SEMICONDUCTOR OPTICAL AMPLIFIER AS A PHASE MODULATOR FOR SHORT-PULSE SYNTHETIC

APERTURE LADAR AND VIBROMETRY

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SEMICONDUCTOR OPTICAL AMPLIFIER AS A PHASE MODULATOR FOR SHORT-PULSE SYNTHETIC APERTURE LADAR AND VIBROMETRY

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SEMICONDUCTOR OPTICAL AMPLIFIER AS A PHASE MODULATOR FOR SHORT-PULSE SYNTHETIC 
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The use of a saturated semiconductor optical amplifier (SOA) as both a phase modulator and an 
amplifier for long range laser radar applications is explored. As will be discussed, this concept 
could reduce the hardware necessary to transmit high bandwidth pulses and allow for the 
transmission of shorter pulses that are less sensitive to the detrimental effects of target motion. 
After reviewing the concepts governing ranging, vibrometry, and synthetic aperture ladar, the 
nature of the phase and amplitude modulation from saturating an amplifier with a high peak 
power Gaussian pulse is explored. The key SOA parameters affecting the modulation of the 
output pulse are addressed and optimized, and their impact on the ideal pulse response of a 
laser radar system is explored. Proof of concept laboratory demonstrations using phase 
modulated pulses to interrogate stationary, vibrating, and translating targets are also presented. 
The concept of using a saturated SOA to enable short pulse synthetic aperture ladar and 
vibrometry is also explored. This research will show that the range resolution of a ladar system
can be optimized by saturating a SOA with a carrier lifetime that is one half the FWHM Gaussian input pulse duration, yielding a substantial improvement in range resolution that is highly insensitive to variations in the input pulse duration and energy.
Dedicated to my husband, my parents, and my brother
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The views expressed in this dissertation are those of the author and do not reflect on the official policy of the Air Force, Department of Defense, or the United States government.
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CHAPTER 1

INTRODUCTION AND BACKGROUND

1.1. Radar Background

The word “radar” is an acronym for Radio Detection And Ranging. As the acronym implies, the primary purpose of a radar system is to detect the presence of an object and provide additional information about that object, such as its location, size, and any other information that may assist in the object’s identification. Radar systems operate by transmitting an electro-magnetic signal, generally at a radio frequency, and detecting the portion of that signal that is returned by the target. An antenna is used to transmit and receive signals. The system must be designed so that the return signal can be distinguished from any noise sources, including those inherent to the system and those resulting from the system’s operational environment.

Antennas transmit signals with an associated beamwidth. By scanning the transmitted beam across an area, radar systems can also be used to generate an image. The resolution of such an image is determined by the beamwidth of the transmitted signal, which is generally approximated as $\lambda/D$, where $\lambda$ is the transmitted wavelength, and D is the effective aperture diameter of the antenna [1]. The resolution in this dimension is often referred to as the cross-range resolution, since it is perpendicular to the range to target. The cross-range resolution of the image can be improved by decreasing the beamwidth of the antenna, which is generally...
achieved by increasing the antenna’s size. As such, there are limits on the cross-range resolution that can be obtained because there are practical limits on how large the antenna can be made.

In the early 1950’s, an alternative approach to improving cross-range resolution, known as Synthetic Aperture Radar (SAR), was proposed by Carl Wiley of the Goodyear Aircraft Corporation [2]. As is implied by the name, SAR is achieved through the synthesis of a larger aperture by transmitting radar signals at various locations. The most practical way of doing this is by mounting the radar on a moving platform, such as an aircraft. The motion of the aircraft then translates the beam across the ground. As depicted in Figure 1.1, the transmitted beam can either move across the ground in a mode known as stripmap SAR, or it can be continually steered to one area in a mode known as spotlight SAR.

![Figure 1.1. Beam positions for stripmap and spotlight SAR modes](image)

SAR processing is based on the Doppler effect. As the radar moves across the area of interest, or as the target moves across the field of view of the radar, the Doppler frequency of the return signal from the target changes. The measured Doppler frequencies are resolved to yield the cross-range location of targets within the beamwidth of the radar. It is important to
note that SAR improves the resolution only in the direction of motion. It does not improve resolution in any other dimension. As such, traditional SAR techniques combined with ranging techniques can be used to create a two dimensional image in range and cross-range.

1.2. Laser Radar Background

As noted above, one way to improve the cross-range resolution of a radar system is to increase the diameter of the transmit antenna. Another way to improve resolution is to decrease the wavelength of the system. Optical wavelengths are on the order of microns and are much shorter than radio wavelengths, which are on the order of meters. This can drastically improve the resolution of the system. The earliest optical radar systems, developed in the late 1960’s and early 1970’s, were referred to as LIDAR systems, which is an acronym for Light Detection and Ranging [3,4,5]. Although it was commonly understood that lasers were used as the source for these systems, some incoherent light ranging techniques using flash lamps had been investigated around the same time period, so the acronym LADAR (Laser Detection And Ranging) was adopted to distinguish between the two approaches [6]. A notional laser radar system is depicted in Figure 1.2.

![Figure 1.2. Notional laser radar system](image-url)
Laser radar systems can employ direct detection techniques, where only the return signal power is measured, or coherent detection techniques, where the return signal power and phase are measured. Coherent laser radar systems offer multiple functions, including ranging, vibrometry, and synthetic aperture imaging. As will be discussed in Chapter 2, ranging applications often involve the transmission of a pulse of light, where the range to an object can be deduced from the round trip travel time of the pulse. Laser vibration sensing, or vibrometry, has been demonstrated as a way to measure the vibrational frequency of an object by monitoring phase changes to determine displacements on the order of the wavelength of light [7]. Since a vibrating target will induce a Doppler shift on the transmitted pulse, it is often referred to as micro-Doppler to distinguish it from other sources of Doppler. The change in phase induced by the vibrating target on each pulse can be measured and tracked to obtain the target’s vibration signature. This modality will also be discussed in Chapter 2.

When the same Doppler principles that are applied to SAR systems are applied to laser radar systems, it is referred to as SAL (Synthetic Aperture Ladar). The earliest SAL experiments took place in the 1970’s when SAR principles were applied to a moving target illuminated with a laser [8,9]. This type of setup is known as Inverse SAL, or ISAL. Laboratory experimentation continued through the 1980’s and many feasibility studies concerning the implementation of SAL systems were completed [10,11,12]. However, for many years hardware limitations prevented a long range demonstration of the technology. Specifically, it was difficult to design sources with both sufficient phase stability and high power for long range operation of the technology. In the 1990’s, long range ground demonstrations were carried out [13,14], and laboratory demonstrations of a translated aperture were carried out in the mid 2000’s [15]. SAL theory will be presented in detail in Chapter 3.
In each of these coherent laser radar modalities, it is advantageous to resolve objects that are closely separated in range. As will be discussed in Chapter 2, this is achieved by increasing the bandwidth of the transmitted pulse, which improves the range resolution of the system. The pulse bandwidth is often increased through the application of linear frequency modulation (LFM), allowing for the transmission of longer pulse durations (on the order of tens of microseconds in duration), and consequently higher pulse energies [7]. However, if the object to be imaged is in motion, it will induce an undesirable Doppler phase shift on the transmitted pulse, which will interfere with the necessary phase measurements for ranging, vibrometry, and synthetic aperture ladar (SAL) [16,17]. As such, it is advantageous to reduce the pulse duration, thereby reducing the amount of time the pulse is in contact with the object of interest, minimizing the detrimental effects of target motion. However, since it becomes more difficult to obtain high pulse energies as the duration of the pulse is decreased, it becomes necessary to use an optical amplifier to increase the pulse energy before it is transmitted [18].

1.3. Optical Amplifiers

As illustrated in Figure 1.2, an optical amplifier is often used to increase the transmitted power of the system and alleviate the power requirements of the laser. One potential optical amplifier is the Semiconductor Optical Amplifier (SOA). Research involving the SOA dates back to the invention of the semiconductor laser in the 1960’s [19]. The SOA became an important component in pulsed optical fiber communications systems in the 1980’s. Although alternative optical amplifiers exhibiting higher gain have been developed, such as the Erbium Doped Fiber Amplifier (EDFA), SOA’s remain an attractive option because of their small size and relative low cost. The architecture of a SOA is virtually identical to that of a semiconductor laser, but with antireflection-coated facets [19]. The application of an injection current provides the carriers necessary for the stimulated emission of photons, which results in gain. Like any gain medium,
the SOA has a specified saturation energy at which the gain experienced by the input pulse has decreased to half its original value. Although operating a SOA in the saturated regime may be required to obtain the desired output power, it is not without consequence. A saturated SOA exhibits the potentially adverse affect of temporal broadening of the optical pulse width [20]. Furthermore, the change in carrier density as the SOA is saturated results in a change in index of refraction experienced by the pulse, which leads to a phase modulation [20]. This effect has also been observed in semiconductor lasers where the drive current is modulated to directly pulse the laser [21,22,23,24].

In general, unwanted phase modulation is detrimental to the operation of an optical system, and steps are often taken to correct the amplifier phase modulation [25,26]. The phase modulation can also be avoided altogether by limiting the input power to avoid saturation, choosing SOA characteristics that limit the saturation, or choosing a different type of amplifier (such as an EDFA) that exhibits less phase modulation [27,28]. Furthermore, as will be discussed in Chapter 2, temporal broadening of the optical pulse width is undesirable because it degrades the range resolution of the pulse. However, because the phase modulation is deterministic and has been modeled [20], this research will investigate the exploitation of the SOA self-phase modulation as bandwidth, thereby allowing the saturated SOA to serve as both an optical amplifier and a phase modulator. Although the use of a SOA as a phase modulator has been investigated for fiber optics communications systems where the injected current was modulated to obtain a desired phase modulation [29,30], and similar techniques have been applied to semiconductor lasers to obtain a desired frequency modulation [31], we investigate the inherent phase modulation of a SOA for a constant injection current as it applies to laser radar systems with pulse durations on the order of nanoseconds. As will be shown, through saturating a SOA and monitoring the transmitted pulse, the self-phase modulation can be
exploited as extra bandwidth to improve the range resolution of the transmitted pulse in spite of the saturation-induced temporal broadening. Such a system would not only be less sensitive to target motion, but also have a less complex system architecture by eliminating the hardware needed to implement LFM.

1.4. Short-Pulse Synthetic Aperture Ladar and Vibrometry

Since this research will explore the use of a SOA to amplify and modulate the transmitted pulse, it will allow for the transmission of shorter pulse durations, which could enable a multi-function short-pulse SAL and vibrometry system. The shorter pulses allow the system to be less sensitive to unanticipated target motion, while the phase modulation from the SOA provides the necessary bandwidth for enhanced range resolution. As discussed above, a SAL system can operate in a spotlight or stripmap mode. In the spotlight mode of operation, the beam is continuously steered to one spot on the ground. As will be shown in Chapter 3, when the ladar is in motion, the angular changes between the object and the ladar lead to changes in the Doppler frequency experienced by the signal, and the cross-range resolution improves as the illumination time of the object increases. The spotlight mode of operation can then achieve the best cross-range resolution since the object is continuously illuminated, but the image area is limited to the size of the beam. In the stripmap mode of operation, the image area is larger, but the object will remain in the beam for a shorter period of time. The cross-range resolution is not as fine as what can be achieved in the spotlight mode, but the image area is much larger. This research will discuss the design of a hybrid mode SAL system, as depicted in Figure 1.3. This mode is a compromise between the spotlight and stripmap modes, as the cross-range resolution will be better than what can be achieved in the stripmap mode, but the image area is larger than what can be achieved in the spotlight mode. Similar hybrid approaches have been developed
for radar systems [32,33]. As will be discussed in Chapter 4, the resolution in the range dimension can be achieved by exploiting the self-phase modulation of a SOA.

Furthermore, this concept will involve a linear array of detectors that will provide diffraction-limited imaging in the y-dimension, as depicted in Figure 1.4. Each detector will form a synthetic aperture in the x-dimension. Additional detectors will serve to increase the area of coverage of the system in the y-direction, keeping in mind that the transmitted power must increase as more detectors are added to the system. This research will present the design and link budget analysis of such a system, as well as a laboratory demonstration of the transmission of pulses with a phase modulation characteristic of the self-phase modulation of a SOA.

Figure 1.3. Depiction of the spotlight, stripmap, and hybrid modes of SAL operation
Since this approach requires coherent detection, it also lends itself to the additional modality of vibration sensing. However, the system must dwell on the target for the full duration of the measurement. As such, the acquisition of vibration data would require a spotlight mode of operation rather than a hybrid or stripmap mode of operation. This research will also explore the requirements and design parameters that would allow the system to have a vibration sensing mode of operation.

1.5. Summary

This dissertation will consist of 7 Chapters, with this introduction and background serving as the first chapter. Chapter 2 will present the general theory of laser radar systems, including vibrometry, and Chapter 3 will discuss synthetic aperture ladar theory. In Chapter 4, the theory and operation of SOA’s will be summarized. The nature of the self-phase modulation
from saturating a SOA with a high peak power Gaussian pulse and the optimal conditions for exploiting this phase modulation as bandwidth will also be analyzed. It will be shown that the range resolution is optimized by choosing a SOA with a carrier lifetime that is one half the input pulse duration, and by maximizing the SOA’s small signal gain and chirp parameter. Furthermore, it will be shown that an input energy one tenth of the SOA saturation energy will provide for significant phase modulation while also allowing the SOA to provide some gain. The simulations in Chapter 4 predict that the range resolution can be improved by a factor of 7.5 when the self-phase modulation from the SOA is exploited as additional bandwidth. Chapter 5 will discuss the operation and point design of a short pulse SAL and vibration sensing system using a SOA as a phase modulator. Chapter 6 will present the laboratory experimental results, which consist of ranging, SAL and vibrometry experiments using waveforms with a phase modulation consistent with the self-phase modulation of a SOA. By exploiting the phase modulation, a range resolution improvement of 7.1 is demonstrated in the laboratory. The conclusions of this work will then be summarized in Chapter 7.
2.1. Introduction

Radar systems can be used to determine the distance to a target, the velocity at which the target is traveling, and to create an image of the target. Using a pulsed radar system, the distance to the target can be determined from the time it takes the pulse to travel to the target and back as

\[ R = \frac{ct_{rt}}{2}, \quad (2.1) \]

where \( c \) is the speed of light and \( t_{rt} \) is the round trip travel time of the pulse. An image of the target can be created by scanning the beam across the target and recording the received power at each location. An image created in this manner will have a cross-range resolution determined by the spot size of the beam, which is limited by diffraction such that

\[ \Delta r_{dl} = \frac{\lambda R}{D_{ra}}, \quad (2.2) \]

where \( \Delta r_{dl} \) is the diffraction limited cross-range resolution, \( \lambda \) is the wavelength, and \( D_{ra} \) is the diameter of the receive aperture. Additional information about the target can be gained if the phase of the return signal is measured. Any relative motion between the target and the ladar platform will induce a Doppler shift upon interaction with the transmitted wave. As will be discussed below, the translational or vibrational velocity of the target can be measured by the
change in phase of the received signal as compared to the transmitted signal. First, the received signal is explored in detail, followed by a discussion of the measurement techniques and associated hardware that are required to measure the phase of the return signal.

2.2. Laser Radar Signal Model

Most laser radar signals can be described as narrow bandpass signals with an associated center frequency and bandwidth. The narrow bandpass signal can be represented using many different notations, beginning with the expression

\[ s(t) = g(t) \cos[\omega_c t + \phi(t)], \]  

(2.3)

where \( s(t) \) is the laser radar signal, \( g(t) \) is the envelope of \( s(t) \), \( \omega_c \) is the center frequency, and \( \phi(t) \) is the instantaneous phase, which could be constant or it could be used to represent a phase modulated signal. Using the trigonometric identity

\[ \cos(A + B) = \cos A \cos B - \sin A \sin B, \]  

(2.4)

the bandpass signal can also be expressed in the canonical form as

\[ s(t) = g_c(t) \cos(\omega_c t) - g_s(t) \sin(\omega_c t), \]  

(2.5)

where \( g_c(t) \) and \( g_s(t) \) are the in-phase and quadrature components, expressed as

\[ g_c(t) = g(t) \cos(\phi(t)) \quad \text{and} \quad g_s(t) = g(t) \sin(\phi(t)). \]  

(2.6)

The significance of this representation will be explored in Section 2.5.2. The complex envelope or phasor of the signal \( s(t) \) is defined as

\[ u(t) = g_c(t) + j g_s(t) = g(t) \exp(j\phi(t)), \]  

(2.7)

which yields a third signal notation of
\[ s(t) = \text{Re}\{u(t)\exp(j\omega_c t)\}. \]  

(2.8)

This exponential form of the signal is often easier to manipulate mathematically than the trigonometric form, and will be used often in the following sections.

### 2.3. The Received Signal

As discussed above, the transmitted signal can be expressed in exponential form as

\[ s_t(t) = u_t(t)\exp(j\omega_c t + \theta(t)), \]  

(2.9)

where \( u_t(t) \) is the transmit pulse envelope, \( \omega_c \) is the carrier frequency, and \( \theta(t) \) represents any phase modulation of the transmitted pulse. If the transmitted signal interacts with a target, the return signal will be delayed by the roundtrip time of the signal found in Equation (2.1). The range to target at a given time \( t \) can be expressed as

\[ R(t) = R_0 + |v_{los}|t, \]  

(2.10)

where \( v_{los} \) line of sight velocity between the target and the transmitter, and \( R_0 \) is the range to target at the initial position of the transmitter. The return signal can then be expressed as

\[
\begin{align*}
    s_r(t) &= u_t(t - t_{rt})\exp[j\omega_c(t - t_{rt}) + \theta(t - t_{rt})] \\
    &= u_t(t - t_{rt})\exp\left[j\omega_c\left(t - \frac{2R(t)}{c}\right) + \theta(t - t_{rt})\right] \\
    &= u_t(t - t_{rt})\exp\left[j\omega_c\left(t - \frac{2R_0 + v_{los}t}{c}\right) + \theta(t - t_{rt})\right] \\
    &= u(t - t_{rt})\exp(j\omega_c t + \theta(t - t_{rt}))( -j2\omega_c \frac{R_0 + v_{los}t}{c}).
\end{align*}
\]  

(2.11)
The second exponential factor represents the change in phase due to the interaction of the transmitted wave with the moving target.

2.4. The Doppler and Micro-Doppler Effects

As will be shown, the final exponential factor in Equation (2.11) represents the Doppler effect. If the target or the ladar system is in motion, the transmitted wave will either be compressed or expanded, resulting in a phase shift of the received signal. This is referred to as a Doppler shift. Although the line of sight velocity, \( v_{los} \), is used to refer to target motion in the derivation above, it can also account for motion of the radar system, or any relative motion between the two.

The Doppler frequency is found by taking the derivative of the Doppler phase factor to yield

\[
 f_D(t) = \frac{1}{2\pi} \frac{d\phi(t)}{dt} = -\frac{2f_c}{c} \frac{d}{dt}(R_0 + |v_{los}|t) = -\frac{2|v_{los}|}{\lambda},
\]

where \( f_c \) is the carrier frequency in hertz. For a platform flying over a target, the line of sight velocity is the component of the velocity in the direction of the ladar motion. As depicted in Figure 2.1, a platform travelling with velocity \( v \) will have a line of sight velocity

\[
 |v_{los}| = |v \cos \theta_A|,
\]

where \( \theta_A \) is the angle between the target and the direction of platform motion. If the platform is directly over the target, the line of sight velocity is zero because \( \theta_A = \pi/2 \). By convention, the time at which the platform is directly above the target is defined as \( t = 0 \). As the platform flies towards the target, where \( t < 0 \), the Doppler shift is positive and the received wave is compressed with respect to the transmitted wave. As the platform flies away from the target,
where $t > 0$, the Doppler shift is negative and the received wave is stretched with respect to the transmitted wave.

If the target is vibrating, the range to target defined in Equation (2.10) must be modified. Assuming the target vibration is sinusoidal in nature, the displacement of the target can be expressed as

$$z_v(t) = a_{Mv} \sin(\omega_v t),$$  \hspace{1cm} \text{(2.14)}

where $z_v(t)$ is the displacement of the target as a function of time, $a_{Mv}$ is the maximum vibrational displacement of the target, and $\omega_v$ is the frequency at which the target is vibrating. The range to target then becomes

$$R(t) = R_0 - v_{los} t + a_{Mv} \sin(\omega_v t).$$  \hspace{1cm} \text{(2.15)}

Since it is smaller than the Doppler shift associated with translational motion, the Doppler shift due to target vibration is often referred to as micro-Doppler. The expression for the return signal in Equation (2.11) then becomes
\[ s_r(t) = u_t(t - t_{rt}) \exp(j\omega_c t + \theta(t - t_{rt})) \]
\[ \times \exp\left(-j2\omega_c \frac{R_0 - v_{lost} t + a_{MV} \sin(\omega_v t)}{c}\right) \]
\[ = u_t(t - t_{rt}) \exp(j\omega_c t + \theta(t - t_{rt})) \exp(-j\phi_t(t)) \exp(-j\phi_v(t)), \]

where \( \phi_t(t) \) is the phase accumulation associated with translation, or
\[ \phi_t(t) = 2\omega_c \frac{R_0 - v_{lost} t}{c}, \]
and \( \phi_v(t) \) is the phase accumulation associated with vibration, or
\[ \phi_v(t) = 2\omega_c \frac{a_{MV} \sin(\omega_v t)}{c}. \]

Methods of detecting the return signal will be discussed in the following section.

2.5. Detection of the Laser Radar Signal

The next step is determining how to detect the received signal in such a way that makes use of the Doppler and micro-Doppler effects to gain information about the target. Laser radar systems can employ several different detection techniques, including direct detection and coherent detection schemes. Direct detection, also known as incoherent detection, is used when the range to target is determined by measuring the received power. Since optical frequencies are too high to be measured directly, the incoherent receiver will detect only the received power. Through the introduction of a master oscillator, coherent detection is a technique used to directly measure a difference frequency. While direct detection requires a much less complicated architecture, coherent detection must be used for any application that requires a phase measurement, including vibration measurements, Doppler measurements, and, as will be discussed in Chapter 3, SAL measurements. As will be shown, coherent detection has other benefits including amplification of the return signal by the master oscillator. However, instabilities in the master oscillator will introduce additional noise to the system, which will be
addressed in Section 2.7. The following sections will explore conventional coherent detection techniques, including a discussion of a technique known as synchronous detection. Finally, the matched filter will be discussed as a way to determine if the received signal is in fact a valid detection.

### 2.5.1. Detection of a Signal

The phasor notation of Poynting’s theorem states that the time averaged power density, assuming free space propagation, is [43]

\[
S = \frac{1}{2} \text{Re}\{E \times H^*\} = \frac{|E|^2}{\eta_0},
\]

(2.19)

where \(S\) is the power density with units [W \cdot m^{-2}], \(E\) is the electric field, \(H^*\) is the conjugate of the magnetic field, and \(\eta_0\) is the intrinsic impedance of free space. The instantaneous received signal power is then

\[
P(t) = \left\langle |E|^2 \right\rangle \frac{1}{\eta_0} A_d,
\]

(2.20)

where \(A_d\) is the area of the detector and the \(\langle \quad \rangle\) brackets are used to denote that the electric field is time averaged by the detector. It follows that the instantaneous signal current is expressed as

\[
i(t) = \rho_i G_d P(t) = \rho_i G_d \frac{\left\langle |E|^2 \right\rangle}{\eta_0} A_d,
\]

(2.21)

where \(\rho_i\) is the responsivity of the detector and \(G_d\) is the internal gain of the detector. This expression describes the measured current for any signal incident on a detector. The following sections will explore specific detection schemes.
2.5.2. Conventional Coherent Detection

The optical setup for coherent detection is depicted in Figure 2.2. The received signal can be expressed as

\[ s_r(t) = u_t(t - t_{rt}) \cos (\omega_c t + \phi_s(t) + \phi_n(t)) \]  \hspace{1cm} (2.22)

where \( u_t(t - t_{rt}) \) is the amplitude of the return signal, \( \omega_c \) is the carrier frequency, \( \phi_s(t) \) is the signal phase, and \( \phi_n(t) \) represents any phase noise affecting the received signal. The master oscillator signal is expressed as

\[ s_{MO}(t) = u_{MO} \cos (\omega_{MO} t + \phi_{MO}(t)), \]  \hspace{1cm} (2.23)

where \( u_{MO} \) is the amplitude of the CW master oscillator signal, \( \omega_{MO} \) is the frequency of the master oscillator and \( \phi_{MO}(t) \) is any phase noise associated with the master oscillator. A beamsplitter is used to combine the received signal and the master oscillator, resulting in the signal

\[ s(t) = u_t(t - t_{rt}) \cos (\omega_c t + \phi_s(t) + \phi_n(t)) + u_{MO} \cos (\omega_{MO} t + \phi_{MO}(t)), \]  \hspace{1cm} (2.24)

When this signal is detected, the square law nature of the detector results in a detected signal expressed as
\[ s_d(t) = \left\langle \left[ u(t - t_{\tau t}) \cos(\omega_c t + \phi_s(t) + \phi_n(t)) + u_{MO} \cos(\omega_{MO} t + \phi_{MO}(t)) \right]^2 \right\rangle, \]  

(2.25)

where the \( \langle \rangle \) brackets again are used to indicate that the signal is time averaged by the detector.

Using trigonometric identities, this expression can be written as

\[ s_d(t) = \left\langle \left[ u_{MO}^2 \cos^2(\omega_{MO} t + \phi_{MO}(t)) + u_t^2(t - t_{\tau t})\cos^2(\omega_c t + \phi_s(t) + \phi_n(t)) \right] \right\rangle \]

\[ + \left\langle 2u_t(t - t_{\tau t})u_{MO} \cos(\omega_{MO} t + \phi_{MO}(t)) \cos(\omega_c + \phi_s(t) + \phi_n(t)) \right\rangle \]

\[ = \left\langle \left[ u_{MO}^2 \cos^2(\omega_{MO} t + \phi_{MO}(t)) + u_t^2(t - t_{\tau t})\cos^2(\omega_c t + \phi_s(t) + \phi_n(t)) \right] \right\rangle \]

\[ + \left\langle u_t(t - t_{\tau t})u_{MO} \cos((\omega_c - \omega_{MO}) t + \phi_s(t) + \phi_n(t) - \phi_{MO}(t)) \right\rangle \]

\[ + \left\langle u_t(t - t_{\tau t})u_{MO}\cos((\omega_c + \omega_{MO}) t + \phi_s(t) + \phi_n(t) + \phi_{MO}(t)) \right\rangle \]. \]  

(2.26)

An optical wavelength of 1.5\( \mu \text{m} \) corresponds to a frequency of \( 2 \times 10^{14} \) Hz. Clearly, a frequency of this magnitude cannot be directly detected due to the insufficient speed of detector technology. The first two terms of Equation (2.26) are cosine squared functions and therefore will time average to half their peak magnitude. Likewise, the final term will time average to zero. If the frequency of the master oscillator is chosen correctly, the third term representing the difference frequency can be directly measured by the detector. The resulting time averaged signal is then expressed as

\[ s_d(t) = \frac{1}{2} u_{MO}^2 + \frac{1}{2} u_t^2(t - t_{\tau t}) \]

\[ + u_t(t - t_{\tau t})u_{MO}\cos((\omega_c - \omega_{MO}) t + \phi_s(t) + \phi_n(t) - \phi_{MO}(t)). \]  

(2.27)

If \( \omega_c=\omega_{MO} \) and the return signal is mixed to baseband, which is referred to as homodyning, the received signal is expressed as

\[ s_d(t) = \frac{1}{2} u_{MO}^2 + \frac{1}{2} u_t^2(t - t_{\tau t}) + u_t(t - t_{\tau t})u_{MO}\cos(\phi_s(t) + \phi_n(t) - \phi_{MO}(t)). \]  

(2.28)
If the amplitude of the MO is much larger than that of the received signal, i.e. \( u_{MO} \gg u_r(t - t_{rt}) \), and if AC coupling is used so that the DC term is subtracted from Equation (2.28), the received signal can be expressed as

\[
s_d(t) = u_r(t - t_{rt}) u_{MO} \cos \left( \phi_s(t) + \phi_n(t) - \phi_{MO}(t) \right)
\]  

(2.29)

Finally, using Equation (2.21), the received signal current is expressed as

\[
i(t) = \frac{\rho_i G_d A_d}{\eta_0} u_r(t - t_{rt}) u_{MO} \cos \left( \phi_s(t) + \phi_n(t) - \phi_{MO}(t) \right).
\]  

(2.30)

2.5.3. Synchronous Detection

One limitation of the detection scheme discussed above is it cannot distinguish between positive and negative Doppler shifts induced by the target or platform motion, since by definition \( \cos(A) = \cos(-A) \). As discussed in Section 2.2, the return signal can be expressed in terms of its in-phase and quadrature components. This can be achieved by using a detector that resolves the signal into its In-phase (I) and Quadrature (Q) components, known as an I/Q detector or synchronous detector. Such a detector is shown in Figure 2.3.

