channel. Plate dies offer the advantages of low cost, ease of production, and rapid installation. The primary
disadvantage is that the sharp changes in flow at the channel-land interface create a stagnation region. The polymer melt in such regions may exhibit thermal degradation if it remains at elevated temperatures for extended periods of time. Therefore, only thermally stable polymers, such as polyolefin (polyethylene and polypropylene), silicone and soft polyvinyl chloride (PVC), can be formed with a plate die. Other commonly extruded and less expensive polymer materials, such as rigid PVC, will degrade in the dead spots [27].

In a streamlined die, the die channel is typically longer with a tapered cross section to better transition the flow from the extruder head to the final shape at the die land, reducing stagnation and producing a more uniform orifice exit velocity [28, 29]. Accordingly, the advantages include producing parts with tighter tolerance requirements and the use of either heat-sensitive or heat-insensitive polymers. Streamlined dies are typically manufactured using wire EDM methods because of the complex channel geometries. The example in Fig. 1.4b has a multi-section channel due
to tooling limitations that prevent cutting a single transitional piece. Disadvantages are increased cost and tooling lead time.

The conventional extrusion process uses a stationary die land to manufacture continuous plastic products with a constant cross-section. A wide range of products are formed through conventional extrusion including pipes, household siding, decorative molding, drinking straws, gutter systems, weatherstripping, and window framing components. However, the conventional extrusion process does not have the flexibility to accommodate a wider range of more complex products that require changes to the sectional profile.

Methods to achieve dramatic changes in die geometry that alter gross cross sectional part shape are uncommon. One development involves linear movement of the sides of a die land orthogonal to the extrusion direction [30]. By coordinating slide motion, continuously variable tapered extrusions are possible. Even with the limited cross-sectional changes that can be achieved by linear movement, this technology is currently used by Toyoda-Gosei for automotive window and door seals [31].

Dies that have the ability to alter their geometry can open the extrusion process to a wider variety of more complex parts, thereby leveraging the associated cost and time savings. General-purpose shape change of an extrusion die, even for relatively simple part geometries like a rectangle-to-parallelogram transition, is well beyond current production methods. Chapter 3 presents an assessment of moveable die components and discusses the design of a die that exhibits gross section changes.

1.3 Spatial Shape Changing Curve Segments

A spherical four-bar mechanism, shown in Fig. 1.5, is comprised of four revolute (R) joints whose rotation axes intersect at a common point. Due to this constraint, any point on the device
maintains a fixed distance from this intersection point during motion. This property is not unique
to spherical four-bar mechanisms and applies to all spherical devices. A universal joint has this
property. The agile-eye, with its capacity to aim a camera about a fixed point faster than the human
eye also shares these properties [32]. Spherical four-bars are used in machine design with a list of
applications that includes flapping wings [33, 34], robotic surgery [35], and spatial re-orientation
devices [36] among others referenced in Mullineux [37]. Additional examples including spherical
mechanisms for aircraft wing deployment, folding archways, and scissor doors are available at the
Mechanical Design 101 website [38].

Serial chains of spherical four-bars may be formed through connecting the coupler of one four-
bar to the base link of the next. Several conditions, seen in Fig. 1.6, are enforced in this investiga-
tion. First, the spherical mechanisms in the chain are identical. Second, the connections between
the mechanisms are identical. Third, the mechanisms are identically actuated. Operating under
these conditions, the connected chain of bodies is observed to form helices of varying diameters

Figure 1.5: A spherical four-bar mechanism’s motion is prescribed by the locations of fixed axes $\vec{G}$
and $\vec{F}$ and moving axes $\vec{n}$ and $\vec{m}$. 
and pitches. Chapter 4 presents a methodology for designing serial chains of spherical four-bar mechanisms subject to these conditions that can achieve a maximum of five target helices. This limit is due to solving the rigid body guidance problem associated with locating the connecting link of mechanisms $i$ and $i+1$ relative to the connecting link of mechanisms $i-1$ and $i$ as seen in Fig. 1.6. The synthesis of spherical four-bar mechanisms for tasks of up to five orientations is found in McCarthy and Soh [39]. Design tools such as SPHINX solve these rigid body guidance synthesis problems [40] and has been enhanced in Ruth and McCarthy [41].

The methodology for designing the serial chains of spherical four-bar mechanisms proceeds as follows. The process begins by specifying a radius and pitch for each design helix. Next, a companion helix is designated for each design helix noting that their radii and pitches are defined
via the synthesis process. Periodically located points on the companion helix are identified as the axis intersection locations for the spherical mechanism. Likewise, periodically placed points on the design helix identify the extensions to the respective mechanism’s couplers. The axis intersection locations being known along multiple companion helices defines a relative orientation along each helix. A spherical mechanism is then designed to produce these relative orientations.

One interest in developing this concept is its potential use in spatial shape-change. In planar shape-change, connected chains of rigid bodies are determined to best approximate sets of curves, as in Murray et al. [7] and Persinger et al. [8]. To advance this concept toward spatial applications, a chain of rigid bodies is needed that approximates spatial curves. The helices proposed in this work provide such a capacity. Another interest in developing this concept is its folding capacity similar to that of origami-based architectures [42, 43, 44]. As the chains designed can achieve helices of desired pitches, a helix with a small pitch may be included in the set of design helices. This produces a chain that coils on itself lying in a nearly flat configuration.

1.4 Organization

This thesis is organized as follows. In Chapter 2, background information on the synthesis of shape-changing mechanisms is introduced. Additionally a graphical synthesis method used to mechanize majority of the dies shown in this thesis is discussed. Finally the equations used to solve for a five orientation task spherical four-bar mechanism are presented. Chapter 3 presents the design and testing of variable geometry dies for polymer extrusion. This includes joint concepts, die design and testing procedures. The results of three experimental tests are provided and future dies to be tested are presented. In Chapter 4, the design of serial chains of spherical four-bar mechanism to achieve design helices is presented. This chapter outlines the design synthesis process, and two example designs. The second example design lies flat when oriented in the first design helix, closely
resembling an origami based device. Chapter 5 concludes the thesis with closing remarks and future work.
CHAPTER II

KINEMATIC SYNTHESIS

This chapter reviews the foundational principles used throughout this work. The references should be consulted for detailed discussion on the topics presented.

2.1 Design and Target Profiles for Mechanism Design

Significant research has been conducted on the synthesis of planar, rigid body mechanisms capable of approximating a set of design curves. Specifically, synthesizing planar mechanisms with constant arc length for open profiles is found in Murray et al. [7], and for close profiles in Persinger et al. [8]. Several of the first variable geometry dies demonstrate shape change with constant arc length. In Shamsudin et al. [1], the process of synthesizing planar, shape-changing, rigid body mechanisms for design profiles with significant differences in arc length is discussed. Where previous work focused on mechanisms capable of approximating design profiles with similar arc length, the introduction of prismatic joints in the synthesis process allows for this added contribution. Future variable geometry die design relies on the ability to change shape between profiles with drastically different arc lengths. This requirement originates from some of the proposed uses of variable geometry dies. Manufacturing of components such as spatulas, knife handles, door stoppers, and radically redesigned licorice, and pool noodles all require significant changes in profile arc length. Although the process can be found in thorough detail in [1], it is summarized here.
A problem is defined by specifying a set of design profiles. These profiles are standardized by introducing a new set of representative coordinate points creating piecewise linear target profiles. The number of points on each target profile is varied so the average distance between successive points on each profile is within a defined tolerance. The segmentation phase creates segments of length and shape that approximate corresponding portions of the target profiles. Two types of segments are used to approximate the target profiles. Fixed length segments with arbitrary curvature and varying length segments with constant curvature. Revolute joints with a range of motion below a defined tolerance are replaced with fused segments which reduces the number of joints in a chain. The synthesis is finalized with a mechanization phase which attempts to lower the degrees-of-freedom (DOF) of the device.

Design profiles may be viewed as piecewise linear curves of \( N_j \) points connected with linear segments. The \( i^{th} \) point on the \( j^{th} \) profile is defined as \( \{a_{ji}, b_{ji}\} \) which gives a piece length at the \( i^{th} \) point of

\[
c_{ji} = \sqrt{(a_{j(i+1)} - a_{ji})^2 + (b_{j(i+1)} - b_{ji})^2},
\]

and an overall arc length of

\[
C_j = \sum_{i=1}^{N_j-1} c_{ji}.
\]

Design profiles can have any number of points spaced at various intervals creating variances in \( c_{ji} \) between points and profiles. Design profiles can also have differences in lengths, \( C_j \). The profiles need to be standardized to aid in the comparison process required for segmentation.

Points on the target profiles are distributed by specifying a desired piece length \( S_d \) and allowing the number of pieces \( m_j \) on the profiles to be calculated as

\[
m_j = \left\lceil \frac{C_j}{S_d} \right\rceil.
\]

\( \lceil C_j/S_d \rceil \) is a ceiling function with the smallest integer no less than \( C_j/S_d \). Provisional target profiles are realized by distributing \( n_j \) points along the design profiles at increments of \( C_j/m_j \). A
The design profile with corresponding target profile points is shown in Fig. 2.1. The $j^{th}$ target profile is a piecewise linear curve comprised of the ordered points $z_{ji} = \{x_{ji}, y_{ji}\}$ for $i = 1, ..., n_j$. The $j^{th}$ profile has $n_j = m_j + 1$ points and an $i^{th}$ piece length of

$$S_{ji} = \|z_{j(i+1)} - z_{ji}\| = \sqrt{(x_{j(i+1)} - x_{ji})^2 + (y_{j(i+1)} - y_{ji})^2}. \tag{2.4}$$

As illustrated in Fig. 2.1, for a provisional target profile, $S_{ji} \leq S_d$ and can only be equal for a section of no curvature long enough to capture a length $S_d$. The provisional target profile has an average piece length of

$$\bar{S}_j = \frac{1}{m_j} \left( \sum_{i=1}^{n_j-1} S_{ji} \right) \tag{2.5}$$

A result of approximating a curve with a piecewise distribution of points determined by arc length is that $S_j$ will be smaller than the desired piece length $S_d$. To remedy this, an error term is introduced as $\epsilon_j = |S_d - \bar{S}_j|$ and the number of points $n_j$ is reduced until the error is minimized. This results in a redistributed target profile of $n_j^*$ points and $m_j^*$ pieces. The final target profile length is determined as

$$S_j = \sum_{i=1}^{m_j^*} S_{ji}, \tag{2.6}$$

Figure 2.1: A design profile (solid) and target profile (dashed) with points distributed at equal arc length distances (adapted from [1]).
and the average length of the pieces on all $p$ profiles is

$$\bar{S}_m = \frac{\sum_{j=1}^{p} S_j}{\sum_{j=1}^{p} m_j^e}. \quad (2.7)$$

### 2.1.1 Segmentation of Target Profiles

With the target profiles containing nearly identical length pieces, the process of identifying segments which approximate the profile edges occurs. The profiles are broken into $q$ segments that approximate corresponding regions on the target profiles. Mean segments or M-segments correspond to fixed length and shape portions of the target profiles. M-segments are embodied by revolute-revolute (RR) links. Constant curvature segments or C-segments are of varying length between target profiles however have a constant curvature. C-segments are embodied by revolute-prismatic-revolute (RPR) links. If only a small relative rotation (below a specified tolerance) is required between a C-segment and an adjacent segment, the two are fused. For example if a C-segment is fused with another C-segment, the combined segment represents an RPR chain.

