THE “45 DEGREE RULE” AND ITS IMPACT ON STRENGTH AND STIFFNESS
OF A SHAFT SUBJECTED TO A TORSIONAL LOAD

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ABSTRACT

THE “45 DEGREE RULE” AND ITS IMPACT ON STRENGTH AND STIFFNESS
OF A SHAFT SUBJECT TO A TORSIONAL LOAD

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Many industrial machines incorporate a multitude of moving and rotating parts necessary for the machinery to perform its intended functions. Rotating machinery, like turbines and compressors, include multiple parts that rotate under heavy loads and high speeds. Shafts are a common medium to transmit these loads and speeds. Quite often, these shafts are required to be stepped to create multiple distinct diameters for carry and located other components. The addition of these steps must be design with care such that a proper radius is selected between two diameters. Parts operating in this field will run for long periods of time and must maintain under multiple start/stop cases and can eventually cause failures.

Rotor and torsional dynamic analyses are completed on most if not all rotors in the turbomachinery field. The 45 degree rule is a method of simplification for modeling
abrupt changes in diameter. This rule of thumb states a line from the lesser of two steps on a shaft can be drawn at a 45 degree angle to the outside diameter of the greater step. The material outside this line can be modeled with zero modulus and actual density. This region of material does not significantly impact the torsional stiffness of the area. The purpose of this research is to find the effect this modeling approach has on the computed strength, stiffness, and overall rotordynamic properties of a rotating shaft. This will also demonstrate the “Best case” shoulder combination with various fillet radii and/or other angle orientation as well as illustrate additional areas this theory may be applicable. Additional considerations will be made for defective or non-homogenous material (e.g., inclusions, cracks, and scratches) that may be contained within the region under consideration and their effect on the overall region’s computed strength and stiffness.

The purpose of this research is split into three claims. The first claim states that after the removal of the material outside the 45° line, neither the strength nor stiffness will be significantly impacted analyzed at various fillet sizes. This claim was proven to be true. After comparing the stress concentration factors for both 90° and 45° geometries, there is no significant (worsening) impact to the stress concentration factor. The stress concentration factor for the 45° design on average reduces the stress concentration factor. The second claim is to prove the assumption of the 45° rule that the stiffness grows or shrinks along the 45° line. This is proven to be false due to the stiffness significantly increasing as the taper of the should progresses from 0° (90°) step to 60°.
The application of the rule will deem whether this difference is significant. The third claim states that if the first claim is proven true, the impact of non-conformities, holes and cuts, inserted into the 45° region will not significantly impact the stress concentration factor. This is proven true, it was seen there is not a significant impact to the stress concentration factor when adding holes or cuts into the 45° region. It can be noted that if the 45° boundary is passed, there will be an increase in stress in the fillet at the hole location.
Dedicated to my family
ACKNOWLEDGEMENTS

I would like to thank my wife, Shelby, for her continued support throughout this research. Special thanks are in order for my advisor, Dr. Thomas Whitney, P.E., for providing the time, resources, and guidance necessary for the work contained herein, and for directing this thesis and bringing it to its conclusion with patience and expertise.
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<table>
<thead>
<tr>
<th>Designator</th>
<th>Definition</th>
</tr>
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<tbody>
<tr>
<td>A</td>
<td>Area</td>
</tr>
<tr>
<td>API</td>
<td>American Petroleum Institute</td>
</tr>
<tr>
<td>D</td>
<td>Larger Diameter</td>
</tr>
<tr>
<td>d</td>
<td>Lesser Diameter</td>
</tr>
<tr>
<td>d_p</td>
<td>Angular Displacement (Point to Point)</td>
</tr>
<tr>
<td>Dia, Ø</td>
<td>Diameter</td>
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<tr>
<td>G</td>
<td>Modulus of Rigidity</td>
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<tr>
<td>In.</td>
<td>Inch</td>
</tr>
<tr>
<td>I</td>
<td>Moment of Inertia</td>
</tr>
<tr>
<td>J</td>
<td>Polar Moment of Inertia</td>
</tr>
<tr>
<td>k</td>
<td>Stiffness</td>
</tr>
<tr>
<td>K_t</td>
<td>Stress Concentration Factor for Normal Stress</td>
</tr>
<tr>
<td>K_t</td>
<td>Theoretical Stress Concentration Factor for Shear Stress</td>
</tr>
<tr>
<td>K_{ts}</td>
<td>Theoretical Stress Concentration Factor for Normal Stress</td>
</tr>
<tr>
<td>K_{ts}</td>
<td>Stress Concentration Factor for Shear Stress</td>
</tr>
<tr>
<td>K_{θ}</td>
<td>Deflection Concentration Factor</td>
</tr>
<tr>
<td>L, l</td>
<td>Length</td>
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<tr>
<td>Lbf</td>
<td>Pound Force</td>
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<tr>
<td>M</td>
<td>Moment</td>
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<tr>
<td>m</td>
<td>Mass</td>
</tr>
<tr>
<td>R, r</td>
<td>Fillet Radius</td>
</tr>
<tr>
<td>N</td>
<td>First Critical speed</td>
</tr>
<tr>
<td>T</td>
<td>Torque</td>
</tr>
<tr>
<td>U</td>
<td>Strain Energy</td>
</tr>
<tr>
<td>UX, UY, UZ</td>
<td>Deflection in Rectangular Coordinates</td>
</tr>
<tr>
<td>x, y, z</td>
<td>Rectangular Coordinates</td>
</tr>
<tr>
<td>ρ</td>
<td>Density</td>
</tr>
<tr>
<td>Θ</td>
<td>Angle Between Lesser Diameter and Shaft Shoulder</td>
</tr>
<tr>
<td>θ_{max}</td>
<td>Maximum Angular Displacement</td>
</tr>
<tr>
<td>θ_{nom}</td>
<td>Nominal Angular Displacement</td>
</tr>
</tbody>
</table>
\( \sigma_{\text{Fail}} \) = Predicted Failure – Principle Stress
\( \sigma_{\text{max}} \) = Maximum Normal Stress
\( \sigma_{\text{nom}} \) = Nominal Normal Stress
\( \sigma_{x-a} \) = Alternating and Mean Bending Stress
\( \tau_{\text{Fail}} \) = Predicted Failure - Shear
\( \tau_{\text{max}} \) = Maximum Shear Stress
\( \tau_{\text{nom}} \) = Nominal Shear Stress
\( \tau_{x-a} \) = Alternating and Mean Torsional Shear Stress
\( \phi \) = Angle of twist
CHAPTER 1
INTRODUCTION

1.1. Research Area

Multiple areas of research are discussed in this manuscript, all relating to what
effect a shaft’s shoulder geometry, or an abrupt change in diameter, plays on the
computed strength and/or stiffness of a shaft or part subjected to a torsional load.
This topic was examined using finite element modeling with a standard geometry,
fixity method, and load configuration with various fillet radii. An analytical model
was constructed to identify the impact the various geometries have on the
computed strength and stiffness as compared to a baseline.

In shaft regions having abrupt changes in diameter, a step change of one-third or
more is considered abrupt [1]. If the relationship in Equation (1) is true, the step
change is considered abrupt, where D is the larger diameter and d is the lesser
diameter.

\[
\text{If } \left( \frac{D}{d} \geq 1.33 \right) \text{ then the Step is Considered Abrupt} \quad (1)
\]

Both abrupt steps in diameter and steps that would not be considered abrupt,
according to Equation (1) have been included in this thesis to show an expanded
view of the stress concentration. It is hypothesized that the actual stiffness of the larger diameter shaft is not significantly different than that calculated using the stepped down diameter. Corbo explains, to account for this in shaft modeling, the effective stiffness diameter for the stiffness calculation should grow or shrink along 45 degree lines as shown in Figure 1 [2]. The material outside the area set by the bisecting 45 degree lines should be accounted for by adding the mass of the “lost” material back as a point load at its associated station. This station is the axial location along the shaft at which the center of gravity of the lost material lies.