![Figure 2.3. Setup for optical synchronous detection](image)

After passing through the half-wave plate, the received signal is polarized at 45 degrees and can be expressed as
\[ s_r(t) = u_t(t - t_{rt})\exp(j\omega_t t + j\theta(t - t_{rt}))\exp\left( j\left( \phi_s(t) + \phi_n(t) \right) \right) \]
\[
\cdot \left[ \frac{1}{\sqrt{2}} \hat{x} + \frac{1}{\sqrt{2}} \hat{y} \right],
\]

where \( \hat{x} \) and \( \hat{y} \) represent the x and y polarizations, \( \phi_s(t) \) represents the signal phase, and the \( \phi_n(t) \) term has been included to represent any phase noise affecting only the received signal.

The CW master oscillator, after passing through the quarter waveplate, is circularly polarized and can be expressed as

\[ s_{MO}(t) = u_{MO}\exp\left( j\omega_{c} t + j\phi_{MO}(t) \right) \left[ \frac{1}{\sqrt{2}} \hat{x} + \frac{1}{\sqrt{2}} \exp\left( j\frac{\pi}{2} \right) \hat{y} \right], \]

where \( u_{MO} \) is the constant amplitude of the master oscillator, \( \phi_{MO}(t) \) represents any phase noise associated with the MO, and the MO is assumed to be at the same frequency as the transmitted signal. The received signal and the MO are first combined by a beamsplitter, followed by a polarizing beamsplitter that directs the x-polarization to one detector and the y-polarization to a second detector. At detectors D₁ and D₂, the two signals are expressed as

\[ s_{Ir}(t) = \langle |s_r(t)\hat{x} + s_{MO}(t)\hat{x}|^2 \rangle 
\]
\[
= \left\langle \left( \frac{1}{\sqrt{2}} u_t(t - t_{rt})\exp(j\theta(t - t_{rt}))\exp(j\omega_{c} t)\exp\left( j\phi_s(t) + j\phi_n(t) \right) \right)^2 \right\rangle \hat{x}
\]

and

\[ s_{Qu}(t) = \langle |s_r(t)\hat{y} + s_{MO}(t)\hat{y}|^2 \rangle 
\]
\[
= \left\langle \left( \frac{1}{\sqrt{2}} u_t(t - t_{rt})\exp(j\theta(t - t_{rt}))\exp(j\omega_{c} t)\exp\left( j\phi_s(t) + j\phi_n(t) \right) \right)^2 \right\rangle \hat{y},
\]

where \( s_{Ir}(t) \) and \( s_{Qu}(t) \) represent the in-phase and quadrature components of the received signal.

Using the identity
\[ |p(t)|^2 = p(t)p^*(t), \]  
\[ (2.35) \]

Equation (2.33) can be written as

\[ s_{tr}(t) = \langle |s_r(t)\hat{x} + s_{MO}(t)\hat{x}|^2 \rangle \]
\[ = \frac{1}{\sqrt{2}} \left( \left( u_t(t - t_{rt}) \exp(j\omega_c t + j\theta(t - t_{rt}) + j\phi_s(t) + j\phi_n(t)) + u_{MO} \exp(j\omega_c t + j\phi_{MO}(t)) \right) \right. \]
\[ \left. \times \left( u_t(t - t_{rt}) \exp(-j\omega_c t - j\theta(t - t_{rt}) - j\phi_s(t) + j\phi_n(t)) + u_{MO} \exp(-j\omega_c t - j\phi_{MO}(t)) \right) \right) \]
\[ = \langle \left( \frac{1}{2} \right) \left( u_t(t - t_{rt}) + u_{MO}^2 \right. \right. \]
\[ \left. \left. + u_t(t - t_{rt})u_{MO} \exp\left( j\theta(t - t_{rt}) + j\phi_s(t) + j\phi_n(t) - j\phi_{MO}(t) \right) \right) \right) \]
\[ = \frac{u_t^2(t - t_{rt}) + u_{MO}^2}{2} \]
\[ + u_t(t - t_{rt})u_{MO} \cos\left( \theta(t - t_{rt}) + \phi_s(t) + \phi_n(t) - \phi_{MO}(t) \right). \]

The polarization vector has been dropped since this is insignificant once the signal has been detected, and the time average brackets have been dropped since the signal is now at baseband and can be directly measured by the detector. The same process can be followed for the quadrature component of the signal. Using Equation (2.35), Equation (2.34) becomes

\[ s_{qr}(t) = \langle |s_r(t)\hat{y} + s_{MO}(t)\hat{y}|^2 \rangle \]
\[ = \langle \left( \frac{1}{\sqrt{2}} \right) \left( u_t(t - t_{rt}) \exp\left( j\omega_c t + j\theta(t - t_{rt}) + j\phi_s(t) + j\phi_n(t) + \frac{j\pi}{2} \right) \right. \right. \]
\[ \left. \left. + u_{MO} \exp\left( j\omega_c t + j\phi_{MO}(t) + \frac{j\pi}{2} \right) \right) \right) \]
\[ \left. \times \left( u_t(t - t_{rt}) \exp\left( -j\omega_c t - j\theta(t - t_{rt}) - j\phi_s(t) - j\phi_n(t) + \frac{j\pi}{2} \right) \right) \right) \]
\[ = \langle \left( \frac{1}{2} \right) \left( u_t(t - t_{rt})^2 + u_{MO}^2 \right. \right. \]
\[ \left. \left. + u_t(t - t_{rt})u_{MO} \exp\left( j\theta(t - t_{rt}) + j\phi_s(t) + j\phi_n(t) - j\phi_{MO}(t) - \frac{j\pi}{2} \right) \right) \right) \]
\[ + u_t(t - t_{rt})u_{MO} \exp\left( -j\theta(t - t_{rt}) - j\phi_s(t) - j\phi_n(t) + j\phi_{MO}(t) + \frac{j\pi}{2} \right). \]
\[
\begin{align*}
\frac{u_t(t-t_{rt})^2}{2} + \frac{u_{MO}^2}{2} + u_t(t-t_{rt})u_{MO}\sin(\theta(t-t_{rt}) + \phi_s(t) + \phi_n(t) - \phi_{MO}(t)).
\end{align*}
\]

If the amplitude of the MO is much larger than that of the received signal such that \(u_{MO} \gg u_t(t-t_{rt})\) and AC coupling is used so that the DC terms are subtracted from Equations (2.37) and (2.38), the first two terms can be ignored. The received signal is then expressed in terms of its in-phase and quadrature components as

\[
s_r(t) = s_{Ir}(t) + js_{Qr}(t)
= u_t(t-t_{rt})u_{MO}\exp(j\theta(t-t_{rt})\exp(j\phi_s(t) + j\phi_n(t) - \phi_{MO}(t)).
\]

Using Equation (2.21), the received currents are found to be

\[
i_{Ir}(t) = \frac{\rho_lG_dA_d}{\eta_0}u_t(t-t_{rt})u_{MO}\cos(\theta(t-t_{rt}) + \phi_s(t) + \phi_n(t) - \phi_{MO}(t))
\]

and

\[
i_{Qr}(t) = \frac{\rho_lG_dA_d}{\eta_0}u_t(t-t_{rt})u_{MO}\sin(\theta(t-t_{rt}) + \phi_s(t) + \phi_n(t) - \phi_{MO}(t)).
\]

The two signals \(i_{Ir}(t)\) and \(i_{Qr}(t)\) can then be compared to determine if the Doppler shift was positive or negative. Since \(\cos(-x) = \cos(x)\), the I term will be the same if the Doppler shift is positive or negative. However, for the Q term, \(\sin(-x) = -\sin(x)\) so a negative Doppler shift will result in a negative Q term. Using the relationships

\[
\cos\left(\theta + \frac{\pi}{2}\right) = \cos\theta \cos\frac{\pi}{2} - \sin\theta \sin\frac{\pi}{2} = -\sin\theta
\]

and

\[
\cos\left(\theta - \frac{\pi}{2}\right) = \cos\theta \cos\frac{\pi}{2} + \sin\theta \sin\frac{\pi}{2} = \sin\theta,
\]

it is evident that for a negative Doppler shift, \(s_{Qr}(t)\) leads \(s_{Ir}(t)\) by \(\pi/2\), and for a positive Doppler shift, \(s_{Qr}(t)\) lags \(s_{Ir}(t)\) by \(\pi/2\). This is illustrated in Figure 2.4.
Figure 2.4. Illustration of I lagging Q (top) for a negative Doppler shift and I leading Q (bottom) for a positive Doppler shift.

In many cases a portion of the transmit signal is recorded to serve as a monitor, as depicted in Figure 2.5. Before the half waveplate, the transmit signal is expressed as

\[ s_t(t) = u_t(t) \exp\left(j \omega_c t + j \theta(t)\right) \left[ \frac{1}{\sqrt{2}} \hat{x} + \frac{1}{\sqrt{2}} \hat{y} \right]. \]  

(2.41)

Before the quarter waveplate, the MO is identical to Equation (2.32), or

\[ s_{MO}(t) = u_{MO} \exp\left(j \omega_c t + j \phi_{MO}(t)\right) \left[ \frac{1}{\sqrt{2}} \hat{x} + \frac{1}{\sqrt{2}} \exp\left(j \frac{\pi}{2}\right) \hat{y} \right]. \]  

(2.42)

By following the same process as Equations (2.33) through (2.38), the I and Q components of the monitor signal are found to be
\[ s_{Im}(t) = u_t(t)u_{MO}\cos(\theta(t) - \phi_{MO}(t)) \]

and

\[ s_{Qm}(t) = u_t(t)u_{MO}\sin(\theta(t) - \phi_{MO}(t)). \]

and the monitor signal can be expressed as

\[ s_m(t) = s_{Im}(t) + js_{Qm}(t) = u_t(t)u_{MO}\exp(j\theta(t) - j\phi_{MO}(t)). \] (2.44)

The role of the monitor signal will be discussed in the following section.

---

2.5.4. The Matched Filter

Once the laser radar signal has been received and demodulated, it can be compared to a known signal (i.e. the monitor signal) in a process referred to as matched filtering. In the frequency domain, the matched filter process can be expressed as

![Figure 2.5. Synchronous detection of the transmitted signal, \(s_t(t)\), which serves as a monitor, and the received signal, \(s_r(t)\).](image-url)
\[ Y(\omega) = H(\omega)S(\omega), \]  

where \( Y(\omega) \) is the spectrum of the output of the matched filter, \( S(\omega) \) is the spectrum of the signal at the input of the matched filter, and \( H(\omega) \) is the transfer function associated with the matched filter. We wish to find an \( H(\omega) \) that will maximize the system’s signal-to-noise ratio, SNR, at a particular time \( t_n \). The power of the signal at time \( t_n \) will be

\[ |s_o(t_n)|^2 = |\mathcal{S}^{-1}[H(\omega)S(\omega)]|^2 = \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)H(\omega)\exp(j\omega t_n) d\omega \right|^2. \]  

The noise is assumed to be Gaussian with a power spectral density of \( N_o/2 \) watts per hertz. The output noise power is then found to be

\[ n_p = \frac{N_o}{4\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega. \]  

The expression for SNR is then

\[ SNR_{MF} = \frac{|s_o(t)|^2}{n_p} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)H(\omega)\exp(j\omega t_n) d\omega \frac{N_o}{4\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega. \]  

The SNR can be maximized using the Schwarz inequality, which states that

\[ \left| \int_{-\infty}^{\infty} A(\omega)B(\omega) d\omega \right|^2 \leq \int_{-\infty}^{\infty} |A(\omega)|^2 d\omega \int_{-\infty}^{\infty} |B(\omega)|^2 d\omega, \]  

given \( B(\omega) = KA^*(\omega) \), where \( K \) is an arbitrary constant \([34]\). If \( A(\omega) = S(\omega)\exp(j\omega t_n) \) and \( B(\omega) = H(\omega) \), it follows from Equation (2.49) that

\[ \left| \int_{-\infty}^{\infty} S(\omega)H(\omega)\exp(j\omega t_n) d\omega \right|^2 \leq \int_{-\infty}^{\infty} |S(\omega)|^2 d\omega \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega. \]  

Substituting the right hand side of Equation (2.50) into the numerator of Equation (2.48) yields

\[ SNR_{MF} \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |S(\omega)|^2 d\omega \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega \frac{N_o}{4\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega = \frac{2}{N_o} \int_{-\infty}^{\infty} |S(\omega)|^2 d\omega. \]  

This is only true under the condition of Equation (2.49), or
\[ H(\omega) = KS^*(\omega)\exp(-j\omega t_n). \] (2.52)

Note that the maximum SNR can also be expressed as

\[ SNR_{MF} \leq \frac{2}{N_o} \int_{-\infty}^{\infty} |S(\omega)|^2 d\omega = \frac{2E}{N_o}, \] (2.53)

where \( E \) is the energy in the input signal, or

\[ E = \int_{-\infty}^{\infty} s^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} S^2(\omega) d\omega. \] (2.54)

The impulse response of the matched filter is then the inverse Fourier transform of Equation (2.52), or

\[ h(t) = KS^*(t - t_o), \] (2.55)

which means the impulse response of the matched filter is a delayed mirror image of the conjugate of the input signal. As shown in the previous section, the transmitted signal can be monitored and stored to serve as the matched filter. The matched filter output then becomes

\[ y(t) = \mathcal{F}^{-1}\{S_r(\omega)S_t^*(\omega)\} = \int_{-\infty}^{\infty} s_r(\tau)s_t^*(\tau - t)d\tau, \] (2.56)

where \( s_r(t) \) and \( s_t(t) \) are the received and transmitted signals, respectively, and \( S_r(\omega) \) and \( S_t(\omega) \) are the Fourier transforms of the received and transmitted signals, respectively, and the constant \( K \) has been ignored. So the matched filter is a correlation of the transmitted and received signals and therefore looks for signals similar to what was transmitted. This is helpful in determining if the received signal is a valid detection of an object, or if it was random noise, because a random noise signal will not correlate well with the known transmitted signal.

### 2.6. Range Processing

For a stationary target and a stationary platform (i.e. no translational or vibrational motion), the received signal found in Equation (2.38) can be expressed as
\[ s_r(t) = u_t(t - t_{rt})u_{MO}\exp(j\theta(t - t_{rt}))\exp(j\phi_s(t)), \]  
\hspace{1cm} (2.57) 

where \( u_t(t - t_{rt}) \) is the time delayed signal envelope, \( u_{MO} \) is the amplitude of the master oscillator, \( \theta(t - t_{rt}) \) is the time delayed signal phase modulation, and \( \phi_s(t) \) is the signal phase due to interaction with the target. Here, we have assumed no phase noise (i.e. \( \phi_n(t) = 0 \) and \( \phi_{MO}(t) = 0 \)), and the signal phase is found from Equations (2.17) and (2.18) to be

\[ \phi_s(t) = 2\omega_c \frac{R_0}{c}, \]  
\hspace{1cm} (2.58) 

since the line of sight velocity, \( v_{los} \), and the vibrational frequency, \( \omega_v \), are zero. The monitor signal was found in Equation (2.44) to be

\[ s_m(t) = u_t(t)u_{MO}\exp(j\theta(t)), \]  
\hspace{1cm} (2.59) 

where we have again assumed no phase noise. Range processing can be carried out by substituting Equations (2.57) and (2.59) into the matched filter expression found in Equation (2.56) to yield

\[ s_M(t) = u_{MO}^2\exp(-j\omega_c t_{rt})\mathcal{R}_s(t - t_{rt}), \]  
\hspace{1cm} (2.60) 

Note that the matched filter output contains a shifted version of the autocorrelation of the baseband transmitted signal, or

\[ \mathcal{R}_s(t - t_{rt}) = \int_{-\infty}^{\infty} u_t(\tau - t_{rt})\exp(j\theta(\tau - t_{rt})) \]
\[ \times u_t^*(\tau - t)\exp(-j\theta(\tau - t))d\tau. \]  
\hspace{1cm} (2.61) 

Since the autocorrelation function is centered at time \( t_{rt} \), the location of the autocorrelation in time yields the range of the object according the relationship of Equation (2.1) The shape and width of the autocorrelation depends on the envelope and phase modulation of the transmitted signal. The degree to which the returns from two targets can be resolved in range depends on the range resolution of the system, which will be discussed in Section 2.8.2.
2.7. Pulsed Vibrometry

In Section 2.4, the return signal for a vibrating target was derived. Vibrometry systems can be designed for a continuous wave (CW) or a pulsed source. Pulsed systems are generally desired for longer range applications due to their high peak power and their ability to measure range to target as well as vibration. However, pulsed systems require some additional considerations, as detailed below.

2.7.1. Pulsed Vibrometry Operation

When a laser vibrometer is operated in a pulsed mode, the vibration of the object of interest is sampled according to the PRF of the laser. In order to satisfy the Nyquist criteria, the PRF of the laser must be at least two times the maximum vibrational frequency of the target, or

\[ PRF > 2f_{v_{\text{max}}}, \]  

where \( f_{v_{\text{max}}} \) is the object’s maximum frequency of vibration. Furthermore, as found in Equation (2.16), the vibration of the target affects the phase of the returned signal. As such, the maximum phase change that can be tracked between pulses is \( 2\pi \). This means that the path length change between pulses cannot exceed one wavelength. Since the optical path length, OPL, experienced by the pulse is equal to twice the displacement of the target, it can be expressed as

\[ OPL = 2\Delta z_v = 2V_{avg} \cdot PRI, \]  

where \( V_{avg} \) is the average vibrational velocity of the vibrating target, and PRI is the pulse repetition interval, or the time between pulses. The maximum vibrational velocity that can be measured by the system, which is referred to as the ambiguous velocity, is found where \( OPL = \lambda \), or
\[ V_{amb} = \frac{\lambda}{2 \cdot PRI} = \frac{\lambda}{2} PRF, \]  

(2.64)

where the PRF is the pulse repetition frequency of the system. As such, the PRF must be high enough to measure the largest vibrational velocity of interest.

Another concern for a pulsed system is range ambiguities. If a second pulse is transmitted before the return from the first pulse is received, there will be ambiguity as to whether the measured return signal is from the first pulse, or from a closer object intercepted by the second pulse. To prevent range ambiguities altogether, the PRF of the system must satisfy the equation

\[ PRF < \frac{c}{2 R_{\text{max}}}, \]  

(2.65)

where \( R_{\text{max}} \) is the maximum range expected for any object in the field of view. However, if reasonable assumptions can be made about the range to any object of interest, the ambiguities can be resolved in post processing and this requirement can be relaxed.

2.7.2. Vibrometry Processing

Assuming there is no translational motion of the platform or the target, the received signal in Equation (2.38) for a vibrating target located at range \( R_p \) is expressed as

\[ s_r(t) = u_t(t - t_{rt})u_{MO}\exp\left(j\theta(t - t_{rt})\right)\exp\left(j\phi_s(t) + j\phi_n(t) - \phi_{MO}(t)\right), \]  

(2.66)

where the signal phase is found from Equation (2.16) to be

\[ \phi_s(t) = \left( -\frac{4\pi}{\lambda} R_p - \frac{4\pi}{\lambda} a_p \sin(\omega_p(t - t_{rt})) \right) \]  

(2.67)

when \( v_{los} = 0 \). Assuming the Doppler shift due to the vibration is approximately constant over the pulse duration, the phase due to vibration can be approximated as
\[ \phi_v(t) = \frac{4\pi}{\lambda} a_v \sin(\omega_v(t - t_{rt})) \approx \phi_v, \quad (2.68) \]

Assuming the MO phase noise and signal phase noise are slowly varying, they are also assumed to be constant over the pulse duration such that

\[ \phi_{MO}(t - t_{rt}) \approx \phi_{MO_{rt}} \quad \text{and} \quad \phi_n(t) \approx \phi_n, \quad (2.69) \]

and the received signal in Equation (2.66) becomes

\[
\begin{align*}
    s_r(t) &= u_t(t - t_{rt}) u_{MO} \exp(j\theta(t - t_{rt})) \\
    &\times \exp\left(-j\frac{4\pi}{\lambda} R_p\right) \exp(-j\phi_v) \exp(-j\phi_{MO_{rt}}) \exp(j\phi_n). \\
\end{align*}
\]

The monitor signal is found from Equation (2.44) to be

\[
\begin{align*}
    s_m(t) &= u_t(t) u_{MO} \exp(j\theta(t) - j\phi_{MO}), \\
\end{align*}
\]

where the MO phase noise is assumed to be slowly varying and is designated as

\[ \phi_{MO}(t) \approx \phi_{MO_t}. \quad (2.72) \]

Using Equation (2.56), the matched filter output is found to be

\[
\begin{align*}
    s_M(t) &= \int_{-\infty}^{\infty} s_r(\tau) s_r^*(\tau - t) d\tau \\
    &= \int_{-\infty}^{\infty} u_t(\tau - t_{rt}) u_{MO} \exp(j\theta(\tau - t_{rt})) \exp\left(-j\frac{4\pi}{\lambda} R_p\right) \exp(-j\phi_v) \exp(j\phi_n) \\
    &\times \exp(-j\phi_{MO_{rt}}) u_t^*(\tau - t) u_{MO} \exp(-j\theta(\tau - t)) \exp(j\phi_{MO_t}) \\
    &= \int_{-\infty}^{\infty} u_t(\tau - t_{rt}) \exp(j\theta(\tau - t_{rt})) u_t^*(\tau - t) \exp(-j\theta(\tau - t)) \\
    &\times u_{MO}^2 \exp(-j\phi_v) \exp\left(-j\frac{4\pi}{\lambda} R_p\right) \exp(-j(\phi_{MO_{rt}} - \phi_{MO_t})) \exp(j\phi_n) d\tau. \\
\end{align*}
\]

Note that the factor containing the phase noise is the difference in the phase noise at times \( t \) and \( t_{rt} \). The matched filter output can be simplified to

31
\[ s_M(t) = u_{MO}^2 \exp(-j\phi_n) \exp\left(-j\frac{4\pi}{\lambda} R_p\right) \exp\left(-j(\phi_{MOt_{rt}} - \phi_{MOt})\right) \exp(j\phi_n) \]
\[
\times \mathcal{R}_{s_t}(t - t_{rt}).
\] 

(2.74)

where the matched filter output contains a shifted version of the autocorrelation of the baseband transmitted signal, or

\[
\mathcal{R}_{s_t}(t - t_{rt}) = \int_{-\infty}^{\infty} u_t(\tau - t_{rt})\exp(j\theta(\tau - t_{rt})) \]
\[
\times u_t^*(\tau - t)\exp(-j\theta(\tau - t))d\tau.
\] 

(2.75)

Since the MO phase noise affects both the monitor and the signal, the matched filter processing reduces the effect of the MO phase to the difference in the phase noise at time \( t \) and time \( t_{rt} \), or \( \phi_{MOt_{rt}} - \phi_{MOt} \). The phase noise affecting only the signal is not reduced by the matched filter process. Assuming the signal phase noise is small and the MO phase noise is slowly varying, the phase of the matched filter output is dominated by the vibrational phase, which has a sinusoidal variation with a frequency equal to the vibrational frequency of the target. By interrogating the target with multiple pulses, the vibrational frequency of the target can be determined from the frequency at which the phase history varies, i.e. by taking a Fourier transform of the phase history. It should be noted that since the vibrational frequency is found through a Fourier transform, the frequency resolution of the system is governed by the time duration of the data set, or

\[
\Delta f_v = \frac{1}{T_{dwell}},
\] 

(2.76)

where \( T_{dwell} \) is the length of time the system dwells on one object collecting vibration data.

In cases where the signal phase noise is large, it is often advantageous to employ a modified matched filter technique that will be shown to reduce the impact of the signal phase noise. Consider the transmission of two monitored pulses. The monitor for the first pulse (\( s_{m1} \))
is recorded at time $t_1$, and the return from the first pulse ($s_{r1}$) is recorded at time $t_2$. The monitor for the second pulse ($s_{m2}$) is recorded at time $t_3$, and the return from the second pulse ($s_{r2}$) is received and recorded at time $t_4$. The two recorded monitor pulses can be expressed as

$$s_{m1}(t - t_1) = u_t(t - t_1)u_{MO}\exp\left(j\theta(t - t_1) - j\frac{4\pi}{\lambda}R_p\right)\exp(-j\phi_{MOT_1}) \quad (2.77)$$

and

$$s_{m2}(t - t_3) = u_t(t - t_3)u_{MO}\exp\left(j\theta(t - t_3) - j\frac{4\pi}{\lambda}R_p\right)\exp(-j\phi_{MOT_3}), \quad (2.78)$$

where the MO phase noise is assumed to be approximately constant over the pulse duration.

The first received pulse can be expressed as

$$s_{r1}(t - t_2) = u_t(t - t_2)u_{MO}\exp\left(j\theta(t - t_2) - j\frac{4\pi}{\lambda}R_p\right)$$

$$\times \exp(j\phi_{v1})\exp(j\phi_{nt_2})\exp(j\phi_{MOT_2}), \quad (2.79)$$

where the change in phase due to vibration over the pulse duration is assumed to be approximately constant, or

$$\phi_v(t - t_3) \approx \phi_{v1}, \quad (2.80)$$

and that the MO phase noise is also constant over the pulse duration and can be designated as

$$\phi_n(t - t_2) \approx \phi_{nt_2}. \quad (2.81)$$

The second received pulse can be expressed as

$$s_{r2}(t - t_4) = u_t(t - t_4)u_{MO}\exp\left(j\theta(t - t_4) - j\frac{4\pi}{\lambda}R_p\right)$$

$$\times \exp(j\phi_{v2})\exp(j\phi_{nt_4})\exp(j\phi_{MOT_4}), \quad (2.82)$$

where the change in phase due to vibration over the pulse duration is again assumed to be approximately constant, or

$$\phi_v(t - t_4) \approx \phi_{v2}, \quad (2.83)$$
and that the change in phase noise over the pulse duration is approximately constant, or

\[ \phi_n(t - t_4) \approx \phi_{nt_4}. \] (2.84)

A matched filter can be applied to two subsequent received pulses such that

\[
s_{M1}(t) = \int_{-\infty}^{\infty} s_{r1}(\tau) s_{r2}^*(\tau - t) d\tau
\]
\[
= \int_{-\infty}^{\infty} u_t(\tau - t_2) u_{MO} \exp \left( j \theta (\tau - t_2) - j \frac{4\pi}{\lambda} R_p \right) 
\]
\[
\times \exp(\jmath \phi_{\nu_1}) \exp(\jmath \phi_{\nu_2}) \exp(\jmath \phi_{\nu_4}) \exp(\jmath \phi_{\nu_5}) d\tau
\]
\[
\times \exp(\jmath \phi_{\nu_4}) \exp(\jmath \phi_{\nu_5}) d\tau
\]
\[
= \exp\left( -j(\phi_{nt_4} - \phi_{nt_2}) \right) \exp\left( -j(\phi_{\nu_4} - \phi_{\nu_5}) \right) \exp(\jmath [\phi_{\nu_1} - \phi_{\nu_2}])
\]
\[
\times \int_{-\infty}^{\infty} u_t(\tau - t_2) u_t^*(\tau - t - t_4) u_{MO}^2 \exp(\jmath \theta (\tau - t_2)) \exp(-j\theta (\tau - t_4)) d\tau
\]
\[
= \exp\left( -j(\phi_{nt_4} - \phi_{nt_2}) \right) \exp\left( -j(\phi_{\nu_4} - \phi_{\nu_5}) \right) \exp(\jmath [\phi_{\nu_1} - \phi_{\nu_2}])
\]
\[
\times R_r (t + [t_2 - t_4]),
\] (2.85)

where

\[
R_r (t + [t_2 - t_4]) = \int_{-\infty}^{\infty} u_t(\tau - t_2) u_t^*(\tau - t - t_4) u_{MO}^2 \exp(\jmath \theta (\tau - t_2)) \exp(-j\theta (\tau - t_4)) d\tau,
\] (2.86)

Note that the signal phase noise factor is now a function of the difference in the phase noise at times \( t_4 \) and \( t_2 \). Similarly, a matched filter can be applied to the two monitor pulses such that

\[
s_{M2}(t) = \int_{-\infty}^{\infty} s_{m1}(\tau) s_{m2}^*(\tau - t) d\tau
\]
\[
= \int_{-\infty}^{\infty} u_t(\tau - t_1) u_{MO} \exp \left( j \theta (\tau - t_1) - j \frac{4\pi}{\lambda} R_p \right) \exp(\jmath \phi_{\nu_1})
\] (2.87)
× u^*_t(\tau - t - t_3) u_{MO} \exp\left(-j\theta (\tau - t - t_3) - j\frac{4\pi}{\lambda} R_p \right) \exp(-j\phi_{MOt_3}) d\tau

= \exp\left(-j(\phi_{MOt_3} - \phi_{MOt_1})\right)

\times \int_{-\infty}^{\infty} u_t(\tau - t_1) u^*_t(\tau - t - t_3) \exp(j\theta (\tau - t_1)) \exp(-j\theta (\tau - t - t_3)) u^2_{MO} d\tau

= \exp\left(-j(\phi_{MOt_3} - \phi_{MOt_1})\right) R_m(t - [t_1 - t_3]),

where

\[ R_m(t - [t_1 - t_3]) = \int_{-\infty}^{\infty} u_t(\tau - t_1) u^*_t(\tau - t - t_3) \]

\[ \times \exp(j\theta (\tau - t_1)) \exp(-j\theta (\tau - t - t_3)) u^2_{MO} d\tau. \]

Finally, a matched filter can be applied to Equations (2.87) and (2.85) such that

\[ s_{M3}(t) = \int_{-\infty}^{\infty} s_{M2}(\tau) s^*_M(\tau - t) d\tau \]

\[ = \int_{-\infty}^{\infty} \exp\left(-j(\phi_{MOt_3} - \phi_{MOt_1})\right) R_m(\tau - [t_1 - t_3]) \exp\left(j(\phi_{nt_4} - \phi_{nt_2})\right) \]

\[ \times \exp\left(j(\phi_{MOt_4} - \phi_{MOt_2})\right) \exp(-j[\phi_{v1} - \phi_{v2}]) R^*_r(t + [t_2 - t_4]) d\tau \]

\[ = \exp\left(j(\phi_{nt_4} - \phi_{nt_2})\right) \exp(-j[\phi_{v1} - \phi_{v2}]) \]

\[ \times \exp\left(-j([\phi_{MOt_3} - \phi_{MOt_1}] - [\phi_{MOt_4} - \phi_{MOt_2}])\right) \]

\[ \times \int_{-\infty}^{\infty} R_m(\tau - [t_1 - t_3]) R^*_r(\tau - t + [t_2 - t_4]) d\tau. \]

Note that this final matched filter output is now a function of the difference in the signal phase noise for each measurement rather than the instantaneous value of the phase noise. So for high PRF's, the difference in the phase noise from pulse to pulse is small, and the measured phase noise can be greatly reduced. After this process is carried out for multiple pulses, the data is processed by observing the phase variation of Equation (2.89) for each pulse pair, which can again be accomplished through a Fourier transform of the phase history.
Figure 2.6. Signal phase used in the comparison of the single and multiple matched filtering techniques. The received phase (red) consists of the 100 Hz vibration signal, the MO phase noise (green) and the signal phase noise (blue).