M-segments are identical between target profile, therefore the number of pieces in each M-segment between target profiles is the same. For a series of target profiles of similar arc length, segmentation could be accomplished with only M-segments. C-segments contain a different number of pieces between target profiles and thus must absorb the difference in the number of pieces between target profiles when a significant arc length difference exists.

### 2.1.2 Segment Properties

The number of pieces on the $e^{th}$ segment of the $j^{th}$ profile is $m_j^e$ which corresponds to $n_j^e = m_j^e + 1$ points defining the segment. Additionally, the number of pieces in the M-segments on all the target profiles is the same, i.e., $m_1^e = m_2^e = \ldots = m_p^e$. To satisfy the difference in pieces between the target profiles, the number of pieces in a C-segment on the $j^{th}$ profile must satisfy $\sum_{e=1}^{q} m_j^e = m_j^e$. 

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Finally, the points that lie between different segments are termed segmentation points and include the end points of the target profile. The index of a segmentation point at the start of the $e^{th}$ segment on the $j^{th}$ profile is designated as $k^e_j$. The index of the next successive segmentation point is thus $k^{e+1}_j$. This point correlates to the final point on the corresponding segment and the first point on the successive segment.

The same number of segments $q$ and their corresponding position in a target profile must be maintained for all other target profiles. A design vector $\vec{V}$ is used to define the number and type of segments. For example, $\vec{V} = [MMCMC]$ defines $q = 5$ segments in the corresponding order M-segment, M-segment, C-segment, M-segment, C-segment. At this time a method of determine the optimal number and type of segments is lacking, rather this is a decision made by the designer from intuition and experimentation. Additionally, an automated method of cycling through different options to produce better design profile matching could be implemented.

2.1.3 M-Segments

The process of creating M-segments begins by shifting the $e^{th}$ segment on each target profile to a common location so an M-segment can be determined that best approximates the profile segments. The rigid body transformation of

$$\vec{Z}^e_{ji} = A^e_j \vec{z}_{ji} + \vec{d}^e_j, \hspace{2mm} i = k^e_j, ..., k^{e+1}_j, \hspace{2mm} j = 2, ..., p,$$

(2.8)

where

$$A^e_j = \begin{bmatrix} \cos \theta^e_j & -\sin \theta^e_j \\ \sin \theta^e_j & \cos \theta^e_j \end{bmatrix}, \hspace{2mm} \vec{d}^e_j = \begin{bmatrix} x^e_j \\ y^e_j \end{bmatrix}$$

(2.9)

is used to shift the segments from all the target profiles to align with the segment from the first profile. This is shown graphically in part (a) and (b) of Fig. 2.2. A method to determine $\theta^e_j$ and $\vec{d}^e_j$ to minimize the sum of the distances between points on corresponding segments is beyond the scope of this introduction and can be found in Shamsudin et al [1]. A new piecewise linear curve is
Figure 2.2: The target segments are translated to the location of the first profile segment (a). The segments are translated with a displacement and rotation found through a distance minimizing function (b). An M-segment is determined which best approximates the corresponding points of each segment (c) (adapted from [1]).

determined with points corresponding to the geometric center of the set of \( p \) points in the reference segment \( \vec{z}_{1i} \) and the shifted segment \( \vec{Z}_{ji} \). The corresponding M-segment is thus defined by the points

\[
\vec{Z}_{mi}^e = \frac{1}{p} \left( \vec{z}_{1i} + \sum_{j=2}^{p} \vec{Z}_{ji} \right) \quad i = k_j^e, \ldots, k_j^{e+1}. \tag{2.10}
\]
2.1.4 C-Segments

A constant curvature C-segment is created using discrete, signed, radius of curvature values at each point along the segment for all target profiles [45]. The radius of curvature for the $i^{th}$ point on the $j^{th}$ profile is $r_{ji}$ and is formed by finding the arc that passes through $\vec{z}_{ji}$ and the neighboring points $\vec{z}_{j(i-1)}$ and $\vec{z}_{j(i+1)}$ [46]. For each point on the $e^{th}$ segment, the radius of curvature is calculated as

$$ r_{ji} = \left| \frac{\vec{z}_{j(i+1)} - \vec{z}_{j(i-1)}}{2 \sin \theta} \right| i = k_j^e + 1, \ldots, k_j^{e+1} - 1 \quad j = 1, \ldots, p, $$  \tag{2.11}

where $\theta$ is the angle $\angle \vec{z}_{j(i-1)} \vec{z}_{ji} \vec{z}_{j(i+1)}$ [47]. The first and last point of a segment $e$ are not included as there is no prior or successive point to evaluate a radius. To determine the sign of the radius of curvature, a direction vector $\vec{P}_{ji}$ extending from $\vec{z}_{j(i-1)}$ to $\vec{z}_{ji}$ is defined as

$$ \vec{P}_{ji} = \begin{cases} x_{ji} - x_{j(i-1)} \\ y_{ji} - y_{j(i-1)} \end{cases}. $$  \tag{2.12}

If the determinate $|\vec{P}_{ji} \times \vec{P}_{j(i+1)}|$ is positive, the radius of curvature is designated positive [48]. Finally the radius of curvature of the $e^{th}$ segment is determined from the average radius of successive pieces in a given segment. This is determined as

$$ \bar{r}_e = \frac{1}{\sum_{j=1}^{p} \frac{1}{m_j^e - 1} \left( \sum_{j=1}^{p} \left( \sum_{i=k_j^e+1}^{k_j^{e+1}-1} r_{ji} \right) \right) }. $$  \tag{2.13}

Average piece lengths of the $e^{th}$ segment on all the P profiles is

$$ \bar{s}_e = \sum_{j=1}^{p} \frac{1}{m_j^e} \left( \sum_{i=k_j^e+1}^{k_j^{e+1}-1} s_{ji} \right). $$  \tag{2.14}

As with M-segments, C-segments are placed on each profile using a distance minimizing function. A visualization of fitting an average radius to three target profile segments with varying arc length is shown in Fig. 2.3. The corresponding number of pieces in each segment is also shown.
Figure 2.3: Three target profile segments of different length are shown in (a). These were chosen to be approximated as C-segments of the same radius (b) (adapted from [1]).

2.1.5 Joining the Chain

The segments are synthesized individually for corresponding locations on the target profiles. As a result of their positions being optimized individually, there is no constraint forcing the endpoints of the segments to coincided. The segments will primarily be connected with revolute joints, and therefore must be adjusted slightly to align the segmentation points. As shown in Fig. 2.4, $\vec{z}_{j_i}^{e} \neq \vec{z}_{j_i}^{e+1}$, $\forall i = k_j^e$, $e = 2, ..., q$ and $j = 1, ..., p$.

Each point on the $j^{th}$ profile is relocated by

$$\vec{z}_{j_i}^{e} = \tilde{A}_j^e \vec{z}_{j_i}^e + \tilde{d}_j^e, \quad i = 1, ..., n_j^*, \quad e = 1, ..., m_j, \quad (2.15)$$

where

$$\tilde{A}_j^e = \begin{bmatrix} \cos \tilde{\theta}_j^e & -\sin \tilde{\theta}_j^e \\ \sin \tilde{\theta}_j^e & \cos \tilde{\theta}_j^e \end{bmatrix}, \quad \tilde{d}_j^e = \begin{bmatrix} \tilde{x}_j^e \\ \tilde{y}_j^e \end{bmatrix}. \quad (2.16)$$

The constraint of $\vec{z}_{j_i}^{e} = \vec{z}_{j_i}^{e+1}$ = $\vec{z}_{j_i}$, $\forall i = k_j^e$, $e = 2, ..., q$, and $j = 1, ..., p$ arises from forcing the endpoints of adjacent segments to be coincident. As in Persinger et al, [8], the segmentation points
Figure 2.4: The segmentation points do not align because the segments were optimized individually (a). The segmentation points are aligned by repositioning each segment (b) (adapted from [1]).

are united through a constrained numerical optimization to determine $\tilde{z}_{ej_i}$, $e = 2, ..., q$, $\tilde{\theta}_j$ and $\tilde{d}_j$.

The relocation results in a chain of segments with coinciding segmentation points like that shown in Fig. 2.4(b).

### 2.1.6 Fused Segments

Reducing the number of segments and joints in a chain also reduces the complexity of the chain. Therefore joints can be removed, resulting in two segments being fused together, if the approximation error is not greatly effected. The importance of a revolute in a segmented chain can be quantified by maximum angular displacement between the target profiles. If this range of motion falls below a certain tolerance, the joint can be removed and replaced with a fused joint, creating one compound segment.
Recall that $\vec{P}_{j,i}$ is the direction vector that extends from $\vec{z}_{(i-1)}$ to $\vec{z}_{ji}$. The direction vector that points from the preceding point to toward the $e^{th}$ segmentation point on the $j^{th}$ profile is $\vec{P}_{ke,j}$, and $\vec{P}_{ke,j+1}$ is a vector from the $e^{th}$ segmentation point to the next point. These vectors define a relative angle between segments for each target profile. Therefore, the relative joint angle at the $e^{th}$ segment on the $j^{th}$ profile can be designated $\sigma_{e}^{j}$ and the range of motion can be found as

$$\Delta \sigma_{e} = \max \{ |\sigma_{e}^{j} - \sigma_{e}^{l}| \} \quad j = 1, ..., p, \quad l = 1, ... p, \quad l \neq j. \quad (2.17)$$

If this range of motion falls below a certain tolerance, the joint can be replaced with a fused joint. In Fig. 2.5(a), a revolute joint is circled to draw attention to its small range of motion between the three target profile orientations. In Fig. 2.5(b), the joint is replaced with a fused joint, shown as a blue square. This creates a combined C-segment and M-segment. This concept will be used often during die design to help reduce the kinematic complexity of segment chains. As will be shown later, joint concepts for variable geometry dies require adequate space to prevent material leakage. As designs become cluttered with joints, low range of motion joints are removed.
2.2 Design of Planar Mechanisms

Determining segment type and shape through the previously described synthesis process results in multiple DOF devices capable of approximating design curves. Mechanization involves adding rigid constraints to the links and joints to create a mechanism with reduced degrees of freedom. Typically, a single degree of freedom device capable of actuating smoothly between the design profiles is preferred for simplicity in control [49, 50]. Mechanization to approximate target profiles may be treated as individual motion generation synthesis for each segment in the target profile. Therefore, a single DOF device can be designed for at most exact matching of five design profiles [39] by adding additional links and fixed pivots. However, this does not guarantee that the device will be capable of actuating between the profiles, or that the profiles will be on the same branch of the mechanism. For smaller design tasks of two to three profiles, variations of graphical synthesis can be implemented. Graphical synthesis is an approach to determining link lengths and pivot locations using geometric constraints to mechanize a chain. This is typically done using Computer Aided Design Software (CAD).

