NATURAL CONVECTION IN A POROUS MEDIUM SATURATED BY
NANOFUID

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I dedicate this thesis to my Grandma.

I also dedicate it to my family. Without their sacrifices, love, blessings, and constant support this graduate study would just have remained a dream.
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A boundary layer analysis is presented for the natural convection heat and mass transfer flow past along two different geometries i.e., a vertical cone and an isothermal sphere in a Non-Darcy porous medium saturated with a nanofluid. A co-ordinate transformation is used to transform the governing equations into non-dimensional non-similar boundary layer equations. These equations are then solved numerically using implicit finite difference method (Keller-box method). Numerical solutions for heat transfer rate, mass transfer rate and friction factor have been presented for parametric variations of the buoyancy ratio parameter $N_r$, Brownian motion parameter $N_b$, thermophoresis parameter $N_t$ and Lewis number $L_e$. The dependency of the local friction factor, surface heat transfer rate (Nusselt number), and mass transfer rate (Sherwood number) on these parameters has been discussed.

**Keywords:** Nanofluid, Natural Convection, Porous Medium
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NOMENCLATURE

\[ \begin{align*}
D_B & \quad \text{Brownian diffusion coefficient} \\
D_T & \quad \text{thermophoretic diffusion coefficient} \\
f & \quad \text{rescaled nano-particle volume fraction} \\
g & \quad \text{gravitational acceleration vector} \\
k_m & \quad \text{effective thermal conductivity of the porous medium} \\
K & \quad \text{permeability of porous medium} \\
L_e & \quad \text{Lewis number} \\
N_t & \quad \text{Buoyancy Ratio} \\
N_b & \quad \text{Brownian motion parameter} \\
N_t & \quad \text{thermophoresis parameter} \\
N_u & \quad \text{Nusselt number} \\
P & \quad \text{pressure} \\
q'' & \quad \text{wall heat flux} \\
Ra_x & \quad \text{local Rayleigh number} \\
S & \quad \text{dimensionless stream function} \\
T & \quad \text{temperature} \\
T_W & \quad \text{wall temperature} \\
T_\infty & \quad \text{ambient temperature attained as y tends to infinity} \\
U & \quad \text{reference velocity} \\
Pr & \quad \text{Prandtl number} \\
A & \quad \text{non-Darcy parameter}
\end{align*} \]
Re  Reynolds number
u,v  Darcy velocity components
(x,y)  Cartesian coordinates

Greek Symbols:

\( \alpha_m \)  thermal diffusivity of porous medium
\( \beta \)  volumetric expansion coefficient of fluid
\( \varepsilon \)  porosity
\( \eta \)  dimensionless distance
\( \theta \)  dimensionless temperature
\( \mu \)  viscosity of fluid
\( \rho_f \)  fluid density
\( \rho_p \)  nano-particle mass density
\( (\rho c)_f \)  heat capacity of the fluid
\( (\rho c)_m \)  effective heat capacity of porous medium
\( (\rho c)_p \)  effective heat capacity of nano-particle material
\( \tau \)  parameter defined by equation (5) and (25)
\( \phi \)  nano-particle volume fraction
\( \phi_w \)  nano-particle volume fraction at wall surface
\( \phi_\infty \)  ambient nano-particle volume fraction attained
\( \psi \)  stream function
CHAPTER I

INTRODUCTION

1.1 Overview

The study of convective heat transfer in nanofluids is gaining a lot of attention. Nanofluids are solid-liquid composite materials consisting of solid nanoparticles or nanofibers with sizes typically of 1-100 nm suspended in liquid. From the recent studies, a small amount (<1% volume fraction) of Cu nanoparticles with ethylene glycol or carbon nanotubes dispersed in oil is reported to increase the inherently poor thermal conductivity of the liquid by 40% and 150%, respectively [1,2]. High concentrations (>10%) of particles is required to achieve such enhancement in case of conventional particle-liquid suspensions. High concentrations lead to amplified problems of stability. Some results in this rapidly evolving field include a surprisingly strong temperature dependence of the thermal conductivity [3] and a three-fold higher critical heat flux
compared with the base fluids [4]. Feasibility of nanofluids in nuclear applications by improving the performance of any water-cooled nuclear system that is heat removal limited has been studies by Kim et al. [5] at the Nuclear Science and Engineering Department of the Massachusetts Institute of Technology (MIT). Possible applications include pressurized water reactor (PWR) primary coolant, standby safety systems, accelerator targets, plasma diverters, etc. [6]. Nanofluids, where heat transfer can be reduced or enhanced at will, can be utilized where straight heat transfer enhancement is very important as in many industrial applications, nuclear reactors, transportation as well as electronics and biomedicine. Studies indicate that nanofluids have the potential to conserve 1 trillion Btu of energy for U.S. industry by replacing of cooling and heating water with nanofluid. For the U.S. electric power industry, using nanofluids in closed-loop cooling cycles could save about 10–30 trillion Btu per year (equivalent to the annual energy consumption of about 50,000–150,000 households). The related emissions reductions would be approximately 5.6 million metric tons of carbon dioxide; 8,600 metric tons of nitrogen oxides; and 21,000 metric tons of sulfur dioxide [7]. In geothermal power, energy extraction from the earth’s crust involves high temperatures around 500\(^0\)C to 1000\(^0\)C, nanofluids can be employed to cool the pipes exposed to such high temperatures. When drilling, nanofluids can serve in cooling the machinery and equipment working in high temperature environment. Nanofluids could be used as a working fluid to extract energy from the earth core [8]. Fluids like Engine oils, automatic transmission fluids, coolants, lubricants etc. used in various automotive applications have inherently poor heat transfer properties. Use of nanofluids by simply adding nanoparticles to these fluids could result in better thermal management [9]. Nanofluids can be used for
cooling of microchips in computers or elsewhere. They can be used in various biomedical applications like cancer therapeutics, nano-drug delivery, nanocryosurgery, cryopreservation etc.