![Figure 1: Method for Modeling Abrupt Change in Diameter](image)

This concept has generally been accepted as a rule of thumb for rotor modeling [3], and has not yet been analyzed on the part’s strength. Additionally, determining the effect of applying this rule to a part’s strength has not been analyzed. This thesis will discuss the resulting stress concentration factors from applying this rule in the
shaft shoulder design beyond modeling. Also, this will discuss the production of parts with non-compliances such as notches and/or holes within the removed 45 degree region, and what effect this has on the resulting part’s strength and/or stiffness.

All claims are summarized in Section 1.3.

1.2. Problem Motivating Work

To be described in Chapter 2, the current literature surrounding abrupt changes in diameter in regards to rotating machinery is not yet fully developed. A potential application of this work is the ability to remove material from a region without affecting the part strength. The removal of the extra material can be very valuable where an unbalance is present and needs to be rectified by removing material 180° from the “heavy side”. Additionally, this allows the designer ability to strategically remove mass for weight reduction and/or rotordynamic/torsional “tuning” [4].

The American Petroleum Institute, API, addresses requirements for machines used in the petroleum, chemical and gas industry services. API 617 states in 2.5.4 that the proper evaluation of stress concentration factors shall be included in the design of stressed parts, including sizing fillets to limit the stress concentration factor. There is a special note for shaft section changes to be an area of concern when designing rotating elements [4].
1.3. Claims Motivating Research

The claims motivating the research for this thesis to be proven or disproven are,

1. After the removal of the material outside the 45° line, neither the strength nor stiffness will be significantly impacted analyzed at various fillet sizes.

2. Prove the assumption made by the Corbo and the 45° rule that the stiffness grows or shrinks along the 45° line.

3. If the first claim is proven true, the impact of non-conformities, holes and cuts, inserted into the 45° region will not significantly impact the stress concentration factor.

These claims are discussed in detail in the coming Chapters and the final results presented in Section 7.1

1.4. Application and General Principle

Application of this principle can be applied to rotordynamic “tuning” of rotating element assemblies as used in turbomachinery and other mechanical systems.

Tuning includes subtracting material from strategic locations to raise or lower a particular mode shape without significantly impacting the strength of the assembly and shaft.

As described by Vance [1], one of the objectives of a rotordynamic analysis is to “calculate balance correction masses and locations from measured vibration data.”

After the production of a rotating assembly or component, an operator, conducting balance corrections, must remove material from the rotating part without inducing
additional stress risers to decrease the amount of unbalance the rotating element or parts experience during operation or testing.

1.5. Common Definitions and Terminology

It is typical for machinery to contain rotating components (e.g., gears, impellers, bearings.) These elements are attached to or mounted on a shaft coupled to an input power source, many times an engine, motor or turbine, and an output source of additional machinery or a process fluid. A shaft is typically loaded axially, torsionally and/or transversely and is most often constructed of various types of steels. Shafts will be designed with shoulders, or steps, constituting its outside profile used for the slinging of oil or positioning of rotating components. A shaft, as described by Budynas [5], is a rotating element, typically of a circular cross section used to transmit power or motion through an axis of rotation to an attached element (e.g., pulley, flywheel, impeller). Shaft shoulders, discussed next, are an efficient and typical way of axially locating these rotating bodies. Shaft geometry and diameters must be designed and materials should be selected with sufficient strength to resist loading stresses. Materials used in industrial machinery are typically low carbon, hot rolled or cold formed steel and will be analyzed in accordance with the finite element section of ASTM A 668 Standard Specification for Steel Forgings, Carbon and Alloy, for General Industrial Use [6].
A fillet, located between diameter changes at the transition from a lesser diameter to a larger diameter, is commonly found in shaft machine design practices. A shaft shoulder in this thesis shall be defined as one or more fillets connecting multiple diameters. For example, the Finite Element Analysis, FEA, conducted in Chapter 4 used two distinct diameters and fillet radius sizes ranging from 0.03125 inches to 0.5000 inches. See the Appendix for stress concentration factor values and their associated fillet radii generated from this research. The stress concentration factor is generally dependent on the smallest of the shaft diameters [7]. This thesis focuses on a single fillet, connecting two diameters as seen in Figure 2.

Holes are typically found in areas of shafts for lifting and handling purposes, either impeller, gear, etc., installation or removal, and can be added post machining for balancing of the components or rotating assembly. This thesis will not include the effect uniaxial and transverse holes have on the stress concentration factor, but will
be examined when located in the 45° offset region of the shaft shoulder. Examples of these holes are shown in Figure 3.

![Figure 3: Description of Transverse and Uniaxial Holes](image)

Stress concentration is an area of high stress, greater than that estimated by elementary stress formulas typically used in the design of members having a constant cross section or gradual change in cross section [7]. The work in this thesis analyzes abrupt changes in a member's cross section. The presence of shoulders, grooves, holes, and other features will result in a change from the simple stress distribution and areas of high stress may occur. A stress concentration factor, Equations (1) and (2), is a ratio used to measure the difference between the general stress formula in Equation (3) and the actual peak stress in the body. Equation (3) shows the nominal shear stress for a part region of circular geometry, the type used for the remainder of this analysis.
For normal stress (in tension or bending),

\[ K_{ts} = \frac{\sigma_{\text{max}}}{\sigma_{\text{nom}}} \quad (2) \]

For shear stress (torsion),

\[ K_t = \frac{\tau_{\text{max}}}{\tau_{\text{nom}}} \quad (3) \]

Where for a round shaft in torsion, \( T \),

\[ \tau_{\text{nom}} = \frac{T}{J} = \frac{T \left( \frac{d}{2} \right)}{\pi \cdot \frac{d^4}{32}} = \frac{16T}{\pi \cdot d^3} \quad (4) \]

There is an abundance of measured stress concentration factors widely available in tables/charts as seen in *Peterson’s Stress Concentration Factors* by Pilkey [7] and *Marks Standard Handbook for Mechanical Engineers* [8]. These charts have been developed, as described by Pilkey, analytically from the elasticity theory, computationally from the finite element method and/or experimentally using photo elasticity or strain gages. The results discussed in Chapter 4 were computed using the finite element method.

The critical speed of a rotating element is the speed at which the element will rotate and create a large amount of vibration close to the assembly or part’s natural frequency. This vibration is from either a forced response or unbalance/instabilities inherent in the part. Natural frequencies or the critical speed can be changed by modifying the material or increasing or decreasing the mass of the part [1].

**1.6. Limitations and Assumptions**

Stress concentration factors should be used as a basis for design, but should not be relied on solely and should be combined with fundamental engineering principals.
when determining sizing criteria. Pilkey [7] explains the theoretical basis for the factors. Using the theory of elasticity factors in analysis and/or design was considered the best practice. These estimations and formulations are obtained using assumptions such as isotropic and homogenous material. These analyses do not account for non-uniform materials, defects and discontinuities, or directional effects that are present in an actual material. This assumption will later be disregarded and computations will be conducted using simulated defects in section 5.2. The assumptions described above, isotropic and homogenous material, were used in the development of the stress concentration factors generated from this research.

Uniform loading has also been assumed over the course of the research in the modeling and in the analysis technique. In practical design cases, starting torque and/or sudden loading in excess of the steady state torque is possible during operation.

1.7. Summary of Research Methodology

The goal of this research, as previously stated, is to determine how the region of material outside the 45° line, as shown in Figure 1, influences the computed strength and stiffness of the region of interest of the part. The analysis will start with solid modeling of the part with the geometries as shown in Figures 4 and 5.
Figure 4: Stepped Shaft Geometry Outline

Figure 5: Sloped Shaft Geometry Outline
The analyses were conducted using the Solidworks Modeling and Simulation suite. The ratios $r/d$ and $D/d$ are varied, and the stress concentration factors are calculated using finite element analysis. Equations (2) and (3), for the normal and shear stress concentration factors, were used to develop the non-dimensional stress concentration factor. The stepped shaft, shown in Figure 4 with a 90° step, and sloped, shown in Figure 5 with a 45° slope. To be consistent with the industry naming convention, “The 45° Rule”, the 45° slope is measured 90° from the lesser diameter, $d$, and would measure 135° if measured from the same datum as the 90° step. The two model versions were then compared to determine any influence the outside boundary material has on the part. The comparison did not show significant impact with or without the material outside the 45° boundary and the areas showing slight variation maintained a similar slope and line curvature. Due to this technique, the stress and deflection design equations, discussed in Chapter 4, may remain the same for both geometries; trends for stepped and sloped shafts are not significantly different. The stress concentration factors will vary between the two designs with their geometry configurations remaining constant.