The primary challenge of designing a variable geometry die for polymer extrusion is the design of the joints. During the joint design phase of this research, dies capable of radical motion were developed, however incorporated limited prescribed design profiles. Often times to determine the feasibility of new joints, design profiles are only determined based on the motion capable with that joint. Therefore, forms of graphical synthesis have primarily been used to reduce the DOF of dies incorporating new joint designs. For completeness, a three position graphical synthesis method used to mechanize many of the dies presented in this text will be described.

The design presented in the following example represents a portion of the process used to design the variable die shown in Fig. 3.26 of Chapter 3. That die, capable of producing knife handles, uses
a 2 DOF mechanism. This example shows the process of designing the mechanism responsible for a single DOF variation which creates the upper portion of the handle.

Three design profiles, shown in Fig.2.6(a), are specified for the upper portion of the ergonomic knife handle grip. The profiles have fixed ends that do not connect to form a closed loop. Via an ad-

hoc approach, a five bar mechanism was designed which comes close to approximating the design profiles, Fig. 2.6(b). The fixed link is chosen to connect the fixed end points of the design profiles. The blue circles represent revolute joints, the cyan components represent a constant curvature segment with a prismatic joint (C-segment) and the blue square represents a fused joint. The fused joint combines a constant radius C-segment with an M-segment and allows no relative motion between successive components.

The design profiles represent the internal flow path that forms an extruded part, therefore mechanismization must result in all the components being outside of the loop formed by the kinematic chain.
The joints between links will not be designed until later in the process. Rather they are modeled as point on point contact between successive links. The unconstrained mechanism with one fixed link is shown in Fig. 2.7. The mechanism at this step has two DOF and will need to be constrained by

![Diagram of unconstrained mechanism](image)

**Figure 2.7:** The unconstrained mechanism has two degrees of freedom and will need an additional link and fixed revolute joint.

The addition of a binary link and fixed pivot. The binary link can be attached with a revolute joint to any non-fixed link to reduce the mechanism by a single DOF. Arbitrarily choosing a link in the mechanism does not guarantee that the device will be capable of actuating between the desired positions without passing through a branch defect. A branch defect occurs when a mechanism becomes stationary and the mechanical advantage reduces to zero [51]. From experience, an extension to the C-segment is chosen for the location of a revolute joint. The device is positioned in the three target profile orientations, corresponding to three revolute joint locations. This process completed for all three target profiles is shown in Fig. 2.8. These three points are used to define an arc and corresponding center point. The center point is the location of the fixed pivot and the binary link
has a length corresponding to the radius of this arc. The addition of this link reduces the mechanism to a single degree of freedom and allows smooth actuation between the design profiles.

The highly constrained joint design space of variable geometry dies dictates many of the decision made during this process. Determining the correct links to constrain, location on links for a revolute joint, and optimal segmentation to promote joint design is an iterative process. Planar mechanism design is a well-established area of research, this research focuses on the design of variable geometry die components. Thus, ad-hoc techniques such as this variation of graphical synthesis are deemed sufficient at this point in die design. More detail on the challenges of joint design is provided in Chapter 3.
2.3 Design of Spherical Four-Bar Mechanisms

In Chapter 4, spherical four-bar mechanisms are used to create serial chains capable of achieving design helices. As will be shown later, each design helix introduces a motion task for each spherical mechanism in the chain to achieve. As with planar four-bar mechanisms, it is possible to design spherical four-bar mechanisms for up to five prescribed motion tasks. McCarthy and Soh [39] describe the algebraic method of designing spherical four-bar mechanisms for up to five motion tasks.

A spherical RR chain consists of a floating link connected to a crank by a revolute joint, with the crank connected to the ground by a revolute joint. Both axes intersect, and the angular length of the crank is given as $\rho$. Two RR chains can be connected to form a single DOF four-bar linkage like the one shown in Fig. 1.5, noting that all four axes of rotation intersect at a single point.

Let $F$ be a fixed reference frame defined at the joint center and $M$ be a moving reference frame on the floating link of the RR chain. Due to the constraints imposed by the joint axes intersecting at a common point, $M$ moves in pure rotation around the axis intersection point. The unit vector that passes through the fixed pivot axis is denoted by $\vec{G}$ and the unit vector that passes through the moving axis in $M$ is $\vec{n}$. Defined in $F$, the coordinates of this axis becomes

$$\vec{N} = A\vec{n}, \tag{2.18}$$

with $A$ representing the rotation of $M$ relative to the fixed frame $F$. The angular length of the crank is constant during rotation satisfying

$$\vec{G} \cdot \vec{N} = |\vec{G}||\vec{N}| \cos(\rho). \tag{2.19}$$

Let the orientations of the frame $M_j$ be defined by the rotations $A_j$ with $j = 1, \ldots, 5$. With $\vec{N}_j$ being the moving axis orientation in the fixed frame $F$, Eq. 2.19 must be satisfied for all rotations

$$\vec{G} \cdot \vec{N}_j = |\vec{G}||\vec{N}_j| \cos(\rho) \quad j = 1, \ldots, 5. \tag{2.20}$$
The $j = 1$ equation can be subtracted from the remaining four equations to obtain

$$
\vec{G} \cdot \left( \vec{N}_j - \vec{N}_1 \right) = 0 \quad j = 2, ..., 5.
$$

(2.21)

It is important to note that this equation defines the perpendicular bisector to each segment $\vec{N}_j - \vec{N}_1$. Eq. 2.21 states that all of these planes pass through the unit vector $\vec{G}$. Equation 2.21 provides four equations in the unknowns of both the fixed and moving axes $\vec{G}$ and $\vec{N}$. Adding the constraint of requiring both these axes to be unit vectors provides an additional two equations. This gives a total of 6 equations in 6 unknowns leading to the maximum of 5 prescribed orientations for a spherical RR chain. Methods of solving for these unknowns can be found in McCarthy and Soh [39]. Up to 6 real solutions are possible, however real solutions are not guaranteed for every set of target orientations.
CHAPTER III

VARIABLE GEOMETRY DIES FOR POLYMER EXTRUSION

3.1 Desirable Design Attributes

The following list of desirable design attributes was compiled from die design references[2, 22, 23] and in conjunction with numerous extrusion die design engineers having years of production experience [52].

3.1.1 Leakage

Variable geometry dies contain joints to allow for the relative movement of the parts. These joints need to minimize or control die leakage. Surface on surface contact with manufacturing to tight fitting tolerances is one way to obtain this desired characteristic. Creating movement with a surface sliding across another surface restricts the leak path [53]. More versatile motion may be achieved by creating joints that have an edge on surface contact. To prevent damage to the extruded part, material that is leaked must be channeled away from the extrudate [54].

3.1.2 Die Land

Die land provides a uniform cross-section passage just prior to the die exit, which relaxes the viscoelastic stresses in the melt before leaving the die [2]. Any grooves or notches along the die land
will disrupt flow and will likely reduce the dimensional stability and surface quality of the extruded part. Thus, moveable die features should not interrupt a continuous die land.

### 3.1.3 Die Exit Plane

All components that form the die exit should lie on a common plane. A uniform die exit plane will help to maintain constant velocity throughout the flow field. Maintaining a uniform velocity throughout the profile section ensures higher quality surface finish [2]. If die exit occurs on multiple planes, extrudate curling can occur from the increased shear stress on the longer portion of the die land.

### 3.1.4 Die Exit Flow Area

The volumetric flow rate produced by an extruder screw of diameter $D$ with rotational speed $N$ is

$$Q = \frac{\pi^2 D^2 H N \cos \theta \sin \theta}{2}$$  \hspace{1cm} (3.1)

where details of the metering section of the extrusion screw include the height of the thread (or flight) $H$, and lead angle $\theta$ [55]. The average velocity of the extrudate is

$$v = \frac{Q}{A_d}$$  \hspace{1cm} (3.2)

which is nominally selected as the extrusion line speed. Note from Eqs. (3.1) and (3.2) that dramatic changes in die exit area $A_d$ may require adjustments to the extruder line. At a constant screw speed $N$ a reduction in die exit area will increase the velocity $v$ of the extrudate. Accordingly, a reduction of die exit could be compensated by slowing the screw speed, increasing line speed, or opening bypass flow ports that keep a constant effective die area. Process development is required to resolve these issues and are left for future work. The initial variable geometry dies will limit changes in $A_d$ to 20%.
3.1.5 Actuation

The forces required to actuate the moving components of the die should not significantly vary throughout the actuation sequence. That is, the linkages formed by the moving components involved in the shape change must not approach kinematic singularities. Single DOF systems can be controlled via a single actuator. Controlling the actuation of a system becomes more complex as the degrees of freedom increase, but expands the envelope of the die exit shapes. For actuation simplicity, single dof systems are preferred but multiple dof systems are viable.

3.2 Joint Design Concepts

Variable geometry dies are comprised of different interconnected parts that form the die exit. The connecting joints must allow relative movement, including pure rotation (revolute joints) and pure translation (prismatic joints). Considering the attributes listed in the previous section, the following joint designs have been generated.

3.2.1 The Crescent Joint

The crescent joint permits rotation between adjoining die components. This joint, shown in Fig. 3.1, uses a curved tongue and groove arrangement that approximates a revolute joint. The parts that comprise the joint contain geometric features that do not vary along one dimension, which facilitates manufacture by milling or through Electric Discharge Machining (EDM). Design of a backer plate must be such that extrudate flow through the gap and groove is sealed off. Surface on surface contact between the two components is maintained throughout motion. Small clearances are specified between the tongue and groove to restrict leakage. The length of the arc is based on the desired motion limits. The radius of curvature can be made small, reducing the necessary concave fillet in the extruded profile. While the crescent joint has desirable attributes, it is not possible to
Figure 3.1: A crescent joint allows rotation between die components with minimum leak paths provided the gap and groove are obstructed from the flow.

form a true revolute joint at a corner on the extruded part. This concept is illustrated in Fig. 3.2.

The radius of curvature on a crescent joint is dictated by the material properties and machining

Figure 3.2: True revolute motion is achieved when a corner is formed between two moving links (a). A crescent joint is really a prismatic joint with a small radius of curvature that approximate revolute motion (b).
precision. While other joints are introduced that do allow for true revolute motion, the crescent joint and interlacing joint (next section) use this approximated revolute property.

### 3.2.2 The Interlacing Joint

The interlacing joint is shown in Fig. 3.3 and is similar to the crescent joint. A series of interlocking plates are stacked to form the joint that permits relative rotation between adjacent parts. Small clearances between the plates and surface-to-surface contact produce highly restrictive leakage paths. Additionally, the joint has a high out-of-plane strength as the plates support each other. However, manufacturing complexity is high as the plates may need to be individually machined and stacked into an assembly.