1.2 Literature Review

1.3 Scope and Approach

The present work has been undertaken in order to analyze the natural convection past a vertical cone and an isothermal sphere embedded in a porous medium saturated by a nanofluid (non-Darcy model). The effects of Brownian motion and thermophoresis are included for the nanofluid. Numerical solutions of the boundary layer equations are obtained using Keller Box Scheme and discussion is provided for several values of the nanofluid parameters governing the problem.
CHAPTER II

NATURAL CONVECTIVE BOUNDARY LAYER FLOW ALONG A VERTICAL CONE EMBEDDED IN A NON-DARCY POROUS MEDIUM SATURATED WITH NANOFUID

2.1 Mathematical formulation and analysis

We consider the steady free convection boundary layer flow past a vertical cone with semi-vertical angle $\Omega$ placed in a nanofluid saturated non-Darcy porous medium. The co-ordinate system is selected such that x-axis is aligned with slant surface of the cone.

We consider the axi-symmetric problem. We consider at $y=0$, i.e. at wall the temperature $T$ and the nano-particle fraction $\phi$ take constant values $T_W$ and $\phi_W$, respectively. The ambient values of $T$ and $\phi$ as $y$ tends to infinity, are denoted by $T_\infty$ and $\phi_\infty$, respectively. The Oberbeck-Boussinesq approximation is employed. We consider the
porous medium whose porosity is denoted by $\varepsilon$ and permeability by $K$. Homogeneity and local thermal equilibrium in the porous medium is assumed.

Figure 2.1 Flow model and co-ordinate system

We now make the standard boundary layer approximation based on a scale analysis and write the governing equations.

\[
\frac{\partial}{\partial x} (ru) + \frac{\partial}{\partial y} (rv) = 0
\]

\[
\frac{\partial u}{\partial y} + \frac{\rho_f \mu}{\rho} \frac{\partial}{\partial y} (u^2) = \frac{(1 - \phi_w) \rho_f \beta g K \cos \Omega}{\mu} \cdot \frac{\partial T}{\partial y} - \frac{(\rho_p - \rho_f) g K \cos \Omega}{\mu} \cdot \frac{\partial \phi}{\partial y}
\]

\[
u \cdot \frac{\partial T}{\partial x} + v \cdot \frac{\partial T}{\partial y} = \alpha_m \frac{\partial^2 T}{\partial y^2} + \tau \left[ D_B \frac{\partial \phi}{\partial y} \frac{\partial T}{\partial y} + \frac{D_T}{T_\infty} \left( \frac{\partial T}{\partial y} \right)^2 \right]
\]
\[
\frac{1}{\varepsilon} \left( u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} \right) = D_B \frac{\partial^2 \phi}{\partial y^2} + \left( \frac{D_T}{T_\infty} \right) \frac{\partial^2 T}{\partial y^2}
\]  
(4)

Where

\[
\alpha_m = \frac{k_m}{(\rho c)_f}, \quad \tau = \frac{\varepsilon(\rho c)_p}{(\rho c)_f}
\]  
(5)

Here \(\rho_f\), \(\mu\) and \(\beta\) are the density, viscosity and volumetric volume expansion coefficient of the fluid while \(\rho_p\) is the density of the particles. The gravitational acceleration is denoted by \(g\). We have introduced the effective heat capacity \((\rho c)_m\) and effective thermal conductivity \(k_m\) of the porous medium. The coefficients that appear in equations (3) and (4) are the Brownian diffusion coefficient \(D_B\) and the thermophoretic diffusion coefficient \(D_T\).

The boundary conditions are taken to be

\[
v = 0, \quad T = T_w, \quad \phi = \phi_w, \quad \text{at} \quad y = 0,
\]  
(6)

\[
u = v = 0, \quad T \to T_\infty, \quad \phi \to \phi_\infty, \quad \text{as} \quad y \to \infty
\]  
(7)

We introduce a stream function \(\psi\) defined by

\[
u = -\frac{1}{r} \frac{\partial \psi}{\partial y}, \quad v = \frac{1}{r} \frac{\partial \psi}{\partial x}
\]  
(8)

so that Equation (1) is satisfied identically. We are then left with the following three equations.
\[
\frac{1}{r} \frac{\partial^2 y}{\partial y^2} + \frac{\rho \alpha K}{\mu} \cdot \frac{\partial}{\partial y} \left( \frac{1}{r} \frac{\partial y}{\partial y} \right)^2 = \frac{(1-\phi_w) \rho \alpha K \cos \Omega}{\mu} \frac{\partial T}{\partial y} - \frac{(\rho_p - \rho \alpha)}{\mu} \frac{\partial \phi}{\partial y} \tag{9}
\]

\[
\frac{1}{r} \frac{\partial y}{\partial y} \frac{\partial T}{\partial x} - \frac{1}{r} \frac{\partial y}{\partial x} \frac{\partial T}{\partial y} = \alpha_m \frac{\partial^2 T}{\partial y^2} + r \left[ D_b \frac{\partial \phi}{\partial y} \frac{\partial T}{\partial y} + \left( \frac{D_T}{T_w} \right) \frac{\partial^2 T}{\partial y^2} \right] \tag{10}
\]

\[
\frac{1}{\varepsilon} \left( \frac{1}{r} \frac{\partial y}{\partial y} \frac{\partial \phi}{\partial x} - \frac{1}{r} \frac{\partial y}{\partial x} \frac{\partial \phi}{\partial y} \right) = D_b \frac{\partial^2 \phi}{\partial y^2} + \left( \frac{D_T}{T_w} \right) \frac{\partial^2 T}{\partial y^2} \tag{11}
\]

