Signed-Measure Valued Stochastic Partial Differential Equations with Applications in 2D Fluid Dynamics

by

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Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy

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CASE WESTERN RESERVE UNIVERSITY

May 2012
CASE WESTERN RESERVE UNIVERSITY
SCHOOL OF GRADUATE STUDIES

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Date of Defense: March 22, 2012
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Acknowledgements

There are several people whom I would like to thank for their support. Without
their help and guidance, I certainly would not have been successful at my time here
at Case. First and most importantly, I would like to thank to my research advisor,
Prof. Peter Kotelenez, for the endless support. His patience and guidance throughout
the process is something I never took for granted. I was very lucky to have him to
turn to for ideas and guidance. He also taught me a lot of life lessons, and he was
the reason I decided on the Case math program and my field of studies.

Prof. Daniela Calvetti has been a wonderful supporter of the graduate students
at Case, but she has been an especially strong advocate for me. I am indebted to her
for the innumerable ways that she has benefited my education and career. During
my time at Case, Prof. Calvetti has put lots of time and energy to create a great
environment for the graduate studies. I am very grateful for this, as it fostered very
strong friendships with my fellow students and our professors.

Prof. Elizabeth Meckes, Prof. Marshall Leitman, Prof. Manfred Denker, and
Prof. Peter Kotelenez put forth a lot of energy and time into serving on my thesis
committee. I would like to thank them for their effort and time to evaluate my thesis.

Chris Butler started me on this journey through mathematics. He gave me my
first exposure to the math department when he hired me as an S.I. in my second year
of undergrad. I am very grateful to him for giving me this opportunity.

Diane Robinson, Jeanne Jurkovich and Gaythreesa Lewis are the unsung heroines
of the math department. They have taken care of so many issues for me during the
last seven years. I am very appreciative of all their efforts.

The graduate students at Case are so much more than just students in the same
program. We are united by the same difficult trials and stresses throughout our years here. They have provided me with friendship and motivation, as well as distraction when I needed it. Thank you for all the memories at Yost.

My family always took the time to listen to my excitement and my frustration. I cannot thank them enough for the support that they provide me.

Finally, my biggest supporter is my girlfriend, Emily. She has been by my side throughout my entire time in the Ph.D. program. She has seen my biggest successes and hardest failures, and she has always provided support and an ear to let me vent. If there is anyone who is more excited for me to be completing my Ph.D., it is she. I will always be indebted to her for the patience she has shown me during the difficult periods of the program. I dedicate this thesis to her.
Abstract

Signed Measure Valued Stochastic Partial Differential Equations

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We note the interesting phenomenon that the Kantorovich-Rubinstein metric is not complete on the space of signed measures. Consequently, we introduce a new metric with a useful partial completeness property. With this metric, a general result about the Hahn-Jordan decomposition of solutions of stochastic partial differential equations is shown. These general results are applied to the smoothed Stochastic Navier-Stokes equations. As an application, we derive that the vorticity of the fluid is conserved for a solution of the Stochastic Navier-Stokes equations.
1 Introduction

The behavior of a fluid in motion is a phenomenon that intrigues and fascinates mathematicians and untrained observers alike. Historically, this fascination led to the development of the field of fluid dynamics. From Archimedes, Isaac Newton, and Leonhard Euler to George Stokes, Claude Navier, and Olga Ladyzhenskaya, many of the great mathematical minds contributed to the development of the field. To completely discuss even the major contributions to fluid dynamics would extend well beyond the scope of this work.\footnote{For a full introduction to the field of mathematical fluid dynamics, refer to the classic texts of [Lad69] or [CM93].} Rather, the Euler and Navier-Stokes equations shall provide the sufficient fluid dynamic basis for the following analysis.

To introduce these equations, consider an incompressible fluid\footnote{An incompressible fluid is a fluid which has a divergence free velocity field.} in motion in $\mathbb{R}^2$. To understand the evolution of the fluid in time, one examines the behavior of the velocity field and the vorticity. The vorticity is a scalar that represents the tendency of a fluid to rotate at a given point. For a rigid body, the vorticity is twice the angular momentum. Denote the velocity field by $U(r,t) = (U_1(r,t),U_2(r,t))^T \in \mathbb{R}^2$ and the vorticity of the fluid by $\mathcal{X}(r,t) \in \mathbb{R}$ where $T$ denotes transpose, $r \in \mathbb{R}^2$ and $t \geq 0$. Assume that the vorticity satisfies:

$$\frac{\partial}{\partial t} \mathcal{X}(r,t) = \nu \Delta \mathcal{X}(r,t) - \nabla \cdot (U(r,t) \mathcal{X}(r,t))$$

$$\mathcal{X}(r,t) = \text{curl } U(r,t) = \frac{\partial U_2}{\partial r_1} - \frac{\partial U_1}{\partial r_2}, \quad \nabla \cdot U \equiv 0 \quad (1.1)$$

where $\nu$ is the kinematic viscosity of the fluid, $\Delta$ is the Laplacian, $\nabla$ is the gradient, and $\cdot$ is the inner product on $\mathbb{R}^2$. For the case of an ideal or inviscid fluid, $\nu = 0$, and these equations represent the Euler equations in vorticity form. For $\nu > 0$, these
The condition, $\nabla \cdot U \equiv 0$, is known as the incompressibility condition, and it has several ramifications for (1.1). For the Euler equation, the vorticity must be conserved along particle paths in the fluids. That is, if $r(t, r_0)$ is the position of a fluid particle under (1.1) at $t \geq 0$ with position $r_0$ at $t = 0$, then

$$X(r(t, r_0), t) = X(r_0, 0).$$

(1.2)

This follows as (1.1) represents the differential form of the continuity equation$^3$ for the vorticity, as remarked by [Lon88].

Another consequence of the incompressibility condition is that one can explicitly express the velocity field in terms of the vorticity distribution as follows.$^4$

$$U(r, t) = \int K(r - q)X(q, t) dq$$

(1.3)

Here $\int (\cdot) dq$ denotes integration over $\mathbb{R}^2$ with respect to Lebesgue measure and $K(\cdot)$ is the Biot-Savart kernel which for $r = (r_1, r_2) \in \mathbb{R}^2$ is given by:

$$K(r) = \nabla^\perp \left( \frac{1}{2\pi} \ln(|r|) \right) = \frac{1}{2\pi |r|^2} (-r_2, r_1)$$

(1.4)

where $\nabla^\perp = (-\frac{\partial}{\partial r_2}, \frac{\partial}{\partial r_1})$ and $|r|^2 = r_1^2 + r_2^2$. Solving the Euler or Navier-Stokes equations encounters several difficulties due in part to the singularity of the Biot-Savart kernel at 0. Rather than explicitly solve the equations, many researchers sought to use (1.1) as a way to simulate flows. To create a numerical method for the Euler equations, Alexandre Chorin developed the regularized point-vortex method

$^3$See Lemma 6.1 for more details.

$^4$Refer to Lemma 6.2 for the derivation.
The methods introduced in [Cho73] followed the introduction of vortex simulation in [Ros31].

The point-vortex method in [Cho73] transforms (1.3) to a more convenient form by implementing two techniques. The first is to use mollifiers to avoid the singularity of the Biot-Savart kernel. That is, one replaces $K(\cdot)$ with $K_\delta(\cdot)$ where $K_\delta(r)$ is a smooth function agreeing with $K(r)$ for $0 < \frac{1}{\delta} < |r| < \delta$ and $\delta > 0$. Consequently, one has the following smoothed form of the velocity field.

$$ U_\delta(r, t) = \int K_\delta(r - q) \mathcal{X}(q, t) dq $$

The second technique assumes the initial vorticity, $\mathcal{X}(r, 0)$, has $N \in \mathbb{N}$ point vorticies with nonzero intensities, $a_k \in \mathbb{R}$, and positions, $r_0^k \in \mathbb{R}^2$. Thus, for $k = 1, \ldots, N$

$$ \mathcal{X}(r, 0) = \sum_{k=1}^{N} a_k \delta_{r_0^k} $$

where $\delta_x$ represents the point-measure concentrated at $x$. Assume the positions of the point-vortices satisfy the following system of differential equations. For $i = 1, \ldots, N$:

$$ \frac{dr^i}{dt} = \sum_{k=1}^{N} a_k K_\delta(r^i - r^k), \quad r^i(0) = r_0^i. $$

It follows that the empirical process associated with (1.7), given by

$$ \mathcal{X}_N(t) := \sum_{k=1}^{N} a_k \delta_{r_0^k(t)}, $$

is a weak solution\(^5\) of the Euler-equation, (1.1), with the smoothed Biot-Savart ker-

\(^5\)In this article, weak solutions shall always be in the sense of partial differential equations (hereafter PDE).
This means that if \( \phi \) is a smooth function from \( \mathbb{R}^2 \) to \( \mathbb{R} \) with compact support and \( \langle \cdot, \cdot \rangle \) is the duality between generalized functions and smooth functions, then we have the following differential:

\[
d < X_N(t), \phi > = \langle X_N(t), (U_{\delta,N} \cdot \nabla)\phi \rangle dt
\]  

(1.9)

where

\[
U_{\delta,N}(r,t) := \int K_\delta(r-q)X_N(dq,t).
\]  

(1.10)

Further, Chorin devises and implements a sampling algorithm to approximate arbitrary vortex distributions by discrete vortex distributions of the form (1.6). As remarked in [Cho73], the algorithm is successful as (1.7) is a rectangular quadrature rule to (1.3). Also, the approximation of initial vortex distributions leads to approximations in the general vortex distribution.

The techniques developed in [Cho73] generate significant progress in the case of Euler’s equation. However, the Navier-Stokes equation still provides difficulties due to the viscosity term (the term involving \( \nu \) in (1.1)). One approach is to examine the random point vortex method and the Stochastic Navier-Stokes equations. The random point vortex method modifies (1.7) to be a stochastic ordinary differential equation (hereafter SODE) rather than a deterministic differential equation. [MP82], [GHL90], [Lon88], [Kot95], [Ami00] all consider (1.7) with a stochastic driving term of the following form. For \( i = 1, \ldots, N \),

\[
\partial t^i r = \sum_{k=1}^{N} a_k K_\delta(r^i(t) - r^k(s))dt + \sqrt{2\nu}dm^i(r_N(t), t)
\]  

(1.11)

\(^6\)Since this property is an essential part of the new analysis presented in this work, a proof is provided in (3.40)-(3.42)
where (1.11) is written in differential form. The $m^i(\cdot)$ are continuous, square-integrable martingales (with respect to an underlying filtration and probability space) that may depend on the positions of the point vortices, $r_N(t) = (r^1(t), \ldots, r^N(t))^T$. The benefit of such an approach is two-fold.

- From a fluid dynamic viewpoint, one can interpret (1.11) as particle motion with random perturbation. As a result, the SODE system represents a model of turbulence in fluid flows.
- As in [Cho73] for the Euler equation, the empirical process associated with (1.11) for a discrete initial vorticity is formally a weak solution of a SPDE. This SPDE represents a random perturbation of the Navier-Stokes equations. Thus, to find solutions for an arbitrary initial vorticity, it suffices to derive a method to approximate initial vorticities by discrete vorticities.

To show the second remark, one applies Itô’s formula\(^8\) to the empirical process associated with (1.11), $\mathcal{X}_N(t)$. This yields the following for $\phi$ smooth and with compact support.

\[
d < \mathcal{X}_N(t), \phi > = \langle \mathcal{X}_N(t), (U_{8,N} \cdot \nabla)\phi \rangle dt \\
+ \nu \sum_{i=1}^{N} a_i \sum_{k,l=1}^{2} \partial_{kl}^2(\phi(r^i(t)))d \langle < m^i_k(r_N, t), m^i_l(r_N, t) > > \\
+ \sqrt{2\nu} \sum_{i=1}^{N} a_i \nabla \phi(r^i(t)) \cdot dm^i(r_N, t)
\]

(1.12)

where $\partial_{kl}^2$ denotes the second derivative with respect to the spatial coordinates

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\(^7\)We call such a SPDE a Stochastic Navier-Stokes Equation (hereafter SNSE).

\(^8\)For Itô’s formula, see Theorem 6.16 in the appendix.
\( r_k, r_l \) and \(< m_k^i(r_N, t), m_l^j(r_N, t) >\) denotes the mutual quadratic variation process of the one-dimensional components of \( m^i(\cdot)\).\(^9\)

In [MP82], [Lon88], and [GHL90] (as well as numerous other approaches), the authors choose \( m^i(r_N(t), t) \) as independent 2-dimensional Brownian motions, \( \beta^i(t) \), for \( i = 1, \ldots, N \). This reduces (1.12) to the following:

\[
d < \mathcal{X}_N(t), \phi > = < \mathcal{X}_N(t), (U_{\delta,N} \cdot \nabla)\phi > dt \\
+ < \mathcal{X}_N(t), \nu \Delta \phi > dt + \sqrt{2\nu} \sum_{i=1}^{N} a_i \nabla \phi(r^i(t)) \cdot d\beta^i(t).
\]  
(1.13)

Note that if the stochastic term did not appear in (1.13), the equation would represent the weak form of the smoothed Navier-Stokes equations. Hence, (1.13) represents a SNSE.

In both [Lon88] and [GHL90], the authors focus on the numerical aspects of the algorithms following from [Cho73]. The authors in [GHL90] show that the point vortex algorithm for the 2D Euler equation is consistent and stable. Further, the authors derive a second order estimate on the error. Long examines the random point vortex method as a numerical algorithm in [Lon88]. Here the author proves a nearly optimal rate of convergence according to the Central Limit Theorem. Yet, in [MP82], the authors establish an interesting advance in the general theory. The authors show the following with some additional hypotheses. As the number of vortices, \( N \), tends to infinity, the empirical process, \( \mathcal{X}_N(t) \), will approximate weak solutions of (1.1).\(^10\)

We call limits where \( N \to \infty \) continuum-limits, and such limits play a central role in the later analysis.

The results of [MP82], [Lon88], and [GHL90] provide important advances to the

\(^9\)See (6.14) for the definition.

\(^10\)See Chapter 4 for a more comprehensive review of [MP82].
theory of SNSE. Yet, the choice of \( m^i(r_N(t), t) = \beta^i(t) \) has several drawbacks. In [Kot95], the author summarizes the limitations of using independent Brownian motions in (1.11) as follows:

- In (1.13), the position of each particle is perturbed by its own “fluctuation force,” \( \beta^i(t) \). This creates an identification that is preserved in the stochastic term, yet disappears in the deterministic term. Consequently, the identification prevents (1.13) from representing a smoothed signed-measure valued SNSE. (1.13) represents a SNSE, but only in a formal way.

- The choice of \( m^i(r_N(t), t) = \beta^i(t) \) yields fluctuation forces which are state independent (i.e. do not depend on \( r_N(t) \)). From the physical interpretation, it is more desirable that the fluctuation forces are state dependent.

- Choosing \( m^i(r_N(t), t) \) as independent Brownian motions rather than spatially correlated increases the high singularity to an associated SPDE on signed-measures.

In order to address these limitations, [Kot95] introduces correlation functions, \( \hat{\Gamma}_\epsilon(r, q) \), in [Kot95] to describe the fluctuation in the motion of the particles. Here \( \hat{\Gamma}_\epsilon : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathcal{M}_{2 \times 2} \) where \( \mathcal{M}_{2 \times 2} \) is the set of \( 2 \times 2 \) matrices over \( \mathbb{R} \). This leads to the removal of the taggings and fluctuation forces that are state dependent, spatially correlated, and driven by Brownian sheets. In this method, the following choice for the square-integrable martingales is made.

\[
m^i(r_N, t) := \int_0^t \int \hat{\Gamma}_\epsilon(r^i(s), p) w(dp, ds)
\]

(1.14)

where \( w(dp, dt) = (w_1(dp, dt), w_2(dp, dt))^T \) is standard space-time Gaussian white
noise differentials.\(^{11}\) With this choice of \(m^i(r_N, t)\), \((1.12)\) becomes the following.

\[
\begin{align*}
\nu < X_N(t), \Delta \phi > dt + \nu < X_N(t), \nabla \cdot (U_\delta \cdot \nabla) \phi > dt \\
+ \nu < X_N(t), \Delta \phi > dt + \sqrt{2\nu} \left\langle X_N(t), \int \hat{\Gamma}_\epsilon(t, p) w(dp, dt), \nabla \phi \right\rangle \\
\end{align*}
\]

Note that integrating \((1.15)\) by parts in the sense of generalized functions shows that \(X_N(t)\) is a weak solution of the following smoothed, signed-measure valued Stochastic Navier-Stokes Equation:

\[
dX(t) = \left[ \nu \Delta X - \nabla \cdot (U_\delta X) \right] dt - \sqrt{2\nu} \nabla \cdot \left( X \int \hat{\Gamma}_\epsilon(\cdot, p) \right) w(dp, dt). \\
\]

As in [Cho73] and [MP82], the author in [Kot95] derives continuum-limit results to generalize the solution of \((1.16)\) from discrete initial vorticities to arbitrary initial vorticities. Recall from \((1.2)\), it is a natural consequence that the vorticity is conserved. Consequently, the author requires the conservation of vorticity in solutions of \((1.16)\) as an additional condition for solving any SNSE. It follows from the construction in [Kot95] that for discrete initial vorticities, the vorticity is conserved. When passing to the continuum limit, [Kot95] employs a product Wasserstein metric. However, restricting the analysis to this metric, the author could not show the preservation of the Hahn-Jordan decomposition in the continuum limit.

In [Ami00], the author extends the results of [Kot95] by analyzing a SODE system similar to \((1.11)\). The author includes a term driven by a Poisson random measure. Such a term is useful as a correction factor to accommodate for unusual physical phenomenons involving the kinematic viscosity. In [Ami07], the author revisits the

\(^{11}\)See Definition 6.15 for formulation.
stochastic vortex theory. Here, Amirdjanova remarks on the difficulties that incompleteness imposes on the analysis. However, in [Ami07], the author can only show the existence and uniqueness of the SODE for positive measures. As a result of the work in [Kot95] and [Ami07], several questions naturally arise.

- One of the standard choices for a metric in SPDE theory is not complete on the space of signed-measures. Can one alter the definition of the metric to yield completeness on this space, but without destroying the usefulness of this metric in the analysis?

- The continuum-limit in [Kot95] does not show conservation of vorticity. With an alternative choice of metric, can one show conservation of vorticity in the continuum-limit?

Seeking answers to these questions led to the original work derived in this work. The analysis created new results in several areas, and we divide this work into chapters based on these areas.

- **Chapter 2: Completeness on the Space of Signed Measures**
  For a complete, separable metric space, we show that the space of finite, Borel signed measures is not complete under the Kantorovich-Rubinstein metric. Further, the natural extension to a product metric is shown to be incomplete on the signed measures. Instead, a new metric is introduced using a quotient-space inspired approach. Further, a useful partial completeness property is derived for the metric.

- **Chapter 3: Hahn-Jordan Decomposition for Signed-Measure Valued SPDEs**
  We consider a general class of signed-measure valued SPDE and the Hahn-Jordan decomposition of a solution. Similar to the argument in (1.12), the
empirical process for an associated SODE-system will satisfy the SPDE. Work in [Kot10] shows that the Hahn-Jordan decomposition for an initial signed-measure is preserved provided that either the initial measure is discrete or that the coefficients of the SODE are sufficiently smooth. Using the partially-complete metric from Chapter 2, we generalize the Hahn-Jordan result to the case where the coefficients of the SODE are only assumed to be Lipschitz. 12

- **Chapter 4: Smooth SNSE**

  The results of the general signed-measure valued SPDEs from Chapter 3 apply to the smoothed version of the SNSE, (1.16). From this, we establish that the vorticity is conserved in the SNSE. 13

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12This result is from the work in [KS12b].
13This result is from the work in [KS12a].
2 Completeness and Signed Measures

In this chapter, the standard metrics on the signed measures are introduced, most notably the Kantorovich-Rubinstein metric. It is shown that this metric fails completeness on the signed measures despite being complete on positive measures. Identifying a relationship between signed measures and diagonal sets inspires a new metric similar to the quotient-space metric. With the goal of completeness in mind, this metric fails convergence for arbitrary Cauchy sequences of signed measures. However, the metric satisfies a more useful partial-completeness result. A Cauchy sequence of signed measures converges if and only if the following property holds. The Hahn-Jordan decompositions of a subsequence converge to a limit which can be identified as a signed measure. Such a result is very desirable in the subsequent analysis. One can conclude a pair of measures represents the Hahn-Jordan decomposition of a signed measure by the convergence properties of the metric.