Figure 3.3: An interlacing joint allows rotation between die components and exposes no gaps or grooves during actuation.

### 3.2.3 The Corner Joint

A corner joint, shown Fig. 3.4, exhibits pure rotation that forms sharp corners on the profile edge. It is generated by placing a curved tongue and groove outward of the die flow, such that the center of rotation lies on the die land surface. Like the crescent joint, the parts that comprise
the corner joint contain geometric features that do not vary along one dimension, which reduces manufacturing difficulties. The drawback to the corner joint is that it relies on edge-to-surface contact or edge-to-edge contact to form a sealing interface. Thus, leakage between the components should be expected. Additionally, a plate must be incorporated to eliminate the leakage path through the gap and groove as seen in Fig. 3.4.

![Diagram of a corner joint](image)

Figure 3.4: A corner joint allows rotation between die components and creates an extruded part with a true corner.

### 3.2.4 The Recessed Revolute Joint

A recessed revolute joint permits rotation between adjoining die components, where the center of rotation is on the corner edge of the extruded part. Like the corner joint, movement of sharp corners on the extruded part are possible. The recessed joint places the axis of a pin collinear with the desired corner as shown in Fig. 3.5. The pin joint is recessed into the die land and a taper is incorporated to allow the extrudate to fill the sharp corner of the die exit. The recessed joint relies
on edge-on-surface contact to form a sealing surface. Because the pin is recessed, the contact edge is unsupported and any deflection can open a leakage path. Additionally, the dimensions of the taper must be specified according to the limits of the EDM process.

Figure 3.5: A recessed joint forms a corner and allows rotation between die components however lacks strength near the exit of the die.

3.2.5 The Sliding Joint

A sliding joint may either allow linear translation between adjoining die components or be formed along a constant radius arc to provide curved translation. A tongue and groove is incorporated as shown in Fig. 3.6. The joint exhibits surface-on-surface contact to impede leakage. Manufacturing of both straight and curved grooves is a straightforward process. A drawback is the interface along the die land, which may adversely affect the surface finish of the extruded part.
3.3 Variable Geometry Dies

After selection of an arbitrary target part-shape, the joints presented in the previous section were combined in moveable die designs for experimental evaluation. The designs were completed in SolidWorks as is the modeling standard throughout this work. Two of the die’s exits change from a rectangular shape to a parallelogram with 20° of actuation. The exit areas in each configuration are approximately 625 mm². The third die with a similar profile incorporates a prismatic joint, and is discussed in section 3.3.3.

3.3.1 The Crescent Die

Considering manufacturing and die leakage, the crescent joint was deemed as the best option when a sharp corner is not required. A die containing four crescent joints was designed as shown in Fig. 3.7. Four, nearly identical links with a length of 25-mm and a 25-mm land form the die exit. These moveable links were surrounded by an outer ring and placed between a backer and cover plate. The center of the crescent joints on link 1 were pinned to the backer plate preventing any motion relative to the extruder. Links 2 and 4 are able to pivot, whereas link 3 serves as a
Figure 3.7: This moveable die contains four crescent joints and is shown in the limit positions. The backer plate can be seen through the links.

-floating coupler. The opening in the backer plate was designed by observing the orifice created by the moving links between their limit positions. To restrict flow through each crescent joint, the groove was machined 0.10-mm wider than the tongue to permit a 0.05-mm clearance along each sliding surface. An axial clearance of 0.04-mm was used between the moving links and the backer and cover plates. The length of the tongue was selected to allow the backer plate to conceal the gaps created in the groove. An actuation bar was fastened to link 2. In the rectangular configuration of Fig. 3.7a, the die exit area is 637 mm$^2$. In the parallelogram configuration of Fig. 3.7b, the die exit area is 594 mm$^2$. The manufactured die is shown in Fig. 3.8.

3.3.2 The Crescent and Corner Die

Since a crescent joint is unable to form a sharp angle in the extruded profile, the corner joint was evaluated as the best option to create this feature. A die containing a corner joint and three crescent joints was designed as shown in Fig. 3.9. As with the previous die, the moving links were surrounded by an outer ring and placed between a backer and cover plate, and link 1 was fixed to the
A moveable die containing four crescent joints was fabricated for testing.

Figure 3.8: A moveable die containing four crescent joints was fabricated for testing.

backer plate. A corner joint is used to connect links 1 and 4. Crescent joints were used to connect links 1 and 2, links 2 and 3, and links 3 and 4. As before, an actuation bar was fastened to link 2 and an axial clearance of 0.04-mm was used between the moving links and the backer and cover plates. To assess the effect of a different crescent joint clearance, the groove was machined 0.25-mm wider than the tongue to permit a 0.13-mm clearance along each sliding surface. The manufactured die is shown in Fig. 3.10.

3.3.3 The Crescent, Corner and Prismatic Die

A third die was created to assess the performance of a prismatic joint. A die containing a corner joint, a prismatic joint and three crescent joints was designed as shown in Fig. 3.11. As with the other dies, link 1 was fixed to the backer plate. A corner joint is used to connect links 1 and 5. Crescent joints were used to connect links 1 and 2, links 2 and 3, and links 3 and 4. The prismatic joint was created between links 4 and 5. As before, an actuation bar was fastened to link 2. Being a two-DOF mechanism, a second actuation method is required. Gear teeth were formed.
between links 4 and 5. The rotation of the gear on link 4 is accomplished with a hex-head shaft and external allen key. When rotated the prismatic pair extends and retracts. This two DOF die has four maximum actuation configurations corresponding to a rectangular or parallelogram configuration and a prismatic joint being open or closed. This die has a minimum and maximum exit area ranging between $625 \text{ mm}^2$ to $715 \text{ mm}^2$. The manufactured die is shown in Fig. 3.12.

### 3.4 Experimental Assessment

Experimental tests were conducted on the three previously described dies. The same test procedures were used on the three dies with the exception of an added protocol for the prismatic die to account for the added degree of freedom. The same methodology to analyze the test results was used for the three dies.

The following test plan was formulated to assess the repeatability of the extruded parts produced by the dies under different motion schemes.
Figure 3.10: A moveable die containing one corner and three crescent joints was fabricated for testing.

T-1. The dies were repeatedly moved between limiting positions with a 1.0 second actuation, followed by a static period of 5.0 seconds between actuations.

T-2. The dies were repeatedly moved between limiting positions with a 5.0 second actuation, followed by a static period of 5.0 seconds between actuations.

T-3. The dies were repeatedly moved between limiting positions with a 1.0 second actuation, followed by a static period of 20.0 seconds between actuations to observe changes in part shape development when isolated from die actuation.

T-4. The actuation handle was continually oscillated between its limits with a cycle time of 2.0 seconds.

T-5. The actuation handle was continually oscillated between its limits with a cycle time of 10.0 seconds.
3.4.1 The Crescent Die

Two temperature insensitive polymers were extruded through the crescent die: soft polyvinyl chloride (PVC) and thermoplastic vulcanizate (TPV). Figure 3.14 shows the crescent die installed on an extruder with a screw having \( D = 63.5 \) mm, \( H = 4.83 \) mm, and \( \theta = 17.65^\circ \). The die was actuated between the limits seen in Fig. 3.7, creating distinctly different profiles. No effect of leakage through the crescent joint interfaces was observed on the extruded part. After completing a series of tests over several hours, the die was disassembled showing that plastic melt had seeped into the joint clearance, but less than 50\% of the land as shown in Fig. 3.13. PVC was tested with a 61 rpm screw speed and a 142°C melt temperature, resulting in die pressures that fluctuated between 2.48 and 2.62 MPa. Equations (3.1) and (3.2) predict a nominal line speed of 2.6 m/min. A slower line speed of 1.8 m/min was initially selected as the profile did not appear to fill the entire die exit at the calculated line speed. This difference is likely associated to the viscous losses of the extruded material. These tests were repeated for line speeds of 1.5 m/min and 1.2 m/min.
Cross sections associated with the two extreme die configurations were scanned using a profile measurement system. The scanner provided the coordinates \((X_i, Y_i)\) for \(i = 1, \ldots, N\) points around the perimeter of the profile. The number of scanned points, \(N\), was not the same for each profile and they were not consistently spaced around the perimeter. Six representative scans from the parallelogram configuration of T-1 are superimposed in Fig. 3.15 to illustrate the repeatability of the extrusion die.

To numerically analyze the profiles, the coordinates were repositioned to center them relative to the origin, \(x_i = X_i - \bar{X}\) and \(y_i = Y_i - \bar{Y}\), where

\[
\bar{X} = \frac{1}{n} \sum_{i=1}^{N} X_i, \quad \text{and likewise for } \bar{Y}.
\]

and likewise for \(\bar{Y}\). The points were ordered according to the angle \(\phi\) between a radial line to each point and a line to a reference feature on each profile. That is, \(\phi_i = \tan^{-1}(y_i/x_i)\).

To facilitate comparison among the profiles, the same number of defining points are distributed around the perimeter to represent the originally scanned points. Representative coordinates \(\tilde{x}_j, \tilde{y}_j\)
were placed at radial angles $\theta_j$ with increments of $\Delta \theta$, such that $j = 1, \ldots, 360\degree / \Delta \theta$. For each $\theta_j$, the number of points $x_i, y_i$ that lie within the sector $\theta_j - \Delta \theta/2$ and $\theta_j + \Delta \theta/2$ were determined and designated as $m_j$. The average radius for the $j^{th}$ region was calculated as

$$\bar{r}_j = \frac{1}{m_j} \sum_{i=k+1}^{k+m_j} \sqrt{x_i^2 + y_i^2},$$

(3.4)

where

$$k = \sum_{i=1}^{j-1} m_i.$$  

(3.5)

The coordinates of the representative profile at each $\theta_j$ are

$$\tilde{x}_j = \bar{r}_j \cos \theta_j,$$  

(3.6)

$$\tilde{y}_j = \bar{r}_j \sin \theta_j.$$  

(3.7)

In this fashion, representative perimeter points were generated for each scanned profile.
An additional process further aligned the profiles for comparison. The first profile was selected as a reference and each point on the other profiles were shifted by a rigid body transformation in the plane. Murray et al. [7] developed a closed-form method to determine the translation and rotation such that the sum of the distances between each point on the non-reference profile and the corresponding point on the reference profile is minimized. A direct, point-by-point distance comparison was made to assess the repeatability of the profile dimensions.

For a part with nominal dimensions of $25 \times 25$ mm, a tolerance of $\pm 250$ microns ($\pm 0.25$ mm) is typical for an extrusion process. For several scans of both rectangular (R) and parallelogram
Figure 3.15: A comparison of six profiles that were taken from different phases of actuation on the same part.