Figure 2.7. A comparison of the single and multiple matched filter techniques. (a) The phase history using the single matched filter shows a large slowly varying phase noise and a signal phase that is smaller in magnitude. (b) The Fourier transform of the phase history shows relatively small peaks at the vibrational frequency of 100 Hz. (c) The phase history of the multiple matched filter technique shows a much smaller phase noise magnitude resulting in (d) much larger peaks at the vibrational frequency.
Note that this modified matched filter technique provides information about the vibrational frequency of the object, but it does not provide range information. The range term, $R_p$, disappears when the two signal pulses are compared in the first matched filter operation. The original data can then be processed twice to yield the information of interest. The data can first be processed using the conventional single matched filter (i.e., Equation (2.73)) to yield range information. The raw data can then be processed using the multiple matched filter technique, thereby reducing the phase noise and improving the vibrometry processing.

The results from simulated data processed with both techniques are provided in Figure 2.6 Figure 2.7. This simulation assumes a vibrational frequency of $f_v = 100$ Hz. The MO phase noise, signal phase noise, and the resulting signal phase used in this simulation are shown in Figure 2.6. The data was simulated for a range to target of 30 km and a PRF of 10 kHz. After performing both matched filter techniques on a series of pulses, the phase histories of the matched filter outputs are shown in Figure 2.7 (a) and (b) for the single and multiple matched filter techniques, respectively. For the single matched filter technique, the slowly varying phase noise is much larger in amplitude than for the multiple matched filter technique. As a result, after performing a Fourier transform on the phase history, the peaks at the vibrational frequency of 100 Hz are much higher in magnitude when the multiple matched filter technique is used.

2.8. Performance Metrics

The performance of the radar system can be analyzed in many different ways. This section will explore the signal-to-noise ratio (SNR) and range resolution metrics.
2.8.1. Introduction to the Signal-to-Noise Ratio

An important characterization of any laser radar system is the SNR. The received power must be sufficiently higher than the noise power to yield useful information. In this section, a generic laser radar system as depicted in Figure 2.8 is assumed. The laser source is assumed to be CW, and any amplitude, phase, or frequency modulation are accounted for in the second block. The optical signal is then transmitted. The received optical signal is directed through the I/Q demodulator by the transmit receive switch. The I and Q signals are then incident on optical detectors and converted to electronic signals. An electronic amplifier then boosts the electronic signals before they are quantized into discrete amplitude levels by the digitizer. The next section will derive an expression for the received optical power. This will be followed by a discussion of the noise sources in an optical system. Finally, the SNR will be derived.

Figure 2.8. Generic laser radar system for SNR derivation
2.8.1.1. Received Power

The received power for a monostatic laser radar system can be found from the radar range equation, which is expressed as [1]

\[ P_r = \frac{P_T G_T}{4\pi R^2} \cdot \frac{\sigma}{4\pi R^2} \cdot A_{ra} \cdot \eta_{sys} \cdot \eta_{atm}, \]  

(2.90)

where \( P_T \) is the transmitted power, \( R \) is the range to target, \( G_T \) is the transmitter antenna gain, \( \sigma \) is the effective target cross-section, \( A_{ra} \) is the area of the receive aperture, \( \eta_{sys} \) accounts for loss in the system, and \( \eta_{atm} \) accounts for loss due to the atmosphere. This expression is detailed below, using the geometry illustrated in Figure 2.9.

![Figure 2.9. Geometry of laser radar and target](image)

The transmitter antenna gain is defined as

\[ G_T = \frac{4\pi}{\theta_{BW}^2}, \]  

(2.91)

where \( \theta_{BW}^2 \) is the transmitter beamwidth, which is defined as \( \lambda/D \) for a laser radar system. The effective target cross-section is defined as
\[ \sigma = \frac{4\pi}{\Omega} \rho dA, \]  
(2.92)

where \( \Omega \) is the scattering solid angle of the target in steradians, \( \rho \) is the target reflectivity, and \( dA \) is the target area. The target is assumed to be larger than the laser beam with a Lambertian scattering distribution. Thus, the target area is equal to the cross-sectional area of the laser beam, or

\[ dA = \frac{\pi R^2 \theta_{BW}^2}{4}, \]  
(2.93)

and the scattering solid angle for a Lambertian target is \( \pi \). Equation (2.90) then becomes

\[ P_r = \frac{P_T \rho A_{ra}}{4R^2} \eta_{sys} \eta_{atm}, \]  
(2.94)

which represents the received power for a monostatic laser radar system when the target is larger than the transmitted beam. The noise sources are evaluated in the next section.

2.8.1.2. Receiver Noise Sources

There are several noise terms that are inherent in the receiver, including dark current noise, shot noise, and thermal noise. Dark current noise is present when the detector operates in a reverse bias mode, which increases its sensitivity. However, operating under a reverse bias yields an undesirable dark current, which flows through the device even when it is not under illumination. The mean squared dark current noise can be expressed as [35]

\[ \langle i_{dk}^2 \rangle = 2qI_{dk}B, \]  
(2.95)

where \( q \) is the electron charge, and \( B \) is the bandwidth of the detector.

Shot noise occurs because the photons in an optical signal arrive at the detector randomly in time, rather than uniformly in time, causing the received signal power to fluctuate randomly about an average value of \( P_{sig} \). Furthermore, the detector will have a quantum
efficiency, $\eta_d$, with which it converts photons to photoelectrons. The resulting statistical fluctuations in the signal power, referred to as shot noise, is the fundamental lower limit of noise experienced by a photodetector, it cannot be eliminated from the system. The mean square current noise, or shot noise, seen by the detector is found to be

$$\langle i_{sh}^2 \rangle = 2\eta_d q^2 B \frac{P_{\text{sig}}}{hf}.$$  \hspace{1cm} (2.96)

In a coherent detection system, there will also be shot noise associated with the master oscillator, which can be expressed as

$$\langle i_{lo}^2 \rangle = 2\eta_d q^2 B \frac{P_{\text{MO}}}{hf}.$$  \hspace{1cm} (2.97)

The shot noise from the signal and master oscillator will be included in the SNR.

The thermal noise, or Johnson noise, in a receiver is associated with the blackbody radiation of resistive components in the detector, which cause random electron motion. The time averaged mean square thermal noise current is found to be

$$\langle i_{th}^2 \rangle = \frac{4kT\Delta f}{R}.$$  \hspace{1cm} (2.98)

At low frequencies a noise spectrum that is roughly proportional to the reciprocal of the detected frequency will be observed. This low frequency noise generally consists of current or voltage fluctuations with frequency components below 10 kHz [36]. This noise was first observed in vacuum tubes in the mid 1920’s, and can be observed in almost any electronic device, including resistors, semiconductor devices, detectors, and photodetectors to name only a few. This noise was only understood after the introduction of the quantum 1/f theory in 1975 [37]. In general, 1/f noise is attributed to the scattering of electrons, resulting in low frequency quantum fluctuations in current. For this analysis, it is assumed that any contributions due to
the 1/f noise will be avoided by using a heterodyne detection technique rather than a homodyne detection technique, thereby keeping the detected signal out of the low frequency regime.

2.8.1.3. Digitization Noise

The digitizer produces a digital signal from the analog signal. This involves breaking up the input electronic signal into discrete amplitude levels. The digitized signal will be slightly different than the analog signal depending on how many quantization levels are possible. This difference is described as quantization noise. Assuming the input signal is AC-coupled, and thus has no DC offset, the signal is assumed to have a voltage excursion of ±V volts. If the signal is quantized into a total of M levels, the spacing in volts between adjacent levels is then

\[ a = \frac{2V}{M}. \]  

(2.99)

The quantization noise arises because a quantized value of \( A \) could be the result of an input analog signal between \( A - a/2 \) and \( A + a/2 \) volts. Since the quantization error \( \varepsilon_q \) varies between ±a/2 volts, the mean-squared value of \( \varepsilon_q \) is found by calculating its statistical expectation, or

\[ E(\varepsilon_q^2) = \frac{1}{a} \int_{-a/2}^{a/2} \varepsilon_q^2 \, d\varepsilon = \frac{a^2}{12}. \]  

(2.100)

This yields an rms voltage error of \( a/\sqrt{12} \). The peak signal value in volts is \( aM/2 \), yielding a quantization power SNR of

\[ SNR_q = \left( \frac{aM/2}{a/\sqrt{12}} \right)^2 = \sqrt{3}M. \]  

(2.101)

This is the maximum obtainable SNR for the digitized signal, assuming the analog signal at the input to the digitizer is noiseless. As discussed in the previous section, there are noise sources
that will impact the signal before it reaches the digitizer. This will be taken into account in the
SNR derivation of the following section.

2.8.1.4. Signal-to-Noise Ratio

The signal-to-noise ratio at each detector in Figure 2.8 is the ratio of the mean squared
signal current to the mean squared noise current, or

\[
SNR_i = \frac{i_{\text{sig}}^2}{i_{bk}^2 + i_{dk}^2 + i_{sn}^2 + i_{\text{th}}^2}.
\]

(2.102)

As detailed in Appendix A, the mean squared signal current can be expressed as

\[
\langle i_{\text{sig}}^2 \rangle = 2\rho_i^2 P_r P_{MO},
\]

(2.103)

where \( \rho_i \) is the responsivity of the detector, \( P_{MO} \) is the master oscillator power, and \( P_r \) is the
received power. The responsivity of the detector is defined as

\[
\rho_i = \frac{\eta_d q}{hf},
\]

(2.104)

where \( \eta_d \) is the quantum efficiency of the detector, \( q \) is the electron charge, \( h \) is Plank’s
constant, and \( f \) is the optical frequency. Using the terms derived in the previous sections, and
plugging them into Equation (2.102), the SNR at the detector can be expressed as

\[
SNR_i = \frac{2\rho_i^2 P_r P_{MO}}{2qP_{\text{solar}}\rho_i B + 2qI_{dk}B_e + 2q(P_r + P_{MO})\rho_i B + \frac{4kTB}{R}}.
\]

(2.105)

It is assumed that all losses occurring in the modulation block of Figure 2.8 are accounted for in
the \( \eta_{\text{sys}} \) variable. If the pulse bandwidth has been increased through phase modulation and
matched filtering techniques are used to detect the pulse, the SNR is increased by the pulse
compression gain, PCG, which is defined as the product of the pulse bandwidth and pulse
duration, or

\[
PCG = B_p \tau_p.
\]

(2.106)

The SNR then becomes
As depicted in Figure 2.8, the signal then passes through an electronic amplifier. Any noise in the amplifier is quantified by its Noise Figure (NF), where

\[ NF = \frac{SNR_{in}}{SNR_{out}}. \]  

The SNR after the electronic amplifier becomes

\[ SNR_{ea} = \frac{2\rho_i^2 P_r P_{MO}}{2qP_{solar}\rho_i B + 2ql_{dk}B_e + 2q(P_r + P_{MO})\rho_i B + \frac{4kTb}{R}} \cdot PCG \cdot \frac{1}{NF_{ea}}. \]  

As discussed in the previous section, the digitizer has an inherent quantization noise. This noise was expressed in terms of the mean square voltage. To find the SNR after the digitizer, the input SNR must also be expressed in terms of the mean square voltage before the digitizer noise is included. Assuming this is an amplified detector, this is described by the transimpedance gain of the detector, \( G_t \), which is the voltage generated by an input current, in units of V/A. The overall SNR is then found to be

\[ SNR = \frac{2\rho_i^2 P_r P_{MO} G_t \cdot PCG}{\left(2qP_{solar}\rho_i B + 2ql_{dk}B_e + 2q(P_r + P_{MO})\rho_i B + \frac{4kTb}{R}\right) G_t \cdot NF_{ea} + \frac{q^2}{12}}. \]  

where the transimpedance gain has been applied to both the signal and the noise and \( a^2/12 \) is the mean squared voltage quantization noise found in Equation (2.100).

Since the master oscillator amplifies the received signal, it is advantageous to increase the power of the master oscillator. If the master oscillator is powerful enough, the shot noise from the master oscillator will dominate all of the other noise sources at the detector, significantly simplifying the expression found in Equation (2.110). The resulting SNR, known as the shot noise limited SNR, is found to be
\[ SNR_{sn} = \frac{2\rho_i^2 P_r P_{MO} G_t \cdot PCG}{2 q P_0 \rho_i B G_t \cdot NF_{ea} + \frac{a^2}{T^2}}. \]  \hspace{1cm} (2.111)

If the shot noise also dominates the quantization noise of the digitizer, this expression can be further reduced to

\[ SNR_{sn} = \frac{\rho_i P_r \cdot PCG}{q B \cdot NF_{ea}}. \]  \hspace{1cm} (2.112)

By substituting Equations (2.94) and (2.104) into Equation (2.112), the shot noise limited SNR can be expressed as

\[ SNR_{sn} = \frac{P_t \rho_t A \eta_d \cdot \frac{1}{NF_{ea}} \cdot \eta_{atm} \eta_{sys} \cdot PCG}{\pi R^2 h B \cdot \eta_{sys} \cdot \eta_{atm}}. \]  \hspace{1cm} (2.113)

2.8.2. The Ideal Pulse Response and Range Resolution

Range resolution is the minimum distance between two targets at which the returns from both targets can be resolved. For a transform-limited pulse, the minimum distance that two objects can be separated in range and yield return signals that do not overlap in time is

\[ \Delta r = \frac{c \tau_p}{2}, \]  \hspace{1cm} (2.114)

where \( \tau_p \) is the pulse length and \( c \) is the speed of light. However, if the pulse is frequency modulated, it can be resolved even if the return pulses overlap in time. This is evident in the following example. If a rectangular pulse is transmitted, such that

\[ s(t) = \text{rect} \left( \frac{t}{T} \right), \]  \hspace{1cm} (2.115)

the Fourier transform yields a pulse spectrum of
\[ S(\omega) = 3 \left[ \text{rect}\left(\frac{t}{\tau}\right) \right] = \tau \text{Sa}\left(\frac{\omega \tau}{2}\right) = \tau \text{sinc}(\nu \tau). \quad (2.116) \]

The first zero of the pulse spectrum will occur at \( \nu = 1/\tau_p = B_p \), where \( B_p \) is the bandwidth of the pulse. By evaluating the pulse in the frequency domain, the range resolution becomes

\[ \Delta r = \frac{c}{2B_p}. \quad (2.117) \]

So the range resolution improves as the bandwidth of the pulse increases.

The above is a specific example of a transmitted rectangular pulse and is often used as an approximation of the range resolution. As discussed above, the matched filter technique is used to detect the received pulse. The impulse response of the matched filter, found in Equation (2.55), represents the Ideal Pulse Response, IPR, of the system because it is the best case scenario where the received signal is perfectly matched to the matched filter. For an arbitrary pulse shape, the range resolution of the system is determined by the width of the IPR. The IPR is often characterized in terms of its Peak-to-Sidelobe Level Ratio (PSLR) and Integrated Sidelobe Level Ratio (ISLR). The PSLR is the ratio of the power of the main lobe of the IPR to the power of the largest sidelobe of the IPR, as depicted in Figure 2.10. The ISLR is the ratio of the energy contained in the main lobe of the IPR to the energy contained in the sidelobes.

As discussed in the introduction, a range image can be generated by scanning the laser beam over an area and accumulating the range measurements as a function of the laser radar location. Large sidelobes in the IPR will appear as secondary target returns, so the PSLR is essentially a measure of the amount of ghosting present in the system. Since the return pulse has a finite amount of energy, as more energy is located in the sidelobes of the IPR, less energy is located in the main lobe. This will increase the background level of the range image, so that the ISLR can be used as a measure of the contrast of the range image. As will be seen, the SNR,
PSLR and ISLR will be used to characterize the short-pulse SAL design that is explored in this research.

![Ideal Pulse Response](image)

**Figure 2.10. PSLR of the Ideal Pulse Response**

2.9. Conclusion

This chapter has presented the theory of operation and performance metrics for a laser radar system for ranging and vibrometry. The laser radar signal model has been developed, and the Doppler and Micro-Doppler effects have been considered. Detection techniques including conventional heterodyne detection, synchronous (I/Q) detection, and the role of the matched filter have been discussed. Processing techniques for ranging and vibrometry were also presented. Finally, performance metrics such as the SNR, the IPR, and the range resolution were derived. The next chapter will discuss the theory of operation of synthetic aperture ladar systems.
CHAPTER 3
SYNTHETIC APERTURE LADAR THEORY

3.1. Introduction

This chapter will summarize the key ideas of synthetic aperture ladar (SAL), including the theory of operation and several performance measures. As will be shown, synthetic aperture techniques improve the cross-range resolution in the direction of motion to less than the footprint of the transmitted beam. After discussing the Doppler effect, which is exploited to improve the cross-range resolution, the signal model, SAL processing techniques, potential modes of operation and associated performance metrics will be presented.

3.2. Doppler Measurements and Cross-range Resolution

As depicted in Figure 3.1, consider a laser radar located at point P, travelling with a constant velocity, \( v \), in the x-direction. If two point targets, A and B, are each located within the beamwidth of the system at a range of \( R \), the returns from each target will have a different relative velocity with respect to the radar. The return from target A will have a relative velocity that is the projection of the platform velocity along the line of sight to the target, or

\[
v_A = v \cos \theta_A. \tag{3.1}\]

Similarly, the return from target B will have a relative velocity of

\[
v_B = v \cos(\theta_A + \delta \theta). \tag{3.2}\]
Following the Doppler frequency definition in Equation (2.12), the Doppler frequency for each target is

\[ f_{DA} = \frac{2v_A}{\lambda} = \frac{2v}{\lambda} \cos \theta_A \]  

(3.3)

and

\[ f_{DB} = \frac{2v_B}{\lambda} = \frac{2v}{\lambda} \cos(\theta_A + \delta \theta). \]  

(3.4)

The difference in Doppler frequencies between the two returns is then found as

\[ \delta f_D = f_{DA} - f_{DB} = \frac{2v}{\lambda} \left[ \cos \theta_A - \cos(\theta_A + \delta \theta) \right] \]

\[ = \frac{2v}{\lambda} \left[ \cos \theta_A - (\cos \theta_A \cos \delta \theta - \sin \theta_A \sin \delta \theta) \right]. \]  

(3.5)

Assuming \( \delta \theta_A \ll \theta_A \), the approximations

\[ \cos \delta \theta \cong 1 \quad \text{and} \quad \sin \delta \theta \cong \delta \theta \]  

(3.6)

can be used to express Equation (3.5) as

\[ \delta f_D \cong \frac{2v \delta \theta}{\lambda} \sin \theta_A. \]  

(3.7)

The Doppler frequencies can be found using a Fourier transform. If the targets are observed over a time \( T_{sa} \), then the frequency resolution of the Fourier transformed signal will be approximately \( 1/T_{sa} \). The smallest frequency difference that can be measured by the system is then

\[ \delta f_{D_{\text{min}}} = \frac{1}{T_{sa}}. \]  

(3.8)

From Figure 3.1, the cross-range separation of the two targets is seen to be

\[ \Delta r_x = 2R \sin(\delta \theta / 2) \approx R \delta \theta. \]  

(3.9)

Substituting Equation (3.9) into Equation (3.7) yields an angular resolution of
\[
\delta f_D = \frac{2v\Delta r_x}{\lambda R} \sin \theta_A.
\] (3.10)

Using Equation (3.8), the expression for the minimum cross-range separation becomes

\[
\Delta r_x = \frac{\lambda R}{2vT_{sa} \sin \theta_A}.
\] (3.11)

The observation time is related to the synthetic aperture length, \(D_{sa}\), by the platform velocity, or

\[
T_{sa} = \frac{D_{sa}}{v}.
\] (3.12)

The cross-range resolution can then be expressed in terms of the synthetic aperture length as

\[
\Delta r_x = \frac{\lambda R}{2D_{sa} \sin \theta_A}.
\] (3.13)

**Figure 3.1.** Geometry for a laser radar system moving in the x-direction with velocity \(v\), and illuminating objects A and B.
3.3. The Received Signal

As discussed in Chapter 2, the signal received by a laser radar system for a single transmitted pulse was found in Equation (2.11) to be

\[ s_r(t) = u(t - t_{rt}) \exp \left[ j \omega_c \left( t - \frac{2R_p}{c} \right) + \theta(t - t_{rt}) \right], \]

(3.14)

where \( u(t - t_{rt}) \) is the envelope of the return signal, delayed by the roundtrip travel time \( t_{rt} = 2R_p/c \), and \( R_p \) is the range to the object. The variable \( \omega_c \) represents the carrier frequency and \( \theta(t - t_{rt}) \) represents any phase modulation of the transmitted pulse, delayed by the roundtrip travel time. As the ladar moves over the target area, the range to the target is changing as a function of the ladar location. As depicted in Figure 3.2, the target, located at point p, and the ladar are separated by a range of \( R_{p0} \left( y_p^2 + z_p^2 \right)^{1/2} \).

Figure 3.2. SAL geometry for a target located at point p and range \( R_{p0} \) from the platform.
The range to the object can then be expressed as a function of time such that

\[ R_p(t) = \sqrt{(vt - x_p)^2 + y_p^2 + z_p^2} = R_{p0} \sqrt{1 + \frac{(vt - x_p)^2}{R_{p0}^2}}. \]  

(3.15)

where the translation has been limited to the \( x \) dimension. Assuming \( R_p \) is large with respect to all other terms, the series expansion \((1 + x)^n \approx 1 + \frac{1}{2} x\) can be used to yield

\[ R_p(t) \approx R_{p0} + \frac{v^2 t^2}{2R_{p0}} - vt \frac{x_p}{R_{p0}} + \frac{x_p^2}{2R_{p0}}. \]  

(3.16)

Note that the \( t^2 \) term implies that the range to target varies approximately quadratically as the data is collected. Equation (3.16) can also be written as

\[ R_p(u) \approx R_{p0} + \frac{u^2}{2R_{p0}} - u \frac{x_p}{R_{p0}} + \frac{x_p^2}{2R_{p0}}, \]  

(3.17)

where \( u = vt \) designates the location of the platform when the pulse is transmitted. Equation (3.14) becomes

\[ s_r(t, u) = u(t - t_{rt}) \exp(j\omega_t t + j\theta(t - t_{rt})) \exp(-j\phi_t(u)), \]  

(3.18)

where

\[ \phi_t(u) = \frac{4\pi}{\lambda} \left( R_{p0} + \frac{u^2}{2R_{p0}} - u \frac{x_p}{R_{p0}} + \frac{x_p^2}{2R_{p0}} \right) \]  

(3.19)

is the phase accumulation associated with translational motion. Since each pulse is transmitted from a different location, Equation (3.18) is a function of the two coupled variables \( t \) and \( u \), hence the notation \( s_r(t, u) \). The fast-time variable \( t \) measures the range to target, and the slow-time variable \( u = vt \) yields information about the position of the target in the cross-range \( (x) \) dimension. Since the phase accumulation associated with translational motion in Equation (3.19) is a function of the range to target, the received phase will also change quadratically as the data is collected. Because frequency is related to phase by a derivative, or
\[ v = \frac{1}{2\pi} \frac{d\phi}{dt}, \] (3.20)

A quadratic change in phase is equivalent to a linear change in frequency. So the received signal experiences a frequency chirp as the transmitter moves over the target. The SAL dataset consists of the summation of the returns from each target, or

\[
s_{SA}(t, u) = \sum_p u_p \left( t - \frac{2R_p(u)}{c} \right) \exp \left[ j\omega_c \left( t - \frac{2R_p(u)}{c} \right) - \theta \left( t - \frac{2R_p(u)}{c} \right) \right], \quad (3.21)
\]

where \( R_p \) represents the range to each target for a given position along the synthetic aperture, which is found according to Equation (3.16). As discussed in Chapter 2, each return signal can be processed individually to yield information about the range to each target. As will be discussed in the following section, the return signals from each position along the synthetic aperture can also be coherently processed together, and the frequency chirp as a function of synthetic aperture position can be determined to yield information about the cross-range location of each target.

![Real Part of the SAL Data Set](image)

**Figure 3.3.** An example SAL dataset for a LFM transmit pulse as a function of fast-time (horizontal axis) and slow-time (vertical axis)
Equation (3.21) is solved assuming a 10 $\mu$s LFM transmit pulse, a single target (i.e. $p = 1$) located at range $R_{p0} = 30$ km and $x_{p0} = 0$, and a platform velocity of 200 m/s, to yield the example SAL dataset shown in Figure 3.3. Each column of the plot represents the return from a single pulse at a given point along the synthetic aperture. The vertical axis representing the duration of each return pulse is referred to as the fast-time axis, and the horizontal axis representing when the pulse was transmitted is referred to as the slow-time axis.