2.1 Definitions of Metrics

Let \((S, \psi)\) be a complete, separable metric space with countable, dense set \(T\). One typically takes \(S\) as \(\mathbb{R}^d\) for \(d \in \mathbb{N}\) and \(\psi\) as the Euclidean metric; however, the following results hold in more generality. Define \(\varrho(r, q) := \psi(r, q) \wedge 1\) where \(r, q \in S\) and \(\wedge\) denotes minimum. By [Mun00], \(\varrho\) induces the same topology as \(\psi\). Consequently, it follows that \((S, \varrho)\) is also a complete, separable metric space. Denote \(M_f(S)\) as the space of finite, Borel measures on \((S, \varrho)\). Many of the later applications require a metric on \(M_f(S)\) to derive certain a priori estimates. To introduce such metrics, the Wasserstein distance must be defined. For \(\mu, \nu \in M_f(S)\), we define \(C(\mu, \nu)\) as the set of joint representations of \(\mu\) and \(\nu\). This means that \(Q \in C(\mu, \nu)\) is a measure on the
product space, $S \times S$, that satisfies $Q(A \times S) = \mu(A)\nu(S)$, $Q(S \times B) = \mu(S)\nu(B)$ for arbitrary Borel sets $A,B$ of $S$. Define the Wasserstein distance for $\mu, \nu \in M_f(S)$ as

$$\mathcal{W}_1(\mu, \nu) := \inf_{Q \in \mathcal{C}(\mu, \nu)} \int \rho(r, q) Q(dr, dq).$$

(2.1)

By [Dud02], the Wasserstein distance defines a metric on the probability measures in $M_f(S)$, which is denoted by $\mathcal{P}_1(S)$.

Although widely used, the above definition of the Wasserstein distance is not advantageous in general. In particular, the Wasserstein distance does not define a metric on $M_f(S)$ unless one only considers measures of equal mass. The famous Kantorovich-Rubinstein Theorem provides an alternative representation of the Wasserstein metric in terms of a dual norm. This dual norm is used to define a metric on the set of finite, Borel, signed measures. One needs several definitions prior to stating the theorem. Denote the space of all Lipschitz continuous functions from $S$ into $\mathbb{R}$ by $C_L(S, \mathbb{R})$, and denote the space of all uniformly bounded, Lipschitz functions from $S$ into $\mathbb{R}$ by $C_{L,\infty}(S, \mathbb{R})$. One can endow $C_{L,\infty}(S, \mathbb{R})$ with the norm $\|\cdot\|_{L,\infty}$ where $\|\cdot\|_{L,\infty} = \|\cdot\|_{\infty} \vee \|\cdot\|_L$, $\vee$ denotes maximum and

$$\|f\|_{\infty} := \sup_{s \in S} |f(s)|, \quad \|f\|_L := \sup_{r,q \in S, r \neq q} \left\{ \frac{|f(r) - f(q)|}{\rho(r, q)} \right\}. \quad \text{(2.2)}$$

**Theorem 2.1 Kantorovich-Rubinstein Theorem**

For $\mu, \nu \in \mathcal{P}_1(S)$, define

$$\gamma_f(\mu, \nu) := \sup_{\|f\|_{L,\infty} \leq 1} \left| \int f(r)(\mu - \nu)(dr) \right|$$

(2.3)

then, $\gamma_f(\mu, \nu) = \mathcal{W}_1(\mu, \nu)$.  

12
\( \gamma_f(\mu, \nu) \) is often written as \( \gamma_f(\mu - \nu) \) due to the linearity of the definition. The original statement of the Kantorovich-Rubinstein Theorem has the supremum taken over \( \|f\|_L \leq 1 \). This case is shown as a step in the proof.

**Proof:** See Theorem 6.3 for the complete proof.

Clearly, one can extend the definition of \( \gamma_f \) to \( M_f(S) \) by (2.3). The spaces, \((\mathcal{P}_1(S), \mathcal{W}_1)\) and \((M_f(S), \gamma_f)\) inherit properties from the underlying metric space, \((S, \varrho)\).

**Theorem 2.2** For a complete, separable metric space \((S, \varrho)\), the spaces, \((\mathcal{P}_1(S), \mathcal{W}_1)\) and \((M_f(S), \gamma_f)\), are also complete and separable.

**Proof:** Define

\[
M_{f,d} := \left\{ \mu := \sum_{i=1}^{N} a_i \delta_{t_i}, N \in \mathbb{N}, t_i \in T \text{ and } a_i \in [0, \infty) \cap \mathbb{Q}, \ i = 1, \ldots, N \right\}.
\]

(2.4)

By Theorem 6.7, \( M_{f,d} \) forms a countable, dense set in \( M_f(S) \) for \( \gamma_f \). Similarly, restricting \( M_{f,d} \) to elements with \( \sum_{i=1}^{N} a_i = 1 \) is a dense set for \((\mathcal{P}_1, \mathcal{W}_1)\).

To show completeness, we follow the argument in [Kot08]. From the definition of \( \gamma_f \), it is clear that \( \gamma_f \) represents a functional norm on the dual of \( C_{L,\infty}(S, \mathbb{R}) \). Denote this dual by \( C_{L,\infty}^*(S, \mathbb{R}) \), and note that \( M_f(S) \) is a cone in \( C_{L,\infty}^*(S, \mathbb{R}) \). To show that \((M_f(S), \gamma_f)\) is complete, it suffices to show that it is closed in the norm topology. Suppose \( \mu_n \in M_f(S) \) converges in the norm topology to a limit \( \mu \). As \( C_{L,\infty}^*(S, \mathbb{R}) \) is a Banach space, convergence in the norm topology implies convergence in the weak-* topology. By the Riesz Representation Theorem, it follows that \( \mu \) must be a finite, Borel measure, and thus \( \mu \in M_f(S) \). The other case follows by the Kantorovich-Rubinstein Theorem. \( \square \)
2.2 Incompleteness Issues

Denote the space of finite, Borel signed measures on \((S, \varrho)\) by \(M_{f,s}(S)\). From (2.3), the definition of \(\gamma_f(\mu, \nu)\) easily extends from \(M_f(S)\) to \(M_{f,s}(S)\). The extension, 

\((M_{f,s}(S), \gamma_f)\), is a normed vector space, and \(\gamma_f\) is called the Kantorovich-Rubinstein distance. Due to its convenient dual form, the Kantorovich-Rubinstein distance is used throughout the analysis. One might expect that a theorem similar to Theorem 2.2 holds for \((M_{f,s}(S), \gamma_f)\). However, the following result shows that the signed measures under \(\gamma_f\) do not have the same properties.

**Theorem 2.3** \(M_{f,s}(S)\) is not complete with respect to \(\gamma_f\), but \((M_{f,s}(S), \gamma_f)\) is separable.

**Proof:** Fix \(s \in S\) and \(\forall k \in \mathbb{N}\) choose \(t_k \in T\) such that \(\varrho(t_k, s) < \frac{1}{k^2}\). For all \(n \in \mathbb{N}\), define \(\mu_n \in M_{f,s}(S)\) by:

\[
\mu_n = \sum_{k=1}^{n} (-1)^k \delta_{t_k}.
\]

We verify that \(\{\mu_n\}_{n=1}^{\infty}\) is Cauchy with respect to \(\gamma_f\). For \(m, n \in \mathbb{N}\) with \(m \geq n\) and \(f \in C_{L,\infty}(S, \mathbb{R})\) with \(\|f\|_{L,\infty} \leq 1\):

\[
\left| \int f(r)(\mu_n(dr) - \mu_m(dr)) \right| = \left| \sum_{k=1}^{n} (-1)^k f(t_k) - \sum_{k=1}^{m} (-1)^k f(t_k) \right| \\
= \left| \sum_{k=m+1}^{n} (-1)^k f(t_k) \right| \\
\leq \sum_{k=m+1}^{n-1} |f(t_k) - f(t_{k+1})| \leq \sum_{k=m+1}^{n-1} \varrho(t_k, t_{k+1})
\]

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Thus, \( \gamma_f(\mu_n - \mu_m) := \sup_{\|f\|_{L,\infty} \leq 1} \left| \int f(r)(\mu_n - \mu_m)(dr) \right| \rightarrow 0 \) as \( n, m \rightarrow \infty \).

Consequently, \( \{\mu_n\}_{n=1}^{\infty} \) is Cauchy with respect to \( \gamma_f \). Suppose there exists a finite signed measure, \( \nu \), such that \( \gamma_f(\mu_n - \nu) \rightarrow 0 \) as \( n \rightarrow \infty \). Then, \( \forall f \in C_{L,\infty}(S, \mathbb{R}) \) with \( \|f\|_{L,\infty} \leq 1 \), \( \int f(r)(\mu_n - \nu)(dr) \rightarrow 0 \) as \( n \rightarrow \infty \). Yet, consider \( f \equiv 1 \in C_{L,\infty}(S, \mathbb{R}) \). For this choice of \( f \),

\[
\left| \int f(r)(\mu_n - \nu)(dr) \right| = \left| \int 1(r)(\mu_n - \nu)(dr) \right| = \left| \sum_{k=1}^{n} (-1)^k - \nu(S) \right|
\]

\[
= \begin{cases} 
|1 + \nu(S)| & \text{if } n = 2\ell + 1 \text{ for some } \ell \in \mathbb{N} \\
|\nu(S)| & \text{if } n = 2\ell \text{ for some } \ell \in \mathbb{N}.
\end{cases}
\]

Consequently, such a \( \nu \in M_f(S) \) cannot exist. To show that \( (M_{f,s}(S), \gamma_f) \) is separable, consider the following:

\[
M_{f,s,d} := \left\{ \mu := \sum_{i=1}^{N} a_i \delta_{t_i}, N \in \mathbb{N}, t_i \in T \text{ and } a_i \in \mathbb{Q} \ i = 1, \ldots, N \right\} \quad (2.5)
\]

Theorem 6.7 shows that \( M_{f,s,d} \) is dense in \( M_{f,s}(S) \) under \( \gamma_f \). Since \( M_{f,s,d} \) is countable, the claim follows.

\[ \square \]

The Kantorovich-Rubinstein distance, \( \gamma_f \), fails completeness on \( M_{f,s}(S) \), but its restriction to \( M_f(S) \) is complete. From this remark, one might believe that using the Hahn-Jordan decomposition of a signed measure may yield completeness on \( M_{f,s}(S) \).
For convenience, the Hahn-Jordan decomposition is stated in the current notation.

**Proposition 2.4  Hahn-Jordan Decomposition**

For \( \nu \in M_{f,s}(S) \), there exists \( \nu^+, \nu^- \in M_f(S) \) such that \( \nu = \nu^+ - \nu^- \) and \( \nu^+ \perp \nu^- \).

Further, \( \nu^+, \nu^- \) are unique up to \( \nu \)-null sets.

**Proof:** Refer to [Fol99] for the proof.

Using the Hahn-Jordan Decomposition and the completeness of \((M_f(S), \gamma_f)\), the product space of measures may satisfy completeness for \( M_{f,s}(S) \). Define the product space of measures and signed measures as:

\[
\hat{M}_f(S) := \{ \hat{\mu} = (\mu_1, \mu_2) : \mu_1, \mu_2 \in M_f(S) \} \\
\hat{M}_{f,s}(S) := \{ \hat{\nu} = (\nu_1, \nu_2) : \nu_1, \nu_2 \in M_{f,s}(S) \}. 
\]

(2.6)

Call \( \hat{M}_f(S) \) (respectively, \( \hat{M}_{f,s}(S) \)) the space of measure pairs (respectively, signed measure pairs). Note that \( \hat{M}_{f,s}(S) \) is a real vector space with componentwise addition and scalar multiplication. Hence, \( \hat{M}_f(S) \) is a cone in \( \hat{M}_{f,s}(S) \) as \( \hat{M}_f(S) \) is closed under componentwise addition, but only scalar multiplication for positive scalars is well-defined. Define the product metric for \( \hat{\mu} = (\mu_1, \mu_2), \hat{\nu} = (\nu_1, \nu_2) \in \hat{M}_{f,s}(S) \):  

\[
\hat{\gamma}_f(\hat{\mu}, \hat{\nu}) := \gamma_f(\mu_1, \nu_1) + \gamma_f(\mu_2, \nu_2) 
\]

(2.7)

From Theorem 2.2 and Theorem 2.3, it follows that \((\hat{M}_{f,s}(S), \hat{\gamma}_f)\) is a separable metric space, and restricted to \( \hat{M}_f(S) \), \( \hat{\gamma}_f \) is complete.

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2.3 New Results

Identify a signed measure, $\mu \in \mathcal{M}_{f,s}(S)$, with the measure pair $\mu^\pm := (\mu^+, \mu^-) \in \mathcal{M}_{f,s}(S)$ by the representation $\mu = \mu^+ - \mu^-$ where $\mu^\pm$ is the Hahn-Jordan decomposition of $\mu$. The following example shows that $\hat{\gamma}_f$ does not, in general, preserve the Hahn-Jordan decomposition in limits.

Example 2.5 Suppose $x, y \in S$ and there exist two sequences of elements $\{v_n\}_{n \geq 1}$ and $\{u_n\}_{n \geq 1}$ in $S$ such that $u_n \neq v_n$, $u_n \neq x$, and $v_n \neq x \forall \ n \in \mathbb{N}$. Further, assume that $v_n \to x$ and $u_n \to x$ as $n \to \infty$ under the metric $\rho$. Now, consider the signed measure $\mu_n = \delta_y + \delta_{v_n} - \delta_{u_n}$. For sufficiently large $n$, one can identify $\mu_n$ with $\mu_n^\pm := (\delta_y + \delta_{v_n}, \delta_{u_n})$. We claim that $\mu_n^\pm$ converges in $\hat{\gamma}_f$ to $(\delta_y + \delta_x, \delta_x)$. To establish this, it suffices to show that $\delta_{u_n} \to \delta_x$ under $\gamma_f$. The other terms follow analogously.

$$\gamma_f(\delta_{u_n}, \delta_x) = \sup_{\|f\|_{L,\infty} \leq 1} \left| \int f(r)(\delta_{u_n} - \delta_x)(dr) \right|$$
$$= \sup_{\|f\|_{L,\infty} \leq 1} |f(u_n) - f(x)| \leq \rho(u_n, x) \to 0 \text{ as } n \to \infty$$

Consequently, we conclude that $\mu_n^\pm \to (\delta_y + \delta_x, \delta_x)$ as $n \to \infty$ whereas $(\delta_y, 0)$ is the Hahn-Jordan decomposition of the signed measure $(\delta_y + \delta_x, \delta_x)$, and $0$ is the measure which assigns $0$ to all Borel sets of $S$. \hfill \Box

As a result, $(\mathcal{M}_{f,s}(S), \hat{\gamma}_f)$ is not a complete space. Indeed, one cannot identify limits under $\hat{\gamma}_f$ as signed measures due to the Hahn-Jordan representations. However, the set of signed measures plays a special role in the product space. One can identify the signed measures with a quotient space under $\hat{\mathcal{M}}_f(S)$. Let

$$\hat{\mathcal{D}}_s := \{(\mu, \nu) \in \hat{\mathcal{M}}_{f,s}(S) : \nu = \mu \text{ in } \mathcal{M}_{f,s}(S)\}.$$
\( \hat{D}_s \) is a closed subspace of the vector space, \( \hat{M}_{f,s}(S) \), and one can consider the quotient space, \( \hat{M}_{f,s}(S)/\hat{D}_s \). Denote the quotient map that sends a signed measure pair, \( \hat{\mu} \), to its equivalence class, \( \pi(\hat{\mu}) \), by \( \pi: \hat{M}_{f,s}(S) \longrightarrow \hat{M}_{f,s}(S)/\hat{D}_s \). With this quotient space, note that if \( \hat{\nu} = (\nu_1, \nu_2), \hat{\mu} = (\mu_1, \mu_2) \in \hat{M}_{f,s}(S) \), then

\[
\pi((\mu_1, \mu_2)) = \pi((\nu_1, \nu_2)) \iff (\mu_1, \mu_2) - (\nu_1, \nu_2) \in \hat{D}_s
\]

\[
\iff (\mu_1 - \nu_1, \mu_2 - \nu_2) = (\eta, \eta) \text{ where } \eta \in M_{f,s}(S)
\]

\[
\iff \mu_1 - \nu_1 = \mu_2 - \nu_2 \text{ in } M_{f,s}(S)
\]

\[
\iff \mu_1 - \mu_2 = \nu_1 - \nu_2 \text{ in } M_{f,s}(S)
\]

\[
\iff \hat{\mu} \text{ and } \hat{\nu} \text{ define the same signed measure.}
\]

This important relationship between the signed measures and the quotient space is crucial in the following analysis. Identifying a signed measure with its equivalence class avoids the Hahn-Jordan preservation issues apparent in Example 2.5. Furthermore, the relationship provides the correct setting for a well-defined, Kantorovich-Rubinstein-type metric on the space of signed measures. Prior to defining such a metric, we make one more observation regarding the Hahn-Jordan decomposition in the quotient-space setting. Let \( \hat{\mu}^\pm = (\mu^+, \mu^-) \) be the Hahn-Jordan decomposition of a signed measure \( \mu \). Note that if also \( \mu = \mu_1 - \mu_2 \) where \( (\mu_1, \mu_2) \in \hat{M}_f(S) \), then we have that for some \( \eta \in M_{f,s}(S) \):

\[
\mu = \mu^+ - \mu^- = \mu_1 - \mu_2
\]

\[
\iff \eta := \mu_2 - \mu^- = \mu_1 - \mu^+
\]

\[
\iff \mu_2 = \mu^- + \eta \quad \mu_1 = \mu^+ + \eta.
\]
tion for the case when \( \mu_1 = \mu_2 = \mu \) by setting \((\mu, \mu)^\pm = (0, 0)\). This implies that for all \( \hat{\mu} \in \hat{\mathcal{M}}_f(S) \) that

\[
\pi(\hat{\mu}) = \hat{\mu}^\pm + \hat{D}_+ \tag{2.10}
\]

where \( \hat{D}_+ := \{ (\nu, \rho) \in \mathcal{M}_f(S) : \nu = \rho \text{ in } \mathcal{M}_f(S) \} \). Note that since \( \hat{D}_+ \) is closed in \( \hat{\mathcal{M}}_f(S) \), so is \( \pi(\hat{\mu}) \).

**Proposition 2.6** The following defines a metric on the space of signed measures under its identification with the quotient space, \( \hat{\mathcal{M}}_{f,s}(S)/\hat{D}_s \). For \( \hat{\mu}, \hat{\nu} \in \hat{\mathcal{M}}_f(S) \)

\[
\lambda(\pi(\hat{\mu}), \pi(\hat{\nu})) := \inf_{\hat{\eta} \in \hat{D}_+} [\hat{\gamma}_f(\hat{\mu}^\pm - \hat{\nu}^\pm - \hat{\eta}) \vee \hat{\gamma}_f(\hat{\nu}^\pm - \hat{\mu}^\pm - \hat{\eta})]. \tag{2.11}
\]

**Proof:** We obviously have symmetry: \( \lambda(\pi(\hat{\mu}), \pi(\hat{\nu})) = \lambda(\pi(\hat{\nu}), \pi(\hat{\mu})) \).

Now, note that if \( \pi(\hat{\mu}) = \pi(\hat{\nu}) \), then \( \hat{\mu}^\pm = \hat{\nu}^\pm \), and the infimum on the right side of (2.11) is 0. If \( \lambda(\pi(\hat{\mu}), \pi(\hat{\nu})) = 0 \), then there exist a sequence \( \{ \hat{\eta}_n \} \subset \hat{D}_+ \) such that \( \hat{\eta}_n \longrightarrow (\hat{\mu}^\pm - \hat{\nu}^\pm) \) and \( \hat{\eta}_n \longrightarrow (\hat{\nu}^\pm - \hat{\mu}^\pm) \) from which it follows that \( \hat{\mu}^\pm = \hat{\nu}^\pm \) and \( \pi(\hat{\mu}) = \pi(\hat{\nu}) \).