(P) shapes under the test conditions outlined above, the average absolute deviation between representative points $\overline{d}$, the standard deviation std(d), the 95% confidence interval, CI, and the percent area beyond a typical tolerance $\Delta A/A$, is given in Table 3.1. The repeatability values as seen by the 95% CI values compare favorably with conventional extrusion processes. Repeatability of the rectangular and parallelogram profiles were similar.

The effect of line speed on the profile shape is seen in Fig. 3.16, where six profiles are superimposed with the die exit. As seen, the final profile shape is greatly influenced by line speed. For each speed, however, repeatability values were similar to those provided in Table 3.1. Die exit design
Table 3.1: A comparison between extrusion profiles created with the crescent die (in microns) shows a repeatability comparable to conventional extrusion processes.

<table>
<thead>
<tr>
<th>Test</th>
<th>Shape</th>
<th>$\overline{d}$</th>
<th>std($d$)</th>
<th>95% CI</th>
<th>$\Delta A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T-1</td>
<td>P</td>
<td>114</td>
<td>139</td>
<td>±272</td>
<td>0.20%</td>
</tr>
<tr>
<td>T-1</td>
<td>R</td>
<td>106</td>
<td>132</td>
<td>±257</td>
<td>0.05%</td>
</tr>
<tr>
<td>T-2</td>
<td>P</td>
<td>134</td>
<td>66</td>
<td>±161</td>
<td>0.02%</td>
</tr>
<tr>
<td>T-2</td>
<td>R</td>
<td>111</td>
<td>141</td>
<td>±276</td>
<td>0.18%</td>
</tr>
<tr>
<td>T-3</td>
<td>P</td>
<td>135</td>
<td>124</td>
<td>±242</td>
<td>0.16%</td>
</tr>
<tr>
<td>T-3</td>
<td>R</td>
<td>148</td>
<td>97</td>
<td>±272</td>
<td>0.09%</td>
</tr>
</tbody>
</table>

guidelines to achieve a desired part profile depend on line speed. Creating these guidelines is left for future work.

3.4.2 The Corner Die

The Corner Die was tested using the PVC temperature insensitive material. As the results from the crescent test showed little deviation between the two materials tested, a single material was deemed satisfactory. A similar observation was made for line speed resulting in only two line speeds being tested: 1.5 m/min and 2.0 m/min. The extruder setup and test procedures were identical for the corner die and crescent die tests. The physical parameters of the corner die are the same as the crescent die with the exception of one joint being replaced by a corner joint. Therefore, the screw speed, barrel temperature, pressure and line speed were similar to that of the crescent die test. Similarly, four cross sections of the part extruded by the corner die in the parallelogram orientation are superimposed for comparison in Fig. 3.17.

The same profile analysis utilized for the crescent die was used on the corner die. The results are shown in Table 3.2 The corner die functioned satisfactorily and produced an extruded part with profiles that exhibited repeatability similar to the crescent die. Note that a significant amount of plastic leaked though the corner joint and into the void behind the die exit links and the outer ring.
Figure 3.16: A comparison of profile shape at different line speeds shows that 1.5 m/min produced the most accurately sized part compared to the die exit (dark profile).

(Fig. 3.18). This plastic melt remained pliable and did not affect actuation limits. Managing the leakage through the joint and through the void are the focus of future work.

3.4.3 The Prismatic Die

While producing an extruded part with significantly different sections, the prismatic die had additional design issues. Physical stops must be included in the prismatic actuation, as the operator often overextended the joint. The gear teeth were made with a diametral pitch of 32 to provide a fine adjustment resolution. These teeth were too small, and with the manufactured clearances were able to skip to the next tooth. Regardless of these design issues, testing of the die was conducted until joint failure. Only the T-2 test at the preliminary speed was conducted with the prismatic die. This
die had two degrees of freedom, the first controlling the die orientation (parallelogram or rectangle), and the second controlling the prismatic joint. Therefore testing in four orientations was conducted using the T-2 actuation scheme. Scans obtained while the die was fully deflected and the prismatic joint was fully extended are shown in Fig. 3.19.

Again the repeatability of the die was analyzed using the previously described method. The prismatic joint position is introduced in this comparison with “O” describing the open position and “C” describing the closed position. Open refers to the prismatic joint being moved to its most extended position. The results of this comparison are shown in Table 3.3.
Table 3.2: A comparison between extrusion profiles created with the corner die (in microns) shows similar repeatability to the parts created with the crescent die.

<table>
<thead>
<tr>
<th>Test</th>
<th>Shape</th>
<th>$\overline{d}$</th>
<th>std($d$)</th>
<th>95% CI</th>
<th>$\Delta A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T-1</td>
<td>P</td>
<td>118</td>
<td>70</td>
<td>±137</td>
<td>0.04%</td>
</tr>
<tr>
<td>T-1</td>
<td>R</td>
<td>90</td>
<td>38</td>
<td>±74.5</td>
<td>0.00%</td>
</tr>
<tr>
<td>T-2</td>
<td>P</td>
<td>139</td>
<td>165</td>
<td>±323</td>
<td>0.40%</td>
</tr>
<tr>
<td>T-2</td>
<td>R</td>
<td>136</td>
<td>148</td>
<td>±290</td>
<td>0.19%</td>
</tr>
<tr>
<td>T-3</td>
<td>P</td>
<td>71</td>
<td>80</td>
<td>±157</td>
<td>0.03%</td>
</tr>
<tr>
<td>T-3</td>
<td>R</td>
<td>88</td>
<td>99</td>
<td>±194</td>
<td>0.00%</td>
</tr>
</tbody>
</table>

Figure 3.18: Material leaked through the corner joint however the range of motion was not affected.

3.5 Additional Die Designs

The experimental assessment of these preliminary dies show promise for the proposed variable geometry die concept. While the crescent die and corner die performed well, the design issues associated with the prismatic die need more attention. Additionally, new concepts with large area changes and drastic profile variation are intended goals in this research.
Figure 3.19: A comparison of five profiles captured after actuation phases during prismatic die testing shows less repeatability attributed to the lack of a hard stop in the prismatic joint.

3.5.1 Prismatic Die Redesign

Several design flaws were identified during the prismatic die test. The gear pitch used to actuate the prismatic joint was too small. This resulted in the teeth skipping during high torque actuation. The lack of hard stops at the extents of the prismatic joint made it difficult to identify when the die had reached its designed motion limit ultimately leading to the die failure. The crescent joint between link 2 and 3 in Fig. 3.11 was designed with too small of an arc length and came apart when the prismatic joint was actuated past its designed motion limit. Finally gaps resulting in large amounts of leakage were exposed when the prismatic joint was extended too far. A second version of the prismatic die was designed to address these issues. This die will be fabricated and tested to
Table 3.3: A comparison between extrusion profiles created with the prismatic die (in microns) shows less repeatability than the previous dies.

<table>
<thead>
<tr>
<th>Test</th>
<th>Shape</th>
<th>Prismatic</th>
<th>$\bar{d}$</th>
<th>std($d$)</th>
<th>95% CI</th>
<th>$\Delta A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T-2</td>
<td>P</td>
<td>O</td>
<td>378</td>
<td>470</td>
<td>±921</td>
<td>2.27%</td>
</tr>
<tr>
<td>T-2</td>
<td>R</td>
<td>C</td>
<td>674</td>
<td>655</td>
<td>±1300</td>
<td>5.42%</td>
</tr>
<tr>
<td>T-2</td>
<td>R</td>
<td>O</td>
<td>830</td>
<td>1020</td>
<td>±1990</td>
<td>7.90%</td>
</tr>
<tr>
<td>T-2</td>
<td>P</td>
<td>C</td>
<td>497</td>
<td>497</td>
<td>±1100</td>
<td>3.45%</td>
</tr>
</tbody>
</table>

determine if the solutions fully addressed the observed issues. The second version of the prismatic die is shown in Fig. 3.20.

Figure 3.20: A second version of the prismatic die incorporates several changes to address problems with the first version.

The upper limit hard stop and shield to prevent flow through the crescent joint during extreme positions are shown in Fig. 3.20a. The hard stop is achieved through contact between a plate on the internal link and outer link of the prismatic joint. The crescent joint arc length, shown in Fig. 3.20b, was increased by 45° compared with the first prismatic die. Finally, the lower limit hard stop and
large pitch gears (chosen at 22 teeth/in) are shown in Fig. 3.20c. The lower limit hard stop is achieved by eliminating the teeth from the outer gear at a prescribed arc angle to prevent motion past a certain position.

3.5.2 Straight Prismatic Die

The assumption that a high level of extruder control will be required for variable geometry dies with large area changes has driven preliminary die design. The added control of either the screw or pull speed is required to account for the velocity change of the material which is indirectly related to the die exit area, as shown in Eq. 3.1 and 3.2. Three dies using common components were designed to experimentally quantify this effect. The three dies exhibit a 100% area change using a single DOF straight prismatic joint. The exit profiles in each die are designed to replicate a typical trim component for an automotive application. The exit profile of the three dies is shown in Fig. 3.21. The first die is a direct area change accomplished by choking the flow with a reduction in exit area as in previously tested designs and is shown in Fig. 3.22. The second die opens an alternate leak path during actuation keeping the effective exit area constant. This die will evaluate the validity of a controlled leak concept to account for large area changes. In theory, the leaked material would be recycled, ground and re-extruded. The third die will attempt to verify the constraint of a common

Figure 3.21: The exit profile changes 100% and is designed to replicate a typical trim component on an automobile windshield.
die exit plane by using a “scraping” feature to change the part profile. This die will not leak material and will actuate with a 100% area change.

Figure 3.22: A variable area straight prismatic die with an area change of 100% will help determine how to control an extrusion process with large area changes.

The outer housing, pre-loaded spring block, actuation screw, top fixed block and hard stops are common among the three described dies. Additionally the die exit profile creating the part is common allowing for comparison between parts produced by the dies. The pre-loaded spring block contains two die springs to keep components tight and reduce leakage despite component wear and fabrication tolerances. A large bolt threaded into the outer ring will be used to actuate the prismatic joint. The second die opens a leak path creating the same part with no effective exit area change. This die is shown in Fig. 3.23.

The sliding block and end condition block are replaced to create this constant area straight prismatic die. Despite the prismatic joint position, the effective exit area is constant. The material that is leaked will be recycled and re-extruded. The leak path is angled away from the desired part to avoid interference between the separated flow paths.
The final die has a multi-level exit plane rather than a common exit plane and is shown in Fig. 3.24. The requirement of a common exit plane has been an assumption dictating die design during this research. Again the sliding block and end condition block are replaced to accommodate this design. A secondary fixed block is introduced to create the maximum part profile over the internal land of the die. The internal profile stretching the width of the die land remains unchanged during actuation.

Figure 3.24: A die that accomplishes the same area change by placing a forming tool directly in the flow outside of the die land will validate the common exit plane assumption.
3.5.3 Knife Handle Die

A knife handle, shown in Fig. 3.25a, was identified as a product capable of illustrating the potential for a variable geometry extrusion dies. The cross-sectional profiles along the length of the knife handle are shown in Fig. 3.25b. The expectation is that this extruded product could be produced at higher rates and with less cost than an injection molded part.