Proceeding with the analysis we introduce the following dimensionless variables:

\[
\eta = \frac{\psi}{x} \cdot Ra_x^{1/2}
\]

\[
Ra_x = \frac{(1-\phi_w) \rho \alpha K \cos \Omega (T_w - T_x)}{\mu \cdot \alpha_m}
\]

\[
S = \frac{\psi}{\alpha_m \cdot r \cdot Ra_x^{1/2}}
\]

\[
\theta = \frac{T - T_x}{T_w - T_x}
\]

\[
f = \frac{\phi - \phi_w}{\phi_w - \phi_x}
\]

We assume that

\[
\xi = \frac{x}{L}
\]

\[
r = x \sin \Omega
\]

Where, \( L \) is the slant height of cone having apex angle of \( 2\Omega \).
Substituting the expressions in equation (12) into the governing equations (9)-(11) we obtain the following transformed equations:

\[ S^* (1 + A \cdot S') - \theta' + N_r \cdot f' = 0 \quad (13) \]

\[ \theta^* + \frac{3}{2} S \theta' + N_b \cdot f' \cdot \theta' + N_f (\theta')^2 = \xi \left[ S' \frac{\partial \theta}{\partial \xi} - \theta' \frac{\partial S}{\partial \xi} \right] \quad (14) \]

\[ f^* + \frac{3}{2} L_c \cdot S \cdot f' + \frac{N_f}{N_b} \theta^* = L_c \cdot \xi \left[ S' \frac{\partial f}{\partial \xi} - f' \frac{\partial S}{\partial \xi} \right] \quad (15) \]

Where the parameters are defined as following:

\[ N_r = \frac{\left( \rho_p - \rho_{f_e} \right) (\phi_W - \phi_e)}{\rho_{f_e} \beta (T_W - T_\infty) (1 - \phi_e)} , \]

\[ N_b = \frac{\varepsilon (\rho \mathbf{x})_f D_B (\phi_W - \phi_e)}{\rho \mathbf{x}_f \alpha_m} , \]

\[ N_f = \frac{\varepsilon (\rho \mathbf{x})_f D_T (T_W - T_\infty)}{\rho \mathbf{x}_f \alpha_m T_\infty} , \]

\[ L_c = \frac{\alpha_m}{\varepsilon \cdot D_B} , \]

\[ A = \frac{2 \cdot \rho_{f_e} \star K \cdot \alpha_m \cdot \text{Ra}_{\star}^{\frac{1}{2}}}{\mu \cdot x^{\frac{1}{2}}} \quad (16) \]

The transformed boundary conditions are:

\[ \eta = 0 : \quad S = 0, \quad \theta = 1, \quad f = 1 \]
\[ \eta \to \infty : \quad S' = 0, \quad \theta = 0, \quad f = 0 \quad (17) \]
2.2 Calculation of related parameters

The Heat Transfer rate is given by:

\[ q_w = -k_f \frac{\partial T}{\partial y} \bigg|_{y=0} \]

The heat transfer coefficient is given by:

\[ h = \frac{q_w}{(T_w - T_\infty)} \]

Local Nusselt number is given by:

\[ Nu_x = \frac{h \cdot x}{k_f} = -Ra_x^\frac{1}{2} \cdot \theta'(\xi,0) \] (18)

The Mass Transfer rate is given by:

\[ N_w = -D \frac{\partial \phi}{\partial y} \bigg|_{y=0} = h_m (\phi_w - \phi_\infty) \]

Where \( h_m = \) mass transfer coefficient,

\[ N_w = -D \cdot (\phi_w - \phi_\infty) f' \cdot Ra_x^\frac{1}{2} \frac{f'(\xi,0)}{x} \]

Sherwood number is given by:

\[ Sh = \frac{h_m \cdot x}{D} = -Ra_x^\frac{1}{2} \cdot f'(\xi,0) \] (19)

The local friction factor is given by \( Cf_x : \)

\[ \tau_w = \mu \frac{\partial u}{\partial y} \bigg|_{y=0} \]
\[ Cf_x = \frac{\tau_w}{\left( \frac{\rho U^2}{2} \right)} = \frac{2 \cdot Ra^3 \cdot \xi^3(\xi,0)}{Re^2_x \cdot Pr} \quad (20) \]

2.3 Numerical solution

Keller Box Method:

The resulting nonlinear system of partial differential equations is solved numerically by Keller-Box method which is an implicit finite difference method. The method allows for non-uniform grid discretion and converts the differential equations into algebraic ones which are then solved using Thomas algorithm. Thomas algorithm is essentially the result of applying Gauss elimination to the tri-diagonal system of equations. The number of grid points in both directions affects the numerical results. To obtain accurate results, a mesh sensitivity study was performed.
2.4 Results and discussions

Equations (13-15) were solved numerically to satisfy the boundary conditions (17) for parametric values of $Le$, $N_r$ (buoyancy ratio number), $N_b$ (Brownian motion parameter) and $N_t$ (thermophoresis parameter) using implicit finite difference method (Keller-box method). Tables 1-5 indicate results for wall values for the gradients of velocity, temperature and concentration functions which are proportional to the friction factor, Nusselt number and Sherwood number, respectively. From Table 1-3, we notice that as $N_r$ and $N_t$ increase, the friction factor increases whereas the heat transfer rate (Nusselt number) and mass transfer rate (Sherwood number) decrease. As $N_b$ increases, the friction factor and surface mass transfer rates increase whereas the surface heat transfer rate decreases. Results from Table 4 indicate that as $Le$ increases, The heat and mass transfer rates increase. From Table 5, we observe that as the non-Darcy parameter “A” increases, the heat and mass transfer rates decrease.

