SPIN HALL EFFECT OF VORTEX BEAMS

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ABSTRACT

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This thesis employs the angular spectrum method to analyze the reflection and transmission of a vortex beam at air-material interface. General expressions of the reflected and transmitted fields are presented in momentum space. Except for some special angles of incidence, such as Brewster angle and critical angle, the model is generally valid. Thereafter, an operator method is utilized to calculate the spin Hall effect of a vortex beam in momentum space. It is shown that, compared with its counterpart at an air-normal material interface, the spin Hall effect of vortex beam remains unchanged at an air-metamaterial (with double negative index) interface. The weak measurement technique is adopted to detect and measure the spin Hall effect of light (SHEL). The theory and experiment are in good agreement for partial reflection of a Gaussian beam at an air-glass interface. The design of an experiment to measure the SHEL of a vortex beam at an air-metamaterial interface is also described; this will be pursued in the future.
For parents, elder sister, and for ephemeral youth
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<th>Description</th>
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<tr>
<td>GH</td>
<td>Goos-Hänchen</td>
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<tr>
<td>TIR</td>
<td>Total Internal Reflection</td>
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<td>IF</td>
<td>Imbert-Fedorov</td>
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<td>SHEL</td>
<td>Spin Hall Effect of Light</td>
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<tr>
<td>LCP</td>
<td>Left-handed Circularly Polarized</td>
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<tr>
<td>RCP</td>
<td>Right-handed Circularly Polarized</td>
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<tr>
<td>HWP</td>
<td>Half Wave Plate</td>
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<td>CCD</td>
<td>Charge-Coupled Device</td>
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\begin{itemize}
\item $l$: topological charge of the vortex beam
\item $\hbar$: Planck constant
\item $k$: vacuum wave number
\item $[\alpha, \beta]^T$: Jones vector
\item $\theta_i$: incident angle
\item $r_p$: reflection coefficient of the $p$ polarized light
\item $r_s$: reflection coefficient of the $s$ polarized light
\item $t_p$: transmission coefficient of the $p$ polarized light
\item $t_s$: transmission coefficient of the $s$ polarized light
\item $n$: refractive index
\item $w(z)$: beam width at $z$
\item $w_0$: beam width at $z = 0$
\item $L_p^{|l|}$: Laguerre polynomial
\item $z_R$: Rayleigh range
\item $\lambda$: vacuum wavelength
\item $p$: momentum density
\item $E$: electric field
\item $H$: magnetic field
\item $\omega$: frequency of the light
\item $j$: angular momentum density
\item $G$: transfer function
\end{itemize}
1.1 Background and motivation

Light reflection and transmission at an air-medium interface has long been the topic of ongoing investigation and contention. From the perspective of classical electrodynamics, the phase, amplitude, and propagation direction of the reflected and transmitted plane waves are determined by the Fresnel relations and Snell’s law. However, for beam with finite width (such as Gaussian, vortex, and Bessel beams), transverse and longitudinal shifts occur upon reflection and transmission. Newton first predicted the existence of the longitudinal shift of beam [1]. It was not until three centuries later that Goos and Hännchen observed and measured the longitudinal shift in total internal reflection (TIR) [2]. Hence, the longitudinal shift is called Goos-Hächen (GH) shift. Subsequently, Fedorov and Schilling conjectured that circularly polarized light will shift perpendicular to the incident plane during TIR [3, 4]. In 1972, Imbert observed the transverse beam shift in TIR [5]. The transverse beam shift after reflection was named Imbert-Fedorov (IF) shift [6–12].

It is worth mentioning that the GH and IF shifts were initially observed in TIR. This is reasonable because the GH and IF shifts are larger in TIR than partial reflection. With advances in theory and measuring technique, scientists observed and detected the GH and IF shifts in partial reflection [14, 15]. The relations given by Artmann and Schilling for GH and IF shifts in TIR are widely acknowledged [4, 16]. However, scientists have different opinions for the IF shift in partial
reflection. This topic has been hotly debated by Onoda et al. [8, 9], Bliokh et al. [12] and Li et al. [17, 18].

The dispute about the IF shift has been resolved through the use of quantum weak measurement technique [15]. Hosten and Kwiat have measured the spin Hall effect of light (SHEL) at an air-glass interface and have offered a rigorous explanation for the split value of the spin. The experimental results have corroborated Bliokh and Bliokh’s theory [12] and invalidated Onoda et al.’s explanation [8]. Research by Hosten and Kwiat indicated that the polarization model used by Onoda et al. is more rigorous mathematically but contradictory to the physical reality [15]. Afterwards, Qin et al. have also measured the SHEL at an air-glass interface [19]. Although SHEL and IF have different physical pictures, they are, in essence, the end result of spin-orbital angular momentum conversion. Thus, we can safely conclude that the transverse shift of light beam in partial reflection has been successfully resolved.

With the development of the weak measurement technique, another category of shift, angular shift, has been predicted theoretically and measured experimentally [20–23]. The angular shift happens in the incident plane and perpendicular to the incident plane, and are called GH angular and IF angular shifts, respectively. GH and IF shifts are constant, while the angular GH and IF shifts increase linearly with the propagation distance. Hence, they stem from dissimilar physical mechanisms. GH and IF shifts come from spin-orbital angular momentum conversion. To satisfy the \( z \)-component angular momentum conservation law, reflected and transmitted beams must possess external transverse orbital angular momentum. The transverse orbital angular momentum constitutes the root of GH and IF shifts. The angular GH and IF shifts are essentially diffractive phenomena. They result from the transverse linear momenta and are governed by the transverse linear momentum conservation law.
Metamaterials have unique modulation capability over the electromagnetic radiation since they are composed of micro-structure units which are smaller than the wavelength [24–27]. Metamaterials which may have negative permittivity and permeability simultaneously, are called double negative metamaterials [26]. Double negative metamaterials have many unique properties, like negative refraction, inverse Doppler effect, inverse rotational Doppler effect, and inverse Cherenkov radiation [28–31]. Recent research indicated that beam shifts in metamaterials are distinctively different from normal materials. For instance, the GH shift at an air-metamaterials interface is reversed compared to air-normal material [32]. By employing multi-layer metamaterials, the reversed GH shift can be magnified significantly [33]. In addition, Luo et al. have demonstrated that the IF is not reversed in metamaterials which can be explained by the unreversed angular momentum in metamaterials [34].

The recent research literature in GH and IF shifts in metamaterials has stimulated our interest in investigating the IF shift of vortex beams in metamaterials. The vortex beam has a spiral wave-front which results in an orbital angular momentum along the propagation axis. Each photon has an angular momentum equal to $l\hbar$, where $l$ is the topological charge of the vortex beam and $\hbar$ is the Planck constant. The angular momentum is expected to complicate the IF shift and make it more interesting. First, the origin of the IF shift may be different. For instance, internal angular momentum can be converted to external angular momentum after reflection or transmission, which is orbital-orbital angular momentum conversion. Secondly, it remains an open question whether the IF shift is reversed after reflection. These issues are important in many aspects. In theory, it deepens our understanding of the linear and angular momentum of light beam in materials. In practice, the IF shifts are closely related to the properties of materials. We can inversely derive the structure parameter of the materials. Recent progress has substantiated our predictions. For example, SHEL has been used to count the number of graphene layers [35] and measure the magneto-optical constant of
Fe films [36]. The IF shift can also be modulated by changing the structure parameter of materials, such as the thickness of the sample and the index of the materials. Novel optical switches, optical modulators, and optical logic devices can be designed based on these principles. In addition, the IF shift is of the scale of the wavelength. Hence, it may find some applications in ultra-sensitive detection [46].

In this thesis, we endeavor to solve the IF shifts, also known as SHEL of the vortex beam in metamaterials, both theoretically and experimentally. We employ weak measurement technique to measure the SHEL of a vortex beam at an air-glass interface. Then, we present an experiment design to measure the SHEL of vortex beam at an air-metamaterial interface.

