UV PROPERTIES OF
GALILEONS

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Submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy

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August 2015
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Although General Relativity has survived and led to predictions which have been confirmed, it is essential to construct alternative theories of gravity to test General Relativity against, and to be able to differentiate General Relativity from theories which may have similar phenomenology but are fundamentally distinct. Alternative gravitational theories would be able to help make clear what tests of General Relativity should be preformed to distinguish it from competitors. Giving the graviton a mass is one way to modify General Relativity. In the decoupling limit of Massive Gravity the scalar mode of the graviton decouples and becomes the relevant degree of freedom we will consider at low energies. The Galileon is this decoupled scalar field and it is believed to violate Unitarity above its strong coupling scale, \( \Lambda \) (Chapter 3).

Taking advantage of the Galileon Duality (Chapter 4), the work presented within demonstrates a way to define the quantum Galileon operator in a way that accounts for the non-perturbative effects of the Vainshtein Mechanism. This Vainshtein Mechanism creates a region of strong coupling whereby normal perturbative calculations are no longer valid. We explicitly show how to calculate the spectral density for the Wightman function for this Galileon and show that at large energies, \( \sqrt{\mu} \), it grows as \( \rho(\mu) \sim e^{\mu^{3/5}/\Lambda^{6/5}} \). In terms of the spectral density, the requirement of Unitarity states \( \rho(\mu) > 0 \) which is true for our definition. Therefore, once accounting for the Vainshtein Mechanism, this Galileon operator appears to not break Unitarity. This is consistent with the UV completion mechanism known as Classicalization.
An interesting consequence when accounting for the Vainshtein Mechanism is that we have an ill-defined position space Wightman function. The exponential growth of the spectral density signals non-localizability within the theory (Chapter 5). However, like gravity, this non-localizability should inform us not to treat Galileons as strictly localizable field theories but as gravitational theories which is conceptually consistent when viewing the Galileon as a decoupling limit of some Massive Gravity theory.
Chapter 1

Introduction

The field of Cosmology attempts to answer questions regarding the Universe as a whole. In particular, theoretical cosmologists attempt to study the origin, evolution, and fate of the universe. It turns out that physics at all scales become relevant for these questions. At micro scales where particle physics dominates, we have quantum fluctuations which dominate the universe at very early times. We can even see the imprint of these quantum fluctuations as the temperature fluctuations in the Cosmic Microwave Background (CMB) \[1\]. On the other hand, as we look out in the night sky at large scales gravity is the dominant force. It is therefore intuitive to see why cosmologists try and look for physics that can give us a continuous understanding from small scale physics to large scale physics.

It is instructive to take some time and see how different physical theories connect. Newtonian Mechanics was the seed that birthed what we know of as physics, along with his theory of gravity, Newtonian Gravity. Maxwell and his studies created the theory of Electromagnetism. Through understanding the double slit experiment,
Quantum Mechanics became the natural extension to Newtonian Mechanics while Special Relativity was a consequence of Electromagnetism. By tying Special Relativity and Newtonian Gravity, Einstein was able to construct the very successful modern theory of gravity - General Relativity. We have also been able to connect Special Relativity with Quantum Mechanics which gives us Quantum Field Theory. So by knowing Quantum Field Theory and General Relativity, one can know almost everything there is to know (so far) about our understanding of the laws of physics. This is summed up in Figure 1.1.

General Relativity [2] has passed every observational test we have conducted. Starting with the three classical tests Einstein proposed to test his theory, the perihelion precession of Mercury [3], the deflection of starlight by the Sun [4], and the gravitational redshift of light [5], it was shown that General Relativity belonged as a formidable theory of gravity. Some modern tests of General Relativity include light travel time delays from radar signals caused by the time dilation felt near the Sun which agrees with General Relativity within a few percent [6], frame dragging which is a slight shift in the precession of test masses around rotating bodies [7, 8], and gravitational energy loss from binary pulsars which is an indirect detection of gravitational waves [9, 10]. There is ongoing research in hopes to find direct detections of gravitational waves [11]. Interestingly, tests of General Relativity are lacking in the far-infrared range [12] which turns out to be an interesting regime for the class of gravitational theories which we are interested in this work.

Although General Relativity has shown no observational flaws, it is important to create alternative forms of gravity that predict uniquely different phenomena than
Figure 1.1: A cartoon diagram depicting how different laws of physics are connected and built out of each other. \( h, c, e, \) and \( G \) stand for the relevant scales of quantum mechanics, special relativity, electromagnetism, and gravity respectively as Planck’s constant, the speed of light, the electric charge, and Newton’s gravitational constant. Different physical theories can incorporate any number of these scales (and therefore their physics). The ‘Theory of Everything’ would in principle incorporate all scales of all physics we know. Although it is interesting to note that the scales in this cartoon are just a representation of all scales in physics. In principle there could be more scales we have not accounted for or can comprehend yet simply because we have not witnessed or conducted any experiment that has shed light on that aspect of reality.
General Relativity to test General Relativity against, particularly at large distances. These alternative theories would help us understand and conduct tests that would be able to distinguish General Relativity from other gravitational theories. Late-time cosmic acceleration, Section 1.1, and the cosmological constant tuning issue, Section 1.2, are a few modern examples that hint at looking for natural extensions to General Relativity in the far-infrared. We quickly review these concepts below.

1.1 Late-Time Cosmic Acceleration

In the early 1900’s, Hubble discovered that the further nebula were from us in space the faster they were receding from us [13]. The consequence of this was a physical justification for the Friedmann equation which looks at an expanding, homogeneous, and isotropic universe with (in spherical coordinates and including a spatial curvature term, $\kappa$),

$$
 ds^2 = -c^2 dt^2 + a(t)^2 \left( \frac{dr^2}{1 - \kappa r^2} + r^2 d\theta^2 + r^2 \sin(\theta)^2 d\phi^2 \right),
$$

(1.1)

where $a(t)$ is known as the scale factor. Applying this metric to General Relativity we can find the dynamics of the scale factor as a function of the energy density of the universe through

$$
 \left( \frac{\dot{a}(t)}{a(t)} \right)^2 = H^2 = H_0^2 \left( \frac{\Omega_m}{a(t)^3} + \frac{\Omega_r}{a(t)^4} + \frac{\Omega_\kappa}{a(t)^2} + \Omega_\Lambda \right),
$$

(1.2)
where \( m, r, \kappa, \) and \( \Lambda \) stand for matter, radiation, spatial curvature, and cosmological constant respectively and the \( \Omega \)'s are density parameters defined to be

\[
\Omega_i = \frac{\rho_i}{\rho_c},
\]  

(1.3)

where \( \rho_c \) is the critical density which is the density of a flat universe (\( \kappa = 0 \)). In the late 1990’s two groups, the Supernova Cosmology Project, \[^{14}]\), and the High-z Supernova Search Team, \[^{15}]\), using Type 1a supernova discovered evidence for cosmic acceleration by measuring \( \Omega_\Lambda > 0 \). This implies that not only are objects moving away from us, but they are increasing their speed due to the cosmological constant. This was an unexpected result as the notion that gravity would be \textit{decelerating} the expansion was the dominant hypothesis at the time. Today, the recent Planck Collaboration \[^{16}^{17}^{18}]\) has quoted

\[
\begin{align*}
\Omega_m & \approx 0.3098 \pm 0.0062 \\
\Omega_\Lambda & \approx 0.6911 \pm 0.0062 \\
\Omega_\kappa & \approx 0.000 \pm 0.005,
\end{align*}
\]  

(1.4) (1.5) (1.6)

which states that about 70% of the energy density of the universe is composed of what is known as \textit{dark energy}. \( \Omega_m \) is composed of both \textit{luminous} matter, the matter in the universe we can directly observe, and \textit{dark matter} which is a hypothetical form of matter that primarily only interacts with gravity. \( \Omega_r \approx 10^{-5} \) and is sometimes neglected. This model of the universe is known as the \( \Lambda \text{CDM} \) model which stands for
Lambda Cold Dark Matter. The word ‘cold’ is in the name to differentiate between a relativistic dark matter (which would be in $\Lambda_r$) and non-relativistic which we have in $\Lambda_m$.

1.2 The Cosmological Constant and Tuning

So we see that current observations require us to include a cosmological constant contribution to the universe. From a strict General Relativity standpoint, there is nothing wrong with this. We may write the Einstein-Hilbert action of General Relativity without a cosmological constant,

$$S_{E.H.} = \int d^4x \sqrt{-g} \frac{M_{pl}^2}{2} R,$$

where $M_{pl}$ is the Planck Mass and $R$ is the standard Ricci scalar. If we add a cosmological constant, $\Lambda$, to this we have

$$S_{E.H.,\Lambda} = \int d^4x \sqrt{-g} \left[ \frac{M_{pl}^2}{2} R - M_{pl}^2 \Lambda \right].$$

Since $\Lambda$ is a constant we still have $\nabla_\mu G^{\mu\nu} = 0$. However, when we add matter fields into the mix we end up encountering what most would say is a problem. Here we review this problem in the context of [19, 20, 21, 22]. Let us consider a toy action for matter being

$$S_m[\phi, \psi] = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} (\partial \phi)^2 - \frac{1}{2} (\partial \psi)^2 - V(\phi, \psi) \right],$$
where

\[ V(\phi, \psi) = V_0 + \frac{\lambda}{4}(\phi^2 - \nu^2)^2 + \frac{\bar{g}}{2}\phi^2\psi^2. \]  \hspace{1cm} (1.10)

So when we add gravity our total action to consider is

\[ S = S_{\text{E.H.}} + S_{\text{m}} \]
\[ = \int d^4x \sqrt{-g} \frac{M^2_{\text{pl}}}{2}R + \int d^4x \sqrt{-g} \left[ -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\psi)^2 - V(\phi, \psi) - M^2_{\text{pl}}\Lambda \right], \]

\hspace{1cm} (1.11)

where we have put the cosmological constant contribution in with matter to better conceptualize the problem. Now, looking at the equations of motion we find

\[ R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{M^2_{\text{pl}}}T_{\mu\nu} - g_{\mu\nu}\Lambda, \]

\hspace{1cm} (1.12)

where \( T_{\mu\nu} \) is the stress-energy tensor defined to be

\[ T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\partial S_{\text{m}}}{\partial g^{\mu\nu}}. \]

\hspace{1cm} (1.13)

If we are just to consider the ground state - the state with the lowest energy, all derivative contributions from \( T_{\mu\nu} \) will vanish so we will be left with

\[ T_{\mu\nu}^{\text{gs}} = -g_{\mu\nu}V_{\text{min}}(\phi_0, \psi_0), \]

\hspace{1cm} (1.14)
where $\phi_0$ and $\psi_0$ are in their ground states. This contribution to the stress energy effectively is another contribution to the cosmological constant, $\Lambda$. We can define an effective cosmological constant

$$\Lambda_{\text{eff}} = \Lambda + \frac{V_{\text{min}}(\phi, \psi)}{M_{\text{pl}}^2}. \quad (1.15)$$

Now, as is done in [20], if we further consider that $\phi$ and $\psi$ are in thermal equilibrium, at some temperature $T$, we replace $\psi^2$ with the thermal average $\langle \psi^2 \rangle_T = aT^2$ with $a$ being an unimportant proportionality constant and defining $g \equiv a\bar g$ we have $V(\phi, \psi) \to V_{\text{eff}}(\phi)$ where

$$V_{\text{eff}}(\phi) = V_0 + \frac{\lambda}{4}(\phi^2 - v^2)^2 + \frac{g}{2}T^2\phi^2$$

$$= \left( V_0 + \frac{\lambda v^4}{4} \right) + \left( \frac{g}{2}T^2 - \frac{\lambda}{2}v^2 \right) \phi^2 + \frac{\lambda}{4}\phi^4$$

$$= \left( V_0 + \frac{\lambda v^4}{4} \right) + \frac{\lambda v^2}{2} \left( \frac{T^2}{T^2_{\text{crit}}} - 1 \right) \phi^2 + \frac{\lambda}{4}\phi^4$$

$$= \left( V_0 + \frac{\lambda v^4}{4} \right) + \frac{1}{2}m^2(T)\phi^2 + \frac{\lambda}{4}\phi^4, \quad (1.16)$$

where we have defined $T^2_{\text{crit}} = \lambda v^2/g$ and a temperature dependent effective mass,

$$m^2(T) \equiv \lambda v^2 \left( \frac{T^2}{T^2_{\text{crit}}} - 1 \right). \quad (1.17)$$
$T_{\text{crit}}$ is the critical temperature at which a phase change will occur. We, therefore, have in this scenario an effective cosmological constant

$$\Lambda_{\text{eff}} = \Lambda + \frac{V_{\text{eff}}(\phi_{\text{min}})}{M_{\text{pl}}^2}, \quad (1.18)$$

where $\phi_{\text{min}}$ is defined to be

$$\frac{\partial V_{\text{eff}}(\phi)}{\partial \phi} \bigg|_{\phi = \phi_{\text{min}}} = 0, \quad (1.19)$$

which gives us

$$m^2(T)\phi_{\text{min}} + \lambda\phi_{\text{min}}^3 = 0. \quad (1.20)$$

Now we see the importance of the critical phase transition temperature. If $T > T_{\text{crit}}$ then we have $\phi_{\text{min}} = 0$ and therefore $V_{\text{eff}}(\phi) = V_0 + \lambda v^4/4$ and we are left with

$$\Lambda_{\text{eff}} = \Lambda + \frac{V_0}{M_{\text{pl}}^2} + \frac{\lambda v^4}{4M_{\text{pl}}^2} \quad \text{for } T > T_{\text{crit}}. \quad (1.21)$$

So we see that the inclusion of matter fields induces contributions to the cosmological constant. Now, once the phase transition occurs, when $T < T_{\text{crit}}$ we have $m^2(T) < 0$ and thus

$$\phi_{\text{min}} = v, \quad (1.22)$$
which gives us $V(\phi_{\text{min}}) = V_0$ which leads to

\[ \Lambda_{\text{eff}} = \Lambda + \frac{V_0}{M_{\text{pl}}^2} \quad \text{for } T < T_{\text{crit}}. \] (1.23)

The point is, if we focus on (1.21) one might say we can just define $V_0 = -\lambda v^4/4$ to make $\Lambda_{\text{eff}} = \Lambda$. However, when comparing this to (1.23) we still will have a new contribution to $\Lambda_{\text{eff}}$. This is saying that in a phase transition, we will always pickup a nonzero contribution to the vacuum energy density, $\Lambda$, from the ground state of the matter fields. So with phase transitions we will pick up an induced energy density

\[ \Lambda_{\text{eff}} = \Lambda + \Lambda_{\text{ind}} \]

\[ \rightarrow \rho_{\text{eff}} = \rho + \rho_{\text{ind}}, \] (1.24)

where the energy densities are defined to be $\rho_i \equiv M_{\text{pl}}^2 \Lambda_i$. As discussed in [20, 22] when looking at the Standard Model we have schematically

\[ \rho_{\text{eff}} = \rho + \rho_{\text{EW}} + \rho_{\text{QCD}} + \ldots \]

\[ \rho_{\text{eff}} = \rho + \rho_{\text{matter}}, \] (1.25)

where we have just grouped all vacuum energy densities from all phase transitions into one $\rho_{\text{matter}}$. The tuning issue arises when we try and put in physical numbers into this expression. The right hand side of (1.25) are quantities we calculate through various Quantum Field Theoretic procedures. Although naively these quantities are
infinite, various ambiguous regularization schemes produce rough estimates usually in the range from $\rho_{\text{matter}} \approx 10^{44} eV^4$ ([20] [21]) to $\rho_{\text{matter}} \approx 10^{112} eV^4$ ([22]).

For instance as described in [22] in the semi-classical quantum gravity context where we treat the matter fields as having some quantum contribution but spacetime as classical, we have for the field equations in a quantum vacuum

$$G_{\mu\nu} = \frac{1}{M_{pl}^2} T_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{M_{pl}^2} \langle 0 \mid \delta T_{\mu\nu} \mid 0 \rangle ,$$

(1.26)

where $\delta T$ is our quantum contribution to the matter fields. Using Lorentz Invariance we may write $\langle 0 \mid \delta T_{\mu\nu} \mid 0 \rangle = \langle 0 \mid \hat{\rho}_{\text{vac}} \mid 0 \rangle g_{\mu\nu}$. Now if we just take a free scalar field as our matter content we know its energy density is just a summation of oscillators with energy density $\omega/2$ so we have

$$\langle 0 \mid \hat{\rho}_{\text{vac}} \mid 0 \rangle = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \sqrt{k^2 + m^2}.$$

(1.27)

Going into spherical coordinates for the momentum integrals we end up with an energy density for an effective cosmological constant to be

$$\rho_{\text{eff}} = \rho + \frac{1}{4\pi^2} \int_0^\infty dk \ k^2 \sqrt{k^2 + m^2}.$$

(1.28)

The trouble comes when taking this integral to infinity, it diverges. Therefore, what is usually done is to put in a cut-off scale, $\Lambda_{UV}$, which will help us regulate the
divergences. That is instead of looking at the above integral we look at

$$\int_{0}^{\Lambda_{UV}} dk \ k^2 \sqrt{k^2 + m^2} = \frac{\Lambda_{UV}^4}{4} + \frac{m^2 \Lambda_{UV}^2}{4} + \frac{m^4}{16} \ln \left( \frac{m^2 e^{1/2}}{4 \Lambda_{UV}^2} \right) + O \left( \frac{1}{\Lambda_{UV}^2} \right), \quad (1.29)$$

where we have focused on the large $\Lambda_{UV}$ limit. This gives us for our cosmological constant energy density

$$\rho_{eff} \approx \rho + \frac{\Lambda_{UV}^4}{16 \pi^2} + \frac{m^2 \Lambda_{UV}^2}{16 \pi^2} + \frac{m^4}{64 \pi^2} \ln \left( \frac{m^2 e^{1/2}}{4 \Lambda_{UV}^2} \right). \quad (1.30)$$

Now, if we could trust Quantum Field Theory up to any arbitrary high energy we would be obligated to take the $\Lambda_{UV} \rightarrow \infty$ limit in this expression. However, it is expected that new physics or understandings must come into play at the Planck scale since above that scale we cannot trust perturbative Quantum Field Theory calculations. Therefore, it is argued that above the Planck scale this calculation cannot be trusted so we impose that the cut-off scale should be around the Planck scale, that is to say we say $\Lambda_{UV} \approx M_{pl}$. This gives us

$$\rho_{eff} \approx \rho + \Lambda_{UV}^4 \approx \rho + M_{pl}^4 \approx \rho + 10^{112} eV^4. \quad (1.31)$$

A red flag may be raised at this point since these estimates are different by a factor of $10^{68}$ and there is still discussion in the literature as to how best approximate the value of $\rho_{matter}$. The point is, with either of these values we still have a tuning issue. The left hand side of (1.25) is a quantity we measure using various experimental and observational techniques. We measure $\rho_{eff} \approx 10^{-11} eV^4 \quad (21, 22)$. This brings us to
the uncomfortable result

\[ 10^{-11} eV^4 = \rho + 10^{44} eV^4 , \tag{1.32} \]

where we are even using the conservative \( \rho_{\text{matter}} \) estimation. Thus we see the *tuning problem*. In order for our calculations to agree with our observations, we need to pick a \( \rho \), or cosmological constant \( \Lambda \), to be accurate to \( 10^{55} \) decimal places! Although, *in principle*, this doesn’t *have* to be a problem, it is very uncomfortable to need such a fine tuning without deeper physics to justify it.

### 1.3 Beyond General Relativity

In an attempt to explore and potentially help understand these phenomena, cosmologists have created alternative theories of General Relativity. As stated above, the success of solar system tests on General Relativity make this endeavor difficult as any modification of General Relativity must at least look remarkably like General Relativity at small scales.

#### 1.3.1 Self-Acceleration

One modification of General Relativity is known as DGP for its creators Dvali, Gabadadze, and Porrati [23, 24, 25, 26]. Here one embeds a four dimensional space
into a higher five dimensional space as in

\[
S = \int d y d x \, \frac{M_5^3}{4} \sqrt{-G} R^{(5)}(G) + \int d^4 x \, \frac{M_5^2}{2} \sqrt{-g} R(g) + \mathcal{L}_m(g, \phi_i), \tag{1.33}
\]

where \(M_5\) is the five dimensional Planck scale, \(G\) is the five dimensional metric, and \(\mathcal{L}_m(g, \phi)\) is our matter Lagrangian characterized by matter fields \(\phi_i\). It was shown that this model has the ability to *self-accelerate* without the need to include a cosmological constant \([23, 24, 25]\). This is the idea that a positive cosmological constant can naturally arise in these models through an effective mass given to the four dimensional graviton, \(m = M_5^3/M_{pl}^2\), which gives a modified Friedmann equation (see \([27]\) for a nice review)

\[
H^2 - mH = 0, \tag{1.34}
\]

where \(H = \dot{a}/a\) is the Hubble scale. We see that \(a(t)\) now accelerates due to this effective mass. However, this branch of solutions has shown to possibly admit ghost-like instabilities \([28, 29]\) although it may be possible, within the framework of \(F(R)\) theories, to overcome this \([30]\). Nonetheless, the concept that the universe could be accelerating without the need to introduce a cosmological constant ‘by hand’ is appealing and the DGP model has given evidence that such physics is plausible. Once more, at small distances DGP looks like standard four dimensional General Relativity, it is only at large scales that it differs from General Relativity therefore agreeing with Solar System tests.
1.3.2 Degravitation

With the concept of ‘self-accelerating’ modified gravitational models which can potentially give a ‘natural’ explanation to the appearance of a cosmological constant it is justified to see if modified gravity can also make statements about the cosmological constant tuning problem. The concept of Degravitation [31, 32, 33, 34, 35] which supposes that the cosmological constant really is large and only appears to be small through a weakening of gravity with the cosmological constant at infrared scales has been another area of interest.

In the context of [33] this can be seen by taking the usually General Relativity field equations

\[ M_{\text{pl}}^2 G_{\mu\nu} = T_{\mu\nu} , \]  

(1.35)

and supposing a modification of Newton’s constant of the form

\[ M_{\text{pl}}^2 \left( 1 + F(L^2\Box) \right) G_{\mu\nu} = T_{\mu\nu} , \]  

(1.36)

where \( L \) is some finite length scale and \( F \) has the property that

\[ \text{if} \quad \alpha \gg 1 \quad \text{then} \quad F(\alpha) \rightarrow 0 , \]  

(1.37)

\[ \text{if} \quad \alpha \ll 1 \quad \text{then} \quad F(\alpha) \gg 1 . \]  

(1.38)

This effectively creates a filter that says, if we think of \( \Box \) as \( k^2 \), that high energy
modes effectively feel normal General Relativity and it is only the low frequency modes that produce a modification to General Relativity.

Before jumping to gravity, let us review an example of a massive vector field $A_\mu$. This example is taken from [36] and shows the point very cleanly. We begin by considering the dynamics of a massive vector $A_\mu$ with a scalar $\phi$ with some source $J^\mu$ governed by

$$\int d^4x \ - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \ - \frac{1}{2} m^2 A_\mu A^\mu - mA^\mu \partial_\mu \phi - \frac{1}{2} (\partial \phi)^2 + A_\mu J^\mu \ . \quad (1.39)$$

We integrate out $\phi$ by replacing it with its value from its equations of motion. That is, the equation of motion for $\phi$

$$\Box \phi = -m \partial_\mu A^\mu \ , \quad (1.40)$$

gives us

$$\phi = -\frac{m}{\Box} \partial_\mu A^\mu \ . \quad (1.41)$$

Replacing $\phi$ with (1.41) back into the action (1.39) gives us

$$S = \int d^4x \ - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \ - \frac{1}{2} m^2 (A_\mu A^\mu + \partial_\mu A^\mu \frac{1}{\Box} \partial_\nu A^\nu) + A_\mu J^\mu \ , \quad (1.42)$$

and noticing that $A_\mu A^\mu + \partial_\mu A^\mu \frac{1}{\Box} \partial_\nu A^\nu = -\frac{1}{2} F_{\mu\nu} \frac{1}{\Box} F^{\mu\nu}$ after integrating by parts ne-
glecting boundary terms we have

\[ S = \int d^4x \left( 1 - \frac{m^2}{\Box} \right) F_{\mu\nu} + A_\mu J^\mu. \] (1.43)

We see that integrating out the \( \phi \) field has created a nonlocal Lagrangian however it has the desired ‘filter’ effect of the source. That is, the equation of motion for \( A_\mu \) gives us

\[ \left( 1 - \frac{m^2}{\Box} \right) \partial_\mu F^{\mu\nu} = -J^\nu. \] (1.44)

The point is for momenta \( k \gg m \) the filtering contribution is effectively zero so we recover usual Maxwell however when \( k \ll m \) we effectively see a weakened source.

So we see that by adding a mass to a vector field we can create this ‘degravitation’ effect. The same phenomena occurs when we give the graviton a mass. As discussed in [36] in the context of linearized Fierz-Pauli linearized massive gravity [37] one way to write the action of a massive graviton is

\[ S = \int d^4x \mathcal{L}_0 - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) + \kappa h_{\mu\nu}T^{\mu\nu} \] (1.45)

where \( \mathcal{L}_0 \) is the Lagrangian for the massless graviton and \( \kappa = M_{\text{Pl}}^{-1} \). We can introduce vector St"{u}kelberg fields [38] \( A_\mu \) through \( h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu \) and introducing

\[ 17 \]
an auxiliary scalar field \( N = \frac{1}{2} h + \partial_\mu A^\mu \) we end up with \[ S = \int d^4 x \ L_0 + m^2 \left[ -\frac{1}{2} h_{\mu\nu} h^{\mu\nu} + \frac{1}{4} h^2 + A_\mu \Box A^\mu + N(h - N) \right. \]
\[ - A^\mu (\partial_\mu h - 2 \partial^\nu h_{\mu\nu} + 2 \partial_\mu N)] + \kappa h_{\mu\nu} T^{\mu\nu}, \] 
and after integrating out the vector field we have
\[ S = \int d^4 x \frac{1}{2} h_{\mu\nu} \left( 1 - \frac{m^2}{\Box} \right) \mathcal{E}^{\mu\nu,\alpha\beta} h_{\alpha\beta} - 2 N \frac{1}{\Box} (\partial_\mu \partial_\nu h^{\mu\nu} - \Box h) + \kappa h_{\mu\nu} T^{\mu\nu}, \] 
where
\[ \mathcal{E}^{\mu\nu}_{,\alpha\beta} = \left( \eta^{(\mu}_{\alpha} \eta^{\nu)}_{\beta} - \eta^{\mu\nu} \eta_{\alpha\beta} \right) \Box - 2 \partial^{(\mu}_{\alpha} \partial^{\nu)}_{\beta} \eta_{\alpha\beta} + \partial^{\mu} \partial^{\nu} \eta_{\alpha\beta} + \partial_{\alpha} \partial_{\beta} \eta^{\mu\nu}. \]

Hence see the same \( \left( 1 - \frac{m^2}{\Box} \right) \) filtering term created by the mass of the graviton, thus creating a degravitational effect.

### 1.4 The Galileon

We see that the concept of giving the graviton a mass can help us conceptualize potential phenomenological explanations for both cosmic acceleration and the cosmological constant tuning problem through self-acceleration and degravitation respectively. An interesting consequence to these types of modified gravity theories is that in their low energy decoupling limit, a scalar field called the Galileon dominates the dynamics. For instance, let us take the full nonlinear massive gravity theory with a dynamics
metric $g$ and reference metric $f$ as \cite{39, 40, 27} (see \cite{27, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50}) for discussions on ghost-like instabilities.

\begin{equation}
S_{\text{mGR}} = \int d^4x \sqrt{-g} \frac{M_{p}^2}{2} \left( R[g] + \frac{m^2}{2} (\alpha_2 \mathcal{L}_2[\mathcal{K}] + \alpha_3 \mathcal{L}_3[\mathcal{K}] + \alpha_4 \mathcal{L}_4[\mathcal{K}]) \right),
\end{equation}

where

\begin{equation}
\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \sqrt{g} \Gamma^\mu_\nu \, f, \quad (1.51)
\end{equation}

and

\begin{align*}
\mathcal{L}_2[\mathcal{K}] &= \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon_{\nu_1 \nu_2} \mu_3 \mu_4 \mathcal{K}_{\mu_1 \nu_1} \mathcal{K}_{\mu_2 \nu_2}, \\
\mathcal{L}_3[\mathcal{K}] &= \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon_{\nu_1 \nu_2 \nu_3} \mu_4 \mathcal{K}_{\mu_1 \nu_1} \mathcal{K}_{\mu_2 \nu_2} \mathcal{K}_{\mu_3 \nu_3}, \\
\mathcal{L}_4[\mathcal{K}] &= \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon_{\nu_1 \nu_2 \nu_3 \nu_4} \mathcal{K}_{\mu_1 \nu_1} \mathcal{K}_{\mu_2 \nu_2} \mathcal{K}_{\mu_3 \nu_3} \mathcal{K}_{\mu_4 \nu_4},
\end{align*}

where $\epsilon$ is the Levi-Civita tensor. Massive Gravity, has, along with the usually two degrees of freedom in massless gravity, three additional degrees of freedom. We can explicitly see these degrees of freedom by looking in the Stükelberg language where we will just consider a Minkowski reference metric and split $g_{\mu \nu} = \eta_{\mu \nu} + H_{\mu \nu}$ \cite{27}, we have

\begin{equation}
H_{\mu \nu} = h_{\mu \nu} + \frac{2}{m} \partial_\mu A_\nu + \frac{2}{m} \partial_\nu A_\mu - \frac{1}{M_{p}^3 m^2} \partial_\mu A^\alpha \partial_\nu A_\alpha \\
- \frac{2}{M_{p}^3 m^3} \partial_\mu A^\alpha \partial_\nu \pi + \frac{2}{m^2} \partial_\mu \partial_\nu \pi - \frac{1}{M_{p}^3 m^4} \partial_\mu \partial_\alpha \pi \partial_\nu \partial_\alpha \pi \quad (1.55)
\end{equation}
where \( h_{\mu
u} \) is the usual tensor degree of freedom in massless General Relativity, \( A_\mu \) are the vector degrees of freedoms, and \( \pi \) is the scalar degree of freedom called the \textit{Galileon}. It is this field and its dynamics we are interested in in this work. We can define an energy scale \( \Lambda_3 = (M_{\text{pl}} m^2)^{1/3} \) and take a decoupling limit where we have \( M_{\text{pl}} \to \infty \) and \( m \to 0 \) while keeping \( \Lambda_3 \) fixed. Focusing on the interactions between \( h \) and \( \pi \) and doing a shift to further decouple \( h \) from \( \pi \) \( (h \to \tilde{h}(h, \pi)) \) we end up with (see [27])

\[
\mathcal{L} = -\frac{1}{4} \tilde{h}^{\mu\nu} \epsilon_{\alpha\beta} \tilde{h}_{\alpha\beta} - \frac{1}{4} \sum_{n=2}^{5} \frac{c_n}{\Lambda_3^{3(n-2)}} \frac{(\partial \pi)^2}{(6-n)!} \mathcal{L}_{n-2}[\pi] \\
+ \frac{(\alpha_3 + 4\alpha_4)}{2\Lambda_3^3} \tilde{h}^{\mu\nu} \left[ -6(\partial^\alpha \partial_\mu \pi)(\partial^\beta \partial_\nu \pi)(\partial_\alpha \partial_\beta \pi) + 6(\partial^\alpha \partial_\mu \pi)(\partial_\alpha \partial_\beta \pi) \partial_\nu \pi \right] \\
+ 3(\partial_\mu \partial_\nu \pi)(\partial^\alpha \partial_\beta \pi)(\partial_\alpha \partial_\beta \pi) - 3(\partial_\mu \partial_\nu \pi)(\partial_\alpha \partial_\beta \pi) (\Box \pi)^2 - 3(\partial^\alpha \partial_\beta \pi)(\partial_\alpha \partial_\beta \pi) \eta_{\mu\nu} \\
+ 2(\partial^\alpha \partial_\beta \pi)(\partial^\gamma \partial_\alpha \pi)(\partial^\beta \partial_\gamma \pi) + (\Box \pi)^3 \right] . 
\]  

(1.56)

where \( \mathcal{L}_{n-2}[\pi] \) has the same epsilon structure as in \([1.52], [1.53], \) and \([1.54] \) and the \( c_n \)'s are just functions of the \( \alpha \)'s in \([1.50] \). Although this Lagrangian looks unwieldy, we will just focus on the terms which are completely decoupled from \( h \), that is the \( \mathcal{L}_n[\pi] \) terms. With the epsilon structure it can be shown that these operators only have second order equations of motion thus avoiding ghost-list instabilities. They also have a symmetry of \( \pi \to \pi + c + b_\mu x^\mu \) where \( c \) and the \( b \)'s are constants. Galileons coming from the low energy limit of the full-nonlinear Massive Gravity theory \([1.50] \) is not unique to \([1.50] \). For instance, Galileons are known to come from the decoupling limit of DGP \([51, 52, 53] \).
With well motivated reasons to consider the dynamics of Galileon theories, we will focus on the most general form of the Galileon theory [54, 55, 56] which can be written in the form, in \( d \) dimensions,

\[
S[\pi] = \sum_{n=2}^{d+1} \frac{1}{\Lambda_\alpha} S_n[\pi] = \sum_{n=2}^{d+1} \frac{1}{\Lambda_\alpha} \int d^d x \ c_n \mathcal{L}_n(\pi)
\]

\[
= \sum_{n=2}^{d+1} \frac{1}{\Lambda_\alpha} \int d^d x \ c_n \epsilon^{\mu_1 \ldots \mu_d} \epsilon^{\nu_1 \ldots \nu_d} \pi \prod_{j=1}^{n-1} (\partial_{\mu_j} \partial_{\nu_j} \pi) \prod_{k=n}^{d} \eta_{\mu_k \nu_k},
\]

(1.57)

where \( \epsilon \) is the Levi-Civita tensor, the \( c_n \)'s are arbitrary coefficients, \( \alpha = n - 2 + \frac{dn}{2} - d \), \( \Lambda \) is some dimensionful number (with Massive Gravity in mind it can be conceptualized as \( \Lambda \sim \Lambda_3 = (M_{pl} m^2)^{1/3} \)) and we will only be considering flat space-time, therefore our metric is simply \( \eta_{\mu\nu} \). For instance, in 4D we have a total of four interactions,

\[
\mathcal{L}_2(\pi) = (\partial \pi)^2, \tag{1.58}
\]

\[
\mathcal{L}_3(\pi) = (\partial \pi)^2 \Box \pi, \tag{1.59}
\]

\[
\mathcal{L}_4(\pi) = (\partial \pi)^2 ((\Box \pi)^2 - (\partial_{\mu} \partial_{\nu} \pi)^2), \tag{1.60}
\]

\[
\mathcal{L}_5(\pi) = (\partial \pi)^2 ((\Box \pi)^3 - 3 \Box \pi (\partial_{\mu} \partial_{\nu} \pi)^2 + 2(\partial_{\mu} \partial_{\nu} \pi)^3). \tag{1.61}
\]

Galileons, once again, have a symmetry of \( \pi \rightarrow \pi + c + b_{\mu} x^{\mu} \) where both \( c \) and \( b \) are constants and their higher derivative structure is such that they do not admit higher than second order equations of motion which can be see by the epsilon structure,
which is important to avoid ghost-like instabilities \[49, 50\]. We can observe this shift symmetry by looking at this epsilon structure. For convenience we will work in four dimensions however the argument holds in all dimensions and we’ll just look at $S_3[\pi]$.

So let us consider

\[ S_3[\pi] = \int d^4 x \ c_3 \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon^{\nu_1 \nu_2} \mu_3 \mu_4 (\partial_{\mu_1} \partial_{\nu_1} \pi)(\partial_{\mu_2} \partial_{\nu_2} \pi) \pi \ . \tag{1.62} \]

Now when we do the shift we have

\[
\pi \rightarrow \pi + c + b_\mu x^\mu , \tag{1.63}
\]

\[
(\partial_{\mu_1} \partial_{\nu_1} \pi) \rightarrow (\partial_{\mu_1} \partial_{\nu_1} \pi) , \tag{1.64}
\]

which gives us

\[ S_3[\pi] = \int d^4 x \ c_3 \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon^{\nu_1 \nu_2} \mu_3 \mu_4 (\partial_{\mu_1} \partial_{\nu_1} \pi)(\partial_{\mu_2} \partial_{\nu_2} \pi)(\pi + c + b_\mu x^\mu) . \tag{1.65} \]

Now we will integrate by parts and neglecting boundary terms we have

\[ S_3[\pi] = - \int d^4 x \ c_3 \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon^{\nu_1 \nu_2} \mu_3 \mu_4 (\partial_{\mu_1} \pi)(\partial_{\mu_2} \partial_{\nu_2} \pi)(\pi + c + b_\mu x^\mu) \\
- \int d^4 x \ c_3 \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon^{\nu_1 \nu_2} \mu_3 \mu_4 (\partial_{\nu_1} \pi)(\partial_{\mu_2} \partial_{\nu_2} \pi)(\partial_{\mu_1} \pi + b_\mu) . \tag{1.66} \]

Now, since the Levi-Civita tensor is anti-symmetric, that is it picks up a negative sign when any two indices are switched and $(\partial_{\mu_1} \partial_{\mu_2} \partial_{\nu_2} \pi)$ is symmetric under such a switch, we have an anti-symmetric tensor contracted with a symmetric tensor which
identically results in zero. Therefore, the first term in the above expression is zero.