Figures 2.2-2.4 indicate that as $N_r$ increases, the velocity decreases and the temperature and concentration increase. Similar effects are observed from Figures 2.5-2.10 as $N_t$ and $N_b$ vary. Figure 2.11 illustrates the variation of velocity within the boundary layer as $Le$ increases. The velocity increases as $Le$ increases. From Figures 2.12 and 2.13, we observe that as $Le$ increases, the temperature and concentration within the boundary layer decrease and the thermal and concentration boundary later thicknesses decrease. Figures 2.14-2.16 indicate that as the non-Darcy parameter “A” increases, the velocity increases whereas the temperature and concentration within the boundary layer decrease.
Table 1. Effects of $N_r$ on $S''(0)$, $-\theta'(0)$ and $-\dot{f}(0)$ for $A=0$, $N_b=0.3$, $N_t=0.1$ and $Le=10$

<table>
<thead>
<tr>
<th>$N_r$</th>
<th>$S''(0)$</th>
<th>$-\theta'(0)$</th>
<th>$-\dot{f}(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-0.2907717</td>
<td>0.5712565</td>
<td>2.804848</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.0192688</td>
<td>0.5582566</td>
<td>2.694939</td>
</tr>
<tr>
<td>0.3</td>
<td>0.2285638</td>
<td>0.5453198</td>
<td>2.579612</td>
</tr>
<tr>
<td>0.4</td>
<td>0.4522196</td>
<td>0.5310432</td>
<td>2.458157</td>
</tr>
<tr>
<td>0.5</td>
<td>0.6485776</td>
<td>0.5160794</td>
<td>2.329314</td>
</tr>
</tbody>
</table>

Table 2. Effects of $N_t$ on $S''(0)$, $-\theta'(0)$ and $-\dot{f}(0)$ for $A=0$, $N_b=0.2$, $N_t=0.5$, and $Le=10$

<table>
<thead>
<tr>
<th>$N_t$</th>
<th>$S''(0)$</th>
<th>$-\theta'(0)$</th>
<th>$-\dot{f}(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.5880982</td>
<td>0.5561268</td>
<td>2.288450</td>
</tr>
<tr>
<td>0.2</td>
<td>0.5941838</td>
<td>0.5315452</td>
<td>2.251458</td>
</tr>
<tr>
<td>0.3</td>
<td>0.6042663</td>
<td>0.5084322</td>
<td>2.225397</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6175116</td>
<td>0.4870419</td>
<td>2.209107</td>
</tr>
<tr>
<td>0.5</td>
<td>0.6338583</td>
<td>0.4667582</td>
<td>2.201233</td>
</tr>
</tbody>
</table>

Table 3. Effects of $N_b$ on $S''(0)$, $-\theta'(0)$ and $-\dot{f}(0)$ for $A=0$, $N_t=0.5$, $N_t=0.3$, and $Le=10$

<table>
<thead>
<tr>
<th>$N_b$</th>
<th>$S''(0)$</th>
<th>$-\theta'(0)$</th>
<th>$-\dot{f}(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.455639</td>
<td>0.5349890</td>
<td>1.981256</td>
</tr>
<tr>
<td>0.2</td>
<td>0.6042663</td>
<td>0.5084322</td>
<td>2.225397</td>
</tr>
<tr>
<td>0.3</td>
<td>0.6813646</td>
<td>0.4742139</td>
<td>2.311157</td>
</tr>
<tr>
<td>0.4</td>
<td>0.7381298</td>
<td>0.4400252</td>
<td>2.356310</td>
</tr>
<tr>
<td>0.5</td>
<td>0.7856572</td>
<td>0.4070248</td>
<td>2.385364</td>
</tr>
</tbody>
</table>
Table 4. Effects of Le on $S''(0)$, $-\theta'(0)$ and $-f'(0)$ for $A=0$, $N_b=0.3$, $N_r=0.5$, and $N_t=0.1$

<table>
<thead>
<tr>
<th>Le</th>
<th>$S''(0)$</th>
<th>$-\theta'(0)$</th>
<th>$-f'(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.2290318</td>
<td>0.4542890</td>
<td>0.4505144</td>
</tr>
<tr>
<td>10</td>
<td>0.6485776</td>
<td>0.5160794</td>
<td>2.329314</td>
</tr>
<tr>
<td>100</td>
<td>3.4110026</td>
<td>0.5359154</td>
<td>7.893836</td>
</tr>
<tr>
<td>1000</td>
<td>12.1292734</td>
<td>0.5425866</td>
<td>25.34372</td>
</tr>
</tbody>
</table>

Table 5. Effects of $A$ on $S''(0)$, $-\theta'(0)$ and $-f'(0)$ for $N_b=0.3$, $N_r=0.5$, $N_t=0.1$ and $Le=10$

<table>
<thead>
<tr>
<th>A</th>
<th>$S''(0)$</th>
<th>$-\theta'(0)$</th>
<th>$-f'(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.6485776</td>
<td>0.5160794</td>
<td>2.329314</td>
</tr>
<tr>
<td>0.1</td>
<td>0.6091557</td>
<td>0.5092353</td>
<td>2.297508</td>
</tr>
<tr>
<td>1.0</td>
<td>0.4107432</td>
<td>0.4670312</td>
<td>2.095359</td>
</tr>
<tr>
<td>10</td>
<td>0.1287693</td>
<td>0.3444217</td>
<td>1.532180</td>
</tr>
<tr>
<td>100</td>
<td>0.0260661</td>
<td>0.2172511</td>
<td>0.9562534</td>
</tr>
</tbody>
</table>
Figure 2. Effects of $N_r$ on velocity profiles