1.2 Recent progress

Considerable progress has been made since the discovery of GH and IF shifts. These findings fall into the following four main categories:

- Formulation of rigorous expressions for GH and IF shifts, and reasonable explanations.
- Extension to other beams, such as vortex beam, Bessel beam, nonparaxial beam.
- Investigation of GH and IF shifts in micro/nano structures.
- Development of new methods to measure the GH and IF shifts, and potential applications.

As discussed above, controversies have abounded concerning the formulae of GH and IF shifts, which have now been resolved. For instance, Artmann’s solution for the GH shift is as follows [16]. The incident beam is decomposed into a spectrum of plane waves. Therefore, the reflection coefficient and phase shift of each component have been calculated. After an inverse Fourier transform of the reflected plane waves, the expression for GH shift has been found. This method is called the “static phase” method. The expression for GH shift is proportional to the first order derivative of
the phase

\[ d = -\frac{1}{k} \left( |\alpha|^2 \frac{\partial \phi_{\nu p}}{\partial \theta_i} + |\beta|^2 \frac{\partial \phi_{\nu s}}{\partial \theta_i} \right), \]  

(1.1)

where \( \phi_{\nu p} \) and \( \phi_{\nu s} \) are the phases of \( p \) and \( s \) polarized reflected beams, \( \theta_i \) is the incidence angle, \( [\alpha, \beta]^T \) is the Jones matrix, and \( k \) is the propagation constant. Equation (1.1) is however invalid at the vicinity of critical angle. Hence, Renard has proposed a new explanation [37]. The mechanism of the GH shift has been interpreted from the viewpoint of energy flux. According to Renard’s theory, the energy penetration lies at the core of the GH shift during TIR. While Renard’s theory seems more reasonable, it has a major problem when the incident angle is much larger than the critical angle. To alleviate this problem, Lai et al. have started from Horowitz’s diffraction integral method [38]. They have successfully solved the GH shift by giving two expressions one around the critical angle and the other away from the critical angle. The theory proposed by Lai et al. is supported by subsequent experiments. Since Lai et al.’s expressions are too complex to use, researchers are more inclined to use a simplified version, Artmann’s formula.

There are many disputes concerning the IF shifts as well. They are resolved by Hosten and Kwiat’s experiment and theory [15]. IF and GH shifts stem from spin-orbit conversion and diffraction, respectively, which are now widely accepted by the scientific community. The longitudinal component of the optical field also play an important role in the IF shift.

The GH and IF shifts of some special beams have recently attracted considerable attention. Fedoseyev has predicted the existence of the IF shift of vortex beam [39]. His theoretical findings indicate that the IF shift of vortex beam is proportional to the topological charge \( l \). Subsequently, Dasgupta and Gupta have confirmed his theory by experiment [40]. In 2010, Merano has introduced the GH shift, angular GH shift, and angular IF shift of a vortex beam [41]. Okuda and Sasada have also investigated the IF shift and beam deformation in TIR [42, 43]. The GH and IF shifts of Bessel beam have also received much attention. Aiello et al. have explored the non-diffracting Bessel
beam [44]. They found that the angular GH and IF shifts are irrelevant to the topological charge of Bessel beam. This finding can be explained by the fact that angular shifts have their roots in diffraction. Nordblad has investigated the non-paraxial Bessel beam [45]. His findings indicate that GH and IF shifts are significantly affected by $E_z$ field near grazing incidence. However, GH and IF shifts of nonparaxial vortex beam still await further research.

Recent progress on GH and IF shifts also can be seen in many micro/nano structures, such as metallic thin films, photonic crystals, and metamaterials [46–49]. Merano et al. have investigated the GH shift at an air-metal interface [14]. This reveals a negative GH shift for $p$ polarized light. The absolute value of the GH shift of $p$ polarized light is larger than that of $s$ polarized light. Similar findings can be found in the infrared region. Luo et al. have investigated the IF shift in multi-layer structures, thin films, and metamaterials. More research can be found in spherical surfaces [50], cylindrical surfaces [51], structures with surface plasmon excitation [52], and topological insulators [53].

More recently, researchers are using methods other than weak measurement to measure the GH and IF shifts. Prajapati has proposed Stokes polarimetry measurements and interferometry to measure the GH and IF shifts [54, 55]. Wang et al. indirectly measured the IF shifts by finding the cross polarization ratio [56].

1.3 Outline of this thesis

We endeavor to explore the SHEL of the vortex beam in metamaterials in this thesis. The unreversed SHEL will hopefully confirm the unreversed angular momentum in metamaterials. This thesis has four Chapters. In this Chapter, we have introduced the GH and IF shifts and have elaborated on our motivation to investigate the IF shift of vortex beam in metamaterials. We have also reviewed recent progress in this area. In Chapter II, we provide a brief introduction to the vortex beam. Thereafter, we explore the spin Hall effect of vortex beam in metamaterials. Comparison
is made between an air-normal material interface and an air-metamaterial interface. In Chapter III, we present the principle of weak measurements and provide the experimental results. Chapter IV concludes this thesis.
CHAPTER II

ORBITAL HALL EFFECT: THEORY

2.1 Vortex beam and angular momentum

When Dirac explored the existence condition of the magnetic monopole in the early 1930s, he predicted a possible phase singularity in electromagnetic field [57]. But the experimental investigation of phase singularity was initially stimulated and initiated by acoustic waves. Nye and Berry collected and analyzed the scattering ultrasound from a coarse surface [58]. They found that when the signal phase changed by $2\pi$, the signal intensity was zero. Their research results provoked a series of investigations on optical singularity and vortices. Vaughan and Willets reported the first laser beam with phase singularity, using a high power Krypton-Argon laser [59]. When the output laser interfered with the mirror laser, a dark intensity hole formed in the interference, which is called an optical vortex. This phenomenon is explained by the interference of Hermite-Gaussian (HG)$_{01}$ and HG$_{10}$ modes. Except for direct output of vortex from multi-mode laser, Soskin also confirmed that optical vortex can be produced through diffraction. When a Gaussian beam is projected into a fork shaped diffraction grating, an optical vortex is generated [60]. At present, this is the most common method to generate vortex beams.

Research continued to expand on optical vortex. Allen et al. have first associated the orbital angular momentum with optical vortex in 1992 [61]. The analysis based on the expression of the Laguerre-Gaussian (LG) mode and the definition of linear and angular momenta, shows that the
orbital angular momentum of a single photon can be given as \((l + \sigma)\hbar\), where \(\sigma\) is an integer that describes the circular polarization state.

The analysis is presented in the following paragraphs. The expression of the vectorial electrical field is given by

\[
E(x, y, z) = \left[ \alpha \hat{x} + \beta \hat{y} + \frac{i}{nk} \left( \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} \right) \hat{z} \right] u(x, y, z) \exp[inkz] \tag{2.1}
\]

where \(n\) is the refractive index, \(k\) is the wave number in vacuum, and \([\alpha, \beta]^T\) is the Jones vector with \(|\alpha|^2 + |\beta|^2 = 1\). The polarization of the electric field is characterized by two parameters:

\(\sigma = 2\text{Im}[\alpha^*\beta]\), \(\chi = 2\text{Re}[\alpha^*\beta]\). \(\sigma = 0\) represents linearly polarized light, \(\sigma = +1\) represents left-handed circularly polarized light, and \(\sigma = -1\) represents right-handed circularly polarized light. \(\chi = 1\) means the long axis the polarization ellipse makes a 45 degree angle with the \(x\) axis, while \(\chi = -1\) means the long axis the polarization ellipse makes a 135 degree angle with the \(x\) axis. \(\chi = 0\) means the light is circularly polarized or \(x\) polarized or \(y\) polarized. Note that these parameters are valid for uniform polarization. For non-uniform polarization, we need other parameters to characterize the polarization.