So doing another integration by parts with the $\partial_{\nu_1}$ derivative and noticing the term that would once again create three derivatives on $\pi$ is zero we end up with

$$S_3[\pi] = \int d^4x \, c_3 \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon_{\nu_1 \nu_2} \mu_3 \mu_4 \pi (\partial_{\mu_2} \partial_{\nu_2} \pi)(\partial_{\nu_1} \partial_{\mu_1} \pi),$$

(1.67)

which is the same as (1.62) thus proving the shift symmetry once using the same logic to any term in any dimension.

Although coming from modified theories of General Relativity, in this work we will treat Galileons alone in their own right. However, in Chapter 5, we give evidence that even when treating Galileons outside of a gravitational context, it may be the case that we must treat them gravitationally to make sense of them.

1.4.1 The Vainshtein Mechanism

This higher derivative structure of the Galileon has a profound property known as the Vainshtein Mechanism [57]. The Vainshtein Mechanism is sometimes described as a screening mechanism of the Galileon fifth force. Meaning, when the Galileon (just focusing on $\mathcal{L}_3(\pi)$ as an example) couples to matter, say a spherically symmetric point source, then this new Galileon force, $F_g(r)$, scales relative to the usual Newtonian gravity force, $F_N(r)$, as, [27]

$$\frac{F_g(r)}{F_N(r)} \sim \left( \frac{r}{r_\star} \right)^{3/2},$$

(1.68)
where \( r_* \) is known at the \textit{Vainshtein radius}

\[
r_* = \frac{1}{\Lambda} \left( \frac{M}{4\pi M_{\text{pl}}} \right)^{1/3},
\]

\[ (1.69) \]

where \( M \) is the mass of the source and \( M_{\text{pl}} \) is the Planck mass. The point is, looking at \[(1.68)\], within the Vainshtein radius, \( r \ll r_* \), we have the fifth Galileon force being highly suppressed. This is essential since if \( F_g(r) \) was not suppressed, we would have observed its effects in the solar system. For instance, suppose \( \Lambda \) is of the order of the Hubble parameter today then this Galileon force on the Earth is 12 orders of magnitude lower than Newton given the mass of the Sun as the source for the Vainshtein Mechanism \cite{27}. This remarkable phenomena is what keeps Galileons within the realm of possibility in regards to tests of General Relativity. The fifth force that Galileons produce is simplify undetectable at solar system scales.

\subsection*{1.4.2 Quantum Considerations}

When considering how Galileons act at the quantum level however, we run into complications. For instance, when asking the tree level two to two scattering amplitude, \( \mathcal{A}(2 \rightarrow 2) \), with center of mass energy \( E \) for Galileons we arrive at (see Chapter 3)

\[
\mathcal{A}(2 \rightarrow 2) \sim \frac{E^6}{\Lambda^6}.
\]

\[ (1.70) \]

This might not seem like an issue until we conceptualize this amplitude as the probability of such a scattering event occurring. Since this amplitude depends on energy,
once $E \sim \Lambda$ and greater, this amplitude breaks Unitarity (see the Appendix \ref{Ann}). The usual resolution to this problem is to introduce heavier degrees of freedom which preserve Unitarity. For instance, if we scatter two fermions with mass $m$ the total cross section as a function of the center of mass energy $E$ will go as \(^{58}\)

$$\sigma(E^2) \sim \frac{(E^2 - m^2)^2}{E^2}. \quad (1.71)$$

However, assuming locality, it was shown in \(^{59}\) that this cross section cannot grow faster than

$$\sigma(E^2) < \ln (E^2)^2, \quad (1.72)$$

in order to preserve Unitarity. This is known as the Froissart Bound. The resolution to this is Wilsonian Completion (see Chapter \ref{Ch3}) where the introduction of heavier gauge bosons, the $W$’s and $Z$’s with heavier masses, schematically labeled as $M$, will correct the cross section to be

$$\sigma(E^2) \sim \frac{(E^2 - m^2)^2}{E^2(1 + (E^2 - m^2)/M^2)}, \quad (1.73)$$

which preserves Unitarity. We see that if we only consider energies $E \ll M$ that \(^{1.73}\) is well approximated by \(^{1.71}\). This is saying that as long as we are at energies below the mass scale of the heavier degrees of freedom than we may effectively ignore the heavier fields dynamics. This is the essence of Effective Field Theory which we will discuss further in Chapter \ref{Ch3}. 

\[\]
Using this Wilsonian Completion logic we then would want to say that the Galileon scale $\Lambda$ is really the mass scale of some heavier degree of freedom that no longer can be neglected and once accounted for, the amplitude would no longer break Unitarity. In this context, $\Lambda$ is called the cut-off of the theory. That is, Galileons shouldn’t be trusted above this energy scale $\Lambda$. However, the Vainshtein Mechanism has something to say about this scale. If we take, for instance, a cubic Galileon

$$\mathcal{L} = -\frac{1}{2} (\partial \pi)^2 - \frac{1}{\Lambda^3} \Box \pi (\partial \pi)^2,$$  \hspace{1cm} (1.74)

and see how perturbations, $\delta \pi$, behave around some classical background $\pi_0$, the dynamics for $\delta \pi$ will be governed by

$$\mathcal{L} = -\frac{1}{2} Z^{\mu \nu}(\pi_0) \partial_\mu \delta \pi \partial_\nu \delta \pi - \frac{1}{\Lambda^3} \Box \delta \pi (\partial \delta \pi)^2,$$  \hspace{1cm} (1.75)

where $Z$ is now some effective metric which is dependent on the classical background solution $\pi_0$. Now, if we rescale $\delta \pi \rightarrow \sqrt{Z} \delta \pi$ where here $Z$ stands for the magnitude of $Z^{\mu \nu}$, then we find

$$\mathcal{L} = -\frac{1}{2} (\partial \delta \pi)^2 - \frac{1}{(\sqrt{Z} \Lambda)^3} \Box \delta \pi (\partial \delta \pi)^2$$

$$= -\frac{1}{2} (\partial \delta \pi)^2 - \frac{1}{\Lambda_\ast^3} \Box \delta \pi (\partial \delta \pi)^2,$$  \hspace{1cm} (1.76)

where we have defined a new cutoff scale for these perturbations $\Lambda_\ast = \sqrt{Z} \Lambda$. That is, if we were to scatter two $\delta \pi$ particles that live on this effective metric we would
find, at tree level,

\[ \mathcal{A}(2 \rightarrow 2) \sim \frac{E^6}{\Lambda^6}, \quad (1.77) \]

which now breaks Unitarity at a new energy scale, \( \Lambda_s \). This points at some subtleties with this apparent cut-off energy scale. For instance, taking the example from [27] if we consider a background configuration well within the Vainshtein radius we’ll have

\[ \Lambda_s = \sqrt{Z} \Lambda \sim \sqrt{\frac{\pi_0'}{\Lambda^3 r}} \Lambda \sim \left( \frac{M}{M_{pl}} \frac{1}{r \Lambda} \right)^{1/4} \Lambda \sim 10^7 \Lambda, \quad (1.78) \]

where like in [27] we are considering the Vainshtein region created by the Earth at the surface of the Earth as an example. So, through this field redefinition, we see there is ambiguity in the cut-off scale of the Galileon. This conceptually points towards the idea that we should not think of \( \Lambda \) (or \( \Lambda_s \)) as the true cut-off of the theory. Although Unitarity appears to be broken in (1.70) and (1.77) it might be the case that there are effects that naive perturbative scattering amplitudes are not accounting for. This work attempts to explore exactly these possible phenomena.

We want to see the quantum effects of this unique, non-perturbative, Vainshtein Mechanism effect on Galileons. However, we will not think of the Vainshtein Mechanism as coming from an external source, as we have been considering so far, but as the strong coupling scale of the Galileon. That is, at some energy scale, \( E \), all Galileon operators contribute equally - we may not treat any operator as a perturbative correction to any other. The Vainshtein radius at which this occurs is (see
The important difference with \( r_*(E) \) with usual energy and distance relationships is that this strong coupling, or Vainshtein radius, _increases_ with energy. Larger energies probe larger distances, this is fundamentally an Ultraviolet/Infrared (UV/IR) mixing that we shall see in this work is essential to understanding UV quantum properties of Galileons \[60\]. We will explicitly calculate the spectral density (see Chapter 2) for the Wightman function for a special Galileon which is dual to a free field. We will show that this spectral density \( \rho(\mu) \sim e^{\mu^{3/5}/\Lambda^{8/5}} \) which is always positive, therefore Unitary, and diverges which signals an inherit non-localizability within the theory.

In Chapter 2 we review important mathematical tools required to understand the work done here. If one is familiar with this equipment there is no need to go into depth. This chapter is written for individuals wanting to understand this work from a pedagogical point of view. Chapter 3 discusses what it means for a theory to be UV complete and important differences between different high energy physical mechanisms. This chapter gives evidence to support the notion that Galileons may not fit into the standard Wilsonian Completion ideology of UV completions and that a recently proposed UV mechanism, Classicalization, might better fit with how Galileons truly behave in the UV. Chapter 4 reveals a very unique property of a special Galileon. That is, one can pick special values for the \( c_n \) coefficients and see that this Galileon is classically dual to a free field. This duality not only is essential to
explicitly calculate the Wightman function for this special Galileon but its existence at all points towards important consequences the Vainshtein mechanism must have in order to achieve such a duality. The take away message is that Galileons possess some type of non-localizability which is caused by the Vainshtein mechanism. This is not conceptually dissimilar to the non-localizability black holes possess in graviton scattering. Chapter 5 reviews and discusses localizability and non-localizability within field theories and some of its consequences. We end with a short summary of what we have learned from this work in the Conclusion.
Chapter 2

A Bit on Quantum Field Theory

This work revolves around Quantum Field Theory and a new mathematical approach that potentially may be able to accurately describe some theories in the UV. In this chapter we review basic concepts of Quantum Field Theory as preliminaries to better understand this work. The best way to understand Quantum Field Theory is to first understand some fundamentals of quantum and field theory independently.

2.1 Quantum Mechanics

Quantum mechanics fundamentally differs from classical mechanics in four general fashions,

- It is indeterministic in terms of the position and momentum of particles. This is incorporated in the fact that the position and momentum of a ‘particle’ no longer commute, that is to say $[\hat{x}, \hat{p}] = i\hbar$ (where $[\hat{a}, \hat{b}] \equiv \hat{a}\hat{b} - \hat{b}\hat{a}$). This is not to be confused with quantum mechanics being fundamentally deterministic. We
can fully determine the wavefunction, $\psi(t, \vec{x})$, of a quantum system which is thought to be the object that encodes all the information we can know.

- Quantum systems are often quantized. An example being the energy of a particle does not take on continuous values, rather discrete values that the system must ‘jump’ between. For an infinite square well the energy values of the wavefunction become $E = \frac{\hbar^2 \pi^2 n^2}{2mL^2}$ with the point being that $n \in \mathbb{Z}$ and $n \not\in \mathbb{R}$.

- The concepts of a wave and particle are no longer independent. Quantum objects have a wave-particle duality that ‘mixes’ these two concepts together.

- Quantum mechanics lets us determine the probabilities of certain events occurring. These probability amplitudes of going from some initial state $i$ to some final state $f$, $P(i \rightarrow f)$, are constructed from complex valued amplitudes, $\mathcal{A}(i \rightarrow f)$, through $P(i \rightarrow f) = |\mathcal{A}(i \rightarrow f)|^2$. If there are some number of intermediate states, $n$, between states $i$ and $f$ then we say $\mathcal{A}(i \rightarrow f) = \sum_n \mathcal{A}(i \rightarrow n) \mathcal{A}(n \rightarrow f)$. This is fundamentally different from the composition law for classical probabilities which state $P(i \rightarrow f) = \sum_n P(i \rightarrow n) P(n \rightarrow f)$.

A standard and useful example which we will build upon is the simple harmonic oscillator. The Hamiltonian for this system is

$$\hat{H} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 \quad (2.1)$$

$$= -\frac{\hbar^2}{2m} \nabla^2 + \frac{1}{2} m\omega^2 \hat{x}^2, \quad (2.2)$$
with the time independent Schrodinger equation,

$$\hat{H} \psi_n(\vec{x}) = E_n \psi_n(\vec{x}).$$  \hspace{1cm} (2.3)$$

We can define two operators

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \hat{x} + i \sqrt{\frac{1}{2m\omega\hbar}} \hat{p},$$  \hspace{1cm} (2.4)$$

and its complex conjugate

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \hat{x} - i \sqrt{\frac{1}{2m\omega\hbar}} \hat{p},$$  \hspace{1cm} (2.5)$$

to be able to write the Hamiltonian and quantum states as

$$\hat{H} = \frac{-\hbar^2}{2m} \nabla^2 + \frac{1}{2} m\omega^2 \hat{x}^2 \rightarrow \hat{H} = \hbar \omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$$  \hspace{1cm} (2.6)$$

$$\psi_n(\vec{x}) \rightarrow | n \rangle.$$  \hspace{1cm} (2.7)$$
The information of the system is then encoded in the fact that

\[
[a, \hat{a}^\dagger] = 1 , \quad (2.8)
\]

\[
\hat{a}|n\rangle = \sqrt{n}|n-1\rangle , \quad (2.9)
\]

\[
\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle , \quad (2.10)
\]

\[
\langle n|m \rangle = \delta_{nm} , \quad (2.11)
\]

\[
\hat{a} |0\rangle = 0 , \quad (2.12)
\]

\[
\hat{H}|n\rangle = E_n |n\rangle = \hbar \omega \left( n + \frac{1}{2} \right) |n\rangle , \quad (2.13)
\]

where (2.12) defines the ground state |0\rangle. This is known as first quantization where the energies of these particles are quantized. When shifting to quantum field theory, we will have a second quantization picture where these \( \hat{a} \) and \( \hat{a}^\dagger \) operators will not be adding or subtracting energy from a particle but adding or subtracting particles from the vacuum.

### 2.1.1 Coherent States

In this section we review coherent states which we will use later in Section 5.4.2.

These states are formally defined to be eigenstates of the annihilation operator, \( \hat{a} \). That is

\[
\hat{a}|\alpha\rangle = \alpha |\alpha\rangle . \quad (2.14)
\]
By writing $\left| \alpha \right\rangle$ in terms of the complete $\left| n \right\rangle$ states we can say

$$\hat{a} \left| \alpha \right\rangle = \sum_n c_n \hat{a} \left| n \right\rangle = \sum_n c_n \sqrt{n} \left| n-1 \right\rangle = \alpha \sum_n c_n \left| n \right\rangle ,$$

which gives us

$$c_{n+1} = \frac{\alpha}{\sqrt{n+1}} c_n ,$$

which leads to

$$c_n = \frac{\alpha^n}{\sqrt{n!}} c_0 .$$

This lets us write

$$\left| \alpha \right\rangle = c_0 \sum_n \frac{\alpha^n}{\sqrt{n!}} \left| n \right\rangle = c_0 \sum_n \frac{\alpha^n (\hat{a}^{\dagger})^n}{n!} \left| 0 \right\rangle .$$

By insisting that these coherent states are normalized, $\langle \alpha | \alpha \rangle = 1$, we are led to

$$\left| \alpha \right\rangle = e^{-\frac{1}{2}\alpha^2} \sum_n \frac{\alpha^n (\hat{a}^{\dagger})^n}{n!} \left| 0 \right\rangle = e^{-\frac{1}{2}\alpha^2} e^{\alpha \hat{a}^{\dagger}} \left| 0 \right\rangle .$$
2.1.1.1 Completeness

We can derive a completeness condition for coherent states by looking at (using $d^2\alpha \equiv dR(\alpha)d\Im(\alpha)$)

$$\int d^2\alpha \ | \alpha\rangle\langle\alpha| = \sum_n \sum_m \frac{1}{\sqrt{n!m!}} |n\rangle\langle m| \int d^2\alpha \ \alpha^n(\alpha^*)^m e^{-|\alpha|^2} .$$  \hspace{1cm} (2.20)$$

Now writing the complex variable $\alpha = re^{i\theta}$ we have

$$\int d^2\alpha \ | \alpha\rangle\langle\alpha| = \sum_n \sum_m \frac{1}{\sqrt{n!m!}} |n\rangle\langle m| \int_0^\infty dr \ r^{n+m+1}e^{-r^2} \int_0^{2\pi} d\theta \ e^{i(n-m)\theta} .$$

The $\theta$ integral simply gives us $2\pi\delta_{nm}$ which leaves us with

$$\int d^2\alpha \ | \alpha\rangle\langle\alpha| = \sum_n \frac{\pi}{n!} |n\rangle\langle n| \int_0^\infty dr \ 2r^{2n+1}e^{-r^2} .$$  \hspace{1cm} (2.21)$$

Now once we do a change of variables to $r^2 = t$ we have

$$\int d^2\alpha \ | \alpha\rangle\langle\alpha| = \sum_n \frac{\pi}{n!} |n\rangle\langle n| \int_0^\infty dt \ t^n e^{-t} .$$  \hspace{1cm} (2.22)$$

Noticing that this $t$ integral is just the definition of the Gamma Function, $\Gamma[n+1]$,

$$\Gamma[n+1] = \int_0^\infty dt \ t^n e^{-t} = n! ,$$  \hspace{1cm} (2.23)$$
we are left with
\[
\int d^2 \alpha \ | \alpha \rangle \langle \alpha | = \sum_n \pi \ | n \rangle \langle n | = \pi .
\] (2.24)

or, rearranging,
\[
1 = \frac{1}{\pi} \int d^2 \alpha \ | \alpha \rangle \langle \alpha |
\] (2.25)

If we have two states, |ψ₁⟩ and |ψ₂⟩, we may say
\[
\langle \psi_1 | \psi_2 \rangle = \langle \psi_1 | 1 | \psi_2 \rangle = \frac{1}{\pi} \int d^2 \alpha \ \langle \psi_1 | \alpha \rangle \langle \alpha | \psi_2 \rangle = \frac{1}{\pi} \int d^2 \alpha \ (\psi_1(\alpha))^* \psi_2(\alpha) ,
\] (2.26)

2.1.1.2 Coordinate Representation

If we consider a position space bra ⟨⃗x | on | α⟩,
\[
\langle \vec{x} | \alpha \rangle = e^{-\frac{1}{2} |\alpha|^2} \langle \vec{x} | e^{\alpha \hat{a}^\dagger} | 0 \rangle = e^{-\frac{1}{2} |\alpha|^2} \langle \vec{x} | e^{\alpha \sqrt{\frac{\hbar}{2m \omega}} \hat{x} - i \sqrt{\frac{\hbar}{2m \omega}} \hat{p}} | 0 \rangle ,
\] (2.27)

Now, taking advantage of the Baker-Campbell-Hausdorff formula [61], where as long as operators \(\hat{A}\) and \(\hat{B}\) commute with their commutator \([\hat{A}, \hat{B}]\) then
\[
e^{\hat{A}} e^{\hat{B}} = e^{\hat{A} + \hat{B}} e^{\frac{1}{2}[\hat{A}, \hat{B}]},
\] (2.28)
and with the ground state of the harmonic oscillator [62],

\[ \langle \vec{x} | 0 \rangle = \psi_0(x) = \left( \frac{m\omega}{\pi \hbar} \right)^{1/4} e^{-\frac{m\omega}{2\pi\hbar} x^2} , \quad (2.29) \]

we may say

\[ \langle \vec{x} | \alpha \rangle = \left( \frac{m\omega}{\pi \hbar} \right)^{1/4} e^{-\frac{1}{2} |\alpha|^2} e^{-\frac{1}{4} \alpha^2} e^{\alpha \sqrt{\frac{m\omega}{\hbar}} x} e^{-\sqrt{\frac{\hbar}{2m\omega}} \alpha} e^{-\frac{m\omega}{2\pi\hbar} x^2} . \quad (2.30) \]

Using the derivative as the translation operator, that is

\[ e^{-\alpha \sqrt{\frac{\hbar}{2m\omega}} \frac{\partial}{\partial x}} f(x) = f\left(x - \alpha \sqrt{\frac{\hbar}{2m\omega}} \right) , \quad (2.31) \]

we have

\[ \langle \vec{x} | \alpha \rangle = \left( \frac{m\omega}{\pi \hbar} \right)^{1/4} e^{-\frac{1}{2} |\alpha|^2} e^{-\frac{1}{4} \alpha^2} e^{\alpha \sqrt{\frac{m\omega}{\hbar}} x} e^{-\frac{m\omega}{2\pi\hbar} x^2} . \quad (2.32) \]

### 2.1.1.3 Classically Quantum

Coherent states have the interesting property of being the most classical state a quantum state can be. That is, when considering an arbitrary state \( | \psi \rangle \), the Heisenberg Uncertainty Principle states

\[ \Delta x_\psi \Delta p_\psi \geq \frac{\hbar}{2} . \quad (2.33) \]
where

\[
\Delta x_\psi = \sqrt{\langle \psi \mid \hat{x}^2 \mid \psi \rangle - \langle \psi \mid \hat{x} \mid \psi \rangle^2}, \quad (2.34)
\]
\[
\Delta p_\psi = \sqrt{\langle \psi \mid \hat{p}^2 \mid \psi \rangle - \langle \psi \mid \hat{p} \mid \psi \rangle^2}. \quad (2.35)
\]

When looking at coherent states, we can use (2.4) and (2.5) to write

\[
\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger), \quad (2.36)
\]
\[
\hat{p} = -i \sqrt{\frac{m\omega \hbar}{2}} (\hat{a} - \hat{a}^\dagger), \quad (2.37)
\]

which gives us

\[
\Delta x_\alpha = \sqrt{\frac{\hbar}{2m\omega}}, \quad (2.38)
\]
\[
\Delta p_\alpha = \sqrt{\frac{m\omega \hbar}{2}}. \quad (2.39)
\]

This gives us

\[
\Delta x_\alpha \Delta p_\alpha = \frac{\hbar}{2}, \quad (2.40)
\]

which says that coherent states have the minimal uncertainty that quantum mechanics will allow. Therefore we say these states are semi-classical states. This will be important later when considering Classicalization, a relatively new idea of how some theories may self-unitarize.
2.1.2 Special Relativity - Klein-Gordon Equation

In the search for a theory that accounts for everything we know, we must be able to combine Einstein’s special relativity with quantum mechanics. The Klein-Gordon equation appropriately incorporates these two fundamental ideas. Setting $\hbar = 1 = c$, special relativity tells us that the energy of a massive free particle is

$$E^2 = p^2 + m^2 . \quad (2.41)$$

Quantum mechanics tells us

$$\hat{E} = i \frac{\partial}{\partial t} , \quad (2.42)$$

$$\hat{p}_i = -i \frac{\partial}{\partial x^i} . \quad (2.43)$$

We now promote (2.41) to an operator equation,

$$\quad (\hat{E}^2)\psi(t, \vec{x}) = (\hat{p}^2 + m^2)\psi(t, \vec{x}) , \quad (2.44)$$

and using (2.42) and (2.43) we are lead to,

$$- \frac{\partial^2}{\partial t^2} \psi(t, \vec{x}) = - \frac{\partial^2}{\partial x^2} \psi(t, \vec{x}) + m^2 \psi(t, \vec{x}) . \quad (2.45)$$

Rearranging we have

$$\left( - \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - m^2 \right) \psi(t, \vec{x}) = 0 . \quad (2.46)$$
We can massage this Klein-Gordon equation into a more convenient form by introducing Einstein notation where

\[
x^\mu \equiv \begin{pmatrix} t \\ \vec{x} \end{pmatrix}, \quad x_{\mu} \equiv \begin{pmatrix} -t \\ \vec{x} \end{pmatrix}.
\]

(2.47)

We can transform from an upstairs index to a downstairs index using the Minkowski metric,

\[
\eta_{\mu\nu} \equiv \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

(2.48)

by

\[
\eta_{\mu\nu} x^\nu = x_\mu.
\]

(2.49)

We may now write the Klein-Gordon equation, (2.46), as

\[
(\eta^{\mu\nu} \partial_\mu \partial_\nu - m^2) \psi(t, \vec{x}) = 0 \equiv (\Box - m^2) \psi(t, \vec{x}),
\]

(2.50)

where the right hand side defines \(\Box\). The Klein-Gordon equation, then, accurately describes a free particle in a special relativistic fashion. This will be key in quantum field theory.
We can derive (2.50) by the principle of least action. We define an action,

\[
S[\psi] = \int dt \ L[\psi] = \int dt \int d^3x \ L[\psi] = \int d^4x \ -\frac{1}{2} \eta^\mu_\nu \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} m^2 \psi^2 , \tag{2.51}
\]

where \( S \) is the action, \( L \) is the Lagrangian being integrated over some time interval, and \( L \) is the Lagrangian density being integrated over some spacetime interval. We vary the wavefunction \( \psi \) by some small perturbation \( \delta \psi \), that is to say we let \( \psi \to \psi + \delta \psi \) and insist the action, \( S[\psi] \), does not change, that is \( S[\psi + \delta \psi] - S[\psi] = 0 \). This leads us to

\[
S[\psi + \delta \psi] - S[\psi] = \int d^4x \ - \eta^\mu_\nu \partial_\mu \psi \partial_\nu \delta \psi - m^2 \psi \delta \psi \\
= \int d^4x \ [\Box - m^2] \delta \psi = 0 , \tag{2.52}
\]

where in the last equation we integrated by parts assuming the boundary term is zero. Since this should be true for any arbitrary \( \delta \psi \) we impose

\[
(\Box - m^2) \psi = 0 , \tag{2.53}
\]

which is exactly (2.50).
2.2 Field Theory - ‘String’ Theory

Field theory attempts to describe the behavior of an entire field - a function with a value at every point in space and time. A simple example (in 2D) to understand what this means is to think of a string between two points, Figure 2.1. We want to be able to describe the motion of this string, that is, at every position $x$ and time $t$ we want to know the location of every infinitesimal segment of the string. We have two boundary conditions,

\begin{align}
\phi(t,0) &= 0 , \\
\phi(t,L) &= 0 ,
\end{align}

which state the ends of the string are not allowed to be in motion. The two key components to field theory are now

- $x$ is no longer the coordinate of the system, it is a label for the coordinate which is now $\phi$. 

Figure 2.1: A string. $\phi(t,x)$ is the field that describes the displacement of the string from its zero (the dotted line).
- We have an infinite number of coordinates since the label of the coordinates, \( x \),
takes on an infinite amount of values. Another way to say this is there are an
infinite infinitesimal points on the string.

The string obeys the wave equation of motion,

\[
- \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \phi(t, x) + \frac{\partial^2}{\partial x^2} \phi(t, x) = 0 ,
\]  

(2.56)

where \( v \) is the velocity of the wave, being defined here as \( v = \sqrt{T/\mu} \) (with \( T \) being
the tension of the string and \( \mu \) being the mass per length) which will now be set to
one. Given some initial ‘position’ \( \phi(0, x) \) and some initial ‘velocity’ \( \dot{\phi}(0, x) \), (where \( \dot{\cdot} \)
means a derivative with respect to time) we can know the state of the system at any
future time \( t \). This is what is to be expected since \( F = m\ddot{x} \) is a linear second order
differential equation which only requires two initial conditions to specify its future
state.

We also can derive (2.56) starting from the action,

\[
S[\phi] = \int d^4x - \frac{1}{2} \eta^{\mu \nu} \partial_\mu \phi \partial_\nu \phi .
\]  

(2.57)

Once again, the classical equations of motion for \( \phi \) are then computed by the principle
of least action which states that \( \phi \) must satisfy

\[
\frac{\delta S}{\delta \phi} = 0 ,
\]  

(2.58)
which gives us back (2.56) with \( v = 1 \). A solution to this equation is the plane wave,

\[
\phi(t, x) = Ae^{-i\omega t + ikx} ,
\]

where \( A \) is a constant, \( \omega \) is the energy, and \( k \) is the wavevector associated with momentum in the \( x \) direction. The equations of motion then impose that

\[
(\omega^2 - k^2) Ae^{-i\omega t + ikx} = 0 ,
\]

which gives us the ‘on-shell condition’ that \( \omega^2 = k^2 \).

### 2.3 Quantum Field Theory

With this brief introduction of quantum mechanics and field theory, we can go into certain main aspects of quantum field theory. Here we will just be concerned with a free quantum field, therefore we want to incorporate the Klein-Gordon equation, (2.50). However, instead of it applying to a wavefunction, \( \psi(t, \vec{x}) \), we will have it apply to a field, \( \phi(t, \vec{x}) \), which is the coordinate which is labeled by \( x \).

First we choose a decomposition of \( \phi(t, \vec{x}) \). Doing a Fourier transform in the three spacial directions we have,

\[
\phi(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3f(k)} e^{i\vec{k}\cdot\vec{x}} \tilde{\phi}(t, \vec{k}) ,
\]

where \( f(k) \) will reveal its importance shortly. We now decompose \( \tilde{\phi}(t, \vec{k}) \) into a
contribution with positive energy, \( \omega \), and negative energy, that is,

\[
\tilde{\phi}(t, \vec{k}) = a_1(\vec{k})e^{-i\omega t} + a_2(\vec{k})e^{i\omega t},
\]

where \( a_1(\vec{k}) \) is associated with positive energy and \( a_2(\vec{k}) \) is associated with negative energy. \[1\] We now impose that \( \phi \) is real by insisting \( \phi = \phi^* \). This requirement demands

\[
\phi(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3 f(k)} e^{i\vec{k} \cdot \vec{x}} \left( a_1(\vec{k})e^{-i\omega t} + a_2(\vec{k})e^{i\omega t} \right)
\]

\[
= \phi^*(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3 f(k)} e^{-i\vec{k} \cdot \vec{x}} \left( a_1^*(\vec{k})e^{i\omega t} + a_2^*(\vec{k})e^{-i\omega t} \right)
\]

\[
= \int \frac{d^3k}{(2\pi)^3 f(k)} e^{i\vec{k} \cdot \vec{x}} \left( a_1^*(-\vec{k})e^{i\omega t} + a_2^*(-\vec{k})e^{-i\omega t} \right),
\]

where in the last line we simply do a change of variables from \( k \rightarrow -k \). We now associate

\[
a_1(\vec{k}) = a_2^*(-\vec{k}),
\]

\[
a_2(\vec{k}) = a_1^*(-\vec{k}),
\]

\[1\] It may seem that the \( e^{-i\omega t} \) factor has negative energy. However, by looking at the time-dependent Schrodinger equation we have

\[
i \frac{\partial \psi}{\partial t} = E\psi,
\]

which is describing a particle with positive energy \( E \). When solved, we have

\[
\psi \sim e^{-iEt},
\]

which is the same exponential factor in (2.62).
which lets us replace $a_2$ with $a_1^*$ giving us

$$\phi(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3 f(k)} e^{i\vec{k} \cdot \vec{x}} \left( a_1(\vec{k}) e^{-i\omega t} + a_1^*(\vec{k}) e^{i\omega t} \right), \quad (2.67)$$

and by doing the same change of variables from $k \rightarrow -k$ in the second term and defining a momentum four-vector,

$$k^\mu \equiv \left( \begin{array}{c} \omega \\ \vec{k} \end{array} \right), \quad (2.68)$$

so that $k^\mu x_\mu = -\omega t + \vec{k} \cdot \vec{x} \equiv k \cdot x$ we are led to

$$\phi(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3 f(k)} \left( a(\vec{k}) e^{i\vec{k} \cdot \vec{x}} + a^*(\vec{k}) e^{-i\vec{k} \cdot \vec{x}} \right). \quad (2.69)$$

By imposing (2.69) solves the Klein-Gordon equation (2.50) we are led to,

$$(\Box - m^2) \phi(t, \vec{x}) = (\omega^2 - \vec{k}^2 - m^2) \phi(t, \vec{x}) = 0, \quad (2.70)$$

which gives us an ‘on-shell condition’

$$\omega_\vec{k} = \sqrt{\vec{k}^2 + m^2}. \quad (2.71)$$
2.3.1 Canonical Quantization

We now want to quantize this field. The process of canonical quantization is the standard procedure used to quantize objects. We first take our coordinate, $\phi$, and promote it to an operator, that is to say

$$\phi(t, \vec{x}) \rightarrow \hat{\phi}(t, \vec{x}) = \int \frac{d^3 k}{(2\pi)^3 f(k)} \left( \hat{a}_k e^{ik \cdot x} + \hat{a}^+_k e^{-ik \cdot x} \right). \quad (2.72)$$

We now define its canonical momentum, $\pi$, through

$$\pi \equiv \frac{\partial \mathcal{L}[\phi]}{\partial \dot{\phi}} = \dot{\phi}, \quad (2.73)$$

where $\mathcal{L}[\phi]$ is (2.51). In the Galileon case, since they are theories with higher derivatives, we notice that their conjugate momentum quickly becomes relatively complicated. However, for a free field, the conjugate momentum is simple. We now impose that this coordinate $\phi$ does not commute with its momentum $\pi$ (at equal times) analogous to classical quantum mechanics. Therefore we have,

$$[\hat{\phi}(x), \hat{\pi}(y)] \bigg|_{x^0 = y^0} = i\delta^{(3)}(x-y). \quad (2.74)$$

This commutator gives us

$$[\hat{\phi}(x), \hat{\pi}(y)] \bigg|_{x^0 = y^0} = \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 p}{(2\pi)^3} \frac{2i\omega_k}{f(k)f(p)} [\hat{a}_k, \hat{a}^+_p] e^{ik \cdot x} e^{-ip \cdot y} \bigg|_{x^0 = y^0}, \quad (2.75)$$
Knowing that we want to arrive at $i\delta^{(3)}(x - y)$, we can see the commutator of the $a$ and $a^\dagger$ must be proportional to $\delta^{(3)}(\vec{k} - \vec{p})$ to get rid of one of the momentum integrals. With this hindsight, we can combine the $f(k)$ and $f(p)$ contributions to just $f(k)^2$. With this we choose

$$f(k) = \sqrt{2\omega_k} , \quad (2.76)$$

to cancel the $2\omega_k$ upstairs. Therefore we notice

$$\left[ \hat{a}_{\vec{k}}, \hat{a}_{\vec{p}}^\dagger \right] = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p}) , \quad (2.77)$$

which leads us to our desired result $(2.74)$. Thus, we have arrived at the quantized version of a free scalar field,

$$\hat{\phi}(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_k}} \left( \hat{a}_k e^{ik \cdot x} + \hat{a}_k^\dagger e^{-ik \cdot x} \right) . \quad (2.78)$$

We now introduce the vacuum, $|0\rangle$, the state with no particles. This is defined to be the state such that

$$\hat{a}_{\vec{k}} |0\rangle = 0 , \quad (2.79)$$
analogous to the ground state in (2.12). We can construct particle states using the \( \hat{a}^\dagger \)'s by

\[
\hat{a}^\dagger_k \ket{0} = \ket{\vec{k}}.
\] (2.80)

This is a state of a particle with 3-momentum \( \vec{k} \). Since we know its momentum exactly, we have no information on its position due to the Heisenberg Uncertainty Principle. Likewise, we can construct two-particle states

\[
\hat{a}^\dagger_k \hat{a}^\dagger_p \ket{0} = \ket{\vec{k}, \vec{p}}.
\] (2.81)

Given that the vacuum is normalized to one, that is \( \langle 0 | 0 \rangle = 1 \), we have

\[
\langle \vec{k} | \vec{p} \rangle = \langle 0 | \hat{a}_k \hat{a}_p^\dagger \ket{0}
\]
\[
= \langle 0 | \hat{a}_k \hat{a}_p^\dagger \ket{0} - 0
\]
\[
= \langle 0 | \hat{a}_k \hat{a}_p^\dagger \ket{0} - \langle 0 | \hat{a}_p^\dagger \hat{a}_k \ket{0}
\]
\[
= \langle 0 | \left[ \hat{a}_k, \hat{a}_p^\dagger \right] \ket{0}
\]
\[
= (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p}) \langle 0 | 0 \rangle
\]
\[
= (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p}).
\] (2.82)

### 2.3.2 Relativistic States

Another subtlety to deal with is relativistic states, that is we want to define a state \( | k \rangle \) which is only subtlety different from the notation of \( | \vec{k} \rangle \) through \( k \), the relativistic
four-momentum and \(\vec{k}\), the ordinary three-momentum. We first notice,

\[
\int \frac{d^3p}{(2\pi)^3} \langle q \mid \vec{p} \rangle \langle \vec{p} \mid \vec{k} \rangle = \int \frac{d^3p}{(2\pi)^3}(2\pi)^3 \delta(\vec{q} - \vec{p})(2\pi)^3 \delta(\vec{p} - \vec{k}) \tag{2.83}
\]

\[
= (2\pi)^3 \delta(\vec{q} - \vec{k}) \tag{2.84}
\]

\[
= \langle q \mid \vec{k} \rangle. \tag{2.85}
\]