Figure 2.3 Effects of $N_r$ on temperature profiles
Figure 2.4 Effects of $N_r$ on volume fraction profiles

$N_r=0.1, 0.2, 0.3, 0.4, 0.5$

Figure 2.5 Effects of $N_t$ on velocity profiles

$N_t=0.1, 0.2, 0.3, 0.4, 0.5$
Figure 2.6 Effects of $N_t$ on temperature profiles

Figure 2.7 Effects of $N_t$ on volume fraction profiles
Figure 2.8 Effects of $N_b$ on velocity profiles

Figure 2.9 Effects of $N_b$ on temperature profiles
Figure 2.10 Effects of $N_b$ on volume fraction profiles

Figure 2.11 Effects of $L_e$ on velocity profiles
Figure 2.12 Effects of $L_e$ on temperature profiles

Figure 2.13 Effects of $L_e$ on volume fraction profiles
Figure 2.14 Effects of $A$ on velocity profiles

Figure 2.15 Effects of $A$ on temperature profiles
Figure 16. Effects of A on volume fraction profiles

- $N_b = 0.3$
- $N_r = 0.5$
- $N_t = 0.1$
- $Le = 10$

Figure 2.16 Effects of A on volume fraction profiles

$A = 0, 0.1, 1.0, 10, 100$
CHAPTER III

NATURAL CONVECTIVE BOUNDARY LAYER FLOW ALONG AN
ISOTHERMAL SPHERE EMBEDDED IN A NON-DARCY POROUS MEDIUM
SATURATED WITH NANOFUID

3.1 Mathematical formulation and analysis

We consider the steady free convection boundary layer flow past a sphere placed in a nano-fluid saturated non-Darcy porous medium. The co-ordinate system is selected such that x measures the distance along the surface of the sphere from the stagnation point and y measures the distance normal to the surface of the sphere. The radius of sphere is a.

We consider the two-dimensional problem. We consider at y=0, the temperature T and the nano-particle fraction φ take constant values Tw and φw, respectively. The ambient values of T and φ, attend as y tends to infinity, are denoted by T∞ and φ∞.
respectively. The Oberbeck-Boussinesq approximation is employed. Homo-geneity and local thermal equilibrium in the porous medium is assumed. We consider the porous medium whose porosity is denoted by $\varepsilon$ and permeability by $K$.

We now make the standard boundary layer approximation based on a scale analysis and write the governing equations.

\[
\frac{\partial}{\partial x} (ru) + \frac{\partial}{\partial y} (rv) = 0
\]

\[
\frac{\partial u}{\partial y} + \frac{\rho_{\infty} K^*}{\mu} \cdot \frac{\partial}{\partial y} (u^2) = \frac{(1 - \phi_a) \rho_{\infty} \beta g K \sin \left( \frac{x}{a} \right)}{\mu} \frac{\partial T}{\partial y} - \frac{(\rho_p - \rho_{\infty}) g K \sin \left( \frac{x}{a} \right)}{\mu} \frac{\partial \phi}{\partial y}
\]

\[
u \cdot \frac{\partial T}{\partial x} + v \cdot \frac{\partial T}{\partial y} = \alpha_m \frac{\partial^2 T}{\partial y^2} + \tau \left[ D_B \frac{\partial \phi}{\partial y} \frac{\partial T}{\partial y} + \frac{D_T}{T_\infty} \left( \frac{\partial T}{\partial y} \right)^2 \right]
\]
\[
\frac{1}{\varepsilon} \left( u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} \right) = D_B \frac{\partial^2 \phi}{\partial y^2} + \left( \frac{D_T}{T_\infty} \right) \frac{\partial^2 T}{\partial y^2}
\]  

(24)

Where

\[
\alpha_m = \frac{k_m}{(\rho c)_f}, \quad \tau = \frac{\varepsilon (\rho c)_p}{(\rho c)_f}
\]  

(25)

The boundary conditions are taken to be

\[
v = 0, \quad T = T_w, \quad \phi = \phi_w, \quad \text{at} \quad y = 0,
\]  

(26)

\[
u = v = 0, \quad T \to T_\infty, \quad \phi \to \phi_\infty, \quad \text{as} \quad y \to \infty
\]  

(27)

We introduce a stream function \( \psi \) defined by

\[
u = \frac{1}{r} \frac{\partial \psi}{\partial y}, \quad v = -\frac{1}{r} \frac{\partial \psi}{\partial x}
\]  

(28)

So that Equation (21) is satisfied identically. We are then left with the following three equations.

\[
\frac{1}{r} \frac{\partial^2 \psi}{\partial y^2} + \frac{\rho_{pg} K^*}{\mu} \frac{\partial}{\partial y} \left( \frac{1}{r} \frac{\partial \psi}{\partial y} \right)^2 = \frac{(1-\phi_\infty) \rho_{pg} \beta g K \sin \left( \frac{x}{a} \right)}{\mu} \frac{\partial T}{\partial y} - \frac{(\rho_p - \rho_{pg}) g K \sin \left( \frac{x}{a} \right)}{\mu} \frac{\partial \phi}{\partial y}
\]

(29)

\[
\frac{1}{r} \frac{\partial \psi}{\partial y} \frac{\partial T}{\partial x} - \frac{1}{r} \frac{\partial \psi}{\partial x} \frac{\partial T}{\partial y} = \alpha_m \frac{\partial^2 T}{\partial y^2} + \tau \left[ D_B \frac{\partial \phi}{\partial y} \frac{\partial T}{\partial y} + \left( \frac{D_T}{T_\infty} \right) \frac{\partial^2 T}{\partial y^2} \right]
\]