Equation (2.1) is incomplete since we do not know the \(u(x, y, z)\). The envelope \(u(x, y, z)\) satisfies the paraxial wave equation:

\[
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 2ik\frac{\partial}{\partial z} \right] u(x, y, z) = 0. \tag{2.2}
\]

In a circular resonator, the solution of the paraxial wave equation is the Laguerre-Gaussian (LG) mode:

\[
u_{pl}(r, \varphi, z) = \frac{C_{pl}}{w} \left( \frac{\sqrt{2}r}{w} \right)^{|l|} L_p^{|l|} \left( \frac{2r^2}{w^2} \right) \exp \left[ -\frac{kr^2}{2(z_R + iz)} - i(2p + |l + 1|)\phi + il\varphi \right], \tag{2.3}
\]

where \(C_{pl} = \sqrt{2\pi^{-1}p!/(p + |l|)!}\) is a normalization constant which ensures that the integral of \(|u|^2\) in the \(x - y\) plane equals to 1. \(w = w_0\sqrt{1 + (z/z_R)^2}\) is the beam radius at \(z\), \(w_0\) is the beam waist,
Figure 2.1: Intensity and wavefront of vortex beam. (a) Beam profile of the Gaussian beam; (b) beam profile of the vortex beam with charge $l = 1$; (c) beam profile of the vortex beam with charge $l = -1$; (d) wavefront of the Gaussian beam; (e) wavefront of the vortex beam with charge $l = 1$; (f) wavefront of the vortex beam with charge $l = -1$.

\[ \varphi = \arctan \frac{y}{x} \] is the polar angle in a polar coordinate system, \( \phi = \arctan \frac{z}{z_R} \) is the Gouy phase which is a result of diffraction, and \( z_R = \frac{nkw_0^2}{2} \) is the Rayleigh range. \( L_p^{|l|} \) denotes the Laguerre polynomial.

We now attempt to illustrate the phase and intensity of the LG mode. In Fig. 2.1, we plot the phase and intensity of vortex beam and Gaussian beam. The beam waist \( w_0 = 20\lambda \), where \( \lambda \) is the wavelength in vacuum which ensures that the vortex beam is paraxial. The radial index \( p = 0 \). Figures 2.1(a) and 2.1(d) are the intensity and phase front of a Gaussian beam at \( z = z_R \), respectively. In the near field, the wave front is paraboloid. In the far field where \( z \gg z_R \), the shape of the phase front of Gaussian can be deemed as a spherical surface.

Figures 2.1(b) and 2.1(e) are the intensity and wavefront of vortex beam with \( l = 1 \). There is a dark hole in the intensity pattern, which increases in size as beam propagates. The wavefront is
Figure 2.2: Transverse energy flux of vortex beam. (a) Transverse energy flux and beam intensity pattern of vortex beam with $l = 1$; (b) transverse energy flux and beam intensity pattern of vortex beam with $l = -1$.

a left-handed spiral surface. At $r = 0$, the phase is uncertain, which is a singular point. A vortex beam with $l = -1$ has the same intensity pattern as $l = 1$, as shown in Fig. 2.1(c), but the wavefront is a right-handed spiral surface, which can be seen in Fig. 2.1(f).

We proceed to discuss the orbital angular momentum of the vortex beam. The linear momentum density of a monochromatic wave in a homogeneous and isotropic medium is: 

$$p = \frac{1}{2} \varepsilon \mu \text{Re}[E^* \times H],$$

where $\varepsilon$ is the permittivity, $\mu$ is the permeability, $E$ is the electric field, $*$ is the conjugate operator of a complex number, and $H$ is the magnetic field. We substitute $H$ with $\frac{1}{i \omega \mu} \nabla \times E$, where $\omega$ is the angular frequency, and get the following equation:

$$p = \frac{\varepsilon}{2 \omega} \text{Im} [E^* \times (\nabla \times E)].$$

(2.4)

We insert Eqs. (2.1) and (2.3) into the above equation, and get the expression for linear momentum density:

$$p = \frac{\varepsilon}{2 \omega} \left[ \frac{k z r}{z^2 + z_R^2} |u|^2 \hat{r} + \left( \frac{l}{r} |u|^2 - \frac{\sigma}{2} \frac{\partial |u|^2}{\partial r} \right) \hat{\varphi} + k |u|^2 \hat{z} \right].$$

(2.5)
The above equation indicates that the transverse linear momentum of a vortex beam is nonzero. A plot of the transverse energy flux $S$, which is proportional to $p$, is given in Fig. 2.2. When $l = 1$, the energy propagates anticlockwise. When $l = -1$, the energy propagates clockwise. According to the definition of the angular momentum $j = r \times p$, the angular momentum per unit length is given by [61]

$$J = \int \int j \, r \, dr \, d\varphi = \frac{\varepsilon}{2\omega} (l + \sigma) \hat{z}. \quad (2.6)$$

We find that the vortex beam has a longitudinal angular momentum.

The angular momentum per photon can be derived in the following way. We first find out the photon number per unit length in $z$ direction

$$N = \frac{W}{\hbar \omega} = \frac{\varepsilon}{2} \int \int |E|^2 r \, dr \, d\varphi = \frac{\varepsilon}{2\hbar \omega}. \quad (2.7)$$

Then the single photon angular momentum is given by

$$j = \frac{J}{N} = (l + \sigma) \hbar. \quad (2.8)$$

The angular momentum consists of orbital angular momentum $l \hbar$ and spin angular momentum $\sigma \hbar$.

In free space, orbital angular momentum does not interact with spin angular momentum.

### 2.2 Interference of vortex beam with plane wave

Inspection of the interference pattern between a vortex beam and a plane wave is a convenient way to verify the topological charge $l$. In general, there are two kinds of interferometer setups. Figures 2.3(a) and 2.3(b) demonstrate the Mach-Zehnder interferometer with $\theta = 0$ and $\theta \neq 0$, where $\theta$ is the angle between the vortex beam and plane wave. A laser beam is split into two in the first beam cube. The first beam reaches a spatial filter and collimation lens to generate a plane wave. The other beam is reflected by a mirror. It passes through a spiral phase plate. Its wavefront is modulated and a vortex beam is generated. The plane wave and the vortex beam are combined
Figure 2.3: Mach-Zehnder interferometer. (a) $\theta = 0$; (b) $\theta \neq 0$, where $\theta$ is the angle between the vortex beam and the plane wave.

using the second beam cube, which is polarized is the $x$ direction. The interacting waves of Figs. 2.3(a) and 2.3(b) can be expressed as

$$E_x = u_l(r, \phi, z) e^{ikz} + e^{ikz};$$

(2.9)

Setup (a):

$$E_x = u_l(r, \phi, z) e^{ikz} + e^{i[k(\sin \theta x + \cos \theta z)]}.$$  

(2.10)

The interference patterns in the far field are plotted in Fig. 2.4. Figures 2.4(a) and 2.4(b) correspond to interference of the setup in Fig. 2.3(a). Figures 2.4(c) and 2.4(d) correspond to interference of the setup in Fig. 2.3(b). We assume that $w_0 = 10\lambda$, $l = \pm 2$. The absolute value and the plus/minus sign of the topological charge can be easily inferred from the interference patterns.

### 2.3 Spin Hall effect of vortex beam

In this Section, we first give a brief introduction of vortex beam. Thereafter, we investigate the Hall effect of vortex beam in a normal material and a metamaterial.
Figure 2.4: Interference between a vortex beam and a plane wave. (a) Interference pattern of a vortex beam with $l = 2$ with a plane wave, $\theta = 0$; (b) Interference pattern of a vortex beam with $l = -2$ with a plane wave, $\theta = 0$; (c) Interference pattern of a vortex beam with $l = 2$ with a plane wave, $\theta \neq 0$; (d) Interference pattern of a vortex beam with $l = 2$ with a plane wave, $\theta = 0$.