Therefore, since this is true for any \(\vec{q}\) and \(\vec{k}\), it must be that

\[
\int \frac{d^3p}{(2\pi)^3} \mid \vec{p} \rangle \langle \vec{p} \mid = 1. \tag{2.86}
\]

As described in [63], we want each individual piece in (2.86) to be Lorentz invariant. However, as it stands neither \(\int d^3p\) or \(\mid \vec{p} \rangle \langle \vec{p} \mid\) is Lorentz invariant since they only depend on the three-momentum, \(\vec{p}\). The argument goes to look at

\[
\int \frac{d^4p}{(2\pi)^3} \theta(p^0) \delta(p^2 + m^2) = \int \frac{d^3p}{(2\pi)^3} \int_0^\infty dp_0 \delta(-p_0^2 + \vec{p}^2 + m^2) \tag{2.87}
\]

\[
= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\sqrt{\vec{p}^2 + m^2}} \tag{2.88}
\]

\[
= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}}, \tag{2.89}
\]

which is Lorentz invariant since \(d^4p\) and \(p^2\) are Lorentz invariant. Therefore, looking back at (2.86), we have

\[
1 = \int \frac{d^3p}{(2\pi)^3} \mid \vec{p} \rangle \langle \vec{p} \mid = \int \frac{d^3p}{(2\pi)^3} \frac{2\omega_{\vec{p}}}{2\omega_{\vec{p}}} \mid \vec{p} \rangle \langle \vec{p} \mid = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}}(\sqrt{2\omega_{\vec{p}}}) \mid \vec{p} \rangle \langle \vec{p} \mid (\sqrt{2\omega_{\vec{p}}})
\]

\[
= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \mid p \rangle \langle p \mid = 1, \tag{2.90}
\]
which defines our relativistic states as

\[ |p\rangle = \sqrt{2\omega_p} |\vec{p}\rangle = \sqrt{2\omega_p \hat{a}_{\vec{p}}^\dagger} |0\rangle , \]  

(2.91)

with the condition,

\[ \langle p | k \rangle = \sqrt{2\omega_p} \sqrt{2\omega_k} (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{k}) \]  

(2.92)

\[ = (2\pi)^3 2\omega_k \delta^{(3)}(\vec{p} - \vec{k}) . \]  

(2.93)

### 2.3.3 Normal Ordering

Normal ordering is a prescription telling us to ignore the commutation relation between \(a\) and \(a^\dagger\) and put all \(a\)'s to the right and all the \(a^\dagger\)'s to the left. This procedure effectively neglects contractions of fields in the coincidence limit which are unphysical infinite contributions. As an example consider some operator, \(\hat{O}\),

\[ \hat{O} = a_{\vec{k}_1} a_{\vec{k}_2} a_{\vec{k}_3}^\dagger a_{\vec{k}_4}^\dagger . \]  

(2.94)

The expectation value for this operator, \(\langle 0 | \hat{O} | 0 \rangle\), can be determined taking advantage of the commutation relation between \(\hat{a}\) and \(\hat{a}^\dagger\)

\[ \hat{a}_{\vec{k}_2} \hat{a}_{\vec{k}_1}^\dagger = \hat{a}_{\vec{k}_1}^\dagger \hat{a}_{\vec{k}_2} + (2\pi)^3 \delta^{(3)}(\vec{k}_1 - \vec{k}_2) . \]  

(2.95)
Therefore we have,

\[
\langle 0 \mid \hat{O} \mid 0 \rangle = \langle 0 \mid a_{\vec{k}_1} a_{\vec{k}_2} a_{\vec{k}_3}^\dagger a_{\vec{k}_4}^\dagger \mid 0 \rangle \\
= \langle 0 \mid a_{\vec{k}_1} \left( a_{\vec{k}_3}^\dagger \hat{a}_{\vec{k}_2} + (2\pi)^3 \delta^{(3)}(\vec{k}_2 - \vec{k}_3) \right) a_{\vec{k}_4}^\dagger \mid 0 \rangle \\
= \langle 0 \mid a_{\vec{k}_1} \hat{a}_{\vec{k}_3}^\dagger \hat{a}_{\vec{k}_2} a_{\vec{k}_4}^\dagger \mid 0 \rangle + \langle 0 \mid a_{\vec{k}_1} a_{\vec{k}_4}^\dagger \mid 0 \rangle (2\pi)^3 \delta^{(3)}(\vec{k}_2 - \vec{k}_3) \\
= \langle 0 \mid a_{\vec{k}_1} \hat{a}_{\vec{k}_3}^\dagger \hat{a}_{\vec{k}_2} \left( \hat{a}_{\vec{k}_4}^\dagger + (2\pi)^3 \delta^{(3)}(\vec{k}_2 - \vec{k}_4) \right) \mid 0 \rangle \\
+ (2\pi)^6 \delta^{(3)}(\vec{k}_1 - \vec{k}_4) \delta^{(3)}(\vec{k}_2 - \vec{k}_3) \\
= \langle 0 \mid a_{\vec{k}_1} \hat{a}_{\vec{k}_3}^\dagger \mid 0 \rangle (2\pi)^3 \delta^{(3)}(\vec{k}_2 - \vec{k}_4) + (2\pi)^6 \delta^{(3)}(\vec{k}_1 - \vec{k}_4) \delta^{(3)}(\vec{k}_2 - \vec{k}_3) \\
= (2\pi)^6 \left( \delta^{(3)}(\vec{k}_1 - \vec{k}_3) \delta^{(3)}(\vec{k}_2 - \vec{k}_4) + \delta^{(3)}(\vec{k}_1 - \vec{k}_4) \delta^{(3)}(\vec{k}_2 - \vec{k}_3) \right).
\]

The result is momentum conservation. If we conceptualize \( a_{\vec{k}_3}^\dagger a_{\vec{k}_4}^\dagger \mid 0 \rangle \) as two particles with momentum \( \vec{k}_3 \) and \( \vec{k}_4 \) being created from the vacuum and \( \langle 0 \mid a_{\vec{k}_1} a_{\vec{k}_2} \mid 0 \rangle \) as destroying two particles with momentum \( \vec{k}_1 \) and \( \vec{k}_2 \) then in order to conserve momentum, it must be the case that either \( \vec{k}_1 = \vec{k}_3 \) while \( \vec{k}_2 = \vec{k}_4 \) or \( \vec{k}_1 = \vec{k}_4 \) while \( \vec{k}_2 = \vec{k}_3 \) which is what the delta functions are telling us.

Now let us consider what the normal ordering operation, which says to put all \( \hat{a} \)'s to the right and all \( \hat{a}^\dagger \)'s to the left, does to \( \hat{O} \). We have

\[
\langle 0 \mid : \hat{O} : \mid 0 \rangle = \langle 0 \mid : a_{\vec{k}_1} a_{\vec{k}_2} a_{\vec{k}_3}^\dagger a_{\vec{k}_4}^\dagger : \mid 0 \rangle \equiv \langle 0 \mid a_{\vec{k}_3}^\dagger a_{\vec{k}_4}^\dagger a_{\vec{k}_1} a_{\vec{k}_2} \mid 0 \rangle = 0.
\]

We see the normal ordered product of \( \hat{O} \), which is an operator constructed out of polynomials of \( \hat{a} \)'s and \( \hat{a}^\dagger \)'s with no constant contribution, is simply zero. Now, let
us consider a more complicated operator, $O_\alpha$,

$$O_\alpha = e^{a\hat{a} + a^{\dagger}\hat{a}^{\dagger}}.$$ (2.97)

If we look at the normal ordered product of $O_\alpha$ we have

$$\langle 0 | : O_\alpha : | 0 \rangle = \langle 0 | : e^{a\hat{a} + a^{\dagger}\hat{a}^{\dagger}} : | 0 \rangle$$

$$= \langle 0 | : \sum_{n=0}^{\infty} \frac{1}{n!} (\alpha\hat{a} + \alpha^{\dagger}\hat{a}^{\dagger})^n : | 0 \rangle$$

$$= \langle 0 | : \left( 1 + \sum_{n=1}^{\infty} \frac{1}{n!} (\alpha\hat{a} + \alpha^{\dagger}\hat{a}^{\dagger})^n \right) : | 0 \rangle$$

$$= 1 + \langle 0 | : \sum_{n=1}^{\infty} \frac{1}{n!} (\alpha\hat{a} + \alpha^{\dagger}\hat{a}^{\dagger})^n : | 0 \rangle$$

$$= 1 ,$$ (2.98)

since $\langle 0 | : \sum_{n=1}^{\infty} \frac{1}{n!} (\alpha\hat{a} + \alpha^{\dagger}\hat{a}^{\dagger})^n : | 0 \rangle$ is a normal ordered product of a polynomial constructed out of $\hat{a}$’s and $\hat{a}^{\dagger}$’s. Now, let us consider a second operator, $O_\beta$,

$$O_\beta = e^{\beta\hat{a} + \beta^{\dagger}\hat{a}^{\dagger}} ,$$ (2.99)

where we also have

$$\langle 0 | : O_\beta : | 0 \rangle = 1 .$$ (2.100)
If we normal order the sandwiched $\mathcal{O}_\alpha \mathcal{O}_\beta$ we have

$$\langle 0 : \mathcal{O}_\alpha \mathcal{O}_\beta : | 0 \rangle = \langle 0 : e^{\alpha \hat{a} + \alpha^* \hat{a}^\dagger} e^{\beta \hat{a} + \beta^* \hat{a}^\dagger} : | 0 \rangle$$

$$= \langle 0 : \left(1 + \sum_{n=1}^{\infty} \frac{1}{n!} (\alpha \hat{a} + \alpha^* \hat{a}^\dagger)^n\right) \left(1 + \sum_{m=1}^{\infty} \frac{1}{m!} (\beta \hat{a} + \beta^* \hat{a}^\dagger)^m\right) : | 0 \rangle$$

$$= 1 + \langle 0 : \left(\sum_{n=1}^{\infty} \frac{1}{n!} (\alpha \hat{a} + \alpha^* \hat{a}^\dagger)^n\right) : | 0 \rangle$$

$$+ \langle 0 : \left(\sum_{m=1}^{\infty} \frac{1}{m!} (\beta \hat{a} + \beta^* \hat{a}^\dagger)^m\right) : | 0 \rangle$$

$$+ \langle 0 : \left(\sum_{n=1}^{\infty} \frac{1}{n!} (\alpha \hat{a} + \alpha^* \hat{a}^\dagger)^n\right) \left(\sum_{m=1}^{\infty} \frac{1}{m!} (\beta \hat{a} + \beta^* \hat{a}^\dagger)^m\right) : | 0 \rangle$$

$$= 1 . \quad (2.101)$$

However, let us consider the more interesting example, $\langle 0 : \mathcal{O}_\alpha :: \mathcal{O}_\beta : | 0 \rangle$, where we are normal ordering $\mathcal{O}_\alpha$ separately from $\mathcal{O}_\beta$. This is not equal to one because of the Baker-Campbell-Hausdorff formula (2.28). Using this, we have

$$\langle 0 : \mathcal{O}_\alpha :: \mathcal{O}_\beta : | 0 \rangle = \langle 0 : e^{\alpha \hat{a} + \alpha^* \hat{a}^\dagger} :: e^{\beta \hat{a} + \beta^* \hat{a}^\dagger} : | 0 \rangle$$

$$= \langle 0 : e^{\alpha^* \hat{a}^\dagger} e^{\alpha \hat{a}} e^{\beta^* \hat{a}^\dagger} e^{\beta \hat{a}} : | 0 \rangle$$

$$= \langle 0 : e^{\alpha \hat{a}} e^{\beta^* \hat{a}^\dagger} : | 0 \rangle$$

$$= \langle 0 : e^{\alpha \hat{a} + \beta^* \hat{a}^\dagger} : | 0 \rangle e^{\frac{1}{2} \alpha \beta^*}$$

$$= \langle 0 : e^{\beta^* \hat{a}^\dagger + \alpha \hat{a}} : | 0 \rangle e^{\frac{1}{2} \alpha \beta^*}$$

$$= \langle 0 : e^{\beta^* \hat{a}^\dagger} e^{\alpha \hat{a}} : | 0 \rangle e^{\alpha \beta^*}$$

$$= e^{\alpha \beta^*} . \quad (2.102)$$
We may use this result as a generating function for a whole host of expectation values.

For instance consider

\[
\langle 0 \mid \hat{a}^n e^{\alpha \hat{a} + \alpha^* \hat{a}^\dagger} \hat{\beta}^m e^{\beta \hat{a} + \beta^* \hat{a}^\dagger} \mid 0 \rangle
\]

\[
= \left( \frac{\partial}{\partial \alpha} \right)^n \left( \frac{\partial}{\partial \beta^*} \right)^m \langle 0 \mid e^{\alpha \hat{a} + \alpha^* \hat{a}^\dagger} e^{\beta \hat{a} + \beta^* \hat{a}^\dagger} \mid 0 \rangle
\]

\[
= \left( \frac{\partial}{\partial \alpha} \right)^n \left( \frac{\partial}{\partial \beta^*} \right)^m e^{\alpha \beta^*} .
\] (2.103)

We can even consider arbitrary functions of \( \hat{a} \) and \( \hat{a}^\dagger \), \( F_1(\hat{a}, \hat{a}^\dagger) \) and \( F_2(\hat{a}, \hat{a}^\dagger) \), and ask

\[
\langle 0 \mid F_1(\hat{a}) e^{\alpha \hat{a} + \alpha^* \hat{a}^\dagger} F_2(\hat{a}^\dagger) e^{\beta \hat{a} + \beta^* \hat{a}^\dagger} \mid 0 \rangle
\]

\[
= \hat{F}_1 \left( \frac{\partial}{\partial \alpha} \right) \hat{F}_2 \left( \frac{\partial}{\partial \beta^*} \right) \langle 0 \mid e^{\alpha \hat{a} + \alpha^* \hat{a}^\dagger} e^{\beta \hat{a} + \beta^* \hat{a}^\dagger} \mid 0 \rangle .
\] (2.104)

We will be using an analogy of this formula later on.

### 2.4 The Feynman Propagator

The Feynman propagator will help us systematically analyze many quantum field theoretic systems. Let us start with the Klein-Gordon equation, (2.50), but with some source term, \( J(x) \), that is, let us look at,

\[
(\Box x - m^2)\phi(x) = J(x) .
\] (2.105)

The source \( J(x) \) represents an external source. In principle, \( J(x) \) could depend on the field \( \phi(x) \) in interacting theories where it is common to then use perturbation theory.
For now we will assume $J(x)$ is external, meaning it does not depend on $\phi(x)$.

We will define a general solution to this differential equation as

$$\phi(x) = \phi_0(x) + \phi_F(x), \quad (2.106)$$

where $\phi_0(x)$ is the homogeneous solution, that is it solves the free Klein-Gordon equation,

$$(\Box - m^2)\phi_0(x) = 0, \quad (2.107)$$

and $\phi_F(x)$ involves the Green’s function for the Klein-Gordon equation, that is

$$\phi_F(x) = \int d^4y \, G_F(x - y)J(y), \quad (2.108)$$

where $G_F(x - y)$ is the Feynman propagator which is the solution to

$$(\Box - m^2)G_F(x - y) = \delta^{(4)}(x - y). \quad (2.109)$$

To see that our construction of $\phi(x)$ actually solves the sourced Klein-Gordon equa-
tion, (2.105), we simply notice,

$$
(\Box_x - m^2)\phi(x) = (\Box_x - m^2)\phi_0(x) + (\Box_x - m^2)\phi_F(x)
$$

$$
= 0 + \int d^4y \, (\Box_x - m^2)G_F(x-y)J(y)
$$

$$
= \int d^4y \, \delta^{(4)}(x-y)J(y)
$$

$$
= J(x) , \quad (2.110)
$$

We notice that the derivatives of $\Box_x$ do not apply to $J(y)$ since $\Box_x$ only applies to the $x$ variable. We now will aim to find a convenient and useful formula for $G_F(x - y)$. Let us first Fourier transform $G_F(x - y)$ to get

$$
G_F(x - y) = \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot (x-y)} \tilde{G}_F(k) , \quad (2.111)
$$

and now imposing that $G_F(x - y)$ solves (2.109), we find

$$
(\Box_x - m^2)G_F(x - y) = \delta^{(4)}(x-y)
$$

$$
\int \frac{d^4k}{(2\pi)^4} \tilde{G}_F(k)(\Box_x - m^2)e^{ik \cdot (x-y)} = \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot (x-y)}
$$

$$
-(k^2 + m^2)\tilde{G}_F(k) = 1 . \quad (2.112)
$$

From here we are tempted to simply say,

$$
\tilde{G}_F(k) = \frac{-1}{k^2 + m^2} , \quad (2.113)
$$
but this is too quick and the reason why is subtle. To best understand the subtleties of this we are going to start by looking at a different scenario.

2.4.1 Matter

Let us suppose we stumble across an equation of the form

$$\omega \tilde{f}(\omega) = 1,$$  \hspace{1cm} (2.114)

where we want to solve for $\tilde{f}(\omega)$. This is an analogous situation we have found ourselves in with (2.112). At first glance, the answer seems obvious, that $\tilde{f}(\omega) = 1/\omega$ but this fails to take into account that there are actually two different scenarios we must consider. The first and obvious scenario is when $\omega \neq 0$ at which point $\tilde{f}(\omega) = 1/\omega$ like we expect. However, when we consider when $\omega = 0$ the answer gets more complicated. When $\omega = 0$ it would seem that in order for (2.114) to remain true, $\tilde{f}(\omega)$ seems like it would have to be some type of infinity to compensate. To better understand what is going on let us give a small imaginary part to omega, that is, let us consider

$$(\omega \pm i\epsilon) \tilde{f}(\omega) = 1,$$  \hspace{1cm} (2.115)
where there is an implicit $\epsilon \to 0$ limit and $\epsilon > 0$. Considering (2.115) we can simply solve for $\tilde{f}(\omega)$,

$$\tilde{f}(\omega) = \frac{1}{(\omega \pm i\epsilon)} . \quad (2.116)$$

Now we need to understand what to do with this $\pm i\epsilon$ factor. One way to do this is to separate $\tilde{f}(\omega)$ into its real and imaginary parts by

$$\tilde{f}(\omega) = \frac{(\omega \mp i\epsilon)}{(\omega \pm i\epsilon)(\omega \mp i\epsilon)} = \frac{\omega}{\omega^2 + \epsilon^2} \mp \frac{\epsilon}{\omega^2 + \epsilon^2} , \quad (2.117)$$

and by taking the $\epsilon \to 0$ limit we find

$$\lim_{\epsilon \to 0} \left[ \frac{\omega}{\omega^2 + \epsilon^2} \right] = P \left( \frac{1}{\omega} \right) , \quad (2.118)$$

$$\lim_{\epsilon \to 0} \left[ \frac{i\epsilon}{\omega^2 + \epsilon^2} \right] = i\pi\delta(\omega) , \quad (2.119)$$

which gives us

$$\tilde{f}(\omega) = \frac{1}{\omega} \mp i\pi\delta(\omega) . \quad (2.120)$$

However, a non-obvious fact is that the condition of causality will enforce that we pick either $+i\epsilon$ or $-i\epsilon$. To see this let us conceptualize $\tilde{f}(\omega)$ to be a function of energy, $\omega$, that was obtained from a Fourier transform of some function of time, $f(t)$,
that is say,

\[ f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{f}(\omega). \]  

(2.121)

Now by insisting (2.116) we have

\[ f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\omega \pm i\epsilon}, \]  

(2.122)

where, once again, the \( e^{-i\omega t} \) is representing that we are considering a particle with positive energy. This is where causality comes into play. \( f(t) \) is some function of time, for instance, a function representing a pulse that is switched on at \( t = 0 \). Causality, then, would enforce that this function of time, \( f(t) \), shouldn’t be able to influence anything \emph{before} the pulse is turned on, that is to say that causality requires that

\[ f(t) = 0 \text{ for } t < 0. \]  

(2.123)

Using Cauchy’s complex integration logic \[64\] we can see the consequences of this causality condition \(2.123\). Let us consider a complex \( \omega \), that is to say \( \omega = \omega_R + i\omega_I \). This gives us

\[ f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega_R t} e^{i\omega_I t}}{\omega \pm i\epsilon}. \]  

(2.124)
Now, since we want to consider negative times, let us replace $t \rightarrow -|t|$ to get

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{(2\pi)} \frac{e^{i\omega |t|}e^{-\omega |t|}}{\omega \pm i\epsilon}. \quad (2.125)$$

Causality forces us to impose that (2.125) must be zero. However, to ensure convergence of this integral through the $e^{-\omega |t|}$ piece we need to only consider the positive $\omega_I$ half-plane. Cauchy says that the integral will have a non-zero contribution from every pole within this positive $\omega_I$ half-plane. So, causality forces us to choose the sign of $\epsilon$ to be the one such that there are no poles in the positive $\omega_I$ half-plane. This informs us that to keep causality we must have $+i\epsilon$.

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{(2\pi)} \frac{e^{-i\omega t}}{\omega + i\epsilon} \rightarrow \text{Causal}, \quad (2.126)$$

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{(2\pi)} \frac{e^{-i\omega t}}{\omega - i\epsilon} \rightarrow \text{Non-Causal}. \quad (2.127)$$

This is summed up in Figure 2.2.

### 2.4.2 Antimatter

We can do this analogous work with a particle of negative energy, that is let us start with

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{(2\pi)} \frac{e^{i\omega t}}{\omega \pm i\epsilon}, \quad (2.128)$$
Figure 2.2: For matter, we need to pick $+i\epsilon$ to ensure there are no poles in the positive $\omega_I$ half-plane to enforce $f(t) = 0$ for negative $t$ values.

where we have just let $\omega \rightarrow -\omega$ in the exponential compared to (2.122). Again considering a complex $\omega$ we have

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega R t} e^{-\omega I t}}{\omega \pm i\epsilon}.$$  \hspace{1cm} (2.129)

By still requiring causality, (2.123), we need to consider negative times $t \rightarrow -|t|$ which leads us to

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega R |t|} e^{\omega I |t|}}{\omega \pm i\epsilon},$$  \hspace{1cm} (2.130)

and insist that (2.130) is zero. By looking at the $e^{\omega |t|}$ we see we must now consider the negative $\omega_I$ half-plane to ensure convergence. Therefore, causality enforces that there cannot be a pole in this negative half-plane which forces us to pick the $-i\epsilon$. 

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Figure 2.3: For antimatter, we need to pick $-i\epsilon$ to ensure there are no poles in the negative $\omega_I$ half-plane to enforce $f(t) = 0$ for negative $t$ values.

That is to say, for particles with negative energy,

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega t}}{\omega + i\epsilon} \rightarrow \text{Non-Causal} \ , \quad (2.131)$$

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega t}}{\omega - i\epsilon} \rightarrow \text{Causal} \ . \quad (2.132)$$

This is summed up in Figure 2.3. If we write a particle with negative energy, $e^{i\omega t}$, in terms of a particle with positive energy, $e^{-i\omega t}$, we can choose to think of antimatter as a particle with positive energy going backward in time or a particle with negative energy going forward in time. That is to say,

$$e^{i\omega t} = e^{-i(-\omega)t} \rightarrow \text{Particle going forward in time with negative energy} \ ,$$

$$e^{i\omega t} = e^{-i\omega(-t)} \rightarrow \text{Particle going backward in time with positive energy} \ .$$
2.4.3 Back to Feynman

We left off with (2.112),

\[-(k^2 + m^2) \tilde{G}_F(k) = 1.\]  \hspace{1cm} (2.133)

By separating this out in terms of \( \omega \) and \( \vec{k} \) we have

\[-(-\omega^2 + \vec{k}^2 + m^2) \tilde{G}_F(k) = 1,\]  \hspace{1cm} (2.134)

and knowing we will be integrating over \( \omega \) there will be a point where we must consider when \( \omega^2 = \vec{k}^2 + m^2 \), or when the factor multiplying \( \tilde{G}_F(k) \) is zero. Therefore we will consider the Feynman prescription of instead of considering (2.112), we will consider

\[-(-\omega^2 + \vec{k}^2 + m^2 - i\epsilon) \tilde{G}_F(k) = 1,\]  \hspace{1cm} (2.135)

which is simply solved to

\[\tilde{G}_F(k) = \frac{-1}{(-\omega^2 + \vec{k}^2 + m^2 - i\epsilon)}.\]  \hspace{1cm} (2.136)

To see the consequences of this Feynman prescription, we will rewrite

\[\frac{-1}{(-\omega^2 + \vec{k}^2 + m^2 - i\epsilon)} = \frac{1}{(\omega + \sqrt{\vec{k}^2 + m^2} - i\epsilon)(\omega - \sqrt{\vec{k}^2 + m^2} + i\epsilon)}.\]  \hspace{1cm} (2.137)
At first glance it might seem (2.137) is not true since
\[
\frac{1}{(\omega + \sqrt{k^2 + m^2} - i\epsilon)(\omega - \sqrt{k^2 + m^2} + i\epsilon)} = \frac{1}{\omega^2 - \tilde{k}^2 - m^2 + 2\sqrt{k^2 + m^2}i\epsilon + \epsilon^2}.
\]

However, since there is an implicit \( \epsilon \to 0 \) limit, the \( \epsilon^2 \) piece is infinitesimally small compared to the \( \epsilon \) piece so we disregard it and we can define \( 2\sqrt{k^2 + m^2}\epsilon = \tilde{\epsilon} \) where we see that \( \tilde{\epsilon} > 0 \). Now, since the factors out from of \( \tilde{\epsilon} \) do not change the functional form we can redefine \( \tilde{\epsilon} \to \epsilon \). Therefore we are led to our result, (2.137). Now, looking at the real space Feynman propagator, we have (2.111)

\[
G_F(x - y) = \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot (x - y)} \tilde{G}_F(k)
\]

\[
= \int \frac{d^4k}{(2\pi)^4} \left( -e^{ik \cdot (x - y)} \right)
\]

\[
= \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{(2\pi)} \frac{e^{-i\omega(x^0 - y^0)}e^{i\tilde{k} \cdot (\tilde{x} - \tilde{y})}}{(\omega + \sqrt{\tilde{k}^2 + m^2} - i\epsilon)(\omega - \sqrt{\tilde{k}^2 + m^2} + i\epsilon)}
\]

(2.138)

where in the second line we used (2.136) and in the last line we used (2.137) and split things up into \( \omega \) and \( \tilde{k} \) separately and \( x^0 \) stands for the time component of \( x^\mu \) and similarly for \( y^0 \). The interesting physics comes in when we evaluate this \( \omega \) integral. Like above, we will use Cauchy’s theorem to get a handle on what’s going on. We first need to consider two different scenarios, one when \( x^0 > y^0 \) and the other when \( x^0 < y^0 \).
2.4.4 When $x^0 > y^0$

When considering $x^0 > y^0$ we will replace $(x^0 - y^0) \rightarrow |x^0 - y^0|$. This lets (2.138) become

$$G_F(x - y) = \int \frac{d^3 k}{(2\pi)^3} \int \frac{d\omega}{(2\pi)} \frac{e^{-i\omega|x^0 - y^0|}e^{i\vec{k} \cdot (\vec{x} - \vec{y})}}{(\omega + \sqrt{k^2 + m^2 - i\epsilon})(\omega - \sqrt{k^2 + m^2 + i\epsilon})}, \quad (2.139)$$

and by considering a complex $\omega$, we are lead to

$$G_F(x - y) = \int \frac{d^3 k}{(2\pi)^3} \int \frac{d\omega}{(2\pi)} \frac{e^{-i\omega|\vec{k}|(x^0 - y^0)|}e^{i\omega|x^0 - y^0|}e^{i\vec{k} \cdot (\vec{x} - \vec{y})}}{(\omega + \sqrt{k^2 + m^2 - i\epsilon})(\omega - \sqrt{k^2 + m^2 + i\epsilon})}, \quad (2.140)$$

For convergence, we need to look in the negative $\omega_I$ half-plane where we see we have a non-zero pole contribution from $(\omega - \sqrt{k^2 + m^2 + i\epsilon})$ which, using Cauchy’s residue theorem, sets $\omega = \omega_\vec{k} = \sqrt{k^2 + m^2}$. This gives us for the $\omega$ integral,

$$G_F(x - y) = \int \frac{d^3 k}{(2\pi)^3} \frac{ie^{-i\omega_\vec{k}|x^0 - y^0|}e^{i\vec{k} \cdot (\vec{x} - \vec{y})}}{2\omega_\vec{k}^2} = i \int \frac{d^3 k}{(2\pi)^3} \frac{e^{i\vec{k} \cdot (x - y)}}{2\omega_\vec{k}}, \quad (2.141)$$

where it is understood that the $\omega$ component of $k^\mu$ is on-shell, that is $\omega = \omega_\vec{k}$.

2.4.5 When $x^0 < y^0$

When considering $x^0 < y^0$ we will replace $(x^0 - y^0) \rightarrow -|x^0 - y^0|$. This lets (2.138) become

$$G_F(x - y) = \int \frac{d^3 k}{(2\pi)^3} \int \frac{d\omega}{(2\pi)} \frac{e^{i\omega|x^0 - y^0|}e^{i\vec{k} \cdot (\vec{x} - \vec{y})}}{(\omega + \sqrt{k^2 + m^2 - i\epsilon})(\omega - \sqrt{k^2 + m^2 + i\epsilon})}, \quad (2.142)$$

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and by considering a complex \( \omega \), we are lead to

\[
G_F(x - y) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{(2\pi)^3} \frac{e^{i\omega_R(x^0 - y^0)}e^{-\omega I|x^0 - y^0|}e^{i\vec{k} \cdot (\vec{x} - \vec{y})}}{(\omega + \sqrt{k^2 + m^2 - i\epsilon})(\omega - \sqrt{k^2 + m^2 + i\epsilon})}, \tag{2.143}
\]

For convergence, we need to look in the positive \( \omega_I \) half-plane where we see we have a non-zero pole contribution from \((\omega + \sqrt{k^2 + m^2 - i\epsilon})\) which, using Cauchy’s residue theorem, sets \( \omega = -\omega^* = -\sqrt{k^2 + m^2} \). This gives us for the \( \omega \) integral,

\[
G_F(x - y) = \int \frac{d^3k}{(2\pi)^3} \frac{ie^{-i\omega e|x^0 - y^0|}e^{i\vec{k} \cdot (\vec{x} - \vec{y})}}{2\omega^*_k}. \tag{2.144}
\]

Since \((x^0 - y^0) = -|x^0 - y^0|\) we replace \(|x^0 - y^0|\) with \(-(x^0 - y^0)\) and we do a change of variables and let \( k \to -k \) to obtain

\[
G_F(x - y) = i \int \frac{d^3k}{(2\pi)^3} \frac{e^{-ik \cdot (x - y)}}{2\omega^*_k}, \tag{2.145}
\]

where, once again, it is understood that the \( \omega \) component of \( k^\mu \) is on-shell, that is \( \omega = \omega^*_k \).

### 2.4.6 Time-Ordering

The idea that we needed to consider two different time order scenarios naturally leads us to define a ‘time-ordering’ operator, \( T \), that can incorporate this fact for us. We define \( T \) with two arbitrary operators \( \hat{A}(x) \) and \( \hat{B}(y) \) as

\[
T(\hat{A}(x)\hat{B}(y)) = \theta(x^0 - y^0)\hat{A}(x)\hat{B}(y) + \theta(y^0 - x^0)\hat{B}(y)\hat{A}(x), \tag{2.146}
\]

67
where $\theta(t) = 1$ for $t > 0$ and $\theta(t) = 0$ for $t < 0$. Since we read operator products acting on some ‘ket’ state, $|\phi\rangle$, we read from right to left. So $T$ simply instructs us to define operator products in the order at which they occur. With this, we can incorporate both time scenarios of the Feynman propagator in one equation as

$$G_F(x - y) = i\theta(x^0 - y^0) \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik \cdot (x-y)}}{2\omega_k} + i\theta(y^0 - x^0) \int \frac{d^3k}{(2\pi)^3} \frac{e^{-ik \cdot (x-y)}}{2\omega_k},$$

(2.147)

Interestingly we see that positive energy solutions are carried forward in time and negative energy solutions are carried backward in time. Therefore, we have for our official Feynman propagator,

$$G_F(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{-e^{ik \cdot (x-y)}}{\sqrt{2} \omega_k^2 + m^2 - i\epsilon}.$$  

(2.148)

### 2.5 Two-Point Function

We can connect this Feynman propagator with free field operator vacuum expectation values. For example let us look at a quantized scalar field, sandwiched between the vacuum, $|0\rangle$,

$$\langle 0 | \hat{\phi}(x) | 0 \rangle = \int \frac{d^3k}{(2\pi)^3} \sqrt{2\omega_k} \langle 0 | (\hat{a}_k e^{-ik \cdot x} + \hat{a}_k^\dagger e^{ik \cdot x}) | 0 \rangle = 0 .$$

(2.149)

This equals zero because our definition of the vacuum, makes the $\hat{a}_k$ contribution zero and the $\hat{a}_k^\dagger$ term creates $\langle 0 | \vec{k} \rangle = 0$ from (2.82). Likewise, an $n$-point
function, such as

\[ \langle 0 | \hat{\phi}(x_1)\hat{\phi}(x_2) \ldots \hat{\phi}(x_n) | 0 \rangle = 0 \quad \text{for } n \text{ odd} \quad (2.150) \]

However, a useful object to construct is known as the Wightman function, \( \langle 0 | \hat{\phi}(x)\hat{\phi}(y) | 0 \rangle \). The Wightman function gives us

\[
\langle 0 | \hat{\phi}(x)\hat{\phi}(y) | 0 \rangle = \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_k}} \int \frac{d^3p}{(2\pi)^3 \sqrt{2\omega_p}} e^{ik \cdot x} e^{-ip \cdot y} \langle k | p \rangle \\
= \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_k}} \int \frac{d^3p}{(2\pi)^3 \sqrt{2\omega_p}} e^{ik \cdot x} e^{-ip \cdot y} (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p}) ,
\]

which, up to a factor of \( i \), is exactly what we have for the \( x^0 > y^0 \) piece of the Feynman propagator, (2.141). Similarly, we can construct another Wightman function with \( x \) and \( y \) interchanged to obtain

\[
\langle 0 | \hat{\phi}(y)\hat{\phi}(x) | 0 \rangle = \int \frac{d^3k}{(2\pi)^3} e^{-ik \cdot (x-y)} \frac{e^{-ik \cdot (x-y)}}{2\omega_k^3} ,
\]

which, as expected, is the other piece of the Feynman propagator. Now with our definition of time-ordering, (2.146), we can relate these Wightman functions with the
Feynman propagator through

\[ i\langle 0 \mid T(\hat{\phi}(x)\hat{\phi}(y)) \mid 0 \rangle = \theta(x^0 - y^0)\langle 0 \mid \hat{\phi}(x)\hat{\phi}(y) \mid 0 \rangle + \theta(y^0 - x^0)\langle 0 \mid \hat{\phi}(y)\hat{\phi}(x) \mid 0 \rangle \]

\[ = i\theta(x^0 - y^0) \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik \cdot (x-y)}}{2\omega_k} + i\theta(y^0 - x^0) \int \frac{d^3k}{(2\pi)^3} \frac{e^{-ik \cdot (x-y)}}{2\omega_k} \]

\[ = G_F(x - y) . \quad (2.153) \]

### 2.6 Interacting Theory - Spectral Density

So far we have only considered a ‘free’ theory, that is a theory with no interactions. We noticed we can view a free theory as an action defined in (2.51). An ‘interacting’ theory is a field theory with higher order interaction terms in its Lagrangian. These generic interaction terms traditionally take the form \( V(\tilde{\phi}) \) when considering polynomials of the field. In an action the generic structure becomes

\[ S[\tilde{\phi}] = \int d^4x - \frac{1}{2}(\partial\tilde{\phi})^2 - \frac{1}{2}m^2\tilde{\phi}^2 - V(\tilde{\phi}) , \quad (2.154) \]

where we used a shorthand notation \((\partial\tilde{\phi})^2 \equiv \eta^{\mu\nu}\partial_{\mu}\tilde{\phi}\partial_{\nu}\tilde{\phi}\) and we are using \(\tilde{\phi}\) to explicitly point out we are dealing with an interacting theory \((V(\tilde{\phi}) \neq 0)\). The equations of motion which are given to us by minimizing the action, \(\delta S[\tilde{\phi}] = 0\),

\[ \partial_{\mu} \left[ \frac{\partial L[\tilde{\phi}]}{\partial (\partial_{\mu}\tilde{\phi})} \right] = \frac{\partial L[\tilde{\phi}]}{\partial \tilde{\phi}} , \quad (2.155) \]
gives us

\[(\Box - m^2)\tilde{\phi} = \frac{\partial V(\tilde{\phi})}{\partial \tilde{\phi}}.\] (2.156)

This new equation will no longer lead to the simple quantized free scalar field, (2.78).