(30)

\[
\frac{1}{\varepsilon} \left( \frac{1}{r} \frac{\partial \psi}{\partial y} \frac{\partial \phi}{\partial x} - \frac{1}{r} \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial y} \right) = D_B \frac{\partial^2 \phi}{\partial y^2} + \left( \frac{D_T}{T_\infty} \right) \frac{\partial^2 T}{\partial y^2}
\]

(31)
Now we introduce the following dimensionless variables:

\[
\eta = \frac{y}{x} \cdot Ra_x^{1/2}
\]

\[
Ra_x = \frac{(1 - \phi_x) \rho_f \beta_g K x (T_w - T_\infty)}{\mu \cdot \alpha_m}
\]

\[
S = \frac{\psi}{\alpha_m \cdot r \cdot Ra_x^{1/2}}
\]

\[
\theta = \frac{T - T_\infty}{T_w - T_\infty}
\]

\[
f = \frac{\phi - \phi_x}{\phi_w - \phi_x}
\]

Here, we assume that

\[
\xi = \frac{x}{a}
\]

\[
r = a \cdot \sin\left(\frac{\sqrt{x}}{a}\right)
\]

Substituting the expressions in equation (32) into the governing equations (29)-(31) we obtain the following transformed equations:

\[
S^* (1 + A \cdot S') - \sin \xi (\theta' + N_r \cdot f') = 0
\]

\[
\theta^* + \frac{1}{2} \theta' + N_b \cdot f' \cdot \theta' + N_c (\theta')^2 = \xi \left( S^* \frac{\partial \theta}{\partial \xi} - \theta' \left( \cot \xi + \frac{\partial S}{\partial \xi} \right) \right)
\]

\[
f^* + \frac{1}{2} L_e \cdot f' + \frac{N_e}{N_b} \theta'' = L_e \cdot \xi \left( S^* \frac{\partial f}{\partial \xi} - f' \left( \cot \xi + \xi \cdot \frac{\partial S}{\partial \xi} \right) \right)
\]
Where the parameters are defined as:

\[ N_x = \frac{(\rho_p - \rho_{fr})(\phi_W - \phi_o)}{\rho_{fr}\beta(T_W - T_x)(1 - \phi_x)} , \]

\[ N_p = \frac{\varepsilon(\rho x)_p D_B(\phi_W - \phi_o)}{(\rho x)_m} , \]

\[ N_t = \frac{\varepsilon(\rho x)_p D_T(T_W - T_x)}{(\rho x)_m \alpha_m T_x} , \]

\[ L_c = \frac{\alpha_m}{\varepsilon \cdot D_B} , \]

\[ A = \frac{2 \cdot \rho_{fr} K^* \alpha_m \cdot Ra_x}{\mu \cdot x} \]

\[ \text{Re}_x = \frac{\rho \cdot x}{\mu} \]

\[ \text{Pr} = \frac{\mu}{\rho \cdot \alpha_m} \]  

(36)

The transformed boundary conditions are:

\[ \eta = 0 : \quad S = 0, \quad \theta = 1, \quad f = 1 \]

\[ \eta \to \infty : \quad S' = 0, \quad \theta = 0, \quad f = 0 \]  

(37)
3.2 Calculation of related parameters

The Heat Transfer rate is given by:

\[ q_w = -k_f \frac{\partial T}{\partial y} \bigg|_{y=0} \]

The heat transfer coefficient is given by:

\[ h = \frac{q_w}{(T_w - T_e)} \]

Local Nusselt number is given by:

\[ Nu_x = \frac{h \cdot x}{k_f} = -Ra_x^{1/2} \cdot \theta'(\xi,0) \]  \hspace{1cm} (38)

The Mass Transfer rate is given by:

\[ N_w = -D \frac{\partial \phi}{\partial y} \bigg|_{y=0} = h_m (\phi_w - \phi_\infty) \]

Where \( h_m = \) mass transfer coefficient,

\[ N_w = -D \cdot (\phi_w - \phi_\infty) f' \frac{Ra_x^{1/2}}{x} \]

Sherwood number is given by:

\[ Sh = \frac{h_m \cdot x}{D} = -Ra_x^{1/2} \cdot f'(\xi,0) \]  \hspace{1cm} (39)

The local friction factor is given by \( \text{Cf}_x \):

\[ \text{Cf}_x = \frac{\Delta P}{L \cdot \rho \cdot u^2 / 2} \]
\[ \tau_w = \mu \cdot \frac{\partial u}{\partial y} \bigg|_{y=0} \]

\[ C_f = \frac{\tau_w}{\frac{\rho U^2}{2}} \frac{2 \cdot Ra^2 \cdot S^n(\xi,0)}{Re_x^2 \cdot Pr} \]  \hspace{1cm} (40)

3.3 Numerical solution

Keller Box Method:

The resulting nonlinear system of partial differential equations is solved numerically by Keller-Box method that is an implicit finite difference method. The method allows for non-uniform grid discretion and converts the differential equations into algebraic ones that are then solved using Thomas algorithm. Thomas algorithm is essentially the result of applying Gauss elimination to the tri-diagonal system of equations. The number of grid points in both directions affects the numerical results. To obtain accurate results, a mesh sensitivity study was performed.
3.4 Results and discussions

Equations (33-35) were solved numerically to satisfy the boundary conditions (37) for the parametric values of $L_e$, $N_r$ (buoyancy ratio number), $N_b$ (Brownian motion parameter), $N_t$ (thermophoresis parameter) using finite difference method.

Figures 3.2-3.7 indicate that as $N_r$ increases, the velocity, heat transfer rate (Nusselt number) and mass transfer rate (Sherwood number) decreases and the skin-friction coefficient, temperature and concentration increases.

Figures 3.8-3.11 indicate that as $N_b$ increases, the velocity, temperature, mass transfer rate (Sherwood number) and the skin-friction coefficient increases whereas heat transfer rate (Nusselt number) and concentration decreases.