Figure 2.5: Hall effect in bulk metal and glass.
2.3.1 Introduction to Hall effect of electrons and photons

The GH and IF shifts are distinctively different. The eigenmodes of the GH shift are $s$ and $p$ modes, while the eigenmodes of the IF shift are left-handed circularly polarized (LCP) light and right-handed circularly polarized (RCP) light. Now linearly polarized light can be visualized as being split into a LCP part and a RCP part after reflection and transmission at an air-medium interface. This effect is called spin Hall effect of light (SHEL). SHEL is very much like the Hall effect of charges in metal and semiconductors. For instance, in a bulk conductor, the magnetic field exerts different Lorentz forces on electrons and ions. Thus, electrons will accumulate on one side of the conductor, while ions accumulate on the other side of the conductor. Figure 2.5 demonstrates the Hall effect of electrons in metal and the SHEL of photons in glass. In Fig. 2.5(a), a bulk conductor is connected to the positive and negative sides of the battery. A uniform magnetic field $\mathbf{B}$ is applied in the $-x$ direction. According to the Lorentz force formula $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, electrons will accumulate at the top of the metal bulk, which means ions will accumulate in the bottom of the metal bulk. In Fig. 2.5(b), linearly polarized light with spin-up and spin-down photons impinge onto the air-glass surface. To meet the angular momentum conservation in $z$ direction, spin-up photons shift to the $-y$ direction, while spin-down photons shift to $y$ direction. The split direction along $y$ is perpendicular to the direction of the gradient of the refractive index $z$.

2.3.2 General solution of reflected and transmitted fields at air-material interface

To study the Hall effect of a vortex beam, we need to develop a general model to formulate the electric field. In this Section, we adopt the angular spectrum method to analyze the reflected and transmitted fields. The main idea is as follows:
• We decompose the vortex beam into a spectrum of angular plane waves. The plane wave component is the eigenstate of linear momentum $p_z$. So, this method is essentially carried out in momentum space.

• We analyze the reflection and transmission of each angular plane wave component. Each wave component has distinctive direction, and have thus different reflection and transmission coefficients. We expand the Fresnel coefficients to the first order by using Taylor series.

• We recompose the reflected and transmitted fields in spatial coordinates.

A detailed analysis can be seen in Ref. [62]. We just restate the results here as follows. The incident field $\tilde{E}_i$, reflected field $\tilde{E}_r$, and transmitted field $\tilde{E}_t$ can be express as

$$\tilde{E}_i = \left[ \alpha \hat{x}_i + \beta \hat{y}_i - \frac{1}{k} (\alpha k_{x_i} + \beta k_{y_i}) \hat{z}_i \right] \tilde{u}_i,$$  \hspace{1cm} (2.11)

$$\tilde{E}_r = \left[ \alpha \left( r_p - \frac{\partial r_p k_{x_r}}{\partial \theta_i} \right) + \beta \left( r_s + r_p \right) \cot \theta_i \frac{k_{y_r}}{k} \right] \tilde{u}_r \hat{x}_r + \left[ \beta \left( r_s - \frac{\partial r_s k_{x_r}}{\partial \theta_i} \right) - \alpha \left( r_s + r_p \right) \cot \theta_i \frac{k_{y_r}}{k} \right] \tilde{u}_r \hat{y}_r - \frac{1}{k} (\alpha r_p k_{x_r} + \beta r_s k_{y_r}) \tilde{u}_r \hat{z}_r,$$  \hspace{1cm} (2.12)

$$\tilde{E}_t = \left[ \alpha \left( t_p + \eta \frac{\partial t_p k_{x_t}}{\partial \theta_t} \right) + \beta \left( t_s - \eta t_s \right) \cot \theta_t \frac{k_{y_t}}{k} \right] \tilde{u}_t \hat{x}_t + \left[ \beta \left( t_s + \eta \frac{\partial t_s k_{x_t}}{\partial \theta_t} \right) + \alpha \left( \eta t_p - t_s \right) \cot \theta_t \frac{k_{y_t}}{k} \right] \tilde{u}_t \hat{y}_t - \frac{1}{n_k} (\alpha t_p k_{x_t} + \beta t_s k_{y_t}) \tilde{u}_t \hat{z}_t,$$  \hspace{1cm} (2.13)

$$\tilde{u}_i = \frac{C_0 w_0}{2} \left[ w_0 (-i k_{x_i} + \text{sgn}[l] k_{y_i}) \right]^{\frac{|l|}{2}} \exp \left[ \frac{-w_0^2 (k_{x_i}^2 + k_{y_i}^2)}{4} \right],$$  \hspace{1cm} (2.14)

where $[\alpha, \beta]^T$ is the Jones matrix, $[\hat{x}_{i,r,t}, \hat{y}_{i,r,t}, \hat{z}_{i,r,t}]^T$ are unit vector in incident, reflected, and transmitted coordinate systems, respectively. $\tilde{u}_{r,t} = \tilde{u}_i \left( \gamma_{r,t} k_{x_{r,t}}, k_{y_{r,t}} \right)$, $u_i$ is the envelope of the incident vortex beam, $\gamma_r = -1$, $\gamma_t = \eta = \cos \theta_t / \cos \theta_i$, where $\theta_t$ is the transmitted angle. $r_{p,s}, t_{p,s}$
are the reflection and transmission coefficients of \( p \) and \( s \) polarized light. \( \theta_i \) is the incidence angle. The transfer function of reflected and transmitted fields are:

\[
G_r = \exp \left[ -\frac{i \pi}{2k} (k_{x_r}^2 + k_{y_r}^2) \right] \quad \text{and} \quad G_t = \exp \left[ -\frac{i \pi}{2nk} (k_{x_t}^2 + k_{y_t}^2) \right].
\]

\( \hat{k}_i = (k_{x_i} \hat{x}_i + k_{y_i} \hat{y}_i + k_{z_i} \hat{z}_i) \), \( \hat{k}_r = (k_{x_r} \hat{x}_r + k_{y_r} \hat{y}_r + k_{z_r} \hat{z}_r) \), \( \hat{k}_t = (k_{x_t} \hat{x}_t + k_{y_t} \hat{y}_t + k_{z_t} \hat{z}_t) \) are the wave vectors of the incident, reflected, and transmitted plane waves. They satisfy the following boundary conditions:

\[
k_{x_r} = -k_{x_i}, \quad k_{y_r} = k_{y_i}, \quad k_{z_r} = k_{z_i},
\]
\[
k_{x_t} = k_{x_i}/\eta, \quad k_{y_t} = k_{y_i}, \quad k_{z_t} = nk_{z_i}.
\]

(2.15)

The above equation represents a very important set of relationships. \( k_{x_r} = -k_{x_i} \) and \( k_{y_r} = k_{y_i} \) indicate that the reflected vortex beam changes its vortex charge after reflection. Hence, the orbital angular momentum of the reflected beam is \(-l \hbar\). \( k_{z_t} = nk_{z_i} \) means the transmitted light should have negative phase velocity in metamaterials, as expected in Ref. [26]. The wavefront will change its rotation direction as a result. Note that we present the field in momentum space in Eqs. (2.12) and (2.13). Full expressions for fields in spatial coordinates can be derived from inverse Fourier transform, which can be seen in Ref. [62].