After concluding the finite element analysis and equation derivation, an additional geometry analysis was considered. The additional geometry analysis calculated the stress and stiffness at 30° and 60° instead of only at 45° and 90°. A separate analysis was conducted to determine if it is necessary for the material outside the 45° boundary to be isotropic or homogenous, meaning without defect or
discontinuities, to not significantly inhibit the mechanical properties of the part. Using the same finite element solver and method previously employed, holes and variously sized cuts are added into the boundary of non-contributing material of the stepped shaft, and the resulting stresses and stress concentrations factors examined.
CHAPTER 2

LITERATURE REVIEW

In the book “Machinery Vibration and Rotordynamics” [1], the authors discuss the 45-degree rule and its application for modeling abrupt changes in diameter. An abrupt change in diameter is defined as a step change of one-third or more in diameter. This text limits the use of this “rule” to only modeling techniques and does not extend to application in production parts.

Mark Corbo, M.S., PE, also tells of the 45-degree rule in his paper “Pump Rotordynamics Made Simple” [2]. An excerpt from this paper is:

In shaft regions having abrupt changes in cross-section, the actual stiffness of the larger diameter shaft is somewhat less than that calculated using its actual diameter. To account for this, the effective stiffness diameter should be assumed to grow or shrink along the 45 degree lines... The diameters that bisect the 45 degree lines should be utilized as the effective stiffness diameters. The material outside the limits set by the inner and outer stiffness diameters should be accounted
for by applying concentrated masses and inertias at the appropriate stations.

Mr. Corbo’s research uses the 45-degree rule to calculate the effective stiffness of the part. This does not take into account what effect this modeling technique has on the computed strength and associated stress while subjected to a load.

Research conducted by Rexnord Industries [9] states shaft fractures resulting from fatigue type torsional stress are most often found at a 45 degree angle to the shaft axis. Also, a fracture occurring along a plane 45 degrees to the axis of the shaft originated at a keyway and progressed along the 45 degree line. This research implies the area of high stress is along the 45 degree line by indicating the fracture plane direction after originating at a stress concentration.

In the prior research and prior art search, there was no significant research found directly relating the 45-degree rule of thumb to its effect on computed strength. Additionally, the prior research did not produce information defining how this modeling technique influences computed stiffness.

Pilkey [7] has in the book *Peterson’s Stress Concentration Factors* many charts and tables describing various loading, geometries and part configurations with plotted stress concentration factor values obtained from numerical, experimental and finite
element methods. The work discussed in this book was referenced for the graphs and charts generated for this research. The values obtained by Peterson are only estimates based on photo elastic solutions for 3-dimensional circumferential grooves and are only generated over a short range of geometries, \(0.25 \leq \frac{t}{r} \leq 4.0\).

Research conducted by Tipton and Sorem generated stress concentration factors and graphs for both stepped and tapered shafts using FEA runs to calculate their data. Their research showed that tapering a shaft can reduce the stress concentration factor in the fillet significantly in a part undergone to bending, tension, and/or torsional loading. The ANSYS FEA package was used to in the study with linear (quadratic) elements loading in both axisymmetric and non-axisymmetric directions. The meshes generated for their analysis were adjusted using automated features for spacing and gradient sizing assuring convergence. This paper identified over 4400 geometries with varying degrees of taper, but only discusses the results of loading due to bending and not torsion [10].
3.1. Summary of Loading

Typically, torsion and bending are the primary types of loading in rotating machinery, principally, turbomachinery, e.g., compressors, turbines, blowers. The following design equations for strength, stiffness, and critical speed are derived from Collins’s et. al. publication on *Mechanical Design of Machine Elements and Machines* [11]. These equations are based on the lesser diameter and will therefore be equal for both 90° and 45° shaft shoulder types. This similarity among the analytical equations relies on the stress concentration factor, previously discussed, to describe any changes in stress experienced by the part under load.

3.2. Strength Based Design Equations

Since the parts studied are all of circular cross-section, the design equations can modeled as a solid circle. The moment of inertia, $I$, about the centroid axis for a solid round shaft of diameter $d$ is,

$$I = \frac{\pi d^4}{64}$$ (5)
The alternating and mean bending stresses, $\sigma_{x-a}$ and $\sigma_{x-m}$, for a part of circular geometry are,

$$\sigma_{x-a} = \frac{M_a \cdot c}{l} = \frac{M_a \cdot \left(\frac{d}{2}\right)}{\pi \cdot \frac{d^4}{64}} = \frac{32 \cdot M_a}{\pi \cdot d^3} \quad (6)$$

$$\sigma_{x-m} = \frac{32 \cdot M_m}{\pi \cdot d^3} \quad (7)$$

Where $M_a$ and $M_m$ are the alternating and mean bending loads, $I$ is the moment of inertia for circular geometry, and $d$ is the lesser diameter of the part.

The polar moment of inertia, $J$, of a solid round shaft is,

$$J = \frac{\pi \cdot r^4}{2} = \frac{\pi \cdot d^4}{32} \quad (8)$$

The mean and alternating torsional shear stresses for a part of circular geometry are,

$$\tau_{x-a} = \frac{T_a \cdot r}{J} = \frac{T_a \cdot \left(\frac{d}{2}\right)}{\pi \cdot \frac{d^4}{32}} = \frac{16 \cdot T_a}{\pi \cdot d^3} \quad (9)$$

$$\tau_{x-m} = \frac{16 \cdot T_m}{\pi \cdot d^3} \quad (10)$$

### 3.3. Stiffness (Deflection) Based Design Equations

For the applications being studied, the shaft deflection is also of interest. The stiffness based design equations are derived with reference from Collins et al [11].

The angle of displacement of a twisted part under a torsional load can be calculated using Equation (11),

$$\theta_{nom} = \frac{T \cdot L}{K \cdot G} = \frac{T \cdot L}{\frac{\pi \cdot d^4}{32} \cdot G} \quad (11)$$
Where $\Theta$ is the angular deflection, $T$ is applied torque, $L$ is shaft length between torque application sites, $G$ is shear modulus of elasticity, and $K$ is equal to $J$, polar moment of inertia, for circular cross sections. For stepped shafts, being the primary focus of this thesis, the individual diameters should be considered as springs in series. From the above equation combined with the principle of added springs in series with each $i^{th}$ distinct diameter of width, $L_i$, Equation (11) becomes,

$$\theta_{nom,step} = \frac{T}{G} \left( \frac{L_1}{J_1} + \frac{L_2}{J_2} + \cdots \frac{L_i}{J_i} \right) = \frac{T}{G} \left( \frac{L_1}{\frac{\pi d_i^4}{32}} + \frac{L_2}{\frac{\pi d_i^4}{32}} + \cdots \frac{L_i}{\frac{\pi d_i^4}{32}} \right)$$

(12)

3.4. Stress Concentration Factor Effect

A non-dimensional relationship can be made for deflection to relate finite element values compared to nominal values providing, a deflection concentration factor,

$$K_\theta = \frac{\theta_{max}}{\theta_{nom}}$$

(13)

The nominal deflection can be calculated using Equation (14),

$$\theta_{nom,step} = \frac{T}{G} \left( \frac{L_1}{\frac{\pi d_i^4}{8}} + \frac{L_2}{\frac{\pi d_i^4}{8}} + \cdots \frac{L_i}{\frac{\pi d_i^4}{8}} \right)$$

(14)

A non-dimensional comparison can also be made for the stress between the two configurations. This is the stress concentration factor, $K_t$,

$$K_t = \frac{r_{max}}{r_{nom}}$$

(15)
The nominal stress is given by the expression for both shaft shoulder configurations,

\[ \tau_{nom} = \frac{\tau a * r}{J} = \frac{\tau a * \left(\frac{d}{2}\right)}{\pi * \frac{d^4}{32}} = \frac{16 * \tau a}{\pi * d^3} \] (16)