A cross-section along the slow-time axis of Figure 3.3 yields the return for a single transmitted pulse at a given position along the synthetic aperture. A cross-section along the fast-time axis of Figure 3.3 shows the return for each point along the synthetic aperture for a given point in each pulse. These cross-sections are depicted in Figure 3.4. Figure 3.4 (a) shows the real part of the received signal for a single pulse. The frequency chirp due to the LFM is evident in the oscillations of the signal. The power of the pulse, shown in Figure 3.4 (b), is constant across the pulse duration, as this simulation assumes a top hat profile with no amplitude modulation. The LFM modulation yields a quadratic phase profile, as seen in Figure 3.4 (c). A cross-section in the slow-time domain is shown in Figure 3.4 (d). As the ladar approaches the target, the Doppler frequency decreases until the ladar passes over the target, at which point the Doppler frequency begins to increase again, resulting in the observed linear frequency modulation. This simulation assumes a uniform illumination of the target area, so the spatial profile of the return seen in Figure 3.4 (e) is constant across the duration of the synthetic aperture. As discussed above, the phase of the return signal varies quadratically with position along the synthetic aperture. This quadratic nature is seen in Figure 3.4 (f).
3.4. SAL Image Formation

Various techniques can be used to process the above SAL dataset to produce an image in range and cross-range. In general a technique employing a two dimensional matched filter can be used [38]. However, in certain circumstances, a simpler technique known as Doppler Beam Sharpening can be used. These techniques will be outlined below.

3.4.1. Two Dimensional Matched Filter

As stated in Equation (3.21) above, the synthetic aperture signal is the summation of each return received over the synthetic aperture length, or
\[ s_{SA}(t, u) = \sum_p u_t \left( t - \frac{2R_p(u)}{c} \right) \exp \left[ j\omega_c \left( t - \frac{2R_p(u)}{c} \right) - \theta \left( t - \frac{2R_p(u)}{c} \right) \right] \]

\[ = \sum_p p \left[ t - \frac{2R_p(u)}{c} \right], \]

where \( p \left[ t - \frac{2R_p(u)}{c} \right] \) is a shorthand notation for the received signal and \( R_p(u) \) was defined in Equation (3.17). As discussed above, the SAL dataset is a function of fast-time, denoted by the variable \( t \), and slow-time, denoted by the variable \( u \). Using the identity

\[ \Im \{ f(t - t_0) \} = F(\omega)\exp(-j\omega t_0), \]

the Fourier transform of Equation (3.22) in the fast-time domain is

\[ s_{SA}(\omega, u) = P(\omega) \sum_p \exp \left[ -j\omega \frac{2R_p(u)}{c} \right] = P(\omega) \sum_p \exp[-j2kR_p(u)], \]

where \( k = \omega/c \) is the wavenumber. Note that this equation is a function of \( \omega \) and \( u \) because the fast-time variable \( t \) has been transformed to the frequency domain, \( \omega \), due to the fast-time Fourier-transform. Next, the slow-time Fourier transform is carried out using the definition [38]

\[ \Im \left\{ \exp \left( -j2k\sqrt{(x_p - u)^2 + z_p^2} \right) \right\} = \exp \left( -j\sqrt{4k^2 - k_u^2z_p} - jk_u x_p \right) \]

to yield the expression

\[ S_{SA}(\omega, k_u) = P(\omega) \sum_p \exp \left( -j\sqrt{4k^2 - k_u^2z_p} - jk_u x_p \right), \]

where we have assumed that \( y_p=0 \) for simplicity. The Fourier transform has now been carried out in both the fast-time and slow-time dimensions such that \( t \) has been transformed to the frequency variable \( \omega \) and \( u \) has been transformed to the spatial frequency variable \( k_u \). This function can be rewritten as

\[ S_{SA}(\omega, k_u) = P(\omega) \sum_p \exp(-jk_x(\omega, k_u)z_p - jk_x(k_u)x_p), \]

(3.27)
where
\[ k_z(\omega, k_u) = \sqrt{4k^2 - k_u^2} \quad \text{and} \quad k_x(k_u) = k_u \] 
are referred to as the synthetic aperture spatial frequency mapping functions [38]. The significance of these equations will be explored below.

As discussed in Chapter 2, fast-time ladar signals are generally detected using a matched filter such that
\[ s_{MF}(t) = \mathcal{Z}^{-1}\{P(\omega)P^*(\omega)\} = \mathcal{Z}^{-1}\{|P(\omega)|^2\}. \] 
Similarly, SAL signals can be detected using a two-dimensional matched filter that is a function of both \( \omega \) and \( k_u \). This is done by considering a signal from a perfect reflector located at the center of the target area such that \( x_p=0 \) and \( z_p=z_c \), where \( z_c \) is the range to the center of the target area. Substituting these values into Equation (3.22) yields the reference signal
\[ s_0(t,u) = p \left( t - \frac{2\sqrt{u^2 + z_c^2}}{c} \right). \] 
Once again using the definition of Equation (3.25), the Fourier transform of the reference signal is found to be
\[ S_0(\omega, k_u) = P(\omega)\exp(-jk_z(\omega, k_u)z_c) = P(\omega)\exp(-jk_z(\omega, k_u)z_c) \cdot \exp(jk_x(\omega, k_u)x_p). \] 
The two-dimensional matched filter is then carried out to yield
\[ F(\omega, k_u) = S_{SA}(\omega, k_u)S_0^*(\omega, k_u) \]
\[ = P(\omega)P^*(\omega) \sum_p \exp(-jk_z(\omega, k_u)z_p - jk_x(k_u)x_p) \exp(jk_z(\omega, k_u)z_c) \]
\[ = |P(\omega)|^2 \sum_p \exp(-jk_z(\omega, k_u)z_p - jk_x(k_u)x_p) \exp(jk_z(\omega, k_u)z_c), \]
The purpose of the image formation is to transform the received data, which is a function of time \( t \) and synthetic aperture position \( u \), into an image that is a function of range \( z \) and cross-
range $x$. However, the spatial frequency mapping function for $k_z$ in Equation (3.28) is a nonlinear function, which will transform evenly spaced data in $k$ and $k_u$ to unevenly spaced in $k_z$. If $k$ and $k_u$ are similar in magnitude, the data must be interpolated to yield evenly spaced data in $k_z$ and $k_x$ so the inverse Fourier transform can be applied. However, for laser radar signals, $k \gg k_u$, so Equation (3.28) becomes

$$k_z(\omega) \approx 2k \quad \text{and} \quad k_x(k_u) = k_u.$$  

Using these substitutions, a two dimensional inverse Fourier transform can be performed on Equation (3.32), yielding the locations of each object in range and cross-range, or

$$f(x,z) = \mathcal{F}^{-1}[F(k_z(\omega), k_x(k_u))].$$  

(3.34)

It should be noted that the first factor of Equation (3.32), $|P(\omega)|^2$, is the output of a conventional matched filter applied in the fast-time frequency domain, as seen in Equation (3.29). This shows that the two dimensional matched filter is separable, and can be accomplished by first applying a matched filter in the fast-time frequency domain, where the fast-time matched filter is defined as

$$S_{FTMF}(\omega) = P^*(\omega),$$  

(3.35)

followed by a matched filter in the slow-time frequency domain, defined as

$$S_{STMF}(\omega, k_u) = \exp(jk_z(\omega, k_u)z_c).$$  

(3.36)

This process is outlined in the block diagram of Figure 3.5.

The dataset presented in Figure 3.3 was processed using the two dimensional matched filtering technique, and the results are shown in Figure 3.6. For a single target, the received signal power is shown in Figure 3.6 (a) as a function of fast-time and slow-time. Note that the power is a constant value because there was no amplitude modulation of the transmitted signal, and for simplicity it is assumed that the target has no affect on the amplitude of the return signal. The result after applying the matched filter in the fast-time domain is shown in Figure
3.6 (b) as a function of fast-time and slow-time. The compression in the fast-time domain due to the application of the matched filter is evident. The result after applying the matched filter in the slow-time domain is presented in Figure 3.6 (c) as a function of range and cross-range position. The signal has now been compressed in the fast and slow-time domains. Note that the range axis shows the range of the object with respect to the range to the center of the target area, \( z_c \) (i.e. an object located at range \( z_c \) shows a range of zero because the match filter assumed a range of \( z_c \)). The same process was applied if Figure 3.6 (d) through (f) for multiple targets. Note that the received signal power in Figure 3.6 (d) now exhibits an interference pattern due to the accumulated returns from multiple targets. After the fast-time matched filter is applied, the target returns are separated in range, but not cross-range, as seen in Figure 3.6 (e). The interference pattern in the second return represents the interference of the returns from two targets with the same range location. After the application of the slow-time matched filter, the positions of three distinct targets can be seen in Figure 3.6 (f).

**Figure 3.5. SAL Image Formation via 2-D Matched Filtering**

![Diagram of SAL Image Formation via 2-D Matched Filtering](Image)
3.4.2. Doppler Beam Sharpening

The Doppler Beam Sharpening (DBS) technique is derived by once again observing the expression for the range to the object of interest in Equation (3.16), restated here as

\[ R_p(t) \approx R_{p0} + \frac{v^2 t^2}{2R_{p0}} - vt \frac{x_p}{R_{p0}} + \frac{x_p^2}{2R_p}. \]  (3.37)

If the quadratic term (the second term) can be ignored, the phase becomes

\[ R_p(u) = R_{p0} - u \frac{x_p}{R_{p0}} + \frac{x_p^2}{2R_{p0}}. \]  (3.38)

where we have made the substitution \( u = vt \). The instantaneous cross-range spatial frequency can be expressed as...
As $R_{p0}$ increases, or as the synthetic aperture size decreases (i.e., $u$ decreases), the quadratic term in Equation (3.37) becomes smaller, and the approximation made in Equation (3.38) becomes valid. This relationship is demonstrated in Figure 3.7, where $R(t)$ in Equation (3.37) is shown for a target at position $x_p = 0$ with $R_0 = 100$ km. A synthetic aperture of 30 m (dotted line) and a synthetic aperture of 2 m (solid line) are shown. The quadratic nature of the range, and therefore the phase, becomes less pronounced with the decrease in synthetic aperture.

Figure 3.7. The change in range to target as a function of synthetic aperture position for a 30 m synthetic aperture (dotted line) and a 2 m synthetic aperture (solid blue line)

For smaller synthetic aperture lengths, the range curvature can be approximated as sinusoidal in nature rather than quadratic. As a result, Doppler Beam Sharpening can be carried out using the steps outlined in Figure 3.8. The received signal was found in Equation (3.22) to be
Using Equation (3.23), the fast-time Fourier transform can be calculated to yield

\[
s_{SA}(\omega, u) = P(\omega) \sum_n \exp \left[ -j\omega \frac{2R_p(u)}{c} \right].
\]  

(3.41)

The conventional matched filter can be applied in the fast-time domain to yield

\[
s_{SA}(t, u) = \mathcal{F}_t^{-1} \{ P(\omega) P^*(\omega) \} \sum_p \exp \left[ -j\omega \frac{2R_p(u)}{c} \right],
\]  

(3.42)

where the notation \( \mathcal{F}_t^{-1} \) indicates that the inverse Fourier transform is taken in the fast-time (t) dimension. As discussed above, the slow-time portion of the signal is sinusoidal in nature, so the position of the target in cross-range can be found by carrying out a slow-time Fourier transform, so that Equation (3.42) becomes

\[
s_{SA}(t, k_u) = \mathcal{F}_t^{-1} \{ P(\omega) P^*(\omega) \} \cdot \mathcal{F}_u \left\{ \sum_p \exp \left[ -j\omega \frac{2R_p(u)}{c} \right] \right\},
\]  

(3.43)

where the notation \( \mathcal{F}_u \) indicates that the Fourier transform is taking in the slow-time \( u \) dimension. As indicated, Equation (3.43) is now a function of time \( t \) and the spatial frequency \( k_u \) because the inverse Fourier transform in the fast-time domain transforms \( \omega \) to \( t \), and the Fourier transform in the slow-time domain transforms \( u \) to \( k_u \). To achieve the position of the target in range \( z \) and cross-range \( x \), the relationships found in Equation (3.39) and the basic range equation can be used such that

\[
z = \frac{ct}{2} \quad \text{and} \quad x = \frac{\lambda R}{4\pi k_u}.
\]  

(3.44)

Note that this technique is only valid when the quadratic portion of the range expression in Equation (3.37) can be ignored. Otherwise, the two dimensional matched filtering technique outlined in Section 3.4.1 must be applied.
The dataset presented in Figure 3.3 was processed using the DBS technique, and the results are shown in Figure 3.9. For a single target, the received signal power is shown in Figure 3.9 (a) as a function of fast-time and slow-time, and the result after applying the matched filter in the fast-time domain is shown in Figure 3.9 (b). These two steps are identical to that of the 2-D matched filter technique in the previous section. For the final step, the slow-time Fourier transform of the dataset in Figure 3.9 (b) is calculated and presented in Figure 3.9 (c) as a function of range and cross-range position. The same process was applied in Figure 3.9 (d) through (f) for multiple targets. Once again, the received signal power and the output after the fast-time matched filter, shown in Figure 3.9 (d) and (e), respectively, are identical to that of Figure 3.8 (d) and (e). After taking the slow-time Fourier transform of the dataset in Figure 3.9 (d), the positions of three distinct targets can be seen in Figure 3.9 (f).
Figure 3.9. Results from the Doppler beam sharpening technique showing the (a) received signal power, (b) output of the fast-time matched filter, and (c) output of the slow-time Fourier transform for a single object. The same process applied to three objects showing (d) the interference pattern of the received power from the three objects, (e) the output of the fast-time matched filter, where an interference pattern is still present in the returns from two objects with the same range position, and (f) the output of the slow-time Fourier transform showing three objects resolved in range and cross-range.

3.4.3. Comparison of 2-D Matched Filter and Doppler Beam Sharpening

In general, the short wavelengths used by SAL systems result in very small synthetic aperture lengths, and DBS techniques are applicable. As shown above, the dataset from Figure 3.3 was processed using both techniques. The results of the two techniques are compared against one another in Figure 3.10 for a dataset consisting of multiple targets. The processed dataset as a function of fast-time and slow-time are seen in Figure 3.10 (a) and (c) respectively. Cross-sections in range and cross-range are shown in Figure 3.10 (b) and (d), respectively. As both techniques involve a fast-time matched filter, the cross-sections in range are identical. In
cross-range, there are subtle differences in the cross-section of each image, but the locations of each target are the same for both techniques.

Figure 3.10. Comparison of the 2-D Matched Filter and DBS techniques for the SAL dataset containing three targets. The images after applying the (a) 2-D matched filter and (c) the Doppler beam sharpening techniques are shown. Cross-sections in the (b) fast-time domain and (d) slow-time domains are compared.

3.5. Modes of Operation

Three modes of operation will be considered here. The spotlight mode of operation has the smallest area of coverage, but the finest cross-range resolution. The stripmap mode of operation has the largest area of coverage and the coarsest cross-range resolution. The hybrid mode of operation is a compromise between the two modes, with a larger area of coverage than the spotlight mode, but a finer cross-range resolution than the stripmap mode. A depiction of all three modes is shown in Figure 3.11.
3.5.1. **Spotlight Mode**

In the spotlight mode of operation, the beam is steered to keep the target area continually in the field of view of the system. The area of coverage corresponds to the spot size of the system at range, or

$$\text{AOC}_{\text{spot}} = \frac{\pi}{4} \left( \frac{\lambda R}{D_{ap}} \right)^2, \quad (3.45)$$
where \( \lambda \) is the wavelength of operation, \( R \) is the operational range, and \( D_{ap} \) is the diameter of the transmit aperture. Theoretically, the synthetic aperture length, \( D_{sa} \), can be as long as desired, and the cross-range resolution will continue to follow Equation (3.13).

### 3.5.2. Stripmap Mode

In the stripmap mode of operation, there is no beam steering, and the beam moves with the motion of the platform. Thus, an object remains in the field of view of the system for a limited period of time. A target will only be illuminated by the system over a synthetic aperture length of

\[
D_{strip_{max}} = D_g,
\]

where \( D_g = \frac{\lambda R}{D_{ap}} \) is the diffraction limited spot size at range. The minimum cross-range resolution is determined by the maximum synthetic aperture length using Equation (3.13) such that

\[
\Delta r_{x_{min}} = \frac{\lambda R}{2v T_{sa_{max}} \sin \theta_A} = \frac{\lambda R}{2D_{sa_{max}} \sin \theta_A} = \frac{\lambda R}{2D_g \sin \theta_A} = \frac{D_{ap}}{2 \sin \theta_A}.
\]

(3.47)

However, the area of coverage can continue indefinitely in the x-dimension as the system continues to fly over the area of interest.

![Figure 3.12. Geometry for determining the minimum cross-range resolution in stripmap SAL](image-url)
3.5.3. Hybrid Mode

In the hybrid mode of operation, the beam is steered to create the desired synthetic aperture length, but it continues to move in the x-direction at a slower rate than in the stripmap case, as depicted in Figure 3.13.

![Figure 3.13. Geometry for determining the minimum cross-range resolution in hybrid SAL](image)

The maximum synthetic aperture length will occur when the beam on the ground has moved a distance of $D_g$. The maximum synthetic aperture length can be found to be

$$D_{sa} = \frac{R_0}{\tan \theta_{A1}} + D_{gx} - \frac{R_0}{\tan \theta_{An}},$$  \hspace{1cm} (3.48)

where $R_0$ is the perpendicular range to the ground plane, $\theta_{A1}$ is the steer angle of the first beam position in the synthetic aperture, and $\theta_{An}$ is the steer angle of the last beam position in the synthetic aperture. These parameters can be set to achieve the desired cross-range resolution.

The system will complete a synthetic aperture image over an area corresponding to one spot size in the synthetic aperture time $T_{sa}$. This will correspond to an area coverage rate of
\[ AOC_{rate} = \frac{\pi}{4} \left( \frac{AR}{D_{ap}} \right)^2 \frac{1}{T_{sa}}. \] (3.49)

The desired cross-range resolution and the desired area of coverage rate have to be balanced in the system design.

### 3.6. Performance Metrics

This section will summarize several performance metrics for a SAL system, including updates to the SNR equation, values for the cross-range resolution, and Doppler ambiguities.

#### 3.6.1. SNR

As outlined in Section 3.1.3, multiple pulses are coherently added together and processed to generate the SAL image. As such, the SNR of the image is partially determined by the number of pulses obtained over the duration of the synthetic aperture. The number of pulses within a synthetic aperture is found to be

\[ N_{pulses} = \frac{PRF \cdot D_{sa}}{v}, \] (3.50)

where \( PRF \) is the pulse repetition frequency of the system, \( D_{sa} \) is the synthetic aperture length, and \( v \) is the velocity of the platform. Assuming that the master oscillator power is sufficiently high, the shot noise due to the master oscillator will be the dominate noise source. The shot noise limited SNR for a single pulse was found in Equation (2.113) and is repeated here for convenience as

\[ SNR_{sn} = \frac{P_t \rho_t A \eta_d}{\pi R^2 h f B} \cdot \frac{1}{NF_{ea}} \cdot \eta_{atm} \eta_{sys} \cdot PCG. \] (3.51)

Using Equation (3.50), the SNR for the synthetic aperture image becomes

\[ SNR_{sa} = \frac{P_t \rho_t A \eta_d}{\pi R^2 h f B} \cdot \frac{1}{NF_{ea}} \cdot \eta_{atm} \eta_{sys} \cdot PCG \cdot \frac{PRF \cdot D_{sa}}{v}. \] (3.52)
For a given synthetic aperture length and platform velocity, the SNR of the image can then be improved by increasing the PRF of the system, which allows more pulses to be coherently processed together to form the image.

### 3.6.2. Cross-Range Resolution

The synthetic aperture lengths that can be achieved for each mode of operation has been summarized above. Figure 3.14 shows the synthetic aperture lengths that are needed to obtain a given cross-range resolution, as derived in Equation (3.13), as a function of range for a broadside scenario, where $\theta_A=\pi/2$. A wavelength of 1.5 $\mu$m and a velocity of 200 m/s are used to generate this plot. These synthetic aperture lengths can be realized in both the spotlight and hybrid modes of operation provided the beam is steered to illuminate the target for the necessary amount of time, $T_{sa}$ (i.e. $D_{sa}/v$). For the stripmap mode of operation, the obtainable cross-range resolution is limited by the size of the transmit aperture, as derived in Equation (3.47). In this case, the cross-range resolution values depicted in Figure 3.14 can only be obtained if the aperture diameter is twice the desired cross-range resolution value for a given range.

![Figure 3.14. Broadside cross-range resolution values as a function of synthetic aperture length and range](image-url)
3.6.3. *Doppler Ambiguities*

The spectrum of a single rectangular pulse, which is a sinc function, is depicted by the red trace in Figure 3.15. By comparison, the spectrum of a series of rectangular pulses consists of a series of peaks that are separated in frequency by the PRF, and follow the same envelope as the spectrum of the single pulse. This is depicted in the black trace of Figure 3.15. When a Doppler shift is present, the spectrum will be shifted by the Doppler frequency. Therefore, the location of the first peak will yield the Doppler frequency that was measured.

![Figure 3.15. The spectrum of a single pulse (red) and a series of pulses (blue)](image-url)
Figure 3.16. Geometry for determining the necessary PRF to avoid Doppler ambiguities

As depicted in Figure 3.16, the transmitted beam will encounter a spread of Doppler frequencies determined by the steer angle and angular spread of the beam. The Doppler frequency at the leading edge of the beam can be expressed as

$$f_{DL} = \frac{2v}{\lambda} \cos \left( \theta_A - \frac{\theta_0}{2} \right),$$

(3.53)

where $\theta_0$ is the beamwidth of the transmit beam, and $\theta_A$ is the steer angle. The Doppler frequency at the trailing edge of the beam is expressed as

$$f_{DT} = \frac{2v}{\lambda} \cos \left( \theta_A + \frac{\theta_0}{2} \right).$$

(3.54)

As depicted in Figure 3.17, in order to prevent Doppler ambiguities, the PRF must be larger than the spread of the Doppler frequencies, or

$$PRF > \Delta f_D = f_{DL} - f_{DT}.$$  

(3.55)

Here, $\Delta f_D = 500 \text{ Hz}$ was simulated. When the PRF is larger than $\Delta f_D$, as depicted in Figure 3.17 (a), the spacing between the red and blue peaks, which are associated with the Doppler
frequencies at the leading and trailing edges of the beam respectively, is smaller than the spacing associated with the PRF. In this case, the frequencies at the leading and trailing edges of the beam are resolved. However, when the PRF is smaller than $\Delta f_D$, as depicted in Figure 3.17 (b), the Doppler frequencies associated with the leading and trailing edges of the beam become ambiguous because the spacing associated with the PRF is now smaller than that associated with $\Delta f_D$. The PRF of the system must then satisfy the condition

$$PRF_{\text{min}} = f_{DL} - f_{DR} = \frac{2v}{\lambda} \left[ \cos \left( \theta_A - \frac{\theta_0}{2} \right) - \cos \left( \theta_A + \frac{\theta_0}{2} \right) \right].$$

(3.56)

Using trigonometric identities, this expression can be simplified to yield

![Figure 3.17. Doppler frequency spectra showing (a) resolved Doppler frequencies of the leading and trailing edges of the pulse (PRF $> \Delta f_0$), and (b) Doppler ambiguity when the Doppler frequencies of the leading and trailing edges of the beam are unresolved (PRF $< \Delta f_0$).](image)
\[ PRF_{\text{min}} = \frac{2v}{\lambda} \left[ \cos \theta_A \cos \frac{\theta_0}{2} + \sin \theta_A \sin \frac{\theta_0}{2} - \cos \theta_A \cos \frac{\theta_0}{2} + \sin \theta_A \sin \frac{\theta_0}{2} \right] \]
\[ = \frac{4v}{\lambda} \sin \theta_A \sin \frac{\theta_0}{2} \approx \frac{2v \theta_0}{\lambda} \sin \theta_A, \]  

(3.57)

where the small angle approximation was made in the final step. Assuming the system transmits a diffraction limited spot size, such that

\[ \theta_0 = \frac{\lambda}{D_{ra}}, \]  

(3.58)

where \( D_{ra} \) is the diameter of the receiver aperture, the minimum PRF can be expressed in terms of the aperture diameter as

\[ PRF_{\text{min}} = \frac{2v}{D_{ra}} \sin \theta_A. \]  

(3.59)

This means that the target must at least be sampled at an interval corresponding to

\[ d_{\text{max}} = \frac{v}{PRF} = \frac{D_{ra}}{2 \sin \theta_A} \]  

(3.60)

to prevent Doppler ambiguities. For the broadside case, at a minimum the system must send out a pulse every time it travels a distance equivalent to one half the transmit diameter.

3.7. Conclusion

This chapter has summarized many of the important aspects in synthetic aperture ladar. A synthetic aperture image will experience fine cross-range resolution in the direction of platform motion, which has been defined here as the x-dimension. The hybrid mode synthetic aperture ladar techniques presented in this chapter allow for coverage in the x-direction that can increase as the platform moves. However, coverage in the y-dimension is limited by the spot size of the system at range. Resolution in the range dimension is determined by the bandwidth of the transmitted pulse. The following chapter will explore the self-phase
modulation resulting from a saturated semiconductor optical amplifier, and the impact this has on the bandwidth of the pulse.
4.1. Introduction

Optical amplifiers are an essential part of laser radar systems since they can amplify the transmitted signal to sufficient power levels for long range applications. However, when operated in the saturation regime, Semiconductor Optical Amplifiers (SOA’s) exhibit a deterministic self-phase modulation that is generally avoided. This chapter will investigate the conditions under which this self-phase modulation could be exploited as signal bandwidth to improve the range resolution of the system. The key SOA parameters affecting the modulation of the output pulse will be addressed and optimized, and their impact on the ideal pulse response of a laser radar system will be explored. This analysis will show that the range resolution of a laser radar system can be optimized by saturating a SOA with a carrier lifetime that is one half the FWHM Gaussian input pulse duration, yielding a substantial improvement in range resolution that is highly insensitive to variations in the input pulse duration and energy. Before this analysis is presented, the theory behind the operation of a SOA will be summarized.

4.2. Semiconductor Optical Amplifier Architecture

A diagram of a SOA is seen in Figure 4.1. As can be seen, the SOA design is almost identical to that of a semiconductor laser, but unlike a laser the SOA has anti-reflection coatings at either facet to prevent reflections of the input signal. Thus, the signal experiences gain in a
single pass through the amplifier, and this type of amplifier is often called a travelling wave amplifier.

![Architecture of a SOA](image)

**Figure 4.1. Architecture of a SOA**

The SOA consists of a p-type and n-type high band gap cladding, surrounding a gain region made up of a lower band gap material. Lower band gap semiconductors have a high index of refraction, which allows the SOA to act as a waveguide, confining the majority of the input signal to the gain region. Gain occurs due to the presence of the injection current. Incident photons in the gain region stimulate the recombination of an electron-hole pair, thus emitting a photon that is identical in phase and polarization to that of the incident photon. The current must be sufficient to allow for a net gain. The next section will discuss the equations that can be used to model a SOA.

**4.3. Semiconductor Optical Amplifier as a Two Level System**

A SOA is generally modeled as a two-level system where carriers occupy one of two discrete energy levels known as the conduction band and the valence band. When carriers recombine, they fall from the higher energy conduction band to the lower energy valence band,
and the excess energy is generally released in the form of a photon. Two such processes are depicted in Figure 4.2.

![Spontaneous Emission and Stimulated Emission](image)

Figure 4.2. Depiction of (a) spontaneous emission and (b) stimulated emission

Spontaneous emission is a random process where an electron in the conduction band decays to the valence band, emitting a photon of random phase and direction. Stimulated emission occurs when an electron interacts with a photon, causing it to decay from the conduction band to the valence band. In this case, the emitted photon is identical in phase and direction to that of the incident photon. Both of these processes are considered radiative processes because a photon is emitted. There are also nonradiative processes that can occur, where the recombination of the electron results in the emission of heat rather than a photon. These radiative and nonradiative processes are the basis of the operation of a SOA and will be included in the SOA model in the following section.