### 2.3.3 Hall effect of vortex beam in normal materials and metamaterials

The centroid of beams in spatial coordinates can be expressed as

\[
\langle x \rangle = \frac{\langle E | x E \rangle}{\langle E | E \rangle},
\]
\[
\langle y \rangle = \frac{\langle E | y E \rangle}{\langle E | E \rangle}.
\]

(2.16)

Since the reflected fields are presented in momentum space in the previous subsection, we need to convert the above equation to momentum space. This can be done using the relation

\[
\langle \mathbf{r} \rangle = \frac{\langle G E | i \partial_{k_z} | G E \rangle}{\langle G E | G E \rangle}.
\]

(2.17)
where \( r_\perp = x\hat{x} + y\hat{y}, \) \( \theta_{k_\perp} = \frac{\partial}{\partial k_x} \hat{x} + \frac{\partial}{\partial k_y} \hat{y}, \) and \( G = \exp \left[ -\frac{iz}{2nk} \left( k_x^2 + k_y^2 \right) \right] \) is the propagation operator. In Eq. (2.17), we have adopted the Dirac notation. For instance, for operator \( \hat{F}, \) \( \langle \tilde{E} | i\partial_{k_\perp} | \tilde{E} \rangle = \sum_{\xi=x,y,z} \int \tilde{E}_\xi \hat{F} \tilde{E}_\xi dk_x dk_y \). The above equation can be easily written to

\[
\langle r_\perp \rangle = \langle \tilde{E} | i\partial_{k_\perp} | \tilde{E} \rangle \langle \tilde{E} | \tilde{E} \rangle + z nk \langle \tilde{E} | k_\perp | \tilde{E} \rangle \langle \tilde{E} | \tilde{E} \rangle \quad (2.18)
\]

We can infer from the above equation that beam has two kinds of shifts: constant shift and angular shift. The first one is a constant, which does not change while the beam propagates. The second one is the angular shift, which increases with the propagation distance \( z \). It also indicates that the angular shift will be reversed in metamaterials since the \( n \) is negative. For the reflected and transmitted vortex beams, the GH shift, IF shift, angular GH shift, and angular IF shift are given through the relations

\[
\langle x_s \rangle = \Delta x_s + z_s \Delta \theta_{x_s}, \quad \langle y_s \rangle = \Delta y_s + z_s \Delta \theta_{y_s}, \quad \text{where} \quad s = r, t. \quad \text{This thesis primarily concerns the IF shift of reflected vortex beam} \Delta y_r. \]

As we have stated in previous Sections, SHEL implies that linearly polarized light is split after reflection at an air-medium interface. We now begin to analyze the SHEL of a vortex beam at an air-normal material interface and an air-metamaterial interface. When the incoming light is \( p \) polarized light (\( \alpha = 1, \beta = 0 \)), the reflected optical field is

\[
\tilde{E}_p^r = \left[ \left( r_p - \frac{\partial r_p k_{xx}}{\partial \theta_i} \right) \hat{x}_r - (r_s + r_p) \cot \theta_i \frac{k_{yx}}{k} \hat{y}_r - r_p \frac{k_{xz}}{k} \hat{z}_r \right] \tilde{u}_r. \quad (2.19)
\]

We write the above equation into a combination of RCP and LCP component \( \tilde{E}_p^r = \tilde{E}_{r+}^p + \tilde{E}_{r-}^p. \)

Hence, the LCP and RCP components can be written as

\[
\tilde{E}_{r\pm}^p = \frac{1}{2} \left[ r_p - \frac{\partial r_p k_{xx}}{\partial \theta_i} \pm i(r_p + r_s) \cot \theta_i \frac{k_{yx}}{k} \right] \left[ \hat{x}_r \pm i\hat{y}_r - \frac{k_{xz}}{k} \pm i\frac{k_{yx}}{k} \hat{z}_r \right] \tilde{u}_r. \quad (2.20)
\]

We substitute the above equation into Eq. (2.18), and get the expression of the spin split of vortex beam

\[
\Delta y_{r\pm}^p = \left[ -\cot \frac{\theta_i}{k} \left( 1 + \frac{r_s}{r_p} \right) - \frac{1}{k} \frac{\partial \ln r_p}{\partial \theta_i} \right] \frac{1}{1 + \left( |l| + 1 \right)^2} \left[ 1 + \left( 1 + \frac{r_s}{r_p} \right)^2 \cot^2 \theta_i + \left( \frac{\partial \ln r_p}{\partial \theta_i} \right)^2 \right], \quad (2.21)
\]
Figure 2.6: Hall effect of $p$ polarized vortex beam. (a) Beam shift of the RCP and LCP components of a vortex beam with $l = 1$ in normal material; (b) beam shift of the RCP and LCP components of a vortex beam with $l = 1$ in metamaterials; (c) beam shift of the RCP and LCP components of a vortex beam with $l = -1$ in normal material; (d) beam shift of the RCP and LCP components of a vortex beam with $l = -1$ in metamaterials.
Figure 2.7: Dispersion and SHEL of vortex beam in metamaterials. (a) Dispersion of the metamaterials; (b) SHEL of the vortex beam with $l = 1$ in metamaterials; (c) SHEL of the vortex beam with $l = -1$ in metamaterials.

where $\theta_0$ is the divergence angle of the incident beam. When the incidence angle is far away from the Brewster angle, the above equation can be simplified as

$$\Delta y_{p}^{\pm} = \pm \frac{\cot \theta_i}{k} \left( 1 + \frac{r_s}{r_p} \right) - \frac{l}{k} \frac{\partial \ln r_p}{\partial \theta_i}$$

(2.22)

The relation between the beam shift and the incidence angle is plotted in Fig. 2.6. We choose $|n| = 1.515$, $\lambda = 405$ nm. We can see from Fig. 2.6 that the SHEL of vortex beam in normal materials and metamaterials are identical. Since the vortex beam has a constant vortex-dependent shift, spin-up and spin-down components have different shift, which can be seen in Fig. 2.6(a).

If there’s no dispersion, the SHEL will be proportional to the wavelength [see Eq. (2.22)]. But for materials with high refractive index, the dispersion is usually very large, which can not be neglected. If the metamaterials has the same absolute value of silicon, we can discuss the effect of dispersion on SHEL. We assume that the absolute value of the refractive index of metamaterials is as shown in Fig. 2.7(a). The SHEL of a vortex beam with topological charge $l = \pm 1$ is shown in Figs. 2.7(b) and (c), respectively. The incidence angle is $\theta_i = 45^\circ$. We can see from Figs. 2.7(b) and (c) that the SHEL of vortex beam is no longer proportional to the wavelength in high dispersion.
materials. If the incident angle is in the vicinity of Brewster angle, the value of SHEL even decreases as wavelength increases. In both figures, the SHEL of vortex beam is no longer symmetric for spin-up and spin-down components. The effect of anomalous dispersion and high loss in SHEL still await further investigation.

Total internal reflection is also a very interesting case in SHEL. The expression of SHEL in TIR is

$$\Delta y_{r\pm}^p = \mp \frac{\cot \theta_i}{k} \left[ 1 + \cos(\phi_{rp} - \phi_{rs}) \right],$$

(2.23)

where $\phi_{rp}$ and $\phi_{rs}$ are the phases of the Fresnel coefficients for $p$ and $s$ polarized beam. A very interesting point is that the SHEL in vortex beam has no vortex-dependent term.

The analysis of SHEL of $s$-polarized vortex beam is very much the same as the $p$-polarized light. Here, we just simply offer the results:

$$\Delta y_{r\pm}^s = \mp \frac{\cot \theta_i}{k} \left( 1 + \frac{r_p}{r_s} \right) - \frac{l}{k} \frac{\partial \ln r_s}{\partial \theta_i}.$$  

(2.24)

2.4 Conclusion

In this Chapter, we have given a brief introduction to vortex beams. The intensity pattern, wavefront, and the angular momentum have been explained theoretically and demonstrated graphically. Two distinctive interference scenarios—parallel beams and tilted beams have been discussed. We have offered general expressions of reflected and transmitted fields of a vortex beam at air-material interface, including normal and double negative metamaterials. The influence of incidence angle and dispersion on the Hall effect have been analyzed.
CHAPTER III

ORBITAL HALL EFFECT: EXPERIMENT

In this Chapter, we will employ the weak measurement technique to measure and detect the
SHEL of vortex beam at an air-glass interface, and show an experiment design for the air-metamaterial
interface.