3.5. Impact to Critical Speed

As described by Vance [1], the first critical speed for some rotors can be approximated by the natural frequency of the model, and converted to revolutions per minute. Equation (17) shows this relationship.

\[ N = \frac{60}{2\pi \sqrt{k/m}} \] (17)

The mass of the region outside the 45° line can be approximated by Equation (18) below,

\[ m = t * \frac{\pi (D^2 - d^2)}{4} * \frac{1}{2} * \rho \] (18)

Where t is the thickness of the region being removed,

\[ t = \frac{D - d}{\tan(45)} = D - d \] (19)

Combining Equation (18) and (19), the mass relationship becomes,

\[ m = \frac{\pi (D^2 - d^2) * (D - d)}{8} * \rho \] (20)

By removing the material outside the 45° boundary, equal to the mass calculated from Equation (20), the first critical speed of the model will increase. This increase is due to the decrease in the part mass with the material stiffness remaining constant. This simplified form will give an estimated impact to the overall critical
speed. This may be important for rotors bound by the specifications listed in section 2.6.2 of API-617 [4]. By having a reduced weight, the damped unbalanced response analysis may indicate an increase in the amplification factor. For amplification factors greater than 2.5, there must be a higher separation margin as described by section 2.6.2.10 [4].
4.1. Analysis Technique

Cosmos Design, now Solidworks simulation 2012 x64 SP1.0, was used for this analysis with FFEPlus solver method employed. Solidworks simulation offers selective mesh refinement to allow the user to receive with higher quality and with a less run time as compared to manual mesh adjusting. This adaptive meshing protocol provides two methods, the P-adaptive and H-adaptive. The P-adaptive method adjusts the polynomial order of the mesh to improve the accuracy of the results. The H-adaptive method refines the size of the mesh to reduce the energy norm error of the model where a smaller mesh is needed [12].

The H-adaptive method is described by Pepper in the following excerpt,

The adaptive grid model can dynamically control grid size (h)—the grid refines and unrefines automatically based on the gradient and error distribution of topography and other key variables (such as velocity and wind power density). The h-adaptive finite element model is ideal for solving problems requiring large-scale calculations over regions where
localized fine meshing is needed and yields far better accuracy than conventional, non-adapting numerical schemes [13].

An H-adaptive method was employed within the simulation suite to converge to within 98% solver target accuracy with a maximum of 5 iterations. This method automatically refines the mesh, using smaller elements, in regions with high error, to be described in Section 4.2 [12]. The H-adaptive method was chosen over the P-adaptive method based on the ability to mesh and solve successfully and quickly. The analysis used a solid mesh with tetrahedral elements, all solid meshes within Solidworks use tetrahedral elements. It was shown by Ramos that when comparing results obtained by simulating a simplified geometry with first and second order tetrahedral and hexahedral elements, the results did not evidence significant differences [14]. All specific geometry details and analysis results are listed in the Appendix. A representative sample of a meshed part is shown in Figure 6. The sample part mesh contains 67,002 nodes and achieved an H-adaptive solver accuracy of 98.6%. Figure 7 illustrates the mesh of the shoulder area.

To ensure the H-adaptive meshing was outputting accurate stress values, an initial comparison was conducted to determine the mesh size within the fillet region, area of interest. The comparison was between the H-adaptive values and manual application of mesh controls within the fillet region. The h-adaptive mesh generated meshes between 6 and 10 times smaller than the smallest feature. After
meshing, A Priori confidence metrics show a high mesh quality with 4 Jacobian points. 99.8% of elements show an aspect ratio of less than 3 with a maximum aspect ratio of 4.5526.

Figure 6: Meshed Part

Figure 7: Meshed Part in the Area of Interest
The Non-dimensional detail drawings were shown in Figures 4 and 5. Figure 8 below illustrates the loading and fixity method employed. The fixed area restrained rotation, axial and translational movement, and growth/shrinkage at the constrained face. The purple lines indicate torsional loading, 2,500 in.-lb., and the green arrows indicate the area of the part that is constrained. The boundary condition fixing the part is located on the left side in Figure 8 and is applied across the entire face. The torque load is applied at the face opposite of the fixity condition on the right face. The torque is applied in a clockwise direction with the large diameter cylindrical section chosen as the axis of rotation.

![Figure 8: Part Fixture and Loading Summary](image)

**4.2. Mesh Sensitivity Analysis**

White [15], defines a sufficient mesh density as one in which the mesh ensures convergence of the analysis within an acceptable tolerance. For this analysis, a mesh’s accuracy will be measured using the ERR function and the design with
dictate what is considered an acceptable error level, but this thesis used a mesh with a maximum error equal to 0.255%. This error, referred to as Energy Norm Error, is determined in the simulation software. The software calculates the stresses at all the Gaussian points and then extrapolates the nodal stresses for each element. A perfectly converged solution should give equal stress values at all common nodes, however, different elements will have stress values unequal at the common node and the program will then average all the stresses at the contributing elements to determine the nodal stress. This results in variation and a measure of this variation provides the accuracy of the model’s solution [16].

The sample geometry used for the mesh sensitivity analysis is shown in Figures 9 and 10 for both shoulder configurations. All mesh elements were of equal dimension in the global mesh size, mesh sizes are indicated by the blue diamonds with sizes along the X-axis in Figure 11. A controlled mesh was added in the fillet region to reduce the accuracy in this critical section. The chart shows a near-linear trend with a minimum slope indicating convergence on the true value. The stress concentration factor estimation given by Equation 3.5 from Pilkey [7], graphed on Figure 11, approximated the expected concentration factor for this geometry and agreed with the FEA results.
Figure 9: Sample Geometry Using Full Shaft Shoulder

Figure 10: Sample Geometry Using 45° Sloped Shaft Shoulder
Using a controlled mesh of 0.03125” in the area of interest, the fillet region, the model’s stress concentration factor deviates 0.009299 from the baseline FEA equation generated by Pilkey’s Equation from prior FEA results and indicating the mesh sensitivity analysis shows an adequate mesh has been applied. Additionally, the low deviation from the calculated value of $K_t$ from Pilkey’s Equation in Chart 3.12 of the referenced text indicates an accurate model is being used to estimate stresses. The comparison of the selected 0.0.3125” mesh and the additional meshes are shown in Figure 11 above.

The equation in Chart 3.12 from Pilkey [7], used as the computational baseline in Figure 11 for referencing the calculations and claims made in this thesis. Equation (21) states,
\[ K_{tn} = C_1 + C_2 \left(2 \frac{t}{D}\right) + C_3 \left(2 \frac{t}{D}\right)^2 + C_4 \left(2 \frac{t}{D}\right)^3 \]  

(21)

Where \( C_1, C_2, C_3, \) and \( C_4 \) are calculated constants for the range \( 0.25 \leq t/r \leq 4.0 \), where \( t \) is \( (D-d)/2 \). Equations (22) through (25) show the relationships.

\[ C_1 = 0.905 + 0.783 \sqrt{t/r} - 0.075 \frac{t}{r} \]  

(22)

\[ C_2 = -0.437 - 1.969 \sqrt{t/r} + 0.553 \frac{t}{r} \]  

(23)

\[ C_3 = 1.557 + 1.073 \sqrt{t/r} - 0.578 \frac{t}{r} \]  

(24)

\[ C_4 = -1.061 + 0.171 \sqrt{t/r} + 0.086 \frac{t}{r} \]  

(25)

Equation (21) should be used for a stepped bar of circular cross section with a circumferential shoulder fillet [7].

The chart generated in Pilkey [7] and Equations (21) through (25) have been plotted against the results generated for this analysis to confirm the error or deviation from the published values does not increase. These results are located in Appendix B.