The solution to (2.156) will produce the ‘interacting’ field equation which depends on the precise form of \(V(\tilde{\phi})\). What we want to define is a two-point function for this interacting field, \(\tilde{\phi}(x), \langle 0 \mid T(\hat{\phi}(x)\hat{\phi}(y)) \mid 0 \rangle\). In the previous sections we relied on the fact that the field we were using was a free field, that is it was a solution to the Klein-Gordon equation, (2.53). Since we no longer have this luxury we must use new techniques to obtain an expression for the interacting two-point function. Let us start by looking at this interacting two-point Wightman function, \(\langle 0 \mid \hat{\phi}(x)\hat{\phi}(y) \mid 0 \rangle\), and use what we know. That is, as discussed in \cite{65}, let us first insert a complete set of energy eigenstates in the interacting Wightman function, \(\langle 0 \mid \hat{\phi}(x)\hat{\phi}(y) \mid 0 \rangle\).

\[
\langle 0 \mid \hat{\phi}(x)\hat{\phi}(y) \mid 0 \rangle = \sum_n \langle 0 \mid \hat{\phi}(x) \mid n \rangle \langle n \mid \hat{\phi}(y) \mid 0 \rangle ,
\] (2.157)

where we have chosen these \(n\) states to be momentum eigenstates of the momentum four-vector and the sum runs over all states. Using translation invariance we may write

\[
\langle 0 \mid \hat{\phi}(x) \mid n \rangle = e^{ip_n \cdot x} \langle 0 \mid \hat{\phi}(0) \mid n \rangle ,
\] (2.158)
and similarly for \( \langle n | \hat{\phi}(y) | 0 \rangle \), which makes (4.1)

\[
\langle 0 | \hat{\phi}(x)\hat{\phi}(y) | 0 \rangle = \sum_n e^{ip_n \cdot (x-y)}|\langle 0 | \hat{\phi}(0) | n \rangle|^2 
\]

\[
= \int \frac{d^4p}{(2\pi)^4} e^{ip \cdot (x-y)} \theta(p^0) 2\pi \rho(-p^2),
\]

where the last line defines for us the Kallen-Lehmann spectral density, \( \rho(-p^2) \), as

\[
\theta(p^0) 2\pi \rho(-p^2) \equiv (2\pi)^3 \sum_n \delta^{(4)}(p - p_n)|\langle 0 | \hat{\phi}(0) | n \rangle|^2.
\]

Now, to make (2.160) more convenient and physically profound we will write

\[
\rho(-p^2) = \int_0^\infty d\mu \rho(\mu) \delta(p^2 + \mu),
\]

which makes the interacting two-point Wightman function,

\[
\langle 0 | \hat{\phi}(x)\hat{\phi}(y) | 0 \rangle = \int_0^\infty d\mu \rho(\mu) \int \frac{d^4p}{(2\pi)^4} e^{ip \cdot (x-y)} \theta(p^0) \delta(p^2 + \mu).
\]

We can now relate the \( p \)-integrals to (2.151) and notice

\[
\langle 0 | \hat{\phi}(x)\hat{\phi}(y) | 0 \rangle = \int_0^\infty d\mu \rho(\mu) \langle 0 | \hat{\phi}(x)\hat{\phi}(y) | 0 \rangle_\mu,
\]
where \( \langle 0 \mid \hat{\phi}(x)\hat{\phi}(y) \mid 0 \rangle_\mu \) is the free Wightman function of mass \( \mu \). As can be seen, the other Wightman function with \( x \) and \( y \) interchanged is simply

\[
\langle 0 \mid \hat{\varphi}(y)\hat{\varphi}(x) \mid 0 \rangle = \int_0^\infty d\mu \, \rho(\mu) \langle 0 \mid \hat{\varphi}(y)\hat{\varphi}(x) \mid 0 \rangle_\mu .
\] (2.165)

We are now ready to consider the time-ordered interacting two-point function, \( \langle 0 \mid T(\hat{\varphi}(x)\hat{\varphi}(y)) \mid 0 \rangle \). We start with the definition of time-ordering, (2.146),

\[
\langle 0 \mid T(\hat{\varphi}(x)\hat{\varphi}(y)) \mid 0 \rangle = \theta(x^0 - y^0)\langle 0 \mid \hat{\varphi}(x)\hat{\varphi}(y) \mid 0 \rangle
+ \theta(y^0 - x^0)\langle 0 \mid \hat{\varphi}(y)\hat{\varphi}(x) \mid 0 \rangle,
\] (2.166)

Now using (2.164) and (2.165) we have

\[
\langle 0 \mid T(\hat{\varphi}(x)\hat{\varphi}(y)) \mid 0 \rangle = \int_0^\infty d\mu \, \rho(\mu) \left[ \theta(x^0 - y^0)\langle 0 \mid \hat{\varphi}(x)\hat{\varphi}(y) \mid 0 \rangle_\mu + \theta(y^0 - x^0)\langle 0 \mid \hat{\varphi}(y)\hat{\varphi}(x) \mid 0 \rangle_\mu \right].
\] (2.167)

and using (2.153) we have

\[
\langle 0 \mid T(\hat{\varphi}(x)\hat{\varphi}(y)) \mid 0 \rangle = -i \int_0^\infty d\mu \, \rho(\mu) G_F(x - y; \mu),
\] (2.168)

where \( G_F(x - y; \mu) \) is the free Feynman propagator of some mass \( \mu \), that is to say,

\[
\langle 0 \mid T(\hat{\varphi}(x)\hat{\varphi}(y)) \mid 0 \rangle = \int_0^\infty d\mu \, \rho(\mu) \int \frac{d^4k}{(2\pi)^4} \frac{ie^{ik(x-y)}}{k^2 + \mu - i\epsilon}.
\] (2.169)
This profound result tells us that any interacting two-point function is just a summation of free two-point functions of mass $\mu$ weighted over some function of $\mu$, $\rho(\mu)$ which depends on the specific interactions. Since unitarity requires $|\langle 0 | \hat{\phi}(0) | n \rangle|^2 \geq 0$ because we require that we do not have negative probabilities, the conclusion is that Unitarity is encoded in the spectral density through

$$ \rho(\mu) \geq 0 \rightarrow \text{Unitarity Requirement} \ . \quad (2.170) $$
Chapter 3

UV Completions

For a Quantum Field Theory to ‘make sense’ we usually impose five criteria

- Finiteness - that all observable quantities return a finite, technically measurable, value.

- Lorentz invariance - that all observers observe the same physics in all inertial reference frames.

- Unitarity - that quantum probabilities are conserved and are non-negative.

- Causality - that knowledge of the present only requires knowledge of the past.

- Locality - that physical information must travel at some finite speed. Together with Lorentz invariance this finite speed becomes the speed of light, $c$, and locality becomes intimately connected with causality.

These criteria may seem obvious but writing down a quantum field theory that obeys all of these is more subtle than one might believe. For instance, a whole set
of systematic processes (Renormalization, normal ordering, etc.) have been created to get around the multitude of infinities that arise when computing quantitative in Quantum Field Theory.

Some theories break Unitarity above some energy scale and the modern perspective is that these theories must only be Effective Field Theories, that is they only approximately describe reality below their cutoff energy scale but fundamentally are not the whole story. To cure these theories, the introduction of heavier fields must come in to preserve Unitarity, known as Wilsonian Completion, described in this chapter. Galileons are thought to not be able to be saved by this Wilsonian Completion and therefore either a new process must come in to save them or some would argue they just simply do not make sense from the get-go.

A novel idea originally proposed by Dvali et al, [66, 67, 68, 69, 70, 71, 72], called Classicalization, states that it may be possible for some theories to admit a UV completion without Wilsonian Completion. Although perturbative calculations suggest Galileon theories are ‘sick’ (break the Unitarity requirement above some energy scale), Classicalization suggests these types of theories are UV complete, that is they are finite, Unitary and Lorentz invariant without the need to add ‘new physics’ or heavier degrees of freedom. We have left out the criteria of ‘causality’ and ‘locality’ in our requirements of UV completion because our work shows Galileons violate micro-locality, and because they are Lorentz invariant, micro-causality. This should not be alarming, however, since marco-locality and macro-causality are still both preserved (see Chapter 5).
3.1 Wilsonian Completion

To understand Classicalization, it is best to first understand its counterpart, Wilsonian Completion. Let us take an example from [73] and say we have a theory of two scalar fields, a heavy field $h(x)$ with mass $M$ and a light field $\ell(x)$ will mass $m$. Let us also say the theory is defined by the action,

$$S[h, \ell] = \int d^4x \left\{ -\frac{1}{2} (\partial \ell)^2 - \frac{1}{2} m^2 \ell^2 - \frac{1}{2} (\partial h)^2 - \frac{1}{2} M^2 h^2 - V(h, \ell) \right\},$$

(3.1)

where the potential, $V(h, \ell)$, is

$$V(h, \ell) = \frac{g_\ell}{4!} \ell^4 + \frac{g_h}{4!} h^4 + \frac{g_{h\ell}}{4} h^2 \ell^2 + \frac{1}{2} \tilde{m} h \ell^2 + \frac{\tilde{g}_h}{3!} M h^3,$$

(3.2)

where $g_\ell, g_h, g_{h\ell}, \tilde{g}_h$ along with all the mass parameters are, in principle at this stage, completely arbitrary. This potential defines, that along with the fields having masses themselves, all the interactions the heavy field, $h$, and the light field, $\ell$, have with each other. We will now ask a scattering question, namely the tree-level amplitude for two light particles to scatter and result with two light particles at the end, $A(2 \rightarrow 2)$. We will also suppose the center of mass energy for these two incoming light particles, $E$, will be much smaller than the mass of the heavy particle, $M$, that is $E \ll M$. Out of all the potential interactions in (3.2) only two terms contribute to this tree-level process,

$$V(h, \ell) \sim \frac{g_\ell}{4!} \ell^4 + \frac{1}{2} \tilde{m} h \ell^2.$$

(3.3)
Figure 3.1: The relevant terms in the potential $V(h, \ell)$ that contribute to a tree-level two-to-two scattering amplitude for the light particles, $\ell$.

This is described in Figure 3.1. We use the $\hat{S}$-matrix to define this quantity through

$$S(2 \to 2) = \langle \text{out} | \hat{S} | \text{in} \rangle ,$$  \hspace{1cm} (3.4)$$

where $| \text{in} \rangle$ is our $\textit{in}$ state defined to be two particles on the vacuum and $| \text{out} \rangle$ is our $\textit{out}$ state, also defined to be two particles on the vacuum, that is

$$| \text{in} \rangle = | k_1, k_2 \rangle = \sqrt{2\omega_{k_1}} \sqrt{2\omega_{k_2}} | \vec{k}_1, \vec{k}_2 \rangle = \sqrt{2\omega_{\vec{k}_1}} \sqrt{2\omega_{\vec{k}_2}} \hat{a}_{\vec{k}_1}^\dagger \hat{a}_{\vec{k}_2}^\dagger | 0 \rangle ,$$  \hspace{1cm} (3.5)$$

$$| \text{out} \rangle = | k_3, k_4 \rangle = \sqrt{2\omega_{\vec{k}_3}} \sqrt{2\omega_{\vec{k}_4}} | \vec{k}_3, \vec{k}_4 \rangle = \sqrt{2\omega_{\vec{k}_3}} \sqrt{2\omega_{\vec{k}_4}} \hat{a}_{\vec{k}_3}^\dagger \hat{a}_{\vec{k}_4}^\dagger | 0 \rangle .$$  \hspace{1cm} (3.6)$$

The $\hat{S}$-matrix is defined to be

$$\hat{S} = T e^{i \int_{-\infty}^{\infty} d^4x \mathcal{L}_{\text{int}}(x)} ,$$  \hspace{1cm} (3.7)$$
where $L_{\text{int}}$ is the interaction component to (3.1) which is all terms that are not the free terms and $T$ is the time-ordering operator defined in (2.146). We can write $S(2 \rightarrow 2) = 1 + iT(2 \rightarrow 2)$ and focus on the $T(2 \rightarrow 2)$ contribution since that is where the physics is. We define the momentum stripped amplitude, $A(2 \rightarrow 2)$, through

$$T(2 \rightarrow 2) = i(2\pi)^4 \delta^{(4)}(k_1 + k_2 - k_3 - k_4)A(2 \rightarrow 2).$$

(3.8)

When the dust settles we see that, [73],

$$A(2 \rightarrow 2) = -g_\ell + \tilde{m}^2 \left[ \frac{1}{s + M^2} + \frac{1}{t + M^2} + \frac{1}{u + M^2} \right],$$

(3.9)

where $s, t,$ and $u$ are the Mandelstam variables [74],

$$s = (k_1 + k_2)^2 = (k_3 + k_4)^2,$$

(3.10)

$$t = (k_1 - k_3)^2 = (k_2 - k_4)^2,$$

(3.11)

$$u = (k_1 - k_4)^2 = (k_2 - k_3)^2.$$  

(3.12)

Now, we want the center of mass energy, $E \ll M$ which is saying let $s \ll M^2$, $t \ll M^2$, and $u \ll M^2$. This approximation leads us to

$$A(2 \rightarrow 2) \approx -g_\ell + \frac{3\tilde{m}^2}{M^2} - \frac{(s + t + u)\tilde{m}^2}{M^4} + \frac{(s^2 + t^2 + u^2)\tilde{m}^2}{M^6} + \mathcal{O}\left(\frac{1}{M^8}\right).$$

(3.13)
and with the identity \( s + t + u = -4m^2 \) we have,

\[
\mathcal{A}(2 \to 2) \approx -g_\ell + \frac{3\tilde{m}^2}{M^2} + \frac{4m^2\tilde{m}^2}{M^4} + \frac{(s^2 + t^2 + u^2)\tilde{m}^2}{M^6} + \mathcal{O}\left(\frac{1}{M^8}\right). \tag{3.14}
\]

What we see now is that this amplitude, \( \mathcal{A}(2 \to 2) \), depends on the center of mass energy, \( E \), once we notice \( s \sim E^2 \). This amplitude is associated with the probability of such a scattering event occurring which we see gets bigger than one once \( s^2 > M^4 \).

The idea that this probability is greater than one violates tree-level Unitarity. We notice, though, that this small energy approximation ignores effects of the heavy field, \( h \). This means that at \( s^2 \sim M^4 \) we should have accounted for the heavier field which means the approximation in (3.13) is violated and we must now consider the full amplitude (3.9). However, the point is, if we restrict our focus on low energies where we can neglect everything of \( \mathcal{O}(1/M^6) \), we can define

\[
V_{\text{eff}}(\ell) = \frac{1}{4!} \left( -g_\ell + \frac{3\tilde{m}^2}{M^2} + \frac{4m^2\tilde{m}^2}{M^4} \right) \ell^4 \equiv \tilde{g}_\ell \frac{\ell^4}{4!}, \tag{3.15}
\]

which is an effective potential that only depends on the light field \( \ell \). We can also construct an effective action that is only a function of the light field \( \ell \),

\[
S_{\text{eff}}[\ell] = \int d^4x \left( -\frac{1}{2}(\partial\ell)^2 - \frac{1}{2}m^2\ell^2 - V_{\text{eff}}(\ell) \right). \tag{3.16}
\]
Now, if we ask the same two-to-two tree-level scattering question, $A_{\text{eff}}(2 \rightarrow 2)$, using this new $S_{\text{eff}}[\ell]$ we will obtain

$$A_{\text{eff}}(2 \rightarrow 2) = -\tilde{g}_\ell,$$

which is the same as (3.13) up to $O(1/M^6)$.

So the logic behind Wilsonian completion is this; if you have a theory that violates unitarity at some scale then to ‘fix’ the physics of the theory, one must consider that they don’t have all the physics, meaning one has not included all degrees of freedom, or fields, into the theory. At some energy scale, usually associated with the mass of the heavier degrees of freedom, new ‘physics’, or fields, must be accounted for in order to preserve unitarity. We conceptualize $S_{\text{eff}}[\ell]$ as some effective theory, that is

$$e^{iS_{\text{eff}}[\ell]} = \int D\ell \ e^{iS[h,\ell]},$$

which is saying $S_{\text{eff}}[\ell]$ is a theory of a light field, $\ell$, obtained from some more fundamental theory $S[h, \ell]$ only after integrating out all heavier field dynamics. As long as your dynamics stay below the energy scale associated with the mass of the heavy particle, $M$, then we say $S_{\text{eff}}[\ell]$ is effectively true. One cannot differentiate between the two theories below the energy scale $M$. This is summed up in Figure 3.2.

Another very simple example to see how additional degrees of freedom may affect dynamics is presented in [75] which we quickly discuss. Let us consider a two field
model with a light field $\phi$ with mass $m$ and a heavy field $\chi$ with mass $M$ governed by

\[ \mathcal{L}[\phi, \chi] = -\frac{1}{2} (\nabla\phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{1}{2} (\nabla\chi)^2 - \frac{1}{2} M^2 \chi^2 - \frac{\lambda}{2} \chi \phi^2 \, . \] (3.19)

We can integrate out the heavier field $\chi$ at tree level by simply replacing $\chi$ with its solution to the classical equations of motion. That is let

\[ \chi \rightarrow \chi_c = \frac{\lambda}{2} \phi^2 \frac{1}{(\Box - M^2)} \, , \] (3.20)

then our effective Lagrangian becomes

\[ \mathcal{L}_{\text{eff}}[\phi] = -\frac{1}{2} (\nabla\phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda^2}{4} \phi^2 \frac{1}{(\Box - M^2)} \phi^2 \, , \] (3.21)
and to make sense of this locally we say let $\Box \ll M^2$ and expand to get

$$L_{\text{eff}}[\phi] = -\frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m^2 \phi^2 + \frac{\lambda^2}{4} \left( \frac{\phi^4}{M^2} + \frac{\phi^2 \Box \phi^2}{M^4} + \cdots \right), \quad (3.22)$$

which is non-renormalizable and therefore if it is to make sense must have some UV completion.

### 3.2 Classicalization

The biggest difference between the usual Wilsonian Completion story and Classicalization is that Classicalization supposes that there is no need to introduce new ‘physics’, or heavier degrees of freedom, into the system to restore unitarity. In fact, Classicalization says the only reason why unitarity appears to be violated in $\hat{S}$-matrix calculations is that for some theories there are an infinite amount of ‘higher order’ terms that become relevant and are not accounted for in perturbative $\hat{S}$-matrix calculations. That is, Wilsonian Completion says a violation of our axioms is because there is new physics that we have not accounted for. Classicalization says that, for some theories, it is the math that fails to capture the already correct physics.

$$\text{Wilsonian Completion} \rightarrow \text{Update the physics} ,$$

$$\text{Classicalization} \rightarrow \text{Update the math} .$$

The generic story of Classicalization originally supposed that, for some theories, scattering amplitudes at high energies are dominated by many soft quanta as opposed to
a few hard quanta. The intermediate state of this classicalizing scattering scenario is
a classical object of quanta, a coherent state, dubbed a ‘classicalon’. Being a coherent
state, we must consider states of all numbers of particles, that is we cannot perturb
assuming only a finite amount of particles contribute.

An intuitive example to understand this logic is to consider the scattering of two
gravitons. We know that General Relativity informs us if any amount of energy is
localized within a small enough region, defined by the Schwarzschild radius \( r_s(E) \),
a black hole will form. So as we consider these two gravitons scattering, we must
at some point consider when the impact parameter, \( b \leq r_s(E) \) where \( r_s(E) \) is the
Schwarzschild radius. At this scale a black hole will form which is a classical con-
figuration of gravitons, or a coherent state of gravitons. No new fields have come
into play to create this ‘classicalon’, it is a result of the strong coupling regime of
gravity. As we consider larger and larger \( E \)’s, the black hole itself becomes larger and
larger. This mixes UV and IR physics, that is higher energies are associated with
larger distances, something which is counter to our usual intuition. This concept is
shown in Figure 3.3.

Classicalization further supposes that the creation of this large classical object is
what ‘saves’ the theory from violating any of our axioms. This classicalon will then
decay into many soft quanta as oppose to a few hard quanta. The energy per particle
decreases as the total energy of the system increases preserving Unitarity.
Figure 3.3: Two scenarios of two gravitons scattering. A blackhole is created whose size is dependent on the incoming center of mass energy. Since $E_2 \gg E_1$, the blackhole on the right would be of greater size than the one on the left. Large energies corresponds to large distances.

3.3 Classicalizing Galileons

So why would we even begin to consider that Galileons might behave in a ‘Classicalizing’ fashion? There are reasons to believe, [76], that Galileons cannot Wilsonian complete. That is, if Galileons break one of our axioms at some energy scale, an introduction of a heavier degree of freedom cannot save us. However, this work does not account for the non-perturbative effects of the Vainshtein Mechanism.

3.3.1 Galileon Scattering

To understand the oddities of the Galileon model let us do another two-to-two scattering example with a cubic Galileon theory,

$$S_g[\pi] = \int d^4x \, - \frac{1}{2} (\partial\pi)^2 - \frac{1}{\Lambda^3} \Box \pi (\partial\pi)^2 ,$$

(3.23)
where $\pi$ is the Galileon field. In the usual $\hat{S}$-matrix prescription we split the Lagrangian into a ‘free’ part $L_{\text{free}}$ and an ‘interacting’ part, $L_{\text{int}}$ through

\begin{align}
L_{\text{free}} &= -\frac{1}{2} (\partial \pi)^2 , \\
L_{\text{int}} &= -\frac{1}{\Lambda^3} \Box \pi (\partial \pi)^2 ,
\end{align}

where (3.24) will define what we mean by a particle as described in (2.78) and (3.25) will define the interactions between the particles through the form of the $\hat{S}$-matrix.

The $\hat{S}$-matrix prescription assumes free asymptotic states meaning our incoming two particles and outgoing two particles can be defined using the free scalar field, so just as in (3.5), we have

\begin{align}
|\text{in}\rangle &= |k_1, k_2\rangle = \sqrt{2\omega_{k_1}} \sqrt{2\omega_{k_2}} |\vec{k}_1, \vec{k}_2\rangle = \sqrt{2\omega_{\vec{k}_1}} \sqrt{2\omega_{\vec{k}_2}} \hat{a}_{\vec{k}_1}^\dagger \hat{a}_{\vec{k}_2}^\dagger |0\rangle , \\
|\text{out}\rangle &= |k_3, k_4\rangle = \sqrt{2\omega_{k_3}} \sqrt{2\omega_{k_4}} |\vec{k}_3, \vec{k}_4\rangle = \sqrt{2\omega_{\vec{k}_3}} \sqrt{2\omega_{\vec{k}_4}} \hat{a}_{\vec{k}_3}^\dagger \hat{a}_{\vec{k}_4}^\dagger |0\rangle .
\end{align}

So for our amplitude we have

\begin{align}
\langle \text{out} | \hat{S} | \text{in}\rangle &= \sqrt{16 \omega_{k_1} \omega_{k_2} \omega_{k_3} \omega_{k_4}} \langle 0 | \hat{a}_{\vec{k}_3} \hat{a}_{\vec{k}_4} T^* e^{i \int d^4x L_{\text{int}}(x) \hat{a}_{\vec{k}_1}^\dagger \hat{a}_{\vec{k}_2}^\dagger} |0\rangle ,
\end{align}

where $T^*$ is the time-ordering operator, (2.146), with an extra step which instructs to operate the derivatives last. That is, for some correlation function with derivatives
on the fields, $T^*$ instructs

$$
\langle 0 \mid T^* \partial_x \hat{\phi}(x) \partial_x \hat{\phi}(x) \partial_y \hat{\phi}(y) \partial_y \hat{\phi}(y) \mid 0 \rangle
= \lim_{x_1 \to x} \lim_{x_2 \to x} \lim_{y_1 \to y} \lim_{y_2 \to y} \partial_{x_1} \partial_{x_2} \partial_{y_1} \partial_{y_2} \langle 0 \mid T \hat{\phi}(x_1) \hat{\phi}(x_2) \hat{\phi}(y_1) \hat{\phi}(y_2) \mid 0 \rangle .
$$

(3.29)

In principle, (3.28) is the full answer to this two-to-two amplitude. However, in practice, we are unable to compute anything meaningful from this expression. What is done is we perturb the $\hat{S}$-matrix and say

$$
\hat{S} = T^* e^{i \int d^4 x \mathcal{L}_{\text{int}}(x)} = T^* \left[ 1 + i \int d^4 x \mathcal{L}_{\text{int}}(x) + \frac{i^2}{2} \int d^4 x \int d^4 y \mathcal{L}_{\text{int}}(x) \mathcal{L}_{\text{int}}(y) + \ldots \right]
\quad = \hat{S}_0 + \hat{S}_1 + \hat{S}_2 + \ldots ,
$$

(3.30)

so now we truncate this series at some point and we just have powers of field operators which we know how to compute. The first term in the perturbed $\hat{S}$-matrix, $\hat{S}_0$, is usually ignored. This term is the trivial, no-interaction, contribution which leads to no information on the actual scattering behavior. The second term, when sandwiched between our in and out states is

$$
\langle \text{out} \mid \hat{S}_1 \mid \text{in} \rangle = i \int d^4 x \sqrt{16 \omega_{\vec{k}_1} \omega_{\vec{k}_2} \omega_{\vec{k}_3} \omega_{\vec{k}_4}}
\times \langle 0 \mid \hat{a}_{\vec{k}_3} \hat{a}_{\vec{k}_4} T^* \left( -\frac{1}{\Lambda^3} \hat{\pi}(x)(\partial \hat{\pi}(x))^2 \right) \hat{a}^\dagger_{\vec{k}_1} \hat{a}^\dagger_{\vec{k}_2} \mid 0 \rangle .
$$

(3.31)
The point is that with three \( \hat{\pi} \)'s there are a total of seven \( \hat{a} \)'s and \( \hat{a}^\dagger \)'s which results in zero. So our first non-zero contribution from our perturbed \( \hat{S} \)-matrix, (3.30), is \( \hat{S}_2 \).

What we want from this contribution is how it depends on the center of mass energy, \( E \), of the two incoming Galileon particles. Using dimensional analysis we know it must go as

\[
\langle \text{out} | \hat{S}_2 | \text{in} \rangle \sim \frac{E^6}{\Lambda^6},
\]

which has a similar energy dependence as (3.13). That is, there is a point where \( E > \Lambda \), and thus this amplitude violates unitarity. However, what we failed to take into account with this conclusion is that \( \langle \text{out} | \hat{S}_2 | \text{in} \rangle \) is actually just one of an infinite amount of contributions to the total amplitude, (3.28). So we can say is that \( \langle \text{out} | \hat{S}_2 | \text{in} \rangle \) breaks tree-level Unitarity but full Unitarity might still be preserved.

To understand why this apparent issue might not be an issue at all, we can simply rescale our variables.

### 3.3.2 Normalizations

Physics is invariant under field redefinitions. For instance, multiplying our field by a constant shouldn’t change any of the physics of the system. Here we will pick a convenient set of normalizations to point out a problem with the above calculation.

First, we will do a field redefinition,

\[
\pi \rightarrow \left( \frac{\Lambda}{E} \right) \pi,
\]
where $\Lambda$ is the usual strong coupling scale and $E$ is the characteristic energy of the situation we are considering. Along with this field redefinition, we will normalize space, time, and the Galileon field by

$$x \to \frac{1}{E} \hat{x},$$  \hspace{1cm} (3.34)$$
$$\partial \to E \hat{\partial},$$  \hspace{1cm} (3.35)$$
$$\pi \to E \hat{\pi}.$$ \hspace{1cm} (3.36)

This gives us for our action, (3.23),

$$S_g[\pi] \to S_g[\hat{\pi}] = \int d^4 \hat{x} - \frac{1}{2} \left( \frac{\Lambda}{E} \right)^2 (\hat{\partial} \hat{\pi})^2 - \hat{\Box} \hat{\pi} (\hat{\partial} \hat{\pi})^2,$$ \hspace{1cm} (3.37)

which makes our free and interacting Lagrangian pieces, (3.24) and (3.25), become

$$\mathcal{L}_{\text{free}} = -\frac{1}{2} \left( \frac{\Lambda}{E} \right)^2 (\hat{\partial} \hat{\pi})^2,$$ \hspace{1cm} (3.38)$$
$$\mathcal{L}_{\text{int}} = -\hat{\Box} \hat{\pi} (\hat{\partial} \hat{\pi})^2.$$ \hspace{1cm} (3.39)

Now, when we computed our $\hat{S}$-matrix amplitude, we perturbed the $\hat{S}$-matrix as in (3.30). This expansion involved an implicit assumption that $\mathcal{L}_{\text{free}} \gg \mathcal{L}_{\text{int}}$. With our normalizations we see that this approximation is only true for a certain energy regime, that is as long as $E \lesssim \Lambda$. We see that once $E \approx \Lambda$, $\mathcal{L}_{\text{free}} \approx \mathcal{L}_{\text{int}}$, and so we may not expand the $\hat{S}$-matrix like we did in (3.30). It is usual to assume this breakdown at the scale $\Lambda$ is due to the breakdown of the entire theory, that is above this scale we need
to Wilsonian Complete by adding heavier degrees of freedom to rescue the theory.

We take a different perspective, that when \( E \sim \Lambda \) the theory does not breakdown, merely the mathematical approximations we have used breakdown. This is saying we are not allowed to expand as we did in (3.30), we must consider every term in the \( \hat{S} \)-matrix since every term is of equal weight. This requires some non-perturbative mathematical scheme to appropriately analyze the dynamics of the \( \hat{S} \)-matrix. With this logic we refer to the energy regime, \( E \geq \Lambda \) as the ‘strong-coupling’ regime, where the interactions must be considered strong and perturbation theory is no longer applicable. So we now see that the breakdown of (3.32) is to be expected and shouldn’t immediately signal a necessity to reconsider the whole theory, the math just simply is not correctly accounting for the new physics (the Vainshtein Mechanism) that occurs at this strong coupling scale.

This is not special to the simple cubic Galileon, (3.23), for a completely generic Galileon in 4D with a field redefinition of \( \pi \rightarrow (\Lambda/E)^{9/4} \pi \), we obtain,

\[
S_g[\tilde{\pi}] = \int d^4\tilde{x} \left( \frac{\Lambda}{E} \right)^{18/5} L_2[\tilde{\pi}] + \left( \frac{\Lambda}{E} \right)^{12/5} L_3[\tilde{\pi}] + \left( \frac{\Lambda}{E} \right)^{6/5} L_4[\tilde{\pi}] + L_5[\tilde{\pi}] ,
\]

which once again informs as that all interactions become of the same order once \( E \sim \Lambda \).

### 3.3.3 Classical Self Energy - Electric Point Charge

Another clue that points towards Galileons being different than standard field theories is that, due to the Vainshtein Mechanism, they admit a finite classical self energy.
In short, since a field has some value at every point in space, it is conceivable that in order to calculate the energy of the entire field we have to look everywhere. This idea of integrating over all space usually results in divergences in the total amount of energy in a field which is usually accounted for by renormalization or regularization. To understand this and how Galileons overcome this, we will start with a more familiar example.

Let us consider a simple static electric charge. Since we have a charge, we know we must also have an electric field. Since the electric field goes as

$$\vec{E}(r) = \frac{1}{4\pi \epsilon_0} \frac{q}{r^2} \hat{r}, \quad (3.41)$$

at every $r$ there is some amount of non-zero electric field contribution to the energy so we must consider all space when calculating the energy of this very simple system. The energy per unit volume, $u$ goes as $|\vec{E}(r)|^2$ which gives us for a total energy $E_{\text{tot}}$,

$$E_{\text{tot}} \sim \int dV \ u \sim \int_0^\infty dr \ r^2 \frac{q^2}{r^4} \sim \int_0^\infty dr \ \frac{q^2}{r^2} \rightarrow \text{Diverges}. \quad (3.42)$$

We see that the 0 in the limit of integration, which is informing us we are looking at a point charge, is what causes this total energy to diverge. So even classical electromagnetism exhibits classical divergences. Let us look at the difference between this scenario and a Galileon.
3.3.4 Classical Self Energy - Galileon Point Mass

Like in the previous example where a point charge will source an electromagnetic field, here we will consider a general Galileon theory which is sourced by a point mass in arbitrary $d$-dimensions. That is we will consider

$$S[\pi] = \sum_{n=2}^{d+1} \frac{1}{A^a} S_n[\pi] + \int d^d x \, \pi \tilde{M} \delta^{(d-1)}(x) \, ,$$

(3.43)

where, once again,

$$\sum_{n=2}^{d+1} \frac{1}{A^a} S_n[\pi] = \sum_{n=2}^{d+1} \frac{1}{A^a} \int d^d x \, c_n \epsilon^{\mu_1 \ldots \mu_d} \epsilon^{\nu_1 \ldots \nu_d} \pi \prod_{j=1}^{n-1} (\partial_{\mu_j} \partial_{\nu_j} \pi) \prod_{k=n}^{d} \eta_{\mu_k \nu_k} \, ,$$

(3.44)

is our general Galileon in $d$-dimensions and $\tilde{M}$ is related to the mass of the point mass, $M$, and the Planck mass, $M_{pl}$, through

$$\tilde{M} = \frac{M}{M_{pl}^{\frac{1}{d-2}}} \, .$$

(3.45)

To get a handle on how to approximate the total self energy of this Galileon field sourced by a point mass, let us first do a field definition and a normalization of our variables like we did in Section 3.3.2

$$\pi \rightarrow \left( \frac{\Lambda}{E} \right)^{\frac{(d+1)d-1}{2(d+1)}} \pi \, ,$$

(3.46)
where, once again, $E$ is the characteristic energy scale of the scenario we are considering. Our action becomes

\[
\sum_{n=2}^{d+1} \frac{1}{\Lambda^\alpha} S_n[\pi] \rightarrow \sum_{n=2}^{d+1} \Lambda^{d+2-\frac{(d+2)n}{d+1}} E^{\frac{(d+1)d-1}{2(d+1)}} \int d^d x \ c_n \epsilon_{\mu_1 \ldots \mu_d} \epsilon^{\nu_1 \ldots \nu_d} \pi \prod_{j=1}^{n-1} (\hat{\partial}_{\mu_j} \hat{\partial}_{\nu_j} \pi) \prod_{k=n}^{d} \eta_{\mu_k \nu_k} ,
\]

(3.47)

and our normalizations are

\[
x = \frac{1}{E} \hat{x} ,
\]

(3.48)

\[
\partial = E \hat{\partial} ,
\]

(3.49)

\[
\pi = E^{\frac{1}{2}(d-2)} \hat{\pi} .
\]

(3.50)

This ends up giving us

\[
\sum_{n=2}^{d+1} \frac{1}{\Lambda^\alpha} S_n[\pi] \rightarrow \sum_{n=2}^{d+1} \left( \frac{\Lambda}{E} \right)^{p_n} \int d^d x \ c_n \epsilon_{\mu_1 \ldots \mu_d} \epsilon^{\nu_1 \ldots \nu_d} \pi \prod_{j=1}^{n-1} (\hat{\partial}_{\mu_j} \hat{\partial}_{\nu_j} \pi) \prod_{k=n}^{d} \eta_{\mu_k \nu_k} .
\]

(3.51)

\[
= \sum_{n=2}^{d+1} \left( \frac{\Lambda}{E} \right)^{p_n} \hat{S}_n[\hat{\pi}] ,
\]

(3.52)

where

\[
p_n = d + 2 - \frac{(d + 2)n}{d + 1} .
\]

(3.53)
The key point is that $p_n > p_{n+1}$. To better understand the importance of this let us consider the general Galileon action term by term,

\[ \sum_{n=2}^{d+1} \left( \frac{\Lambda}{E} \right)^{p_n} \hat{S}_n[\hat{\pi}] = \left( \frac{\Lambda}{E} \right)^{p_2} \hat{S}_2[\hat{\pi}] + \left( \frac{\Lambda}{E} \right)^{p_3} \hat{S}_3[\hat{\pi}] + \ldots + \left( \frac{\Lambda}{E} \right)^{p_d} \hat{S}_d[\hat{\pi}] + \left( \frac{\Lambda}{E} \right)^{p_{d+1}} \hat{S}_{d+1}[\hat{\pi}] . \] (3.54)

We see a general feature of what we saw in (3.37). When $E \ll \Lambda$ we see that $\hat{S}_n[\hat{\pi}] \gg \hat{S}_{n+1}[\hat{\pi}]$. This makes $\hat{S}_2[\hat{\pi}]$ the most dominant term and therefore usual perturbation theory is applicable. When $E \sim \Lambda$ we have that $\hat{S}_n[\hat{\pi}] \sim \hat{S}_{n+1}[\hat{\pi}]$ which is telling us that if we are to consider any operator we must consider all operators to be consistent. Perturbation theory is not applicable in this energy regime thus we call it our strong coupling regime. Finally, when $E \gg \Lambda$ we have $\hat{S}_n[\hat{\pi}] \ll \hat{S}_{n+1}[\hat{\pi}]$ which is saying that the highest order Galileon term is now dominant. Perturbation theory, in the usual sense, is no longer applicable.