3.1 Principle of weak measurement

The weak measurement technique is a widely used method to magnify the SHEL in various
microstructures. The setup is sketched in Fig. 3.1. The light source is a laser light source. The
wavelength depends on the working wavelength of the sample, which can be 405 nm, 543 nm, or
633 nm. A half wave plate (HWP) is used to adjust the polarization axis of the output beam from the
light source, and hence is also capable of adjusting the normalized intensity of the light from 0 to 1.
The HWP can be replaced with an intensity filter, but the modulation capability is limited. Polarizer
P1 is a Glan laser polarizer with high extinction ratio, which can reach to one million. Lens L1 with
focal length $f_1$ is used to focus the light onto the surface of the sample. The sample is placed at the
beam waist of the beam after Lens L1. Lens L2 with focal length $f_2$ is a collimation lens, which
ensures that the beam centroid is not sensitive to the propagation distance between the sample and
the detector. Polarizer P2 is the second polarizer, which lies at the core of this experiment. Its fast
axis is almost orthogonal to that of P1.
Figure 3.1: Sketch of weak measurement setup. Light source is a He-Ne laser, HWP means half wave plate, the focal length of lens L1 is 300 mm, P1 is a Glan laser polarizer, the prism is a glass prism with $n = 1.515$ at 633 nm wavelength, P2 is a Glan laser polarizer, the focal length of lens L2 is 200 mm, CCD means charge-coupled device.
The principle can be explained from the perspective of quantum theory as well as classical electrodynamics. Here, we explain this experiment from the viewpoint of classical electromagnetics.

We take $p$ polarized light as an example. A linearly polarized light is generated from a coherent light source. After passing through a phase plate, diffraction grating, or spatial light modulator (which is not shown in Fig. 3.1), a vortex beam with topological charge $l$ is generated before the HWP. Behind P1, a vortex beam with $p$ polarization is generated. The expression of the optical field after P1 is

$$
\tilde{E}_i = \left[ \hat{x}_i - \frac{k x_i}{k} \hat{z}_i \right] \frac{C_i w_1}{2} \left[ \frac{w_1 (-i k x_i + \text{sgn}[l] k y_i)}{\sqrt{2}} \right]^{[l]} \exp \left[ - \frac{w_1^2 (k x_i^2 + k y_i^2)}{4} \right]. \tag{3.1}
$$

We assume that the P1 is close enough to L1 and that these optical elements are located far beyond the Rayleigh range of the source. The width $w_1$ should be associated with the initial divergence angle $\vartheta_1$ from the light source, and can be expressed as $w_1 = \frac{\lambda}{\pi \vartheta_1}$. After lens L1, the beam divergence angle and the beam radius will change: $\vartheta_2 = \lambda/\pi f_1 \vartheta_1$, $w_2 = f_1 \vartheta_1$. After the reflection from the sample, the electric field is

$$
\tilde{E}_r = \left[ \left( r_p - \frac{\partial r_p}{\partial \theta_i} \frac{k x_i}{k} \right) \hat{x}_r - (r_p + r_s) \cot \theta_i \frac{k y_r}{k} \hat{y}_r - \frac{r_p k x_r}{k} \hat{z}_r \right] \tilde{u}_r, \tag{3.2}
$$

where $\tilde{u}_r = \frac{C_i w_2}{2} \left[ \frac{w_2 (-i k x_r - \text{sgn}[ll] k y_r)}{\sqrt{2}} \right]^{[ll]} \exp \left[ - \frac{w_2^2 (k x_r^2 + k y_r^2)}{4} \right]$. We notice that after reflection, the vortex beam reverses its topological charge.

After reflection, the beam hits onto a second polarizer P2. The angle between P1 and P2 is $90 - \Delta$, where $\Delta$ is a small angle ranging from $-2^\circ$ to $2^\circ$. Since P2 is almost orthogonal to P1, the light intensity after P2 is relatively weak, almost $10^{-6}$ compared to the incident beam. The transfer function of the polarizer P2 is

$$
M_{P2} = \begin{bmatrix}
\sin^2 \Delta & \sin \Delta \cos \Delta \\
\sin \Delta \cos \Delta & \cos^2 \Delta
\end{bmatrix} \tag{3.3}
$$
Figure 3.2: Step-by-step sketch of beam propagation in weak measurement.

The expression for the reflected optical field after the second polarizer P2 is

$$\tilde{E}'_r = r_p \sin^2 \Delta \left[ 1 - \frac{\partial \ln r_p k_{x_r}}{\partial \theta_i} \right] - \left( 1 + \frac{r_s}{r_p} \right) \cot \theta_i \cot \frac{k_{y_r}}{k} \tilde{u}_r \hat{x}_r$$

$$+ r_p \sin \Delta \cos \Delta \left[ 1 - \frac{\partial \ln r_p k_{x_r}}{\partial \theta_i} \right] - \left( 1 + \frac{r_s}{r_p} \right) \cot \theta_i \cot \frac{k_{y_r}}{k} \tilde{u}_r \hat{y}_r$$

$$- r_p \sin^2 \Delta \left( \frac{k_{x_r}}{k} + \cot \frac{k_{y_r}}{k} \right) \tilde{u}_r \hat{z}_r.$$  \hspace{1cm} (3.4)

Then, light beam reaches to second lens L2, which is a long focal length lens. We use the definition of the beam centroid in the previous Chapter and find out the beam centroid after the second lens (see Appendix for details):

$$\Delta Y^p = \frac{A}{B}$$

$$A = -\frac{l}{k} |r_p| \left( \frac{\partial |r_p|}{\partial \theta_i} \right) \sin \Delta + \frac{|r_p||r_s|}{k} \sin(\varphi_s - \varphi_p) \cot \theta_i \sin \Delta \cos \Delta$$

$$- \frac{(|l| + 1)z}{k z R_2} \left[ |r_p|^2 + |r_p||r_s| \cos(\varphi_s - \varphi_p) \right] \cot \theta_i \sin \Delta \cos \Delta$$

$$B = \sin^2 \Delta |r_p|^2 + \frac{|l| + 1}{k^2 w^2} \left[ \sin^2 \Delta \left( \frac{\partial |r_p|}{\partial \theta_i} \right)^2 + \sin^2 \Delta \left( \frac{\partial \varphi_p}{\partial \theta_i} \right)^2 + \sin^2 \Delta |r_p|^2 \right. + \left. \cos^2 \Delta \cot^2 \theta_i \left( |r_p|^2 + |r_s|^2 + 2|r_p||r_s| \cos(\varphi_s - \varphi_p) \right) \right].$$  \hspace{1cm} (3.5)
The above equation takes the complex reflection coefficients into account. So, it can be used in lossy media, thin film etc. If the reflection coefficients are real numbers, then Eq. (A. 7) can be simplified to

\[
\Delta Y_p = \frac{-l}{k} r_p \frac{\partial r_p}{\partial \theta} \sin^2 \Delta \Delta - \frac{(|l|+1)f_2}{k z R_2} \left[ r_p^2 + r_p r_s \right] \cot \theta_i \sin \Delta \cos \Delta \sin^2 \Delta r_p^2 + \frac{(|l|+1)}{2 k z R_2} \cos^2 \Delta \cot^2 \theta_i (r_p + r_s)^2.
\] (3.6)

Depending on parameters $\theta_1$, $f_2$, and $f_1$, the magnification factor ranges from 1000-10000. So, the nano-meter scale SHEL can be measured by a conventional CCD with 1$\mu$m resolution.

As for s polarized incoming light, the analysis is similar. So, we simply state the result here:

\[
\Delta Y_s = \frac{-l}{k} r_s \frac{\partial r_s}{\partial \theta} \sin^2 \Delta \Delta - \frac{(|l|+1)f_2}{k z R_2} \left[ r_s^2 + r_p r_s \right] \cot \theta_i \sin \Delta \cos \Delta \sin^2 \Delta r_s^2 + \frac{(|l|+1)}{2 k z R_2} \cos^2 \Delta \cot^2 \theta_i (r_p + r_s)^2.
\] (3.7)

### 3.2 SHEL at air-glass interface: experiment

In this Section, we will experimentally demonstrate the SHEL of light beam at air-glass interface.

#### 3.2.1 Partial reflection

The experimental setup is the same as previous Section. The incident angle is set to 45°. The focal lengths are: $f_1 = 300$ mm, $f_2 = 200$ mm. The light source is a 633 nm He-Ne laser. We have measured the beam width of the incident beam at several different locations and employed a fitting analysis to find out the divergence angle of the output beam from the laser, see Fig. 3.3. The divergence angle of the incident beam is is $\theta_1 = 1.2mrad$, and the divergence angle of the beam after reflection is $\theta_2 = 6.297mrad$. We have then measured the magnified centroid of the beam.