To measure the accuracy of the model, the ERR: Energy Norm Error function of Solidworks Simulation is employed [12], [10]. In a sense, error plots show how a generated mesh matches the complexity of the model and applied loads. When a mesh matches the model, the reported error from Solidworks will be low. The
energy norm error is plotted as an estimate of the error in the energy norm and not in the stress, but is viewed as a representation of the relative distribution of stress errors in homogenous meshes. The percentage error for each element is calculated by determining the variation in strain energy at an element within a common node and the average stresses of the common node. The percentage error for each element is calculated by Equation (26) from the Solidworks user manual

\[
\eta = \left[ \frac{||e_\sigma||^2}{(||u||^2 + ||e_\sigma||^2)/n} \right]^{1/2} \times 100
\]  

(26)

Where \( \eta \) is the percentage error for each element, \( ||e_\sigma|| \) is the error in energy norm within a common node, \( ||u|| \) is twice the total strain energy, and \( n \) is the total number of elements within the common node [16].

The areas of interest within the error plots are indicated as “A”, “B”, and “C” in Figure 12. These areas are located at the mesh boundary in the studied area and provide a general idea of the region’s accuracy.
The iterations and mesh sizes evaluated in this analysis are 0.015625”, 0.03125”, 0.0625” and 0.09375” and the results can be seen in Figure 13: FEA Reported Error Level for Various Mesh Sizes. Tables 1 and 2 show the Von Mises Stress calculated for each global mesh size and the reported percent error level. The max reported error was not located in the region of interest, but along a split line, and can be disregarded. Figure 13 below uses the adjacent color scale with a max error value of 1%.
The results of 3 of the global mesh sizes are summarized in Tables 1 and 2. Table 1 provides the Von Mises Stress for each of the 3 mesh sizes sampled at each of the nodes “A”, “B”, and “C”. The last column provides the maximum reported energy norm error calculated by the part. This error is higher than acceptable and is located at each face of the part and along the split line where the boundary conditions are present. These areas are not of concern for this analysis and have been disregarded. Table 2 illustrates the energy norm error for the same nodes analyzed in Table 1. For mesh sizes larger than 0.09375”, the error level is
excessive for this analysis and cannot be used. The meshes were selected to be much smaller than the smallest fillet to be analyzed for all configurations.

Table 1: Von Mises Stress for Various Mesh Sizes

<table>
<thead>
<tr>
<th>Mesh Size</th>
<th>Von Mises Stress - PSI</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
</tr>
<tr>
<td>0.015625&quot;</td>
<td>2,148</td>
</tr>
<tr>
<td>.03125&quot;</td>
<td>1,991</td>
</tr>
<tr>
<td>.0625&quot;</td>
<td>2,044</td>
</tr>
<tr>
<td>0.09375&quot;</td>
<td>2,077</td>
</tr>
</tbody>
</table>

Table 2: Energy Norm Error Percent for Various Mesh Size

<table>
<thead>
<tr>
<th>Mesh Size</th>
<th>Energy Norm Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
</tr>
<tr>
<td>0.015625&quot;</td>
<td>0.0257</td>
</tr>
<tr>
<td>.03125&quot;</td>
<td>0.245</td>
</tr>
<tr>
<td>.0625&quot;</td>
<td>2.5</td>
</tr>
<tr>
<td>0.09375&quot;</td>
<td>5.116</td>
</tr>
</tbody>
</table>

The mesh with an acceptable error level and is solved within a reasonable amount of time for this analysis in the area of interest is 0.03125", and is therefore what will be used in this analysis. PVEng [17] states the actual error will be much less than what is stated and some elements will have higher error levels. A sample stress distribution chart generated by the simulation is illustrated in Figure 14. The values listed in table 2 are worst case values and were calculated from the model prior to implementing any finer mesh controls or adaptive meshing techniques. Therefore, the values listed in Appendix A contain an error in strain energy much less than what was included above.
Figure 14: Overall Stress Pattern Seen for 0.03125" Mesh Size

All FEA results have been compared to the Peterson’s Stress Concentration Factor charts published by Pilkey, [7], for the 90° step configuration to verify the results from this analysis. These comparisons can be found in Appendix B. Good correlations between the two versions of data indicating a low amount of error is present in the model being used and accurate data will be generated for the analyses to be conducted in addition to the 90° step, E.G. tapered shafts and defect analysis.

4.3. Stress Flow Line Simulation

A stress flow line simulation or diagram is a conceptual visualization of how the stress through the part “flows” as it passes around a fillet, groove, or obstruction present under a stressed part. The model results in Figures 15 and 16 provide a visual representation of the stress flow within both sloped and stepped shaft
geometries. In the model, the parts are fixed at the left end, with green arrows indicating a boundary condition. The analysis was conducted using a torque value of 2,500 in-lb. located at the end of the small diameter, the purple arrows indicate load and direction. See Figures 9 and 10 for the sample geometry used for this configuration, where D=5.00", d=2.50", and r=0.125".

Figure 15: Force Flow Lines Using Full Shaft Shoulder ($\sigma_{y_{max}}=2,401$ psi)

Figure 16: Force Flow Lines Using 45° Sloped Shaft Shoulder ($\sigma_{y_{max}}=2,244$ psi)
As can be seen from Figures 15 and 16, the stress flow line simulation does not significantly differ between the shaft with a full shoulder versus a 45° sloped shoulder, but the 45° sloped version actually decreases the max stress exhibited by the part.

The previously described equations from chapter 3 can be used to show the following relationships between stepped and slopped shoulder configurations. This demonstrates the numerical distinction between the two models and the basis of this thesis. A non-dimensional comparison between the two configurations is the stress concentration factor, $K_t$. The nominal stress is given by the expression for both configurations,

$$
\tau_{\text{nom,VMs}} = \frac{16 \cdot T}{J} = \frac{16 \cdot T}{\pi \cdot d^3} = \frac{16 \cdot 2,500 \text{ lbf-in}}{\pi \cdot 2.5 \text{ in}^3} = 814 \text{ psi}
$$

(27)

Since the output value from the FEA software calculates Stress according to the Von Mises Criterion, an additional conversion must be made. The Von Mises criterion estimates the yield strength, $\sigma_y$, of a uniaxially load bar. For a case where the bar is loaded in torsion, when $\sigma_y = -\tau_y$, where $\tau_y$ is the yield strength of a bar in torsion, the Von Mises Stress outputted from Solidworks becomes [7],

$$
\tau_y = \frac{\sigma_y}{\sqrt{3}} = 0.577 \cdot \sigma_y
$$

(28)

Using Equation (28), the values from Solidworks become

$$
\tau_{\text{max,Full Shoulder}} = \frac{\sigma_y}{\sqrt{3}} = \frac{2401 \text{ psi}}{\sqrt{3}} = 1386.2 \text{ psi}
$$

(29)
\[
\tau_{\text{max,45° Slope}} = \frac{\sigma_y}{\sqrt{3}} = \frac{2244 \text{ psi}}{\sqrt{3}} = 1295.6 \text{ psi}
\]  

Therefore, the corresponding stress concentration factors for the given geometry of the stress flow simulation becomes,

\[
K_{t,\text{Full Shoulder}} = \frac{\tau_{\text{max}}}{\tau_{\text{nom}}} = \frac{1386.2 \text{ psi}}{814 \text{ psi}} = 1.70
\]  

\[
K_{t,45° \text{ Slope}} = \frac{\tau_{\text{max}}}{\tau_{\text{nom}}} = \frac{2244 \text{ psi}}{814 \text{ psi}} = 1.59
\]

Similarly, a flow line diagram can be created for each part of the displacement of the part under load to determine the part stiffness differences between the two configurations. The following model results, Figures 17 and 18, were generated using the same boundary conditions method, loading, and solver configuration as the stress analysis [7].

Figure 17: Displacement Flow Lines Using Full Shaft Shoulder  
(UY=3.599 x 10^{-4} \text{ in.})
As can be seen from Figures 17 and 18, the displacement flow line simulation does not significantly differ between the shaft with a full shoulder versus a 45° sloped shoulder. This difference is measured by UY in the FEA software. UY is the displacement calculated in the Y direction. The calculated UY shown in the titles of Figures 17 and 18 indicate the maximum deflection in the part. A non-dimensional comparison between the two configurations is the stiffness concentration factor, \( K_\Theta \).