For the triangle inequality, let \( \epsilon > 0, \hat{\nu}, \hat{\mu}, \hat{\beta} \in \hat{\mathcal{M}}_{f,s}(S) \), and choose \( \hat{d}_1, \hat{d}_2 \in \hat{D}_+ \) such that:

\[
\lambda(\pi(\hat{\nu}), \pi(\hat{\mu})) + \frac{\epsilon}{2} > \hat{\gamma}_f(\hat{\nu}^\pm - \hat{\mu}^\pm - \hat{d}_1) \vee \hat{\gamma}_f(\hat{\nu}^\pm - \hat{\mu}^\pm - \hat{d}_1)
\]

\[
\lambda(\pi(\hat{\mu}), \pi(\hat{\beta})) + \frac{\epsilon}{2} > \hat{\gamma}_f(\hat{\mu}^\pm - \hat{\beta}^\pm - \hat{d}_2) \vee \hat{\gamma}_f(\hat{\beta}^\pm - \hat{\mu}^\pm - \hat{d}_2).
\]

Note that \( \hat{d}_1 + \hat{d}_2 \in \hat{D}_+ \). Since for nonnegative constants \( a, b, c, d \), we have that:

\[
(a + b) \vee (c + d) \leq (a \vee c) + (b \vee d),
\]

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it follows that:

\[
\lambda(\pi(\hat{\nu}), \pi(\hat{\beta})) \leq \hat{\gamma}_f(\hat{\nu}^\pm - \hat{\beta}^\pm - (\hat{d}_1 + \hat{d}_2)) \lor \hat{\gamma}_f(\hat{\beta}^\pm - \hat{\nu}^\pm - (\hat{d}_1 + \hat{d}_2))
\]

\[
\leq [\hat{\gamma}_f(\hat{\nu}^\pm - \hat{\mu}^\pm - \hat{d}_1) + \hat{\gamma}_f(\hat{\mu}^\pm - \hat{\beta}^\pm - \hat{d}_2)] \lor [\hat{\gamma}_f(\hat{\mu}^\pm - \hat{\nu}^\pm - \hat{d}_1) + \hat{\gamma}_f(\hat{\beta}^\pm - \hat{\mu}^\pm - \hat{d}_2)]
\]

\[
\leq [\hat{\gamma}_f(\hat{\nu}^\pm - \hat{\mu}^\pm - \hat{d}_1) \lor \hat{\gamma}_f(\hat{\mu}^\pm - \hat{\nu}^\pm - \hat{d}_1)] + [\hat{\gamma}_f(\hat{\mu}^\pm - \hat{\beta}^\pm - \hat{d}_2) \lor \hat{\gamma}_f(\hat{\beta}^\pm - \hat{\mu}^\pm - \hat{d}_2)]
\]

\[
< \lambda(\pi(\hat{\nu}), \pi(\hat{\mu})) + \lambda(\pi(\hat{\mu}), \pi(\hat{\beta})) + \epsilon.
\]

Letting \( \epsilon \to 0 \) yields the triangle inequality.

\[\square\]

The Kantorovich-Rubinstein distance, \( \gamma_f \), and its associated product metric, \( \hat{\gamma}_f \), are used to define the quotient-type metric, \( \lambda \). Consequently, there are several inequalities relating the metrics. These inequalities are used frequently in understanding the role of completeness for \( \lambda \). To state the inequalities, define the mapping \( \Phi : \hat{M}_{f,s}(S) \to M_{f,s} \) by \( \Phi(\hat{\mu}) = \mu_1 - \mu_2 \) where \( \hat{\mu} = (\mu_1, \mu_2) \in \hat{M}_{f,s}(S) \).

**Proposition 2.7** For \( \hat{\mu}, \hat{\nu} \in \hat{M}_{f,s}(S) \),

\[
\lambda(\pi(\hat{\mu}), \pi(\hat{0})) = \hat{\gamma}_f(\hat{\mu}^\pm)
\]

(2.12)

and

\[
\gamma_f(\Phi(\hat{\mu}) - \Phi(\hat{\nu})) \leq \lambda(\pi(\hat{\mu}), \pi(\hat{\nu})) \land \hat{\gamma}_f(\hat{\mu} - \hat{\nu}) \leq \lambda(\pi(\hat{\mu}), \pi(\hat{\nu})) \leq \hat{\gamma}_f(\hat{\mu}^\pm - \hat{\nu}^\pm).
\]

(2.13)
**Proof**: Choosing $\hat{0} \in \hat{D}_+$ for $\hat{\nu}^\pm$ and $\hat{\eta}$ in the definition of $\lambda$ yields

$$
\lambda(\pi(\hat{\mu}), \pi(\hat{0})) := \inf_{\hat{\eta} \in \hat{D}_+} [\hat{\gamma}_f(\hat{\mu}^\pm - \hat{\eta}) \lor \hat{\gamma}_f(-\hat{\mu}^\pm - \hat{\eta})] \leq \hat{\gamma}_f(\hat{\mu}^\pm).
$$

Yet, considering the second term,

$$
\hat{\gamma}_f(-\hat{\mu}^\pm - \hat{\eta}) = \hat{\gamma}_f(\hat{\mu}^\pm + \hat{\eta}) = \gamma_f(\mu^+ + \eta) + \gamma_f(\mu^- + \eta)
$$

\[= \sup_{\|f\|_{L,\infty} \leq 1} \int f(r)(\mu^+ + \eta)(dr) + \sup_{\|f\|_{L,\infty} \leq 1} \int f(r)(\mu^- + \eta)(dr)\]

\[= \mu^+(S) + \eta(S) + \mu^-(S) + \eta(S) = \hat{\gamma}_f(\hat{\mu}^\pm) + 2\eta(S).
\]

The last equalities follow as the supremum is attained if $f \equiv 1$. Finally, we have that

$$
\lambda(\pi(\hat{\mu}), \pi(\hat{0})) \geq \inf_{\hat{\eta} \in \hat{D}_+} \hat{\gamma}_f(-\hat{\mu}^\pm - \hat{\eta}) = \inf_{\hat{\eta} \in \hat{D}_+} \left(\hat{\gamma}_f(\hat{\mu}^\pm) + 2\eta(S)\right) = \hat{\gamma}_f(\hat{\mu}^\pm)
$$

which establishes (2.12).

Take an arbitrary $f \in C_{L,\infty}(S,\mathbb{R})$ with $\|f\|_{L,\infty} \leq 1$, and let $\eta \in M_f(\mathbb{R}^d)$ be arbitrary.

\[
\left| \int f(x)(\Phi(\hat{\mu}) - \Phi(\hat{\nu}))(dx) \right|
\]

\[= \left| \int f(x) \left(\mu^+ - \mu^- - (\nu^+ - \nu^-)\right)(dx) \right|
\]

\[= \left| \int f(x) \left(\mu^+ - \nu^+ - \eta - (\mu^- - \nu^- - \eta)\right)(dx) \right|
\]
\[
\leq \left| \int f(x)(\mu^+ - \nu^- - \eta)(dx) \right| + \left| \int f(x)(\mu^- - \nu^- - \eta)(dx) \right|
\]

\[
\leq \sup_{\|f\|_{L,\infty} \leq 1} \left( \left| \int f(x)(\mu^+ - \nu^- - \eta)(dx) \right| + \left| \int f(x)(\mu^- - \nu^- - \eta)(dx) \right| \right)
\]

\[
\leq \hat{\gamma}_f(\hat{\mu}^\pm - \hat{\nu}^\pm - \hat{\eta})
\]

where \(\hat{\eta} = (\eta, \eta)\). Thus, taking the supremum over \(f \in C_{L,\infty}(S, \mathbb{R})\) with \(\|f\|_{L,\infty} \leq 1\) yields

\[
\gamma_f(\Phi(\hat{\mu}) - \Phi(\hat{\nu})) \leq \hat{\gamma}_f(\hat{\mu}^\pm - \hat{\nu}^\pm - \hat{\eta}).
\]

Since \(\eta \in M_f(\mathbb{R}^d)\) was arbitrary, it follows from the preceding inequality that

\[
\gamma_f(\Phi(\hat{\mu}) - \Phi(\hat{\nu})) \leq \inf_{\hat{\eta} \in D_+} \hat{\gamma}_f(\hat{\mu}^\pm - \hat{\nu}^\pm - \hat{\eta})
\]

and, consequently, we have that

\[
\gamma_f(\Phi(\hat{\mu}) - \Phi(\hat{\nu})) \leq \inf_{\hat{\eta} \in D_+} [\hat{\gamma}_f(\hat{\mu}^\pm - \hat{\nu}^\pm - \hat{\eta}) \lor \hat{\gamma}_f(\hat{\mu}^\pm - \hat{\nu}^\pm + \hat{\eta})] = \lambda(\pi(\hat{\mu}), \pi(\hat{\nu})).
\]

The first inequality in (2.13) now follows as

\[
\gamma_f(\Phi(\hat{\mu}) - \Phi(\hat{\nu})) = \gamma_f(\mu^+ - \mu^- - (\nu^+ - \nu^-)) \leq \gamma_f(\mu^+ - \mu^-) + \gamma_f(\nu^+ - \nu^-) = \hat{\gamma}_f(\hat{\mu} - \hat{\nu}).
\]

The second inequality follows immediately, and the third follows simply by choosing \(\hat{\eta} = (0, 0)\) in (2.11). \(\square\)
The motivation for the work was to construct a metric that is complete on $M_{f,s}(S)$. We next examine whether $\lambda$ is a complete metric on the space of signed measures. Suppose $\{\pi(\hat{\mu}_n)\}_{n \in \mathbb{N}}$ is a Cauchy sequence for $\lambda$ with a subsequence $\{\pi(\hat{\mu}_{n_k})\}$. If $\{\pi(\hat{\mu}_{n_k})\}_{n \in \mathbb{N}}$ converges, then it would follow that $(M_{f,s}(S), \lambda)$ is complete. However, the following analysis shows that this subsequence does not converge in general. Rather, the following derivation yields something that is in a way more powerful than completeness itself.

**Theorem 2.8** A Cauchy sequence $\{\pi(\hat{\mu}_n)_{n \in \mathbb{N}}\}$ converges in $\lambda$ iff a subsequence of measure pairs $\{\hat{\mu}^\pm\}$ is a Cauchy sequence in $(\hat{M}_f(S), \hat{\gamma}_f)$ whose limit, $\hat{\mu}$, satisfies $\hat{\mu} = \hat{\mu}^\pm$. In particular, the limit, $\hat{\mu}$, satisfies the identification of the Hahn-Jordan decomposition of a signed measure.

**Proof**: For the reverse implication, note that by Proposition 2.7,

$$\lambda(\pi(\hat{\mu}_{n_k}), \pi(\hat{\mu})) \leq \hat{\gamma}_f(\hat{\mu}^\pm_{n_k}, \hat{\mu}^\pm) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

So, $\{\pi(\hat{\mu}_n)_{n \in \mathbb{N}}\}$ must converge as it is Cauchy.

For the forward implication, choose a subsequence $\{\pi(\hat{\mu}_{n_k})\}_{k \in \mathbb{N}}$ of $\{\pi(\hat{\mu}_n)_{n \in \mathbb{N}}$ such that

$$\lambda(\pi(\hat{\mu}_{n_k}), \pi(\hat{\mu}_{n_{k+1}})) < 2^{-k} \forall k \in \mathbb{N}.$$ 

Set $\hat{\phi}_1 := \hat{0}$ and by the definition of $\lambda$ choose $\hat{\phi}_k \in \hat{D}_+$ such that

$$\hat{\gamma}_f(\hat{\mu}^\pm_{n_k} - \hat{\mu}^\pm_{n_{k+1}} - \hat{\phi}_k) \lor \hat{\gamma}_f(\hat{\mu}_{n_k} - \hat{\mu}_{n_{k+1}} + \hat{\phi}_k) < 2^{-k} \forall k \in \mathbb{N}.$$ 

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Further, define another sequence in $\hat{D}_+$ by $\hat{\xi}_k := \sum_{i=0}^{k-1} \hat{\phi}_i$, $k \in \mathbb{N}$. The first term in the definition of $\lambda$, (2.11), satisfies

$$\hat{\gamma}_f(\hat{\mu}_{n_k}^\pm + \hat{\xi}_k - \hat{\mu}_{n_{k+1}}^\pm - \hat{\xi}_{k+1}) < 2^{-k} \forall k \in \mathbb{N}.$$ 

Setting $\hat{\zeta}_k := \hat{\mu}_{n_k}^\pm + \hat{\xi}_k \in \hat{M}_f(S)$, we obtain

$$\hat{\gamma}_f(\hat{\zeta}_k - \hat{\zeta}_{k+1}) < 2^{-k}.$$ 

Since $(\hat{M}_f(S), \hat{\gamma}_f)$ is complete and by (2.10), there is a unique $\hat{\zeta} = \hat{\zeta}^\pm + \hat{\theta}$, $\hat{\theta} \in \hat{D}_+$ such that

$$\hat{\zeta}_k = \hat{\mu}_{n_k}^\pm + \hat{\xi}_k \xrightarrow{\hat{\gamma}_f} \hat{\zeta}^\pm + \hat{\theta}.$$ 

Due to this convergence and the argument in Proposition 2.7,

$$\sup_{k \in \mathbb{N}} \hat{\gamma}_f(\hat{\mu}_{n_k}^\pm + \hat{\xi}_k) = \sup_{k \in \mathbb{N}} (\mu_{n_k}^+(S) + \mu_{n_k}^-(S) + 2\xi_k(S)) < \infty,$$

Consequently, the monotone limit $\hat{\xi} := \lim_{k \to \infty} \hat{\xi}_k = \sum_{i=1}^{\infty} \hat{\phi}_i$ exists and $\hat{\xi} \in \hat{D}_+$.

Further, for $m \in \mathbb{N},$

$$\hat{\gamma}_f(\sum_{i \leq m} \hat{\phi}_i - \hat{\xi}) = 2\hat{\gamma}_f(\sum_{i > m} \hat{\phi}_i) \leq \sum_{i > m} 2\phi_i(S) \to 0 \text{ as } m \to \infty.$$ 

Thus,

$$\hat{\gamma}_f(\sum_{i \leq m} \hat{\phi}_i - \hat{\xi}) \to 0 \text{ as } m \to \infty.$$
Therefore, we may write $\hat{\mu}_{nk}^+ + \hat{\xi}_k \to \hat{\zeta}^+ + \hat{\theta} - \hat{\xi}$ and conclude that $\{\hat{\mu}_{nk}^\pm\} \subset \hat{M}_f(S)$ has $\hat{\mu}_{nk}^\pm \to \hat{\zeta}^+ + \hat{\theta} - \hat{\xi} \in \hat{M}_f(S)$. Denote $\hat{\mu} := \hat{\zeta}^+ + \hat{\theta} - \hat{\xi}$. By considering the equivalence class of this limit and using (2.10), there are $\hat{\mu}^\pm \in \hat{M}_f(S)$ and $\hat{\phi} \in \hat{D}_+$ such that $\hat{\mu}^+ + \hat{\phi} = \hat{\zeta}^+ + \hat{\theta} - \hat{\xi}$. Thus, we obtain

$$\hat{\mu}_{nk}^\pm \to \hat{\mu}^\pm + \hat{\phi}. \quad (2.14)$$

To complete the proof of Theorem 2.5, it suffices to show the following lemma:

**Lemma 2.9**

$$\lambda(\pi(\hat{\mu}_{nk}^\pm), \pi(\hat{\mu}^\pm + \hat{\phi})) \to 0 \text{ iff } \hat{\phi} = \hat{0} \quad (2.15)$$

where $\hat{0} = (0, 0)$.

**Proof**: The reverse implication follows from Proposition 2.7 as

$$\lambda(\pi(\hat{\mu}_{nk}^\pm), \pi(\hat{\mu}^\pm)) \leq \hat{\gamma}_f(\hat{\mu}_{nk}^\pm - \hat{\mu}^\pm)$$

and $\hat{\mu}_{nk}^\pm \to \hat{\mu}^\pm$ in $\hat{\gamma}_f$ as $k \to \infty$.

For the forward implication, it follows that if $\hat{\phi} \neq \hat{0}$, then $\hat{\gamma}_f(\hat{\phi}) > 0$. Thus,

$$\lambda(\pi(\hat{\mu}_{nk}^\pm), \pi(\hat{\mu}^\pm)) \geq \inf_{\hat{\eta} \in \hat{D}_+} \hat{\gamma}_f(\hat{\mu}_{nk}^\pm - \hat{\mu}_{nk}^\pm - \hat{\eta}) \vee \inf_{\hat{\eta} \in \hat{D}_+} \hat{\gamma}_f(\hat{\mu}^\pm - \hat{\mu}_{nk}^\pm - \hat{\eta}).$$

Let $\epsilon > 0$, and compute the second term:

$$\inf_{\hat{\eta} \in \hat{D}_+} \hat{\gamma}_f(\hat{\mu}_{nk}^\pm - \hat{\mu}_{nk}^\pm - \hat{\eta}) = \inf_{\hat{\eta} \in \hat{D}_+} \hat{\gamma}_f(\hat{\mu}_{nk}^\pm + \hat{\phi} - \hat{\mu}_{nk}^\pm - \hat{\phi} - \hat{\eta})$$

$$\geq \inf_{\hat{\eta} \in \hat{D}_+} \hat{\gamma}_f(-\hat{\phi} - \hat{\eta}) - \hat{\gamma}_f(\hat{\mu}_{nk}^\pm + \hat{\phi} - \hat{\mu}_{nk}^\pm)$$

$$\geq \inf_{\hat{\eta} \in \hat{D}_+} \hat{\gamma}_f(\hat{\phi} + \hat{\eta}) - \epsilon$$

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for sufficiently large $n$ since, by (2.14), $\hat{\gamma}_f(\hat{\mu}^\pm + \hat{\phi} - \hat{\mu}_{nk}^\pm) \to 0$. Hence,

$$\liminf_{n \to \infty} \inf_{\hat{\eta} \in \hat{D}_+} \hat{\gamma}_f(\hat{\mu}^\pm - \hat{\mu}_{nk}^\pm - \hat{\eta}) \geq \inf_{\hat{\eta} \in \hat{D}_+} \hat{\gamma}_f(\hat{\phi} + \hat{\eta}) = \hat{\gamma}_f(\hat{\phi}) > 0.$$ 

Therefore, we also have

$$\liminf_{n \to \infty} \lambda(\pi(\hat{\mu}_{nk}), \pi(\hat{\mu})) \geq \hat{\gamma}_f(\hat{\phi}) > 0.$$

With Theorem 2.8 established, $\lambda$ is a Kantorovich-Rubinstein-type metric that has a useful form of partial-completeness. One can use the partial-completeness property to conclude that a limit of measure pairs in $(\hat{M}_f(S), \hat{\gamma}_f)$ is the Hahn-Jordan decomposition of a signed measure. This property will allow significant results in the field of Signed Measure Valued SPDE. One can infer the Hahn-Jordan decomposition of a solution by properties of $\lambda$. Note that none of the Wasserstein distance, Kantorovich-Rubinstein distance, or their product metrics can yield such a result directly. Consequently, the new results presented here show a significant advancement in the theory of metrics on signed measures.
3 Signed Measure Valued SPDE

In the following, we examine a general class of signed-measure valued SPDE and an associated system of SODE. The approach is to pass a solution of the SODE system to a solution of the SPDE by Itô’s Formula. For a discrete initial signed measure, it follows that the Hahn-Jordan decomposition of the solution of the SPDE is preserved. However, the extension to arbitrary signed measures requires justification due to incompleteness issues. From [Kot10], the extension holds provided one assumes smoothness on the coefficients of the associated SODE. Using Theorem 2.8, we derive the extension provided the coefficients of the SODE only satisfy Lipschitz conditions. From a stochastic analysis viewpoint, this result is ideal as Lipschitz conditions form the basis for existence and uniqueness of solutions.