The beam profile of s polarized light is plotted in Fig. 3.4. Figures. 3.4(a)-(e) are the experimental results at $\Delta = 0.4^\circ$, $\Delta = 0.2^\circ$, $\Delta = 0^\circ$, $\Delta = -0.2^\circ$, $\Delta = -0.4^\circ$, respectively. Figures. 3.4(f)-(j) are the corresponding theoretical results. The measured results of both p and s polarized incident beams are presented in Fig. 3.5. Both the beam profile and the measured shifts are in good agreement.
with the theoretical predictions. The results for $s$ polarized light is better than the $p$ polarized light. Possible sources of errors might be the fitting error of the incoming beam, the vibration of the setup, the resolution of the rotation stage, etc. As we can see in Fig. 3.5(a), there are two extremal points. At these points, the magnified shifts are maximal, while the first order derivatives are zero. Hence, the systematic error is minimal, which can be seen from Eq. 3.8:

$$\sigma(\Delta Y_r) = \sqrt{(\sigma f_2)^2 \left( \frac{\partial \Delta Y_r}{\partial f_2} \right)^2 + (\sigma z_{R_2})^2 \left( \frac{\partial \Delta Y_r}{\partial z_{R_2}} \right)^2 + (\sigma \theta_i)^2 \left( \frac{\partial \Delta Y_r}{\partial \theta_i} \right)^2 + (\sigma \Delta)^2 \left( \frac{\partial \Delta Y_r}{\partial \Delta} \right)^2}.$$ (3.8)

We may call these points optimal points where magnified shift is maximal while system error is minimal. The results are in agreement with the findings in Ref. [64], which appeared during the time our experiments were being performed. Zhou et al. also investigated the relation between the optimal $\Delta$ and the incident angle.
Figure 3.4: Experimental and theoretical beam profiles of $s$ polarized light after L2. (a) Experimental beam profile at $\Delta = 0.4^\circ$; (b) experimental beam profile at $\Delta = 0.2^\circ$; (c) experimental beam profile at $\Delta = 0^\circ$; (d) experimental beam profile at $\Delta = -0.2^\circ$; (e) experimental beam profile at $\Delta = -0.4^\circ$; (f) theoretical beam profile at $\Delta = 0.4^\circ$; (g) theoretical beam profile at $\Delta = 0.2^\circ$; (h) theoretical beam profile at $\Delta = 0^\circ$; (i) theoretical beam profile at $\Delta = -0.2^\circ$; (j) theoretical beam profile at $\Delta = -0.4^\circ$.

Figure 3.5: Experimental and theoretical magnified beam shifts in partial reflection. (a) Magnified beam shifts of $p$ polarized incident beam at air-glass interface; (b) magnified beam shifts of $s$ polarized incident beam at air-glass interface. The incident angle is $45^\circ$. 

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Figure 3.6: Experimental and theoretical magnified beam shifts in TIR. (a) Magnified beam shifts of $p$ polarized incident beam in TIR; (b) magnified beam shifts of $s$ polarized incident beam in TIR. The incident angle is 45°.

3.2.2 Total internal reflection

Now, we continue to investigate the SHEL at air-glass interface in TIR. In TIR, $|r_p| = |r_s| = 1$. So, the magnified shifts are simplified to

$$\Delta Y^p_r = \frac{\frac{1}{k} \sin(\varphi_s - \varphi_p) \cot \theta_i \sin \Delta \cos \Delta - \frac{(|l|+1)f_2}{k R_2} [1 + \cos(\varphi_s - \varphi_p)] \cot \theta_i \sin \Delta \cos \Delta}{\sin^2 \Delta + \frac{(|l|+1)}{k R_2} \cos^2 \Delta \cot^2 \theta_i [1 + \cos(\varphi_s - \varphi_p)]},$$

$$\Delta Y^s_r = \frac{\frac{1}{k} \sin(\varphi_s - \varphi_p) \cot \theta_i \sin \Delta \cos \Delta + \frac{(|l|+1)f_2}{k R_2} [1 + \cos(\varphi_s - \varphi_p)] \cot \theta_i \sin \Delta \cos \Delta}{\sin^2 \Delta + \frac{(|l|+1)}{k R_2} \cos^2 \Delta \cot^2 \theta_i [1 + \cos(\varphi_s - \varphi_p)]}. \quad (3.9)$$

The first term of the numerator is relatively small. So, $\Delta Y^p_r \approx -\Delta Y^s_r$. The measured results are presented in Fig. 3.6. The theoretical and experimental data have the same trend. But the measured result is about one sixth of the theoretical one. The measurement in TIR is inherently complicated. There are other two SHELs in the incoming surface and emerging surface, which are not considered in this experiment. In addition, there might be multiple reflections within the prism, which may also cause systematic errors.
3.3 SHEL at air-metamaterial interface: design of experiment

In this Section, we describe our progress with the measurement of SHEL of THE vortex beam at an air-metamaterial interface. Owing to the limited time and technical problems, we have not been able to measure the SHEL of vortex beam in metamaterials at the present time. The attempts and efforts are presented in the following paragraphs. We demonstrate our experimental setup in Fig. 3.7, which consists of three parts:

- Part A: Beam circularization
- Part B: Vortex generation and test
- Part C: Weak measurement
Figure 3.8: Two orthogonal cylindrical lenses beam circularization setup. (a) Top view; (b) Front view.
Part A is essentially a beam circularization module. The output beam profile from the diode laser with 405 nm is elliptical. So, we attempt to transform the elliptical profile to a circular one. We have therefore employed a spatial filter and a collimation lens. The micro-pinhole is placed in front of the back focal plane of the microscope lens. This is a very important trick which ensures that we can get Airy disk. If we place the micro-pinhole at the back focal plane. Then the beam profile would still be elliptical. The Airy disk is collimated by a collimation lens. High order lobes of the Airy disk are blocked by another pinhole. The central lobe of the Airy disk is a zero order Bessel function, which can be seen as a Gaussian beam. The beam profile looks perfect [see Fig. 3.9]. However, there is a major drawback of this technique: the beam power is as low as 200 nW. Since the two polarizers are almost orthogonal to each other in quantum weak measurement, the incident beam should have at least 5 mW optical power. Hence, the small beam power 200 nW does not permit us to carry out the weak measurement. We can also replace Part A with another beam circularization setup. For instance, we may position two orthogonal cylindrical lenses with appropriate distances (see Fig. 3.8).

Although the beam intensity is too weak to do the weak measurement, we continue to generate vortex beam. After the beam is reflected on the surface of the first mirror, it is split into two parts. The first part impinges onto another mirror and then passes through a spatial filter. A plane wave is generated after this spatial filter. The other part of the beam impinges onto a mirror and then passes through a spiral phase plate. A vortex beam is generated after this phase plate. Part B is essentially a Mach-Zehnder interferometer. It can be used to test the vortex charge. The profile of the generated vortex beam and the interference patterns between plane wave and vortex beam are presented in Fig. 3.10. Figures. 3.10(a) and (d) are the experimental and theoretical beam profiles of vortex beam with vortex charge \( l = 1 \). Figures. 3.10(b) and (e) are the interference patterns of
Figure 3.9: Beam profile after beam circularization setup.

Figure 3.10: Vortex beam generation and test. (a) Experimental beam profile of vortex beam with $l = 1$; (b) experimental interference pattern of a vortex beam with $l = 1$ with a tilted plane wave; (c) experimental interference pattern of a vortex beam with $l = -1$ with a tilted plane wave; (d) theoretical beam profile of vortex beam with $l = 1$; (e) theoretical interference pattern of a vortex beam with $l = 1$ with a tilted plane wave; (f) theoretical interference pattern of a vortex beam with $l = -1$ with a tilted plane wave.
vortex beam with vortex charge \( l = 1 \), respectively. Figures 3.10(c) and (f) are the interference patterns of vortex beam with vortex charge \( l = -1 \), respectively.