Figure 3.25: A knife handle could be formed by a variable geometry extrusion die.

Two crescent, three corner and two prismatic joints were combined to create a two-dof die capable of forming the knife grip. Although the mechanism produced here was designed in an ad-hoc fashion, techniques in Shamsudin et al. [1] and Khao et al. [56] may be used to support the dimensional synthesis of a mechanism to solve this problem. The die shown in Fig. 3.26 has all moving components labeled, where links 1 through 7 form the die exit. The link labeled 1 is pinned.
to the backer plate. Corner joints connect links 1 and 2, 3 and 4, and 7 and 1. Prismatic joints connect links 2 and 3, and 6 and 7. Crescent joints connect links 4 and 5, and 5 and 6. To constrain the links that form the die exit, bellcranks 8 and 9 are pinned to the backer plate. Links 10 and 11 connect the bellcranks to the actuation links 12 and 13. The handles attached to links 12 and 13 are successively actuated to their full extent and returned to their original position to achieve the knife grip profiles shown in Fig. 3.25b. Further control over the grip size and location of ergonomic features could be controlled by altering the rate at which these handles are actuated with respect to a given extrusion line speed. This knife handle example is to illustrate potential, but there is no immediate plan to prototype and test this die.

Figure 3.26: An extrusion die capable of forming knife handles has 2 DOF and uses 3 joint types.
CHAPTER IV

SERIAL CHAINS OF SPHERICAL FOUR-BAR MECHANISMS TO ACHIEVE DESIGN HELICES

This chapter presents a methodology for achieving prescribed helices with chains of spherical four-bar mechanisms. As initial work in realizing spatial shape change, this discussion focuses on helices as straightforward examples of spatial curves. There are numerous obvious generalizations to this theory including those from selecting a variety of actuation schemes or by connecting any string of one DOF mechanisms.

4.1 Serial Chain Parameter Identification

This section details the process for determining the serial chain parameters given a set of design helices. The design helices identify the locations that points on the serial chain of spherical mechanisms must achieve. When configured at a design helix, the chain includes points that lie on the helix at periodic increments. Moreover, the chain is designed such that the relative orientations between successive links in the chain are the same. With the determination of the parameters of the serial chain, the spherical mechanisms may then be synthesized in the following section.
4.1.1 The Design and Companion Helix

Given the parameterized form for a helix, points along a design helix \( J \) may be identified as

\[
\vec{H}_{ji} = \begin{cases} r_j \cos (i \phi_j) \\ r_j \sin (i \phi_j) \end{cases} b_j i \phi_j
\]  

(4.1)

where \( \phi_j \) is the incremented parameter of helix \( j \), \( r_j \) is the radius of the helix, and \( b_j \) is its pitch. Integer values of \( i \) identify equally spaced points along helix \( j \). Given that the points on the helix are identified relative to links of the chain, a companion helix proves useful in identifying locations for points on the chain itself. Points along the companion helix are defined as

\[
\vec{I}_{ji} = \begin{cases} h_j \cos (i \phi_j + \theta_j) \\ h_j \sin (i \phi_j + \theta_j) \end{cases} b_j (i \phi_j) + J_j
\]  

(4.2)

where \( h_j \) is its radius, \( \theta_j \) is a fixed angular offset relative to design helix \( j \), and \( J_j \) is a fixed axial offset between companion and design helix \( j \). The points on the companion helix identify the joint axes intersection locations for the spherical mechanisms in the chain. These intersections are identified by the \( \vec{I}_{ji} \) locations shown in Fig. 4.1. Note that the successive locations of points on the design and companion helices also identify triangular bodies that represent the serial chain. Additionally, the relative orientation of successive triangles is the same allowing for the later introduction of identical spherical mechanisms at each of these \( \vec{I}_{ji} \) locations. One of these triangles is depicted in Fig. 4.2. The vector \( \vec{d} \) locates a point on the design helix relative to an axis intersection location and the vector \( \vec{k} \) locates the next axis intersection location on the companion helix. These vectors are defined relative to the \( X_{ji}, Y_{ji}, Z_{ji} \) reference frame shown. The vector \( X_{ji} \) is in the direction of \( \vec{k} \) when viewed from the fixed frame and \( Y_{ji} \) is in the plane spanned by \( \vec{d} \) and \( \vec{k} \). A serial chain defined in this way achieves each design helix by altering the rotation at each axis intersection location in an identical fashion.
Figure 4.1: Points on the design and companion helices identify the triangular body shown in fig. 4.2.

Figure 4.2: The triangular body used to identify relative points between design and companion helices.
4.1.2 The Triangular Body Parameters

From the definition of the vectors \( \vec{d} \) and \( \vec{k} \) in the previous subsection, observe that

\[
|\vec{d}| = d = |\vec{H}_{j1} - \vec{I}_{j0}|, \quad (4.3)
\]

\[
e = |\vec{I}_{j1} - \vec{H}_{j1}|, \quad (4.4)
\]

\[
|\vec{k}| = k = |\vec{I}_{j1} - \vec{I}_{j0}| \quad (4.5)
\]

where \( e \) is the length of the third side of the triangular body. Substituting Eqs. 4.1 and 4.2 into Eqs. 4.3, 4.4 and 4.5 respectively,

\[
d^2 = J_j^2 + r_j^2 + h_j^2 + b_j^2 \phi_j^2 - 2r_jh_j \cos(\phi_j - \theta_j) - 2J_jb_j\phi_j, \quad (4.6)
\]

\[
e^2 = J_j^2 + r_j^2 - 2\cos(\phi_j) r_jh_j + h_j^2, \quad (4.7)
\]

\[
k^2 = b_j^2 \phi_j^2 + 2h_j^2 - 2h_j^2 \cos(\phi_j). \quad (4.8)
\]

Given a set of design helices the values of \( r_j \) and \( b_j \) are known. For the design helix \( j = 1 \), the selection of the values \( h_1, \phi_1, \theta_1 \) and \( J_1 \) in Eqs. 4.6-4.8 determine \( d, e \) and \( k \). For the other design helices, \( j = 2 \) for example, \( r_2 \) and \( b_2 \) are known and \( d, e \) and \( k \) must be the same as for \( j = 1 \) (and all other design helices). Thus, Eqs. 4.6-4.8 contain four unknowns from which the values of \( h_2, \phi_2, \theta_2 \) and \( J_2 \) must be determined.

An approach to solving this set of equations (for any remaining \( j \)) is to solve for \( h_j \) in Eq. 4.8,

\[
h_j = \frac{\sqrt{k^2 - b_j^2 \phi_j^2}}{2 - 2\cos(\phi_j)}. \quad (4.9)
\]

Selecting a suitable value for \( \phi_j \) produces a value for \( h_j \). \( \phi_j \) must also be chosen to ensure a real value of \( h_j \). The two remaining unknowns are \( J_j \) and \( \theta_j \) in Eqs. 4.6 and 4.7. Eliminating \( J_j \) between these two equations yields

\[
0 = \left[ e^2 - d^2 + b_j^2 \phi_j^2 + 2h_jr_j \cos(\theta_j) - 2h_jr_j \cos(\phi_j - \theta_j) \right]^2
\]

\[
-4b_j^2 \phi_j^2 \left[ e^2 - h_j^2 + 2h_jr_j \cos(\theta_j) - r_j^2 \right], \quad (4.10)
\]
in terms of the single unknown $\theta_j$. Expanding Eq. 4.10, the half-angle substitution $B_j = \tan(\theta_j/2)$ results in a fourth order polynomial in $B_j$. As roots appear in imaginary pairs, it is possible for this polynomial to have all imaginary solutions. In this case, a new choice of $\phi_j$ in Eq. 4.9 must be tried. A choice of $\phi_j$ resulting in real values of $B_j$ has been found for every case attempted. After calculating the possible $\theta_j$ from usable values of $B_j$, $J_j$ is determined from Eq. 4.7. Due to the existence of superfluous roots, all possible solutions are substituted into Eq. 4.6 to determine the true solutions.

With the parameters $\phi_j$, $\theta_j$, $h_j$ and $b_j$ identified for every design helix, all $\vec{H}_{ji}$ and $\vec{I}_{ji}$ may be determined from Eqs. 4.1 and 4.2. The design parameters describe a serial chain of triangular bodies, like the one shown in Fig. 4.1, where the helix extension points lie on the design helix and the axis intersection locations lie at equal intervals along the companion helix.

### 4.1.3 The Orientations of the Triangular Bodies

As depicted in Fig. 4.1, a connected chain of triangular bodies can be arranged to reach periodically spaced points on a design helix. This is accomplished by making the relative rotation between successive triangular bodies the same. Moreover, using the theory presented in this section, the triangular bodies can be sized and reoriented to reach periodically spaced points on any number of design helices. For the purpose of designing a spherical four-bar mechanism to produce the relative orientations between the triangular bodies, a reference frame is attached to each as shown in Fig. 4.2. Thus, relative to the fixed frame, the $i^{th}$ axis intersection location on the $j^{th}$ design helix has the rotation matrix

$$ A_{ji} = \begin{bmatrix} \vec{X}_{ji} & \vec{Y}_{ji} & \vec{Z}_{ji} \end{bmatrix}. \quad (4.11) $$

As $\vec{X}_{ji}$ aligns with the $\vec{k}$ side of the triangular body,

$$ \vec{X}_{ji} = \frac{\vec{I}_{ji} - \vec{I}_{j(i-1)}}{||\vec{I}_{ji} - \vec{I}_{j(i-1)}||}. \quad (4.12) $$
As $Z_{ji}$ is perpendicular to the plane of the triangular body,

$$
\hat{Z}_{ji} = \frac{\hat{X}_{ji} \times \hat{D}_{ji}}{|\hat{X}_{ji} \times \hat{D}_{ji}|}
$$

(4.13)

where

$$
\hat{D}_{ji} = \hat{H}_{ji} - \hat{I}_{j(i-1)}.
$$

(4.14)

Finally,

$$
\hat{Y}_{ji} = \hat{Z}_{ji} \times \hat{X}_{ji}.
$$

(4.15)

Noting that the relative rotation between successive triangular bodies is the same for any design helix,

$$
T_j = A_{ji}^T A_{j(i+1)} \forall i
$$

(4.16)

### 4.1.4 Aligning the Helices

Equations 4.1 and 4.2 define each helix as having independent locations for the first triangular body as shown in Fig. 4.3. With the pair of helices now discretized as chains of triangular bodies, the initial body of each chain may be aligned as shown in Fig. 4.4(a). This realignment is done for three reasons. First, the true motion produced by the changes in radius and pitch may be seen. Second, the new alignment provides a common body for all helices required for the realization of a physical system. Finally, the coordinates needed to complete the mechanical design in a CAD environment are made explicit.