3.1 Definitions

Prior to stating the SODE-system or SPDE, we state some of the standard assumptions and notation for the stochastic analysis. Consider a probability space, $(\Omega, \mathcal{F}, P)$, and a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ contained in $\mathcal{F}$. Denote $E(\cdot)$ and $\int (\cdot) dP$ as the mathematical expectation with respect to the probability measure. To avoid certain pathological issues that may arise, we make the following standard assumptions on the filtration:

- $\mathcal{F}_0$ contains all the $P$-null sets (and hence so does $\mathcal{F}_t$ for all $t \geq 0$).

- The filtration is right continuous (i.e. $\mathcal{F}_t = \mathcal{F}_{t+} := \cap_{s \geq t} \mathcal{F}_s$).

One reason for making such assumptions on the filtration is for convenience when working with stopping times. A stopping time is a mapping, $\tau : \Omega \rightarrow [s, \infty)$, such
that the event \( \{ \tau \leq t \} \in F_t \) for all \( t \geq 0 \). Assuming that the filtration is right continuous implies that the events \( \{ \tau < t \} \in F_t \) for a stopping time, \( \tau \).

For notation, we need to define the standard \( L_p \) spaces for \( p \geq 1 \). Let \( K \) be a metric space with metric \( \psi \). \( L_{0,F_s}(K) \) is the space of \( K \)-valued \( F_s \)-measurable random variables \( \xi \). \( L_{p,F_s}(K) \subset L_{0,F_s}(K) \) is the set such that for \( \xi \in L_{p,F_s}(K) \), \( E\psi^p(\xi, \eta) < \infty \) where \( \eta \in K \) is an arbitrary fixed element.

Denote, \( C([s,T]; K) \) as the set of continuous functions from \([s,T]\) to \((K,\psi)\) where \( s,T \in [0,\infty) \). \( L_{0,F_\cdot}(C([s,T]; K)) \) is the space of random variables with values in \( C([s,T]; K) \) which are adapted to the filtration \( F_t \). \( L_{p,F}(C([s,T]; K)) \subset L_{0,F}(C([s,T]; K)) \) is the space of \( p \)-th integrable random variables with values in \( C([s,T]; K) \). Finally, \( L_{loc,p,F}(C([s,T]; K)) \) is the space of processes \( \xi(\cdot) \) such that there is a sequence of stopping times \( \{ \tau_n \}_{n \in \mathbb{N}} \) with \( \xi(\cdot \wedge \tau_n) \in L_{p,F}(C([s,T]; K)) \) and \( \tau_n \to \infty \). Such a sequence of stopping times is said to be localizing.

To aid in the notation, let \( \gamma_{f,s}(\cdot,\cdot) \) denote the partially-complete metric from Chapter 2.\(^{14}\) Let \( w_l(dp,dt) \) denote i.i.d. standard Gaussian space-time white noise differentials\(^{15}\) where \( l = 1, \ldots, d \), and define \( w(dp,dt) = (w_1(dp,dt), \ldots, w_d(dp,dt))^T \).

Now, let \( F : \mathbb{R}^d \times M_{f,s}(\mathbb{R}^d) \times [0,\infty) \to \mathbb{R}^d \), \( J_\varepsilon : \mathbb{R}^d \times \mathbb{R}^d \times M_{f,s}(\mathbb{R}^d) \times [0,\infty) \to M_{d \times d} \), where \( \varepsilon \) is the correlation length,\(^{16}\) and \( M_{d \times d} \) denotes the set of \( d \times d \) matrices over \( \mathbb{R} \). We can now state the SODE system in differential form.\(^{17}\) For \( i = 1, \ldots, N \in \mathbb{N} \), and \( s \leq t \leq T \) with \( s,T \in [0,\infty) \)

\[
    dr^i_\varepsilon = F(r^i_\varepsilon(t), \mathcal{Y}(t), t)dt + \int J_\varepsilon(r^i_\varepsilon(t), p, \mathcal{Y}(t), t)w(dp,dt) \tag{3.1}
\]

\(^{14}\)\( \gamma_{f,s} \) was denoted by \( \lambda \) in Chapter 2.

\(^{15}\)See Definition 6.15 for formulation.

\(^{16}\)\( \varepsilon \) represents the distance where the correlation between particles driven by \( w(dp,dt) \) can be observed.

\(^{17}\)See Definition 6.13 for additional background.
with initial conditions:

\[ r^i_\epsilon(s) = q^i \in \mathbb{R}^d, \quad (3.2) \]

and \( \mathcal{Y}(t) \) is a signed-measure valued, adapted process. One may think of the system as representing \( N \) particles moving in a random medium with positions given by \( r^i_\epsilon \). The signed measure input process is represented by \( \mathcal{Y}(t) \) in \( \mathbb{R}^d \), and it provides the mathematical interpretation of the random medium.

### 3.2 Existence and Uniqueness of SODE

To address questions of existence and uniqueness to (3.1), one must specify more conditions on the coefficients, \( F \) and \( J_\epsilon \). Assume that \( F \) and \( J_\epsilon \) are jointly measurable in all arguments. Also, assume that the one dimensional components of \( J_\epsilon \), denoted by \( J_{\epsilon,ij} \) for \( i,j = 1,\ldots,d \), are square integrable with respect to Lebesgue measure, \( dq \), on \( \mathbb{R}^d \). Also, assume the following Lipschitz and growth constants on the coefficients in terms of \( \gamma_{f,s} \) and \( \varrho(r,q) := |r - q| \wedge 1 \) where\(^{18} \) \( |\cdot| \) denotes the Euclidean metric on \( \mathbb{R}^d \) and \( r,q \in \mathbb{R}^d \).

There exists\(^{19} \) a \( c_{F,J} \in (0,\infty) \) such that for all \( r,r_i \in \mathbb{R}^d, \hat{\mu}, \hat{\mu}_i \in \hat{M}_{f,s}(\mathbb{R}^d) \), \( t \geq s \) and \( i = 1,2 \):

\[
|F(r_1,\Phi(\hat{\mu}_1),t) - F(r_2,\Phi(\hat{\mu}_2),t)| \leq c_{F,J} \{ (\hat{\gamma}_f(\hat{\mu}_1) \vee \hat{\gamma}_f(\hat{\mu}_2)) \rho(r_1,r_2) + \gamma_{f,s}(\mu_1,\mu_2) \}
\]

\[ (3.3) \]

\(^{18}\)We write \( \varrho(q) \) for \( \varrho(q,0) \) and \( \varrho(r - q) \) for \( \varrho(r,q) \) without further mention.

\(^{19}\)Throughout this work, we use the notation that the subscripts on a constant indicate dependence on those parameters.
\[
\sum_{k,\ell=1}^{d} \int (J_{\varepsilon, kl}(r_1, p, \Phi(\hat{\mu}_1), t) - J_{\varepsilon, kl}(r_2, p, \Phi(\hat{\mu}_2), t))^2 dp \\
\leq c_{F,J}^2 \{ (\hat{\gamma}_f^2(\hat{\mu}_1) \vee \hat{\gamma}_f^2(\hat{\mu}_2)) \rho^2(r_1, r_2) + \gamma_{f,s}^2(\mu_1, \mu_2) \} 
\]

(3.4)

\[
|F(r, \Phi(\hat{\mu}), t)|^2 + \sum_{k,\ell=1}^{d} \int J_{\varepsilon, kl}^2(r, p, \Phi(\hat{\mu}), t) dp \leq c_{F,J} \hat{\gamma}_f^2(\hat{\mu}). 
\]

(3.5)

The constant \( c_{F,J} \) in (3.3)-(3.5) may also depend on the space dimension, \( d \). Alternatively, if we relax the boundedness assumption on the coefficients in (3.5), we need to impose a linear growth condition. This growth condition is assumed in addition to the corresponding Lipschitz conditions from (3.3)-(3.4) in terms of the Euclidean metric.

There exists a \( c_{F,J} \in (0, \infty) \) such that for all \( r, r_i \in \mathbb{R}^d, \hat{\mu}, \hat{\mu}_i \in \hat{\mathcal{M}}_{f,s}(\mathbb{R}^d), t \geq s \) and \( i = 1, 2 \):

\[
|F(r_1, \Phi(\hat{\mu}_1), t) - F(r_2, \Phi(\hat{\mu}_2), t)| \leq c_{F,J} \{ (\hat{\gamma}_f(\hat{\mu}_1) \vee \hat{\gamma}_f(\hat{\mu}_2)) |r_1 - r_2| + \gamma_{f,s}(\mu_1, \mu_2) \} 
\]

(3.6)

\[
\sum_{k,\ell=1}^{d} \int (J_{\varepsilon, kl}(r_1, p, \Phi(\hat{\mu}_1), t) - J_{\varepsilon, kl}(r_2, p, \Phi(\hat{\mu}_2), t))^2 dp \\
\leq c_{F,J}^2 \{ (\hat{\gamma}_f^2(\hat{\mu}_1) \vee \hat{\gamma}_f^2(\hat{\mu}_2)) |r_1 - r_2|^2 + \gamma_{f,s}^2(\mu_1, \mu_2) \} 
\]

(3.7)

\[
|F(r, \Phi(\hat{\mu}), t)|^2 + \sum_{k,\ell=1}^{d} \int J_{\varepsilon, kl}^2(r, p, \Phi(\hat{\mu}), t) dp \leq c_{F,J}^2 (1 + |r|^2) \hat{\gamma}_f^2(\hat{\mu}). 
\]

(3.8)

With the appropriate Lipschitz assumptions on the coefficients, the following existence and uniqueness result follows the standard techniques in the literature.\(^{20}\) However, due to its importance in the later analysis, the complete proof of the result is given.

\(^{20}\)See [Pro04], [MP80], or [IW89].
The case for positive measures is established in [Kot08]. The following argument follows the work in [Kot08] and generalizes it to signed measures. In the following propositions, we simply write the localizing stopping times \( \{\tau_n\}_{n \geq 1} \) as \( \tau \). Furthermore, let \( \tilde{\rho} \in \{\rho, |\cdot|\} \).

**Proposition 3.1** Suppose either (3.3)-(3.5) or (3.6)-(3.8) holds:

i. Let the initial condition, \( q_s = (q^1_s, \ldots, q^N_s) \) have \( q^i_s \in L^2_{2,F_s}(\mathbb{R}^d) \) for \( i = 1, \ldots, N \), and the signed valued input process have, \( \tilde{Y} \in L_{loc,2,F}(C([s,T]; \mathbb{M}_{f,s}(\mathbb{R}^d))) \). There exists a unique, continuous, adapted solution of (3.1):

\[
r_{\varepsilon}(t, \tilde{Y}, q_s, s) = (r^1_{\varepsilon}(t, \tilde{Y}, q^1_s, s), \ldots, r^N_{\varepsilon}(t, \tilde{Y}, q^N_s, s)) \in L_{loc,2,F}(C([s,T]; \mathbb{R}^d)).
\]

ii. For initial conditions, \( q^i_j \in L^2_{2,F_s}(\mathbb{R}^d), i = 1, \ldots, N, j = 1, 2 \), signed measure valued processes \( \tilde{Y}_j \in L_{loc,2,F}(C([s,T]; \mathbb{M}_{f,s}(\mathbb{R}^d))) \), \( j = 1, 2 \) and a localizing stopping time, \( \tau \):

\[
E \sup_{s \leq t \leq T \land \tau} \tilde{\rho}^2(r^j_{\varepsilon}(t, \tilde{Y}_1, q^j_1, s), r^j_{\varepsilon}(t, \tilde{Y}_2, q^j_2, s)) \leq c_{T,F,J,\tau} \left\{ E\tilde{\rho}^2(q^j_1, q^j_2)1_{\{\tau \geq s\}} + E \int_s^{T \land \tau} \gamma^2_{f,s}(\tilde{Y}_1(u), \tilde{Y}_2(u))1_{\{\tau \geq s\}} du \right\}
\]

(3.9)

iii. For any \( N \in \mathbb{N} \) there is a \( \mathbb{R}^{dN} \)-valued map in the variables \( (t, \omega, \mu, r, s), 0 \leq s \leq t < \infty \) such that for any fixed \( s \geq 0 \):

\[
r_{\varepsilon}(\cdot, \ldots, \cdot, s) : [s,T] \times \Omega \times C([s,T]; \mathbb{M}_{f,s}(\mathbb{R}^d)) \times \mathbb{R}^{dN} \to C([s,T]; \mathbb{R}^{dN}),
\]

(3.10)
such that:

(a) For any $t \in [s, T]$, $r_\varepsilon(t, \cdot, \ldots, \cdot, s)$ is $\overline{G}_{s,t} \otimes F_{M_f,s(\mathbb{R}^d),s,t} \otimes \mathcal{B}^{dN} - \mathcal{B}^{dN}$ measurable. Here, $\overline{G}_{s,t}$ is the completed $\sigma$-algebra generated by $w(dp, du)$ between $s$ and $t$, $F_{M_f,s(\mathbb{R}^d),s,t}$ denotes the $\sigma$-algebra of cylinder sets on $M_{f,s}(\mathbb{R}^d)$, and $\mathcal{B}^{dN}$ denotes the Borel sets on $\mathbb{R}^{dN}$.

(b) The $i$th component of $r_\varepsilon = (r^1_\varepsilon, \ldots, r^N_\varepsilon)$ depends only on the $i$th initial condition, $w(dq, dt)$, and $\tilde{Y} \in L^2(F(C([s,T]; M_{f,s}(\mathbb{R}^d))))$:

$$(r_\varepsilon(t, \cdot, \tilde{Y}, q^i_s, s))^i = r^i_\varepsilon(t, \tilde{Y}, q^i_s, s).$$

Proof: (i). Without loss of generality, we will prove the result on $[0,T]$. For notational convenience, we will suppress the dependence on $\varepsilon$ and the particle index in the argument. Assume (3.3)-(3.5) as the other case will follow similarly. Since the stopping times are localizing, we have that for all $\delta > 0$ and for all $b > 0$, there is a stopping time, $\tau$, and an $\mathcal{F}_0$ measurable set $\Omega_b$ such that:

$$P \left\{ \sup_{0 \leq t \leq T \land \tau} \left( \gamma_f, s(\tilde{Y}_1(t, \omega)) \lor \gamma_f, s(\tilde{Y}_2(t, \omega)) \right) 1_{\Omega_b}(\omega) \right\} \leq b \right\} \geq 1 - \delta.$$ 

Therefore, without loss of generality, we may assume that:

$$c_y := \text{ess sup}_{\omega} \sup_{0 \leq t \leq T \land \tau} \left( \gamma_f, s(\tilde{Y}_1(t, \omega)) \lor \gamma_f, s(\tilde{Y}_2(t, \omega)) \right) < \infty. \quad (3.11)$$

Similarly, the same holds with $\hat{\gamma}_f$ replacing $\gamma_f, s$. To show existence, the approach is to use completeness and the Picard-Lindelöf approximation procedure. To use this approach, we need to show that the approximations lie in the Banach Space
of \( L_{2,F}(C([0,\infty);\mathbb{R}^d)) \). To this end, assume that \( q_j(\cdot \wedge \tau) \in L_{2,F}(C([0,\infty);\mathbb{R}^d)) \) for \( j = 1, 2 \), and define:

\[
\tilde{q}_j(t) := q_j(0) + \int_0^t F(q_j(s), \tilde{Y}_j(s), s)ds + \int_0^t \int \mathcal{J}_\varepsilon(q_j(s), p, \tilde{Y}_j(s), s)w(dp, ds).
\]

(3.12)

For \( j = 1, 2 \),

\[
E \sup_{0 \leq t \leq T} \rho^2(\tilde{q}_j(t \wedge \tau)) \leq 9E \rho^2(q_j(0)) + 9E \sup_{0 \leq t \leq T} \rho^2 \left( \int_0^{t \wedge \tau} F(q_j(s), \tilde{Y}_j(s), s)ds \right)
\]

\[
+ 9E \sup_{0 \leq t \leq T} \rho^2 \left( \int_0^{t \wedge \tau} \int \mathcal{J}_\varepsilon(q_j(s), p, \tilde{Y}_j(s))w(dp, ds). \right)
\]

(3.13)

By Cauchy-Schwarz,\(^{21}\) (3.5), and (3.11), we have that there is a constant \( c_{F,J,T} \) such that:

\[
E \sup_{0 \leq t \leq T} \rho^2 \left( \int_0^{t \wedge \tau} F(q_j(s), \tilde{Y}_j(s), s)ds \right)
\]

\[
\leq E \sup_{0 \leq t \leq T} T \int_0^{t \wedge \tau} |F(q_j(s), \tilde{Y}_j(s), s)|^2ds \leq c_{F,J,T}TE \int_0^{T \wedge \tau} ds \leq c_{F,J,T}.
\]

(3.14)

Applying Doob’s Inequality,\(^{22}\) (3.5), and (3.11) to the random term yields another

\(^{21}\)See Theorem 6.9 for formulation.
\(^{22}\)See Theorem 6.10 for formulation.
constant, \( c_{F,\mathcal{J},T} \), with:

\[
E \sup_{0 \leq t \leq T} \tilde{\rho}^2 \left( \int_0^{t \land \tau} \int \mathcal{J}_\varepsilon(q_j(s), p, \tilde{Y}_j(s), s) \, w(dp, ds) \right)
\]

\[
\leq 4 \sum_{k=1}^d E \int_0^{T \land \tau} \int \left( \sum_{i=1}^d \mathcal{J}_{\varepsilon,ki}(q_j(s), p, \tilde{Y}_j(s), s) \right)^2 \, dpds
\]

(3.15)

\[
\leq 4d^2 \sum_{k,i=1}^d E \int_0^{T \land \tau} \int \mathcal{J}_{\varepsilon,ki}^2(q_j(s), p, \tilde{Y}_j(s), s) \, dpds
\]

\[
\leq c_{F,\mathcal{J},T}.
\]

where we have used (6.15) in the first inequality. Combining (3.13), (3.14), and (3.15) yields that there is a finite constant, \( c_{F,\mathcal{J},T,d,\tau} \), such that:

\[
E \sup_{0 \leq t \leq T} \tilde{\rho}^2(\tilde{q}_j(t \land \tau)) \leq 9E\tilde{\rho}^2(q_j(0)) + c_{F,\mathcal{J},T,d,\tau}.
\]

(3.16)

Thus, we have shown that if the initial condition satisfies \( q_j(0) \in L_{2,\mathcal{F}}(\mathbb{R}^d) \), then \( \tilde{q}_j(t \land \tau) \) defined by (3.12) is in \( L_{2,\mathcal{F}}(C([0, \infty); \mathbb{R}^d)) \). Now, we state estimates to compare terms of the type defined by (3.12). We may assume that \( c_{F,\mathcal{J}} \geq 1 \), and note that \( a \land c \leq (a \land 1)c \) for all \( a \geq 0 \) and \( c \geq 1 \). Thus,

\[
|F(q_1(s), \tilde{Y}(s), s) - F(q_2(s), \tilde{Y}_2(s), s)| \leq c_{F,\mathcal{J}} \tilde{\rho} \left( F(q_1(s), \tilde{Y}_1(s), s), F(q_2(s), \tilde{Y}_2(s), s) \right).
\]

(3.17)
Now, by using Cauchy-Schwarz, (3.11), (3.17), and (3.3), we have that

\[
\tilde{\rho}^2 \left( \int_0^{t \wedge \tau} F(q_1(s), \tilde{Y}_1(s), s) ds \int_0^{t \wedge \tau} F(q_2(s), \tilde{Y}_2(s), s) ds \right)
\]

\[
\leq \left\| \int_0^{t \wedge \tau} F(q_1(s), \tilde{Y}_1(s), s) ds - \int_0^{t \wedge \tau} F(q_2(s), \tilde{Y}_2(s), s) ds \right\|^2
\]

\[
\leq T \int_0^{t \wedge \tau} \left| F(q_1(s), \tilde{Y}_1(s), s) - F(q_2(s), \tilde{Y}_2(s), s) \right|^2 ds
\]

\[
\leq c_{F, \mathcal{J}, \mathcal{Y}} \left( \int_0^{t \wedge \tau} \tilde{\rho}^2(q_1(s), q_2(s)) ds + \int_0^{t \wedge \tau} \gamma_{f,s}^2(\tilde{Y}_1(s), \tilde{Y}_2(s)) ds \right).
\]