In Part C, we attempt to use the weak measurement technique to detect and measure the SHEL of vortex beam at an air-metamaterial interface. The metamaterial sample is composed of co-sputtered silver and silicon-carbide mixture on a glass substrate [63]. In visible region, most plasmonic materials exhibit negative permittivity. The negative effective permeability of silicon-carbide nanoparticles is due to the enhanced magnetic activity near TE resonance. The magnetic flux density is opposite to the direction of the magnetic field of the electromagnetic wave [63]. The working wavelength of the metamaterial sample is 405 nm. That’s why we use a diode laser source.

3.4 Conclusion

In this Chapter, we have explained the principle weak measurement from the perspective of classical electrodynamics. A step-by-step analysis indicated that the beam shifts could be significantly magnified by 1000-10000 times. Hence, a conventional CCD with \( \mu \text{m} \) resolution is good enough to measure the magnified shifts. We have managed to measure the SHEL at air-glass interface in partial reflection and total internal reflection. The experimental results of partial reflection were in good agreement with theoretical predictions and Ref. [64]. But the results of total internal reflection were unsatisfactory. Possible sources of errors might be the multiple reflections within the prism or the neglected SEHL at the incoming surface and emerging surface.
CHAPTER IV

CONCLUSION

In conclusion, we have investigated the spin Hall effect of vortex beam at air-metamaterial interface both theoretically and experimentally. Starting from the angular spectrum method, we have obtained the expressions of reflected and transmitted fields. Based on these formulae, the spin Hall effect of vortex beam at air-metamaterial interface has been presented theoretically. Our research indicated that the spin Hall effect of vortex beam at air-metamaterial interface is the same as the air-normal material interface, assuming these two materials have the same refractive index and loss. The influence of dispersion has also been analyzed in the Drude model. In highly dispersive media, the shift is no longer proportional to the wavelength. By increasing or decreasing the vortex charge, the Hall effect can be modulated.

Experimental efforts have been made to measure the spin Hall effect of vortex beam in metamaterials. We have demonstrated the relation between the magnified shifts and the polarization angle. Similar results have been published while this research has been in progress. We have designed the experiment to measure the SHEL of vortex beam in metamaterials using a 405 nm laser source which is required for the metamaterial; however, it has not been successful thus far due to the beam quality of the diode laser. This work will be pursued in the near future.
BIBLIOGRAPHY


In this Appendix, we will show how to derive the magnified beam shifts. The electric field after the second polarizer $P_2$ [see Fig. 3.2] is

$$\tilde{E}'_{xr} = r_p \sin^2 \Delta \left[ 1 - \frac{\partial \ln r_p k_{xr}}{\partial \theta_i} - \left( 1 + \frac{r_s}{r_p} \right) \cot \theta_i \cot \frac{k_{yr}}{k} \right] \tilde{u}_r,$$

$$\tilde{E}'_{yr} = r_p \sin \Delta \cos \Delta \left[ 1 - \frac{\partial \ln r_p k_{xr}}{\partial \theta_i} - \left( 1 + \frac{r_s}{r_p} \right) \cot \theta_i \cot \frac{k_{yr}}{k} \right] \tilde{u}_r,$$

$$\tilde{E}'_{zr} = -r_p \sin^2 \Delta \left( \frac{k_{xr}}{k} + \cot \frac{k_{yr}}{k} \right) \tilde{u}_r,$$

(A. 1)

where $\tilde{u}_r = \frac{C_l w_2}{2} \left[ \frac{w_2 (-ik_{xr} - \text{sgn}(l)k_{yr})}{\sqrt{2}} \right]_l \exp \left[ -\frac{w_2^2 (k_{xr}^2 + k_{yr}^2)}{4} \right]$.

To simplify the equation, we replace $\frac{\partial \ln r_p}{\partial \theta_i}$ with $a$, and $\left( 1 + \frac{r_s}{r_p} \right) \cot \theta_i \cot \Delta$ with $b$. Then the above equation can be rewritten as

$$\tilde{E}'_{xr} = r_p \sin^2 \Delta \left( 1 - a \frac{k_{xr}}{k} - b \frac{k_{yr}}{k} \right) \tilde{u}_r,$$

$$\tilde{E}'_{yr} = r_p \sin \Delta \cos \Delta \left( 1 - a \frac{k_{xr}}{k} - b \frac{k_{yr}}{k} \right) \tilde{u}_r,$$

$$\tilde{E}'_{zr} = -r_p \sin^2 \Delta \left( \frac{k_{xr}}{k} + \cot \frac{k_{yr}}{k} \right) \tilde{u}_r.$$

(A. 2)

We first find out the energy of the optical field

$$\langle \tilde{E}'_{r} | \tilde{E}'_{r} \rangle = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left( |\tilde{E}'_{xr}|^2 + |\tilde{E}'_{yr}|^2 + |\tilde{E}'_{zr}|^2 \right) dk_{xr} dk_{yr},$$

$$= \left[ 1 + \frac{(|l| + 1)}{k^2 w_2^2} (|a|^2 + |b|^2 + 1) \right] |r_p|^2 \sin^2 \Delta,$$

(A. 3)

where we have used the following formula

$$\int_{0}^{\infty} r^{|l|-1} \exp(-r)dr = (|l| - 1)!.$$  

(A. 4)
We continue to calculate another term $\langle \tilde{E}_r' | i \partial_{k_y} | \tilde{E}_r' \rangle$

$$
\langle \tilde{E}_r' | i \partial_{k_y} | \tilde{E}_r' \rangle = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ \tilde{E}^*_{rx} \left( i \frac{\partial \tilde{E}'_y}{\partial k_{y_l}} \right) + \tilde{E}^*_{ry} \left( i \frac{\partial \tilde{E}'_y}{\partial k_{y_l}} \right) + \tilde{E}^*_{rz} \left( i \frac{\partial \tilde{E}'_z}{\partial k_{y_l}} \right) \right] dk_x,dk_y,
$$

where

$$
\begin{align*}
\text{Re}[a] &= \frac{\partial \ln |r_p|}{\partial \theta_i}, \quad \text{Im}[a] = \frac{\partial \varphi_p}{\partial \theta_i}; \\
\text{Re}[b] &= \left[ 1 + \frac{|r_s|}{|r_p|} \cos(\varphi_s - \varphi_p) \right] \cot \theta_i \cot \Delta, \quad \text{Im}[b] = \frac{|r_s|}{|r_p|} \sin(\varphi_s - \varphi_p) \cot \theta_i \cot \Delta.
\end{align*}
$$

(A. 5)

Therefore, the magnified beam shift is

$$
\Delta Y_p = \frac{\langle \tilde{E}_r' | i \partial_{k_y} | \tilde{E}_r' \rangle}{\langle \tilde{E}_r' | \tilde{E}_r' \rangle} = \frac{A}{B}
$$

$$
A = - \frac{l}{k} |r_p| \frac{\partial |r_p|}{\partial \theta_i} \sin^2 \Delta + \frac{|r_p||r_s|}{k} \sin(\varphi_s - \varphi_p) \cot \theta_i \sin \Delta \cos \Delta
$$

$$
- \frac{(|l| + 1)z}{kz_R} \left[ |r_p|^2 + |r_p||r_s| \cos(\varphi_s - \varphi_p) \right] \cot \theta_i \sin \Delta \cos \Delta
$$

$$
B = \sin^2 \Delta |r_p|^2 + \frac{|l| + 1}{k^2 w_z^2} \left[ \sin^2 \Delta \left( \frac{\partial |r_p|}{\partial \theta_i} \right)^2 + \sin^2 \Delta \left( \frac{\partial \varphi_p}{\partial \theta_i} \right)^2 + \sin^2 \Delta |r_p|^2 
+ \cos^2 \Delta \cot^2 \theta_i \left( |r_p|^2 + |r_s|^2 + 2|r_p||r_s| \cos(\varphi_s - \varphi_p) \right) \right].
$$

(A. 7)