The vectors defining the edges of the initial triangular body aligned with the fixed frame, as shown in Fig. 4.4(b), are

$$
\vec{d} = A_{j1}^T (\vec{H}_{j1} - \vec{I}_{j0}),
$$

(4.17)

$$
\vec{k} = A_{j1}^T (\vec{I}_{j1} - \vec{I}_{j0})
$$

(4.18)
noting that the \( j \) is omitted due to the values of \( \vec{d} \) and \( \vec{k} \) being the same for all \( j \). The equations

\[
\vec{K}_{ji} = T_j^i \vec{K}_0 + \vec{K}_{j(i-1)} \quad \forall i,
\]

\[
\vec{D}_{ji} = T_j^i \vec{d}_0 + \vec{K}_{j(i-1)} \quad \forall i,
\]

(4.19)

(4.20)
determine the remaining \( \vec{d} \) and \( \vec{k} \) vectors relative to the fixed frame.

### 4.2 Spherical Four-Bar Mechanism Parameters

The value of \( T_j \) determined in Eq. 4.16 is the relative orientation between successive triangular bodies when the chain is aligned with design helix \( j \). As a spherical mechanism is used to produce these relative orientations, the values of \( T_j \) identify the orientations in a spherical motion generation problem. The well-established theory associated with spherical motion generation dictates that at most five orientations may be exactly produced by a spherical four-bar mechanism [39]. Thus, a limit of five design helices may be achieved when this device is selected to join the triangular bodies.
The parameters of the spherical four-bar mechanism that define its motion are the angular link
lengths \( \alpha, \beta, \gamma \) and \( \eta \) as seen in Fig. 1.5. These angles may be derived from the directions of the
corresponding fixed and moving axes \( \vec{G}, \vec{F}, \vec{m} \) and \( \vec{n} \). A fixed and moving axis pair may be found
from the solution of the following equations,

\[
\vec{G} \cdot T_1 \vec{n} = \vec{G} \cdot T_j \vec{n}_j \quad j = 2, \ldots, N, \tag{4.21}
\]

\[
\vec{F} \cdot T_1 \vec{m} = \vec{F} \cdot T_j \vec{m}_j \quad j = 2, \ldots, N, \tag{4.22}
\]

As all usable fixed and moving axis pairs in the spherical four-bar mechanism satisfy Eqs. 4.21
and 4.22, the solutions also define all possible values of \( \vec{F}, \vec{m}, \vec{G} \) and \( \vec{n} \).

Finally, because the chain includes multiple copies of the spherical mechanism, the axes for
each mechanism are oriented differently relative to the fixed frame. To determine their respective
directions in the fixed frame,

\[ \vec{G}_{ji} = T_j^{(i-1)} \vec{G} + \vec{K}_{j(i-1)}, \]  

(4.23)

\[ \vec{n}_{ji} = T_j^i \vec{n} + \vec{K}_{j(i-1)}. \]  

(4.24)

Equations 4.23 and 4.24 can be replicated for \( \vec{G} \) replaced by \( \vec{F} \) and \( \vec{n} \) replaced by \( \vec{m} \).

### 4.3 Construction of the Chain

The result of Sections 4.2 and 4.3 is that all parameters needed to assemble the serial chain of spherical mechanism are now known. The first spherical four-bar mechanism aligned with the \( j = 1 \) helix is shown in Fig. 4.5. The first triangular body seen in Fig. 4.5 is the fixed link of the system. Thus, the axes \( \vec{G}_{11} \) and \( \vec{F}_{11} \) are fixed directions. All other axes in the system will change.

![Figure 4.5: The location in space of the first spherical four-bar mechanism when aligned with the \( j = 1 \) helix.](image-url)
as the system moves according to Eq. 4.23. As the fixed link of the system does not establish a connection between spherical mechanisms, its geometry is not as constrained. The addition of the connecting link and second spherical four-bar mechanism is shown in Fig. 4.6. The link and second mechanism are now reproduced and mated to the distal coupler. This process may be repeated as many times as desired to achieve any length for the mechanical chain, noting that the chain has as many degrees of freedom as the number of spherical four-bar mechanisms. Figure 4.7 shows a chain comprised of 24 identical links and mechanisms noting that the relative configuration of all the four-bars is the same. The same chain with the relative configuration of the spherical four-bars at two additional angles results in the three helices shown in Fig. 4.8.

Figure 4.6: The connecting link and the spherical four-bar mechanism are reproduced to define the remainder of the chain.
4.4 Additional Design Considerations

A readily observed concern about the mechanical chain described in this work is its capacity to be actuated. Recall that the distance between each axis intersection point is constant. A shaft segmented by universal joints located at these axis intersection points could transmit torque to each spherical mechanism. The segmented shaft concept relating the universal joints to the triangular bodies resulting from the synthesis methodology is shown in Fig. 4.9. Using this methodology, each spherical four-bar must support a portion of the segmented shaft with its base link and extract the torque needed to create rotation. As the axis of the driving link on each spherical mechanism intersects with the axis of rotation of the segmented shaft, bevel gears offer a potential method of rotating the driving link. Figure 4.10 shows the concept when the spherical four-bars are included in the chain. The image also shows a single spherical mechanism which can be repeated to create a
single degree of freedom chain of spherical mechanisms. Figure 4.11 shows the assembled device from the component portions in Fig. 4.10.

This segmented drive shaft concept raises concern that the axis intersection point of the spherical four-bar and the skew rotation point of the universal joint must align. Additionally, there are practical issues of overcoming the torque required to initiate motion in this complicated scheme. There is also the notion of the maximum drive angle allowable through a universal joint which may be solvable via a properly restrained cable drive. This however presents other issues.

A second idea under consideration is the introduction of a spring into each of the four-bar mechanisms. This has the potential of giving each mechanism the same rest configuration. Thus, the chain will rest at one of its design helices. Although still involving a large number of degrees of freedom, the mechanical system would not need individual actuation of its component four-bars but could be driven through manipulation of its free ends.
Figure 4.9: A segmented shaft connected by universal joints could provide each four-bar mechanism with equal actuation.

4.5 An Example With Three Helices

This section details the design of a serial chain of spherical four-bar mechanisms to reach the three design helices with the pitches and radii listed in Table 4.1. The table also includes the chain parameters resulting from the design calculations.

Table 4.1: A chain is designed in this example to reach the three design helices with the parameters listed.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Helix 1</th>
<th>Helix 2</th>
<th>Helix 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>0.2</td>
<td>3.0</td>
<td>3.0</td>
</tr>
<tr>
<td>$b$</td>
<td>4.0</td>
<td>4.0</td>
<td>0.1</td>
</tr>
<tr>
<td>$h$</td>
<td>0.75</td>
<td>3.53</td>
<td>3.55</td>
</tr>
<tr>
<td>$\phi$ (rad)</td>
<td>0.262</td>
<td>0.200</td>
<td>0.301</td>
</tr>
<tr>
<td>$\theta$ (rad)</td>
<td>0.122</td>
<td>-0.022</td>
<td>0.006</td>
</tr>
<tr>
<td>$J$</td>
<td>0.0875</td>
<td>0.1493</td>
<td>0.0686</td>
</tr>
</tbody>
</table>

The $j = 1$ helix is defined by the radius $r_1 = 0.2$ and the pitch $b_1 = 4.0$. The remaining parameters in Eqs. 4.6-4.8 are selected as $h_1 = 0.75$, $J_1 = 0.0875$ and the angles $\phi_1 = 0.262$ and
\[ \theta_1 = 0.122 \text{ radians. This results in the values } k = 1.065, d = 1.107 \text{ and } e = 0.559. \]

For the \( j = 2 \) helix, with \( r_2 = 3.0 \) and \( b_2 = 4.0 \), the selection of \( \phi_2 = 0.200 \) in Eq. 4.9 results in \( h_2 = 3.53 \). Given these values and the previously determined values of \( d \) and \( e \), Eq. 4.10 results in four values of \( \theta_2: -0.061 \pm 1.282i, 0.050 \text{ and } -0.022 \). Values corresponding to the two real values of \( \theta_2 \) are used in Eq. 4.7 to find two possible values of \( J_2 \). Substitution of the two solutions into Eq. 4.6 reveals \( \theta_2 = -0.022 \) and \( J_2 = 0.1493 \) as the usable pair. A similar process on \( j = 3 \) yields the values shown in Table 4.1.

These parameters are used with Eqs. 4.1 and 4.2 to determine the \( \vec{H}_{ij} \) and \( \vec{I}_{ij} \) for all three design helices at the points \( i = 0, 1, 2 \). The six points (per helix) provide all of the information necessary in Eqs. 4.11- 4.16 to determine the matrix \( T_j \). The three matrices are

\[
T_1 = \begin{bmatrix} 1.000 & -0.046 & 0.013 \\ 0.048 & 0.966 & -0.254 \\ -0.001 & 0.255 & 0.967 \end{bmatrix}, \quad (4.25)
\]
Figure 4.11: Single degree of freedom spherical mechanism chain which can approximate three design helices.

\[ T_2 = \begin{bmatrix} 0.991 & -0.131 & -0.016 \\ 0.127 & 0.981 & -0.151 \\ 0.036 & 0.147 & 0.989 \end{bmatrix} , \quad (4.26) \]

and

\[ T_3 = \begin{bmatrix} 0.955 & -0.294 & -0.034 \\ 0.294 & 0.956 & -0.014 \\ 0.037 & 0.003 & 1.000 \end{bmatrix} . \quad (4.27) \]

To align the initial triangular body of the three chains, Eqs. 4.17 and 4.18 produce \( \vec{d}_0 = [1.065 \ 0.000 \ 0.000] \) and \( \vec{k}_0 = [0.962 \ 0.549 \ 0.000] \). Equations 4.19 and 4.20 provide the coordinates needed to construct the serial chains from a common triangular body as shown in Fig. 4.12.
Substituting $T_1, T_2$ and $T_3$ into Eqs. 4.21 and 4.22 defines the rotation axes needed for the spherical four-bar mechanism. As this is a three orientation synthesis problem, the selection $\vec{n} = [0.250 \ 0.433 \ 0.866]$ and $\vec{m} = [0.250 \ -0.433 \ 0.866]$ results in $\vec{G} = [-0.832 \ -0.393 \ 0.391]$ and $\vec{F} = [-0.782 \ 0.410 \ -0.470]$. The corresponding mechanism link lengths are $\alpha = 95.02$, $\beta = 51.32$, $\gamma = 35.11$ and $\eta = 72.20$.

All remaining points are determined with Eqs. 4.23 and 4.24, allowing the design of the repeated four-bar mechanism and the connection link shown in Fig. 4.13. The segmented shaft connected with universal joints is also incorporated into this design. When this mechanism is copied and mated with the distal coupler, it will maintain a single degree of freedom. Copies of the connection link,
second spherical mechanism, and driving shaft in Fig. 4.13 and assembled according to the helices shown in Fig. 4.12 result in the chain of spherical four-bar mechanisms presented in Fig. 4.14.