(3.18)

Similarly, by using (3.17), (3.4), and (6.15), we have the following:

\[
E \tilde{\rho}^2 \left( \int_0^{t \wedge \tau} \int \mathcal{J}_\varepsilon(q_1(s), p, \tilde{Y}_1(s), s) w(dp, ds), \int_0^{t \wedge \tau} \int \mathcal{J}_\varepsilon(q_2(s), p, \tilde{Y}_2(s), s) w(dp, ds) \right)
\]

\[
\leq E \sum_{k=1}^d \sum_{i=1}^d \int_0^{t \wedge \tau} \left| (\mathcal{J}_{\varepsilon,ki}(q_1(s), p, \tilde{Y}_1(s), s) - \mathcal{J}_{\varepsilon,ki}(q_2(s), p, \tilde{Y}_2(s), s)) w_i(dp, ds) \right|^2
\]

\[
= \sum_{k=1}^d E \int_0^{t \wedge \tau} \left| \sum_{i=1}^d (\mathcal{J}_{\varepsilon,ki}(q_1(s), p, \tilde{Y}_1(s), s) - \mathcal{J}_{\varepsilon,ki}(q_2(s), p, \tilde{Y}_2(s), s)) \right|^2 dp ds
\]

\[
\leq d^2 E \sum_{k,j=1}^d \int_0^{t \wedge \tau} \left| (\mathcal{J}_{\varepsilon,ki}(q_1(s), p, \tilde{Y}_1(s), s) - \mathcal{J}_{\varepsilon,ki}(q_2(s), p, \tilde{Y}_2(s), s)) \right|^2 dp ds
\]

\[
\leq c_{F, \mathcal{J}, \mathcal{Y}, d} E \left( \int_0^{t \wedge \tau} \tilde{\rho}^2(q_1(s), q_2(s)) ds + \int_0^{t \wedge \tau} \gamma_{f,s}^2(\tilde{Y}_1(s), \tilde{Y}_2(s)) ds \right).
\]

(3.19)

It now follows from (3.18), (3.19) and Doob’s Inequality that there exist a constant
\[\hat{c}_{F,J,T,\tau,d}\] such that:

\[
E \sup_{0 \leq t \leq T} \tilde{\rho}^2(\tilde{q}_1(t \wedge \tau), \tilde{q}_2(t \wedge \tau)) \leq \hat{c}_{F,J,T,\tau,d}\left\{ E\tilde{\rho}^2(q_1(0), q_2(0))
\right.
\]

\[
+ E \int_0^T \tilde{\rho}^2(q_1(s \wedge \tau), q_2(s \wedge \tau)) ds + E \int_0^T J_\varepsilon(q_n(s, \tilde{\mathcal{Y}}(s), \tilde{\mathcal{Y}}(s), s) w(dp, ds)
\]

\[
(3.20)
\]

From [Arn92], apply the Picard-Lindelöf procedure. Define iteratively for \(n \in \mathbb{N}\) with \(\tilde{\mathcal{Y}}(\cdot) := \tilde{\mathcal{Y}}_1(\cdot) \equiv \tilde{\mathcal{Y}}_2(\cdot),\)

\[
\tilde{q}_{n+1}(t) := q(0) + \int_0^t F(\tilde{q}_n(s), \tilde{\mathcal{Y}}(s), s) ds + \int_0^t \int \mathcal{J}_\varepsilon(\tilde{q}_n(s), p, \tilde{\mathcal{Y}}(s), s) w(dp, ds)
\]

\[
\tilde{q}_0(t) := q(0).
\]

(3.21)

By (3.20), it follows that with \(c_N := \hat{c}_{F,J,T,\tau,d}:\)

\[
E \sup_{0 \leq t \leq T} \tilde{\rho}^2(\tilde{q}_{n+1}(t \wedge \tau), \tilde{q}_n(t \wedge \tau)) \leq c_N \int_0^T E \sup_{0 \leq t \leq T} \tilde{\rho}^2(\tilde{q}_{n+1}(s), \tilde{q}_n(s)) ds.
\]

(3.22)

Thus, for a constant \(\alpha:\)

\[
E \sup_{0 \leq t \leq T} \tilde{\rho}^2(\tilde{q}_{n+1}(t \wedge \tau), \tilde{q}_n(t \wedge \tau)) \leq \frac{\alpha c_N^2 T^n}{n!}.
\]

(3.23)

From (3.16), \(\tilde{q}_n\) lies in the complete space, \(L_{2,F}(C([0, \infty); \mathbb{R}^d))\). So, by (3.23) it follows that the sequence converges uniformly to a limit, \(\tilde{q}_\infty\), in mean square on the compact interval \([0, T]\). By the construction, \(\tilde{q}_\infty\) is continuous as each \(\tilde{q}_n\) is continuous.
Now, we claim that \( \tilde{q}_\infty \) is a solution of (3.1). By Fatou’s Lemma and (3.23):

\[
E \left( \int_0^T |\tilde{q}_\infty(s) - \tilde{q}_n(s)|^2 ds \right) \leq \lim_{m \to \infty} E \left( \int_0^T |\tilde{q}_m(s) - \tilde{q}_n(s)|^2 ds \right) \to 0 \quad (3.24)
\]
as \( n \to \infty \). Further, by Cauchy-Schwarz and (3.3):

\[
\int_0^T F(\tilde{q}_n(t), Y(t), t) dt \to \int_0^T F(\tilde{q}_\infty(t), Y(t), t) dt \quad (3.25)
\]
as \( n \to \infty \). By Doob’s Inequality and (3.4):

\[
\int_0^T \int \mathcal{J}_\varepsilon(\tilde{q}_n(t), p, Y(t), t) w(dp, dt) \to \int_0^T \int \mathcal{J}_\varepsilon(\tilde{q}_\infty(t), p, Y(t), t) w(dp, dt) \quad (3.26)
\]
as \( n \to \infty \). It follows that \( \tilde{q}_\infty \) is a solution of (3.1) as claimed. To show uniqueness, suppose that both \( \tilde{q}_\infty(\cdot) \) and \( \tilde{r}_\infty(\cdot) \) are solutions of (3.1) with initial condition \( \tilde{q}_\infty(0) = \tilde{r}_\infty(0) = q_0 \). By (3.20)

\[
E \sup_{0 \leq t \leq T} \tilde{\rho}^2(\tilde{q}_\infty(t \wedge \tau), \tilde{r}_\infty(t \wedge \tau)) 
\leq \hat{c}_{F, J, T, \tau, d} \left\{ E \int_0^T \tilde{\rho}^2(\tilde{q}_\infty(s \wedge \tau), \tilde{r}_\infty(s \wedge \tau)) ds + E \int_0^T \gamma_{\tilde{J}, s}(\tilde{Y}(s), \tilde{Y}(s)) ds \right\} 
= \hat{c}_{F, J, T, \tau, d} E \left\{ \int_0^T \tilde{\rho}^2(\tilde{q}_\infty(s \wedge \tau), \tilde{r}_\infty(s \wedge \tau)) ds \right\}. \quad (3.27)
\]

Applying Gronwall’s Inequality\(^ {23} \) yields that:

\[
E \sup_{0 \leq t \leq T} \tilde{\rho}^2(\tilde{q}_\infty(t \wedge \tau), \tilde{r}_\infty(t \wedge \tau)) = 0. \quad (3.28)
\]

\(^ {23} \)See Theorem 6.11.
Thus, we have uniqueness as desired as this clearly implies that \( \bar{q}_\infty(\cdot) = \bar{r}_\infty(\cdot) \) on \([0, T]\).

(ii). Now, (3.9) follows immediately from the previous arguments. In particular, (3.18) and (3.19) are the critical inequalities in establishing the claim. Applying Gronwall’s lemma yields (3.9).

(iii). For a proof of the measurability refer to [Kot10] where the result for measures is established. In fact, with no additional argument, the result in [Kot10] generalizes to the signed measure case.

The SODE system, (3.1), is an \( \mathbb{R}^{dN} \)-valued decoupled SODE system in that the signed-measure process, \( \Upsilon(t) \), does not explicitly depend on the solutions \( r^i(t), i = 1, \ldots, N \). For a more general framework, one can also consider a coupled SODE system. For \( i = 1, \ldots, N \) and \( s \leq t \leq T \) where \( s, T \in [0, \infty) \)

\[
dr^i_\varepsilon = F(r^i_\varepsilon(t), \mathcal{X}_N(t), t)dt + \int J_\varepsilon(r^i_\varepsilon(t), p, \mathcal{X}_N(t), t)w(dp, dt) \tag{3.29}
\]

with

\[
\mathcal{X}_N(t) := \sum_{i=1}^{N} a_i \delta_{r^i_\varepsilon(t)} \tag{3.30}
\]

where \( a_i \in \mathbb{R} \) for \( i = 1, \ldots, N \). Since for each \( \omega \in \Omega \), the initial value, \( \mathcal{X}_N(s) \), is a finite sum of point measures, it must be finite. Due to Proposition 3.1, it follows that \( \mathcal{X}_N(s) \in L_{0,F}(\mathcal{M}_{f,s}(\mathbb{R}^d)) \). Solving the coupled system, (3.29), essentially follows the same lines to show that (3.1) has a unique solution. However, there is an important distinction that changes the argument. The coupled system, (3.29), depends on the number of particles, \( N \), whereas the decoupled system, (3.1), is independent of \( N \).
Consequently, following [Kot08], define the metric \( \varrho_N \) on \( \mathbb{R}^{dN} \) by

\[
\varrho_N(r_N, q_N) := \max_{1 \leq i \leq N} \varrho(r_i, q_i)
\]

(3.31)

where \( r_N := (r^1, \ldots, r^N), q_N := (q^1, \ldots, q^N) \in \mathbb{R}^{dN} \). The following proposition is claimed in [KS12b], and again follows the method for measures established in [Kot08].

**Proposition 3.2** Assume either (3.3)-(3.5) or (3.6)-(3.8) holds as well as \( |X_N|(s) \in L_{0, \mathcal{F}_s}(M_f(\mathbb{R}^d)) \). Then, to each \( q_N(s) \in L_{0, \mathcal{F}_s}(\mathbb{R}^{dN}) \), there exists a unique continuous solution of (3.29), \( q_N(\cdot, q_N(s)) \in L_{0, \mathcal{F}}(C([s, \infty); \mathbb{R}^{dN})) \)

**Proof:** As in the case of the decoupled system, assume without loss of generality that \( s = 0 \). Further, we show the case where we assume (3.3)-(3.5). Note without loss of generality, we can choose a \( b > 0 \) such that

\[
|X_N|(0, \omega, \mathbb{R}^d) := \sum_{i=1}^{N} |a_i| \delta_{r^i}(\mathbb{R}^d) \leq b \text{ a.s.}
\]

Following the argument for the decoupled case, define for \( j = 1, 2, i = 1, \ldots, N \) and \( n \in \mathbb{N} \)

\[
q_{j,n+1}^i(t) := q_{j,n}^i(0) + \int_0^t F(q_{j,n}^i(s), X_{j,n}^i(s), s)ds + \int_0^t \int \mathcal{J}_\varepsilon(q_{j,n}^i(s), p, X_{j,n}^i(s), s)w(dp, ds)
\]

(3.32)

and \( q_{j,0}^i = q_N^i(0) \) where

\[
X_{j,n}(t) := \sum_{i=1}^{N} a_i \delta_{q_{j,n}^i(t)}.
\]

(3.33)

Note that the arguments from (3.18)-(3.20) hold in the coupled case with \( \mathcal{Y}_j(t) = X_{j,n}(t) \). However, to compare the \( q_{j,n}(\cdot) \), one must derive the relationships between the associated empirical measures, (3.33). As these empirical measures are signed
measures, we need the following definitions:

$$X_{j,n}^+(t) := \sum_{i=1}^{N} a_i \mathbf{1}_{\{a_i>0\}} \delta_{q_i^j,n}(t)$$

and

$$X_{j,n}^-(t) := \sum_{i=1}^{N} -a_i \mathbf{1}_{\{a_i<0\}} \delta_{q_i^j,n}(t).$$

These definitions correspond to the Hahn-Jordan decomposition of the empirical measures: $X_{j,n}(t) := X_{j,n}^+(t) - X_{j,n}^-(t)$. Now, one can compute the following:

$$\gamma_{f,s}(X_{1,n}(t), X_{2,n}(t)) \leq \gamma_f \left( \hat{X}_{1,n}^+(t), \hat{X}_{2,n}^+(t) \right)$$

$$= \gamma_f \left( X_{1,n}^+(t), X_{2,n}^+(t) \right) + \gamma_f \left( X_{1,n}^-(t), X_{2,n}^-(t) \right)$$

$$= \left( \sup_{\|f\|_{L,\infty} \leq 1} \int f(s)d(\hat{X}_{1,n}^+(s) - \hat{X}_{2,n}^+(s)) \right)$$

$$+ \left( \sup_{\|f\|_{L,\infty} \leq 1} \int f(s)d(\hat{X}_{1,n}^-(s) - \hat{X}_{2,n}^-(s)) \right)$$

$$= \left( \sup_{\|f\|_{L,\infty} \leq 1} \sum_{i=1}^{N} a_i \mathbf{1}_{\{a_i>0\}} \left[ f(q_i^1,n(t)) - f(q_i^2,n(t)) \right] \right)$$

$$+ \left( \sup_{\|f\|_{L,\infty} \leq 1} \sum_{i=1}^{N} -a_i \mathbf{1}_{\{a_i<0\}} \left[ f(q_i^1,n(t)) - f(q_i^2,n(t)) \right] \right)$$
\[
\leq \sum_{i=1}^{N} a_i \mathbf{1}_{\{a_i > 0\}} \varrho \left( q_{1,n}^i(t), q_{2,n}^i(t) \right) \\
+ \sum_{i=1}^{N} -a_i \mathbf{1}_{\{a_i < 0\}} \varrho \left( q_{1,n}^i(t), q_{2,n}^i(t) \right) \\
= \sum_{i=1}^{N} |a_i| \varrho \left( q_{1,n}^i(t), q_{2,n}^i(t) \right) \\
\leq b \varrho_N \left( q_{1,n}(t), q_{2,n}(t) \right).
\]

Consequently, using the estimates (3.18)-(3.20) and (3.36), we have that:

\[
E \sup_{0 \leq t \leq T} \varrho^2(q_{1,n+1}(t), q_{2,n+1}(t)) \leq c_{F,J,T} \left( E \varrho^2(q_{1}^i(0), q_{2}^i(0)) \right) \\
+ E \int_0^T c_b \varrho_N^2(q_{1,n}(s), q_{2,n}(s)) ds \\
\]

where \(c_{F,J,T}\) and \(c_b\) are nonnegative constants depending on the associated parameters. From which it follows that

\[
E \sup_{0 \leq t \leq T} \varrho_N^2(q_{1,n+1}(t), q_{2,n+1}(t)) \leq c_{F,J,T} \left( E \varrho_N^2(q_{1}^i(0), q_{2}^i(0)) \right) \\
+ E \int_0^T c_b \varrho_N^2(q_{1,n}(s), q_{2,n}(s)) ds \\
\]

Consequently, using the analogous arguments as (3.16)-(3.28), one can derive the existence of a unique, continuous solution by the contraction mapping principle.\(^{24}\)

\(^{24}\)See Theorem 6.12.
Without further mention, assume the measurability and integrability conditions of initial conditions and the Lipschitz and growth conditions needed to derive unique solutions of the SODE systems, (3.1) and (3.29).

3.3 SPDE Results

With the existence and uniqueness of solutions to (3.1) and (3.29), we can now examine a general type of SPDE. As we will soon derive, the empirical processes of (3.1) and (3.29) automatically yield weak solutions of a SPDE. Prior to stating the SPDE, one also needs to define the one and two particle diffusion matrices. These matrices represent a measure of correlations in the stochastic term in (3.1) and (3.29). Define the diffusion matrices as follows. For \( l, k = 1, \ldots, d, i, j = 1, \ldots, N, r^i, r^j \in \mathbb{R}^2, \mu \in \mathbf{M}_{f,s}(\mathbb{R}^d) \)

\[
\tilde{D}_{kl}(r^i, r^j, \mu, t) := \sum_{q=1}^{d} \int \mathcal{J}_{\varepsilon,kq} (r^i, p, \mu, t) \mathcal{J}_{\varepsilon,lq} (r^j, p, \mu, t) dp
\]

and

\[
D(r^i, \mu, t) := \tilde{D}(r^i, r^i, \mu, t).
\]

For \( m \in \mathbb{N} \cup \{0\} \), let \( C^m(\mathbb{R}^d; \mathbb{R}) \) be the space of \( m \) times continuously differentiable functions from \( \mathbb{R}^d \) into \( \mathbb{R} \). Further, let \( C^m_0(\mathbb{R}^d; \mathbb{R}) \) be the subspace of \( C^m(\mathbb{R}^d; \mathbb{R}) \), whose elements together with all their derivatives vanish at infinity. Recall that we denoted the inner product on \( \mathbb{R}^d \) by “•” and \( \partial_k \) and \( \partial^2_{kk} \) the first and second partial derivatives
the spatial coordinates $r_k$ and $r_k, r_{\ell}$, respectively.

In the following, $X_N$ represents the empirical process associated with the SODE system, (3.29). However, the same result will hold for (3.1). Applying Itô’s formula to $\langle X_N(t), \varphi \rangle$ for $\varphi \in C_0^2(\mathbb{R}^d; \mathbb{R})$, yields the following.

$$d\langle X_N(t), \varphi \rangle = d\left( \sum_{i=1}^{N} a_i \delta_{r^i_\varepsilon(t), \varphi(r^i_\varepsilon(t))} \right) = \sum_{i=1}^{N} a_i d\varphi(r^i_\varepsilon(t))$$

$$= \sum_{i=1}^{N} a_i (\nabla \varphi)(r^i_\varepsilon(t)) \cdot d(r^i_\varepsilon) + \frac{1}{2} \sum_{i=1}^{N} a_i \sum_{k,l=1}^{d} \partial^2_{kl} \varphi(r^i_\varepsilon(t)) d << r^i_{\varepsilon,k}(t), r^i_{\varepsilon,l}(t) >>$$

(where $<< \cdot, \cdot >>$ denotes the mutual quadratic variation of the one dimensional components.)

$$= \sum_{i=1}^{N} a_i (\nabla \varphi)(r^i_\varepsilon(t)) \cdot F(r^i_\varepsilon(t), X_N(t)(t), t) dt$$

$$+ \sum_{i=1}^{N} a_i (\nabla \varphi)(r^i_\varepsilon(t)) \cdot \int J_\varepsilon(r^i_\varepsilon(t), p, X_N(t)(t), t) w(dp, dt)$$

$$+ \frac{1}{2} \sum_{i=1}^{N} a_i \sum_{k,l=1}^{d} \partial^2_{kl} \varphi(r^i_\varepsilon(t)) d << r^i_{\varepsilon,k}(t), r^i_{\varepsilon,l}(t) >>$$

(3.40)

Now, note that

$$d << r^i_{\varepsilon,k}(t), r^i_{\varepsilon,l}(t) >> = D_{kl}(r^i_\varepsilon, X_N(t)) dt$$

(3.41)

as any terms with finite total variation in (3.29) do not contribute to (3.41). Combining (3.40) and (3.41) yields:
\[
\begin{align*}
&= \sum_{i=1}^{N} a_i (\nabla \varphi)(r^i_\varepsilon(t)) \cdot F(r^i_\varepsilon(t), \mathcal{X}_N(t), t) dt \\
&\quad + \sum_{i=1}^{N} a_i (\nabla \varphi)(r^i_\varepsilon(t)) \cdot \int J_\varepsilon(\mathcal{X}_N(t), t) w(dp, dt) \\
&\quad + \frac{1}{2} \sum_{i=1}^{N} a_i \sum_{k,l=1}^{d} \partial^2_{kl} \varphi(r^i_\varepsilon(t)) D_{kl}(r^i_\varepsilon(t), \mathcal{X}_N(t), t) dt \\
&= \langle \mathcal{X}_N(t), (\nabla \varphi)(\cdot) \cdot F(\cdot, \mathcal{X}_N(t), t) dt \rangle \\
&\quad + \left\langle \mathcal{X}_N(t), (\nabla \varphi)(\cdot) \cdot \int J_\varepsilon(\cdot, \mathcal{X}_N(t), t) w(dp, dt) \right\rangle \\
&\quad + \left\langle \mathcal{X}_N(t), \frac{1}{2} \sum_{k,l=1}^{d} (\partial^2_{kl} \varphi)(\cdot) D_{kl}(\cdot, \mathcal{X}_N(t), t) dt \right\rangle \\
&= \sum_{k=1}^{d} \langle \mathcal{X}_N(t) F_k(\cdot, \mathcal{X}_N(t), t) dt, \partial_k \varphi \rangle \\
&\quad + \sum_{k=1}^{d} \left\langle \mathcal{X}_N(t) \int J_{\varepsilon,k}(\cdot, \mathcal{X}_N(t), t) w(dp, dt), \partial_k \varphi \right\rangle \\
&\quad + \left\langle \frac{1}{2} \sum_{k,l=1}^{d} D_{kl}(\cdot, \mathcal{X}_N(t), \mathcal{X}_N(t) dt, \partial^2_{kl} \varphi \right\rangle. \tag{3.42}
\end{align*}
\]
Thus, it follows that we have that

\[ d < X_N, \varphi > = \langle \frac{1}{2} \sum_{k,l=1}^{d} \partial^2_{kl}(D_{kl}(\cdot, X_N, t) X_N(t)) dt, \varphi \rangle \]

\[ - \langle \nabla \bullet (X_N(t) F(\cdot, X_N(t), t) dt), \varphi \rangle \]

\[ - \langle \nabla \bullet \left( X_N(t) \int J_{\epsilon}(\cdot, p, X_N(t), t) w(dp, dt) \right), \varphi \rangle. \]

We can now conclude the following:

**Proposition 3.3** \( X_N(\cdot) \) is a weak solution of the signed-measure valued SPDE:

\[
\begin{align*}
    dY &= \left( \frac{1}{2} \sum_{k,l=1}^{d} \partial^2_{kl}(YD_{kl}(\cdot, Y, t)) - \nabla \bullet (YF(\cdot, Y, t)) \right) dt \\
    &\quad - \nabla \bullet \left( Y \int J_{\epsilon}(\cdot, Y, p, t) w(dp, dt) \right)
\end{align*}
\]

(3.44)

with initial condition \( Y_s = X_s = \sum_{i=1}^{N} a_i \delta_{r_i} \) and Hahn-Jordan decomposition \( Y_s^\pm = X_s^\pm \).