### 3.3.4.1 Strong Coupling Radius

There exists a non-trivial distance scale, $r_*$, associated with the strong coupling regime. Let us see how this $r_*$ distance scale scales with the energy, $E$. In the previous section we saw that the strong coupling regime is when all Galileon operators are of the same weight. To find this distance we can look at the energy of the Galileon system,

\[ E = \int d^{d-1}x \mathcal{H}_2[\hat{\pi}] + \sum_{n=3}^{d+1} \frac{1}{\Lambda^\alpha} \int d^{d-1}x \mathcal{H}_n[\pi] , \] (3.55)
where $\mathcal{H}_n[\pi]$ is the Hamiltonian density for the $n$th galileon operator. However, for this argument we will just consider static spherically symmetric solutions so we can say

$$\mathcal{H}_n[\pi] \sim \mathcal{L}_n[\pi] = c_n \epsilon^{\mu_1...\mu_d} \epsilon^{\nu_1...\nu_d} \pi \prod_{j=1}^{n-1} (\partial_{\mu_j} \partial_{\nu_j} \pi) \prod_{k=n}^{d} \eta_{\mu_k \nu_k}, \quad (3.56)$$

where $\mathcal{L}_n[\pi]$ is the Lagrangian density for the $n$th galileon operator. To see the effects of the strong coupling regime we simply need to assume all operator terms of the energy are of the same order, namely

$$E \sim \int dr \, r^{d-2} \mathcal{L}_2[\pi] \sim \frac{1}{\Lambda^a} \int dr \, r^{d-2} \mathcal{L}_n[\pi], \quad (3.57)$$

for all $n$. We now normalize spacetime and the fields like we did in the previous section to strip them of their units,

$$r = r_\ast \hat{r}, \quad (3.58)$$

$$\partial = \frac{1}{r_\ast} \hat{\partial}, \quad (3.59)$$

$$\pi = Z^{\frac{1}{2}(d-2)} \hat{\pi}. \quad (3.60)$$

Applying this normalization to the first term on the right hand side in (3.57) we get an equation for $Z$ being,

$$Z = \left( \frac{E}{r_\ast^{d-3}} \right)^{\frac{1}{d-2}}. \quad (3.61)$$
Doing the same procedure for the last term on the right hand side in (3.57) and using the relation in (3.61) we conclude

\[ E^{1-n/2} = \frac{r^{d+1-2n-dn/2+3n/2}}{\Lambda^{n-2+dn/2-d}}, \]

(3.62)

and solving for \( r^* \) we get

\[ r^* = \frac{1}{\Lambda} \left( \frac{E}{\Lambda} \right)^{\frac{1}{d+1}}. \]

(3.63)

We can see that \( r^* \) does not depend on \( n \) meaning there is only one distance scale at which all operators must be of the same order. The non-intuitive behavior of this scale is that it grows with energy. It is precisely this growth that regulates the quantum theory. However, it is also this growth that has the Galileon admit a finite classical self energy. It is this \( r^* \) behavior that defines the distance where perturbation theory is no longer applicable. As energy increases, this \( r^* \) distance increases, creating a UV/IR mixing, that is high energy physics is now associated with larger distances, a feature which is counter intuitive. This expression would not be of much use unless \( E \) is finite where we show below.

Let us now go back to (3.43),

\[ S[\pi] = \sum_{n=2}^{d+1} \frac{1}{\Lambda^n} S_n[\pi] + \int d^d x \, \pi \tilde{M} \delta^{(d-1)}(x), \]

(3.64)

This point source mass will induce a region \( 0 < r < r^* \) that is inherently non-perturbative as discussed in the previous section. In principle the total energy of the
Galileon field in this scenario is

$$E = \sum_{n=2}^{d+1} \frac{1}{\Lambda^\alpha} \int_0^\infty dr \ r^{d-2} \ L_n[\pi] .$$  \hspace{1cm} (3.65)$$

However, the highest order Galileon operator dominates the field behavior inside the strong coupling region and $S_2[\pi]$ dominates the field behavior outside the strong coupling region. Knowing this, we may approximate the total energy of this system to be

$$E \approx \int_0^{r_*} dr \ r^{d-2} \ L_d[\pi_d] + \int_{r_*}^{\infty} dr \ r^{d-2} \ L_2[\pi_2] ,$$  \hspace{1cm} (3.66)$$

where $\pi_i$ is the classical Galileon field configuration defined by just the $S_i[\pi]$ Galileon operator, that is

$$\left. \frac{\delta S_i[\pi]}{\delta \pi} \right|_{\pi=\pi_i} = 0 .$$  \hspace{1cm} (3.67)$$

where

$$S_i[\pi] = \frac{1}{\Lambda^\alpha} \int d^d x \ L_i[\pi] + \int d^d x \ \pi \bar{M} \delta^{(d)}(x) .$$  \hspace{1cm} (3.68)$$

Note that we are not concerned with $S_{d+1}[\pi]$ because that necessarily has time derivatives on $\pi$ which are zero since we are in a static configuration. Therefore, $S_d[\pi]$ is the highest order non-zero Galileon operator in this scenario. Suppressing irrelevant
numerical factors we have

\[ \pi_d(r) = \left( \tilde{M} \Lambda^{\frac{1}{2}(d^2 - 4)} \right)^{\frac{1}{d-1}} r, \tag{3.69} \]

\[ \pi_2(r) = \frac{\tilde{M}}{r^{d-3}}, \tag{3.70} \]

which leaves us with a total energy of

\[ E \approx \left( \tilde{M}^d \Lambda^{\frac{1}{2}(d^2 - 4)} \right)^{\frac{1}{d-1}} r_* + \frac{\tilde{M}^2}{r_*^{d-3}}, \tag{3.71} \]

which is finite. We now see the difference between a Galileon field being sourced by a point mass and an electromagnetic field sourced by a point charge. Unlike the point charge, Galileons create a non-perturbative regime, the Vainshtein regime, where the strong coupling behavior dominates. This effectively splits the space up into two regimes. One with the normal $1/r^{d-3}$ behavior and a new Vainshtein regime which goes as $r$. Therefore, the divergence of the point charge which occurs from the 0 in the limit of integration of the total energy no longer occurs.

With the concept of a finite classical self energy from this non-perturbative $r_*$ regime, we can actually connect this classical physics to quantum physics through looking at quantum corrections to classical physics.

### 3.3.5 Loop Counting Parameter

Usually we count powers of $\hbar$ when considering quantum corrections. The higher order of $\hbar$, the higher order of quantum corrections we are considering. The relevant
quantity to look at in Quantum Field Theory is the path integral, \( Z \), defined by

\[
Z = \int D\pi \ e^{\frac{i}{\hbar} S[\pi]},
\]

where we have \( \hbar \neq 1 \) for demonstrational purposes. Here we see that the units of the action, \([S]\) are \( \hbar \), \([S] = [\hbar] \). When considering quantum contributions it is usual to organize the contributions in \( \hbar \), that is the higher order of \( \hbar \) a term is, the more ‘quantum’ it is. Let us now go back and consider the complete action of the Galileon field,

\[
S[\pi] = \sum_{n=2}^{d+1} \frac{1}{\Lambda^n} \int d^d x \ c_n \epsilon^{\mu_1 \ldots \mu_d} \epsilon^{\nu_1 \ldots \nu_d} \pi \prod_{j=1}^{n-1} (\partial_{\mu_j} \partial_{\nu_j} \pi) \prod_{k=n}^{d} \eta_{\mu_k \nu_k},
\]

and do the same normalization as in (3.58),

\[
x = r_\ast \hat{x},
\]

\[
\partial = \frac{1}{r_\ast} \hat{\partial},
\]

\[
\pi = Z^{\frac{d}{2}(d-2)} \hat{\pi}.
\]

We are then led to

\[
S[\pi] = \sum_{n=2}^{d+1} \frac{r_{d-2n+2} Z^{\frac{n}{2}(d-2)}}{\Lambda^n} \int d^d \hat{x} \ c_n \epsilon^{\mu_1 \ldots \mu_d} \epsilon^{\nu_1 \ldots \nu_d} \hat{\pi} \prod_{j=1}^{n-1} (\hat{\partial}_{\mu_j} \hat{\partial}_{\nu_j} \hat{\pi}) \prod_{k=n}^{d} \eta_{\mu_k \nu_k},
\]

where \( \hbar \neq 1 \) for demonstrational purposes. Here we see that the units of the action, \([S]\) are \( \hbar \), \([S] = [\hbar] \). When considering quantum contributions it is usual to organize the contributions in \( \hbar \), that is the higher order of \( \hbar \) a term is, the more ‘quantum’ it is. Let us now go back and consider the complete action of the Galileon field,

\[
S[\pi] = \sum_{n=2}^{d+1} \frac{1}{\Lambda^n} \int d^d x \ c_n \epsilon^{\mu_1 \ldots \mu_d} \epsilon^{\nu_1 \ldots \nu_d} \pi \prod_{j=1}^{n-1} (\partial_{\mu_j} \partial_{\nu_j} \pi) \prod_{k=n}^{d} \eta_{\mu_k \nu_k},
\]

and do the same normalization as in (3.58),

\[
x = r_\ast \hat{x},
\]

\[
\partial = \frac{1}{r_\ast} \hat{\partial},
\]

\[
\pi = Z^{\frac{d}{2}(d-2)} \hat{\pi}.
\]
and using our relationship between $Z$ and $E$, (3.61) we have

$$S[\pi] = \sum_{n=2}^{d+1} r_n \frac{E^{n/2}}{\Lambda^n} \int d^d \hat{x} \ c_n \epsilon_{\mu_1...\mu_d} \epsilon_{\nu_1...\nu_d} \hat{\pi} \prod_{j=1}^{n-1} (\hat{\partial}_{\mu_j} \hat{\partial}_{\nu_j} \hat{\pi}) \prod_{k=n}^{d} \eta_{\mu_k \nu_k}. \ (3.78)$$

Finally, by using our relationship between $E$ and $r_*$, (3.63) we have

$$S[\pi] = r_* E \sum_{n=2}^{d+1} \int d^d \hat{x} \ c_n \epsilon_{\mu_1...\mu_d} \epsilon_{\nu_1...\nu_d} \hat{\pi} \prod_{j=1}^{n-1} (\hat{\partial}_{\mu_j} \hat{\partial}_{\nu_j} \hat{\pi}) \prod_{k=n}^{d} \eta_{\mu_k \nu_k}$$

$$= r_* E \hat{S}[\hat{\pi}]. \quad (3.79)$$

$$= \left( \frac{E}{\Lambda} \right)^{\frac{(d+2)}{(d+1)}} \hat{S}[\hat{\pi}] . \quad (3.80)$$

Now looking back at (3.72) and setting $\hbar = 1$ we see

$$Z = \int D\pi \ e^{i \left( \frac{E}{\Lambda} \right)^{\frac{(d+2)}{(d+1)}} S[\pi]}. \quad (3.81)$$

We can see now that our effective loop counting parameter, or our effective $\hbar$ in terms of counting quantum contributions, is

$$\hbar \to \left( \frac{\Lambda}{E} \right)^{\frac{(d+2)}{(d+1)}}. \quad (3.82)$$

This expression is only valid because we have shown that $E$ is actually finite in the Galileon case. Since this is true, we see that as energy increases, our effective loop counting parameter decreases. This means at higher and higher energies quantum mechanical contributions become less relevant. This is completely consistent with, and a prediction of, the Classicalization Mechanism. Physics becomes more and more
classical at high energies.
Chapter 4

Galileon Duality

Dualities, or field redefinitions, can become useful tools both mathematically and conceptually in field theory. Sometimes by doing a field redefinition the mathematics can become much simpler. In this chapter we first review a duality that transforms one Galileon into another Galileon. This was first found in [77] and further developed in [78]. This might, at first, not seem too profound until one notices that a free field is actually a type of Galileon with $c_n = 0$ for $n \geq 3$. This means there is a Galileon with a special choice of coefficients that is dual to just a free field. A free field has no interactions so it must be, even though it wouldn’t appear at first glance, that this special Galileon also has no interactions.

In this chapter we then use this special Galileon field to define a new quantum operator for the Galileon field, $\hat{\pi}$, and compute its Wightman function. We will show that this Wightman function is dominated by many soft quanta just like Classicalization predicts and is ill-defined in position space signaling an inherit gravitational non-locality.
4.1 A Coordinate Transformation

The origin of this Galileon Duality is through a field dependent coordinate transformation,

\[ \tilde{x}^\mu = x^\mu + \frac{\partial^\mu \pi(x)}{\Lambda^\sigma}, \]  

(4.1)

where \( \sigma = d/2 + 1 \). Here \( \pi(x) \) is our Galileon field. We define the dual Galileon field, \( \tilde{\pi}(\tilde{x}) \), through the inverse transformation,

\[ x^\mu = \tilde{x}^\mu - \frac{\tilde{\partial}^\mu \tilde{\pi}(\tilde{x})}{\Lambda^\sigma}. \]  

(4.2)

Some useful consequences of this transformation are that

\[ \int d^d x = \int d^d \tilde{x} \det \left( \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \right) = \int d^d \tilde{x} \det \left( \delta^\mu_\nu - \frac{\tilde{\partial}^\mu \tilde{\partial}_\nu \tilde{\pi}(\tilde{x})}{\Lambda^\sigma} \right), \]  

(4.3)

and defining

\[ \tilde{\Pi}^\mu_\nu \equiv \frac{\tilde{\partial}^\mu \tilde{\partial}_\nu \tilde{\pi}(\tilde{x})}{\Lambda^\sigma}, \]  

(4.4)

we have, in matrix notation,

\[ \int d^d x = \int d^d \tilde{x} \det \left( 1 - \tilde{\Pi} \right). \]  

(4.5)
Similarly we have

\[ \int d^d\bar{x} = \int d^d x \ \det(1 + \Pi) . \]  \hspace{1cm} (4.6)

where once again we have defined

\[ \Pi_{\mu}^{\nu} \equiv \frac{\partial \mu \partial_{\nu} \pi(x)}{\Lambda_{\sigma}} , \]  \hspace{1cm} (4.7)

which leads us to

\[ \det(1 + \Pi) = \frac{1}{\det(1 - \tilde{\Pi})} . \]  \hspace{1cm} (4.8)

By putting (4.1) into (4.2) we have

\[ \partial_{\mu} \pi(x) = \tilde{\partial}_{\mu} \tilde{\pi}(\bar{x}) . \]  \hspace{1cm} (4.9)

By varying (4.1) we have for the field fluctuations,

\[ \partial_{\mu}(\delta \pi - \delta \tilde{\pi}) = 0 , \]  \hspace{1cm} (4.10)

which gives us the relation \( \delta \pi = \delta \tilde{\pi} \). This will be useful when we equate the equations of motion from two Galileon actions. Let us see what happens when we do this transformation on a general Galileon action.
4.2 Classical Map

We will first show the equivalence of two Galileon theories at the classical level. This means that we will show that two seemingly different actions actually produce the same equations of motion. This is what we mean when we say two theories are equivalent at the classical level.

4.2.1 How the Galileon Transforms

We start by writing the most general Galileon in arbitrary \(d\)-dimensions as

\[
S[\pi] = \int d^d x \Lambda^{d+2} \sum_{n=2}^{d+1} c_n \epsilon^{\mu_1 \ldots \mu_d} \epsilon^{\nu_1 \ldots \nu_d} \frac{\pi(x)}{\Lambda^\sigma} \prod_{j=1}^{n-1} \Pi_{\mu_j \nu_j} \prod_{k=n}^{d+1} \eta_{\mu_k \nu_k},
\]

(4.11)

\[
= \int d^d x \sum_{n=2}^{d+1} c_n L_n[\pi(x)],
\]

(4.12)

where, once again, \(\Lambda\) is the strong coupling scale with dimensions of mass, \(\epsilon\) is the Levi-Civita symbol, the \(c_n\)'s are a set of arbitrary constants, \(L_n[\pi(x)]\) is the \(n\)th Galileon Lagrangian density, and we will only be concerned with a flat background metric, \(\eta_{\mu\nu}\). We will take the equations of motion from this action, do the transformation, and find another action that would give us the same equations of motion. Therefore, since two actions give us the same equations of motion, we say they are classically equivalent. After varying \(\pi(x)\) the equations of motion of (4.11) are

\[
\delta S[\pi] = \int d^d x \Lambda^{d+2} \left[ \sum_{n=2}^{d+1} nc_n \epsilon^{\mu_1 \ldots \mu_d} \epsilon^{\nu_1 \ldots \nu_d} \frac{\prod_{j=1}^{n-1} \Pi_{\mu_j \nu_j} \prod_{k=n}^{d+1} \eta_{\mu_k \nu_k}}{\Lambda^\sigma} \right] \delta \pi(x),
\]

(4.13)
which we will now write in shorthand notation by suppressing all index contractions as

\[ \delta S[\pi] = \int d^d x \: \Lambda^{d+2} \left[ \sum_{n=2}^{d+1} n c_n \epsilon \epsilon \Pi^{n-1} \eta^{d-n+1} \right] \frac{\delta \pi(x)}{\Lambda^\sigma}, \tag{4.14} \]

We now notice that, with implied index contractions,

\[
(d - n + 1)! \frac{\partial^{n-1}}{\partial \mu^{n-1}} \det(\eta + \mu \Pi)_{|\mu=0} = \frac{(d - n + 1)!}{d!} \epsilon \epsilon \frac{\partial^{n-1}}{\partial \mu^{n-1}} (\eta + \mu \Pi)|_{\mu=0}^{d-n+1} = \epsilon \epsilon \Pi^{n-1} \eta^{d-n+1}, \tag{4.15}
\]

which lets us write \(4.13\) as

\[ \delta S[\pi] = \int d^d x \: \Lambda^{d+2} \left[ \sum_{n=2}^{d+1} n c_n (d - n + 1)! \frac{\partial^{n-1}}{\partial \mu^{n-1}} \det(\eta + \mu \Pi)|_{\mu=0} \right] \frac{\delta \pi(x)}{\Lambda^\sigma}. \tag{4.16} \]

We now write

\[
\eta + \mu \Pi = \mu(\eta + \Pi) + \eta(1 - \mu) \]
\[
= \frac{\eta + (\mu - 1) \Pi}{\eta - \Pi}, \tag{4.17}
\]

where in the last line we used the duality transformation. The only other two quantities that transform under the duality are the integration measure and \(\delta \pi\) which
trivially transforms as $\delta \pi = \delta \tilde{\pi}$. This transforms our total equations of motion to

$$
\delta S[\tilde{\pi}] = \int d^d \tilde{x} \det(\eta - \tilde{\Pi}) \Lambda^{d+2} \times \left[ \sum_{n=2}^{d+1} nc_n(d - n + 1)! \frac{\partial^{n-1}}{\partial \mu^{n-1}} \det \left( \frac{\eta + (\mu - 1)\tilde{\Pi}}{\eta - \tilde{\Pi}} \right) \right] \delta \tilde{\pi}(\tilde{x}) \Lambda^\sigma \\
= \int d^d \tilde{x} \det(\eta - \tilde{\Pi}) \Lambda^{d+2} \times \left[ \sum_{n=2}^{d+1} nc_n(d - n + 1)! \frac{\partial^{n-1}}{\partial \mu^{n-1}} \det(\eta + (\mu - 1)\tilde{\Pi}) \right] \delta \tilde{\pi}(\tilde{x}) \Lambda^\sigma .
$$

(4.18)

Now, from (4.15), we can write

$$(d - n + 1)! \frac{\partial^{n-1}}{\partial \mu^{n-1}} \det(\eta + (\mu - 1)\tilde{\Pi})|_{\mu=0} = \epsilon \epsilon \sum_{k=0}^{d-n+1} (-1)^k k! (d-n+1)! \tilde{\Pi}^{n+k-1} \eta^{d-n-k+1} .
$$

(4.19)

and using the binomial expansion,

$$(x + y)^r = \sum_{k=0}^r \frac{r!}{k!(r-k)!} x^k y^{r-k} .
$$

(4.20)

we can write (4.15) as

$$(d - n + 1)! \frac{\partial^{n-1}}{\partial \mu^{n-1}} \det(\eta + (\mu - 1)\tilde{\Pi})|_{\mu=0} = \epsilon \epsilon \sum_{k=0}^{d-n+1} (-1)^k k! (d-n-k+1)! \tilde{\Pi}^{n+k-1} \eta^{d-n-k+1} .
$$

(4.21)
Going back to (4.18) we are lead to

\[
\delta S[\tilde{\pi}] = \int d^{d+2} \bar{\Lambda} \left[ \epsilon \sum_{n=2}^{d+1} \frac{\sum_{k=0}^{n-1} (-1)^k (d - n + 1)!}{k!(d - n - k + 1)!} \tilde{\Pi}^{n+k-1} \eta^{d-n-k+1} \right] \frac{\delta \tilde{\pi}(\bar{x})}{\bar{\Lambda}^\sigma},
\]

(4.22)

while still suppressing all index contractions. We now define \( m = n + k \) and then redefine \( n \to k \) and \( m \to n \) to give us

\[
\delta S[\tilde{\pi}] = \int d^{d+2} \bar{\Lambda} \left[ \epsilon \sum_{k=2}^{d+1} \sum_{n=k}^{d+1} c_{nk} \frac{k(-1)^{n-k}(d - k + 1)!}{(n-k)!(d-n+1)!} \tilde{\Pi}^{n-1} \eta^{d-n+1} \right] \frac{\delta \tilde{\pi}(\bar{x})}{\bar{\Lambda}^\sigma}.
\]

(4.23)

Now, to put this in a more convenient form we notice

\[
\sum_{k=2}^{d+1} \sum_{n=k}^{d+1} f(n,k) = \sum_{n=2}^{d+1} \sum_{k=2}^{d+1} f(n,k) \theta(n-k),
\]

(4.24)

to let us write

\[
\delta S[\tilde{\pi}] = \int d^{d+2} \bar{\Lambda} \left[ \epsilon \sum_{n=2}^{d+1} \sum_{k=2}^{d+1} \theta(n-k) c_{nk} \frac{k(-1)^{n-k}(d - k + 1)!}{(n-k)!(d-n+1)!} \tilde{\Pi}^{n-1} \eta^{d-n+1} \right] \frac{\delta \tilde{\pi}(\bar{x})}{\bar{\Lambda}^\sigma}.
\]

(4.25)
Now we define a set of $p_n$’s such that
\[
np_n \equiv (-1)^n \sum_{k=2}^{d+1} \theta(n-k)(-1)^k c_k \frac{k(d-k+1)!}{(n-k)!(d-n+1)!} ,
\] (4.26)
such that (4.25) now becomes
\[
\delta S[\tilde{\pi}] = \int d^d \tilde{x} \Lambda^{d+2} \left[ \epsilon \epsilon \sum_{n=2}^{d+1} np_n \tilde{\Pi}^{n-1} \eta^{d-n+1} \right] \frac{\delta \tilde{\pi}(\tilde{x})}{\Lambda^{d}} ,
\] (4.27)

Equating these equations of motions with the equations of motion for $\pi(x)$, (4.14) and the action from which comes from, (4.11), we see that the equations of motion for $\tilde{\pi}(\tilde{x})$ come from the action
\[
S[\tilde{\pi}] = \int d^d \tilde{x} \Lambda^{d+2} \sum_{n=2}^{d+1} p_n \epsilon^{\mu_1 \ldots \mu_d} \epsilon^{\nu_1 \ldots \nu_d} \tilde{\pi}(\tilde{x}) \tilde{\Pi}^{n-1} \prod_{j=1}^{n-1} \tilde{\Pi}_{\mu_j \nu_j} \prod_{k=n}^{d+1} \eta_{\mu_k \nu_k} ,
\] (4.28)
which is just another general Galileon which is related to our original through the relationship between and $c_n$’s and the $p_n$’s being
\[
p_n = \frac{(-1)^n}{n} \sum_{k=2}^{d+1} (-1)^k c_k \frac{k \Gamma[d-k+2]}{\Gamma[n-k+1] \Gamma[d-n+2]} ,
\] (4.29)
where we have $n! = \Gamma[n+1]$ to account for the $\theta(n-k)$ in (4.26). Therefore, we have our original Galileon, $\pi(x)$, related to another Galileon, $\tilde{\pi}(x)$, by
\[
S[\pi] = \int d^d x \sum_{n=2}^{d+1} c_n \mathcal{L}_n[\pi] = \int d^d x \sum_{n=2}^{d+1} p_n \mathcal{L}_n[\tilde{\pi}] = S[\tilde{\pi}] .
\] (4.30)
4.2.2 A Special Galileon

A free field can be thought of as a Galileon with \( c_n = 0 \) for \( n \geq 3 \). In 4D, (4.29) gives us

\[
\begin{align*}
p_2 & = c_2 , \\
p_3 & = \frac{1}{3}(-6c_2 + 3c_3) , \\
p_4 & = \frac{1}{4}(6c_2 - 6c_3 + 4c_4) , \\
p_5 & = \frac{1}{5}(-2c_2 + 3c_3 - 4c_4 + 5c_5) ,
\end{align*}
\]

and if we choose to canonically normalize our kinetic term \((\mathcal{L}_2)\), we choose \( c_2 = -1/12 = p_2 \) and if we also insist that \( p_3 = p_4 = p_5 = 0 \) we end up with

\[
S_{\text{Galileon}}[\pi] = \int d^4x \left[ -\frac{1}{12}\mathcal{L}_2[\pi] - \frac{1}{6}\mathcal{L}_3[\pi] - \frac{1}{8}\mathcal{L}_4[\pi] - \frac{1}{30}\mathcal{L}_5[\pi] \right]
\]

(4.31)

\[
= \int d^4x \left[ -\frac{1}{12}\mathcal{L}_2[\rho] = S_{\text{free}}[\rho] \right]
\]

(4.32)

where we have set the dual Galileon field \( \tilde{\pi} \to \rho \) to indicate this special choice of \( c_n \)'s. We see that with this choice, this special Galileon field \( \pi \) is dual to a free field, that is a field with no interactions. All interesting and complicated dynamics one would believe (4.31) should posses must in the end be trivial since a free field has no interesting dynamics to speak of. This profound result points towards subtleties to Galileon theories that have not been observed at the perturbative level. We argue that these interesting characteristics in Galileon theories have not been observed because of a breakdown of perturbation theory. This special Galileon will be essential for
us to create a concrete mathematical framework to analyze, at least for this special
Galileon, analytically. We will use this special Galileon field which is dual to a free
field to calculate quantum quantities in a nonperturbative, Vainshtein relevant, fashion
and discuss interesting conclusions of the quantum theory of Galileons using these
calculations. Therefore, in what follows for the rest of the chapter, we will only be
concerned with this special Galileon with the specific coefficients defined in (4.31).

Knowing what we now know, we can write this special Galileon that is dual to a
free field in a more convenient fashion,

\[
S_{\text{Galileon}}[\pi] = \int d^d x \left[ -\frac{1}{12} \mathcal{L}_2[\pi] - \frac{1}{6} \mathcal{L}_3[\pi] - \frac{1}{8} \mathcal{L}_4[\pi] - \frac{1}{30} \mathcal{L}_5[\pi] \right] \\
= \int d^d x \ \det(1 + \Pi) \left( -\frac{1}{2} \right) (\partial^2 \pi). 
\] (4.33)

Which, using (4.6) and (4.9), we can easily see

\[
S_{\text{Galileon}}[\pi] = \int d^d x \ \det(1 + \Pi) \left( -\frac{1}{2} \right) (\partial^2 \pi) = \int d^d x \ -\frac{1}{2} (\partial^2 \rho) = S_{\text{free}}[\rho]. 
\] (4.35)

### 4.2.3 Coupling to Matter

As discussed in [78], we can straightforwardly see how this duality holds under the
matter couplings. For instance, under this transformation a scalar field, \( \chi(x) \) would
transform as

\[
\chi(x) = \chi \left( \tilde{x}^\mu - \frac{\tilde{\partial}^\mu \tilde{\pi}(\tilde{x})}{\Lambda^\sigma} \right) \equiv \tilde{\chi}(\tilde{x}) 
\] (4.36)
where the last equality is simply doing a field transformation which keeps the scalar as a local function of $\tilde{x}$. We define the dual vector field to be

$$
\tilde{V}_\mu(\tilde{x}) = \frac{\delta x^\nu}{\delta \tilde{x}^\mu} V_\nu(x) = \left[1 - \tilde{\Pi}(\tilde{x})\right]_\nu^\mu V_\nu(x) ,
$$

(4.37)

and likewise any tensor transforms as

$$
\tilde{T}_{\mu_1...\mu_n}(\tilde{x}) = \left[1 - \tilde{\Pi}(\tilde{x})\right]_{\nu_1}^{\mu_1} ... \left[1 - \tilde{\Pi}(\tilde{x})\right]_{\nu_n}^{\mu_n} T_{\nu_1...\nu_n} .
$$

(4.38)

Therefore, by looking at some arbitrary scalar coupling we have the dual expression,

$$
\int d^4 x \mathcal{L}(\chi(x), \partial_\mu \chi(x), \pi(x), \partial_\mu \pi(x)) = \int d^d \tilde{x} \det(1 - \tilde{\Pi}) \mathcal{L}\left(\tilde{\chi}(\tilde{x}), \left[\left(1 - \tilde{\Pi}\right)^{-1}\right]^\nu_\mu \tilde{\partial}_\nu \tilde{\chi}(\tilde{x}), \rho(\tilde{x}) - \frac{1}{2} \frac{(\tilde{\partial} \rho(\tilde{x}))^2}{\Lambda^\alpha}, \tilde{\partial}_\mu \rho(\tilde{x})\right) ,
$$

(4.39)

where we have used (4.51). We can even consider an example of some perturbatively renormalizable theory,

$$
S = \int d^4 x \left[-\frac{1}{2} (\partial \pi)^2 - \frac{1}{2} (\partial \chi)^2 - \frac{1}{2} m_\pi^2 \pi^2 - \frac{1}{2} m_\chi^2 \chi^2 - \frac{1}{2} g \pi^2 \chi^2 - \frac{1}{2!} \lambda_\pi \pi^4 - \frac{1}{4!} \lambda \chi^4 \right] ,
$$

(4.40)
and upon doing the duality we get

\[ S = \int d^4x \det(1 - \tilde{\Pi}) \left[ \frac{1}{2} (\partial \rho)^2 - \frac{1}{2} \eta^{\alpha\beta} \left[ (1 - \tilde{\Pi})^{-1} \right]_{\alpha}^{\mu} \left[ (1 - \tilde{\Pi})^{-1} \right]_{\beta}^{\nu} \partial_\mu \tilde{\pi} \partial_\nu \tilde{\pi} \right. \\
- \frac{1}{2} m_{\pi}^2 \left( \rho(x) - \frac{1}{2} \frac{(\partial \rho(x))^2}{\Lambda^3} \right)^2 - \frac{1}{2} m_{\chi}^2 \chi^2 - \frac{1}{2} g \left( \rho(x) - \frac{1}{2} \frac{(\partial \rho(x))^2}{\Lambda^3} \right)^2 \tilde{\chi}^2 \\
- \frac{1}{4!} \lambda_{\pi} \left( \rho(x) - \frac{1}{2} \frac{(\partial \rho(x))^2}{\Lambda^3} \right)^4 - \frac{1}{4!} \lambda_{\chi} \tilde{\chi}^4 \right], \tag{4.41} \]

where since $\tilde{x}$ is a dummy variable we have let $\tilde{x} \to x$ inside the integral. Naively, (4.41) looks extremely non-renormalizable with its many irrelevant operators however since it is dual to a renormalizable theory, it must be that any regularization scheme that is invariant under this duality must admit that (4.41) is perturbatively renormalizable.

### 4.3 Quantum Map

This research revolves around finding a formal quantum definition of Galileon theories. Using this Galileon duality, we can at least define a quantum theory for the special Galileon that is dual to a free field. However, we speculate that the physical characteristics of this special theory also carry over to the more general Galileon theories. We give evidence for this in Section 5.4.1.

We aim to find a formal expression for the full non-perturbative Wightman function $\langle 0 \mid \hat{\pi}(x) \hat{\pi}(x') \mid 0 \rangle$. We will rely heavily on this Galileon duality defined as the coordinate transformations in (4.1) and (4.2). However, to start we first need to define what it is we mean by $\hat{\pi}(x)$. However, in order to define the quantum operator,
\( \ddot{\pi}(x) \), let us first look at just the \textit{classical field}, \( \pi(x) \). Doing a Fourier transform we have

\[
\pi(x) = \int \frac{d^d k}{(2\pi)^d} \pi(k) e^{i k \cdot x}.
\] (4.42)

Now, doing an inverse Fourier transform we have

\[
\pi(k) = \int d^d x \pi(x) e^{-i k \cdot x}.
\] (4.43)

We want to apply the Galileon transformation onto this expression however before we do we must determine how \( \pi(x) \) transforms under the transformation. By doing a field redefinition of

\[
\pi(x) = -\frac{1}{2} x^2 \Lambda^\sigma + \phi(x),
\] (4.44)

\[
\ddot{\pi}(\ddot{x}) = 1/2 \ddot{x}^2 \Lambda^\sigma + \ddot{\phi}(\ddot{x}),
\] (4.45)

our original coordinate transformations, \([4.1]\) and \([4.2]\), become

\[
\ddot{x}^\mu = \partial^\mu \left( \frac{\phi(x)}{\Lambda^\sigma} \right),
\] (4.46)

\[
x^\mu = -\tilde{\partial}^\mu \left( \frac{\ddot{\phi}(\ddot{x})}{\Lambda^\sigma} \right).
\] (4.47)

Now, relating these expressions through a Legendre transform, for instance with La-
grangian and Hamiltonian mechanics we have

\[ \dot{x} = \frac{\partial H}{\partial p}, \quad p = \frac{\partial L}{\partial \dot{x}}, \quad H = p\dot{x} - L. \]  (4.48)

Now, relating

\[ \dot{x} \equiv \tilde{x}, \quad p \equiv x, \quad L \equiv -\tilde{\phi}(\tilde{x}) \Lambda^{\sigma}, \quad H \equiv \phi(x) \Lambda^{\sigma}, \]  (4.49)

(4.48) gives us

\[ \frac{\phi(x)}{\Lambda^{\sigma}} = x_\mu\tilde{x}^\mu + \frac{\tilde{\phi}(\tilde{x})}{\Lambda^{\sigma}}. \]  (4.50)

Transforming back into \( \pi(x) \) and \( \tilde{\pi}(\tilde{x}) \) we have

\[ \frac{\pi(x)}{\Lambda^{\sigma}} = -\frac{1}{2} \tilde{x}^2 - \frac{1}{2} x^2 + x_\mu\tilde{x}^\mu + \frac{\tilde{\pi}(\tilde{x})}{\Lambda^{\sigma}} = -\frac{1}{2}(x^\mu - \tilde{x}^\mu)^2 + \frac{\tilde{\pi}(\tilde{x})}{\Lambda^{\sigma}} = -\frac{1}{2} \left( \tilde{x}^\mu - \frac{\tilde{\partial}\tilde{\pi}(\tilde{x})}{\Lambda^{\sigma}} - \tilde{x}^\mu \right)^2 + \frac{\tilde{\pi}(\tilde{x})}{\Lambda^{\sigma}} \rightarrow \pi(x) = \tilde{\pi}(\tilde{x}) - \frac{1}{2} \frac{(\tilde{\partial}\tilde{\pi}(\tilde{x}))^2}{\Lambda^{\sigma}}. \]  (4.51)

Since we are specifically concerned with the special Galileon that is dual to a free field, \( \rho \), we have

\[ \pi(x) = \rho(\tilde{x}) - \frac{1}{2} \frac{(\tilde{\partial}\rho(\tilde{x}))^2}{\Lambda^{\sigma}}. \]  (4.52)
We are now ready to apply the Galileon transformation onto (4.43) by equating

\[ \int d^d x = \int d^d \tilde{x} \det(1 - \tilde{\Sigma}) , \]  
(4.53)

\[ \tilde{\Sigma}_\mu^\nu \equiv \frac{\tilde{\partial}_\mu \tilde{\partial}_\nu \rho(\tilde{x})}{\Lambda^\sigma} , \]  
(4.54)

\[ \pi(x) = \rho(\tilde{x}) - \frac{1}{2} \frac{(\tilde{\partial} \rho(\tilde{x}))^2}{\Lambda^\sigma} , \]  
(4.55)

\[ x^\mu = \tilde{x}^\mu - \frac{\tilde{\partial}^\mu \rho(\tilde{x})}{\Lambda^\sigma} , \]  
(4.56)

we have

\[ \pi(k) = \int d^d x \pi(x) e^{-ik \cdot x} = \int d^d \tilde{x} \det(1 - \tilde{\Sigma}) \left( \rho(\tilde{x}) - \frac{1}{2} \frac{(\tilde{\partial} \rho(\tilde{x}))^2}{\Lambda^\sigma} \right) e^{-ik \cdot \tilde{x}} e^{ik \cdot \tilde{\partial} \rho(\tilde{x})/\Lambda^\sigma} \]

\[ = \int d^d x \det(1 - \Sigma) \left( \rho(x) - \frac{1}{2} \frac{(\partial \rho(x))^2}{\Lambda^\sigma} \right) e^{-ik \cdot x} e^{ik \cdot \partial \rho(x)/\Lambda^\sigma} \]

\[ = \int d^d x \ U(\rho(x)) e^{-ik \cdot x} e^{ik \cdot \partial \rho(x)/\Lambda^\sigma} , \]  
(4.57)

where

\[ U(\rho(x)) = \det(1 - \Sigma) \left( \rho(x) - \frac{1}{2} \frac{(\partial \rho(x))^2}{\Lambda^\sigma} \right) , \]  
(4.58)

incorporates all polynomial pieces in \( \rho(x) \) which will be sub-dominate to our exponential arguments created from the \( e^{ik \cdot \partial \rho(x)/\Lambda^\sigma} \) contribution therefore we will not be concerned with the precise form of \( U(\rho(x)) \). Now, by doing another Fourier transform
into position space we have,

$$\pi(x) = \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot x} \pi(k) = \int \frac{d^d k}{(2\pi)^d} \int d^d y \ e^{ik \cdot (x-y)} U(\rho(y)) e^{ik \cdot \rho(y)/\Lambda^\sigma}. \quad (4.59)$$

We are now ready to define our Galileon quantum field operator, \( \hat{\pi}(x) \) as

$$\hat{\pi}(x) \equiv \int \frac{d^d k}{(2\pi)^d} \int d^d y \ e^{ik \cdot (x-y)} U(\hat{\rho}(y)) e^{ik \cdot \hat{\rho}(y)/\Lambda^\sigma} :,$$  \quad (4.60)

where \( \hat{\rho}(y) \) is our \( d \)-dimensional free quantum field operator defined in (2.78),

$$\hat{\rho}(y) = \int \frac{d^{d-1} k}{(2\pi)^{d-1}} \frac{1}{\sqrt{2\omega_k}} \left( \hat{a}_k e^{ik \cdot y} + \hat{a}_k^\dagger e^{-ik \cdot y} \right). \quad (4.61)$$

and :: is an instruction to follow the normal ordering prescription for the free field \( a \) and \( a^\dagger \)'s as described in 2.3.3.