Figures 3.14-3.19 indicate that as $N_t$ increases, the velocity, temperature, concentration increases whereas heat transfer rate (Nusselt number), mass transfer rate (Sherwood number) and the skin-friction coefficient decreases.

Figures 3.20-3.25 indicate that as $L_e$ increases, the velocity, the skin-friction coefficient and mass transfer rate (Sherwood number) within the boundary layer increases whereas temperature, concentration, heat transfer rate (Nusselt number) within the boundary layer decreases.

Figures 3.26-3.31 indicate that as the non-Darcy parameter A increases, the temperature and concentration increases whereas the velocity, heat transfer rate (Nusselt number), mass transfer rate (Sherwood number) and skin-friction coefficient within the boundary layer decrease.
Figure 1. Effects of $N_r$ on velocity profiles

Figure 2. Effects of $N_r$ on temperature profiles

Figure 3.2 Effects of $N_r$ on velocity profiles

Figure 3.3 Effects of $N_r$ on temperature profiles
Figure 3.4 Effects of $N_r$ on volume fraction profiles

Figure 3.5 Effects of $N_r$ on skin-friction coefficient
Figure 3.6 Effects of $N_r$ on heat transfer rate

Figure 3.7 Effects of $N_r$ on mass transfer rate
Figure 7. Effects of $N_b$ on velocity profiles
$A=1.0$  
$N_r=0.5$  
$N_t=0.1$  
$Le=10$  
$\xi=0.5$

Figure 8. Effects of $N_b$ on temperature profiles
$A=1.0$  
$N_r=0.5$  
$N_t=0.1$  
$Le=10$  
$\xi=0.5$
Figure 3.10 Effects of $N_b$ on volume fraction profiles

Figure 3.11 Effects of $N_b$ on skin-friction coefficient
Figure 3.12 Effects of $N_b$ on heat transfer rate

Figure 3.13 Effects of $N_b$ on mass transfer rate
Figure 13. Effects of $N_t$ on velocity profiles

$A=1.0$
$N_b=0.3$
$N_r=0.5$
$Le=10$
$\zeta=0.5$

$N_t=0.1, 0.2, 0.3, 0.4, 0.5$

Figure 14. Effects of $N_t$ on temperature profiles

$A=1.0$
$N_b=0.3$
$N_r=0.5$
$Le=10$
$\zeta=0.5$

$N_t=0.1, 0.2, 0.3, 0.4, 0.5$

Figure 3.14 Effects of $N_t$ on velocity profiles

Figure 3.15 Effects of $N_t$ on temperature profiles
Figure 15. Effects of $N_t$ on volume fraction profiles

Figure 16. Effects of $N_t$ on skin-friction coefficient

Figure 3.17 Effects of $N_t$ on skin-friction coefficient
Figure 3.18 Effects of $N_t$ on heat transfer rate

Figure 3.19 Effects of $N_t$ on mass transfer rate
Figure 3.20 Effects of $L_e$ on velocity profiles

Figure 3.21 Effects of $L_e$ on temperature profiles
Figure 21. Effects of Le on volume fraction profiles

Figure 22. Effects of Le on skin-friction coefficient
Figure 3.24 Effects of \( L_e \) on heat transfer rate

Figure 3.25 Effects of \( L_e \) on mass transfer rate
Figure 3.26 Effects of $A$ on velocity profiles

Figure 3.27 Effects of $A$ on temperature profiles
Figure 27. Effects of $A$ on volume fraction profiles

$A = 0, 0.1, 1, 5, 10$

$N_b = 0.3$
$N_r = 0.5$
$N_t = 0.1$
$Le = 10$

Figure 3.28 Effects of $A$ on volume fraction profiles

$A = 0, 0.1, 1, 5, 10$

Figure 28. Effects of $A$ on skin-friction coefficient

$N_b = 0.3$
$N_r = 0.5$
$N_t = 0.1$
$Le = 10$

$S''(\zeta, 0)$

Figure 3.29 Effects of $A$ on skin-friction coefficient

$A = 0, 0.1, 1, 5, 10$
Figure 3.30 Effects of A on heat transfer rate

Figure 3.31 Effects of A on mass transfer rate
CHAPTER IV

CONCLUDING REMARKS

In this study, we presented a boundary layer analysis for the natural convection past a vertical cone and an isothermal sphere embedded in a porous medium saturated with a nanofluid. Numerical results for friction factor, surface heat transfer rate and mass transfer rate have been presented for parametric variations of the buoyancy ratio parameter $N_r$, Brownian motion parameter $N_b$, thermophoresis parameter $N_t$ and Lewis number $L_e$. The results indicate that as $N_r$ and $N_t$ increase, the friction factor increases whereas the heat transfer rate (Nusselt number) and mass transfer rate (Sherwood number) decrease. As $N_b$ increases, the friction factor and surface mass transfer rates increase whereas the surface heat transfer rate decreases. As $L_e$ increases, the heat and mass transfer rates increase. As the non-Darcy parameter $A$ increases, the heat and mass transfer rates decrease.
REFERENCES