### 3.4 Extension by Continuity

Proposition 3.3 inspires the following method and representation. The empirical process associated with (3.29) satisfies (3.44). Thus, one can construct a weak solution of (3.44) provided that the initial signed measure, \( Y_s \), is discrete. One approach to construct solutions for arbitrary signed measures is **Extension by Continuity**. This method approximates arbitrary signed measures by discrete signed measures in an
appropriate metric. One then shows that the solutions associated with the discrete signed measures converge. Finally, one shows that this limit is the solution of (3.44) with the arbitrary initial signed measure.

The first step in the method’s argument is justifying approximations of initial signed measures by discrete signed measures. To show such a statement, it is necessary to derive certain a priori estimates on the empirical processes. These estimates show that the empirical processes associated with solutions of (3.1) or (3.29) can be estimated in terms of the initial signed measures. Again, to simplify notation, we assume that $s = 0$.

**Proposition 3.4** Suppose for $i = 1, 2$, $X_i(0) \in L_{2,\mathcal{F}_0}(M_{f,s}(\mathbb{R}^d))$ are the initial signed measures and $\tilde{Y} \in L_{\text{loc},2,\mathcal{F}}(C((s,T];M_{f,s}(\mathbb{R}^d)))$ is the signed-measure input process. Further, $Y_i(\cdot)$ are the empirical processes, given by (3.30) associated with $X_i(0)$ for $i = 1, 2$. Finally, let $\tau$ be a localizing stopping time for $\tilde{Y}, X_1(0)$ and $X_2(0)$. Then there is a constant $c_{T,F,\mathcal{J},\tau}$ such that

$$E\left(\sup_{0 \leq t \leq T \land \tau} \gamma^2_{f,s}(Y_1(t), Y_2(t))1_{\{\tau > 0\}}\right) \leq c_{T,F,\mathcal{J},\tau} E\left(\gamma^2_f(\tilde{X}^\pm_1(0), \tilde{X}^\pm_2(0))1_{\{\tau > 0\}}\right).$$  

(3.45)

**Proof:** The following proof is claimed in [KS12a] and follows the measure case established in [Kot08]. Suppose that the initial distributions are non-random. Once we establish the result under this case, we can condition on the $\sigma$-algebra, $\mathcal{F}_0$, to establish the general claim. Further, assume that $Y_i(t)$ has Hahn-Jordan decomposition $Y_i^\pm(t)$ with measure $Y_i^\pm(0, \mathbb{R}^d) = m_i^\pm \geq 0$ for $i = 1, 2$. 

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By Proposition 2.7:

\[
E \left( \sup_{0 \leq t \leq T \land \tau} \gamma^2_{f,s}(Y_1(t), Y_2(t)) \right) \leq E \left( \sup_{0 \leq t \leq T \land \tau} \gamma^2_{\hat{f}}(\hat{Y}_1(t), \hat{Y}_2(t)) \right)
\]

\[
\leq 2E \left( \sup_{0 \leq t \leq T \land \tau} \gamma^2_{\hat{f}}(\hat{Y}_1^+(t), \hat{Y}_2^+(t)) \right) + 2E \left( \sup_{0 \leq t \leq T \land \tau} \gamma^2_{\hat{f}}(\hat{Y}_1^-(t), \hat{Y}_2^-(t)) \right).
\]  

(3.46)

We focus on the first term as the second will follow by a similar argument. The following argument is based on a relationship between the Kantorovich-Rubinstein distance and the Wasserstein distance. These details are given in Proposition 6.8. By this proposition, we have the following.

\[
E \sup_{0 \leq t \leq T \land \tau} \gamma^2_{\hat{f}}(\hat{Y}_1^+(t), \hat{Y}_2^+(t))
\]

\[
\leq E \sup_{0 \leq t \leq T \land \tau} \left[ (m_1^+ \wedge m_2^+) \mathbb{W}_1 \left( \frac{Y_1^+(t)}{m_1^+}, \frac{Y_2^+(t)}{m_2^+} \right) + |m_1^+ - m_2^+|^2 \right]
\]

Focus on the first term on the right side.

\[
E \sup_{0 \leq t \leq T \land \tau} \left[ (m_1^+ \wedge m_2^+) \mathbb{W}_1 \left( \frac{Y_1^+(t)}{m_1^+}, \frac{Y_2^+(t)}{m_2^+} \right) \right]^2
\]

\[
= (m_1^+ \wedge m_2^+)^2 E \sup_{0 \leq t \leq T \land \tau} \min_{Q_+ \in C\left( \frac{x_1^{(0)}}{m_1^+}, \frac{x_2^{(0)}}{m_2^+} \right)} \left[ \int \int g(r(t, \hat{Y}, q), r(t, \hat{Y}, \tilde{q})) Q_+(dq, d\tilde{q}) \right]^2
\]

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Now, let $Q_+ \in C(X_1(0) + m_1^+, X_2(0) + m_2^+)$ be arbitrary.\(^{25}\)

\[
E \sup_{0 \leq t \leq T \wedge \tau} \left[ (m_1^+ \wedge m_2^+) W_1 \left( \frac{Y_1^+(t)}{m_1^+}, \frac{Y_2^+(t)}{m_2^+} \right) \right]^2 
\leq (m_1^+ \wedge m_2^+) E \sup_{0 \leq t \leq T \wedge \tau} \left[ \int \int \phi(r(t, \tilde{Y}, q), r(t, \tilde{Y}, \tilde{q})) Q_+(dq, d\tilde{q}) \right]^2 
= (m_1^+ \wedge m_2^+) E \sup_{0 \leq t \leq T \wedge \tau} \int \int \int \int \phi(r(t, \tilde{Y}, q), r(t, \tilde{Y}, \tilde{q})) \phi(r(t, \tilde{Y}, p), r(t, \tilde{Y}, \tilde{p})) Q_+(dp, d\tilde{p}) Q_+(dq, d\tilde{q})
\]

By the Cauchy-Schwarz inequality and Proposition 3.1, we have that:

\[
E \sup_{0 \leq t \leq T \wedge \tau} \int \int \int \int \phi(r(t, \tilde{Y}, q), r(t, \tilde{Y}, \tilde{q})) \phi(r(t, \tilde{Y}, p), r(t, \tilde{Y}, \tilde{p})) Q_+(dp, d\tilde{p}) Q_+(dq, d\tilde{q}) 
\leq \int \int \int \int \sqrt{E \sup_{0 \leq t \leq T \wedge \tau} \phi^2(r(t, \tilde{Y}, q), r(t, \tilde{Y}, \tilde{q}))} 
\times \sqrt{E \sup_{0 \leq t \leq T \wedge \tau} \phi^2(r(t, \tilde{Y}, p), r(t, \tilde{Y}, \tilde{p})) Q_+(dp, d\tilde{p}) Q_+(dq, d\tilde{q})} 
\leq \tilde{c}_{T,F,J,\tau} \int \int \int \int \sqrt{\phi^2(q, \tilde{q})} \sqrt{\phi^2(p, \tilde{p})} Q_+(dp, d\tilde{p}) Q_+(dq, d\tilde{q}).
\]

As $Q_+$ was an arbitrary element of $C(X_1(0) + m_1^+, X_2(0) + m_2^+)$, combining the these results yields

\(^{25}\)Recall the definition from (2.1).
that
\[ E \sup_{0 \leq t \leq T \wedge \tau} \left[ \left( m_1^+ \land m_2^+ \right) \mathbb{W}_1 \left( \frac{X_1^+(0)}{m_1^+}, \frac{X_2^+(0)}{m_2^+} \right) \right]^2 \]
\[ \leq \tilde{c}_{T,F,J,\tau} \left( m_1^+ \land m_2^+ \right)^2 \mathbb{W}_1^2 \left( \frac{X_1^+(0)}{m_1^+}, \frac{X_2^+(0)}{m_2^+} \right) . \]

Yet, by Proposition 6.8, adding \(|m_1^+ - m_2^+|\) to the right side represents a metric that is equivalent to \(\gamma_f(\cdot, \cdot)\). Consequently, it follows that:

\[ E \left( \sup_{0 \leq t \leq T \wedge \tau} \gamma_f^2(Y_1^+(t), Y_2^+(t)) \right) \leq \tilde{c}_{T,F,J,\tau} E \left( \gamma_f^2(X_1^+(0), X_2^+(0)) \right) . \] (3.47)

The term for the negative components of the signed measures follows similarly. Combining these terms yields the claim.

The technique of extension by continuity is employed in [Kot95], [KX99], [MP82], and [KK10]. We shall comment on [MP82] and [Kot95] in Chapter 4 as an application of the general results. However, the other articles have generated general results in signed measure valued SPDE. In [KX99], the authors focus on a more general class of SPDE. In particular, the associated SODE (3.1) (or (3.29)) includes another random term driven by independent Brownian motions as an external noise term. Furthermore, the case allows for the particle weights, \(a_i\), to be time dependent and not constant. However, there are several drawbacks from the approach in [KX99]:

- The authors use that the initial conditions of the weights and the positions are exchangeable in addition to square integrable and measurable.
- The authors assume Lipschitz and growth conditions such as (3.3)-(3.5) (3.6)-(3.8) for every possible representation of a signed measure as the difference of two measures. This assumption is not only difficult to verify, but such a strong assumption implies that the assumptions are not the ideal setting for the
Despite these drawbacks, the authors derive existence and uniqueness result for their SODE system, and similarly pass the solution to a solution of an SPDE by Itô’s Formula.

Following the results from [KX99], [KK10] uses the results on exchangeable pairs to derive limit results for deterministic PDE. In particular, the authors examine the same class of SPDE as in this work, but with a different result in mind. The authors derive macroscopic limit results for solutions of SPDE. That is, the correlation length, \( \varepsilon \), converges to 0, and consequently, the limit of the solutions of SPDE converge to a solution of a deterministic PDE.

### 3.5 Hahn-Jordan Decompositions

Another important work intimately related to this thesis is [Kot10]. In this work, the author examines the same SODE system and class of SPDE, and derives the following interesting result.

**Proposition 3.5** Assume that the coefficients of the SODE systems, (3.1) and (3.29), satisfy either the Lipschitz conditions, (3.3)-(3.4) or (3.6)-(3.7). Then, the particles satisfying (3.1) or (3.29) do not coalesce in finite time with probability one. That is:

\[
\tau_{\infty} := \inf\{t > 0 : r^i_\varepsilon(t) = r^j_\varepsilon(t) \text{ for some } i \neq j\} = \infty \text{ a.s.} \quad (3.48)
\]

With this proposition and some additional hypothesis, the author establishes that a solution to the SPDE, (3.44), preserves the Hahn-Jordan decomposition of the initial empirical process. An important part of the argument in [Kot10] relates solutions
to a certain flow representation. To show this representation, consider the empirical measures associated with (3.1). For convenience, fix $\tilde{Y} \in L_{loc,2,r}(C((s,T]; M_{f,s}(\mathbb{R}^d)))$, and assume that $q \in \mathbb{R}^{dN}$. Without loss of generality, choose $s = 0$ in the following. Consequently, (3.1) has a unique solution:

$$r(t, \omega, q) := r(t, \omega, \tilde{Y}(t), q) = (r^1(t, \omega, \tilde{Y}(t), q), \ldots, r^N(t, \omega, \tilde{Y}(t), q)). \quad (3.49)$$

By Proposition 3.1, $r(t, \omega, q)$ is measurable in $(t, \omega, q)$. Consequently, $$(\omega, q) \mapsto \delta_{r(t,\omega,q)}, \quad \Omega \times \mathbb{R}^d \mapsto M_f(\mathbb{R}^d), \quad (3.50)$$
is $\mathcal{F}_t \otimes \mathcal{B}_d - \mathcal{B}_{M_f(\mathbb{R}^d)}$ measurable, where $\mathcal{B}_{M_f(\mathbb{R}^d)}$ is the Borel $\sigma$ - algebra on $(M_f(\mathbb{R}^d), \gamma_f)$. The following proposition from [Kot10] shows the definition and the importance of the flow representation.

**Proposition 3.6** Suppose $X_0 \in L_{2,F_0}(M_{f,s}(\mathbb{R}^d))$ and $Y(t, \omega) \in M_{f,s}(\mathbb{R}^d)$ is defined by

$$Y(t, \omega) := Y(t, \omega, \tilde{Y}(t), X_0(\omega)) = \int \delta_{r(t,\omega,q)}X_0(dq, \omega). \quad (3.51)$$

Then, $Y(t, \omega)$ is a solution to the SPDE, (3.44), where $\tilde{Y}$ replaces $Y$ in the arguments of $D$, $F$, and $J_\varepsilon$.

**Proof:** The proof is given in [Kot10]. We reproduce it to fill in the details for the argument in this thesis. To simplify notation, denote $r(t, q) = r(t, \omega, q)$ and $m(r(s, q), ds) := \int J_\varepsilon(r(s, p), p, \tilde{Y}(s), s)w(dp, ds)$. Let $\{\eta_n\}_{n \in \mathbb{N}}$ be a complete, orthonormal system in $L_{2,F_0}(\mathbb{R}^d)$, and define $\tilde{\eta}_n = \eta_n I_d$ where $I_d$ is the identity matrix

26The right side is by definition the image of the initial measure $X_0$ under the flow $q \mapsto r(t, \omega, q)$. 

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in $\mathcal{M}_{d \times d}$. Set
\[
\beta^n(t) := \int_0^t \int \hat{\eta}_n(p) w(dp, ds).
\]
The $\beta^n(\cdot)$ are i.i.d. standard $\mathbb{R}^d$-valued Brownian motions by [Kot08]. Also, note that we have the following
\[
\int \mathcal{J}_\varepsilon(r(s,p),p,\tilde{\mathcal{Y}}(s),s)w(dp, ds) = \sum_{n=1}^{\infty} \sigma_n(r(t,q),\tilde{\mathcal{Y}}(t),t)\beta^n(dt)
\]
where
\[
\sigma_n(r,\mu,t) := \int \mathcal{J}_\varepsilon(r,p,\mu,t)\hat{\eta}_n(p) dp.
\]
Let $\phi \in C_0^2(\mathbb{R}^d, \mathbb{R})$. By Itô’s Formula, we obtain that
\[
< \mathcal{Y}(t),\phi > = \int \phi(r)\mathcal{Y}(t,dr) = \int \phi(r) \int \delta_{r(t,q)}(dr)\mathcal{X}_0(dq)
\]
\[
= \int \phi(q)\mathcal{X}_0(dq) + \int \int_0^t (\nabla \phi)(r(s,q)) \bullet F(r(s,q),\tilde{\mathcal{Y}}(s),s)ds\mathcal{X}_0(dq)
\]
\[
+ \int \int_0^t (\nabla \phi)(r(s,q)) \bullet m(r(s,q), ds)\mathcal{X}_0(dq)
\]
\[
+ \frac{1}{2} \sum_{k,l=1}^{d} \int \int_0^t (\partial_{kl}^2 \phi)(r(s,q))D_{kl}(r(s,q),\tilde{\mathcal{Y}}(s),s)\mathcal{X}_0(dq).
\]
Call these terms $I_1, I_2, I_3$ and $I_4$, respectively.

Now, note that
\[
m_k(r(s,q), ds) = \sum_{n=1}^{\infty} \sum_{l=1}^{d} \sigma_{n,kl}(r(s,q),\tilde{\mathcal{Y}}(s),s)\beta^n_l(ds)
\]
which implies that,

\[ I_3(t) = \int \int_0^t (\nabla \phi)(r(s, q)) \cdot m(r(s, q), ds) \mathcal{X}_0(dq) \]

\[ = \sum_{n=1}^{\infty} \sum_{k,l=1}^{d} \int_0^t (\partial_k \phi)(r) \sigma_{n,kl}(r, \tilde{Y}(s), s) \beta_t^n(d) \int \delta_{r(s,q)} \mathcal{X}_0(dq) \]

\[ = \sum_{n=1}^{\infty} \sum_{k,l=1}^{d} \int_0^t (\partial_k \phi)(r) \sigma_{n,kl}(r, \tilde{Y}(s), s) \mathcal{Y}(s, dr) \beta_t^n(d) \]

\[ = \sum_{n=1}^{\infty} \sum_{k,l=1}^{d} \int_0^t < \sigma_{n,kl}(:, \tilde{Y}(s), s) \mathcal{Y}(s), (\partial_k \phi)(::) > \beta_t^n(d) \]

\[ = - \sum_{n=1}^{\infty} \sum_{k,l=1}^{d} \int_0^t < \partial_k (\sigma_{n,kl}(:, \tilde{Y}(s), s) \mathcal{Y}(s)), \phi(::) > \beta_t^n(d) \]

(Integrating by parts in the sense of generalized functions.)

\[ = \int_0^t \sum_{k=1}^{d} \partial_k \left( -\mathcal{Y}(s) \sum_{n=1}^{\infty} \sum_{l=1}^{d} \sigma_{n,kl}(:, \tilde{Y}(s), s) \right) \beta_t^n(ds), \phi(::) > \]

\[ = \int_0^t \nabla \bullet \left( -\mathcal{Y}(s) \int \mathcal{J}_\varepsilon(:, p, \tilde{Y}(s), s) w(dp, ds) \right), \phi( :: ) > . \]
Using similar arguments yields the following

\[ < \mathcal{Y}(t), \phi > = < \mathcal{X}_0, \phi > \]

\[ - < \int_0^t \nabla \cdot \left( \mathcal{Y}(s) F(\cdot, \tilde{\mathcal{Y}}(s), s) ds \right), \phi(\cdot) > \]

\[ - < \int_0^t \nabla \cdot \left( \mathcal{Y}(s) \int \mathcal{J}(\cdot, p, \tilde{\mathcal{Y}}(s), s) w(dp, ds) \right), \phi(\cdot) > \]

\[ + \frac{1}{2} \sum_{k,l=1}^d \int_0^t \partial^2_{kl} \mathcal{Y}(s) D_{kl}(\cdot, \tilde{\mathcal{Y}}(s), s) ds, \phi(\cdot) > . \]

Yet, note that this is a weak form of (3.44) with \( \tilde{\mathcal{Y}} \) replacing \( \mathcal{Y} \) in \( D, F, \) and \( \mathcal{J}_\mathcal{E} \). □

The empirical process associated with the SODE system, (3.1), has such a flow representation given by

\[ \mathcal{X}_N(t, \omega) = \mathcal{X}_N(t, \omega, \tilde{\mathcal{Y}}(t), \mathcal{X}_0(\omega)) = \sum_{i=1}^N a_i \delta_{r^i(t, \omega, \tilde{\mathcal{Y}}(t), \mathcal{X}_0)} = \int \delta_{r(t, \omega, \tilde{\mathcal{Y}}(t), q)} \mathcal{X}_0(dq, \omega). \]

(3.52)

Further, the empirical process has a Hahn-Jordan decomposition given by the following.

\[ \mathcal{X}^\pm(t, \omega) = \mathcal{X}^\pm(t, \omega, \tilde{\mathcal{Y}}(t), \mathcal{X}_0^\pm(\omega)) = \int \delta_{r(t, \omega, \tilde{\mathcal{Y}}(t), q)} \mathcal{X}_0^\pm(dq, \omega). \]

(3.53)

Here \( \mathcal{X}_0 \) is an initial discrete signed measure, with Hahn-Jordan decomposition, \( \mathcal{X}_0^\pm \).