### 4.4 Galileon Wightman Function

Let us take this definition and see what it says for the Wightman function for the Galileon, \( \langle 0 | \hat{\pi}(x) \hat{\pi}(x') | 0 \rangle \). We first say

$$\langle 0 | \hat{\pi}(x) \hat{\pi}(x') | 0 \rangle = \int \frac{d^d k}{(2\pi)^d} \int d^d y \ e^{ik \cdot (x-y)} \int \frac{d^{d'} k'}{(2\pi)^d} \int d^d y' \ e^{ik' \cdot (x'-y')} \times \langle 0 | U(\hat{\rho}(y)) e^{ik \cdot \hat{\rho}(y)/\Lambda^\sigma} :: U(\hat{\rho}(y')) e^{ik' \cdot \hat{\rho}(y')/\Lambda^\sigma} : | 0 \rangle.$$  \quad (4.62)
4.4.1 Translation Invariance

To simplify (4.62) let us consider some operator of $\hat{\rho}$, $O(\hat{\rho})$, and look at

$$\langle 0 | T(O(\hat{\rho}(y))O(\hat{\rho}(y'))) | 0 \rangle = \int D\rho(x) \ O(\rho(y))O(\rho(y'))e^{iS_{\text{free}}[\rho(x)]}$$

$$= \int D\rho(x) \ O(\rho(y - y' + y'))O(\rho(y'))e^{iS_{\text{free}}[\rho(x)]} ,$$

(4.63)

where we are now working in the path integral language. Now if we shift all spacetime points by $-y'$ by doing a field redefinition we have

$$\int D\rho(x) \ O(\rho(y - y' + y'))O(\rho(y'))e^{iS_{\text{free}}[\rho(x)]}$$

$$= \int D\rho(x - y') \ O(\rho(y'))O(\rho(0))e^{iS_{\text{free}}[\rho(x-y')]},$$

(4.64)

and using the invariance of

$$\int D\rho(x - y') = \int D\rho(x),$$

(4.65)

$$S_{\text{free}}[\rho(x - y')] = S_{\text{free}}[\rho(x)],$$

(4.66)

we have

$$\langle 0 | T(O(\hat{\rho}(y))O(\hat{\rho}(y'))) | 0 \rangle = \langle 0 | T(O(\hat{\rho}(y - y'))O(\hat{\rho}(0))) | 0 \rangle .$$

(4.67)
This is known as translation invariance which is the statement that correlations only depend on the difference between spacetime points, not the spacetime points themselves. Knowing this, we may write (4.62) as

$$\langle 0 | \hat{\pi}(x)\hat{\pi}(x') | 0 \rangle = \int \frac{d^d k}{(2\pi)^d} \int d^d y \ e^{i k \cdot (x - y)} \int \frac{d^d k'}{(2\pi)^d} \int d^d y' \ e^{i k' \cdot (x' - y')} \times \langle 0 | : U(\hat{\rho}(y - y')) e^{i k \cdot \partial \hat{\rho}(y - y')/\Lambda^\sigma} :: U(\hat{\rho}(0)) e^{i k' \cdot \partial \hat{\rho}(0)/\Lambda^\sigma} : | 0 \rangle .$$

(4.68)

Now, switching variables from $y$ and $y'$ to $z$ and $z'$ through

$$z = y - y' , \quad (4.69)$$
$$z' = y + y' , \quad (4.70)$$

(4.68) becomes

$$\langle 0 | \hat{\pi}(x)\hat{\pi}(x') | 0 \rangle = \int d^d z \int \frac{d^d k}{(2\pi)^d} \int d^d z' \int \frac{d^d k'}{(2\pi)^d} \ e^{i k \cdot z + i k' \cdot z'} \times e^{-\frac{i z'}{2} (k + k')} e^{-\frac{i z}{2} (k - k')} \times \langle 0 | : U(\hat{\rho}(z)) e^{i k \cdot \partial \hat{\rho}(z)/\Lambda^\sigma} :: U(\hat{\rho}(0)) e^{i k' \cdot \partial \hat{\rho}(0)/\Lambda^\sigma} : | 0 \rangle .$$

(4.71)
Now, doing the $z'$ integral introduces a $\delta^{(d)}(k + k')$ which then makes the $k'$ integral trivial giving us

$$\langle 0 \mid \hat{\pi}(x)\hat{\pi}(x') \mid 0 \rangle = \int d^d z \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot (x-x')} e^{-ik \cdot z}$$

$$\times \langle 0 \mid U(\hat{\rho}(z)) e^{ik \cdot \partial \hat{\rho}(z)} / \Lambda^\sigma :: U(\hat{\rho}(0)) e^{-ik \cdot \partial \hat{\rho}(0)} / \Lambda^\sigma : 0 \rangle. \quad (4.72)$$

Now, using an analogous result as (2.104) we have

$$\langle 0 \mid \hat{\pi}(x)\hat{\pi}(x') \mid 0 \rangle = \int d^d z \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot (x-x')} e^{-ik \cdot z} F(k,z)$$

$$\times \langle 0 \mid e^{ik \cdot \partial \hat{\rho}(z)} / \Lambda^\sigma :: e^{-ik \cdot \partial \hat{\rho}(0)} / \Lambda^\sigma : 0 \rangle, \quad (4.73)$$

where $F(k,z)$ is from the polynomial in $\hat{\rho}$ pieces coming from the $U(\hat{\rho})$’s on the generating function $\langle 0 \mid e^{ik \cdot \partial \hat{\rho}(z)} / \Lambda^\sigma :: e^{-ik \cdot \partial \hat{\rho}(0)} / \Lambda^\sigma : 0 \rangle$. To continue we now focus on this generating function piece. We may write

$$\langle 0 \mid e^{ik \cdot \partial \hat{\rho}(z)} / \Lambda^\sigma :: e^{-ik \cdot \partial \hat{\rho}(0)} / \Lambda^\sigma : 0 \rangle$$

$$= \langle 0 \mid \sum_{n=0}^\infty \frac{1}{n!} \left( \frac{ik^\mu \partial_{\mu} \hat{\rho}(z)}{\Lambda^\sigma} \right)^n :: \sum_{m=0}^\infty \frac{1}{m!} \left( \frac{-ik^\nu \partial_{\nu} \hat{\rho}(0)}{\Lambda^\sigma} \right)^m : 0 \rangle$$

$$= \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{1}{n! m!} \left( \frac{ik^\mu}{\Lambda^\sigma} \right)^n \left( \frac{-ik^\nu}{\Lambda^\sigma} \right)^m (\partial_{\mu})^n (\partial_{\nu})^m \langle 0 \mid \hat{\rho}(z)^n :: \hat{\rho}(0)^m : 0 \rangle. \quad (4.74)$$
The normal ordering prescription effectively makes all divergent coincidence limit contractions zero. This enforces that \( m = n \) which gives us

\[
\langle 0 \mid e^{i k \cdot \hat{\rho}(z) / \Lambda^\sigma} \hat{\rho}^{\prime} e^{-i k \cdot \hat{\rho}(0) / \Lambda^\sigma} \mid 0 \rangle
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left( \frac{k^\mu \partial^\mu}{\Lambda^\sigma} \right)^n \left( \frac{k^\nu \partial^\nu}{\Lambda^\sigma} \right)^n \langle 0 \mid \hat{\rho}(z)^n \hat{\rho}(0)^n \mid 0 \rangle . \tag{4.75}
\]

Using Wick’s Theorem we may say

\[
\langle 0 \mid \hat{\rho}(z)^n \hat{\rho}(0)^n \mid 0 \rangle = n! (\langle 0 \mid \hat{\rho}(z) \hat{\rho}(0) \mid 0 \rangle)^n , \tag{4.76}
\]

which gives us as a convenient form,

\[
\langle 0 \mid e^{i k \cdot \hat{\rho}(z) / \Lambda^\sigma} \hat{\rho}^{\prime} e^{-i k \cdot \hat{\rho}(0) / \Lambda^\sigma} \mid 0 \rangle
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{k^\mu \partial^\mu}{\Lambda^\sigma} \right)^n \left( \frac{k^\nu \partial^\nu}{\Lambda^\sigma} \right)^n \langle 0 \mid \hat{\rho}(z) \hat{\rho}(0) \mid 0 \rangle^n
\]

\[
= e^{\frac{k^\mu k^\nu}{\Lambda^2} \partial^\mu \partial^\nu (0) \hat{\rho}(0) \hat{\rho}(0)} . \tag{4.77}
\]

This gives us for the Wightman function,

\[
\langle 0 \mid \hat{\pi}(x) \hat{\pi}(x') \mid 0 \rangle = \int \frac{d^d z}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} e^{i k \cdot (x-x')} e^{-i k \cdot z} F(k, z) e^{\frac{k^\mu k^\nu}{\Lambda^2} \partial^\mu \partial^\nu (0) \hat{\rho}(z) \hat{\rho}(0)} . \tag{4.78}
\]
We can write \( \langle 0 \mid \hat{\rho}(z)\hat{\rho}(0) \mid 0 \rangle \) in \( d \)-dimensions as

\[
\langle 0 \mid \hat{\rho}(z)\hat{\rho}(0) \mid 0 \rangle = \frac{1}{4\pi^{d/2}} \frac{\Gamma[d/2 - 1]}{(-i\epsilon)^2 + (\vec{z}^2)^{(d-2)/2}} \frac{a_d}{(d-2)} \frac{1}{(-z^0 - i\epsilon)^2 + (\vec{z}^2)^{(d-2)/2}} \cdot
\]

(4.79)

where the last line defines \( a_d \) which will prove to be convenient. Focusing on the derivative contribution to the exponential we have \( \partial_\mu \partial_\nu \langle 0 \mid \hat{\rho}(z)\hat{\rho}(0) \mid 0 \rangle \) where by this expression we really mean

\[
\partial_\mu \partial_\nu \langle 0 \mid \hat{\rho}(z)\hat{\rho}(0) \mid 0 \rangle \rightarrow \lim_{x \rightarrow z} \lim_{y \rightarrow 0} \left[ \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \langle 0 \mid \hat{\rho}(x)\hat{\rho}(y) \mid 0 \rangle \right]
\]

\[
= \frac{a_d}{(d-2)} \lim_{x \rightarrow z} \lim_{y \rightarrow 0} \left[ \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \frac{1}{(x_\nu - y_\nu)} \right]
\]

\[
= a_d \lim_{x \rightarrow z} \lim_{y \rightarrow 0} \left[ \frac{\partial}{\partial x^\mu} \frac{1}{(-x^0 - y^0 - i\epsilon)^2 + (\vec{x} - \vec{y})^2)^{(d-2)/2}} \right]
\]

\[
= a_d \lim_{x \rightarrow z} \lim_{y \rightarrow 0} \left[ \eta_{\mu\nu} \frac{(x_\mu - y_\mu)(x_\nu - y_\nu)}{(-x^0 - y^0 - i\epsilon)^2 + (\vec{x} - \vec{y})^2)^{(d+2)/2}} \right]
\]

\[
= a_d \left( \frac{\eta_{\mu\nu}}{(-x^0 - i\epsilon)^2 + (\vec{z}^2)^{(d+2)/2}} - d \frac{z_\mu z_\nu}{(-x^0 - i\epsilon)^2 + (\vec{z}^2)^{(d+2)/2}} \right)
\]

(4.80)

where we have defined

\[
\frac{1}{z^d} = \frac{1}{(-x^0 - i\epsilon)^2 + (\vec{z}^2)^{d/2}}
\]

(4.81)
When all is said and done we have for the Wightman function for our Galileon operator defined in (4.60) as

\[
\langle 0 | \hat{\pi}(x)\hat{\pi}(x') | 0 \rangle = \int d^dz \int \frac{d^dk}{(2\pi)^d} e^{ik \cdot (x-x')} e^{-ik \cdot z} F(k, z) \exp \left[ \frac{a_d}{\Lambda^{2\sigma}} \left( \frac{k^2}{z^{d-2}} - d \frac{(k \cdot z)^2}{z^{d+2}} \right) \right].
\]

(4.82)

However, this expression has a problem; it breaks one of our fundamental assumptions about a UV complete theory in that for certain \( k \cdot z \) directions this two point function diverges. Some would conclude that this theory simply is sick. To further our progress we notice that although this \textit{position space} two point function diverges (aka does not exist), the \textit{momentum space} two point function for our quantum Galileon operator \textit{does} exist. This is best seen by looking at the spectral density of this Galileon operator. We will show the spectral density, \( \rho(\mu) \), will grow at an exponential rate such that the \( \mu \) integral that must be done diverges. However, since this spectral density is an entire function, we will have no issues defining this Wightman function in momentum space. The divergent \( \mu \) integral is what points to an inherit non-localizability within the theory.

### 4.5 Galileon Spectral Density

To see the connection between position space, momentum space, and this divergence let us consider the spectral density, as discussed in section 2.6 for this Galileon...
operator. Recall, for any interacting theory,

\[\langle 0 | \hat{\phi}(x)\hat{\phi}(y) | 0 \rangle = \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot(x-y)}\theta(k^0)2\pi \rho(-k^2) , \tag{4.83}\]

where we have extended this spectral density in section 2.6 to arbitrary \(d\)-dimensions.

Now, rewriting (4.82) as

\[\langle 0 | \hat{\pi}(x)\hat{\pi}(x') | 0 \rangle = \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot(x-x')} \times \left( \int d^dz e^{-ik\cdot z} F(k, z) \exp \left[ \frac{a_d}{\Lambda^2} \left( \frac{k^2}{z^d} - d \frac{(k \cdot z)^2}{z^{d+2}} \right) \right] \right) . \tag{4.84}\]

we can simply read off the spectral density for our Galileon operator to be

\[2\pi \theta(k^0) \rho(-k^2) = \int d^dz e^{-ik\cdot z} F(k, z) \exp \left[ \frac{a_d}{\Lambda^2} \left( \frac{k^2}{z^d} - d \frac{(k \cdot z)^2}{z^{d+2}} \right) \right] . \tag{4.85}\]

### 4.5.1 4D Analysis

To further our progress we will restrict ourselves to the 4-dimensional case although we conjecture all physical features of this 4-dimensional case should apply to any arbitrary \(d\) dimensional Galileon. We also conjecture that all polynomial pieces coming from \(F(k, z)\) will have only sub-dominant effects on the spectral density so for
simplicity we focus on the simplest term and say

\[
F(k, z) \rightarrow \frac{1}{(4\pi^2) z^2},
\]

\[
\rho(-k^2) \rightarrow \rho_0(-k^2).
\]

(4.86)

(4.87)

We will then focus on the effects of \( \rho_0(-k^2) \) and suppose all physical consequences carry over to the full \( \rho(-k^2) \). This conjecture is solidified in that the exponential contribution to \( \rho_0(-k^2) \) will be the relevant factor in determining high energy dynamics and that factor will be present in all terms when considering the full \( F(k, z) \). So with \( a_4 = 1/(2\pi^2) \) we have in 4-dimensions

\[
2\pi \theta(k^0) \rho_0(-k^2) = \int d^4z \ e^{-ik\cdot z} \frac{1}{(4\pi^2) z^2} \frac{1}{z^2} \exp \left[ \frac{1}{2\pi^2 \Lambda^6} \left( \frac{k^2}{z^4} - \frac{4(k \cdot z)^2}{z^6} \right) \right]
\]

...expanding the exponential...

\[
= \frac{1}{(4\pi^2)} \sum_{n=0}^{\infty} \frac{1}{(2\pi^2 \Lambda^6)^n} \frac{1}{n!} \int d^4z \ e^{-ik\cdot z} \frac{1}{z^2} \frac{k^2}{z^4} - \frac{4(k \cdot z)^2}{z^6} \right)^n
\]

...using the binomial theorem...

\[
= \frac{1}{(4\pi^2)} \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{1}{(2\pi^2 \Lambda^6)^n} \frac{1}{r!} \frac{(-4)^{n-r}}{(n-r)!} \frac{k^{2r}(k \cdot z)^{2n-2r}}{z^{6n-2r+2}}
\]

...organizing...

\[
= \frac{1}{(4\pi^2)} \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(-1)^r}{(2\pi^2 \Lambda^6)^n} \frac{4^{n-r}(-k^2)^r}{r!(n-r)!} \frac{(k \cdot z)^{2(n-r)}}{z^{2(3n-r+1)}}.
\]

(4.88)
4.5.1.1 \( N \)-Particle Phase Space Density

To solve the \( z \) integral let us first look at a seemingly irrelevant quantity,

\[
\left( \langle 0 \mid \hat{\phi}(z)\hat{\phi}(0) \mid 0 \rangle \right)^N .
\] (4.89)

We will now write

\[
\left( \langle 0 \mid \hat{\phi}(z)\hat{\phi}(0) \mid 0 \rangle \right)^N = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot z} \theta(k^0)2\pi \Omega_N(-k^2) ,
\] (4.90)

where this last line appears to be coming from left field, however, just like with the spectral density, \( \rho(-k^2) \), we are simply defining the \( N \)-particle phase space density, \( \Omega_N(-k^2) \), to be

\[
\theta(k^0)2\pi \Omega_N(-k^2) \equiv \prod_{i=1}^{N} \left[ \int \tilde{d}k_i \right] (2\pi)^4 \delta^{(4)} \left( k - \sum_{i=1}^{N} k_i \right) ,
\] (4.91)

where we are also defining

\[
\int \tilde{d}k_i \equiv \int \frac{d^4k_i}{(2\pi)^4} \theta(k_i^0)2\pi \delta(k_i^2 + \mu) .
\] (4.92)

We can see that this definition makes sense by first writing

\[
(2\pi)^4 \delta^{(4)} \left( k - \sum_{i=1}^{N} k_i \right) = \int d^4x \ e^{-i(k - \sum_{i=1}^{N} k_i)\cdot x} ,
\] (4.93)
then (4.90) becomes

$$\int \frac{d^4k}{(2\pi)^4} e^{ik\cdot z} \theta(k^0) 2\pi \Omega_N(-k^2) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot z} \prod_{i=1}^N \left[ \int d\tilde{k}_i \right] (2\pi)^4 \delta^{(4)}(k - \sum_{i=1}^N k_i)$$

$$= \int d^4x \int \frac{d^4k}{(2\pi)^4} e^{i(z-x)\cdot k} \prod_{i=1}^N \left[ \int d\tilde{k}_i \right] e^{ik_i\cdot x}$$

$$= \int d^4x \int \frac{d^4k}{(2\pi)^4} e^{i(z-x)\cdot k} \left( \int d\tilde{k}_i \right) e^{ik_i\cdot x}$$

$$= \int d^4x \int \frac{d^4k}{(2\pi)^4} e^{i(z-x)\cdot k}$$

$$\times \left( \int \frac{d^4k_i}{(2\pi)^4} \theta(k_i^0) 2\pi \delta(k_i^2 + \mu) e^{ik_i\cdot x} \right)^N$$

$$= \int d^4x \int \frac{d^4k}{(2\pi)^4} e^{i(z-x)\cdot k} \left( \int \frac{d^4k_i}{(2\pi)^4} e^{ik_i\cdot x} \right)^N$$

$$= \int d^4x \int \frac{d^4k}{(2\pi)^4} e^{i(z-x)\cdot k} \left( \langle 0 | \hat{\phi}(x) \hat{\phi}(0) | 0 \rangle \right)^N$$

$$= \int d^4x \delta^{(4)}(z - x) \left( \langle 0 | \hat{\phi}(x) \hat{\phi}(0) | 0 \rangle \right)^N$$

$$= \left( \langle 0 | \hat{\phi}(z) \hat{\phi}(0) | 0 \rangle \right)^N , \quad (4.94)$$

which is our desired result. Now, just like with the spectral density, we are going to rewrite

$$\Omega_N(-k^2) = \int_0^\infty d\mu \ \Omega_N(\mu) \delta(k^2 + \mu) , \quad (4.95)$$

which gives us

$$\left( \langle 0 | \hat{\phi}(z) \hat{\phi}(0) | 0 \rangle \right)^N = \int_0^\infty d\mu \ \Omega_N(\mu) \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot z} \theta(k^0) 2\pi \delta(k^2 + \mu) . \quad (4.96)$$
4.5.1.2 Euclidean Analysis

To find a useful expression for $\Omega_N(\mu)$ we may switch to the Euclidean where we say

$$x^0 \rightarrow ix^0,$$

$$k_0 \rightarrow ik_0.$$

This procedure makes the time direction into a space direction. In this framework we have \((4.96)\) become

$$\left( \langle 0 \mid \hat{\phi}(z)\hat{\phi}(0) \mid 0 \rangle \right)^N = \int_0^\infty d\mu \ \Omega_N(\mu) \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik \cdot z}}{k^2 + \mu}, \tag{4.97}$$

where the last factor is the Euclidean form of the Feynman propagator. We note that in the Euclidean there is no difference between time-ordered products and non-time-ordered products. The left hand side of this equation simply becomes

$$\left( \frac{1}{4\pi^2 z^2} \right)^N = \int_0^\infty d\mu \ \Omega_N(\mu) \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik \cdot z}}{k^2 + \mu}. \tag{4.98}$$

Now, using the fact that

$$\frac{1}{A} = \int_0^\infty ds \ e^{-sA}, \tag{4.99}$$

we may write

$$\left( \frac{1}{4\pi^2 z^2} \right)^N = \int_0^\infty ds \ \int_0^\infty d\mu \ \Omega_N(\mu) \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot z} e^{-s(k^2 + \mu)}. \tag{4.100}$$
Doing the $k$ integral gives us

$$
\left( \frac{1}{4\pi^2} \frac{1}{z^2} \right)^N = \frac{1}{2^4\pi^2} \int_0^\infty ds \int_0^\infty d\mu \ \Omega_N(\mu) \frac{e^{-s^2/(4s)} e^{-s\mu}}{s^2} .
$$

(4.101)

Now, looking at (4.91) we have dimensionally $[\Omega_N] = E^{2(N-2)}$ and $[\mu] = E^2$ which gives us functionally

$$
\Omega_N(\mu) = a_N \mu^{N-2} ,
$$

(4.102)

for some unknown $a_N$. This lets us say

$$
\left( \frac{1}{4\pi^2} \frac{1}{z^2} \right)^N = \frac{a_N}{2^4\pi^2} \int_0^\infty ds \int_0^\infty d\mu \ \mu^{N-2} \frac{e^{-s^2/(4s)} e^{-s\mu}}{s^2} ,
$$

(4.103)

and doing the $\mu$ integral we have

$$
\left( \frac{1}{4\pi^2} \frac{1}{z^2} \right)^N = \frac{a_N}{2^4\pi^2} \Gamma[N-1] \int_0^\infty ds \frac{e^{-s^2/(4s)}}{s^{N+1}} ,
$$

(4.104)

where, once again, $\Gamma[n] = (n-1)!$. Finally, doing the $s$ integral gives us

$$
\left( \frac{1}{4\pi^2} \frac{1}{z^2} \right)^N = \frac{4^N a_N}{2^4\pi^2} \Gamma[N] \Gamma[N-1] \frac{1}{z^{2N}} .
$$

(4.105)

Solving for the unknown $a_n$ we end up with the useful expression

$$
\Omega_N(\mu) = \frac{\mu^{N-2}}{(16\pi^2)^{N-1} \Gamma[N] \Gamma[N-1]} .
$$

(4.106)
We will be taking advantage of this expression in a bit. However, for now let us go back to the spectral density for the Galileon.

4.5.1.3 Back to Spectral Density

Looking at (4.88) it will be helpful to consider an integral of the form

\[
\int d^4z \, e^{-\mathbf{i}k \cdot z} \frac{1}{(4\pi)^N} \left( \frac{-i k \cdot z}{z^2} \right)^{2m} = \left( \frac{\partial}{\partial \alpha} \right)^{2m} \int d^4z \, e^{-\mathbf{i}\alpha k \cdot z} \left( \frac{1}{4\pi^2} \frac{1}{z_2^2} \right)^N \Bigg|_{\alpha=1}
\]

\[
= \left( \frac{\partial}{\partial \alpha} \right)^{2m} \theta(k^0) 2\pi \Omega_N(-k^2) \alpha^{2(N-2)} \Bigg|_{\alpha=1}
\]

\[
= \theta(k^0) 2\pi \Omega_N(-k^2) \left( \frac{\partial}{\partial \alpha} \right)^{2m} \alpha^{2(N-2)} \Bigg|_{\alpha=1}
\]

\[
= \theta(k^0) 2\pi \Omega_N(-k^2) \frac{(2N-4)!}{(2N-2m-4)!} \alpha^{2(N-m-2)} \Bigg|_{\alpha=1}
\]

\[
= \theta(k^0) 2\pi \Omega_N(-k^2) \frac{(2N-4)!}{(2N-2m-4)!} \cdot (4.107)
\]

Comparing (4.107) with (4.88) we have

\[
N = 3n - r + 1 \quad (4.108)
\]

\[
m = n - r \quad (4.109)
\]

So going back to (4.88) we now have

\[
2\pi \theta(k^0) \rho_0(-k^2) \left( \frac{-i k \cdot z}{z^2} \right)^{2m} \left( \frac{1}{4\pi^2} \frac{1}{z_2^2} \right)^N \Bigg|_{\alpha=1}
\]

\[
= \frac{2\pi}{4\pi^2} \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(-1)^r (4\pi)^{n-r} (4\pi^2)^{3n-r+1} (-k^2)^r}{r!(n-r)!} \theta(k^0) \Omega_{3n-r+1}(-k^2) \frac{(6n-2r-2)!}{(4n-2)!} \left( \frac{1}{4\pi^2} \frac{1}{z_2^2} \right)^N \Bigg|_{\alpha=1}
\]

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and cleaning up just a bit

\[
\rho_0(-k^2) = \frac{1}{4\pi^2} \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(-1)^r (4)^{n-r}(4\pi^2)^{3n-r+1}(-k^2)^r (6n-2r-2)!}{r!(n-r)! (4n-2)!} \Omega_{3n-r+1}(-k^2).
\]

Switching to the \(\mu\) variable where we have \(-k^2 = \mu\) we get

\[
\rho_0(\mu) = \frac{1}{4\pi^2} \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(-1)^r (4)^{n-r}(4\pi^2)^{3n-r+1}(\mu)^r (6n-2r-2)!}{r!(n-r)! (4n-2)!} \Omega_{3n-r+1}(\mu),
\]

and using (4.106) we are led to

\[
\rho_0(\mu) = \frac{1}{4\pi^2} \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(-1)^r (4)^{n-r}(4\pi^2)^{3n-r+1}(\mu)^r (6n-2r-2)!}{r!(n-r)! (4n-2)!} \Omega_{3n-r+1}(\mu)
\times \left( \frac{\mu^{3n-r-1}}{(16\pi^2)^{3n-r}(3n-r)!(3n-r-1)!} \right)
\]

...doing the \(r\) sum...

\[
= \frac{1}{4\pi^2} \sum_{n=0}^{\infty} \frac{2^n \pi^{\frac{3}{2}-2n} \Gamma \left[ \frac{3n+5}{2} \right]}{\Gamma \left[ 4n-1 \right] \Gamma \left[ n+1 \right] \Gamma \left[ 3n+1 \right]} 2F_1 \left( -3n, -n; \frac{3}{2} - 3n; \frac{1}{4} \right) \left( \frac{\mu^{3n-1}}{\Lambda^{6n}} \right),
\]

(4.110)

where \(2F_1(a, b; c; x)\) is called a hypergeometric function, defined as

\[
2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c) x^n}{\Gamma(a)\Gamma(b)\Gamma(c+n)} \frac{x^n}{n!}.
\]

(4.111)

Since we are interested in the high energy dynamics, we will focus on the large \(\mu\)
(and large \(n\)) limit. Therefore we may say that \(3/2 \ll n\) and replace \(3/2\) with some
Figure 4.1: The ratio between \( _2F_1 \left( -3n, -n; -3n; \frac{1}{4} \right) \) and \( _2F_1 \left( -3n, -n; \frac{3}{2} - 3n; \frac{1}{4} \right) \).

An infinitesimally small number, \( \epsilon \), to give us

\[
_2F_1 \left( -3n, -n; \epsilon - 3n; \frac{1}{4} \right) = a \, _2F_1 \left( -3n, -n; -3n; \frac{1}{4} \right) + O(\epsilon),
\]

where \( a \approx .85 \). A diagram showing the ratio of these two hypergeometrics is shown in Figure 4.1. So now doing this replacement then doing the \( n \) sum in (4.110) we are left with

\[
\rho_0(\mu) \approx \delta(\mu)
\]

\[
+ \frac{a}{4\pi^2} \sum_{n=1}^{\infty} \frac{2^n \pi^{\frac{3}{2} - 2n} \Gamma \left[ 3n - \frac{1}{2} \right]}{\Gamma \left[ 4n - 1 \right] \Gamma \left[ n + 1 \right] \Gamma \left[ 3n + 1 \right]} \, _2F_1 \left( -3n, -n; -3n; \frac{1}{4} \right) (\frac{\mu^{3n-1}}{\Lambda^{6n}})
\]

\[
\approx \delta(\mu) + \frac{a}{4\pi^2} \frac{3\mu^2}{32\Lambda^6} \, _pF_Q \left( \frac{5}{6}, \frac{7}{6}; \frac{3}{4}, \frac{5}{4}, \frac{4}{3}, \frac{5}{3}; \frac{2}{3}, \frac{2}{3}, \frac{3}{2}, \frac{3}{2}; \frac{3}{512\pi^2} \Lambda^6 \right). \tag{4.114}
\]
Looking at large $\mu$ we find

$$\rho_0(\mu) \approx \frac{a}{3\pi \mu} \sqrt{\frac{2}{15}} e^{\left(\frac{5\cdot3^{1/5}}{2^{9/5} \pi^{2/5}}\right) \left(\frac{\mu^3}{\Lambda^5}\right)^{1/5}}. \quad (4.115)$$

So now we may formally write the high energy limit (large $\mu$) of our Galileon two point function to be

$$\langle 0 | \hat{\pi}(x) \hat{\pi}(x') | 0 \rangle = \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot (x-x')} \theta(k^0) 2\pi \rho(-k^2)$$

$$= \int_0^\infty d\mu \, \rho(\mu) \langle 0 | \hat{\phi}(x) \hat{\phi}(x') | 0 \rangle_\mu$$

$$\approx \frac{a}{3\pi} \sqrt{\frac{2}{15}} \int_0^\infty d\mu \, \frac{1}{\mu} e^{\left(\frac{5\cdot3^{1/5}}{2^{9/5} \pi^{2/5}}\right) \left(\frac{\mu^3}{\Lambda^5}\right)^{1/5}} \langle 0 | \hat{\phi}(x) \hat{\phi}(x') | 0 \rangle_\mu, \quad (4.116)$$

which diverges. Therefore, the two point function in position space is ill-defined, however in momentum space, with our spectral density being an entire function, is perfectly well defined. That is, in momentum space, our Galileon operator is, in 4D,

$$\hat{\pi}(k) \equiv \int d^4x \, e^{-ik \cdot x} : U(\hat{\rho}(x)) e^{ik \cdot \hat{\rho}(x)/\Lambda^3} : . \quad (4.117)$$

The spectral density is related to the momentum space two point function through

$$\langle 0 | \hat{\pi}(k) \hat{\pi}(-k) | 0 \rangle = (2\pi)^4 2\pi \rho(-k^2), \quad (4.118)$$

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and so for any given $k$, $\rho(-k^2)$ is an entire, well defined, function so there is no problems or ambiguity defining these operators in momentum space.

### 4.5.2 Connection with Classicalization

We can observe a connection with Classicalization at this stage by asking ‘What $n$ value dominates the sum of the spectral density in (4.113)?’ Considering the high $\mu$ limit we consider the large $n$ limit of (4.113) to say

$$
\rho_0(\mu) \approx \frac{a}{4\pi^2} \sum_{n=1}^{\infty} \frac{2^n \pi^{\frac{3}{2}} 2^{2n} \Gamma[3n - \frac{1}{2}] \Gamma[n + 1] \Gamma[3n + 1]}{\Gamma[4n - 1] \Gamma[n + 1] \Gamma[3n + 1]} \ _2F_1\left(-3n, -n; -3n; \frac{1}{4}\right) \left(\frac{\mu^{3n-1}}{\Lambda^{6n}}\right)
$$

$$
\approx \sum_{n=1}^{\infty} \exp\left[(3n - 1) \ln(\mu) - 6n \ln(\Lambda) - 5n \ln(n) + c\right] \equiv \int dn \exp\left[f(n)\right],
$$

(4.119)

where $c$ is some irrelevant constant and the right hand side defines $f(n)$. We can find the dominate $n$ contribution by doing a saddle point, that is to say the dominant $n$, $n_*$, can be found by

$$
\left.\frac{\partial f(n)}{\partial n}\right|_{n=n_*} = 0,
$$

(4.120)

which will give us

$$
n_*(\mu) = \left(\frac{\mu^3}{\Lambda^6}\right)^{1/5} = \left(\frac{\sqrt{\mu}}{\Lambda}\right)^{6/5}.
$$

(4.121)
Now, interpreting \( n \) to be referring to an \( n \) particle state, as can be seen through our definition of the spectral density, (2.161)

\[
\theta(p^0)2\pi\rho(-p^2) \equiv \sum_n \delta^{(4)}(p - p_n)|\langle 0 | \hat{\pi}(0) | n \rangle|^2 ,
\]

and \( \sqrt{\mu} \) as energy, \( E \), we see that at high energies, our Galileon two point function is dominated by a large amount of particles. The more total energy, the more particles. Even further, if we define an energy per particle, \( \epsilon \) we have

\[
\epsilon = \frac{E}{n^*(E)} = \Lambda \left( \frac{\Lambda}{E} \right)^{1/5} = \frac{1}{r^*(E)} .
\]

So not only does the system prefer many particles at high energies, it prefers for these many particles to be soft meaning the particles have less and less energy as the total energy increases. This is exactly the behavior predicted by the Classicalization Mechanism and we see it here with our formal definition of the Galileon operator which is dual to a free field.