4.6 Origami Based Design

Modern use of the term origami includes folding devices and mechanisms. Serial chains of spherical four-bar mechanisms appear to fold up on themselves, however due to their mechanical thickness and arbitrary joint axis directions, the connection to origami requires additional constraints. By designing the mechanical helices under consideration with a mechanism capable of lying flat when aligned at the first design helix, having 0 pitch, the result is a mechanical device modeling origami. This section develops the added constraints to ensure the joint axes lie in a common plane when the first helix is chosen with \( J_1 = 0 \) and \( b_1 = 0 \). These added constraints reduce the maximum number of possible design helices to three.
4.6.1 Added Design Constraints

Assuming the mechanism lies flat in the XY plane, the relative rotation matrix for the \( j = 1 \) helix is of the form

\[
T_1 = \begin{bmatrix}
\cos (\phi_1) & -\sin (\phi_1) & 0 \\
\sin (\phi_1) & \cos (\phi_1) & 0 \\
0 & 0 & 1
\end{bmatrix}.
\] (4.28)

Note that \( T_1 \), for a flat design helix, is a Z-axis rotation matrix by (the incremental helix parameter) \( \phi_1 \). For the axes of the spherical mechanism to lie in the same (XY) plane, defining an entirely flat configuration of the device, \( \vec{G}_z = 0 \) and \( \vec{F}_z = 0 \) for all points on the first design helix. The moving axis pair \( \vec{N}_{1i} \) and \( \vec{M}_{1i} \) must lie completely in the same plane resulting in a \( z \) component equal to zero when oriented with the \( j = 1 \) helix. Again, Eq. 4.21 is used to solve for the fixed and moving axis pair. Therefore to satisfy

\[
T_1\vec{n} = \begin{bmatrix} N_{1x} \\ N_{1y} \\ 0 \end{bmatrix},
\] (4.29)

\[\text{Figure 4.14: The serial chain of spherical four-bar mechanisms that reaches the three design helices in tab. 4.1.}\]
given the form of $T_1$, $\vec{\bar{n}} = \{n_x \, n_y \, 0\}^T$ in the world reference frame. The restriction of choosing two variables ($G_z = 0$ and $n_z = 0$) reduces the maximum number of possible design helices to three. Subtracting the $j = 2$ and $j = 3$ variations of Eq. 4.21 yields

$$\vec{G}^T \Delta T_{12} \vec{\bar{n}} = \vec{G}^T \Delta T_{23} \vec{\bar{n}} = 0.$$ (4.30)

Considering the elements of $\Delta T_{12}$ as

$$\Delta T_{12} = \begin{bmatrix} a_{12} & b_{12} & c_{12} \\ d_{12} & e_{12} & f_{12} \\ g_{12} & h_{12} & i_{12} \end{bmatrix},$$ (4.31)

expanding Eq. 4.30 for $j = 2$ and $j = 3$ (and recalling $G_z = n_z = 0$),

$$a_{12}G_xn_x + b_{12}G_xn_y + d_{12}G_yn_x + e_{12}G_yn_y = 0,$$ (4.32)

$$a_{23}G_xn_x + b_{23}G_xn_y + d_{23}G_yn_x + e_{23}G_yn_y = 0,$$ (4.33)

$$G_x^2 + G_y^2 - 1 = 0,$$ (4.34)

$$n_x^2 + n_y^2 - 1 = 0.$$ (4.35)
Combining Eqs. 4.32 and 4.33,

\[
\begin{bmatrix}
  a_{12}n_x + b_{12}n_y & d_{12}n_x + e_{12}n_y \\
  a_{23}n_x + b_{23}n_y & d_{23}n_x + e_{23}n_y
\end{bmatrix}
\begin{bmatrix}
  G_x \\
  G_y
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}.
\] (4.36)

To have a non-trivial solution for \( G_x \) and \( G_y \), the matrix in Eq. 4.36 must be singular, and

\[
(a_{12}n_x + b_{12}n_y)(d_{23}n_x + e_{23}n_y) - (a_{23}n_x + b_{23}n_y)(d_{12}n_x + e_{12}n_y) = 0.
\] (4.37)

With the addition of Eq. 4.35, the result is 2 equations in the 2 unknowns \( n_x \) and \( n_y \). Solving produces 2 unique solutions and their conjugate pairs. The unique solutions correlate to suitable moving axis pairs \( \vec{n} \) and \( \vec{m} \). These solutions are substituted into Eq. 4.36 and Eq. 4.34 to determine the fixed axis pair \( \vec{G} \) and \( \vec{F} \).

The process of aligning the initial triangular body for a mechanism with a flat \( J = 1 \) helix is the same as previously described. A three helix spiral design is shown in Fig. 4.16. The mechanisms
are designed the same as previously described however avoiding interference introduces new design challenges.

### 4.6.2 Mechanical Design of a Flat Spherical Four-Bar Mechanism

Designing serial chains of spherical four-bar mechanisms with a flat helix introduces two new design challenges, interference of components and inherent singularities. A property of a spherical four-bar mechanism capable of lying flat is that either the four link angles, supplements of the four link angles or a pair of unchanged link angles with a pair of link angle supplements must add to $2\pi$.

Therefore a spherical four-bar mechanism can be designed such that the links have a constant radius and do not lie on top of one another, resembling the way paper folds in traditional origami. This property is illustrated in Fig. 4.17 with a single spherical four-bar mechanisms designed to lie flat.

![Figure 4.17: The link lengths shown add to $2\pi$ creating a mechanism that can lie flat.](image)

The second design challenge is the singularity a spherical mechanism is in when lying flat. While a flat mechanism is placed at a branch point (singularity), physical solutions for forcing a mechanism into a desired branch exist. Springs and joint angle limiters are two solutions to solve this issue. The chain designed for this example uses joint angle limiters to ensure the device enters
the correct branch during actuation. Two angle limiters are shown in Fig. 4.18 and a third can be seen above in Fig. 4.17. With a mechanism and connecting link designed, the mechanism can be copied to create a chain capable of achieving three design helices. An example serial chain aligned with a flat helix can be seen in Fig. 4.19. The same chain aligned with two additional design helices

Figure 4.18: Allowable joint angles can be limited with physical stops at the desired maximum angle.

Figure 4.19: A full chain with its first design helix having no pitch or axial offset is limited in length to a full circle before the chain interferes with itself.
is shown in Fig. 4.20.

Figure 4.20: A full chain with its first design helix having no pitch or axial offset.
CHAPTER V

CONCLUSIONS AND FUTURE WORK

This thesis presented the kinematic synthesis and mechanical design of shape-changing devices for both planar and spatial applications. Variable geometry dies are a practical example of the application of rigid-body shape-change. The focus of variable geometry dies has been on joint design rather than design from synthesized segmentation. The strong foundation of joint design presented in this thesis promotes the use of synthesized segmentation and mechanization for the design of variable geometry dies as future work. Serial chains of spherical mechanisms are an initial attempt at extending rigid-body shape-change concepts into space and a direct link to origami is observed. The mechanisms presented achieve prescribed helices and represent an initial step toward designing for more complex curve approximation.

5.1 Variable Geometry Die Conclusions

Chapter 3 presented the development of variable geometry dies that enable the extrusion of plastic parts with a varying cross section. Desirable design attributes were identified and potential joint designs that could be used to join the interconnected parts that form the die exit were described. The various joint concept designs were used to create dies that formed a four-sided exit profile. One die consisted of four crescent joints where the internal angles could be altered while the extruder is operating. Experimental observations showed few adverse leakage effects of the moving die exit.
Repeatability results indicate that this moveable die formed parts with measured repeatability that was comparable to tolerances achievable with a conventional extrusion die. Testing was conducted on more complex dies. The corner die incorporated a corner joint between the fixed link and a moving link of the mechanism. This die performed adequately, however significant leakage was observed through the corner joint, but remained within the die outer ring. The leakage did not affect the die motion or extruded part and is a result of the line on surface contact attribute seen in the corner joint design. The prismatic die contained both a corner joint and prismatic joint. Inadequate gear tooth sizing on the prismatic joint, insufficient crescent length, and the lack of a hard stop on the prismatic joint led to operational issues and limited testing.

5.2 Variable Geometry Die Future Work

The issues observed with the prismatic die have been addressed and the necessary components have been redesigned as outlined in section 3.5.1. The second version of this die will be retested with the same procedure to determine the feasibility of using prismatic joints in variable geometry dies.

A set of dies has been designed to determine how large changes in area affect the extrusion process. These dies utilize a straight prismatic joint and are capable of a 100% change in area as outlined in section 3.5.2. The exit profile of these dies were designed to simulate a typical automotive trim component. Additionally, a die that is capable of creating a complex series of profiles that form knife grips was shown. Such a die will require significant testing specific to its suggested usage. Profile position and shape is dependent on the rate and time of actuation and therefore will require intelligent controls for adequate testing. Long term die design includes the introduction of these intelligent controls, and design for specified applications using segmentation techniques.
5.3 Serial Chains of Spherical Four-Bar Mechanisms Conclusions

Chapter 4 presented the kinematic synthesis of serial chains of spherical four-bar mechanisms. The chain is constructed by connecting the coupler of one spherical mechanism to the base link of the subsequent mechanism. The chain is noted to have one degree of freedom per spherical mechanism. Although each mechanism and each connection could be unique, these chains are not synthesized as such. In this work, all of the spherical four-bar mechanisms and the connections between them are the same. Moreover, all of the mechanisms are assumed to be identically actuated. The result is a mechanical chain with a helical structure.

A design methodology shows that the parameters of the spherical four-bars and the connecting links can be determined to match up to five helices. The chains match the helices by including a point on an extension to the coupler that lies on the design helix. Due to the repeated structure of the device, the points on the design helix occur periodically. The methodology takes advantage of the periodic nature of the chain by introducing a companion helix to each design helix. Equally spaced points on this companion helix identify the axis intersection locations of the spherical mechanisms. Despite the differences in radius and pitch between the companion helices, the distances between the axis intersection locations along them are the same. Although the points along any design helix are equally spaced, the spacing is not the same on two different design helices. An example shows the design of one of these systems to reach three helices and some mechanical features of the chain are discussed. Additionally, the methodology of designing a mechanism with all joint axes lying in a common plane when oriented with a 0-pitch helix is presented. Using these added constraints leads to a mechanism that lies perfectly flat when aligned with the first helix. This device is an example of origami-based engineering.
5.4 Future Serial Chain Work

A realizable solid model of the origami concept has been designed and submitted to be constructed using rapid prototyping technology. Several assumptions were made to design these devices including, identical four-bar mechanisms and identical actuation. Removing these constraints allows for more general spatial curves to be achieved, such as polynomial pitched helices or spatial spline curves. This may be an intriguing direction for the research. Ultimately the tie to origami is likely the best direction for this work. Origami is a growing field with numerous applications including space research, material packaging, and artistic design. The proven ability of these devices to lie perfectly flat and actuate into two additional design helices has the most potential to be useful. One possible use is a rapid deployment mechanism capable of conserving space when not being used.
BIBLIOGRAPHY