One would like to conclude that the Hahn-Jordan decomposition of \( \mathcal{X}(t, \omega, \mathcal{X}_0) \) also has such a flow representation for general initial signed measures. However, to show such a result for the arbitrary initial signed measures, [Kot10] adds the assumption that the coefficients of the SODE, (3.1) and (3.29), must be bounded with bounded
derivatives up to an order \( m \) for some \( m \in \mathbb{N} \).

### 3.6 New Results

With this approach in mind, we wish to generalize the results of [Kot10]. With the apriori estimates given in Proposition 3.4, we are now ready to address the role of the Hahn-Jordan decomposition. Proposition 3.5 implies that the Hahn-Jordan decomposition of the empirical process is preserved through the flow representation, (3.53), provided the initial empirical process, \( \mathcal{X}_0 \), is discrete. If the particles driving the SODE system never coalesce, then the weights, \( a_i \), in the definition of \( \mathcal{X}_N(t) \) cannot change values. We now extend this result to arbitrary initial signed measures.

**Corollary 3.7** The Hahn-Jordan decomposition of the initial condition is preserved in the solution of (3.44) for all \( t > 0 \) for arbitrary \( \mathcal{F}_0 \)-measurable \( \mathcal{X}_{0,N} \). That is, for the flow representation,

\[
Y(t, \omega) := Y(t, \omega, \tilde{Y}(t), \mathcal{X}_{0,N}(\omega)) = \int \delta_{r(t, \omega, q)} \mathcal{X}_{0,N}(dq, \omega)
\]  

(3.54)

the Hahn-Jordan decomposition can be found by the flow representation against the Hahn-Jordan decomposition of the initial signed measures

\[
Y^\pm(t, \omega) := Y^\pm(t, \omega, \tilde{Y}(t), \mathcal{X}_{0,N}(\omega)) = \int \delta_{r(t, \omega, q)} \mathcal{X}^\pm_{0,N}(dq, \omega).
\]  

(3.55)

**Proof**: Let \( N \in \mathbb{N} \). As mentioned, (3.55) holds for discrete random initial signed measures, \( \mathcal{X}_{0,N} \). For the general case, choose a sequence, \( \{\mathcal{X}_{0,N}\}_{N \in \mathbb{N}} \) such that

\[
E\hat{\gamma}_f^2(\mathcal{X}_{0,N}, \mathcal{X}_0) \to 0.
\]
Truncating the initial signed measure if necessary, we may assume that the initial signed measures are square integrable with respect to $\hat{\gamma}_f$. By (3.45), for $M \in \mathbb{N}$

$$E\gamma_{f,s}^2(\mathcal{Y}(t, X_{0,N}), \mathcal{Y}(t, X_{0,M})) \leq c_{T,F,J,\tau} E(\hat{\gamma}_f^2(\hat{X}_{0,N}, \hat{X}_{0,M})).$$

Hence, as $N, M \to \infty$

$$E\gamma_{f,s}^2(\mathcal{Y}(t, X_{0,N}), \mathcal{Y}(t, X_{0,M})) \to 0,$$

By passing to a subsequence and relabeling if necessary, we have that $\mathcal{Y}(t, X_{0,N})$ is a Cauchy sequence for $\gamma_{f,s}$. Furthermore, the sequence converges in $\gamma_{f,s}$ as the sequence of measure pairs converges in the product space. Employing Theorem 2.8 completes the proof.

With this result on the Hahn-Jordan decomposition, we wish to show there is a solution of the SPDE, (3.44), for arbitrary adapted initial signed measures. However, to establish such a claim, it is necessary to first derive another a priori estimate. This estimate shows how solutions depend on their input signed measure processes, $\tilde{\mathcal{Y}}(\cdot)$.

**Proposition 3.8** Suppose $\tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2 \in L_{\text{loc},2,\mathcal{F}}(C([s,T]; \mathcal{M}_{f,s}(\mathbb{R}^d)))$. Let $\mathcal{Y}(t, \tilde{\mathcal{Y}}_1)$ and $\mathcal{Y}(t, \tilde{\mathcal{Y}}_2)$ be two solutions of (3.44) with $\mathcal{X}(0, \omega) := \mathcal{Y}(0, \tilde{\mathcal{Y}}_1) = \mathcal{Y}(0, \tilde{\mathcal{Y}}_2)$ and flow representations, (3.54). Then, $\forall T > 0$ there is a positive $c_T < \infty$ such that

$$E \sup_{0 \leq t \leq T} \gamma^2_{f,s}(\mathcal{Y}(t, \tilde{\mathcal{Y}}_1), \mathcal{Y}(t, \tilde{\mathcal{Y}}_2)) \leq E \sup_{0 \leq t \leq T} \hat{\gamma}_f^2(\hat{\mathcal{Y}}^\pm(t, \tilde{\mathcal{Y}}_1), \hat{\mathcal{Y}}^\pm(t, \tilde{\mathcal{Y}}_2))$$

$$\leq c_T \int_0^T E\gamma_{f,s}^2(\tilde{\mathcal{Y}}_1(s), \tilde{\mathcal{Y}}_2(s)) \, ds \leq c_T \int_0^T E\gamma_{f,s}^2(\hat{\mathcal{Y}}^\pm_1(s), \hat{\mathcal{Y}}^\pm_2(s)) \, ds.$$

(3.56)
\textbf{Proof}: Truncating the initial measure, $\mathcal{X}^\pm(0, \omega)$, if necessary, we may without loss of generality assume that

$$ess \sup_{\omega} \sum_{\pm} \gamma_f(\mathcal{X}^\pm(0, \omega)) \leq c < \infty. \quad (3.57)$$

Hence,

$$E \sup_{0 \leq t \leq T} \gamma_{f,s}^2(\mathcal{Y}(t, \tilde{Y}_1), \mathcal{Y}(t, \tilde{Y}_2)) \leq E \sup_{0 \leq t \leq T} \hat{\gamma}_f^2(\hat{\mathcal{Y}}^\pm(t, \hat{Y}_1), \hat{\mathcal{Y}}^\pm(t, \hat{Y}_2)) \quad \text{(by Proposition 2.7)}$$

$$= E \sup_{0 \leq t \leq T} \sup_{\|f\|_{L, \infty} \leq 1} \sum_{\pm} \left[ \left( f(r(t, \tilde{Y}_1, q)) - f(r(t, \tilde{Y}_2, q)) \right) \mathcal{X}^\pm_0(dq) \right]^2$$

$$\leq E \sup_{0 \leq t \leq T} \sum_{\pm} \left| \int g(r(t, \tilde{Y}_1, q), r(t, \tilde{Y}_2, q)) \mathcal{X}^\pm_0(dq) \right|^2$$

$$\leq cT \sum_{\pm} E \int (\mathcal{X}^+_0(\mathbb{R}^d) + \mathcal{X}^-_0(\mathbb{R}^d))^2 \sup_{0 \leq t \leq T} g^2(r(t, \tilde{Y}_1, q), r(t, \tilde{Y}_2, q)) \mathcal{X}^\pm_0(dq)$$

$$\leq \tilde{c}_T E \int_0^T \gamma_{f,s}^2(\tilde{\mathcal{Y}}_1(u), \tilde{\mathcal{Y}}_2(u)) du$$

(by Proposition 3.1 and the boundedness of measures)

$$\leq \tilde{c}_T E \int_0^T \hat{\gamma}_f^2(\hat{\mathcal{Y}}^\pm_1(s), \hat{\mathcal{Y}}^\pm_2(s)) ds$$

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where the last inequality follows from the definition of $\gamma_{f,s}$.

We can now state our general theorem which shows that for any initial signed measure, the SPDE, (3.44), has a solution. Furthermore, the Hahn-Jordan decomposition of the solution of (3.44) is preserved through the flow representation.

**Theorem 3.9**  

i. There is a weak solution of the SPDE, (3.44), with initial condition, $X_0$, and Hahn-Jordan decomposition, $X_0^\pm$.

ii. This solution has the representation

$$X(t) := X(t, X, X_0) = \int \delta_{r(t, \omega, X, q)} X_0(dq). \quad (3.58)$$

Further,

$$X^\pm(t) := X^\pm(t, X, X_0) = \int \delta_{r(t, \omega, X, q)} X_0^\pm(dq). \quad (3.59)$$

Essentially, the theorem states that

$$\int \delta_{r(t, \omega, X, q)} X_0^\pm(dq)$$

is the Hahn-Jordan decomposition $X^\pm(t)$ of $X(t)$ for all $t > 0$. Thus, the Hahn-Jordan decomposition is preserved.

**Proof:** Using the flow representation, one can define recursively

$$Y_0(t) \equiv X_0, \quad Y_n^\pm(t) := \int \delta_{r(t, Y_{n-1}, q)} X_0^\pm(dq). \quad (3.60)$$

Without loss of generality, assume that the total variation of initial signed measure is bounded uniformly in $\omega$. Due to conservation of the intensities by Proposition 3.5,
the same holds for the measures $\mathcal{Y}_n^\pm(t)$. By Proposition 3.8,

$$E \sup_{0 \leq t \leq T} \hat{\gamma}_f^2 \left( \hat{\mathcal{Y}}^\pm_n(t), \hat{\mathcal{Y}}^\pm_m(t) \right)$$

$$\leq c_T \int_0^T E \hat{\gamma}_f^2 \left( \hat{\mathcal{Y}}^\pm_{n-1}(s), \hat{\mathcal{Y}}^\pm_{m-1}(s) \right) ds$$

$$\leq c_T \int_0^T E \sup_{0 \leq s \leq t} \hat{\gamma}_f^2 \left( \hat{\mathcal{Y}}^\pm_{n-1}(s), \hat{\mathcal{Y}}^\pm_{m-1}(s) \right) ds.$$ 

The contraction mapping principle and the completeness of $\hat{\mathbb{M}}_f(\mathbb{R}^d)$ yield a unique adapted $\hat{\mathbb{M}}_f(\mathbb{R}^d)$-valued process $\hat{\mathcal{X}}(\cdot) \in C([0, \infty); \hat{\mathbb{M}}_f(\mathbb{R}^d))$ a.s. such that

$$E \sup_{0 \leq t \leq T} \hat{\gamma}_f^2 \left( \hat{\mathcal{Y}}^\pm_n(t), \hat{\mathcal{X}}(t) \right) \to 0, \text{ as } n \to \infty. \quad (3.61)$$

Setting $\mathcal{X} := \hat{\mathcal{X}}^+ - \hat{\mathcal{X}}^-$, we now define

$$\hat{\mathcal{X}}(t) := \int \delta_{r(t,\mathcal{X},q)} \mathcal{X}_0(dq),$$

and by Proposition 3.7, we have the Hahn-Jordan decomposition

$$\hat{\mathcal{X}}^\pm(t) = \int \delta_{r(t,\mathcal{X},q)} \mathcal{X}_0^\pm(dq). \quad (3.62)$$

Applying Proposition 3.8 yields:

$$E \sup_{0 \leq t \leq T} \hat{\gamma}_f^2 \left( \hat{\mathcal{Y}}^\pm_n(t), \hat{\mathcal{X}}^\pm(t) \right) \leq c_T \int_0^T E \hat{\gamma}_f^2 \left( \hat{\mathcal{Y}}^\pm_{n-1}(s), \hat{\mathcal{X}}^\pm(s) \right) ds \to 0, \text{ as } n \to \infty.$$
By the uniqueness of limit in (3.61), we have \( \hat{X}(t) \equiv \tilde{X}(t) \). Hence, by (3.62), we obtain the desired representation. We note that Proposition 3.6 shows that \( \mathcal{X}(\cdot) \) is a weak solution of (3.44) as \( \mathcal{X}(\cdot) \) has a flow representation.

The results derived in this chapter provide an advancement in the understanding of signed-measure valued SPDE. In particular, Theorem 3.9 yields that there is a solution of the SPDE, (3.44). Furthermore, the Hahn-Jordan decomposition is preserved from the initial signed measure. The proofs of these arguments require using only the minimal hypotheses of Lipschitz coefficients. Such a result is very desirable from an application viewpoint. In particular, in 2D fluid dynamics, Theorem 3.9 can be used to show that the vorticity of the fluid is a conserved quantity.
4 Smooth Stochastic Navier-Stokes

In the previous chapter, the analysis established existence and uniqueness results for a general class of SPDE. As an application of Theorem 2.8, we showed that a solution of the SPDE of the form, (3.44), preserves the Hahn-Jordan decomposition of the initial signed measure. This chapter examines a particular case of (3.44) which has applications to fluid dynamics. We focus on the smoothed stochastic Navier-Stokes equations (SNSE) introduced earlier. Previous results in the literature show existence and uniqueness of solutions for the stochastic Navier-Stokes equations in our context. However, with Theorem 3.9, we establish the new result that the vorticity is conserved pathwise for solutions of the SNSE.

4.1 Formulation

We recall the formulation of the SNSE with smoothed Biot-Savart kernel. The smoothed SNSE are random perturbations of the smoothed Navier-Stokes equations:

\[
\frac{\partial}{\partial t} \mathcal{X}(r, t) = \nu \Delta \mathcal{X}(r, t) - \nabla \cdot (U_\delta(r, t) \mathcal{X}(r, t))
\]  

(4.1)

where

\[
U_\delta(r, t) = \int K_\delta(r - q) \mathcal{X}(q, t) dq
\]

(4.2)

and \( K_\delta(\cdot) \) is the smoothed form of the Biot-Savart kernel. That is, for \( \delta > 0 \) and \( 1/\delta < |r| < \delta \)

\[
K_\delta(r) = K(r) := \frac{1}{2\pi |r|^2} (-r_2, r_1)
\]

where \( r = (r_1, r_2) \in \mathbb{R}^2 \) and \( K_\delta \in C^2(\mathbb{R}^2, \mathbb{R}) \) with bounded derivatives up to order two. Without loss of generality, we can also assume that \( K_\delta(0) = 0 \).
There is extensive literature on random perturbations of the Navier-Stokes equations since the work of [Cho73]. The results of [Cho73], [Lon88], and [GHL90] are more numerical analytic than our focus. Consequently, the discussion of these works in the introduction is sufficient for our purposes. However, the works, [MP82], [Kot95], [AX06] and [Ami07] form a background to the stochastic Navier-Stokes equations in this work.

From [Cho73], the authors of [MP82] approach the SNSE by considering a system of SODE of the form:

\[
dr^i(t) = \sum_{k=1}^{N} a_k K_{\delta}(r^i(t) - r^k(s))dt + \sqrt{2\nu}d\beta^i(t)
\]  

(4.3)

where for \(i = 1, \ldots, N\), \(\beta^i(t)\) are independent two-dimensional Brownian motions. The main consequences from [MP82] are the following continuum-limit results.

- For the case \(\nu = 0\), (4.3) is deterministic. The Euler equation, (4.9), has a unique solution by [Kat67] which we denote as \(X\). The authors establish that there is a sequence, \(\{\delta_N\} \subset (0, \infty)\), such that if \(X_N\) is the empirical process associated with (4.3) and mollifying constant \(\delta_N\):

  \[
  < X_N(t), \phi > \longrightarrow < X(t), \phi >
  \]

  where \(\phi \in C^2_0(\mathbb{R}^2; \mathbb{R})\).

- For \(\nu > 0\), assume half the weights \(a_i = \frac{a}{N} > 0\) and the other half are equal to \(-\frac{a}{N}\). By [GMO88], there exists a solution of the Navier-Stokes equation, (4.9), which we denote \(X(t)\). Again, let \(X_N\) be the empirical process associated with (4.3) and mollifying constant \(\delta_N\). If \(< EX_N(0), \phi > \longrightarrow < X(0), \phi >\), then there
exists a sequence, \( \{\delta_N\} \subset (0, \infty) \), such that

\[
< EX_N(t), \phi > \longrightarrow < X(t), \phi >
\]

where \( \phi \in C_0^2(\mathbb{R}^2; \mathbb{R}) \).

Recall that there are several limitations to using independent Brownian motions as the stochastic term for (4.3). In particular, the noise term is state independent rather than depending on the positions, \( r_i(t) \). In particular, [Kot08] makes the following remark about the work in [MP82]:

By applying Itô’s formula to the empirical process associated with (4.3), and computing the quadratic variation\(^{27}\)

\[
d[< X_N(t), \varphi >] = 2\nu \sum_{\pm} \left[ \sum_{j=1}^{N} \frac{2a^\pm}{N} \{(\nabla \varphi)(r^j(t)) \cdot \beta^j(dt)\} \right]
\]

\[
= 2\nu \sum_{\pm} \sum_{j=1}^{N} \frac{(2a^\pm)^2}{N^2} \left\{ \sum_{k=1}^{2} (\partial_k \varphi)^2(r^j(t)) \right\} dt,
\]

where we used the independence of \( \beta^j(\cdot) \). Hence, for \( t \leq T \)

\[
[< X_N(t), \varphi >] = O\left(\frac{1}{N}, \varphi, T\right). \tag{4.4}
\]

Thus, the empirical vorticity distribution \( X_N(t) \) becomes deterministic as \( N \to \infty \).

Choosing a sequence \( K_{\delta(N)}(r) \longrightarrow K(r) \) and assuming a suitable convergence of the \(^{27}\)In the following, the quadratic variation is denoted by \([\cdot]\).
initial conditions towards the initial condition in (4.9), it should follow that:

\[ < X_N(t), \varphi > \rightarrow < X(t), \varphi >, \quad (4.5) \]

where \( X(t) \) is the solution to (1.1).

To address the drawbacks of using independent Brownian motions, [Kot95] introduces the following correlation functionals. For \( i, j \in \{1, 2\}, \tilde{\Gamma}_{ij, \varepsilon} : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R} \) are symmetric, bounded, Borel measurable functions satisfying,

\[ \int \tilde{\Gamma}_{ii, \varepsilon}^2(r, p) dp = 1, \quad (4.6) \]

and there is a finite positive constant, \( c \), such that \( r, q \in \mathbb{R}^2 \)

\[ \int \left( \tilde{\Gamma}_{ij, \varepsilon}(r, p) - \tilde{\Gamma}_{ij, \varepsilon}(q, p) \right)^2 dp \leq c \varrho(r, q). \quad (4.7) \]

Two examples of correlation functionals are the following:

**Example 4.1** Choose \( \tilde{\Gamma}_{i,ii, \varepsilon}(r, q) := -c_\varepsilon \nabla_1^2 \pi \varepsilon e^{-|r-q|^2/4\varepsilon} \) where \( c_\varepsilon > 0 \) for \( i = 1, 2 \).