The nominal displacement angle for the stepped shaft is calculated by,
\[
\theta_{\text{nom, step}} = \frac{T}{G} \left( \frac{L_1}{\pi \cdot d^4/32} + \frac{L_2}{\pi \cdot d^4/32} + \cdots + \frac{L_i}{\pi \cdot d^4/32} \right)
\]

\[
= \frac{2500 \text{ lbf} \cdot \text{in}}{11,167,902 \text{ psi}} \left( \frac{5 \text{ in}}{\pi \cdot (2.5 \text{ in})^4/32} + \frac{5 \text{ in}}{\pi \cdot (5 \text{ in})^4/32} \right) \cdot \frac{180}{\pi}
\]

\[
= 0.01777^\circ
\]

(33)

Converting the values from the analysis from point displacement to angular displacement is shown in Equations (34), (35), and (36) with variables defined in Figure 19. Where \(d_p\) is an outputted value from the FEA equal to UY.

![Figure 19: Representation of Converting Point Displacement to Angular Displacement](image)

\[
\varphi = \tan^{-1} \left( \frac{d_p}{d/2} \right)
\]

(34)

\[
\varphi_{\text{Full Shoulder}} = \tan^{-1} \left( \frac{d_p}{d/2} \right) = \tan^{-1} \left( \frac{3.599 \times 10^{-4}}{2.5/2} \right) = 0.0165^\circ
\]

(35)

\[
\varphi_{45^\circ \text{ Slope}} = \tan^{-1} \left( \frac{d_p}{d/2} \right) = \tan^{-1} \left( \frac{3.722 \times 10^{-4}}{2.5/2} \right) = 0.0170^\circ
\]

(36)
Therefore, the corresponding stiffness concentration factors for the angle of
deflection for the given geometry of the stress flow simulation becomes,

\[ K_{\theta, \text{Full Shoulder}} = \frac{\theta_{\text{max}}}{\theta_{\text{nom}}} = \frac{0.0165^\circ}{0.01777^\circ} = 0.929 \]  \hspace{1cm} (37) \\
\[ K_{\theta, 45^\circ \text{Slope}} = \frac{\theta_{\text{max}}}{\theta_{\text{nom}}} = \frac{0.0170^\circ}{0.01777^\circ} = 0.957 \]  \hspace{1cm} (38)

4.4. Part Geometry, Loading, and Boundary Condition Summary

The analysis was conducted with sample geometry as shown in Figures 9 and 10. The geometry and part design was specifically designed for the purpose of this research. The simplistic design allows the reader and author to focus solely on the area of interest and allows for fast computational time. See Appendix A for the variable geometry values and the associated concentration factor values.

The boundary condition and load type was conducted in accordance with the illustration shown in Figure 8. Both analysis types were conducted with the assumption of homogenous material and identical loading and boundary condition.

4.5. Analysis & Results

The analysis results developed were split into two groups,

Group 1: Fixed D/d ratio and variable r/d for D/d ratios of: 2.5, 2.0, 1.666, and 1.25.
Group 2: Fixed r/d ratio and variable D/d for r/d ratios of: 0.01, 0.02, 0.03, 0.05, 0.07, 0.10, 0.15, 0.20, and 0.30.

These groups and ratios were selected to encompass a broad range of design criteria. The fillet radii selected include the smallest size deemed practical, 0.03125 in. for Group 1 data and 0.025 in. for Group 2 data. The largest fillet sizes were selected based on the largest radius size capable of fitting between the two diameters. All results are located in the appendix, A1 for group 1 results, and A2 for group 2 results [7].

Shown in Figures 20 and 21, are typical FEA results generated for the 90° step and 45° step respectively. The stresses are visually color-scaled with blue representing the least stressed areas and red representing the highest stressed areas. Each part is currently shown utilizing the same color scale with a maximum Von Mises stress value of 2,346 psi. For this simulation, A 668, Class K steel was used. The mechanical properties are listed below [6].
Table 3: Mechanical Properties of Analyzed Shaft

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tensile Strength</td>
<td>105,000 psi min</td>
</tr>
<tr>
<td>Yield Strength (0.2% offset)</td>
<td>80,000 psi min</td>
</tr>
<tr>
<td>Elongation (2” gauge length)</td>
<td>20% min</td>
</tr>
<tr>
<td>Reduction of area</td>
<td>50% min</td>
</tr>
<tr>
<td>Brinell Hardness</td>
<td>212 min – 269 max</td>
</tr>
<tr>
<td>Poisson’s Ratio</td>
<td>0.29</td>
</tr>
<tr>
<td>Elastic Modulus</td>
<td>29,007,506 psi</td>
</tr>
<tr>
<td>Shear Modulus</td>
<td>11,167,902 psi</td>
</tr>
</tbody>
</table>

Figure 20: FEA Sample Result for 90° Step Shaft
Figure 21: FEA Sample Result for 45° Step Shaft
5.1. Additional Geometry Analysis

Additional geometry considerations include testing the same hypothesis previously researched at additional angles. This was completed at a 30° slope and 60° slope as shown in Figures 22 and 23.

Figure 22: Sample Geometry Using 30° Shaft Shoulder
Results of the additional geometry testing will be discussed in section 6.4.

5.2. Addition of Defects in (Removed) 45 Degree Region

Simulated defects may be added to the outer diameter surface to determine how additional stress risers in the region of interest affect the over stress concentration factor. If the assumption of homogenous material within the region outside the 45° zone is no longer valid, additional analyses can be compiled. Special care was taken to ensure the simulated defect only resided in the area of interest. Failure to do so may improperly influence the total part stress level. The misapplication, making larger or mis-located holes, may also introduce additional stress risers in the design. For this section, the simulated defects used are,
1. 0.25” diameter by 0.75” deep hole (See Figure 24)

2. 0.5” diameter by 0.125” deep hole (See Figure 25)

3. 1.0” by 1.0” Cut, 0.50” Wide. (See Figure 26)

4. 0.5” by 0.5” cut, 0.25” wide. (See Figure 27)

5. 0.5” diameter by 1.5” deep hole (See Figure 29)

6. 1.0” by 2.0” Cut, 0.25” Wide (See Figure 30)

These diameters were chosen to represent a range of standard drill sizes, both small and large from 0.25 in. to 0.5 in, as a general representation of possible defects. As previously stated, the depth of the holes and cuts are cut into the shoulder without crossing the 45° boundary line as shown in Figure 28. Data point 5 listed above intentionally crosses the 45° boundary to determine the impact to the stress once this boundary is crossed. All holes are centered 0.375” away from the face. The cut width is located on the larger diameter, straddling the part’s centerline.

Figure 24: Simulated Defect - 0.25” Diameter by 0.75” Deep Hole
Figure 25: Simulated Defect - 0.5” Diameter by 0.5” Deep Hole

Figure 26: Simulated Defect – 1.0” by 1.0” Cut, 0.50” Wide
Figure 27: Simulated Defect - 0.5” by 0.5” Cut, 0.25” Wide

Figure 28: Sketch Illustrating Defects Contained Within 45° Region
The simulated defects results, with their associated stress concentration factors, are shown in Figure 28. All maximum stress values were taken at the same element/node as the non-defective cases. These values are compared to the 90° step as a baseline and the deviation between the runs has been calculated. The geometry used in Figure 9 is the geometry of the part used for this analysis.
Figure 31 below summarizes the results from the simulated defect analysis. The vertical bars indicate the magnitude of the stress concentration factor, the circles show the deviation from the 90° step baseline and each defect case. As a reference, the 45° slopped baseline is also plotted.