We also can explicitly see that at any arbitrary high energy, Unitarity is still preserved in this Galileon theory. This is simply because \( \rho(\mu) > 0 \) for all \( \mu \). This feature of the spectral density gives further evidence that the apparent breakdown of Unitarity in perturbative scattering calculations is a feature of the breakdown of perturbation theory, not a breakdown of the Galileon theory itself. In the next chapter we will see that this ill-defined real space two point function is a symptom of an inherit nonlocality within the theory which can be dealt with mathematically by an
appropriate smearing of our Galileon operator. Conceptually, however, this should not be worrisome when viewing the Galileon as the decoupling limit of a Massive Gravity theory.
Chapter 5

Localizable and Non-Localizable Field Theories

It is common to assume locality in field theory. In fact, it is usually considered to be one of the fundamental requirements for a field theory to possess. We will see, however, that the exponential growth of the Galileon spectral density is actually a consequence of an inherit non-locality within Galileon theories. In this chapter we will briefly review what it means for a field theory to be localizable and give evidence that Galileons cannot be localizable.

5.1 Locality

In classical quantum mechanics, the position operator does not commute with its associated momentum operator, that is to say $[\hat{x}, \hat{p}] = i\hbar$. In quantum field theory, we are concerned with the field operator, $\hat{\phi}(x)$ and its associated conjugate momentum,
\( \hat{\pi}(x) \), (which is not to be confused with the Galileon field with the same name) defined to be

\[
\pi(x) \equiv \frac{\partial L[\phi]}{\partial \dot{\phi}} .
\] (5.1)

We then impose a canonical commutation relation between \( \hat{\phi} \) and \( \hat{\pi} \) at equal times to be

\[
[\hat{\phi}(x), \hat{\pi}(y)]_{x^0 = y^0} = i\hbar\delta^{(3)}(\vec{x} - \vec{y}) .
\] (5.2)

The physics of this statement is that at equal times, information about \( \hat{\phi}(x) \) cannot propagate to \( \hat{\pi}(y) \) unless we consider \( \hat{\pi}(y) \) to be at the same spacetime point as \( \hat{\phi}(x) \). If they are separated by any distance, these two operators commute. This concept is depicted in Figure 5.1. When we consider a more general situation where \( \hat{\phi}(x) \) and \( \hat{\pi}(y) \) are at two arbitrary different times we have the commutation relation

\[
[\hat{\phi}(x), \hat{\pi}(y)] = 0 \quad \text{if} \quad (x - y)^2 > 0 .
\] (5.3)

We can understand the \( (x - y)^2 > 0 \) since

\[
(x - y)^2 = -c^2(x^0 - y^0)^2 + (\vec{x} - \vec{y})^2 > 0 \\
\rightarrow \frac{(\vec{x} - \vec{y})^2}{(x^0 - y^0)^2} > c^2 ,
\] (5.4)
which says that if $\hat{\phi}(x)$ does not commute with $\hat{\pi}(y)$, then they influence each other, but (5.4) states that the influential information must have traveled faster than $c$ - the speed of light. So, together with Lorentz invariance, locality is the statement (5.3).

### 5.1.1 The Trouble with Galileons

So what’s going on with Galileons? The expectation value of the commutator of a field and its conjugate momentum between two vacuum states is just the difference of two Wightman functions,

$$\langle 0 \mid [\hat{\phi}(x), \hat{\pi}(y)] \mid 0 \rangle = \langle 0 \mid \hat{\phi}(x)\hat{\pi}(y) \mid 0 \rangle - \langle 0 \mid \hat{\pi}(y)\hat{\phi}(x) \mid 0 \rangle$$

$$= \langle 0 \mid \hat{\phi}(x)\hat{\phi}(y) \mid 0 \rangle - \langle 0 \mid \hat{\phi}(y)\hat{\phi}(x) \mid 0 \rangle$$

$$= \frac{\partial}{\partial y^0} \left( \langle 0 \mid \hat{\phi}(x)\hat{\phi}(y) \mid 0 \rangle - \langle 0 \mid \hat{\phi}(y)\hat{\phi}(x) \mid 0 \rangle \right) , \quad (5.5)$$
where in the second line we assume a theory with no higher derivatives so we may say \( \pi(y) = \dot{\phi}(y) \). Since the Galileon is a higher derivative theory, the momentum conjugate to the Galileon, \( \pi \), becomes a bit more complicated. If we consider just a cubic Galileon in 4D we have

\[
S[\pi] = \int d^4x \left( -\frac{1}{2} (\partial \pi)^2 - \frac{1}{\Lambda^3} \Box \pi (\partial \pi)^2 \right),
\]

which, for our purposes, can be written in a more convenient form

\[
S[\pi] = \int d^4x \left( \frac{1}{2} \dot{\pi}^2 - \frac{1}{2} (\nabla \pi)^2 - \frac{1}{\Lambda^3} \ddot{\pi} \dot{\pi}^2 + \frac{1}{\Lambda^3} \dot{\pi} (\nabla \dot{\pi})^2 + \frac{1}{\Lambda^3} \dot{\pi}^2 (\nabla^2 \pi) - \frac{1}{\Lambda^3} (\nabla^2 \pi)(\nabla \pi)^2 \right).
\]

Now, looking at the third term we have

\[
\int dt \left( -\frac{1}{\Lambda^3} \ddot{\pi} \dot{\pi}^2 \right) = -\frac{1}{3\Lambda^3} \int dt \frac{d}{dt} (\dot{\pi}^3) = 0 ,
\]

neglecting boundary terms. Looking at the fourth term, integrating by parts, we have

\[
\int d^4x \left( \frac{1}{\Lambda^3} \ddot{\pi} (\nabla \pi)^2 \right) = \int d^4x \left( -\frac{2}{\Lambda^3} \dot{\pi} (\nabla \pi)(\nabla \dot{\pi}) \right)
= \int d^4x \left( \frac{1}{\Lambda^3} (\nabla \pi) \nabla (\dot{\pi}^2) \right) = \int d^4x \frac{1}{\Lambda^3} \dot{\pi}^2 (\nabla^2 \pi) ,
\]

which lets us write (5.6) as

\[
S[\pi] = \int d^4x \left( \frac{1}{2} \dot{\pi}^2 - \frac{1}{2} (\nabla \pi)^2 + \frac{2}{\Lambda^3} \dot{\pi}^2 (\nabla^2 \pi) - \frac{1}{\Lambda^3} (\nabla^2 \pi)(\nabla \pi)^2 \right). \]
We write the action in this form to be able to easily read off the conjugate momentum to $\pi$, $P$ as

$$P(x) = \frac{\partial L[\pi]}{\partial \dot{\pi}} = \left(1 + \frac{4\nabla^2 \pi}{\Lambda^3}\right) \dot{\pi}.$$ \hspace{1cm} (5.11)

Now by looking at our normal locality condition we will find

$$\langle 0 \mid [\hat{\pi}(x), \hat{P}(y)] \mid 0 \rangle = \frac{\partial}{\partial y^0} \left( \langle 0 \mid \hat{\pi}(x)\hat{\pi}(y) \mid 0 \rangle - \langle 0 \mid \hat{\pi}(y)\hat{\pi}(x) \mid 0 \rangle \right) + \text{higher derivative terms}.$$ \hspace{1cm} (5.12)

The point is, the first contribution to the expectation value of this commutation relation depends on the real space Wightman function for the Galileon. However, in the previous chapter we saw that this is infinite. Therefore, the commutation between the Galileon and its momentum conjugate is meaningless as it stands, the usual definition of locality would say that Galileons are non-localizable. So we can see that so long as the real space Wightman functions for a field exist, we may talk about the commutator between a field and its momentum conjugate and therefore locality in the usual sense.

5.1.2 Micro and Macro Locality

So what can we say about locality for Galileons? Due to the nature of the strong coupling radius, $r_\ast(\sqrt{\mu})$, we have for a given amount of energy, $\sqrt{\mu}$, a regime of strong coupling. The further the spacetime points $x$ and $y$ becomes relative to $r_\ast(\sqrt{\mu})$, 141
Figure 5.2: Two cartoon diagrams with different ratios of $(x - y)^2$ to $r_*(\sqrt{\mu})$. The left diagram depicts correlations considered at spacetime points larger than $r_*(\sqrt{\mu})$ which, the further and further $x$ and $y$ becomes, better and better behaves as a standard localizable field. However, the right diagram is of spacetime points well inside the strong couple $r_*(\sqrt{\mu})$ region where nonlocal effects dominate. This is the quantum effects of the Vainshtein mechanism.

the more these correlations become localized. This is depicted in Figure 5.2. This presents to us a difference between micro-locality and macro-locality. Galileons, as we have presented them, do not satisfy micro-locality (5.3). Our view is that Galileons, however, do satisfy macro-locality,

$$[\hat{\pi}(x), \hat{P}(y)] \to 0 \quad \text{as} \quad (x - y)^2 \gg r_*(\sqrt{\mu}) \quad . \quad (5.13)$$

\section{5.2 Localizable and Non-Localizable}

We can do better and actually characterize how much an operator is non-localizable. Let us consider any arbitrary operator, $\hat{A}(x)$, and define its spectral density, $\rho_A(\mu)$
in the usual manner,

\[ \langle 0 \left| \hat{A}(x)\hat{A}(y) \right| 0 \rangle = \int_0^\infty d\mu \: \rho_A(\mu) \langle 0 \left| \hat{\phi}(x)\hat{\phi}(y) \right| 0 \rangle_\mu . \tag{5.14} \]

We want to know what conditions are required for this \( \mu \) integral to converge which means that this two-point function for the \( \hat{A}(x) \) operator is localized. At large \( \mu \) we have \( \langle 0 \left| \hat{\phi}(x)\hat{\phi}(y) \right| 0 \rangle_\mu \sim e^{-\sqrt{\mu}|x-y|} \). This means, at most, \( \rho_A(\mu) \) can only grow as fast as a linear exponential, that is the spectral density must grow as \( \rho_A(\mu) \sim e^{\sqrt{\mu}|x-y|} \) or slower to ensure convergence of the \( \mu \) integral and thus locality. We can define a quantity that will determine the order of the spectral density, \( \alpha \), to be

\[ \lim_{\mu \to \infty} \left[ \frac{\ln(\ln(\rho_A(\mu)))}{\ln(\sqrt{\mu})} \right] = 2 \alpha \tag{5.15} \]

Applying this definition to \( \rho_A(\mu) \sim e^{\sqrt{\mu}|x-y|} \) we get

\[ \lim_{\mu \to \infty} \left[ \frac{\ln(\ln(\rho_A(\mu)))}{\ln(\sqrt{\mu})} \right] = \lim_{\mu \to \infty} \left[ \frac{\ln(\sqrt{\mu}|x-y|)}{\ln(\sqrt{\mu})} \right] = 1 \equiv 2 \alpha , \tag{5.16} \]

which gives us \( \alpha = 1/2 \). This says that if the order of the spectral density, \( \alpha \leq 1/2 \) then the two point function for the operator in question is localizable. However, if \( \alpha > 1/2 \) then something must be amiss since the \( \mu \) integral will no longer converge. To see some of the subtleties with all this, it is instructive to do some examples.
5.2.1 Strictly Localizable Field Example

The first scenario we will consider is of strictly localizable fields which are the most common fields used in field theory. Strictly localizable fields are defined to have $\alpha < 1/2$. Let us first consider a spectral density which is a polynomial in $\mu$. For instance, consider

$$\rho_A(\mu) = \sum_{n=0}^{N} c_n \mu^n,$$  \hfill (5.17)

where $N$ is finite. Considering the highest order term since that term will grow the fastest, using our definition of the order of the spectral density, $\alpha$, we have

$$\lim_{\mu \to \infty} \left[ \frac{\ln(\ln(\rho_A(\mu)))}{\ln(\sqrt{\mu})} \right] = \lim_{\mu \to \infty} \left[ \frac{\ln(\ln(c_N) + N \ln(\mu))}{\ln(\sqrt{\mu})} \right] = 0 \equiv 2\alpha. \hfill (5.18)$$

So we see in this case, $\alpha = 0$ and therefore any spectral density that is a polynomial in $\mu$ must be strictly localizable. One may worry that this is only true since we truncated the sum. In fact, however, we don’t need to truncate the sum like we did in (5.17), we may consider special cases where $N \to \infty$ such as

$$\hat{A}(x) = \sum_{n=1}^{\infty} \frac{g^n}{n!} : \hat{\phi}(x) : =: e^{g\hat{\phi}(x)} - 1 :. \hfill (5.19)$$

Now let us consider the two-point function,

$$\langle 0 \mid \hat{A}(x)\hat{A}(0) \mid 0 \rangle = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{g^{n+m}}{n!m!} \langle 0 \mid : \hat{\phi}(x)^n : \hat{\phi}(0)^m : \mid 0 \rangle, \hfill (5.20)$$
where once again we have \( \langle 0 | : \hat{\phi}(x)^n : \hat{\phi}(0)^m : | 0 \rangle = n! \left( \langle 0 | \hat{\phi}(x)\hat{\phi}(0) | 0 \rangle \right)^n \) by Wick’s theorem which gives us

\[
\langle 0 | \hat{A}(x)\hat{A}(0) | 0 \rangle = \sum_{n=1}^{\infty} \frac{g^{2n}}{n!} \left( \langle 0 | \hat{\phi}(x)\hat{\phi}(0) | 0 \rangle \right)^n . \tag{5.21}
\]

Going to momentum space and considering the spectral density we have

\[
\theta(k^0)2\pi \rho_A(-k^2) = \int d^4x \ e^{-ik\cdot x} \langle 0 | \hat{A}(x)\hat{A}(0) | 0 \rangle = \sum_{n=1}^{\infty} \frac{g^{2n}}{n!} \left( \langle 0 | \hat{\phi}(x)\hat{\phi}(0) | 0 \rangle \right)^n = \theta(k^0)2\pi \delta(-k^2) + \sum_{n=2}^{\infty} \frac{g^{2n}}{n!} \theta(k^0)2\pi \Omega_n(-k^2) , \tag{5.22}
\]

where in the last line we are again using the \( n \)-particle phase space density defined in \((4.90)\). Going to the \( \mu \) variable we have

\[
\rho_A(\mu) = \delta(\mu) + \sum_{n=2}^{\infty} \frac{g^{2n}}{(16\pi^2)^{n-1}n!(n-1)!(n-2)!} \mu^{n-2} = \delta(\mu) + \frac{g^4}{32\pi^2} {}_pF_q \left( ; 2, 3; \frac{g^2 \mu}{16\pi^2} \right) . \tag{5.23}
\]

Considering the potential divergence of this in the \( \mu \) integral we consider large \( \mu \) to see

\[
\rho_A(\mu) \approx \frac{1}{\sqrt{3}} \left( \frac{2g^4}{\pi \mu^4} \right)^{1/3} e^{\frac{3g^2/3}{\pi^{2/3} \mu^{1/3}}} . \tag{5.24}
\]
So we see even though our operator $\hat{A}(x)$ is made up of an infinite amount of terms of the free field $\hat{\phi}(x)$, we still have an $\alpha = 1/3 < 1/2$ therefore is strictly localizable. The spectral density $\mu$ integral will converge in this case.

### 5.2.2 Non-Localizable Field Example

A simple yet interesting example of a non-localizable field operator is

$$\hat{A}(x) =: \frac{\hat{\phi}(x)}{1 - g\hat{\phi}(x)} := \sum_{n=0}^{\infty} g^n : \hat{\phi}(x)^{n+1} :.$$  \hspace{1cm} (5.25)

The two point function gives us

$$\langle 0 \mid \hat{A}(x)\hat{A}(y) \mid 0 \rangle = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} g^{n+m} \langle 0 \mid :\hat{\phi}(x)^{n+1} ::\hat{\phi}(y)^{m+1} : \mid 0 \rangle$$

$$= \sum_{n=0}^{\infty} g^{2n}(n+1)! \left( \langle 0 \mid \hat{\phi}(x)\hat{\phi}(y) \mid 0 \rangle \right)^{n+1}$$

$$= \langle 0 \mid \hat{\phi}(x)\hat{\phi}(y) \mid 0 \rangle + \sum_{n=1}^{\infty} g^{2n}(n+1)! \left( \langle 0 \mid \hat{\phi}(x)\hat{\phi}(y) \mid 0 \rangle \right)^{n+1}.$$  \hspace{1cm} (5.26)

Now, once again, using the definition of the spectral density and $n$-particle phase space density we have

$$\langle 0 \mid \hat{A}(x)\hat{A}(y) \mid 0 \rangle = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)} \theta(k^0) 2\pi \rho_{\lambda}(-k^2) ,$$

$$\langle 0 \mid \hat{\phi}(x)\hat{\phi}(y) \mid 0 \rangle = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)} \theta(k^0) 2\pi \delta(-k^2) ,$$

$$\left( \langle 0 \mid \hat{\phi}(x)\hat{\phi}(y) \mid 0 \rangle \right)^{n+1} = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)} \theta(k^0) 2\pi \Omega_{n+1}(-k^2) .$$  \hspace{1cm} (5.27)
Going to the $\mu$ variable we have

$$
\rho_A(\mu) = \delta(\mu) + \sum_{n=1}^{\infty} g^{2n}(n+1)!\Omega_{n+1}(\mu)
$$

$$
= \delta(\mu) + \sum_{n=1}^{\infty} \frac{g^{2n}(n+1)!}{(16\pi^2)^n(n-1)!(n-2)!}\mu^{n-1}
$$

$$
= \delta(\mu) + \sum_{n=1}^{\infty} \frac{g^{2n}(n+1)n}{(16\pi^2)^n(n-2)!}\mu^{n-1}
$$

$$
= \delta(\mu) + \frac{g^4\mu(g^4\mu^2 + 96\pi^2g^2\mu + 1536\pi^4)}{65536\pi^8} e^{\frac{g^2}{16\pi^2}\mu}. \tag{5.28}
$$

At large $\mu$ we are left with

$$
\rho_A(\mu) \approx \left( \frac{g^8\mu^3}{65536\pi^8} \right) e^{\frac{g^2}{16\pi^2}\mu}. \tag{5.30}
$$

So we see this operator has $\alpha = 1$ which is in the non-localizable regime. The spectral density $\mu$ integral will not converge for this two point function which we can have anticipated from noticing that (5.26) is asymptotic and is therefore ill-defined. However, this shows the power of the spectral density as a relevant quantity to consider since it is itself an entire function, that is the sum in (5.28) can be completed with no ambiguity. However, we can give some sense to the two point function for these non-localizable fields through the process of smearing.

### 5.3 Smearing

Giddings and Lippert [79, 80] were able to put a bound on locality in gravitational theories that becomes relevant for our discussion. This bound is essentially (5.13) but
Figure 5.3: A depiction of a smeared function centered around a point $x$.

we can see this bound more mathematically using smearing functions. Once again, thinking of Galileons as a low energy decoupling limit of massive gravity, it is of no surprise progress and efforts done in gravity are applicable to our work.

To understand how smearing works, let us define a smeared operator, $\hat{\phi}_s(x)$, to be

$$
\hat{\phi}_s(x) \equiv \int d^4x' \, f(x - x') \hat{\phi}(x'),
$$

where, for now, $f(x - x')$ is just merely some function. This effectively creates a wavepacket centered around $x$. This is depicted in Figure 5.3. The properties of this $f(x - x')$ which will ensure converge of some operator expectation value can also be used to define localizable and non-localizable fields. To understand how this works, we will focus on three different $f(x - x')$’s. The first scenario we will consider when $f(x - x') = \delta^{(4)}(x - x')$. This is the simplest example since we are led right back to $\hat{\phi}_s(x) = \hat{\phi}(x)$. In this ‘null-smearing’ scenario we are led to usually quantum field theory, where operators are localizable down to any arbitrary small distance. That is, the commutator between $\hat{\phi}_s(x)$ and its conjugate momentum $\pi_s(y)$ satisfy (5.3).
Figure 5.4: A depiction of two ‘smeared’ functions using \( f(x - x') = \delta^{(4)}(x - x') \) and \( f(y - y') = \delta^{(4)}(y - y') \). We see that locality defined in (5.3) is completely consistent with these ‘null’ smeared operators.

This is summed up in Figure 5.4.

\[
[\hat{\phi}_s(x), \hat{\pi}_s(y)] = 0 \quad \text{if } (x - y)^2 > 0 . \tag{5.32}
\]

The next scenario is when \( f(x - x') \) is a function of compact support, that is \( f(x - x') \) is only nonzero in some finite region of parameter space. If the two regions of compact support for the two smeared functions do not overlap, then locality is preserved in the usual sense, that is

\[
[\hat{\phi}_s(x), \hat{\pi}_s(y)] = 0 \quad \text{if } (x - y)^2 > r^2 . \tag{5.33}
\]

where \( r \) is the boundary of the region where the support of these two \( f(x - x') \)'s overlap. Inside the region \( r \) locality in the usual sense is no longer applicable. This is summed up in Figure 5.5.

The last scenario is when \( f(x - x') \) has no compact support. That is, \( f(x - x') \)
Figure 5.5: Two different scenarios with smear functions of compact support. The top diagram has \((x - y)^2 > r^2\) where \(r\) is the largest possible region the two supports could overlap. The bottom diagram is when \((x - y)^2 < r^2\) thus we see, shaded, a region where non-local effects occur.
Figure 5.6: A diagram depicting smeared functions $f(x - x')$ and $f(y - y')$ with no compact support. This means the non-local region (the shaded region) will technically never be zero. However, at large enough distances it can be negligible and usual locality becomes a better and better approximation.

is not constrained to be only nonzero in some finite region of parameter space. This means there are operator expectation values where the spectral density grows in $\mu$ so fast that the only smeared functions that will tame the $\mu$ integral have some overlap in their support no matter how far apart $x$ and $y$ are. In these scenarios, locality \textit{fundamentally} does not exist how we have defined it, \textit{however} locality in the usual sense becomes a better and better approximation at large distances and low energies. That is,

$$[\hat{\phi}_s(x), \hat{\pi}_s(y)] \to 0 \quad \text{as} \quad (x - y)^2 \to \infty . \quad (5.34)$$

This is known as macro-locality. This idea is summed up in Figure 5.6.
5.3.1 Galileon Smearing

From our definition of the Galileon operator, our Wightman function (and likely all operator expectation values) falls into this last class of requiring smearing functions with no compact support. We can see this by defining a Galileon smeared operator, \( \hat{\pi}_s(f) \), to be

\[
\hat{\pi}_s(f) \equiv \int \frac{d^4 k}{(2\pi)^4} f(k) \hat{\pi}(k) .
\]  

(5.35)

where, once again,

\[
\hat{\pi}(k) = \int d^4 x \ e^{-ik \cdot x} : U(\hat{\rho}(x)) e^{i k \cdot \hat{\rho}(x)/\Lambda^3} : .
\]  

(5.36)

Now, instead of looking at \( \langle 0 | \hat{\pi}(x) \hat{\pi}(x') | 0 \rangle \), we look at \( \langle 0 | \hat{\pi}_s(f) \hat{\pi}_s(g) | 0 \rangle \), we will find

\[
\langle 0 | \hat{\pi}_s(f) \hat{\pi}_s(g) | 0 \rangle = \int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 k'}{(2\pi)^4} \int d^4 y \int d^4 y' e^{i k \cdot y} e^{i k' \cdot y'} f^*(k) g(k) \times \langle 0 : U(\hat{\rho}(y)) e^{i k \cdot \hat{\rho}(y)/\Lambda^3} :: U(\hat{\rho}(y')) e^{i k' \cdot \hat{\rho}(y')/\Lambda^3} : | 0 \rangle ,
\]  

(5.37)
where \( g(k) \) does not necessarily have to equal \( f(k) \). Now, we can recycle the same math done starting with (4.62) to get

\[
\langle 0 \mid \hat{\pi}_s(f) \hat{\pi}_s(g) \mid 0 \rangle = \int d^4z \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot z} f^*(k)g(k) \\
\times F(k,z) \exp \left[ \frac{1}{2 \pi^2 \Lambda^6} \left( \frac{k^2}{z_+^4} - 4 \frac{(k \cdot z)^2}{z_0^8} \right) \right].
\]

(5.38)

We can now choose \( f(k) \) and \( g(k) \) to be functions that counter the exponential growth. That is, for example, if \( f(k) = g(k) \) then if we choose \( f(k) \) to go like

\[
f(k) \sim e^{-\left(\frac{|k|}{\Lambda}\right)^{3/5}},
\]

(5.39)

we can tame the divergence of the \( k \) integrals and therefore we have a well defined position space observable. However, since \( f(k) \) is a function with no compact support, we find ourselves in the third smeared region where we will lose micro-locality and are left only with macro-locality,

\[
[\hat{\pi}(x), \hat{P}(y)] \to 0 \quad \text{as} \quad (x - y)^2 \gg r_s(E).
\]

(5.40)

### 5.4 Beyond the Duality

We can see that not only is this non-local behavior consistent with the Vainshtein mechanism, local effects at arbitrary small distances are *contradictory* to the Vainshtein mechanism. The Vainshtein mechanism creates an inherit UV/IR mixing. That is, large energies correspond to large distances, which is counter to the usual story of
small distances corresponding to high energies. We see this through the Vainshtein radius,

\[ r_*(E) = \frac{1}{\Lambda} \left( \frac{E}{\Lambda} \right)^{1/5}. \]  

(5.41)

Looking at the spectral density at high energies for the Galileon we found

\[ \rho(E) \sim e^{\left(\frac{E}{\Lambda}\right)^{6/5}} \sim e^{E r_*(E)}, \]  

(5.42)

which grows with energy precisely because \( r_*(E) \) grows with energy. Thus, the exponential growth of the spectral density makes the Wightman function in position space ill-defined therefore it is non-local. Now, if this Galileon Wightman function was local, that would mean \( r_*(E) \) would have to fall off with energy to ensure convergence, however that would violate the properties of the Vainshtein mechanism. Therefore we see that a theory which exhibits the Vainshtein mechanism must have some degree of non-locality.

### 5.4.1 Canonical Quantized Galileon

Canonical Quantization is a systematic procedure with the intent of making some classical field theory into a quantum theory, (see Chapter 2). Constructing the Hamiltonian from the Lagrangian is essential to understanding the dynamics of canonically quantized fields and when we do this with Galileons we see this same non-localizability that is seen with the special Galileon. To understand how this works let us take a
standard classical scalar field theory,

\[ S[\phi] = \int d^4x \, L[\dot{\phi}, \phi] = \int d^4x \, -\frac{1}{2}(\partial\phi)^2 - V(\phi) , \quad (5.43) \]

The Hamiltonian density is constructed from the Lagrangian density by

\[ \mathcal{H}[\pi, \phi] = \pi \dot{\phi} - L[\dot{\phi}, \phi] , \quad (5.44) \]

where, once again,

\[ \pi \equiv \frac{\partial L}{\partial \dot{\phi}} . \quad (5.45) \]

This gives us

\[ \mathcal{H}[\pi, \phi] = \frac{1}{2} \pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) . \quad (5.46) \]

We can go further and define an *interacting representation* which separates the Hamiltonian density into a free, time independent part, $\mathcal{H}_0$, and an interacting time dependent part, $\mathcal{H}_I$. In this example we would have

\[ \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I , \quad (5.47) \]
\[ \mathcal{H}_0 = \frac{1}{2} \pi^2 + \frac{1}{2}(\nabla\phi)^2 , \quad (5.48) \]
\[ \mathcal{H}_I = V(\phi) . \quad (5.49) \]
In this representation, a state $|\psi\rangle$ solves the free equations of motion and evolves through the interaction Hamiltonian,

$$|\psi(t)\rangle = T e^{-i \int d^4 x \mathcal{H}_I(t)} |\psi(0)\rangle.$$  \hspace{1cm} (5.50)

An insightful question to ask, then, is ‘What does the interacting Hamiltonian look like for Galileons?’ The interesting part of this question is that we do not need to restrict ourselves to just the Galileon that is dual to a free field. This is true for all Galileons. However, for simplicity we will just focus on a cubic Galileon in 4D, (5.6).

We saw that the conjugate momentum, $P(x)$, is

$$P = \left(1 + \frac{4 \nabla^2 \pi}{\Lambda^3}\right) \hat{\pi}.$$ \hspace{1cm} (5.51)

This gives us for our Galileon Hamiltonian

$$\mathcal{H} = \frac{1}{2} \frac{P^2}{1 + \frac{4 \nabla^2 \pi}{\Lambda^3}} + \frac{1}{2} (\nabla \pi)^2 + \frac{1}{\Lambda^3} (\nabla^2 \pi) (\nabla \pi)^2.$$ \hspace{1cm} (5.52)

The interesting, non-localizable, feature of this Hamiltonian comes from the first term which, once expanded, has an infinite number of field operator contributions. Expanding this out we have

$$\mathcal{H} = \frac{1}{2} P^2 + \frac{1}{2} (\nabla \pi)^2 + \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n P^2 \left(\frac{4 \nabla^2 \pi}{\Lambda^3}\right)^n + \frac{1}{\Lambda^3} (\nabla^2 \pi) (\nabla \pi)^2.$$ \hspace{1cm} (5.53)
Relating this to (5.47), (5.48), and (5.49) we have for the Galileon interaction Hamiltonian,

$$\mathcal{H}_I = \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n : P^2 \left( \frac{4 \nabla^2 \pi}{\Lambda^3} \right)^n : + \frac{1}{\Lambda^3} : (\nabla^2 \pi) (\nabla \pi)^2 : . \quad (5.54)$$

The second term is just a finite contribution so we will just focus on the operator,

$$\hat{O}(x) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{\Lambda^{3n}} : \hat{P}(x)^2 \left( \nabla^2 \hat{\pi}(x) \right)^n : , \quad (5.55)$$

where we have scaled $\Lambda^3 \to 4\Lambda^3$. We will now just look at the Wightman function for this operator, $\hat{O}$. We will construct its spectral density and see that it grows precisely as the predicted $e^{(E/\Lambda)^{6/5}}$ therefore indicating an inherit non-locality even in the Galileon Hamiltonian. We start with

$$\langle 0 \mid \hat{O}(z) \hat{O}(0) \mid 0 \rangle = \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{n+m}}{\Lambda^{3(n+m)}} \langle 0 \mid \hat{P}(z)^2 \left( \nabla^2 \hat{\pi}(z) \right)^n :: \hat{P}(0)^2 \left( \nabla^2 \hat{\pi}(0) \right)^m :: 0 \rangle . \quad (5.56)$$

To simplify this calculation without losing any of the physics, we are just going to focus on one term of this complicated expectation value, that is we are going to let

$$\langle 0 \mid \hat{P}(z)^2 \left( \nabla^2 \hat{\pi}(z) \right)^n :: \hat{P}(0)^2 \left( \nabla^2 \hat{\pi}(0) \right)^n : 0 \rangle \quad (5.57)$$

$$\to \langle 0 \mid \hat{P}(z)^2 :: \hat{P}(0)^2 :: 0 \rangle \langle 0 \mid \left( \nabla^2 \hat{\pi}(z) \right)^n :: \left( \nabla^2 \hat{\pi}(0) \right)^m :: 0 \rangle . \quad (5.58)$$
Now
\[ \hat{P}(x) = \hat{\pi}(x) , \] (5.59)
since we are in the interacting representation so the field obeys the free equations of motion. Therefore, a subset of terms in the total \( \langle 0 \mid \hat{O}(z)\hat{O}(0) \mid 0 \rangle \) will take the form

\[ \langle 0 \mid \hat{O}(z)\hat{O}(0) \mid 0 \rangle_{\text{simplified}} = \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{n+m}}{\Lambda^{3(n+m)}} \langle 0 \mid :\hat{\pi}(z)^2 ::\hat{\pi}(0)^2 :|0\rangle \langle 0 \mid :\nabla_z^2\hat{\pi}(z)\nabla_0^2\hat{\pi}(0)^m :|0\rangle . \] (5.60)

Taking one step at a time, we first focus on the \( \langle 0 \mid :\nabla_z^2\hat{\pi}(z)\nabla_0^2\hat{\pi}(0)^m :|0\rangle \) contribution where we can say

\[ \langle 0 \mid :\nabla_z^2\hat{\pi}(z)^n ::\nabla_0^2\hat{\pi}(0)^m :|0\rangle = n! \left( \nabla_z^2\nabla_0^2 \frac{1}{4\pi^2} \frac{1}{-(z^0 - i\epsilon)^2 + \vec{z}^2} \right)^n \delta_{mn} \] (5.61)

where Wick’s theorem makes \( m = n \). Now to further simplify this calculation, we will just focus on the easiest term in this and say

\[ \langle 0 \mid :\nabla_z^2\hat{\pi}(z)^n ::\nabla_0^2\hat{\pi}(0)^m :|0\rangle = n! \left( \nabla_z^2\nabla_0^2 \frac{1}{4\pi^2} \frac{1}{-(z^0 - i\epsilon)^2 + \vec{z}^2} \right)^n \delta_{mn} \xrightarrow{\epsilon \to 0} \frac{6^n n!}{\pi^{2n}} \left( \frac{1}{-(z^0 - i\epsilon)^2 + \vec{z}^2} \right) \delta_{mn} . \] (5.62)
Now focusing on \( \langle 0 \mid \hat{\pi}(z)^2 \cdot \hat{\pi}(0)^2 \mid 0 \rangle \) we have

\[
\langle 0 \mid \hat{\pi}(z)^2 \cdot \hat{\pi}(0)^2 \mid 0 \rangle = \lim_{x \to z} \lim_{y \to 0} \langle 0 \mid \hat{\pi}(x) \hat{\pi}(y) \mid 0 \rangle^2
\]

\[
= 2 \lim_{x \to z} \lim_{y \to 0} \left( \frac{\partial}{\partial x^0} \frac{\partial}{\partial y^0} \frac{1}{4\pi^2} \frac{1}{-(x^0 - y^0 - i\epsilon)^2 + (\vec{x} - \vec{y})^2} \right)^2
\]

\[
= 2 \left( \frac{-1}{4\pi^2} \right)^2 \left( \frac{2}{-(z^0 - i\epsilon)^2 + \vec{z}^2} \right)^2 + \frac{8z^0}{\left( -(z^0 - i\epsilon)^2 + \vec{z}^2 \right)^3} \right)^2
\]

\[
= 2 \left( \frac{-1}{2\pi^2} \right)^2 \left( \frac{3z^0 + \vec{z}^2}{\left( -(z^0 - i\epsilon)^2 + \vec{z}^2 \right)^3} \right)^2.
\]  

(5.63)

With all of these simplifications we can now define a subset of the total \( \langle 0 \mid \hat{O}(z)\hat{O}(0) \mid 0 \rangle \) which we will call \( \langle 0 \mid \hat{O}(z)\hat{O}(0) \mid 0 \rangle_{\text{sub}} \) as

\[
\langle 0 \mid \hat{O}(z)\hat{O}(0) \mid 0 \rangle_{\text{sub}} = \frac{1}{8\pi^4} \sum_{n=1}^{\infty} \frac{6^n n!}{\pi^{2n} \Lambda^{6n}}
\]

\[
\times \left( \frac{1}{\left( -(z^0 - i\epsilon)^2 + \vec{z}^2 \right)^3n} \right) \left( \frac{3z^0 + \vec{z}^2}{\left( -(z^0 - i\epsilon)^2 + \vec{z}^2 \right)^3} \right)^2.
\]  

(5.64)

To make progress let us consider the quantity,

\[
\int d^4 z \, e^{-ik\cdot z} \left( \frac{3z^0 + \vec{z}^2}{\left( -(z^0 - i\epsilon)^2 + \vec{z}^2 \right)^3} \right)^2 = \int d^4 z \, e^{-ik\cdot z} \left( \frac{4z^0 + \vec{z}^2}{\left( -(z^0 - i\epsilon)^2 + \vec{z}^2 \right)^3} \right)^2
\]

\[
= \sum_{r=0}^{2} \frac{(-1)^r 4^{2-r} 2!}{r!(2-r)!} \int d^4 z \, e^{-ik\cdot z} \frac{(-z^0)^{2-r}}{\left( -(z^0 - i\epsilon)^2 + \vec{z}^2 \right)^{6-r}}.
\]  

(5.65)
Now, with

\[-z^0{}^2 = \frac{\partial^2}{\partial k_0^2} e^{-ik \cdot z},\]  \hspace{1cm} (5.66)

we can write (5.65) as

\[
\int d^4z \ e^{-ik \cdot z} \left( \frac{3z^0{}^2 + \vec{z}^2}{(-z^0 - i\epsilon)^2 + \vec{z}^2)^3} \right)^2 = \sum_{r=0}^{2} \frac{(-1)^r 4^{2-r} 2!}{r!(2 - r)!} \frac{\partial^{2(r-2)}}{\partial k_0^{2(2-r)}} \int d^4z \ e^{-ik \cdot z} \frac{1}{(-z^0 - i\epsilon)^2 + \vec{z}^2)^{6-r}}
\]

\[
= \sum_{r=0}^{2} \frac{(-1)^r 4^{2-r} 2!(4\pi^2)^{6-r}}{r!(2 - r)!} \theta(k_0) 2\pi \frac{\partial^{2(r-2)}}{\partial k_0^{2(2-r)}} \Omega_{6-r}(-k^2) \],  \hspace{1cm} (5.67)

where in the last line we are using the definition of our \(n\)-particle phase space density.