4.4. Semiconductor Optical Amplifier Model

In general, the operation of the SOA is described by the rate of change of the carriers in the active region. This is determined by the generation and recombination of electron-hole pairs, or
\[
\frac{dN}{dt} = \frac{dN_g}{dt} - \frac{dN_r}{dt},
\]

(4.1)

where \(N\) is the carrier density with units of \([m^{-3}]\), \(dN_g/dt\) represents the rate at which carriers are generated with units of \([m^{-3}/s]\), and \(dN_r/dt\) represents the recombination of carriers with units of \([m^{-3}/s]\). Carriers are generated by the injection current, \(I\). For an active region of volume \(V\), the carrier density generation rate is

\[
\frac{dN_g}{dt} = \frac{I}{qV},
\]

(4.2)

where \(q\) is the electron charge, which converts the current from units of \([\text{Coulombs}/s]\) to carriers per second, or \([s^{-1}]\). The carrier density recombination rate is governed by the emission of photons and can be expressed as

\[
\frac{dN_r}{dt} = \frac{N}{\tau_c} + \frac{g(N)}{\hbar\omega_0} |A(z, t)|^2,
\]

(4.3)

where \(\tau_c\) is the carrier lifetime, \(\hbar\omega_0\) is the photon energy, \(|A(z, t)|^2\) is the power of the output signal, which has a spatial distribution in \(z\) and a temporal distribution in \(t\), and \(g(N)\) is the gain, which is a function of the carrier density and will be defined below. The carrier lifetime is defined as

\[
\frac{1}{\tau_c} = \frac{1}{\tau_{sp}} + \frac{1}{\tau_{nr}},
\]

(4.4)

where \(\tau_{sp}\) is the lifetime due to spontaneous emission and \(\tau_{nr}\) is the lifetime due to the nonradiative recombination. The second term in Equation (4.3) accounts for the stimulated emission of photons and is governed by the gain of the amplifier, which is in terms of the carrier density and has units of \([m^{-3}]\). The rate of recombination due to the gain of the amplifier is governed by the output power of the amplifier, which has units of \([\text{Joules}/s]\). Dividing by the photon energy, which has units of \([\text{Joules}]\), yields the carrier density rate in units of \([m^{-3}/s]\). The carrier density rate of Equation (4.1) becomes
\[ \frac{\partial N(z, t)}{\partial t} = \frac{I}{qV} - \frac{N}{\tau_c} - \frac{g(N)}{\hbar \omega_0} |A(z, t)|^2, \quad (4.5) \]

The gain \( g(N) \) is defined as [20]

\[ g(N) = \Gamma a (N(z, t) - N_0), \quad (4.6) \]

where \( \Gamma \) is the confinement factor, and \( a \) is the gain coefficient, and \( N_0 \) is the carrier density required for transparency. The optical pulse envelope is defined as

\[ A(z, t) = \sqrt{P(z, t)} \exp[j\phi(z, t)], \quad (4.7) \]

where \( P(z, t) \) and \( \phi(z, t) \) are the power and phase of the pulse travelling through the amplifier.

An expression for the gain rate equation is found by taking the time derivative of Equation (4.6) to yield

\[ \frac{\partial g(z,t)}{\partial t} = \frac{\partial}{\partial t} [\Gamma a (N(z,t) - N_0)] = \Gamma a \frac{\partial N(z,t)}{\partial t} - 0. \quad (4.8) \]

Substituting Equation (4.5) into Equation (4.8) yields

\[ \frac{\partial g(z,t)}{\partial t} = \Gamma a \left[ \frac{I}{qV} - \frac{N(z,t)}{\tau_c} - \frac{g(N)}{\hbar \omega_0} |A(z,t)|^2 \right]. \quad (4.9) \]

Equation (4.6) is substituted into the second term of Equation (4.9) to yield the expression

\[ \frac{\partial g}{\partial t} = \frac{\Gamma a I}{qV} - \frac{g(N) + \Gamma a N_0}{\tau_c} - \frac{\Gamma a g(N)}{\hbar \omega_0} |A(z,t)|^2 \]

\[ = \frac{\Gamma a \tau_c}{qV} - \frac{\Gamma a N_0 - g(N)}{\tau_c} - \frac{\Gamma a g(N)}{\hbar \omega_0} |A(z,t)|^2 \quad (4.10) \]

\[ = \frac{\Gamma a N_0 \left( \frac{I \tau_c}{qVN_0} - 1 \right) - g(N)}{\tau_c} - \frac{\Gamma a g(N)}{\hbar \omega_0} |A(z,t)|^2. \]

This expression is simplified to

\[ \frac{\partial g(z,t)}{\partial t} = \frac{g_0 - g(N)}{\tau_c} - \frac{g(N)}{E_{sat}} |A(z,t)|^2 = \frac{g_0 - g(N)}{\tau_c} - \frac{g(N)}{E_{sat}} P(z,t) \quad (4.11) \]

where the small signal gain \( g_0 \) is defined as
\[ g_0 = \Gamma a N_0 \left( \frac{I \tau_c}{qVN_0} - 1 \right) = \Gamma a N_0 \left( \frac{I}{I_0} - 1 \right), \quad \text{(4.12)} \]

\( I_0 \) is the required current for transparency, and the saturation energy is defined as

\[ E_{\text{sat}} = \frac{\hbar \omega_0}{\Gamma a}. \quad \text{(4.13)} \]

As detailed in Appendix B, the propagation of a pulse through a waveguide can be described as

\[ \frac{\partial A(z, t)}{\partial z} = \frac{j \omega_o \Gamma}{2 \bar{n} c} \chi A(z, t) - \frac{1}{2} \alpha_{\text{int}} A(z, t), \quad \text{(4.14)} \]

where \( \alpha_{\text{int}} \) represents internal loss and \( \chi \) is the susceptibility of the SOA, which describes its response to the applied field and is defined as [20]

\[ \chi(N) = -\frac{\bar{n} c}{\omega_0} (\alpha + j) a (N - N_0). \quad \text{(4.15)} \]

Here, \( \bar{n} \) is the effective mode index, and \( \alpha \) is the linewidth enhancement factor, also known as the chirp parameter. The chirp parameter is a dimensionless value that describes the magnitude of the phase change induced by the amplifier. It generally has a value between 4 and 12, and it must be measured for each device [20]. Since the susceptibility and gain are a function of the carrier density, the effect of the carrier density on the pulse as it travels through the SOA is found by substituting Equations (4.6) and (4.15) into Equation (4.14) to yield

\[ \frac{\partial A(z, t)}{\partial z} = \frac{j \omega_o \left( \frac{g(z, t)}{a(N(z, t) - N_0)} \right)}{2 \bar{n} c} \left[ \frac{\bar{n} c}{\omega_0} (\alpha + j) a (N(z, t) - N_0) \right] A(z, t) \]

\[ -\frac{1}{2} \alpha_{\text{int}} A(z, t) = A(z, t) \left( \frac{g(z, t)}{2} - \frac{1}{2} \alpha_{\text{int}} - j \frac{g(z, t) \alpha}{2} \right) \quad \text{(4.16)} \]

\[ = \sqrt{P(z, t)} \exp(j \phi(z, t)) \left( \frac{g(z, t)}{2} - \frac{1}{2} \alpha_{\text{int}} - j \frac{g(z, t) \alpha}{2} \right). \]

The definition of \( A(z, t) \) in Equation (4.7) was used in the final line of Equation (4.16). However, from Equation (4.7) we can also derive the expression
\[ \frac{\partial A(z, t)}{\partial z} = \frac{\partial}{\partial z} \left[ \sqrt{P(z, t)} \exp(j\phi(z, t)) \right] \]

\[ = \frac{\partial}{\partial z} \sqrt{P(z, t)} \exp(j\phi(z, t)) + j \frac{\partial \phi(z, t)}{\partial z} \sqrt{P(z, t)} \exp(j\phi(z, t)). \]

(4.17)

By comparing the final expression of Equation (4.16) and the final expression of Equation (4.17), the following expressions for the pulse power and phase can be deduced:

\[ \frac{\partial \phi(z, t)}{\partial z} = -\frac{g(z, t)\alpha}{2} \quad \text{and} \quad \frac{\partial \sqrt{P(z, t)}}{\partial z} = \sqrt{P(z, t)} \left( \frac{g(z, t)}{2} - \frac{1}{2} \alpha_{int} \right). \]

(4.18)

Applying the power rule to the second expression yields

\[ \frac{\partial \sqrt{P(z, t)}}{\partial z} = \frac{1}{2} \frac{1}{\sqrt{P(z, t)}} \frac{\partial P(z, t)}{\partial z} = \sqrt{P(z, t)} \left( \frac{g(z, t)}{2} - \frac{1}{2} \alpha_{int} \right). \]

(4.19)

Finally, the power and phase of the pulse as it travels through the SOA can be expressed as

\[ \frac{\partial P(z, t)}{\partial z} = P(z, t) (g(z, t) - \alpha_{int}) \]

(4.20)

and

\[ \frac{\partial \phi(z, t)}{\partial z} = -\frac{g(z, t)\alpha}{2}. \]

(4.21)

If the internal loss is insignificant ($\alpha_{int}=0$), Equation (4.20) can be rewritten as

\[ \frac{\partial P(z, t)}{\partial z} \frac{1}{P(z, t)} = g(z, t). \]

(4.22)

Both sides of Equation (4.22) can be integrated over the length of the amplifier to yield

\[ \ln[P(L, t)] - \ln[P(0, t)] = \int_0^L g(z, t) \, dz, \]

(4.23)

where $L$ is the length of the amplifier. Note that $P(L, t)$ represents the output power, or the power at position $z = L$ in the amplifier, and $P(0, t)$ represents the input power, or the power at position $z = 0$ in the amplifier. From Equation (4.23), the output power can be expressed in terms of the input power as
\[ P_{\text{out}}(t) = P_{\text{in}}(t) \exp[h(t)], \quad (4.24) \]

where

\[ h(t) = \int_0^L g(z, t) \, dz. \quad (4.25) \]

Both sides of Equation (4.21) can be integrated over the length of the amplifier to yield

\[ \int_0^L \partial \phi = \phi(L, t) - \phi(0, t) = -\int_0^L \frac{g(z, t)\alpha}{2} \, dz. \quad (4.26) \]

Using Equation (4.25), Equation (4.26) becomes

\[ \phi_{\text{out}}(t) = \phi_{\text{in}}(t) - \frac{\alpha}{2} h(t). \quad (4.27) \]

This expression describes the output phase of a pulse after it travels through the amplifier.

An expression for \( h(t) \) can be found by integrating over Equation (4.11) such that

\[ \frac{\partial}{\partial t} \int_0^L g(z, t) \, dz = \int_0^L \frac{g_0}{\tau_c} \, dz - \int_0^L \frac{g(z, t)}{\tau_c} \, dz - \frac{g(z, t) E_{\text{sat}}}{P(z, t)} \, dz \]

\[ = \frac{g_0 L}{\tau_c} - \int_0^L \frac{g(z, t)}{\tau_c} \, dz - \int_0^L \frac{g(z, t)}{E_{\text{sat}}} P(z, t) \, dz \quad (4.28) \]

Using Equation (4.25), Equation (4.28) becomes

\[ \frac{\partial h(z, t)}{\partial t} = \frac{g_0 L - h(z, t)}{\tau_c} - \int_0^L \frac{g(z, t)}{E_{\text{sat}}} P(z, t) \, dz. \quad (4.29) \]

Assuming the losses are negligible, Equation (4.20) can be substituted into the second term of Equation (4.29) to yield

\[ \frac{\partial h(z, t)}{\partial t} = \frac{g_0 L - h(z, t)}{\tau_c} - \frac{1}{E_{\text{sat}}} \int_0^L \frac{\partial P(z, t)}{\partial z} \, dz. \quad (4.30) \]

The partial derivative with respect to \( z \) and the integration with respect to \( z \) cancel each other out, so \( P(z, t) \) is evaluated at the limits of integration \( z = 0 \) and \( z = L \) to yield

\[ \frac{\partial h(z, t)}{\partial t} = \frac{g_0 L - h(z, t)}{\tau_c} - \frac{1}{E_{\text{sat}}} (P_{\text{out}}(t) - P_{\text{in}}(t)). \quad (4.31) \]

Finally, using Equation (4.24), Equation (4.31) is expressed as
\[
\frac{\partial h(z, t)}{\partial t} = \frac{g_0 L - h(z, t)}{\tau_c} - \frac{P_{in}(t)}{E_{sat}} \left[ \exp(h(z, t)) - 1 \right]. \tag{4.32}
\]

If a Gaussian input pulse is assumed, the input power is expressed as

\[
P_{in}(t) = \frac{E_{in}}{\tau_0 \sqrt{\pi}} \exp\left(-\frac{t^2}{\tau_0^2}\right). \tag{4.33}
\]

The output power and phase of the saturated SOA are described by Equations (4.24) and (4.27), respectively. In order to solve both of these equations, the ordinary differential equation in Equation (4.32) must be solved. Equation (4.32) cannot be solved analytically and must be solved numerically. For this research, the \texttt{ode23t} command in Matlab\textsuperscript{®} was used to numerically solve this differential equation.

4.5. The Saturation Energy and Carrier Lifetime

The saturation energy \(E_{sat}\) is defined as the energy of a short pulse which leads to a reduction in gain to 1/e of its small signal value [39]. When defining the saturation energy, the input pulse duration is assumed to be much shorter than the carrier lifetime so there is no gain recovery. As the input pulse travels through the SOA, the pulse energy increases as the pulse experiences gain. The SOA can start to experience partial saturation for energies less than the saturation energy, and if the pulse energy reaches a level equal to that of the saturation energy, then the gain will have decreased to 1/e of its small signal value. If the gain is defined as \(P_{out}(t)/P_{in}(t)\), Equation (4.24) can be rearranged to yield the expression

\[
G_{sat} = \frac{P_{out}(t)}{P_{in}(t)} = \exp(h(t)). \tag{4.34}
\]

If \(\tau_p \ll \tau_c\), Equation (4.32) can be solved for \(h(t)\) and substituted into Equation (4.34) to verify the definition of the saturation energy. The input pulse, calculated according to Equation (4.33) for \(E_{in} = 1\) mJ and \(\tau_p = 0.1\) ns, is shown by the solid line of Figure 4.3 (a). Equation (4.32) was solved for \(h(t)\) assuming \(\alpha = 8\), \(G_0 = 30\) dB, \(\tau_c = 1\) ns, and \(E_{sat} = 10\) mJ. The result was
substituted into Equation (4.24), and the output pulse power $P_{out}(t)$ is shown by the dotted line of Figure 4.3 (a). The output power was divided by the input power, as in Equation (4.34), to yield the gain plotted in Figure 4.3 (b). As can be seen, the gain is initially at its small signal value of 1000. However, the pulse energy is amplified as it travels through the SOA, and the gain begins to saturate. The output pulse energy (i.e., the area under the curve representing the output pulse power) is plotted as a function of time, shown as the dotted line in Figure 4.3 (b). The dotted blue line drawn from the left axis of Figure 4.3 (b) indicates where the gain has reduced by 1/e of its small signal value of 1000. The dotted blue line then drops down to intersect the output pulse energy at the same point in time, where it can be seen to intercept the right axis of Figure 4.3 (b) at an output energy of 10 mJ. This confirms the simulated saturation energy of 10 mJ.

However, when the pulse duration and carrier lifetime are similar in value, gain recovery occurs as the pulse is travelling through the SOA, thus the pulse experiences more gain. As such, the point at which the gain has reduced to 1/e of its small signal value no longer represents the saturation energy. This is depicted in Figure 4.4 (a), where the input pulse power was calculated
for $E_{in} = 1 \text{ mJ}$ and $\tau_p = 2 \text{ ns}$. The output pulse was found by calculating $h(t)$ assuming $\alpha = 8$, $G_0 = 30 \text{ dB}$, $\tau_c = 1 \text{ ns}$, and $E_{sat} = 10 \text{ mJ}$. The gain and output energy, shown in Figure 4.4 (b), shows that the point at which the gain has reduced to $1/e$ of its small signal value no longer corresponds to the nominal saturation energy.

![Graphs showing input power, output power, and gain vs. time](image)

**Figure 4.4.** An example of the impact of the carrier lifetime on the saturation characteristics of a SOA for $\tau_p \approx \tau_c$

Upon comparing Figure 4.3 and Figure 4.4, it becomes evident that the carrier lifetime has a significant impact on the saturation characteristics of the SOA, and thus the amount of gain experienced by the input pulse. An inspection of Equation (4.34) indicates that the carrier lifetime must affect $h(t)$. It follows from Equation (4.27) that the carrier lifetime must also affect the self-phase modulation experienced by the output pulse. The following section will further explore the effects of varying the parameters of the SOA.

**4.6. Characteristics of the Output Signal of a Saturated Semiconductor Optical Amplifier**

As seen in Equations (4.24), (4.27), and (4.32), the key parameters that determine the phase and magnitude of the pulse after propagating through the SOA are the SOA’s carrier lifetime $\tau_c$, unsaturated gain $G_0 = g_0L$, and chirp parameter $\alpha$, as well as the ratio of the input
pulse energy to the saturation energy of the SOA $E_{in}/E_{sat}$ and the input pulse duration $\tau_p$.

Section 4.6.1 will investigate the impact of the carrier lifetime and input pulse energy for a Gaussian pulse and Section 4.6.2 will address the impact of the unsaturated gain and the chirp parameter.

4.6.1. Effects of the Carrier Lifetime and Input Energy

To explore the effects of the carrier lifetime, as well as the ratio of the input pulse energy to the saturation energy of the SOA, Equation (4.32) was solved for $h(t)$ assuming a 1 ns FWHM Gaussian input pulse. All of the simulations in this section will assume a SOA with a mid-level chirp parameter of $\alpha=8$ and a high unsaturated gain of $G_0=30$ dB. The effects of varying the carrier lifetime are explored in Figure 4.5. The output pulse shape, calculated according to Equation (4.24), is shown in Figure 4.5 (a) for carrier lifetimes of 0.1 ns and 2.0 ns. Each pulse is normalized to compare the pulse shapes against one another, and the dotted line represents the input 1 ns FWHM Gaussian input pulse. As can be seen, the carrier lifetime has a significant impact on the pulse shape. After the leading edge of the input pulse saturates the amplifier, the carriers repopulate the active region while the trailing edge of the pulse propagates through the amplifier. When the carrier lifetime is much smaller than the pulse duration, the gain recovers quickly, allowing the leading and trailing edges of the pulse to experience more gain than the center of the pulse. The increased gain experienced by the leading and trailing edges of the pulse results in a temporal broadening that increases the temporal FWHM of the output pulse as compared to the input pulse. As the carrier lifetime increases this effect becomes more pronounced, until the carrier lifetime becomes so long that the leading edge of the input pulse begins to experience more gain than the trailing edge. This yields the asymmetric pulse shape shown for a carrier lifetime of 2.0 ns. As seen in Figure 4.5 (b), the gain recovery also affects the phase of the output pulse, which is calculated according to Equation (4.27). The pulse is
assumed to have no phase modulation prior to entering the SOA (i.e., $\phi_{in}(t) = 0$). When the carrier lifetime is significantly smaller than the pulse duration, the gain fully recovers, and the self-phase modulation is symmetric, as seen for a carrier lifetime of 0.1 ns in Figure 4.5 (b). As the carrier lifetime increases, the duration of the saturation increases, and the gain does not fully recover during the pulse duration. As a result, the magnitude of the self-phase modulation increases, but the self-phase modulation becomes asymmetric, as shown for a carrier lifetime of 2.0 ns in Figure 4.5 (b). The spectrum of the output pulse, shown in Figure 4.5 (c), is defined to be

$$S_t(\omega) = \left| \Im \left[ \left( P_{out}(t) \right)^{1/2} \exp[j\phi_{out}(t)] \right] \right|^2,$$

(4.35)

Figure 4.5. A summary of the impact of the carrier lifetime on the (a) power, (b) phase, (c) pulse spectra, and (d) IPR of the output pulse. This simulation assumes $\alpha=8$, $G_0=30$ dB, and a 1 ns FWHM Gaussian pulse at the input to the SOA with an energy one tenth the saturation energy.
where $\mathcal{F}\{\cdot\}$ denotes the Fourier transform operation. Increasing the phase modulation has the effect of increasing the bandwidth of the pulse, as indicated by the broadened pulse spectra of Figure 4.5 (c). As the carrier lifetime increases and the self-phase modulation becomes increasingly asymmetric, the spectrum also becomes increasingly asymmetric.

The IPR was calculated using the matched filtering techniques presented in Equation (2.56), which is repeated here as

$$s_M(t) = \int_{-\infty}^{\infty} s_r(\tau)s_\tau^*(\tau - t)d\tau, \quad (4.36)$$

where $s_r(\tau)$ is the received signal, and $s_\tau^*(\tau - t)$ is the time reversed conjugate of the monitor signal. The detection of the signal will be explored in more detail in Section 4.9, for now we assume the signal is detected using the I/Q demodulation techniques described in Chapter 2, and the monitor signal is defined according to Equation (2.44) as

$$s_m(t) = u_{MO}u_t(t)\exp(j\theta(t)) = u_{MO}\sqrt{P_{out}(t)}\exp(j\phi_{out}(t)), \quad (4.37)$$

where $u_{MO}$ is the amplitude of the master oscillator, $u_t(t)$ is the envelope of the transmitted pulse, and $\theta(t)$ is the phase modulation of the transmitted pulse. We are assuming the transmission of a pulse after it propagates through a saturated SOA, so $u_t(t) = \sqrt{P_{out}(t)}$ and $\theta(t) = \phi_{out}(t)$, where $P_{out}(t)$ and $\phi_{out}(t)$ are the output power and phase in Equations (4.24) and (4.27), respectively. Similarly, the received signal is the time delayed version of the monitor, or

$$s_m(t) = u_{MO}\sqrt{P_{out}(t - t_{rt})}\exp(j\phi_{out}(t - t_{rt})), \quad (4.38)$$

where $t_{rt}$ is the roundtrip travel time of the pulse. The IPR for each pulse, calculated according to Equation (4.36), is shown in Figure 4.5 (d). Here, the values of $u_{MO}$ and $t_{rt}$ are inconsequential because the results are normalized with respect to amplitude and time. The resulting IPRs exhibit sidelobes with peak values on the order of -10 dB. Before examining the IPR in more detail, the effect of the input pulse energy will be explored.
Variations in the energy of the input pulse are explored in Figure 4.6, where a carrier lifetime of 2 ns is assumed. Increasing the energy of the input pulse, shown in Figure 4.6 (a), results in a sharper leading edge of the output pulse, which increases the FWHM of the output pulse. Increasing the input energy also has the effect of saturating the amplifier earlier in the pulse duration, thus the remainder of the pulse experiences more gain recovery. This results in an increase in the magnitude the self-phase modulation, as shown in Figure 4.6 (b). The increase in self-phase modulation increases the bandwidth of the pulse, as seen in Figure 4.6 (c). This results in a narrowing of the IPR, as shown in Figure 4.6 (d). Note that the peak sidelobes of each IPR remain around -10 dB.
Figure 4.7. A summary of the (a) FWHM and (b) saturated gain of the output pulse as a function of carrier lifetime and input pulse energy. This simulation assumes $\alpha=8$ and $G_0=30$ dB.

The effect of the carrier lifetime and input pulse energy on the output pulse is summarized in Figure 4.7, which shows the FWHM of the output pulse and the saturated gain as a function of carrier lifetime and the ratio of input pulse energy to saturation energy. As discussed above, for smaller carrier lifetimes, the pulse remains somewhat symmetric, and the FWHM increases as the leading and trailing edges of the pulse experience increased gain with respect to the center of the pulse. However, as the leading edge begins to experience more gain than the trailing edge of the pulse, the FWHM decreases as the output pulse becomes asymmetric. In addition, increasing the pulse energy increases the FWHM of the pulse as the leading edge rises more sharply. These trends are seen in Figure 4.7 (a). Moreover, as shown in Figure 4.7 (b), increasing the carrier lifetime reduces the amount of gain experienced by the pulse as it takes longer for the gain to recover after saturation. Increasing the input pulse energy also decreases the saturated gain experienced by the pulse as the saturation occurs earlier in
the pulse duration. Here, the saturated gain is defined as the ratio of the peak output power to the peak input power.

As discussed in Chapter 2, the PSLR is the ratio of the peak power of the main lobe of the IPR to the peak power of the largest sidelobe of the IPR, while the ISLR is the ratio of the energy contained in the main lobe of the IPR to the energy contained in the sidelobes. This is illustrated again in Figure 4.8 for convenience. The bandwidth of the pulse can be defined by the temporal width of the IPR 3 dB down from its maximum value such that

\[
B = \frac{1}{\Delta \tau_{3dB}},
\]

(4.39)
in which case the range resolution defined in Equation (2.117) becomes
The effect of the carrier lifetime and the input pulse energy on the IPR is summarized in Figure 4.9. Figure 4.9 (a) shows the range resolution, as calculated in Equation (4.40), as a function of carrier lifetime. As the carrier lifetime initially increases, the increased gain recovery time increases the magnitude and duration of the self-phase modulation, which narrows the IPR and improves the range resolution. However, as the carrier lifetime and thus the magnitude of the
self-phase modulation continues to increase, the self-phase modulation becomes increasingly asymmetric. This asymmetry results in a temporal broadening of the IPR, which in turns degrades the range resolution. An optimum carrier lifetime balances the narrowing of the IPR due to the increased self-phase modulation with the temporal broadening of the IPR due to the asymmetric self-phase modulation. In Figure 4.9 (a), the optimum carrier lifetime is seen to be one half the FWHM of the input pulse, or 0.5 ns. For reference, the carrier lifetime required to obtain an optimum range resolution is calculated for a range of input pulse durations, as shown in Figure 4.9 (b). As can be seen, the optimum carrier lifetime over the pulse durations of interest is always one half the FWHM of the input pulse.

![Figure 4.10. The impact of the carrier lifetime on the ISLR and PSLR of the IPR. This simulation assumes $E_{in}/E_{sat}=0.1$, $\alpha=8$ and $G_0=30$ dB](image)

The IPR is also characterized by the ISLR and PSLR, shown in Figure 4.10. The ISLR decreases slightly as the carrier lifetime increases, indicating that an increase in carrier lifetime causes a small amount of energy within the IPR to spread outside of the 3 dB width of the IPR. The PSLR also shows an overall decreasing trend with an increasing carrier lifetime, indicating
that the peak sidelobe levels slightly increase for carrier lifetimes greater than 0.4 ns. In any
event, the variations in the ISLR and PSLR with carrier lifetime are not significant and are seen to
be on the order of 1.6 dB, and less than 1 dB, respectively.

Figure 4.11. A summary of the impact of the carrier lifetime on the relative gains of a pulse after travelling through the SOA. The (a) power, (b) phase, (c) pulse spectra, and (d) IPR of the output pulse are shown. This simulation assumes $\alpha=8$, $G_0=30$ dB, and a 1 ns FWHM Gaussian pulse at the input to the SOA with an energy one tenth the saturation energy. Results for the optimum carrier lifetime of 0.5 ns are presented.

Figure 4.11 further demonstrates the effect of the carrier lifetime on the saturated gain. The pulse shapes shown in Figure 4.11(a) are normalized to the pulse with the maximum output power (i.e. for $\tau_c = 0.1$ ns) to show the decrease in gain as the carrier lifetime increases. The effects of the decreasing pulse energies are evident in the spectra, shown in Figure 4.11 (c), and the IPRs, shown in Figure 4.11 (d) assuming a master oscillator amplitude of unity. Results for
the optimum carrier lifetime of 0.5 ns are included. The IPR for $\tau_c = 0.5$ ns is a compromise between the phase symmetry for shorter carrier lifetimes, and the increased magnitude of the self-phase modulation for longer carrier lifetimes, as seen in Figure 4.11 (b). Note that for a 1 ns FWHM Gaussian input pulse, the pulse emerging from the SOA in this simulation has broadened to a FWHM of 1.6 ns. However, due to the self-phase modulation, the IPR has a 3dB width of 0.24 ns, corresponding to a range resolution of 3.6 cm.

Figure 4.12. The impact of $E_{in}/E_{sat}$ on the (a) range resolution, which is defined as the 3 dB temporal width of the IPR, and the saturated gain, as well as (b) the ISLR and PSLR of the IPR. This simulation assumes $\tau_c = 0.5$ ns, $\alpha = 8$ and $G_0 = 30$ dB
The range resolution and saturated gain values for $\frac{E_{in}}{E_{sat}}$ ranging from 0 to 1 are shown in Figure 4.12 (a). Beyond an $\frac{E_{in}}{E_{sat}}$ value of approximately 0.1, there is very little improvement in the range resolution, though the gain continues to steadily decrease. In this general region of interest, the PSLR and ISLR change by less than 4 dB and 3 dB, respectively, as shown in Figure 4.12 (b).

This analysis shows that to optimize range resolution, the optimum carrier lifetime of the SOA is one half the FWHM of the input pulse. Furthermore, a trade-off exists between the increased self-phase modulation resulting from increasing the energy of the input pulse with respect to the saturation energy of the SOA, and the decreased gain experienced by the pulse. The ISLR and PSLR of the IPR do not show significant variations with carrier lifetime and input pulse energy.