[28]
(a) Matlab code referred for fourth order runge-kutta method:

```matlab
function [wi, ti] = rk4 ( RHS, t0, x0, tf, N )

%RK4       approximate the solution of the initial value problem
%           x'(t) = RHS( t, x ),   x(t0) = x0
% using the classical fourth-order Runge-Kutta method - this
% routine will work for a system of first-order equations as
% well as for a single equation
%
calling sequences:
% [wi, ti] = rk4 ( RHS, t0, x0, tf, N )
% rk4 ( RHS, t0, x0, tf, N )
%
inputs:
% RHS     string containing name of m-file defining the
%          right-hand side of the differential equation;
%          the m-file must take two inputs - first, the value of
%          the dependent variable; second, the value of
%          the initial variable
% t0      initial value of the independent variable
% x0      initial value of the dependent variable(s)
% (          a row vector containing all initial values
% tf      final value of the independent variable
% N       number of uniformly sized time steps to be taken
% to advance the solution from t = t0 to t = tf
%
output:
% wi      vector / matrix containing values of the approximate
% solution to the differential equation
% ti      vector containing the values of the independent
% variable at which an approximate solution has been
% obtained
%
neqn = length ( x0 );
ti = linspace ( t0, tf, N+1 );
wi = [ zeros( neqn, N+1 ) ];
wi(1:neqn, 1) = x0';
```
\[ h = \frac{\text{tf} - \text{t0}}{N}; \]

\[
\text{for } i = 1:\text{N} \\
\quad k1 = h * \text{feval}(\text{RHS}, \text{t0}, \text{x0}); \\
\quad k2 = h * \text{feval}(\text{RHS}, \text{t0} + h/2, \text{x0} + k1/2); \\
\quad k3 = h * \text{feval}(\text{RHS}, \text{t0} + h/2, \text{x0} + k2/2); \\
\quad k4 = h * \text{feval}(\text{RHS}, \text{t0} + h, \text{x0} + k3); \\
\quad \text{x0} = \text{x0} + \left( k1 + 2*k2 + 2*k3 + k4 \right) / 6; \\
\quad \text{t0} = \text{t0} + h; \\
\]

\[
\text{wi(1:neqn,i+1)} = \text{x0}'; \\
\text{end;}
\]

(b) Matlab code referred for Thomas Algorithm:

```matlab
function x = thomas(varargin)
% THOMAS    Solves a tridiagonal linear system
%           x = THOMAS(A,d) solves a tridiagonal linear system using the very
%            efficient Thomas Algorithm. The vector x is the returned answer.
%           A*x = d;    /  a1  b1   0   0   0   ...   0   \   / x1   \   / d1
%          |  c1  a2  b2   0   0   ...   0   |   | x2   |   | d2
%          |  0   c2  a3  b3   0   ...   0   |   | x  |   | d3
%          |   :   :   :   :   :    :    :   |   | x4   |   | d4
%          |  0   0   0   0   \  cn-2  an-1  \ bn-1 |   |    |   |   \\
%          \  0   0   0   0   0   \  cn-1  \ an /   \ xn /   \ dn
% % - The matrix A must be strictly diagonally dominant for a stable
% solution. - This algorithm solves this system on (5n-4)
% multiplications/divisions and
% (3n-3) subtractions.
% x = THOMAS(a,b,c,d) where a is the diagonal, b is the upper
% diagonal, and c is
% the lower diagonal of A also solves A*x = d for x. Note that a
% is size n
% while b and c is size n-1.
% If size(a)=size(d)=[L C] and size(b)=size(c)=[L-1 C], THOMAS
% solves the C
% independent systems simultaneously.
% %
% % ATTENTION : No verification is done in order to assure that A is a
% tridiagonal matrix.
```
% If this function is used with a non tridiagonal matrix it will produce wrong results.

[a,b,c,d] = parse_inputs(varargin{:});

% Initialization
m = zeros(size(a));
l = zeros(size(c));
y = zeros(size(d));
n = size(a,1);

%1. LU decomposition

% L = / 1 \   
%   | l1 1 |   | m1 r1 |
%   | l2 1 |   | m2 r2 |
%   | : : : |   | : : : |
%   \ ln-1 1 /   \ mn /
% ri = bi -> not necessary
m(1,:) = a(1,:);
y(1,:) = d(1,:);

%2. Forward substitution (L*y=d, for y)
for i = 2 : n
    i_1 = i - 1;
    l(i_1,:) = c(i_1,:)./m(i_1,:);
    m(i,:) = a(i,:) - l(i_1,:).*b(i_1,:);
    y(i,:) = d(i,:) - l(i_1,:).*y(i_1,:);
end

%3. Backward substitutions (U*x=y, for x)

x(n,:) = y(n,:)./m(n,:);
for i = n-1 : -1 : 1
    x(i,:) = (y(i,:) - b(i,:).*x(i+1,:))./m(i,:);
end

function [a,b,c,d] = parse_inputs(varargin)
if nargin == 4
a = varargin{1};
b = varargin{2};
c = varargin{3};
d = varargin{4};

elseif nargin == 2
    A = sparse(varargin{1});
a = diag(A);
b = diag(A,1);
c = diag(A,-1);
d = varargin{2};
else
    error('Incorrect number of inputs.')
end

(c) Matlab code referred for Shooting Technique:

% Shooting method example of solving boundary value ODE
% % d^2 T
% % ----- + c (Tinf - T) = 0
% % dx^2
% % which models the temperature distribution in a rod
% % Will be solved using Euler's method
%
% define constants

Ta   = 40;   % left side temperature
Tb   = 200;  % right side temperature
Tinf = 20;   % ambient temperature
c    = 0.01;  % constant

L      = 10;  % length of bar
x      = 0;   % start position
deltax = 0.01; % increment in position for numerical procedure

T = Ta;    % initial condition
z = input('Enter starting slope: ');

count = 1;
Tvect(count) = T;
xvect(count) = x;

while x < L
    dTdx = z;
dzdx = -c * (Tinf - T);

    T = T + dTdx * deltax;  % Euler's method
    z = z + dzdx * deltax;

    x = x + deltax;
end

Tvect
x = x + deltax;

count = count + 1;
xvect(count) = x;  % store values for plotting
Tvect(count) = T;
end;

fprintf('The end temperature is: %f\n',T);

plot(xvect,Tvect);