Therefore, also using our definition for the spectral density, that is

\[
\langle 0 \mid \hat{O}(z)\hat{O}(0) \mid 0 \rangle_{\text{sub}} \equiv \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot z} \theta(k_0) 2\pi \rho_{\text{sub}}(-k^2) , \hspace{1cm} (5.68)
\]

we have for the spectral density,

\[
\rho_{\text{sub}}(-k^2)
\]

\[
= \delta(-k^2)
\]

\[
+ \frac{1}{4\pi^4} \sum_{n=2}^{\infty} \frac{6^n n!(4\pi^2)^{3n}}{\pi^{2n} \Lambda^{6n}} \Omega_{3n}(-k^2) \left[ \sum_{r=0}^{2} \frac{(-1)^r 4^{2-r} (4\pi^2)^{6-r}}{r!(2 - r)!} \frac{\partial^{2(r-2)}}{\partial k_0^{2(2-r)}} \Omega_{6-r}(-k^2) \right]. \hspace{1cm} (5.69)
\]
Focusing in the high energy regime, $k_0 \gg |\vec{k}|$, and looking at the $r$ sum we have

\[
\frac{\partial^2(2-r)}{\partial k_0^{2(2-r)}} \Omega_{6-r}(k_0^2) = \frac{1}{(16\pi^2)^{6-r-1}(6-r-1)!(6-r-2)!} \frac{\partial^2(2-r)}{\partial k_0^{2(2-r)}} k_0^{8-2r} = \frac{(8-2r)!}{4!(16\pi^2)^{6-r-1}(6-r-1)!(6-r-2)!} k_0^4,
\]

which gives us

\[
\sum_{r=0}^{2} \frac{(-1)^r 4^{2-r} (4\pi^2)^{6-r} \partial^2(2-r)}{r!(2-r)!} \frac{\partial^2(2-r)}{\partial k_0^{2(2-r)}} \Omega_{6-r}(k_0^2) = \frac{\pi^2}{128} k_0^4.
\]

Going back to the spectral density we have

\[
\rho_{\text{sub}}(-k^2) = \delta(-k^2) + \frac{1}{512\pi^2} k_0^4 \sum_{n=2}^{\infty} \frac{6^n n!(4\pi^2)^{3n}}{\pi^{2n} \Lambda^{6n}} \Omega_{3n}(k_0^2).
\]

Once again, writing

\[
\Omega_{3n}(k_0^2) = \frac{1}{(16\pi^2)^{3n-1}(3n-1)!(3n-2)!} k_0^{6n-4},
\]

and doing the $n$ sum we have

\[
\rho_{\text{sub}}(-k^2) = \delta(-k^2) + \frac{3}{2048\pi^2} \left( \frac{k_0}{\Lambda} \right)^6 \left[ _{10/3}F_{9/3} \left( 2; \frac{2}{3}, 1, \frac{4}{3}, \frac{4}{3}, \frac{5}{3}; \frac{1}{1776\pi^2} \left( \frac{k_0}{\Lambda} \right)^6 \right) - 1 \right],
\]

(5.74)
which at large $k_0$ gives us

$$
\rho_{\text{sub}}(-k^2) \approx \frac{1}{768\sqrt{5\pi}^{6/5}} \left( \frac{k_0}{\Lambda} \right)^{18/5} e^{\frac{\sqrt{5}}{6\pi^{2/5}}} \left( \frac{k_0}{\Lambda} \right)^{6/5} .
$$

(5.75)

So we see this same exponential growth of the spectral density for this interacting Hamiltonian as we did we the Wightman function of our definition of the Galileon operator, $\hat{\pi}(x)$. This points towards a universal exponential growth, and therefore non-localizability, within all Galileon theories, not just the special Galileon which is dual to a free field. Also, with this positive exponential growth, we see that Galileons do not break Unitarity, they simply become non-perturbative and the Vainshtein Mechanism must be accounted for.

### 5.4.2 An Aside - Semi-Classical Cross Section

In [81], Alberte and Bezrukov were able to approximate the cross section from going from a few hard quanta to many soft quanta with a total energy $E$ and particle number $N$. They, like us, were interested in the viability of Classicalization so they wanted to see the cross section’s behavior in this few to many quanta scenario. They had other, non-Galileon, theories in mind, such as the DBI action,

$$
S[\phi] = \int d^4x \frac{\Lambda^4}{2} \sqrt{1 - 2e^{-3(\partial\phi)^2}} ,
$$

(5.76)
where $\epsilon = \pm 1$. In the $\epsilon = -1$ case they see the precise behavior predicted by Classicalization, that is

$$\sigma(E, N) \sim \left( \frac{N_{\text{crit}}}{N} \right)^{3N},$$

(5.77)

where $N_{\text{crit}} \sim (E/\Lambda)^{4/3}$. So we see that the cross section is dominated by an amount of particles, $N_{\text{crit}}$, with grows with the total energy of the system. This is saying that at higher and higher energies the system prefers more and more particles, all with an energy, $\epsilon$,

$$\epsilon \sim \frac{E}{N_{\text{crit}}} \sim \Lambda \left( \frac{\Lambda}{E} \right)^{1/3},$$

(5.78)

which, as Classicalization predicts, decreases with the total energy of the system, $E$. We repeated this process for Galileons and we see the same qualitative results consistent with the work done here. For background see Appendix 7.2.

We start with the total scattering cross section with a few particle initial state to final state of total energy $E$ and particle number $N$ being dominated by

$$\sigma(E, N) \sim e^{W(E, N)},$$

(5.79)

where

$$W(E, N) = ET - N\theta - 2\Im[S(T, \theta)],$$

(5.80)
with $T$ and $\theta$ being the Legendre variable to $E$ and $N$ respectively and where the field satisfies the conditions

$$\frac{\delta S}{\delta \phi} = -iJ \left( \delta^{(4)}(x) - \frac{\partial}{\partial t} \delta^{(4)}(x) \right), \quad (5.81)$$

$$\phi_i(\vec{k}) = \frac{a^*_{-\vec{k}}}{\sqrt{2\omega_{\vec{k}}}} e^{i\omega_{\vec{k}}t} \quad t \to -\infty, \quad (5.82)$$

$$\phi_f(\vec{k}) = \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left( b_{\vec{k}} e^{i\omega_{\vec{k}} T - \theta} + b^*_{-\vec{k}} e^{i\omega_{\vec{k}} t} \right) \quad t \to +\infty, \quad (5.83)$$

$$E = \int d^3k \omega_{\vec{k}} \beta^*_{\vec{k}} \beta_{\vec{k}} e^{i\omega_{\vec{k}} T - \theta}, \quad (5.84)$$

$$N = \int d^3k \beta^*_{\vec{k}} \beta_{\vec{k}} e^{i\omega_{\vec{k}} T - \theta}, \quad (5.85)$$

where $J \to 0$ to simulate a few amount of initial particles, $a$ and $b$ are complex numbers, $\omega_{\vec{k}} = \sqrt{k^2 + m^2}$, and $\beta$ is the final dominate coherent state. We have an energy jump at $x = 0$ which can be seen from

$$E = -\int d^4x \frac{\delta S}{\delta \phi} \dot{\phi} = iJ(\dot{\phi}(0) + \ddot{\phi}(0)), \quad (5.86)$$

therefore we need a singularity at the origin in either $\dot{\phi}(0)$ or $\ddot{\phi}(0)$. Referring to [81, 82], in the $g^2 N \ll 1$ limit, where $g$ is our semi-classical variable being sent to zero, we can approximate

$$2\Im[S(T, \theta)] \approx e^{-\theta} \int d^3k \ a^*_{\vec{k}} a_{\vec{k}} e^{i\omega_{\vec{k}} T} \equiv e^{-\theta} I(T), \quad (5.87)$$

where we have switched to Euclidean time, $\tau$, where $\tau = -it$ and the $a_{\vec{k}}$'s are defined in (7.38). Now using (7.48) and (7.49) we can relate $T$ to $E$ and $\theta$ to $N$ and we can
define an energy per particle, $E/N$ to give us

$$
\epsilon = \frac{I'(T)}{I(T)} ,
$$

(5.88)

$$
\theta = -\ln(N) + \ln(I(T)) ,
$$

(5.89)

where the prime denotes a derivative with respect to $T$. So then for the cross section we have approximately \[81 \ [82\]

$$
\sigma(E, N) = e^{N\ln(g^2 N) - N + Nf(\epsilon)} ,
$$

(5.90)

where we have defined a function $f(\epsilon)$

$$
f(\epsilon) = \epsilon T(\epsilon) - \ln(g^2 I(T)) .
$$

(5.91)

So, as described in \[81\], to find an expression for the cross section of a few particles to many, one first needs to find the Euclidean equations of motion singular at either $\dot{\phi}(0)$ or $\ddot{\phi}(0)$ with the asymptotics

$$
\phi(\vec{k}) = \frac{a_{\vec{k}}}{\sqrt{2\omega_{\vec{k}}}} e^{-\omega_{\vec{k}}\tau} \quad \tau \to \infty ,
$$

(5.92)

then, for a given $T$, one needs to extremize $I(T)$ for all values of $a_{\vec{k}}$ which, using (5.88), which gives $\epsilon$ for a given $T$, and then use (5.90) and (5.91) to find an expression for the cross section.
5.4.2.1 Galileon Calculation

To diversify ourselves we will now show an example with a general Galileon,

\[ S = \int d^4 x \left\{ - \frac{1}{2} (\partial \phi)^2 - \frac{c_1}{\Lambda^3} \Box \phi (\partial \phi)^2 - \frac{1}{4} \frac{c_2}{\Lambda^6} (\Box \phi)^2 (\partial \phi)^2 \\
+ \frac{1}{2} \frac{c_2}{\Lambda^6} (\Box \phi) (\partial_\mu \phi) (\partial_\nu \phi) (\partial_\rho \phi) + \frac{1}{4} \frac{c_2}{\Lambda^6} (\partial_\mu \partial_\nu \phi) (\partial_\rho \partial_\sigma \phi) (\partial_\phi)^2 \\
- \frac{1}{2} \frac{c_2}{\Lambda^6} (\partial_\mu \phi) (\partial_\nu \partial_\alpha \phi) (\partial_\rho \partial_\sigma \phi) (\partial_\phi) \right\}. \tag{5.93} \]

We have introduced order one variables \( c_1 \) and \( c_2 \) in front of the cubic and quartic Galileon terms respectively. The equations of motion in \( O(4) \) symmetry give us

\[ \frac{6c_2 \rho}{\lambda^6} \left( \frac{d \phi}{d \rho} \right)^3 + \frac{6c_1 \rho^2}{\lambda^3} \left( \frac{d \phi}{d \rho} \right)^2 + \rho^3 \left( \frac{d \phi}{d \rho} \right) + A = 0, \tag{5.94} \]

where \( A \) is a constant and we have Wick rotated into complex time making \( \rho^2 = \tau^2 + x^2 \) with \( \tau = -it \). Solving (5.94) gives us

\[ \frac{d \phi}{d \rho} = -\frac{c_1}{3c_2} \Lambda^3 \rho - \left[ \frac{N_2}{9c_2 \left( 2N_1 + 2 \sqrt{N_1^2 + 4N_2^3} \right)^{1/3}} \lambda^3 \rho \right] \tag{5.95} \]

\[ + \left[ \frac{\left( \frac{1}{2} N_1 + \frac{1}{2} \sqrt{N_1^2 + 4N_2^3} \right)^{1/3}}{18c_2} \lambda^3 \rho \right], \]

where

\[ N_1 = 324c_1 c_2 - 432c_3^3 - \frac{972A c_2^2}{\Lambda^3 \rho^4} \tag{5.96} \]

\[ N_2 = 18c_2 - 36c_1^2. \tag{5.97} \]
(5.86) tells us we need a field configuration that has a single or double derivative being singular at the origin. (5.96) has a singularity in $\frac{d^2 \phi}{d\rho^2}$ when $N_1^2 + 4N_3^2 = 0$. This condition leads to

$$A = \frac{3c_1c_2 + \sqrt{2}(2c_1^2 - c_2)^3 - 4c_1^3}{9c_2^2} \Lambda^3 R_s^4$$

$$= B \Lambda^3 R_s^4 \, ,$$

(5.98) (5.99)

where $R_s$ is the specific $\rho$ value that makes $\frac{d^2 \phi}{d\rho^2}$ singular and the last equality defines the variable $B$. Since we need the double derivative to be singular at the origin, we will shift $\rho$ to

$$\rho^2 \rightarrow (\tau + R_s)^2 + \vec{x}^2 \, .$$

(5.100)

In order to take advantage of $I(T)$ in (5.87) we need to find $a_\vec{k}$ which can be done using

$$\phi(\tau, \vec{k}) = \int d^3 x \, e^{-i\vec{k} \cdot \vec{x}} \phi(\tau, \vec{x}) \, .$$

(5.101)

This leads to

$$a_\vec{k} = \frac{2\pi^2 B \Lambda^3 R_s^4}{\sqrt{2\omega_\vec{k}}} e^{-\omega_\vec{k} R_s} \, .$$

(5.102)
\( I(T) \), after extremizing over all \( R_s \), becomes

\[
I(T) = \frac{2^{11}\pi^5 B^2}{3^8} \Lambda^6 T^6 . \tag{5.103}
\]

The last piece we need is \( f(\epsilon) \) which can be found to be

\[
f(\epsilon) = 6 - \ln \left( g^2 \frac{2^{17}\pi^5 B^2}{3^2} \Lambda^6 \left( \frac{N}{E} \right)^6 \right) . \tag{5.104}
\]

In the instanton semi-classical approximation we see that

\[
\sigma(E, N) = \left( \frac{N_{\text{crit}}}{N} \right)^N , \tag{5.105}
\]

with

\[
N_{\text{crit}}^5 = c^5 \left( \frac{E}{\Lambda} \right)^6 , \tag{5.106}
\]

\[
c^5 = \frac{3^2 e^5}{2^{17}\pi^5 B^2} . \tag{5.107}
\]

(5.105) exhibits behavior exactly predicted by Classicalization. As the incoming energy, \( E \), increases, the cross section prefers to create more particles, as opposed to a few particles with high energy. Many soft quanta restore unitarity. In [83] they conceptualize the maximum amount of energy per particle to be

\[
E_{\text{max}} = \frac{E}{N_{\text{crit}}} . \tag{5.108}
\]
As shown in [83], the inverse of this $E_{\text{max}}$ is the $r_*$ classical length,

$$\frac{1}{E_{\text{max}}} = \frac{c}{\Lambda} \left( \frac{E}{\Lambda} \right)^{1/5} = r_* .$$  \hspace{1cm} (5.109)

This classical $r_*$ length may now be thought of as the effective size of a Galileon particle. If a Galileon becomes more energetic than $1/r_*$ then this semi-classical approximation suggests that the field prefers to create another particle instead of giving more energy to the original particle. This gives a natural explanation as to why the scattering process would prefer more particles with less energy as opposed to a few particles with a lot of energy.
Chapter 6

Conclusions

This work aimed to provide some insight on how to possibly incorporate the Vainshtein Mechanism, in a consistent manner, into a quantum description of Galileons. We noticed that the Vainshtein Mechanism creates a distance scale, \( r_\ast \), that grows with the energy of the system. This growth creates a UV/IR mixing that essentially regulates the theory. However, a consequence of the growth of this strong coupling scale, \( r_\ast \), is that the two point Wightman function, in position space, is ill-defined. We best see this through the spectral density which grows exponentially,

\[
\rho(\mu) \sim e^{\sqrt{\mu r_\ast} (\sqrt{\mu})} \sim e^{(\sqrt{\frac{\mu}{\Lambda}})^{6/5}},
\]

(6.1)

where \( \Lambda \) is the strong coupling scale for the Galileon theory. This growth is dominated by an amount of Galileon particles, \( n_\ast \), that goes as

\[
n_\ast = E r_\ast(E) = \left( \frac{E}{\Lambda} \right)^{6/5},
\]

(6.2)
which states at larger and larger energies, the system prefers more and more particles.

This is exactly in line with the predictions of Classicalization \[66, 67, 68, 69, 70, 71, 72\].

Furthermore, we can define an energy per particle, \(\epsilon\),

\[
\epsilon = \frac{E}{n_*} = \frac{1}{r_*(E)},
\]

(6.3)

and since, once again, the Vainshtein radius grows with energy, we see that the energy per particle decreases as the total energy of the system increases. Once again, this is completely consistency with Classicalization.

The fact that the Wightman function in position space is ill-defined points towards an inherit non-localizability within the theory (see Chapter 5). However, this feature should not be worrisome when conceptualizing the Galileon as a decoupling limit of some massive gravity theory, that is to say that Galileons should be thought of as gravitational theories as opposed to strictly localizable theories. Since throughout this process we have kept Lorentz Invariance, this breaking of micro-locality also means we break micro-causality. However, once again, this should not be worrisome since when considering spacetime points far from the Vainshtein/Classicalization radius \(r_*\), we recover macro-locality and macro-causality.

Of equal importance is the notion that Galileons break Unitarity above their strong coupling scale, \(\Lambda\), is now in question. We see this through the Unitarity requirement that \(\rho(\mu) > 0\). Once again, looking at large \(\mu\) (which is the limit in question in regards to Unitarity), the positive, exponential growth of the spectral density suggests that Galileons do not break Unitarity at the scale \(\Lambda\) once accounting
for the non-perturbative Vainshtein Mechanism.

A next step in furthering our understanding would be to obtain an expression for the full non-perturbative $n$-point function for the Galileon that is dual to a free field to ensure the positivity of the spectral density exists and posses this same non-localizability effect. It would be very enlightening both physically and mathematically to also develop a framework to account for the Vainshtein effect for all Galileon theories.

It would also be worthwhile to further investigate the implications of a violation of micro-locality and micro-causality. In principle, if there is some logical inconsistency with theories that violate micro-locality and micro-causality, then some might take the perspective that we have only shifted the problem of Galileons violating Unitarity to Galileons violating micro-locality and micro-causality. This is an interesting viewpoint and should be further investigated.
Chapter 7

Appendix

7.1 Optical Theorem

Here we briefly review some Unitary consequences on scattering amplitudes as discussed in [75]. The $\hat{S}$-matrix takes some initial state $|i\rangle$ to some final state $|f\rangle$ through

$$|f\rangle = \hat{S} |i\rangle .$$  \hspace{1cm} (7.1)

We want the norm of the final state to equal the normal of the initial state, that is

$$\langle f | f \rangle = \langle i | \hat{S}^\dagger \hat{S} | i \rangle \equiv \langle i | i \rangle ,$$ \hspace{1cm} (7.2)

which is true if the $\hat{S}$-matrix is Unitary, that is

$$\hat{S}^\dagger \hat{S} = 1 .$$  \hspace{1cm} (7.3)
We can deduce some consequences of this restriction on the \( \hat{S} \)-matrix by first decomposing the \( \hat{S} \)-matrix into

\[
\hat{S} = 1 + i\hat{T} .
\]

(7.4)

This is a useful decomposition since the \( \hat{S} \)-matrix describes scattering amplitudes and the 1 in (7.4) is the uninteresting null-scattering contribution. Unitary, (7.3), then imposes

\[
i(\hat{T}^\dagger - \hat{T}) = \hat{T}^\dagger \hat{T} ,
\]

(7.5)

and if we define our momentum stripped amplitudes, \( \mathcal{A}(i \rightarrow f) \) to be

\[
\langle f \mid \hat{T} \mid i \rangle = (2\pi)^4 \delta^{(4)}(p_i - p_f)\mathcal{A}(i \rightarrow f) ,
\]

(7.6)

then sandwiching (7.5) between some initial and final state will give us

\[
i(2\pi)^4 \delta^{(4)}(p_i - p_f)(\mathcal{A}^*(f \rightarrow i) - \mathcal{A}(i \rightarrow f)) = \langle f \mid \hat{T}^\dagger \hat{T} \mid i \rangle .
\]

(7.7)

Using the completeness of the Hilbert space of states we may say

\[
1 = \sum_X \int \prod_{j \in X} \frac{d^3 p_j}{(2\pi)^3 2E_j} \frac{1}{X} \langle X \mid \rangle \rangle ,
\]

(7.8)
which once inserted into the right hand side of (7.7) we have The Optical Theorem

\[ \mathcal{A}(i \rightarrow f) - \mathcal{A}^*(f \rightarrow i) = i \sum_X \int \prod_{j \in X} \frac{d^3p_j}{(2\pi)^3} \frac{1}{2E_j} (2\pi)^4 \delta^{(4)}(p_i - p_X) \mathcal{A}(i \rightarrow X) \mathcal{A}^*(f \rightarrow X) , \]

(7.9)

and if, for instance we consider our initial and final state to both be a two particle state we have

\[ 2\Im(\mathcal{A}(2 \rightarrow 2)) = \sum_X \int \prod_{j \in X} \frac{d^3p_j}{(2\pi)^3} \frac{1}{2E_j} (2\pi)^4 \delta^{(4)}(p_i - p_X) |\mathcal{A}(2 \rightarrow X)|^2 . \]

(7.10)

7.1.1 Partial Wave Unitarity Bound

We can see that the right hand side of (7.10) looks at all possible intermediate states in the scattering process. Therefore we can say that

\[ 2\Im(\mathcal{A}(2 \rightarrow 2)) \geq \frac{\pi \delta(E_i - E_f)}{E_{CM}} |\mathcal{A}(2 \rightarrow 2)|^2 . \]

(7.11)

Now, as in [75], considering an elastic collision and decomposing our amplitudes into partial waves where

\[ \mathcal{A}(\theta) = 16\pi \sum_{j=0}^{\infty} (2j + 1)P_j(\cos(\theta))a_j , \]

(7.12)
where the $P_j$’s are the Legendre polynomials, (7.11) gives us

$$32\pi \sum_{j=0}^{\infty} (2j + 1) P_j(\cos(\theta)) \Im(a_j) \geq 16\pi^2 \frac{\delta(E_i - E_f)}{E_{CM}} \sum_{j=0}^{\infty} (2j + 1) P_j(\cos(\theta)) |a_j|^2 .$$

(7.13)

The point is, since $|a_j| \geq \Im(a_j)$, $|a_j|$ cannot be arbitrarily large. Thus Unitarity imposes an energy bound on scattering amplitudes.

### 7.2 Semi-Classical Approximation

Alberte and Bezrukov, [83], set out to answer the question ‘What is the semi-classical cross section for a theory going from a few hard quanta to many soft quanta?’ To best approximate this they wanted to find an expression for the cross section of the form

$$\sigma(E, N) = \int da_1 da_2 ... da_n \, e^{F(a_1, a_2, ..., a_n)/g^2} ,$$

(7.14)

where, for now, the $a_i$’s are placeholders for actual physical quantities we will shortly get to. The point is, we can approximate this integral in the semi-classical regime, $g \to 0$, by the saddle point approximation where we say

$$\sigma(E, N) \approx e^{F(a_1^*, a_2^*, ..., a_n^*)} ,$$

(7.15)
where $a^*_i$ is the value of the $a_i$ variable that extremizes the function $F$, that is

$$
\frac{\partial F}{\partial a_i} \bigg|_{a_i=a^*_i} = 0 .
\tag{7.16}
$$

To start we have for the formal cross section $\sigma(E,N) = \sum_{\text{final}} |\langle \text{final} | \hat{S} | \text{initial} \rangle|^2$,

$$
\sigma(E,N) = \sum_{\text{final}} |\langle \text{final} | \hat{S} | \text{initial} \rangle|^2 ,
\tag{7.17}
$$

where $\hat{S}$ is the $\hat{S}$-matrix and the sum is running over all possible final states with a final energy $E$ and total particle number $N$. To project out just these final states with $E$ and $N$ projection operators are used, that is to say they write

$$
| \text{final} \rangle = P_E P_N | f \rangle ,
\tag{7.18}
$$

where $P_E$ and $P_N$ are formally the projection operators for a given energy $E$ and total number of particles $N$ respectively. In order to get this cross section of the useful form (7.14) they use the coherent state representation (see Chapter 2 for a review of coherent states). Essentially we have

$$
\text{Quantum Mechanics} \rightarrow \text{Quantum field Theory}
\begin{align*}
| \alpha \rangle &= e^{-\frac{1}{2} |\alpha|^2} e^{\alpha \hat{a}^\dagger} | 0 \rangle
\rightarrow
| \alpha \rangle &= e^{-\frac{1}{2} \int d^3k \ |\alpha_k|^2 e^{\int d^3k \ \alpha_k \hat{a}_k^\dagger} | 0 \rangle ,
\end{align*}
\tag{7.19}
$$

since in Quantum Field Theory we now have an infinite amount of oscillators characterized by their momentum $\vec{k}$. 

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So, before considering the quantity in question, let’s first look at the quantity \[ \langle \beta \mid \hat{U} \mid \alpha \rangle , \] (7.20)

where \( \alpha \) and \( \beta \) are both coherent states and \( \hat{U} \) is the unitary time evolution operator. Now, using the analogous completion relation from quantum mechanics for the coordinate representation, that is

\[
\begin{align*}
\text{Quantum Mechanics} & \rightarrow \text{Quantum field Theory} \\
\int dx \ | x \rangle \langle x | = 1 & \rightarrow \int D\phi \ | \phi \rangle \langle \phi | = 1 ,
\end{align*}
\]

(7.21)

we can insert this unity twice to project some initial and final states. So they say

\[
\langle \beta \mid \hat{U} \mid \alpha \rangle = \int D\phi_i D\phi_f \langle \beta \mid \phi_f \rangle \langle \phi_f \mid \hat{U} \mid \phi_i \rangle \langle \phi_i \mid \alpha \rangle .
\]

(7.22)

Now, writing,

\[
\langle \phi_f \mid \hat{U} \mid \phi_i \rangle = \int_{\phi_i}^{\phi_f} D\phi \ e^{iS[\phi]} ,
\]

(7.23)

and noticing that \( \langle \phi_i \mid \alpha \rangle \) and \( \langle \beta \mid \phi_f \rangle \) are the Quantum Field Theory equivalents of (2.32) we may write,

\[
\langle \phi_i \mid \alpha \rangle = e^{B_i (\phi_i, \alpha_k)} ,
\]

(7.24)

\[
\langle \beta \mid \phi_f \rangle = e^{B_f (\phi_f, \beta_k^*)} ,
\]

(7.25)
where

\[ B_i(\phi_i, \alpha_{\vec{k}}) = -\frac{1}{2} \int d^3k \, \alpha_{\vec{k}}^* \alpha_{-\vec{k}} - \frac{1}{2} \int d^3k \, \omega_{\vec{k}} \phi_i(\vec{k}) \phi_i(-\vec{k}) + \int d^3k \, \sqrt{2\omega_{\vec{k}}} \alpha_{\vec{k}} \phi_i(\vec{k}) \, , \]  
\[ (7.26) \]

\[ B_f(\phi_i, \beta^*_{\vec{k}}) = -\frac{1}{2} \int d^3k \, \beta^*_{\vec{k}} \beta^*_{-\vec{k}} - \frac{1}{2} \int d^3k \, \omega_{\vec{k}} \phi_f(\vec{k}) \phi_f(-\vec{k}) + \int d^3k \, \sqrt{2\omega_{\vec{k}}} \beta^*_{\vec{k}} \phi_f(\vec{k}) \, , \]  
\[ (7.27) \]

where \( \phi_i(\vec{k}) \) and \( \phi_f(\vec{k}) \) are the Fourier Transforms of the initial and final asymptotic states and \( \omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2} \). Therefore we can write (7.20) in the form (7.14),

\[ \langle \beta | \hat{U} | \alpha \rangle = \int D\phi_i D\phi_f \int_{\phi_i}^{\phi_f} D\phi \, e^{B_i(\phi_i, \alpha_{\vec{k}})} e^{B_f(\phi_f, \beta^*_{\vec{k}})} e^{iS[\phi]} \, . \]  
\[ (7.28) \]

As discussed in [81, 82] to obtain the kernel of the \( \hat{S} \)-matrix (the quantity we’re interested in) we need to substitute

\[ B_i(\phi_i, \alpha_{\vec{k}}) \rightarrow B_i(\phi_i, \alpha_{\vec{k}} e^{-i\omega_{\vec{k}} T_i}) \, , \]  
\[ (7.29) \]

\[ B_f(\phi_f, \beta^*_{\vec{k}}) \rightarrow B_f(\phi_f, \beta^*_{\vec{k}} e^{i\omega_{\vec{k}} T_f}) \, . \]  
\[ (7.30) \]

Although we are interested in having a coherent state as our final state, we are only interested in a few particles as our incoming state. Defining a number operator \( \hat{N} \) as \( \hat{N} = \hat{a}^\dagger \hat{a} \) we have \( \langle \alpha | \hat{N} | \alpha \rangle = |\alpha|^2 \) we see that the number of particles of our
incoming state correlates with $\alpha$. Therefore, we will pick our initial state in (7.17) as

$$|\text{initial}\rangle = e^{J(\phi(0)+\dot{\phi}(0))} |0\rangle,$$  \hspace{1cm} (7.31)

where we will in the end be sending $J \to 0$. There is a significant amount of work done in the literature \cite{81, 86, 87} on how different choices of initial states will affect the outcome of this semi-classical approximation for the cross section, however it is believed that in the semi-classical approximation most choices for the intital state will not effect the qualitative results therefore we have picked an initial state that will be convenient for our purposes. When the dust settles we have for our formal expression for the cross section \cite{81, 82}

$$\sigma(E, N) = \int d\beta_k^* d\beta_k^* d\xi d\eta D\phi D\phi' \exp \left( -\int d\beta_k^* \beta_k^* e^{i\omega_k \xi + i\eta} + iE\xi + iN\eta \right. $$

$$\left. + B_i(\phi, 0) + B_f(\phi_f, \beta_k^* + B_i^*(\phi', 0) + B_f^*(\phi_f', \beta_k) + iS[\phi] - iS[\phi'] \right. $$

$$\left. + J\phi(0) + J\phi'(0) + J\dot{\phi}(0) + J\dot{\phi'}(0) \right).$$ \hspace{1cm} (7.32)

Lets dissect the physics from this expression. The $\beta$ integrals are instructing to look over all final coherent states with the Lagrange Multipliers $\xi$ and $\eta$ constraining the final states to have a specific energy $E$ and total number of particles $N$ (the projection operators $P_E$ and $P_N$). The $B_i$’s and $B_f$’s are the initial and final boundary conditions
respectively being,

\[ B_i(\phi_i, 0) = -\frac{1}{2} \int d^3k \, \omega_k \phi_i(\bar{k}) \phi_i(-\bar{k}) , \] (7.33)

\[ B_f(\phi_f, \beta^*_k) = -\frac{1}{2} \int d^3k \, \beta^*_k \beta^*_{-k} e^{-2i\omega_k T_f} - \frac{1}{2} \int d^3k \, \omega_k \phi_f(\bar{k}) \phi_f(-\bar{k}) \]
\[ + \int d^3k \, \sqrt{2\omega_k} e^{i\omega_k T_f} \beta^*_k \phi_f(-\bar{k}) , \] (7.34)

where \( T_f \) is some final late time. Now we make some assumptions on these variables and how they scale with our ‘\( \hbar \) parameter’, \( g \), in order to properly use the form of (7.14) to obtain this expression in the semi-classical approximation. With this in mind we scale

\[(\phi, \phi', \beta, \beta^*, J, E, N, \Lambda^3) \rightarrow \frac{1}{g}(\phi, \phi', \beta, \beta^*, J, \frac{\epsilon}{g}, \frac{n}{g}, \frac{1}{\lambda^3}) , \] (7.35)

where we have in mind we will be doing this calculation with Galileons so \( \Lambda \) is the original parameter in the Galileon action, the strong coupling scale, and \( g\Lambda^3 = \lambda^3 \).

This is to ensure in this rescaling, using a cubic Galileon as an example,

\[ S = \int d^4x \, -\frac{1}{2}(\partial \pi)^2 - \frac{1}{2} \Lambda^3 \Box \pi(\partial \pi)^2 \rightarrow \frac{1}{g^2} \int d^4x \, -\frac{1}{2}(\partial \pi)^2 - \frac{1}{g\Lambda^3} \Box \pi(\partial \pi)^2 \]
\[ = \frac{1}{g^2} \int d^4x \, -\frac{1}{2}(\partial \pi)^2 - \frac{1}{\lambda^3} \Box \pi(\partial \pi)^2 , \]

therefore \( S \rightarrow S/g^2 \) which gives us the scaling we want when looking at (7.14).

Therefore we see that \( \epsilon = g^2E, n = g^2N, j = gJ, \) and \( \lambda \) are our semi-classical variables. That is we assume they all stay finite in the classical \( g \rightarrow 0 \) limit. We
see that this limit is naturally a high energy and high number of particle limit, the precise limit we are interested in. With these assumptions we can write

$$\sigma(E, N) \sim \int d\beta_k^* d\beta_k d\xi d\eta D\phi D\phi' e^{F/\sigma^2},$$  \hspace{1cm} (7.36)$$

which is of the precise form (7.14). Now, the saddle point equations for \(\phi\), \(\phi_i(\vec{k})\), \(\phi_f(\vec{k})\), \(E\), and \(N\), give us [81]

\[
\frac{\delta S}{\delta \phi} = -iJ \left( \delta^{(4)}(x) - \frac{\partial}{\partial t} \delta^{(4)}(x) \right), \hspace{1cm} (7.37)
\]

\[
\phi_i(\vec{k}) = \frac{a^*_{-\vec{k}}}{\sqrt{2\omega_{\vec{k}}}} e^{i\omega_{\vec{k}}t}, \hspace{1cm} (7.38)
\]

\[
\phi_f(\vec{k}) = \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left( b_{\vec{k}} e^{\omega_{\vec{k}}(T-\theta-i\omega_{\vec{k}}t)} + b_{-\vec{k}}^* e^{i\omega_{\vec{k}}t} \right), \hspace{1cm} (7.39)
\]

\[
E = \int d^3k \omega_k \beta_k^* \beta_k e^{\omega_{\vec{k}}(T-\theta)}, \hspace{1cm} (7.40)
\]

\[
N = \int d^3k \beta_k^* \beta_k e^{\omega_{\vec{k}}(T-\theta)}, \hspace{1cm} (7.41)
\]

where a conjuncture is used, [82], that the saddle point values of \(\xi\) and \(\eta\) are imaginary so we replace \(T = i\xi\) and \(\theta = -i\eta\). We see from (7.37) that we have a source only at \(x = 0\) which corresponds to an energy jump. We can see this by looking at

\[
E = \int dt \frac{dE}{dt} = \int d^4x \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \dot{\phi} - L \right)
\]

\[
= \int d^4x \left( \frac{d}{dt} \frac{\partial L}{\partial \phi} \right) \dot{\phi} + \frac{\partial L}{\partial \phi} \ddot{\phi} - \frac{dL}{dt}, \hspace{1cm} (7.42)
\]
and with the equations of motion being

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = -\partial_i \frac{\partial L}{\partial \partial_i \phi} + \frac{\partial L}{\partial \phi} - \frac{\delta S}{\delta \phi}, \tag{7.43}
\]

and by rewriting

\[
\frac{dL}{dt} = \frac{\partial L}{\partial \phi} \dot{\phi} + \frac{\partial L}{\partial \dot{\phi}} \ddot{\phi} + \frac{\partial L}{\partial \partial_i \phi} \partial_i \dot{\phi}, \tag{7.44}
\]

which gives us

\[
E = -\int d^4 x \frac{\delta S}{\delta \phi} \dot{\phi} = iJ(\dot{\phi}(0) + \ddot{\phi}(0)). \tag{7.45}
\]

Now, we want to eventually take the \( J \to 0 \) limit to impose a small number of initial particles. Therefore, we will need to have a singularity in either \( \dot{\phi}(0) \) or \( \ddot{\phi}(0) \) to keep a finite energy. We will see that for our Galileon example we will impose the singularity in \( \ddot{\phi}(0) \). By using our saddle point solutions, \( (7.37), (7.38), \) and \( (7.39) \) we have the dominate contribution to the cross section being

\[
\sigma(E, N) \sim e^{W(E,N)}, \tag{7.46}
\]

where

\[
W(E, N) = \frac{1}{g^2} (\epsilon T - n \theta - 2\Im[S(\phi)]) = ET - N\theta - 2\Im[S(\phi)]. \tag{7.47}
\]

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So the problem of calculating \((7.32)\) reduces to solving the classical boundary value problem for the field, \((7.37), (7.38),\) and \((7.39)\). However, since we are interested in a few particle initial state, \((7.31)\), we are only concerned with a solution that is singular at \(x^\mu = 0\). Then, placing this solution into \((7.47)\) and using the relations

\[
T = \frac{\partial W}{\partial E},
\]

\[
\theta = \frac{\partial W}{\partial N},
\]

one can obtain an expression for the scattering cross section. For further details, refer to \([81, 82, 84, 85, 86, 87]\).
Bibliography