As can be seen from Figure 31, there is no significant change in stress when defects are introduced, significant defined as greater than the difference between the 90° step and 45° slope. There is a significant impact when the 0.5” hole by 1.5” deep is added, indicating when the 45° is crossed, there will be an impact to the stress.
6.1. Overall Results Discussion

Compiling the results of this research, it can be concluded the region of material outside the 45° boundary does not significantly affect the strength or stiffness of the region of interest, this region being the area of high stress located in the radii extending from the lesser diameter to the greater diameter. In retrospect, by removing this material, the stress concentration is slightly decreased on average by 4%. This not only confirms the original hypothesis, but adds additional consideration to the practice of removing this material to further decrease the stress in this region. As demonstrated in the stress flow line simulation of Section 4.3, there is no significant change between the two models regarding the stiffness, or angle of twist, of the stressed parts and confirms the industry’s practice of there being no need to consider this region during modeling. All concentration factor values and significance criteria may be influenced by the application of this rule in a real-world product.
6.2. Standard Deviation Results

To determine the difference between the two shaft models, plots of the deviation between the two shafts were generated. These charts show the magnitude of the deviation at various geometry conditions. The charts can be seen in Figures 32 – 35, the deviation of $K_t$ between the different shaft versions is very low, typically less than 4% of the total stress concentration factor value. The regions with standard deviations greater than 4%, deviation typically in excess of 0.10 are located in areas with small radius sizes, or low ratio between the radius and lesser diameter. Table 3 illustrates a table that indicates when $(D-d)/r$ becomes greater than 16.0, the standard deviation increases from 5% to as much as 15% greater than the original stress concentration factor value. Close attention should be paid to fillet sizing and it can be noted that the region outside 45° may not significantly impact the stress concentration factor with an appropriately sized shoulder fillet.

Figure 32: Deviation between the 45° and the 90° Shaft Models Versions for D/d=2.5
Figure 33: Deviation between the 45° and the 90° Shaft Models for D/d=2.0

Figure 34: Deviation between the 45° and the 90° Shaft Models for D/d=1.666
Figure 35: Deviation between the 45° and the 90° Shaft Models for D/d=1.25

Table 4: (D-d)/r Recommended Practice

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It can be noted from Figure 36, that as the ratio of the larger diameter to the lesser diameter increases, the deviation between the two step configurations exponentially decreases. The least amount of deviation exists when $r=0.75”$ and $d=2.5”$. For this selection $D$ ranges from $2.25”$ to $3.00”$, the maximum deviation is $0.008$. This selection also exhibited the lowest stress concentration factor of $1.15$.

In general, the deviation decreases and the radius size between the two diameters increases. However, the stress concentration factor is significantly less, and the standard deviation greater, for cases in **bold** from Table 4.

![Figure 36: Average Standard Deviation for an analyzed r/d for All D/d analysis Points](image)
6.3. **Complete Stress Concentration Charts**

On the following pages, Figures 37 and 38 are full stress concentration charts generated from this research and listed with various D/d and varying r/d and r/d with varying D/d at both sloped and stepped shaft shoulder configurations. Figure 37 represents the data for all Group 1 configurations, and Figure 38 represents the data for all Group 2 configurations. All data used to create these charts is located in Appendix A.1 and A.2.

The two graphs show two different stories of the formation of stress concentration in the shoulder at various geometries with the introduction of a fillet in the corner of the diameters. Figure 37 indicates the ratio of the radius to the lesser diameter, r/d, has a larger influence on the stress concentration factor that the change in step size between the diameters, D/d. The ratios from D/d=2.5 to D/d=1.25 follow a similar trend. The deviation between the 90° step and 45° is greater when r/d>0.063. After this point, the deviation can be considered negligible and there will not be any significant change in strength between the shaft geometries. Contrary to original thoughts, the stress concentration factor of the 45° step is significantly less than that of the 90° step and can be used in design as a method of reducing stress in shoulder regions when smaller fillets are present.

Figure 38 indicates that as the difference between the lesser and larger diameters increases, the radius of the fillet between should also increase, or the deviation
between the 45° and 90° steps is significantly larger. Figure 38 also agrees with
Figure 37 in regards to the practice of removing the 45° region as an effective
method of reducing the stress concentration factor. The design equations
discussed in Chapter 3 indicate that if the lesser diameter remains constant, the
stress within the fillet should not increase at various greater diameters. Figure 38
shows that for small D/d ratios and for the smallest fillets, there is an impact to the
stress concentration within the fillet region when this was thought to remain
constant.
Figure 37: Fixed D/d versus Varying r/d for Both Configurations
Figure 38: Fixed r/d versus Varying D/d for Both Shaft Configurations
6.4. Additional Geometry Results Discussion

The results of the additional geometry angles for the stress concentration factors are in Figure 39 below, and indicate the maximum elemental stress found in each additional geometry analysis. It can be seen that as the angle increases, the stress level decreases. This occurrence is analogous to increasing the fillet size on the 90° step and creating a gradual transition between the two diameters. The 90° step may also be considered as 0° and the remaining angles measured from the baseline or without slope or angle. The bullet points represent the data points analyzed with a best fit polynomial curve estimating the stress concentration factor for addition angles. The Von Mises stress calculated is the maximum stress located in the fillet of each shoulder.

![Figure 39: Additional Geometry Stress for 30°, 45°, 60° and 0° (or 90°) Shoulder Angles](image)

\[ y = -1E-04x^2 + 0.0016x + 1.6987 \]
Using Equations (39) through (42), the following simulations were conducted for 30° and 60° slope angles. These analyses show how the angle influences the part stiffness. Figures 40 and 41 below indicated the displacement flow lines when subjected to the same 2,500 in-lb. load previously employed. The UY values listed in the titles is the maximum deflection of the part for each of the slope cases.

Figure 40: Displacement Flow Lines Using 60° Sloped Shaft Shoulder (UY=3.840 x 10^{-4} in.)
Figure 41: Displacement Flow Lines Using 30° Sloped Shaft Shoulder (UY=3.666 x 10^{-4} in.)

The results from Equation (29), \( \varphi_{Full\ Shoulder} = 0.0165^\circ \), will continue to be the baseline from which the other conditions are compared.

The following equations show the angle of twist for the 60° and the 30° conditions,

\[
\varphi_{60^\circ\ Slope} = \tan^{-1} \left( \frac{d_p}{d/2} \right) = \tan^{-1} \left( \frac{3.840 \times 10^{-4}}{2.5/2} \right) = 0.0176^\circ 
\]

(39)

\[
\varphi_{30^\circ\ Slope} = \tan^{-1} \left( \frac{d_p}{d/2} \right) = \tan^{-1} \left( \frac{3.666 \times 10^{-4}}{2.5/2} \right) = 0.0168^\circ 
\]

(40)

Therefore, the associated stiffness concentration factors are,

\[
K_{\theta,60^\circ\ Slope} = \frac{0.0176^\circ}{0.01777^\circ} = 0.991 
\]

(41)

\[
K_{\theta,30^\circ\ Slope} = \frac{0.0168^\circ}{0.01777^\circ} = 0.946 
\]

(42)

The results of the additional geometry analysis angles for the stiffness concentration factors are in Figure 42 below, and indicate the maximum elemental stiffness found in each additional geometry analysis. It can be seen that as the
angle increases, the stiffness level also increases. This indicates the assumption stated in the beginning of this thesis and the basis for the 45° that the stiffness grows or shrinks along the 45° line is incorrect. It cannot be assumed the stiffness with grow or shrink along this line due to the significant increase in the stiffness concentration when compared to the 90° step, below shown as 0°, and the 45° slope.

Figure 42: Additional Geometry Stiffness Tested at 30°, 45°, 60° and 0° (or 90°)

Although the difference between the angles of the step are limited, the percent deviation from the baseline is 1.86%, 3.36%, and 6.48% respectively for 30°, 45°, and 60° shoulder steps. These differences are very low in comparison to the Energy Error Norm, with exception to the 60° version, and it can therefore be assumed
there is no contribution to stiffness at any angle as long as the step to a larger diameter occurs, to as high as 6.48% if the slope angle is 60°.

6.5. Analysis with Simulated Defects Discussion

It should also be noted from Chapter 5, the region outside the 45° boundary does not need to be homogenous or without defect for the 45° rule to apply. The analysis conducted showed little to no deviation in the stress concentration factor, as compared to the stepped baseline, with the addition of drilled holes and/or “cuts” in the 45° region.

This is an area for future work to determine how this method of design can be leveraged to “tune” a shaft rotordynamically or to provide fixture holes and/or balance holes to assist in production and/or transit of the part. While this research simulated holes and cuts into the outer diameter, simulations where scratches, gouges, or sharp corners generating from the areas of high stress near the fillet region were not included.
7.1. Overall Results Discussion

The original claims, or hypothesis, made by this thesis are,

1. After the removal of the material outside the 45° line, neither the strength nor stiffness will be significantly impacted analyzed at various fillet sizes.