4.6.2. Effects of the Unsaturated Gain and Chirp Parameter

To explore the effects of the unsaturated gain $G_0 = g_0 L$ and the chirp parameter $\alpha$ on the output pulse, Equation (4.3) is once again solved assuming a 1 ns FWHM Gaussian input pulse. Since the range resolution does not significantly improve for values of $\frac{E_{in}}{E_{sat}}$ larger than 0.1, this section will assume an input energy one tenth the saturation energy of the SOA. The optimum carrier lifetime of 0.5 ns will also be assumed. Figure 4.13 addresses the impact of the unsaturated gain $G_0$, for a chirp parameter of 8. As seen in Figure 4.13 (a), which was calculated according to Equation (4.24), increasing the unsaturated gain of the amplifier increases the gain experienced by the leading edge of the pulse, which results in sharper leading edge and increased asymmetry of the output pulse. Increasing the gain also increases the self-phase modulation experienced by the pulse, which was calculated according to Equation (4.27) and shown in Figure 4.13 (b). This increases the bandwidth of the pulse, which is evident by the
broader spectrum in Figure 4.13 (c), which was calculated using Equation (4.35). This narrows the width of the IPR seen in Figure 4.13 (d), which was calculated using Equation (4.36).

The effect of varying the chirp parameter is explored in Figure 4.14, where an unsaturated gain of 30 dB is assumed. As indicated by Equations (4.24) and (4.27), the chirp parameter only affects the phase of the pulse and has no affect on the pulse amplitude. This is reflected in Figure 4.14 (a), where the pulse profiles are identical for $\alpha = 2$ and $\alpha = 12$. However, the significant effect on the magnitude of the self-phase modulation is evident in Figure 4.14 (b). The increased self-phase modulation broadens the pulse spectrum and narrows the IPR, as seen in Figure 4.14 (c) and (d) respectively.

![Figure 4.13. A summary of the impact of the unsaturated gain on the (a) power, (b) phase, (c) pulse spectra, and (d) IPR of the output pulse. This simulation assumes $\alpha=8$, $E_{in}/E_{sat}=0.1$, $\tau_c=2.0$ ns, and a 1 ns FWHM Gaussian pulse at the input to the SOA.](image)
Next, the FWHM and saturated gain as a function of the unsaturated gain are shown in Figure 4.15. Increasing the unsaturated gain also increases the FWHM of the output pulse as the gain experienced by the leading and trailing edges of the pulse increases with respect to the gain experienced by the center of the pulse. The saturated gain as a function of unsaturated gain is shown in Figure 4.15 (b). Since the chirp parameter affects only the phase and not the magnitude of the pulse, it has no affect on the FWHM or gain of the output pulse.

Finally, the effect of the unsaturated gain and the chirp parameter on the IPR is summarized in Figure 4.16. As seen in Figure 4.16 (a), the range resolution improves with
increases in the unsaturated gain or chirp parameter. The ISLR shows less than 1 dB in variation as the unsaturated gain increases, although an increase in chirp parameter from 2 to 12 can cause the ISLR to decrease by roughly 2.5 dB. The PSLR is shown only for cases where there are discernable peaks. For instance, in Figure 4.14 (d) the IPR has no discernable peaks when $\alpha = 2$. As can be seen, the PSLR shows no more than 2 dB of variation as the unsaturated gain is increased.

![Figure 4.15](image.png)

**Figure 4.15.** A summary of the (a) FWHM and (b) saturated gain of the output pulse as a function of the unsaturated gain. This simulation assumes $\tau_c = 0.5$ ns and $E_{in}/E_{sat} = 0.1$. The FWHM and saturated gain are not affected by the chirp parameter.

This analysis shows that in order to optimize the range resolution and saturated gain experienced by the pulse, it is advantageous to choose a SOA with maximum unsaturated gain. The unsaturated gain values of a SOA are typically between 10 dB and 30 dB [40]. This analysis has also shown that the range resolution can be optimized by maximizing the chirp parameter. Although the range resolution improves significantly and the PSLR decreases slightly as the chirp parameter is increased, it does have the negative impact of slightly decreasing the ISLR. However, this is only a slight disadvantage as compared to the significant range resolution improvement.
Figure 4.16. The impact of the saturated gain and chirp parameter on the (a) range resolution, which is defined as the 3 dB width of the IPR, (b) ISLR of the IPR, and (c) PSLR of the IPR. This simulation assumes $\tau_c=0.5$ ns and $E_{in}/E_{sat}=0.1$. 
4.7. Comparison of the Amplifier Phase Modulation to LFM and Transform Limited Waveforms

The above analysis has demonstrated the characteristics of SOA’s and their impact on the IPR of a laser radar system. Based on the analysis presented above, the SOA parameters summarized in Table 4.1 are assumed for a 1 ns FWHM Gaussian input pulse. The carrier lifetime of 0.5 ns was found to optimize the range resolution of the system. A high unsaturated gain of 30 dB is assumed. An input energy such that $E_{in}/E_{sat} = 0.1$ is assumed since the range resolution does not significantly improve beyond this value, as shown in Figure 4.12 (a). This allows for significant phase modulation while also providing a saturated gain of approximately 18.2 dB, as shown in Figure 4.9 (a) and Figure 4.12 (a). If the source cannot emit pulses with sufficient energy to saturate the amplifier, it is assumed that additional optical amplifiers could be employed prior to the SOA in order to achieve the necessary input energy, provided that the additional amplifiers are not saturated and do not result in any appreciable phase modulation prior to the saturated SOA. A chirp parameter of 8 is assumed to be reasonable as it is between the minimum value of 2 and the maximum value of 12 cited in the literature [41].

Table 4.1. Assumed SOA Parameters and Resulting Output Pulse Parameters

<table>
<thead>
<tr>
<th>SOA Parameters</th>
<th>Output Pulse Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Variable</strong></td>
<td><strong>Designation</strong></td>
</tr>
<tr>
<td>Input Pulse FWHM</td>
<td>$\tau_p$</td>
</tr>
<tr>
<td>Carrier Lifetime</td>
<td>$\tau_c$</td>
</tr>
<tr>
<td>Unsaturated Gain</td>
<td>$G_0$</td>
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<tr>
<td>Ratio of Input Energy to Saturation Energy</td>
<td>$E_{in}/E_{sat}$</td>
</tr>
<tr>
<td>Chirp Parameter</td>
<td>$\alpha$</td>
</tr>
</tbody>
</table>
Figure 4.17. IPRs for an amplifier modulated pulse, a 20 µs LFM pulse, and a transform limited Gaussian pulse, all with a range resolution of 3.6 cm. The highest sidelobe level for the amplifier modulated IPR is 2 dB higher than that of the traditional LFM pulse.

As previously discussed, the primary motivation for this work is to investigate methods that enable the transmission of high bandwidth pulses that are also shorter in duration than those typically used, thereby minimizing the detrimental effects due to target motion. Figure 4.17 offers a comparison of the IPR obtained using the parameters outlined in Table 4.1 to those obtained using the longer duration LFM pulses that are typically employed. Here, a 20 µs LFM pulse with a chirp rate of $1.8 \times 10^{14}$ Hz/s was simulated, yielding a 3 dB range resolution of 3.6 cm [34]. The IPR for a transform limited Gaussian pulse with a 0.17 ns FWHM, yielding a range resolution of 3.61 cm, is also included for reference. The characteristics of each pulse are summarized in Table 4.2. In each case, the 3 dB widths of the main lobe are identical. The ISLR
for the amplifier modulated pulse is degraded by about 2.4 dB as compared to the transform limited pulse, and by about 0.9 dB as compared to the LFM pulse. The largest sidelobe is about 2.5 dB higher for the SOA-modulated pulse than the traditional LFM pulse. Overall, the use of SOA’s in short pulse laser radar systems appears to be highly promising.

Table 4.2. Comparison of amplifier modulated, LFM, and transform limited Gaussian pulses

<table>
<thead>
<tr>
<th></th>
<th>Amplifier Modulated</th>
<th>LFM</th>
<th>Transform Limited</th>
</tr>
</thead>
<tbody>
<tr>
<td>FWHM</td>
<td>1.6 ns</td>
<td>20 µs</td>
<td>0.17 ns</td>
</tr>
<tr>
<td>∆R</td>
<td>3.6 cm</td>
<td>3.6 cm</td>
<td>3.6 cm</td>
</tr>
<tr>
<td>ISLR</td>
<td>2.36 dB</td>
<td>3.90 dB</td>
<td>4.79 dB</td>
</tr>
<tr>
<td>PSLR</td>
<td>10.8 dB</td>
<td>13.3 dB</td>
<td>n/a</td>
</tr>
</tbody>
</table>

4.8. Consequence of Input Pulse Variations

A pulsed source will experience some variation in pulse duration and energy. The effect on the IPR of a ±10% variance in both pulse duration and energy is summarized in Figure 4.18. In each case, it is assumed that the output of the SOA has been monitored and is used as the matched filter, therefore the matched filter accounts for the variations in pulse duration and energy. Assuming the SOA parameters outlined in Table 4.1, a ±10% change in pulse duration for a 1 ns FWHM Gaussian pulse affects the range resolution by ±10%. For a pulse duration of 1 ns, a ±10% change in input energy affects the range resolution by ±2%. Note that the SOA is much more sensitive to changes in pulse duration than variations in pulse energy. However, even for the variations in pulse duration, the amplifier has little impact on the IPR since a transform limited pulse with variations of ±10% in pulse duration would also experience variations of ±10% in range resolution, as determined by Equation (2.111). This suggests that
for a reasonably stable source a monitor may not be necessary, as the amplifier does not introduce significant additional variations in the IPR.

If a single matched filter is constructed using the SOA parameters outlined in Table 4.2, the matched filter output for pulses with varying pulse durations can be calculated as shown in Figure 4.19 (a). In this case, the matched filter will not account for variations in the transmitted pulse. The solid line represents the IPR, when the matched filter is perfectly matched to the received signal for $\tau_p = 1$ ns. We see that variations in pulse duration result in an asymmetric matched filter output, but little change in the 3 dB temporal width of the matched filter output as compared to the IPR. As indicated in Figure 4.19 (b), a change in pulse duration of ±10% yields a 0.7 dB variation in ISLR and a 0.36 cm variation in range resolution. It can also be deduced from Figure 4.19 (a) that variations in the sidelobe levels are much less than 1 dB, and therefore any change in the PSLR is considered insignificant. However, it is evident that the peak of the IPR shifts slightly in time as the duration of the output pulse varies, which will yield
an error in the range measurement. This is summarized in Figure 4.20, which shows that range
e errors on the order of a negligible ±0.55 cm could be expected for ±10% variations in the pulse
duration.

Figure 4.19. Effect of variations in input pulse duration on the matched filter output for the SOA
parameters outlined in Table 1. In each case, the matched filter is fixed and assumes an input pulse
duration of $\tau_p=1$ ns.

![Figure 4.19](image-url)
The impact due to ±10% variations in the pulse energy, as shown in Figure 4.21 (a), is much less noticeable. Once again, the matched filter will not account for variations in the transmitted pulse. The solid line represents the IPR, when the matched filter is perfectly matched to the received signal for $E_{in}/E_{sat} = 0.1$. According to Figure 4.21 (b), this yields a negligible 0.01 dB variation in ISLR and a similarly negligible 0.10 cm variation in range resolution. Therefore, mismatches between the return signal and the matched filter due to variations in the pulse energy or duration at the input to the SOA do not produce significant variations in the matched filter output. This confirms that for a stable pulsed source with duration and energy variations on the order of ±10%, there is little error introduced if the matched filter is constructed for the ideal case rather than from monitoring the variations in each transmitted pulse. This offers the opportunity to design greatly simplified laser radar systems.

Figure 4.20. Range error as a function of variations in input pulse duration for the SOA parameters outlined in Table 4.2.
Figure 4.21. Effect of variations in input energy on the matched filter output for the SOA parameters outlined in Table 1. The matched filter is fixed and assumes an input pulse energy of $E_{in}/E_{sat} = 0.1$. 
Figure 4.22. (a) The range resolution, which is defined as the 3 dB width of the IPR, as function of pulse duration and chirp parameter for $E_{in}/E_{sat}=0.1$. The carrier lifetime is optimized for each pulse duration. (b) The range resolution of the temporally broadened asymmetric output pulse as a function of pulse duration if the self phase modulation is not exploited as additional bandwidth.

For now, we will assume that the desired range resolution is 15 cm. The motivation for this requirement will be discussed in Section 5.2 of Chapter 5. The above analysis assumes a 1 ns FWHM Gaussian input pulse, with the self-phase modulation improving the range resolution beyond the 15 cm requirement. The use of the SOA as both an amplifier and a phase modulator can also serve to relax the source requirements by allowing the use of a longer input pulse duration that is modulated to provide the necessary 15 cm range resolution. To explore this option, Equation (4.32) was solved for various pulses with the same energy but different pulse...
durations. The range resolution of the resulting transmitted pulse as a function of the input pulse duration is shown in Figure 4.22 (a) for $G_0=30$ dB and $E_{in}/E_{sat} = 0.1$. The solid lines represent the calculations for a carrier lifetime of 0.5 ns, and the dashed lines represent calculations where the carrier lifetime was optimized to be one half of each pulse duration, as found in Figure 4.9 (b), representing the best case scenario for range resolution. As expected, the required pulse duration is heavily dependent on the chirp parameter of the SOA. For a chirp parameter of $\alpha = 8$, an input pulse with a FWHM of 3.5-4.1 ns would meet the 15 cm range resolution requirement, depending on the duration of the carrier lifetime. A SOA with $\alpha = 12$ would allow an input pulse FWHM duration of 4.9-6.1 ns. Figure 4.22 (b) represents the range resolution of the asymmetric temporally broadened pulse when the self-phase modulation of the SOA has been ignored, assuming the carrier lifetime has been optimized. This indicates the range resolution improvement that can be expected by exploiting the self-phase modulation as bandwidth. Our analysis has shown that a range resolution of 3.6 cm can be expected by saturating an amplifier with a 1 ns FWHM Gaussian input pulse with $E_{in}/E_{sat} = 0.1$, $G_0=30$ dB, $\alpha = 8$, and $\tau_c = 0.5$ ns. In Figure 4.22 (b), we see that the range resolution of this output pulse would be 27.0 cm if the phase modulation was not exploited as additional bandwidth. This shows that an improvement factor of 7.5 can be obtained by exploiting the self-phase modulation of the SOA.

Figure 4.23 shows the ISLR and PSLR as a function of the input pulse duration and $\alpha$. As seen in Figure 4.23 (a), the ISLR increases slightly for pulse durations less than 0.5 ns, but remains relatively stable as the input pulse duration increases beyond 0.5 ns. The PSLR shows very little change with pulse duration, as seen in Figure 4.23 (b). Note that there are no discernable peaks in the IPR for $\alpha = 2$.  

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Figure 4.23. The (a) ISLR of the IPR, and (b) PSLR of the IPR as a function of chirp parameter and pulse duration. These simulations assume $E_{\text{in}}/E_{\text{sat}}=0.1$. The optimum carrier lifetime is used for each pulse duration.
4.9. SOA as a Phase Modulator in a Multi-Function Coherent Laser Radar System

One potential system architecture using the SOA to amplify and phase modulate the transmitted pulse is depicted in Figure 4.24. A continuous wave laser serves as the master oscillator of the system and as a seed to the pulsed laser. A SOA is used to amplify the pulse, and a beamsplitter directs a portion of the signal to an I/Q demodulator, where it is decomposed into its in-phase (I) and quadrature (Q) components. The pulse then propagates through the transmit/receive switch to the target area. The return pulse, after passing through the transmit/receive switch, is directed to an I/Q demodulator. As discussed in CHAPTER 2, the return pulse and the stored monitor pulse can then be compared to yield information about the target. As discussed above, if the pulsed laser is stable, it may not be necessary to monitor the pulse before it is transmitted.

It is advantageous to improve the range resolution not only in laser radar systems dedicated to ranging applications, but also in multi-function laser radar systems that collect vibrometry and/or SAL data. Sections 4.9.1 through 4.9.3 will explore the use of the self-phase modulation in ranging, vibrometry, and SAL applications.

4.9.1. Ranging

The calculations for range processing were outlined in Equations (2.57) through (2.59). Here, we use the pulse envelope and phase modulation associated with Equations (4.24) and (4.27), so the transmitted signal is expressed as

\[
\tilde{s}_t(t) = \sqrt{P_{\text{out}}(t)} \exp(j \omega_c t + j \phi_{\text{out}}(t)) \left[ \frac{1}{\sqrt{2}} \tilde{x} + \frac{1}{\sqrt{2}} \exp \left( j \frac{\pi}{2} \right) \tilde{y} \right] \\
= s_b(t) \exp(j \omega_c t) \left[ \frac{1}{\sqrt{2}} \tilde{x} + \frac{1}{\sqrt{2}} \exp \left( j \frac{\pi}{2} \right) \tilde{y} \right],
\]

(4.41)
where $\omega_c$ is the carrier frequency, $s_b(t) = \sqrt{P_{out}(t)}\exp(j\phi_{out}(t))$ represents the baseband version of the transmitted signal, and where the signal has right-hand circular polarization due to the quarter waveplate in the transmit/receive switch in Figure 4.24. Upon interaction with a target, the polarization of the return signal becomes left-hand circular due to the change in direction of propagation, and the received signal is delayed by the roundtrip time $t_{rt} = 2R_p/c$. After passing through the quarter waveplate, the signal polarization becomes linear and the signal can be expressed as

$$\tilde{s}_r(t) = \sqrt{P_{out}(t-t_{rt})}\exp(j\phi_{out}(t-t_{rt}))\exp(j\omega_c(t-t_{rt}))$$

$$= s_b(t-t_{rt})\exp(j\omega_c(t-t_{rt}))\tilde{x}. \quad (4.42)$$

Figure 4.24. Basic architecture utilizing SOA as an amplifier and a phase modulator
For simplicity, it is assumed that the target has no affect on the envelope of the signal. The polarizing beamsplitter directs the received signal to the signal I/Q demodulator. As detailed in Section 2.5.3, the I/Q demodulator combines the signal and MO, and decomposes them into the in-phase and quadrature components $s_{I_r}(t)$ and $s_{Q_r}(t)$, respectively. Assuming homodyne detection (i.e., the MO and the received signal are at the same carrier frequency) the composite received electrical signal is expressed as

$$s_r(t) = s_{I_r}(t) + js_{Q_r}(t) = u_{MO} s_b(t - t_{rt}) \exp(-j \omega_c t_{rt}),$$

(4.43)

where $u_{MO}$ is the amplitude of the CW master oscillator. As in Section 2.5.3, we have assumed $u_{MO} \gg u_t(t - t_{rt})$, and that AC coupling is used to filter out any DC terms. The monitor signal is also combined with the master oscillator in the monitor I/Q demodulator, yielding the expression

$$s_m(t) = u_{MO} s_b(t).$$

(4.44)

Assuming conventional matched filtering techniques are used to detect the received pulse, the output of the matched filter for a single pulse is

$$s_M(t) = \int_{-\infty}^{\infty} s_r(\tau) s_m^*(\tau - t) d\tau = \mathfrak{F}^{-1}\{S_r(\omega)S_m^*(\omega)\},$$

(4.45)

where $s_r(t)$ is the received signal, the matched filter $s_m^*(t - t)$ is the time reversed conjugate of the monitor signal, $S_r(\omega)$ and $S_m(\omega)$ are the Fourier transforms of the received and monitor signals, respectively, and $\mathfrak{F}^{-1}$ denotes the inverse Fourier transform operation. As detailed in Section 2.6, range processing is carried out by substituting Equations into Equation (2.48) such that

$$s_M(t) = u_{MO}^2 \exp(-j \omega_c t_{rt}) R_{sb}(t - t_{rt}).$$

(4.46)

Note that the matched filter output contains a shifted version of the autocorrelation of the baseband transmitted signal, or
\[ R_{sb}(t - t_{rt}) = \int_{-\infty}^{\infty} s_b(\tau - t_{rt})s_b^*(t - \tau) d\tau. \] (4.47)

This processing was carried out for a series of simulated pulses assuming the amplifier characteristics detailed in Table 4.1 and a target at a range of 30 km. The results are seen in Figure 4.25, where the fast-time axis represents the pulse duration, and the slow-time axis represents the series of pulses. The received signal power of the data set is seen in Figure 4.25 (a). After applying the matched filter, the range compressed image calculated according to Equation (4.46) is shown in Figure 4.25 (b). The real part of the receive signal and the range compressed image are shown in Figure 4.25 (c) and (d), respectively, where the phase modulation is evident in the fluctuations of the signal.

Figure 4.25. The simulated (a) received signal power, (b) range compressed signal power, (c) real part of the received signal, and (d) real part of the range compressed signal as a function of fast-time and slow-time for a stationary target
4.9.2. Vibrometry

As discussed in Chapter 2, a vibrating target introduces an additional phase factor that can be exploited to yield further information about the nature of the target. For a vibrating target located at range \( R_p \) and cross-range location of \( x_p = 0 \), with no translational motion (i.e., \( v_t = 0 \)), the received signal (after the I/Q demodulator) was found in Equation (2.66) to be expressed as

\[
s_r(t) = u_{MO} s_b(t - t_{rt}) \exp\left(-j \frac{4\pi}{\lambda} R_p\right) \exp\left(-j \frac{4\pi}{\lambda} a_v \sin(\omega_v t)\right)
\]

(4.48)

where \( a_v \) is the maximum vibrational displacement of the target toward the ladar system, \( \omega_v \) is the frequency of the assumed sinusoidal vibration, and we have ignored the phase noise terms. Equations (4.48) and (4.44) can be substituted into Equation (4.45), yielding

\[
s_M(t) = \exp\left(-j \frac{4\pi}{\lambda} R_{p0}\right) \int_{-\infty}^{\infty} u_{MO}^2 s_b(\tau - t_{rt}) s_b^*(\tau - t) \exp\left(-j \frac{4\pi}{\lambda} a_v \sin(\omega_v(\tau))\right) d\tau
\]

(4.49)

As detailed in Section 2.7.2, the Doppler shift due to the vibration is assumed to be approximately constant over the pulse duration such that \( \phi_v(t) \approx \phi_v \), and Equation (4.49) can be simplified to yield

\[
s_M(t) = u_{MO}^2 \exp(-j \phi_v) \exp(-j \frac{4\pi}{\lambda} R_p) R s_b(t - t_{rt}).
\]

(4.50)

Once again, the matched filter output contains a delayed version of the autocorrelation of the baseband signal. Furthermore, the phase of the matched filter output has a sinusoidal variation with a frequency equal to the vibrational frequency of the target. As detailed in Section 2.7.2, by interrogating the target with multiple pulses, the vibrational frequency of the target can be determined from the frequency at which the phase history varies, i.e. by taking a Fourier
transform of the phase history. In this manner, the range to target as well as the vibrational frequency of the target can be determined.

Figure 4.26 shows the results of this processing for a vibrating target with a vibrational frequency of 100 Hz. The power of the received signal as a function of fast-time and slow-time is shown in Figure 4.26 (a), and the power of the received signal after the matched filter has been applied is shown in Figure 4.26 (b). Similarly, the real part of the received signal as a function of fast-time and slow-time is shown in Figure 4.26 (c), and the real part of the received signal after the matched filter has been applied is shown in Figure 4.26 (d). The pattern in Figure 4.26 (c) and (d) represents the interference of the phase modulation from the amplifier with the phase variations due to the vibrating target.

Figure 4.26. The simulated (a) received signal power, (b) range compressed signal power, (c) real part of the received signal, and (d) real part of the range compressed signal as a function of fast-time and slow-time for a vibrating target.
4.9.3. SAL

For a ladar that is translating with respect to a stationary target with no vibration (i.e., \( \omega_v = 0 \)), the received signal from Equation (3.9) is expressed as

\[
s_r(t, u) = u(t - t_{rt})u_{MO} \exp(j \theta (\tau - t_{rt})) \\
\times \exp \left( -j \frac{4\pi}{\lambda} \left( R_{p0} + \frac{u^2}{2R_{p0}} - u \frac{x_p}{R_{p0}} + \frac{x_p^2}{2R_{p0}} \right) \right),
\]

(4.51)

where \( \nu_t t \). This signal can be processed using the 2-D matched filter technique outlined in Section 3.4.1. A simulated SAL signal is seen in Figure 4.27 for a synthetic aperture length of 75 cm and a range of 30 km. A single object in the center of the target area is assumed. The received signal power is shown in Figure 4.27 (a), the range-compressed output power of the fast-time matched filter is shown in Figure 4.27 (b), and the output power of the slow-time
matched filter is shown in Figure 4.27 (c). The real part of each of these signals is shown in Figure 4.27 (d) through (f), where the phase modulation in the fast and slow-time domains is evident in the oscillations of the signal. These signals are explored in more detail in Figure 4.28.

A cross-section of the received signal in the fast-time domain is shown in Figure 4.28 (a). The amplitude and phase modulation due to the SOA is evident in the envelop and oscillations of the signal. The phase of the fast-time signal is seen in Figure 4.28 (b), which matches the phase calculated for the signals above. A cross-section of the signal in the slow-time domain is shown in Figure 4.28 (c). As discussed in Chapter 3, the chirp of the signal is due to the changes in Doppler frequency as the platform moves with respect to the object of interest. The chirp of the slow-time signal results in a quadratic phase, as seen in Figure 4.28 (d).

![Figure 4.28. Cross-sections of the signal presented in Figure 4.18 in (a) fast-time and (c) slow-time. The phase of the signal in (b) fast-time and (d) slow-time are also presented.](image-url)
A cross-section of the power of the return signal is shown in Figure 4.29 (a). After the full 2-D matched filter is applied, cross-sections of the IPR in fast-time and Figure 4.29 (c) and (d), respectively. As expected, the profile of the fast-time IPR matches the amplifier modulated IPR in Figure 4.17.

![Figure 4.29. A fast-time cross-section of the received signal is shown in (a). Cross-sections of the IPR are shown in (b) range (fast-time) and (c) cross-range (slow-time).](image)

4.10. Conclusion

This analysis has shown that the self-phase modulation induced by operating a SOA in the saturated regime can be exploited as additional bandwidth to improve the range resolution of a laser radar system. Furthermore, the range resolution of a laser radar system can be optimized by saturating a SOA with a carrier lifetime that is one half the FWHM Gaussian input pulse duration, yielding a substantial improvement in range resolution that is highly insensitive to variations in the input pulse duration and energy. For the simulations presented here, the range resolution was improved by a factor of 7.5 by exploiting the self-phase modulation as additional bandwidth. The amplifier modulated signal was also shown to be comparable to LFM waveforms. The concept of using a SOA as a phase modulator in multi-function laser radar systems was also discussed. The following chapter will explore the concept of using a saturated SOA to enable short-pulse SAL and vibrometry.
CHAPTER 5
SHORT PULSE SYNTHETIC APERTURE LADAR AND VIBROMETRY

5.1. Introduction

In Chapter 4, the concept of using a saturated SOA as both an amplifier and a phase modulator was presented in detail. This chapter will explore how this concept could enable the design of a synthetic aperture ladar and vibrometry system utilizing pulses that are much shorter in duration than those typically employed.

5.2. Requirements

The requirements for this system design are outlined in Table 5.1. It is assumed that any object of interest will fall within a 30 meter by 30 meter field of view, so the system must be able to create an image of an area of this size. It is assumed that the platform will be moving with a velocity of 200 m/s. Most platforms will not have an aperture size larger than 30 cm, setting the limit on the diameter of the real aperture. The resolution requirements are derived from the Johnson criteria, which are summarized in Table 5.2 [42]. Here, the resolution is specified in terms of line pairs, where one line pair is equivalent to two pixels. For an object of size $x$, the required resolution is [43]

$$\Delta r = \frac{x}{2n}$$

(5.1)

where $n$ is the number of line pairs. So for a target that is on the order of 8 feet (2.4 meters) in size, a resolution $\leq 15$ cm in any dimension is desired for identification purposes. Using
Equation (2.114), a range resolution of 15 cm corresponds to a transform limited pulse length of 1 ns.