2. Prove the assumption made by the Corbo and the 45° rule that the stiffness grows or shrinks along the 45° line.

3. If the first claim is proven true, the impact of non-conformities, holes and cuts, inserted into the 45° region will not significantly impact the stress concentration factor.

Claim 1 is proven to be true. After comparing the stress concentration factors for both 90° and 45° geometries, there is no significant (worsening) impact to the stress concentration factor. The stress concentration factor for the 45° design on average reduces the stress concentration factor.

Claim 2 is proven to be false. Referring to Figure 42, it can be noted the stiffness does increases as the slope angle increases. When the slope has increased to 45°,
the stiffness also increases by 3.4%. Depending on the designer utilizing this rule, this deviation may not be considered significant.

Claim 3 is proven to be true. After the insertion of “Simulated Defects” in Section 5.2, it was seen there is not a significant impact to the stress concentration factor when adding holes or cuts into the 45° region. It can be noted that if the 45° boundary is passed, there will be an increase in stress in the fillet at the hole location.

As can be seen in Figures 69 through 74 in Appendix B, the results calculated in this analysis do not significantly vary from the published results for this type of should geometry and loading. This correlation brings strong confidence to the information stated in claims 1, 2, and 3.

7.2. Future Work

This thesis focused on finite element modeling to calculate the stress concentrations presented. Future work can be completed to test parts and validate the FEA results. These tests should include an analysis for both the additional geometry analysis and the inclusion of defects of various geometries.

The effect of removing this material on the critical speed of a rotor was explained in Chapter 1, but not tested or analyzed. This effect will be used to increase the critical
speed by removing mass from the 45° region. This technique is heavily dependent on the rotor size and operating speed. For large rotors with small steps, this technique may prove to be inefficient.

An additional area for future work is surrounding the results of Figure 38. The design equations from Chapter 3 show that the nominal stress is not a function of the greater diameter. Additional work may be completed to understand why at small diameter ratios, D/d, and smaller fillet sizes the stress concentration deviates.


[6] ASTM A668/A 668M – 04 Standard Specification for Steel Forgings, Carbon and Alloy, for General Industrial Use. ASTM International. 100 Barr Harbor Drive, PO Box C700, West Conshohocken, PA 19428-2959, United States


APPENDIX A:

FINITE ELEMENT ANALYSIS RESULTS

A.1. $r/d$ Versus $K_t$ Data

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<th>$r/d$</th>
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Figure 43: $r/d$ FEA Data – D/d=2.5

$$y(90) = -339.59x^3 + 178.82x^2 - 31.903x + 4.0776$$

$$y(45) = -308.93x^3 + 156.06x^2 - 26.036x + 3.5667$$

Figure 44: $r/d$ FEA Chart – D/d=2.5
Figure 45: r/d FEA Data – D/d=2.0

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$y(90) = -576.49x^3 + 246.31x^2 - 36.56x + 4.1032$

$y(45) = -658.58x^3 + 263.42x^2 - 34.86x + 3.7475$

Figure 46: r/d FEA Chart – D/d=2.0

90°, D/d=2
45°, D/d=2
Poly. (90°, D/d=2)
Poly. (45°, D/d=2)
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Figure 47: r/d FEA Data – D/d=1.429

\[
y(90) = -3634.3x^3 + 861.96x^2 - 74.27x + 4.7604
\]

\[
y(45) = -1607.5x^3 + 474.44x^2 - 46.926x + 3.8813
\]

Figure 48: r/d FEA Chart – D/d=1.429
### Table: Nominal Stress vs. Von Mises Stress

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### Figure 49: r/d FEA Data – D/d=1.25

The diagram shows the relationship between the variables y(90°) and y(45°) for different values of r/d and D/d. The equations for y(90°) and y(45°) are provided:

- \( y(90°) = -20401x^3 + 3036.6x^2 - 159.07x + 5.5428 \)
- \( y(45°) = -2203.8x^3 + 583.45x^2 - 51.565x + 3.8616 \)

### Figure 50: r/d FEA Chart – D/d=1.25

The chart illustrates the data from the table, with lines representing the polynomial fits for both 90° and 45°. The chart is labeled with the equations for the polynomial fits, which are the same as those provided in the table.
### A.2. D/d Versus Kt Data

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<th>D/d</th>
<th>r/d</th>
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**Figure 51:** D/d FEA Data – r/d=0.01

**Figure 52:** D/d FEA Chart – r/d=0.01

\[ y(90) = 0.5843x^3 - 4.0385x^2 + 9.0345x - 4.0937 \]

\[ y(45) = 0.4434x^3 - 2.9234x^2 + 6.1547x - 2.0952 \]
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### Equation

- $y(90) = 0.2217x^3 - 1.5887x^2 + 3.7379x - 0.797$
- $y(45) = 0.2656x^3 - 1.7884x^2 + 3.8632x - 0.8743$

### Figures

- **Figure 53**: D/d FEA Data – r/d=0.02
- **Figure 54**: D/d FEA Chart – r/d=0.02
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y(90) = 0.1865x^3 - 1.3106x^2 + 2.9826x - 0.3183

y(45) = 0.2182x^3 - 1.4639x^2 + 3.1573x - 0.5316

Figure 55: D/d FEA Data – r/d=0.03

Figure 56: D/d FEA Chart – r/d=0.03
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### Equations

- \( y(90) = 0.1437x^3 - 0.9958x^2 + 2.2445x - 0.0162 \)
- \( y(45) = 0.1474x^3 - 0.9862x^2 + 2.1151x + 0.0311 \)

### Figures

**Figure 57**: D/d FEA Data – r/d=0.05

**Figure 58**: D/d FEA Chart – r/d=0.05
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\[ y(90) = 0.0557x^3 - 0.4094x^2 + 0.9958x + 0.7033 \]

\[ y(45) = 0.1013x^3 - 0.6843x^2 + 1.4859x + 0.3426 \]

Figure 59: D/d FEA Data – r/d=0.07

Figure 60: D/d FEA Chart – r/d=0.07

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\[
\begin{align*}
\text{Poly. (90\degree, r/d=0.10)} & : y(90) = 0.0512x^2 - 0.3694x^2 + 0.8613x + 0.7121 \\
\text{Poly. (45\degree, r/d=0.10)} & : y(45) = -0.0998x^3 + 0.6372x^2 - 1.2947x + 2.1305
\end{align*}
\]

Figure 61: D/d FEA Data – r/d=0.10

Figure 62: D/d FEA Chart – r/d=0.10
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Figure 63: D/d FEA Data – r/d=0.15

\[
y(90) = 0.0271x^3 - 0.1783x^2 + 0.3905x + 0.9511 \\
y(45) = 0.0231x^3 - 0.1632x^2 + 0.3758x + 0.9925
\]

Figure 64: D/d FEA Chart – r/d=0.15
Figure 65: D/d FEA Data – r/d=0.20

Figure 66: D/d FEA Chart – r/d=0.20
Figure 67: D/d FEA Data – r/d=0.30

Figure 68: D/d FEA Chart – r/d=0.30
APPENDIX B:

NATION FEA RESULTS COMPARED TO PETERSON’S $K_t$ CHARTS

The below charts compare the data analyzed for this thesis and compares it to the current industry data. Figure 67 shows all of the data summarized and Figures 68 through 72 show the individual comparisons between the data generated for this thesis and Chart 3.12 from *Peterson’s Stress Concentration Factors* to verify the new results [1].
Figure 69: Stress Concentration Factors, $K_t$, With Data from Nation and Pilkey
Figure 70: $D/d=1.111$ $K_t$ Data - Pilkey Compared to Nation

Figure 71: $D/d=1.25$ $K_t$ Data - Pilkey Compared to Nation
Figure 72: $D/d=1.666$ $K_t$ Data - Pilkey Compared to Nation

Figure 73: $D/d=2.00$ $K_t$ Data - Pilkey Compared to Nation
Figure 74: D/d=2.5 $K_t$ Data - Pilkey Compared to Nation