Table 5.1. System Performance Requirements

<table>
<thead>
<tr>
<th>Variable</th>
<th>Designation</th>
<th>Value</th>
</tr>
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<tbody>
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<td>Field of View</td>
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<td>Cross-Range Resolution</td>
<td>$\Delta r_c$</td>
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</tr>
<tr>
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<td>$\leq$ 15 cm</td>
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<td>Operational Range</td>
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<tr>
<td>Real Aperture Diameter</td>
<td>$D_{ra}$</td>
<td>$&lt;$ 30 cm</td>
</tr>
<tr>
<td>Vibrational Velocity</td>
<td>$V_v$</td>
<td>$&gt;$ 5 mm/s</td>
</tr>
<tr>
<td>Vibrational Frequency</td>
<td>$f_v$</td>
<td>0-2 kHz</td>
</tr>
<tr>
<td>Frequency Resolution</td>
<td>$\Delta f_v$</td>
<td>1 Hz</td>
</tr>
</tbody>
</table>

Table 5.2. Johnson Criteria (from [42])

<table>
<thead>
<tr>
<th>Broadside View</th>
<th>Detection</th>
<th>Orientation</th>
<th>Recognition</th>
<th>Identification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Truck</td>
<td>0.90</td>
<td>1.25</td>
<td>4.5</td>
<td>8.0</td>
</tr>
<tr>
<td>M-48 Tank</td>
<td>0.75</td>
<td>1.20</td>
<td>3.5</td>
<td>7.0</td>
</tr>
<tr>
<td>Stalin Tank</td>
<td>0.75</td>
<td>1.20</td>
<td>3.3</td>
<td>6.0</td>
</tr>
<tr>
<td>Centurion Tank</td>
<td>0.75</td>
<td>1.20</td>
<td>3.5</td>
<td>6.0</td>
</tr>
<tr>
<td>Half-Track</td>
<td>1.00</td>
<td>1.50</td>
<td>4.0</td>
<td>5.0</td>
</tr>
<tr>
<td>Jeep</td>
<td>1.20</td>
<td>1.50</td>
<td>4.5</td>
<td>5.5</td>
</tr>
<tr>
<td>Command Car</td>
<td>1.20</td>
<td>1.50</td>
<td>4.3</td>
<td>5.5</td>
</tr>
<tr>
<td>Soldier (standing)</td>
<td>1.50</td>
<td>1.80</td>
<td>3.8</td>
<td>8.0</td>
</tr>
<tr>
<td>105 Howitzer</td>
<td>1.00</td>
<td>1.50</td>
<td>4.8</td>
<td>6.0</td>
</tr>
</tbody>
</table>
SAL is most useful at ranges where the diffraction limited resolution no longer meets the Johnson criteria. Figure 5.1 shows the diffraction limited resolution as a function of range for various aperture sizes. In the best case scenario, i.e. an aperture of 30 cm, the diffraction limited resolution no longer meets the 15 cm requirement derived above at ranges beyond 30 km.

Figure 5.1. Diffraction-limited resolution as a function of range and real aperture size

5.3. Short Pulse Synthetic Aperture Ladar Concept

The short pulse SAL concept involves operation in the hybrid mode to increase coverage in the along-track direction while maintaining a suitable cross-range resolution. As discussed in Chapter 3, this will limit the cross-range resolution that can be obtained, but it will also allow the system to cover a larger area. Coverage in the cross-track direction will be obtained by transmitting an elliptical beam of light that is diffraction limited along-track, and some multiple of the diffraction limit in the cross-track, as depicted in Figure 5.2. This can be obtained by using a cylindrical lens. The beam in Figure 5.2 is depicted as rectangular for simplicity.
As shown in Figure 5.2, the size of the transmitted beam on the ground in the x-dimension, $D_{gx}$, is the diffraction limited spot size, or

$$D_{gx} = \frac{\lambda R}{D_{ra}},$$  \hspace{1cm} (5.2)

where $D_{ra}$ is the diameter of the receive/transmit aperture, which will now be referred to as the real aperture to distinguish it from the synthetic aperture. The size of the spot on the ground in the y-dimension, $D_{gy}$, is some multiple of the diffraction limit, or
\[ D_{gy} = N_{DL}D_{gx} = N_{DL} \frac{\lambda R}{D_{ra}}, \]

where \( N_{DL} \) represents the multiplicative factor. Sampling in the \( y \)-direction occurs by arranging the linear array such that each detector has a field of view that is matched to a diffraction limited spot size on the ground, as depicted notionally in Figure 5.3 for three detectors. The receiver aperture for each detector is the full area of the lens.

Figure 5.3. Fields of view for each detector in the linear area

A depiction of the system operating in the hybrid mode is shown in Figure 5.4. The length of the synthetic aperture is denoted as \( D_{sa} \). The dotted ellipses depict the field of view of every other detector in the linear array. Each detector will create a synthetic aperture image. The system will have a diffraction limited resolution in the \( y \)-dimension, a resolution determined by the length of the synthetic aperture in the \( x \)-dimension, and a resolution determined by the bandwidth of the transmitted pulse in the \( z \)-dimension.
5.4. Short Pulse Vibration Sensing Concept

Since the concept for the short pulse SAL system involves measuring phase to obtain synthetic aperture data, the system could be modified to obtain the phase measurements necessary for acquiring vibration data. Ideally, the system would transmit a two dimensional array of diffraction-limited beams to sample the target area. Such an array could be created by introducing a phase plate into the system architecture [44]. The return from each diffraction-limited spot would be matched to a separate detector in a two dimensional detector array. This concept is depicted in Figure 5.5. The top portion of the figure depicts a two dimensional detector array where only the leftmost column is being actively used for SAL operation. An elliptical beam of light is transmitted to the target area, and each detector is matched to a diffraction limited spot. For the vibration sensing operation, some of the detectors in the linear array are no longer active, but the active detectors create a sparsely populated two dimensional array. Each detector is matched to a diffraction limited spot in the target area. The number of
beams and their spacing must be carefully considered. While it would be ideal to transmit enough beams to fully populate and sample the entire target area, this would require more transmitted power than will likely be possible. As such, the number of transmitted beams will be limited by the available transmitted power, and they should be spaced to sample the entire target area.

![Detector array arrangements for SAL and vibration sensing modes](image)

**Figure 5.5. Detector array arrangements for SAL and vibration sensing modes**

Another difference between the SAL and vibration sensing approaches is that the system must dwell on the same target area for the duration of the measurement to obtain a vibration signature. This means that the system will need to operate in a spotlight mode when acquiring vibration signatures, as depicted in Figure 5.6.
5.5. Performance Metrics

In this section, performance metrics for this system design will be derived based on the metrics derived for generic SAL and vibrometry in Chapters 2 and 3.

5.5.1. Vibe SNR

The data collection process for vibrometry operation involves the coherent addition of pulses over the dwell time. The vibrometry SNR is then increased by the number of pulses collected, or

\[
SNR_v = SNR_{ra} \cdot \frac{PRF \cdot D_{dwell}}{v},
\]  

(5.4)

where \(SNR_{ra}\) is the SNR for a single pulse, as derived in Equation (2.113), and \(D_{dwell} = vT_{dwell}\), where \(v\) is the velocity of the platform and \(T_{dwell}\) is the amount of time the platform dwells on...
the object of interest. When operating with multiple detectors, as depicted in Figure 5.7, the SNR is decreased by the number of detectors, $N_d$, or

$$SNR_v = SNR_{ra} \cdot \frac{PRF \cdot D_{dwell}}{v} \cdot \frac{1}{N_d}.$$  \hspace{1cm} (5.5)

Using the single pulse shot noise limited SNR derived in Equation (2.113), the complete expression is then

$$SNR_v = \frac{P_i \rho e A_{ra} \eta_d}{\pi R^2 h_f B} \cdot \frac{1}{NF_{ea}} \cdot \eta_{atm} \eta_{sys} \cdot PCG \cdot \frac{PRF \cdot D_{dwell}}{v} \cdot \frac{1}{N_d}.$$  \hspace{1cm} (5.6)

5.5.2. SAL SNR

A depiction of the SAL system is shown in Figure 5.7, where the black dotted lines represent the cross-range resolution, $\Delta r_x$, of the system.

![Figure 5.7. Geometry for calculating the SNR as a function of the the cross-range resolution cell](image-url)
Because SAL processing improves resolution in the direction of motion, the synthetic aperture SNR is often expressed in terms of the SNR for one cross-range resolution cell, or

\[
SNR_{sa} = SNR_{ra} \cdot \frac{PRF \cdot D_{sa}}{v} \cdot \frac{\Delta r_x}{D_{dl}},
\]

(5.7)

where the ratio \(\Delta r_x/D_{dl}\) represents the ratio of transmitted power that falls within one cross-range resolution cell. The synthetic aperture length was found in Equation (3.13) to be

\[
D_{sa} = \frac{\lambda R}{2\Delta r_x \sin \theta_s}.
\]

(5.8)

Here, the squint angle \(\theta_s\) is assumed to be approximately 90 degrees over the synthetic aperture.

Finally, as was done in the previous section for vibrometry, the SNR for SAL operation will be decreased by the number of detectors. Since each detector is matched to a diffraction limited spot on the ground, and assuming that the FOV of each detector overlaps by one half the diffraction limited spot size, as depicted in Figure 5.8, the number of detectors can be expressed as

\[
N_d = 2 \frac{D_{gy}}{D_{dl}} - 1
\]

(5.9)

This yields a final expression for the synthetic aperture SNR of

\[
SNR_{sa} = \frac{P_t \rho_c A_{ra} \eta_d}{\pi R^2 h f B} \cdot \frac{1}{N F_e a} \cdot \eta_{atm} \eta_{sys} \cdot PCG \cdot \frac{PRF \cdot D_{sa}}{v} \cdot \frac{\Delta r_x}{D_{dl}} \cdot \frac{1}{N_d}
\]

(5.10)

where the atmospheric transmittance is related to the atmospheric efficiency as

\[
\eta_{atm} = T_2^{atm}
\]

(5.11)
5.5.3. Coverage Time

Figure 5.9 (a) shows the position of the beam on the ground as it completes one synthetic aperture length. The beam is moving across the ground at a rate of

\[ V_g = \frac{2D_{dt}}{T_{sa}}. \]  

(5.12)

As illustrated in Figure 5.9 (b), the time it will take to cover the required FOV will be

\[ T_{FOV} = \frac{D_{gy} + 2D_{dt}}{V_g} = \frac{T_{sa}(D_{gy} + 2D_{dt})}{2D_{dt}}, \]  

(5.13)

Where the diffraction limited beam size is defined as \( D_{dt} = \lambda R / D_{ra} \).
5.6. Range Considerations

This section will discuss the range limitations of a system of this architecture. Although the range is ultimately limited by the amount of transmit power available, this section will discuss two other factors that can also limit the operational range of the system. First, the curvature of the earth provides a fundamental limit on the operation of the system because the system can only operate as far as the horizon. A second consideration is the coherence diameter of the atmosphere. As will be shown, the coherence diameter decreases with range. The limitations this places on the operational range of the system will be explored in detail below.

Figure 5.9. Geometry used to determine the amount of time required to cover the FOV of interest

(a) 

(b) 

$T_{SA}$ 

$D_{dl}$ 

$T_{FOV}$ 

$D_{dl}$ 

$30 \text{ m}$ 

$D_{dl}$
5.6.1. *Horizon*

The curvature of the earth provides a fundamental limit on the operational range of a system, as depicted in Figure 5.10.

![Diagram showing horizon effect on operational range](image)

*Figure 5.10. Geometry for determining the effect of the horizon on the maximum operational range*

![Graph showing maximum range vs. altitude](image)

*Figure 5.11. Maximum operational range as dictated by the horizon*
The maximum range of the system is the point at which the transmitted beam is tangential to the earth’s surface. According to the Pythagorean theorem, this yields the relationship

\[ R_0^2 + R_e^2 = (R_e + h_p)^2, \]  
(5.14)

where \( R_e \) is the radius of curvature of the earth, 6378 km, and \( h_p \) is the altitude of the system. Solving this equation yields the maximum operational range of the system as a function of altitude, shown in Figure 5.11.

5.6.2. Coherence Diameter

A concern for synthetic aperture processing is the atmospheric coherence diameter, \( r_0 \). This measures the diameter over which the atmosphere can be considered constant. The three possible scenarios are outlined in Figure 5.12. In the first scenario, \( r_0 \) is larger than the synthetic aperture, which is ideal because each pulse received over the synthetic aperture sees the same atmosphere. In the second scenario, \( r_0 \) is larger than the real aperture, but smaller than the synthetic aperture. Since all of the pulses received over the synthetic aperture are coherently processed together, a changing atmosphere will be manifested as a slowly varying phase noise in the synthetic image. It is assumed that such a phase noise could be corrected using autofocus techniques [45]. The third scenario depicts an \( r_0 \) that is smaller than the real aperture, which is a much more difficult problem to correct. For this work, it is assumed that the system must operate in regimes II or III, meaning the atmospheric coherence diameter will influence the maximum real aperture diameter possible for the system.
The coherence diameter is defined as [46]

\[ r_0 = \left[ 423k \int_0^R C_n^2 \left( h_p + \frac{z}{R} (h_t - h_p) \right) \left( \frac{Z}{R} \right)^{5/3} dz \right]^{-3/5}, \]  

(5.15)

where \( k \) is the wavenumber, \( C_n^2 \) is the turbulence profile, \( h_p \) is the altitude of the platform, and \( h_t \) is the altitude of the target. Here, the Hufnagel Valley 5/7 turbulence profile is used, which is defined as

\[ C_n^2(h) = 0.00594 \left( \frac{v_w}{27} \right)^2 (10^{-5} h)^{10} \exp \left( -\frac{h}{1000} \right) + 2.7 \times 10^{-16} \exp(-h/1500) \]

(5.16)

+ \( A_0 \exp(-h/100) \).

For the Hufnagel Valley 5/7 turbulence profile, the wind speed \( v_w \) is 21m/s, and the turbulence speed \( A_0 \) is \( 1.7 \times 10^{-14} \text{ m}^{-2/3} \) [46]. The resulting atmospheric coherence diameter as a function of altitude and range is shown in Figure 5.13 (a).
Figure 5.13. The atmospheric coherence diameter (a) as a function of range and altitude and (b) as a function of range for altitudes of 20 kft and 30 kft.

The coherence diameters for the altitudes of interest, 20 – 30 kft, are seen in Figure 5.13 (b). The synthetic aperture length as a function of range is shown by the dotted red line. Note that the coherence diameter is smaller than the synthetic aperture length beyond a range of 60 km for an altitude of 20 kft, and beyond a range of 70 km for an altitude of 30 kft. Beyond these ranges, the system begins operating in Region II. At an altitude of 20 kft, the atmosphere would limit a 30 cm diameter real aperture to an operational range of 115 km. At an altitude of 30 kft, a 30 cm aperture would be limited to a range of 162 km. A 20 cm aperture would not be limited by the atmosphere for these ranges.
5.7. Requirements Flowdown

Figure 5.14 presents a summary of how the requirements in Table 5.1 impact the synthetic aperture ladar operation of this system. The range resolution, $\Delta r$, was found in Equation (2.117) to be

$$\Delta r = \frac{c}{2B_p},$$

(5.17)

where $c$ is the speed of light and $B_p$ is the bandwidth of the pulse. Choosing a desired range resolution determines the necessary pulse bandwidth, which affects the pulse compression gain of the system and also determines the necessary detector bandwidth, both of which are included in the SNR equation derived in Equation (5.10). The size of the real aperture, $D_{ra}$, factors directly into Equation (5.10). The altitude of the platform, $h_p$, affects the atmospheric transmission, which affects the atmospheric efficiency, $\eta_{atm}$, found in Equation (5.10). The field of view requirement sets the required value for the diameter of the beam on the ground in the y-dimension $D_{gy}$, as seen in Figure 5.4. From Equation (5.9), $D_{gy}$ can be expressed as

$$D_{gy} = \frac{N_dD_{dt} + 1}{2},$$

(5.18)

where $D_{dt} = \lambda R/D_{ra}$ is the diffraction limited spot size. So for a given range and aperture size, the FOV requirement determines the number of detectors, $N_d$, that are necessary, which factors into Equation (5.10). Finally, using Equation (5.8), the cross-range resolution can be expressed as

$$\Delta r_x = \frac{\lambda R}{2D_{sa}\sin \theta_A},$$

(5.19)

where $D_{sa} = vT_{sa}$ is the length of the synthetic aperture, $v$ is the velocity of the platform, and $T_{sa}$ is the time it takes to complete the synthetic aperture. Therefore, for a given range and
velocity, the cross-range resolution requirement determines $T_{sa}$, which factors into the SNR of Equation (5.10) and the coverage time, $T_{FOV}$, of Equation (5.13).

For a given SNR, range, velocity, and PRF Equation (5.10) can be solved to yield the required transmit power. Note that a trade-off exists between the transmit power and the PRF of the system since increasing the PRF can decrease the required transmit power, keeping in mind that the PRF must satisfy the range and Doppler ambiguity requirements, as discussed in Chapter 3. The transmit power requirement is also limited by the fact that this application requires the transmission of shorter pulses, on the order of a nanosecond in duration.

---

**Figure 5.14. Flow of requirements for SAL operation**
For a given real aperture diameter, Equation (5.13) can be solved to yield the amount of
time it takes to collect the data for the desired field of view. This contains an inherent trade-off
because increasing the size of the transmit diameter reduces the required transmit power, but
increases the amount of time it takes to collect the data.

The remaining requirements outlined in Table 1 are the vibrational velocity, frequency,
and frequency resolution. The impact of these requirements are outlined in Figure 5.15. The
required vibrational velocity, which was found in Equation (2.64) to be

\[ V_{amb} = \frac{\lambda}{2} \text{PRF}, \]  

helps determine the required PRF for vibrometry operation. The required vibrational
frequencies also help determine the required PRF, as found in Equation (2.62), which is repeated
here as

\[ \text{PRF} = 2f_{v_{max}} \]  

The required frequency resolution was found in Equation (2.76) to be

\[ \Delta f_{v} = \frac{1}{T_{dwell}}, \]  

which determines the amount of time the ladar must dwell on the object of interest. The PRF
and \( T_{dwell} \) factor into the SNR equation derived in Equation (5.6). This yields a trade-off
between the required transmit power and the maximum number of detectors that can be
accommodated, which determines the extent to which the FOV can be sampled for vibration
measurements.
5.8. Predicted Performance

This section will summarize the predicted performance of the short pulse SAL and vibrometry system. First, the required PRF will be derived based on the vibration and SAL requirements. In section 5.8.2, the coverage time as a function of the real aperture size will be investigated. Finally, in section 5.8.3 the required power as a function of altitude and real aperture size will be presented.

5.8.1. Required PRF

As found in Equation (2.62), the PRF for vibrometry must be at least twice the highest vibrational frequency of the target, or

\[ \text{PRF} > 2f_{v_{\text{max}}} \]  \hspace{1cm} (5.23)
The maximum vibrational frequency in Table 5.1 is specified to be 2 kHz, so to satisfy the Nyquist criteria the PRF must be at least 4 kHz. Furthermore, to prevent ambiguities with respect to velocity, it was found in Equation (2.64) that the PRF must satisfy

$$PRF > \frac{2v_{v_{\text{max}}}}{\lambda}$$

(5.24)

where $v_{v_{\text{max}}}$ is the maximum vibrational velocity of the object. In Table 5.1, the vibrational velocity is specified to be at least 5 mm/s, so the PRF must be at least 6.7 kHz to prevent velocity ambiguities.

For SAL operation, it was found in Equation (3.59) that to prevent Doppler ambiguities the PRF must satisfy

$$PRF > \frac{2v_{D}}{D_{ra} \sin \theta_{A}}.$$  

(5.25)

For this analysis, 10 cm is assumed to be the smallest practical real aperture size. Assuming a platform velocity $v = 200$ m/s and broadside operation such that $\theta_{A} \approx 90$ degrees, a PRF of at least 4 kHz is required to prevent Doppler ambiguities.

Using these calculations as a guide, a PRF of 10 kHz will be assumed for the remainder of these discussions. This will meet the requirements with respect to velocity and Doppler ambiguities, while allowing the vibrational frequency to be sampled above the Nyquist frequency. As found in Equation (2.65), a PRF of 10 kHz will yield range ambiguities of

$$R_{amb} = \frac{c}{2 \cdot PRF} = 15 \text{ km}.$$  

(5.26)

It is assumed that the range to any object of interest will be known to within 15 km, so these range ambiguities can be tolerated and corrected in post processing.
5.8.2. Coverage Time

The amount of time required to cover the desired field of view is calculated according to Equation (5.13), where $D_{gy}$ is defined by the FOV to be 30 meters. The coverage time as a function of range and real aperture size is presented in Figure 5.16. As can be seen, there is very little change in the coverage time as the range is increased. However, the coverage time increases by nearly a factor of three as the real aperture size is increased by the same amount.

![Figure 5.16. Time required to cover the field of view, $T_{FOV}$, as a function of range and real aperture diameter, $D_{ra}$](image)

5.8.3. Required Power

The atmospheric transmittance as a function of range is calculated according to Beer’s Law, or [46]
\[ T_{atm} = \exp \left( - \int \alpha_{atm}(z) \, dz \right) , \] (5.27)

where \( \alpha_{atm} \) is the atmospheric absorption as a function of range, which is found using the FASCODE® software for mid latitude summer condition with no aerosols. The resulting two-way transmittance as a function of range and altitude is shown in Figure 5.17.

Assuming each detector requires a synthetic aperture SNR of 1, Equation (2.90) was solved to yield the required transmitted peak power. For this simulation, the parameters listed in Table 5.3 were assumed, the noise factor of the electronic amplifier was ignored, and the
synthetic aperture size was adjusted to yield the required 10 cm cross-range resolution at each range. The atmospheric efficiencies calculated in Figure 5.17 were used. The results as a function of altitude and range are depicted in Figure 5.18 for aperture diameters of 20 cm and 30 cm.

### Table 5.3. Parameters for SNR Calculations

<table>
<thead>
<tr>
<th>Variable</th>
<th>Designation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>SNR per detector</td>
<td>SNR&lt;sub&gt;d&lt;/sub&gt;</td>
<td>1</td>
</tr>
<tr>
<td>Cross-Range Resolution</td>
<td>( \Delta r_x )</td>
<td>10 cm</td>
</tr>
<tr>
<td>Target Backscattering Coefficient</td>
<td>( \rho_t )</td>
<td>0.1</td>
</tr>
<tr>
<td>Detector Quantum Efficiency</td>
<td>( \eta_d )</td>
<td>0.5</td>
</tr>
<tr>
<td>Pulse Bandwidth</td>
<td>( B_p )</td>
<td>3.1 GHz</td>
</tr>
<tr>
<td>Pulse Duration</td>
<td>( \tau_p )</td>
<td>1.6 ns FWHM</td>
</tr>
<tr>
<td>Detector Bandwidth</td>
<td>( B_d )</td>
<td>3.1 GHz</td>
</tr>
<tr>
<td>Detector quantum efficiency</td>
<td>( \eta_d )</td>
<td>0.5</td>
</tr>
<tr>
<td>Optical system efficiency</td>
<td>( \eta_{sys} )</td>
<td>0.5</td>
</tr>
<tr>
<td>Pulse Repetition Frequency</td>
<td>PRF</td>
<td>10 kHz</td>
</tr>
<tr>
<td>Platform velocity</td>
<td>( V )</td>
<td>200 m/s</td>
</tr>
</tbody>
</table>

The atmospheric limitations, combined with the required transmitted power, sets the bounds for the aperture diameter and maximum operational range of the system, which is summarized for the altitudes of interest in Figure 5.19. The stars in Figure 5.19 (a) represent the operational range limitation due to the atmosphere.
Figure 5.18. Required peak transmitted power for a real aperture diameter of (a) 30 cm and (b) 20 cm assuming a PRF of 10 kHz and a cross-range resolution of 10 cm.

Figure 5.19. Required peak transmit power as a function of range for (a) a 30 cm diameter real aperture and (b) a 20 cm diameter real aperture. The stars represent the maximum operational range due to the atmospheric coherence diameter.
5.9. **Power Considerations**

There are several ways the system architecture could be altered to reduce the amount of power required for the source. The methods outlined below could be used alone or in combination with one another to relax the requirements of the transmitter for the system.

5.9.1. **Multiple SOA’s**

Multiple SOA’s could be used to boost the power of the pulsed laser before modulating the phase, as depicted in Figure 5.20. This is a modified depiction of Figure 4.18 where only the transmit portion of the system is shown. In this depiction, two SOA’s are used to boost the power of the pulsed laser, and a final SOA is saturated to provide the phase modulation and some amplification of the pulse. In this case, the first low power SOA has a saturation energy that is larger enough to prevent saturation as the pulse is amplified. The second SOA operates in the same manner but will be designed for higher power operation with an even higher saturation energy than the first SOA. This architecture reduces the power requirements of the source.

![Figure 5.20](image)

**Figure 5.20.** One potential system architecture using multiple booster SOA’s to increase the power before the saturated SOA

5.9.2. **Multiple Sources**

To further reduce the power requirements for a single source, the system could be designed with multiple sources. Each source would transmit through a separate SOA and would have a FOV adjacent to the next source. This is depicted in Figure 5.21 for two sources, where
the red ellipses represent the detector FOV’s for one source and the blue ellipses represent the
detector FOV’s for a second source. By covering the 30 m X 30 m FOV with more than one
source, the power required from each source is reduced by the number of sources used.

Figure 5.21. One potential system architecture using multiple sources, each with its own saturated SOA,
to cover the desired FOV while relaxing the power requirements for each source.

5.9.3. Multiple Passes

Another way to reduce the amount of power required for the source is to cover the 30
m X 30 m FOV with multiple passes, as depicted in Figure 5.22. Here, as the platform travels
from position $x_1$ to position $x_3$, it covers half the FOV. At position $x_3$ the platform steers the
beam to begin covering the second half of the FOV, and completes the coverage when it reaches
position $x_4$. In this manner, the amount of power required from the source can be
approximately divided by the number of passes taken to cover the entire area of interest.
However, it must be kept in mind that the range to target is continuing to increase with each
pass, so the number of passes that can be taken will eventually be power limited by the system’s
maximum operational range.
5.10. Conclusion

This chapter has discussed some of the issues that should be considered for a short-pulse SAL and vibrometry system enabled by employing a SOA as an amplifier and a phase modulator. The short-pulse SAL and vibrometry concept has been presented, and the performance metrics that are unique to this operation were explored. Factors that would limit the range of such a system, such as the coherence diameter, were explored. Some notional requirements were discussed, and their effect on the system design was investigated. Using atmospheric models, the required transmit power was derived, and some system configurations that may relax these power requirements were also presented. The next chapter will present proof of concept laboratory demonstrations where pulses with a phase modulation that is characteristic of a saturated SOA were used to interrogate stationary, vibrating, and translating objects.
CHAPTER 6
LABORATORY DEMONSTRATION

6.1. Introduction

The amplifier modulated signal presented in Chapter 4 was considered for SAL and vibrometry applications in Chapter 5. As a proof of concept, laboratory demonstrations transmitting pulses with phase modulation characteristic of a saturated SOA were carried out using stationary, vibrating, and translating objects. The experimental results are presented in this chapter.

6.2. Laboratory Setup

The optimal SOA characteristics and resulting output pulse characteristics for a 1 ns FWHM Gaussian input pulse presented in Table 4.1 of Chapter 4 are repeated here in Table 6.1 for convenience. The amplitude and phase of the amplifier modulated signal with these characteristics are shown in Figure 6.1 (a) and (b), respectively. In Chapter 4, the 3 dB temporal width of the IPR of this pulse was found to be 0.24 ns, yielding a bandwidth of 4.2 GHz. To satisfy Nyquist, the signal must be sampled at a rate of 8.4 GHz. To ease the requirements on the digitizers used to sample the return signal, the signal presented in Chapter 4 was simulated through the amplitude and phase modulation of a CW source, as depicted in Figure 6.2, allowing the transmitted pulse to be slowed down by a factor of 25, greatly reducing the bandwidth of
the received pulse. This section will first discuss the amplitude and phase modulation
techniques, followed by a discussion of the laboratory setup used to carry out the SAL and
vibrometry demonstrations.

Table 6.1. Assumed SOA Parameters and Resulting Output Pulse Parameters

<table>
<thead>
<tr>
<th>SOA Parameters</th>
<th>Output Pulse Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variable</td>
<td>Designation</td>
</tr>
<tr>
<td>Input Pulse FWHM</td>
<td>$\tau_p$</td>
</tr>
<tr>
<td>Carrier Lifetime</td>
<td>$\tau_c$</td>
</tr>
<tr>
<td>Unsaturated Gain</td>
<td>$G_0$</td>
</tr>
<tr>
<td>Ratio of Input Energy to Saturation Energy</td>
<td>$E_{in}/E_{sat}$</td>
</tr>
<tr>
<td>Chirp Parameter</td>
<td>$\alpha$</td>
</tr>
</tbody>
</table>

Figure 6.1. (a) Amplitude and (b) phase of the amplifier modulated signal presented in Table 6.1 and simulated in the laboratory demonstration
6.2.1. Amplitude Modulation

The amplitude modulation of the CW source was accomplished by pulsing a Brimrose® 500 MHz Acousto-Optic Modulator (AOM). The drive signal to the AOM originated from one channel of a Tektronix® Arbitrary Waveform Generator (AWG), which was passed through an electronic amplifier before being applied to the AOM, as depicted in Figure 6.2. To pulse the output signal, the drive signal seen in Figure 6.4 was applied to the AOM. The first part of the signal is oscillating at a frequency of 500 MHz, which is the resonant frequency of the AOM. This effectively turns on the AOM, allowing the light to pass through it. Note that the wavelength of the light passing through the AOM is shifted by 500 MHz, the significance of which will be discussed in Section 6.2.3. To simulate the asymmetric pulse profile shown in Figure 6.1, the sharp leading edge of the pulse was obtained by turning on the AOM as fast as possible. The trailing edge of the pulse was accomplished by turning off the AOM more slowly, hence the slowly decreasing amplitude of the signal in Figure 6.4. To turn off the AOM, rather than turning off the drive signal, a drive signal with a frequency other than the resonant frequency of the AOM was transmitted. This minimized any ringing of the amplifier that would occur from turning the drive signal on and off at a very high rate. Since the AOM has no response to the off resonant frequency, no light passes through, and the resulting amplitude modulation creates a pulse with the desired profile.
Figure 6.3. The amplitude modulation was accomplished by pulsing the AOM, with the drive signal originating from one channel of an AWG prior to passing through an electronic amplifier.

Figure 6.4. The drive signal sent to the AOM was at the resonant frequency of 500 MHz for a short period of time before switching to an off resonant frequency, which pulsed the output signal. An off resonant frequency was used instead of turning off the drive signal to minimize ringing of the electronic amplifier.

6.2.2. Phase Modulation

The phase modulation was accomplished using an EOSpace® Lithium Niobate phase modulator (PM). Note that the phase modulation must have a maximum value of approximately 8 radians, which corresponds to approximately 2.5 \( \pi \) radians, as seen in Figure 6.1. Many phase modulators are designed to oscillate quickly between 0 and \( \pi \) radians, where the phase modulation is obtained by increasing the voltage of the drive signal applied to the phase modulator. Every phase modulator has a specified \( V_\pi \), which corresponds to the amount of
voltage that must be applied to generate $\pi$ radians of phase excursion. However there is a limit to the drive voltage that can be applied without damaging the device, thereby limiting the amount of phase modulation the device can provide. This application required the use of a device designed with a low $V_\pi$ allowing for 8 radians of phase modulation without damaging the device. For this demonstration, an EOspace low $V_\pi$ Lithium Niobate phase modulator with a $V_\pi$ of 2.7 Volts was chosen. As depicted in Figure 6.5, the drive signal to the phase modulator originated from the second channel of the AWG, which was passed through an electronic amplifier before being applied to the phase modulator. The signal shown in Figure 6.6 was generated by the AWG. The phase modulation occurred after the amplitude modulation, but both signals originated from the same AWG, so the phase signal was delayed slightly with respect to the amplitude signal to correct for this. As can be seen, the phase signal closely matches the profile of the desired phase in Figure 6.1 (b). Although the output signal is being pulsed, the drive signal to the phase modulator was continually repeated because turning off the drive signal resulted in a ringing of the electronic amplifier, which was detrimental to the signal.

Figure 6.5. The electronic drive signal to the phase modulator originated from a second channel of the AWG and passed through an electronic amplifier before being applied to the phase modulator.
6.2.3. System Setup

The laboratory demonstration was accomplished using the setup detailed in Figure 6.7. This system was intended to simulate the system architecture presented in Figure 4.17 of Chapter 4. The source is a CW 1.55 µm NP Photonics fiber laser. The first beam splitter (BS) picks off a portion of the beam to serve as a master oscillator (MO). A Brimrose® 500 MHz acousto-optic modulator (AOM) serves as an amplitude modulator, as discussed in Section 6.2.1, where the drive signal originates from one channel of the AWG. The AOM also serves to offset the transmitted signal from the LO slightly in frequency, allowing for heterodyne detection. A second channel of the AWG drives the phase modulator (PM) used to simulate the phase modulation of a saturated SOA. Before transmission, a portion of the signal is picked off to serve as a monitor. Upon receipt, the return signal is intercepted by a transmit receive switch (Tx/Rx). Two free space I/Q demodulators are used to detect the monitor and received signal, each containing a quarter waveplate (λ/4) and a half waveplate (λ/2) to yield a circularly polarized LO and a signal/monitor polarized at 45 degrees. The I/Q demodulator combines the LO and signal/monitor with a beamsplitter and then separates the signal into I and Q components using a polarized beamsplitter (PBS), as detailed in Chapter 2. The I and Q
components of the monitor are measured using fiber-coupled 12-GHz 1544-B New Focus® detectors (D₁ and D₂), and the I and Q components of the signal are measured using identical detectors (D₃ and D₄). A 4-channel Acqiris® DC282 Digitizer was used to collect the data at a rate of 2 gigasamples/second. An output pulse with the amplitude and phase characteristics in Figure 6.1 was transmitted. Note that the components within the dashed box of Figure 6.7 are simulating the components within the dashed box of Figure 4.17.

![Figure 6.7. Laboratory setup used to simulate self-phase modulation in Gaussian pulses.](image)

6.2.4. Timing

As discussed in Section 6.1, laboratory demonstrations were carried out using a stationary object, a translating object, and a vibrating object. There is a limit to how long the
digitizers can acquire data before they run out of memory. For the stationary and vibrating objects, the digitizers can be turned on and off to acquire the desired amount of data. However, for a translating object, the digitizers need to be triggered so they start collecting data only when the object is translating through the beam. This was accomplished using the optical trigger depicted in Figure 6.8. A CW He-Ne laser was placed in the path of the translating object, which for this experiment was a pendulum. The He-Ne was directed to a detector, and the electronic output of the detector was placed through an electronic amplifier and then to the event input of the AWG, which is an input that can be used as a trigger. Channel 1 and 2 of the AWG were set to trigger off of the down-slope of the event input, so when the He-Ne broke the beam and the measured signal dropped, the AWG began transmitting the drive signals to the AOM and the PM. The AWG also has a marker output, which was also triggered off the event input, sending a signal to trigger the digitizer. The digitizer would start collecting data when it received the signal from the AWG. This allowed the AWG to serve as the master clock for the entire experiment.

![Figure 6.8. Setup for the optical trigger](image-url)
6.3. Ranging Demonstration

This setup was used to obtain range data from a stationary corner cube. An example of the power, phase, and IPR of one monitored pulse is shown in Figure 6.9 after being digitally mixed to baseband. The theoretical data is represented by the dashed line. As can be seen, the transmitted pulse closely approximates the characteristics of the simulated pulse. The transmitted pulse has a FWHM of approximately 40 ns, or 25 times the 1.6 ns pulse in Table 6.1. Two sets of IPR’s are plotted in Figure 6.9 (c). The dotted lines represent the theoretical IPR’s, where the broad IPR is that of the asymmetric pulse envelope with no self phase modulation, and the narrow IPR is that of the asymmetric pulse envelope when the self phase modulation is measured and exploited as bandwidth. The 3 dB temporal width of the IPR is improved by a factor of 7.5 when the phase modulation is exploited. The solid lines represent the experimentally measured IPR’s for each case, where an improvement factor of 7.2 is demonstrated.

A series of 500 return pulses was collected at a PRF of 6.6 kHz from the stationary corner cube, as seen in Figure 6.10. The real part of the received signal as a function of fast and slow-time is seen in Figure 6.10 (a). When comparing the real part of the signal in Figure 6.10 (a) to the theoretical case presented in Figure 4.25 (a), it is evident that some additional structure exists in the real data. This is due to phase noise in the system. This indicates that the multiple matched filtering technique described in Section 2.7.2 will be necessary to reduce the phase noise for the vibrating target. After the matched filter is applied in the range domain, the range compressed image is shown in Figure 6.10 (b). The range compressed image is centered at approximately 21.5 ns, yielding a range measurement of 3.225 meters, which is in agreement with the range of 3.226 meters measured with a tape measure. However, since no steps have been taken to reduce the phase noise, it is still evident. In Figure 6.10 (c), the dotted lines
represent the theoretical IPR’s with and without self phase modulation. The solid lines represent the experimental IPR’s with and without self phase modulation, where the experimental IPR with phase modulation is simply a cross section of the range compressed image in Figure 6.10 (b). Experimentally, an improvement factor of 7.1 is demonstrated, as compared to the theoretical improvement factor of 7.5.

Figure 6.9. Experimental (solid line) and theoretical (dashed line) results for the (a) pulse profile, (b) phase modulation, and (c) IPR of the monitor signal for Gaussian pulses with a phase modulation characteristic of a saturated SOA. The experimental and theoretical broad IPR’s for a transmitted asymmetric pulse with no phase modulation are also shown to demonstrate the improvement from exploiting the phase modulation.

For comparison, the results using the multiple matched filter technique are presented in Figure 6.11. The real part of the signal presented in Figure 6.11 (a) is identical to that of Figure 6.10 (a). After the multiple matched filter technique has been applied, the phase noise in Figure 6.11(b) is dramatically reduced as compared to Figure 6.10 (b). However, since the multiple
matched filter causes a loss of the range information, as discussed in Section 2.7.2, the range compressed data is now centered in the middle of the data set rather, than offset by the round trip travel time. Although the range information has been lost, the reduction in phase noise shows that the multiple matched filter technique will allow for the processing of vibrometry data using this laboratory setup.

Figure 6.10. (a) The real part of the received signal from a stationary corner cube as a function of fast-time and slow-time, (b) the real part of the range compressed signal after the fast-time matched filter is applied, (c) a cross-section of the range compressed signal showing the theoretical (dashed line) and measured (solid line) IPR. The IPR of the output pulse without phase modulation (dotted line) is also shown.

Figure 6.11. (a) The real part of the received signal from a stationary corner cube as a function of fast-time and slow-time and (b) the real part of the range compressed signal after the multiple matched filter technique is applied.
6.4. Vibrometry Demonstration

This setup was also used to obtain data from a corner cube attached to a Hardy vibration calibration stand. A series of 500 return pulses at a PRF of 6.6 kHz was collected from the corner cube vibrating at a frequency of 848 Hz, as seen in Figure 6.12. The real part of the received signal as a function of fast and slow-time is seen in Figure 6.12 (a). The data was first processed using a single matched filter to measure the range. After the matched filter is applied in the range domain, the range compressed image is shown in Figure 6.12 (b). As before, the range compressed signal is centered at approximately 13 ns, yielding a range of 1.95 meters. A cross-section of the range compressed image is shown in Figure 6.12 (c), where the fast-time axis has now been centered about the middle of the IPR. The solid line is the measured data and the dashed line is the theoretical data.

Figure 6.12. (a) The real part of the received signal from a vibrating corner cube as a function of fast-time and slow-time, (b) the real part of the range compressed signal after a single fast-time matched filter is applied, and (c) a cross-section of the range compressed signal showing the theoretical (dashed line) and measured (solid line) IPR. The IPR of the output pulse without phase modulation (dotted line) is also shown.

Next, the data was processed using the multiple matched filter technique, as shown in Figure 6.13. The real part of the signal in Figure 6.13 (a) is identical to that presented in Figure
6.12 (a), but the range compressed signal presented in Figure 6.13 (b) has now lost the translation yielding the range information. As discussed in Section 2.7.2, the phase history of the range compressed data is analyzed to yield the vibrational frequency of the target. The phase of a slow-time cross-section of the range compressed signals for each matched filter technique is presented in Figure 6.14. As can be seen, the multiple matched filter technique dramatically reduces the phase noise, and the sinusoidal vibration phase becomes much more evident.

Figure 6.13. (a) The real part of the received signal from a vibrating corner cube as a function of fast-time and slow-time and (b) the real part of the range compressed signal after the multiple matched filter technique is applied.

Figure 6.14. Comparison of the slow-time phase for the single matched filter (red) and multiple matched filter (blue) techniques, showing the dramatic reduction in phase noise for the multiple matched filter technique.
Figure 6.15. Processed vibrometry data for a corner cube vibrating at 1075 Hz (top), 3001 Hz (middle) and 848 Hz (bottom). The data was obtained using pulses with a phase modulation characteristic of a saturated SOA.

Vibration data was collected in this manner with the calibration stand set to 1075 Hz, 3001 Hz, and 848 Hz. The Fourier transform of the phase history is presented in Figure 6.15 for the single matched filter and multiple matched filter techniques. The measured vibration frequencies match the known values as reported by the Hardy vibration calibration stand. However, the peaks for the multiple matched filter technique are very high, and the peaks for
the single matched filter technique are almost indiscernible in some cases. This has demonstrated the usefulness of the multiple matched filter technique.

6.5. SAL Demonstration

This setup was also used to obtain data from a translating target, which was a corner cube attached to a pendulum that travelled through the center of the transmit beam. For this data set, a 3 inch telescope was used. The data was processed using the 2-D matched filtering technique outlined in Chapter 3, and the results are presented in Figure 6.16. Figure 6.16 (a) shows the received signal power as a function of fast and slow-time. Figure 6.16 (b) is the result after the fast-time matched filter is applied, showing compression in the fast-time domain. Figure 6.16 (c) is the result after the slow-time matched filter is applied, yielding the location of the corner cube in fast-time (range) and slow-time (cross-range). The results confirm that the target is located in the center of the target area. A cross-section of the matched filter output in fast-time and slow-time are shown in Figure 6.16 (d) and (e), respectively. Here, the dotted lines represent the theoretical results. As can be seen, the experimental results closely match the theoretical results, although the experimental results experience higher noise levels.

6.6. Conclusion

These proof of concept experiments have shown that pulses with the phase and amplitude modulation expected from a saturated SOA can be collected and range compressed from stationary, vibrating, and translating targets to improve range resolution. The range compressed data via single and multiple matched filtering techniques has agreed well with the theoretical simulation.
Figure 6.16. Processed SAL data for a corner cube attached to a pendulum. The (a) real part of the received signal, (b) real part of the output of the fast-time matched filter, and (c) real part of the output of the subsequently applied slow-time matched filter are presented. A cross-section of the 2-D matched filter output is shown in (d) fast-time and (e) slow-time for both experimental (solid line) and theoretical (dotted line) data. The data was obtained using pulses with a phase modulation characteristic of a saturated SOA.
CHAPTER 7
CONCLUSIONS

The use of a semiconductor optical amplifier (SOA) as both an amplifier and a phase modulator has been explored. Although operating a SOA in the saturated regime is often required to obtain the desired output power, it was shown in Chapter 4 that the saturation results in phase effects and temporal broadening of the optical pulse width. Although temporal broadening of the optical pulse reduces its bandwidth, it was shown that the phase modulation is deterministic and can be exploited as additional bandwidth. In general, a pulsed source can be used to saturate the amplifier, and the transmitted pulse can be monitored. However, it was also shown in Chapter 4 that monitoring the pulse may not be necessary for a stable pulsed source.

Improving the range resolution is desired for multiple coherent laser radar functions, including ranging, vibrometry, and synthetic aperture imaging. The use of a SOA as a phase modulator in each of these modalities was explored in Chapter 4. Since the pulse bandwidth is often increased through the application of linear frequency modulation (LFM), the IPRs of the self-phase modulated waveform and the LFM waveform were compared and shown to be comparable to one another in terms of the PSLR and ISLR. Experimental results were presented in Chapter 6 showing the range compression of transmitted pulses with a phase modulation characteristic of that from a saturated SOA for stationary, vibrating, and translating targets.
Since the transmission of shorter pulse durations reduces the amount of time the pulse is in contact with the object of interest, it minimizes the detrimental effects of target motion. This is particularly advantageous for SAL applications, where target motion will cause blurring of the SAL image. As such, the use of a SOA as both an amplifier and a phase modulator could be an enabling technology for a short-pulse SAL system, which could also acquire vibrometry data. This concept was explored in Chapter 5.

This research has shown how a SOA can be optimized to serve as both a phase modulator and an amplifier for long range laser radar applications. The following new ideas have been presented:

1. For a Gaussian input pulse, the range resolution is optimized for a carrier lifetime that is one half the FWHM input pulse duration.
2. There is little improvement in the range resolution for input pulse energies greater than 10% of the saturation energy.
3. The SOA-modulated output pulse and resulting IPR is highly insensitive to variations in the input pulse.
4. There is a substantial improvement in range resolution when the self-phase modulation from a SOA is exploited as bandwidth.

Since SOA’s are low cost and small in size, the use of a SOA as a phase modulator has many advantages. The carrier lifetimes of a SOA are generally on the order of several hundred picoseconds in duration, which makes them ideal to provide the optimized range resolution for transmitted pulses on the order of a nanosecond in duration. Furthermore, this approach allows a single component to be used for two purposes, amplification and phase modulation, eliminating the hardware necessary for traditional phase modulation techniques, like LFM. The simulations in Chapter 4 predicted an improvement in the range resolution of 7.5 by exploiting
the self-phase modulation of the SOA, and an improvement of 7.1 was demonstrated in the lab.
Finally, this approach allows for the transmission of shorter pulse durations, which minimize the sensitivity of the system to target motion.
REFERENCES


APPENDIX A

DERIVATION OF THE MEAN SQUARE SIGNAL CURRENT

The instantaneous Poynting vector is defined as

\[ S = E \times H, \]  

(A.1)

where \( S \) is the instantaneous power density, \( E \) is the electric field, and \( H \) is the magnetic field. If the instantaneous electric and magnetic fields are expressed as

\[ E_s = |E| \cos(\omega_s t + \phi_E(t)) \hat{x} \]

and

\[ H_s = |H| \cos(\omega_s t + \phi_H(t)) \hat{y}, \]

(A.2)

the instantaneous Poynting vector is found to be

\[ S = E \times H = |E||H| \cos(\omega_s t + \phi_E(t)) \cos(\omega_s t + \phi_H(t)) (\hat{x} \times \hat{y}). \]  

(A.3)

Using the identity

\[ \cos(u)\cos(v) = \frac{1}{2} [\cos(u + v) + \cos(u - v)], \]  

(A.4)

Equation (A.3) becomes

\[ S = \frac{1}{2} |E||H|[\cos(\phi_E(t) - \phi_H(t)) + \cos(2\omega_s t + \phi_E(t) + \phi_H(t))] \cdot (\hat{x} \times \hat{y}). \]  

(A.5)

The time averaged Poynting vector can be found from Equation (A.5) to be
〈\(S\)〉 = \(\frac{|E||H|}{2}\) \(\cos(\phi_E(t) - \phi_H(t)) \cdot (\hat{x} \times \hat{y})\). \hspace{1cm} (A.6)

Upon inspection, Equation (A.6) can also be written as

\[
\langle S \rangle = \frac{1}{2} \text{Re}\{(|E|\exp(j\phi_E(t))\hat{x}) \times (|H|\exp(-j\phi_H(t))\hat{y})\}
\]

\[
= \frac{1}{2} \text{Re}\{E_S \times H_S^\ast\}. \hspace{1cm} (A.7)
\]

Consider the received signal found in Equation (2.22) to be

\[
E_r(t) = u_t(t - t_{rt}) \exp(j\omega_c t + j\theta(t - t_{rt})) \exp(j(\phi_s(t) + \phi_n(t))) \hat{x}
\]

and

\[
H_r(t) = \frac{u_t(t - t_{rt})}{\eta_0} \exp(j\omega_c t + j\theta(t - t_{rt})) \exp(j(\phi_s(t) + \phi_n(t))) \hat{y},
\]

where \(\eta_0\) is the free space intrinsic impedance. In this case, Equation (A.7) becomes

\[
\langle S_r \rangle = \frac{1}{2} \frac{|E_r(t)|^2}{\eta_0} = \frac{1}{2} \frac{u_t^2(t - t_{rt})}{\eta_0}. \hspace{1cm} (A.9)
\]

The received power is then found to be

\[
P_r = \frac{1}{2} \frac{u_t^2(t - t_{rt})}{\eta_0} A_d, \hspace{1cm} (A.10)
\]

where \(A_d\) is the area of the detector. The MO signal was found in Equation (2.23) to be

\[
E_{MO}(t) = u_{MO} \exp(j\omega_c t + \phi_{MO}(t)). \hspace{1cm} (A.11)
\]

The power of the MO signal is then

\[
P_{MO} = \frac{1}{2} \frac{u_{MO}^2}{\eta_0} A_d. \hspace{1cm} (A.12)
\]

The received signal current was found in Equation (2.30) to be
\[ i_{\text{sig}}(t) = \frac{\rho_t A_d}{\eta_0} u_t (t - t_{rt}) u_{MO} \cos \left( \phi_s(t) - \phi_M(t) \right) \]  

(A.13)

The mean square signal current, is found from Equation (A.13) to be

\[ \langle i_{\text{sig}}^2 \rangle = \left( \frac{\rho_t A_d}{\eta_0} \right)^2 \left( \frac{u_t^2 (t - t_{rt}) u_{MO}^2}{2} \right). \]  

(A.14)

Using the relationships found in Equations (A.10) and (A.12), the mean square signal current can be simplified to yield the following expression:

\[ \langle i_{\text{sig}}^2 \rangle = \frac{1}{2} \left( \frac{\rho_t A_d}{\eta_0} \right)^2 \left( \frac{2\eta_0 P_r}{A_d} \right) \left( \frac{2\eta_0 P_{MO}}{A_d} \right) = 2\rho_t^2 P_r P_{MO}. \]  

(A.15)
APPENDIX B

PROPAGATION OF A PULSE THROUGH A WAVEGUIDE

The propagation of the electromagnetic wave within the amplifier is described by the wave equation

\[ \nabla^2 E - \frac{\varepsilon}{c^2} \frac{\partial^2 E}{\partial t^2} = 0 \tag{B.1} \]

where \( c \) is the speed of light and \( \varepsilon \) is the dielectric constant. The electric field is assumed to be of the form

\[ E(x, y, z, t) = \hat{x} \frac{1}{2} \{ F(x, y)A(z, t)\exp[j(k_0 z - \omega_0 t)] \} + \text{c.c.}, \tag{B.2} \]

where \( F(x, y) \) is the waveguide mode distribution, and \( A(z, t) \) is the slowly varying envelope associated with the optical pulse travelling through the SOA. Substituting Equation (B.2) into Equation (B.1) yields the expression

\[ \frac{1}{2} A(z, t) \left[ \frac{\partial^2 F(x, y)}{\partial x^2} + \frac{\partial^2 F(x, y)}{\partial y^2} \right] + \frac{1}{2} F(x, y) \left[ \frac{\partial^2 A(z, t)}{\partial z^2} + 2jk_0 \frac{\partial A(z, t)}{\partial z} - k_0^2 A(z, t) \right] \]

\[ - \frac{\varepsilon}{2c^2} F(x, y) \left[ \frac{\partial^2 A(z, t)}{\partial t^2} - 2j\omega_0 \frac{\partial A(z, t)}{\partial t} + \omega_0^2 A(z, t) \right] = 0. \tag{B.3} \]

Using the slowly varying envelope approximation, the second derivatives with respect to \( t \) and \( z \) can be ignored, yielding
\[
A(z, t) \left[ \frac{\partial^2 F(x, y)}{\partial x^2} + \frac{\partial^2 F(x, y)}{\partial y^2} \right] + F(x, y) \left[ 2j k_0 \frac{\partial A(z, t)}{\partial z} - k_0^2 A(z, t) \right] \\
+ \frac{\epsilon}{c^2} F(x, y) \left[ 2j \omega_0 \frac{\partial A(z, t)}{\partial t} - \omega_0^2 A(z, t) \right] = 0
\]  
(B.4)

The first term of Equation (B.4) takes the form of the following eigenvalue problem

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) F(x, y) + \frac{\omega_0^2}{c^2} n^2(x, y, \omega_0) F(x, y) = k_0^2 F(x, y)
\]  
(B.5)

where \( F(x, y) \) is normalized such that

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^2(x, y) \, dx \, dy = 1.
\]  
(B.6)

Substituting Equation (B.5) into Equation (B.4) yields

\[
A(z, t) \left[ -\frac{\omega_0^2}{c^2} n^2(x, y, \omega_0) F(x, y) \right] + F(x, y) \left[ 2j k_0 \frac{\partial A(z, t)}{\partial z} \right] \\
+ \frac{\epsilon}{c^2} F(x, y) \left[ 2j \omega_0 \frac{\partial A(z, t)}{\partial t} - \omega_0^2 A(z, t) \right] = 0.
\]  
(B.7)

Equation (B.7) can be rearranged to yield

\[
-F(x, y) A(z, t) \frac{\omega_0^2}{c^2} [n^2(x, y, \omega_0) - \epsilon] \\
+ F(x, y) \left[ 2j k_0 \frac{\partial A(z, t)}{\partial z} + \frac{2j \omega_0 \epsilon}{c^2} \frac{\partial A(z, t)}{\partial t} \right] = 0.
\]  
(B.8)

Dividing Equation (B.8) by the quantity \( 2j k_0 \) yields the expression

\[
F(x, y) \left[ \frac{\partial A(z, t)}{\partial z} + \frac{\omega_0 \epsilon}{k_0 c^2} \frac{\partial A(z, t)}{\partial t} \right] = -F(x, y) A(z, t) j \frac{\omega_0^2}{2k_0 c^2} [n^2(x, y, \omega_0) - \epsilon].
\]  
(B.9)

After using the definition \( k_0 = \tilde{n} \omega_0 / c \), Equation (B.9) becomes

\[
F(x, y) \left[ \frac{\partial A(z, t)}{\partial z} + \frac{\epsilon}{\tilde{n} c} \frac{\partial A(z, t)}{\partial t} \right] = -F(x, y) A(z, t) j \frac{\omega_0}{2 \tilde{n} c} [n^2(x, y, \omega_0) - \epsilon].
\]  
(B.10)

Multiplying through by \( F(x, y) \) and making use of Equation (B.6) yields
\[
\frac{\partial A(z,t)}{\partial z} + \frac{1}{n_c} \frac{\partial A(z,t)}{\partial t} = \int_{-\infty}^{\infty} F^2(x,y)\varepsilon(x,y)dxdy
\]

\[
= -A(z,t)j\frac{\omega_0}{2n_c} \int_{-\infty}^{\infty} F^2(x,y)\{n^2(x,y) - \varepsilon(x,y)\}dxdy.
\]

(B.11)

The dielectric constant is defined as

\[
\varepsilon = n_b^2(x,y) + \chi
\]

(B.12)

where \(\chi\) is the susceptibility and can be approximated as

\[
\chi \approx \frac{n_c}{\omega_0}(\alpha + j)a(N - N_0)
\]

(B.13)

in the active region of the SOA. Note that \(\chi\) is not a function of \(x\) and \(y\). If \(n^2(x,y) \approx n_b^2(x,y)\), Equation (B.11) becomes

\[
\frac{\partial A(z,t)}{\partial z} + \frac{1}{n_c} \frac{\partial A(z,t)}{\partial t} = -A(z,t)j\frac{\omega_0}{2n_c} \int_{-\infty}^{\infty} F^2(x,y)dxdy
\]

(B.14)

where the limits of integration have been limited to the active region of the SOA of width \(w\) and length \(d\). It can be shown that

\[
\int_{-\infty}^{\infty} F^2(x,y)\varepsilon(x,y)dxdy \approx \tilde{n}^2,
\]

(B.15)

where \(\tilde{n}\) is. Equation (B.14) becomes

\[
\frac{\partial A(z,t)}{\partial z} + \frac{1}{v_g} \frac{\partial A(z,t)}{\partial t} = -A(z,t)j\frac{\omega_0}{2n_c} \int_{0}^{d} \int_{0}^{w} F^2(x,y)dxdy,
\]

(B.16)

where \(v_g \approx c/\tilde{n}\) is the group velocity. Finally, using the definition of the confinement factor...
\[
\Gamma = \int_0^w \int_0^d F^2(x, y) \, dx \, dy \left/ \int_{-\infty}^{\infty} F^2(x, y) \, dx \, dy \right.
\]

Equation (B.16) then becomes

\[
\frac{\partial A(z, t)}{\partial z} + \frac{1}{v_g} \frac{\partial A(z, t)}{\partial t} = -j \frac{\omega_0 \chi \Gamma}{2\tilde{\eta}c} A(z, t).
\]

This expression describes the propagation of a pulse through a waveguide.