**Example 4.2** Choose \( \tilde{\Gamma}_{i,ii, \varepsilon}(r, q) := \frac{1}{\sqrt{2\pi \varepsilon}} \exp \left( -\frac{|r-q|^2}{4\varepsilon} \right) \) for \( i = 1, 2 \)

Kotelenez defines the following matrix of correlation functionals by

\[ \hat{\Gamma}_\varepsilon(r, p) := \begin{pmatrix} \tilde{\Gamma}_{11, \varepsilon}(r, p) & 0 \\ 0 & \tilde{\Gamma}_{22, \varepsilon}(r, p) \end{pmatrix}. \quad (4.8) \]

The author considers the following stochastic Navier-Stokes equation:

\[ d\mathcal{X}(t) = [\nu \Delta \mathcal{X} - \nabla \cdot (U_\delta \mathcal{X})] dt - \sqrt{2\nu} \nabla \cdot \left( \mathcal{X} \int \hat{\Gamma}_\varepsilon(\cdot, p) \right) w(dp, dt), \quad (4.9) \]
and the associated SODE system. For $i = 1, \ldots, N$,

$$dr_i(t) = \sum_{k=1}^{N} a_k K_\delta(r_i(t) - r_k(t))dt + \sqrt{2\nu} \int \hat{\Gamma}_\varepsilon(r_i(t), p) w(dp, dt)$$  \hspace{1cm} (4.10)$$

where $a_k \in \mathbb{R}$ for $k = 1, \ldots, N$. The following conservation condition is taken as a requirement in [Kot95]. Fix $a^+, a^- > 0$, and define $a = a^+ + a^-$. A solution of (4.9) must have

$$\mu \in M_{f,s} \text{ with } \mu^\pm(\mathbb{R}^2) = a^\pm \text{ where } a^+, a^- > 0$$  \hspace{1cm} (4.11)$$

With these assumptions, the author in [Kot95] shows that the smoothed Stochastic Navier-Stokes Equations have a solution for any adapted initial signed measure.

Finally, we discuss the works of [Ami00], [Ami07] and [AX06]. Recall that [Ami00] generalized (4.10) so that the vorticity could include jump processes driven by Poisson random measures. The author reexamines the work in [Ami07], and presents the results in the case of only positive measures due to completeness issues on the space of signed measures. Finally, Amirdjanova derives a diffusion approximation to the vorticity model with jump processes. In [AX06], the authors verify a form of exponential tightness for the stochastic vorticity. Such a result is then used to derive a macroscopic limit theorem. That is, the authors show that as the magnitude of the stochastic term in (4.10) tends to zero, the solutions of the stochastic equations tend to a solution of a deterministic equation.

With these results in mind, we now state the problem we wish to analyze in this work. Following [Kot95], we use the same the SODE system and conditions as in (4.10) to analyze the SNSE. The issue of existence and uniqueness for the smoothed stochastic Navier-Stokes equations follows from general results. The existence and uniqueness results from Chapter 3 yield solutions to the SNSE. What is of greater
interest is the issue of conservation of vorticity. Recall that it is a natural consequence that if $\mathcal{X}$ satisfies (1.1), one should expect that the vorticity is conserved along particle paths. Thus, one has that

$$\mathcal{X}^\pm(\mathbb{R}^2, t, \mathcal{X}_0) = \mathcal{X}^\pm(\mathbb{R}^2, 0, \mathcal{X}_0) = \mathcal{X}_0^\pm(\mathbb{R}^2) = a^\pm$$

where $\mathcal{X}_0$ is the initial signed measure. As in [Kot95], we make the assumption throughout the rest of this chapter that a solution, $\mu$, of the SNSE, (4.9), must satisfy

$$\mu^\pm(\mathbb{R}^2) = a^\pm$$

where $a^+$, $a^-$ and $a = a^+ + a^-$.  

To establish the vorticity claim, we must establish existence and uniqueness for (4.10). However, as we will soon show, (4.9) is a special form of the general SPDE from Chapter 3. Consequently, we apply the results from the previous chapter to the SNSE.

**Theorem 4.3** To each $\mathcal{F}_0$-adapted initial condition $q_N(0) \in \mathbb{R}^{2N}$, (4.10) has a unique $\mathcal{F}_t$-adapted solution, $q_N(\cdot) \in C([0, \infty); \mathbb{R}^{2N})$ a.s.

**Proof:** By Proposition 3.2, it suffices to verify that the coefficients satisfy the desired Lipschitz and boundedness conditions, (3.3)-(3.5) or (3.6)-(3.8). Clearly, the boundedness properties follow from the definition of the correlation functionals and the boundedness of $K_\delta$. For the Lipschitz conditions, if $\{\tilde{\phi}_n\}_{n \in \mathbb{N}}$ is a complete orthonormal system in $L_{2,\mathcal{F}_0}(\mathbb{R}^2)$ then, define

$$\phi_n := \begin{pmatrix} \tilde{\phi}_n & 0 \\ 0 & \tilde{\phi}_n \end{pmatrix}.$$
We have

\[
\int_0^t \int \hat{\Gamma}_\epsilon(r, p) w(dp, ds) = \sum_{n=1}^\infty \int \hat{\Gamma}_\epsilon(r, p) \phi_n(p) dp \beta^n(t),
\]

where \( \beta^n(t) := \int_0^t \int \phi_n(p) w(dp, ds) \) are i.i.d. \( \mathbb{R}^2 \)-valued standard Brownian motions by [Kot08]. It follows from the definition of the correlation function that:

\[
\sum_{n=1}^\infty \left[ \int \left( \hat{\Gamma}_\epsilon(r, p) - \hat{\Gamma}_\epsilon(q, p) \right) \tilde{\phi}_n(p) dp \right]^2 = \int \left( \hat{\Gamma}_\epsilon(r, p) - \hat{\Gamma}_\epsilon(q, p) \right)^2 dp \\
\leq c \varrho^2(r, q).
\]

Now, we note that the drift coefficient can be represented by \( F(\chi_N(t), r) : \mathcal{M}_{f,s} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) where \( F(\mu, r) := K_\delta * \mu(r) \) and \( * \) denotes convolution. For \( \mu_1, \mu_2 \in \mathcal{M}_{f,s}, r_1, r_2 \in \mathbb{R}^2 \) we have the following:

\[
|F(\mu_1, r_1) - F(\mu_2, r_2)| \leq |F(\mu_1, r_1) - F(\mu_1, r_2)| + |F(\mu_1, r_2) - F(\mu_2, r_2)| \\
\leq \left| \int K_\delta(r_1 - q) \mu_1(dq) - \int K_\delta(r_2 - q) \mu_1(dq) \right| \\
+ \left| \int K_\delta(r_2 - q) \mu_1(dq) - \int K_\delta(r_2 - q) \mu_2(dq) \right| \\
\leq c_\delta \int \varrho(r_1 - q, r_2 - q) \mu_1(dq) + c_K \int (\mu_1(dq) - \mu_2(dq)) \\
\leq c_{K,\delta} \left( a \varrho(r_1, r_2) + \hat{\gamma}_f(\hat{\mu}_1^\pm, \hat{\mu}_2^\pm) \right)
\]
where \( \hat{\mu}^\pm = (\mu^+, \mu^-) \) and \( \mu^+, \mu^- \) is the Hahn-Jordan decomposition of \( \mu \) and we have used the assumption that \( K_\delta \) is bounded with bounded derivatives.

\[ \square \]

### 4.2 New Results

The next result is the first step in showing conservation of vorticity for solutions of (4.9). It shows that the empirical process associated with (4.10) is a solution of (4.9). To show conservation of vorticity, we provide a similar argument as for extension by continuity. We show that the vorticity is conserved if the initial signed measure is discrete. After establishing this result, the general case will follow by an approximation result.

**Theorem 4.4**

i. The empirical process, \( \mathcal{X}_N(t) \), with Hahn-Jordan decomposition \( \mathcal{X}_N^\pm(t) \) is a weak solution of (4.9).

ii. Conservation of Vorticity for Discrete Initial Signed Measures

\[ \mathcal{X}_N^\pm(\mathbb{R}^2, t) = \mathcal{X}_N^\pm(\mathbb{R}^2, 0) = a^\pm \text{ a.s. for all } t \geq 0 \]

**Proof:**

i. Note that the one particle diffusion matrix associated with (4.10) is given by the following. For \( i, j \in \{1, \ldots, N\} \) and \( k, l = 1, 2 \)

\[
D(r^i, \mu, t)_{kl} := D(r^i, r^i, \mu, t)_{kl} = 2\nu \sum_{q=1}^{2} \int \hat{\Gamma}_{kq,\varepsilon}(r^i, p) \hat{\Gamma}_{lq,\varepsilon}(r^i, p) dp. \tag{4.13}
\]
Since $\hat{\Gamma}_\varepsilon$ is diagonal, this expression is 0 if $k \neq l$. Otherwise, it is given by

$$D(r^i, \mu, t)_{kk} = 2\nu \int \hat{\Gamma}_{kk, \varepsilon}^2(r^i, p).$$  \hfill (4.14)

Applying (4.6) yields that:

$$D(r^i, \mu, t) = 2\nu I_2$$

where $I_2 \in \mathcal{M}_{2 \times 2}$ is the identity matrix. The statement now follows immediately from Proposition 3.3.

ii. In Theorem 4.3, we verified that the coefficients to the SODE system satisfy the Lipschitz conditions. By Proposition 3.5, it follows that the particles $r^i(t), r^j(t)$ for $i \neq j$ do not hit in finite time. Consequently, the Hahn-Jordan decomposition of the empirical processes must remain as the following form:

$$X^+_N(t) = \sum_{a_i > 0} a_i \delta_{r^i(t)} \quad \text{and} \quad X^-_N(t) = \sum_{a_i < 0} -a_i \delta_{r^i(t)}.$$

These measures must remain on disjoint supports. Consequently, we have the claim.

We now extend the result of Theorem 4.4 to arbitrary adapted initial conditions. The existence of a weak solution for (4.9) is shown in [Kot95] and [Kot08]. However, due to its importance to the argument for conservation of vorticity, we provide the details according to the previous results in the thesis.

**Theorem 4.5**  \hspace{1cm} \bullet \hspace{0.5cm} There is a weak solution of the SNSE, (4.9), with initial condition, $\mathcal{X}_0$, and Hahn-Jordan decomposition, $\mathcal{X}_0^\pm$.  

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\section{Conservation of Total Vorticity}

\[ \mathcal{X}^\pm(\mathbb{R}^2, t, \mathcal{X}_0) = \mathcal{X}^\pm(\mathbb{R}^2, 0, \mathcal{X}_0) = a^\pm \text{ a.s..} \quad (4.15) \]

\textbf{Proof:} In Theorem 4.3, it was shown that the coefficients of (4.10) satisfy the Lipschitz and boundedness properties. By Theorem 4.4, it follows that for discrete initial signed measures, (4.9) has a weak solution. Consequently, by Theorem 3.9, we have a weak solution of (4.9) for arbitrary initial signed measures.

To show conservation of vorticity for arbitrary initial signed measures, we employ the flow representation of the solution, \( \mathcal{X}(\cdot) \), from Theorem 3.9.

\[ \mathcal{X}(t) := \mathcal{X}(t, \mathcal{X}, \mathcal{X}_0) = \int \delta_{r(t,\omega,\mathcal{X},q)} \mathcal{X}_0(dq). \quad (4.16) \]

Further,

\[ \mathcal{X}^\pm(t) := \mathcal{X}^\pm(t, \mathcal{X}, \mathcal{X}_0) = \int \delta_{r(t,\omega,\mathcal{X},q)} \mathcal{X}_0^\pm(dq). \quad (4.17) \]

Choose a sequence of discrete signed measures, \( \{\mathcal{X}_{N,0}\}_{N \geq 1} \subset \tilde{\mathcal{M}}_{f,s,d}(\mathbb{R}^2) \), such that \( \tilde{\mathcal{X}}_{N,0} \rightarrow \tilde{\mathcal{X}}_0 \) in \( \hat{\gamma}_f \). Recall that (4.9) is a special case of the general class of the SPDE analyzed in the previous chapter. Consequently, one can apply (3.4) to the solutions of the SNSE to conclude:

\[ E \left( \sup_{0 \leq t \leq T \wedge \tau} \gamma_{f,s}^2(\mathcal{X}(t, \mathcal{X}_{M,0}), \mathcal{X}(t, \mathcal{X}_{N,0})) \right) \leq c_{T,F,J,\tau} E \hat{\gamma}_f^2(\tilde{\mathcal{X}}_{N,0}, \tilde{\mathcal{X}}_{M,0}). \quad (4.18) \]

By passing to a subsequence and relabeling indices if necessary it follows that \( \mathcal{X}(\cdot, \mathcal{X}_{N,0}) \) is a Cauchy sequence for \( \gamma_{f,s} \). Furthermore, it must converge by the existence of the solution \( \mathcal{X}(\cdot, \mathcal{X}_0) \). By Theorem 2.8, we have that the limit, \( \mathcal{X}(\cdot, \mathcal{X}_0) \), must be in Hahn-Jordan form. The Hahn-Jordan positive and negative vorticities are conserved.

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for the discrete case by Theorem 4.4, and we have conservation of vorticity, (4.15).

Thus, we have established that the vorticity is a conserved quantity for the smooth stochastic Navier-Stokes equations. The arguments generated by showing this claim also have important applications to understanding the Hahn-Jordan decompositions for fluids. Such a conclusion represents an important advancement to fluid dynamics. Furthermore, the results established in Theorem 4.5 also establish the existence and uniqueness of solutions to the smoothed stochastic Navier-Stokes.
5 Conclusion

The new research presented in this thesis provide significant advancements to the fields of fluid dynamics and stochastic differential equations. The advancements, while in diverse areas, followed a specific trend of theoretical development and application. We briefly recall the main results.

- In chapter two, the analysis focuses on the incompleteness of certain metrics on the signed measures. The first important result involves identifying a relationship between the Kantorovich-Rubinstein metric and quotient spaces. Recall that one can identify the signed measures with a quotient space on the space of product measures. With this identification, we define a quotient-type metric on the signed measures. The main result involves completeness and this quotient-type metric. Although the metric does not satisfy completeness on the signed measures, it satisfies a very useful partial-completeness result. One can use the convergence properties of the metric to conclude a limit is in Hahn-Jordan form.

- In chapter three, we introduce a general class of signed-measure valued SPDE and their associated SODE system. The focus of this thesis was the role of the Hahn-Jordan decomposition for solutions. The quotient-metric from the previous chapter yields a powerful new result for the Hahn-Jordan decomposition. Applying the quotient-metric and the product metric in conjunction yields that the Hahn-Jordan decomposition of the initial signed measure is preserved in solutions. This represents a significant advancement as previous results assumed smoothness of the coefficients of the SODE. From a stochastic viewpoint, the result is ideal as the conditions are the same as those required for existence and uniqueness.
• In chapter four, the general results from chapter three apply to the smoothed SNSE. In the general literature, questions of conservation of certain quantities related to fluid dynamics naturally arise. Of great importance is that dealing with the conservation of vorticity. With the analysis of the Hahn-Jordan decomposition established in the previous chapter, we show that the vorticity of a fluid is conserved. As this question is important from both the fluid dynamic and stochastic analytic approaches, this advancement in the fields is a significant one.

Each of these results provide a significant advancement to the current research in the fields of fluid dynamics and stochastic partial differential equations. However, with these advancements, new questions arise that one would like to answer. Although numerous paths follow, we discuss only the following, yet very important possible future work.

In order to show that a solution of the SNSE exists, one must solve the singular vortex SODE with unsmoothed kernel.

\[
\begin{align*}
\frac{d}{dt}(r^i(t)) &= \sum_{j=1}^{N} a_j K(r^i(t) - r^j(t)) dt + \sqrt{2\nu} \int_{\hat{\Gamma}_e} \hat{\Gamma}_e(r^i(t), p) w(dp, dt)
\end{align*}
\] (5.1)

with \( r^i(s) = r^i_s \) for \( i = 1, \ldots, N \). To show that solutions of (5.1) exist on \([s, \infty)\), one needs to show that the point vortices satisfying (5.1) do not coalesce. Several approaches to (5.1) use a stochastic driver of independent Brownian motions instead of the correlation functionals, \( \hat{\Gamma}_e \). The works of [Tak85], [Osa85], and [FM07] show that point vortices satisfying (5.1) never coalesce under various conditions on the vortex intensities, \( a_j \). Takanobu shows the result when all the intensities, \( a_j \), have the same sign in [Tak85]. Osada shows the result for arbitrary intensities, \( a_j \) in
[Osa85], but must employ abstract techniques from PDE theory that disguise the physical interpretation. Yet, [FM07] employs a technique to preserve the physical interpretation called path-clustering. To use this technique, [FM07] must have vortex intensities that satisfy the following. For all $I \subset \{1, \ldots, N\}$, $\sum_{i \in I} a_i \neq 0$.

The results of [FM07] show potential for extension to the case where correlation functionals drive (5.1). To generalize the arguments of [FM07], one must address two issues.

- The stochastic flow is the mapping that sends an initial condition, $x \in \mathbb{R}^{2N}$, to the solution of (5.1) with this initial condition. In the case of independent Brownian motions, the stochastic flow is differentiable, and the Jacobian has a determinant that is identically 1. In the case of correlation functionals, the determinant is no longer identically 1, and must be controlled to use the results of [FM07].

- To generate solutions of (5.1), one uses solutions of (4.10) with smoothing parameter $\delta$ and lets $\delta \rightarrow 0$ in a certain sense. In the case of independent Brownian motions as drivers, the diffusion term is independent of $\delta$. Yet, in the case of the correlation functionals, the diffusion term depends on $\delta$. This fact presents a difficulty in adapting the arguments from [FM07].

Suppose that one can address these issues and prove that a solution of (5.1) exists on $[s, \infty)$. Itô’s Formula can be applied to yield solutions to the stochastic Navier-Stokes equations for discrete initial signed measures. However, as in chapter three and four, it is the continuum-limit that poses the most difficulty. A priori estimates must be derived for the singular Biot-Savart kernel which provides a significant barrier to the general solution of the stochastic Navier-Stokes equations. Consequently, if one
can establish these estimates, then one can solve the true stochastic Navier-Stokes equations. This will be a significant advancement for the fields of fluid dynamics and stochastic analysis.
6 Appendix

In this appendix, we collect many of the standard results and notations for the development of the new results. We separate the statements into two categories: Fluid Dynamics and Stochastics.

6.1 Fluid Dynamics

Lemma 6.1  Let \( r(t, r_0) \) be a path of a fluid particle in \( \mathbb{R}^2 \) under the Euler equations,

\[
\frac{\partial}{\partial t} \mathcal{X}(r, t) = -\nabla \cdot (U(r, t)\mathcal{X}(r, t))
\]

\[
\mathcal{X}(r, t) = \text{curl } U(r, t) = \frac{\partial U_2}{\partial r_1} - \frac{\partial U_1}{\partial r_2}, \quad \nabla \cdot U \equiv 0,
\]

with position \( r_0 \) at \( t = 0 \). Here \( U(r, t) = (U_1(r, t), U_2(r, t))^T \) is the velocity field, \( \mathcal{X}(r, t) \) is the vorticity, \( \nabla \) is the gradient and \( \cdot \) is the inner product on \( \mathbb{R}^2 \). Then, for all \( t \geq 0 \), it follows that

\[
\mathcal{X}(r(t, r_0), t) = \mathcal{X}(r_0, 0).
\]

Proof: Define \( f : [0, \infty) \longrightarrow \mathbb{R} \) by \( f(t) = \mathcal{X}(r(t, r_0), t) \). Computing the ordinary derivative yields the following. As \( U(r, t) \) is the velocity field for the path of the fluid particle, it follows that:

\[
\frac{df}{dt} = \left( \frac{\partial \mathcal{X}}{\partial t} + (\nabla \mathcal{X}) \cdot U \right)_{t(t, r_0)}
\]

\[
= \left( \frac{\partial \mathcal{X}}{\partial t} + (\nabla \cdot U)\mathcal{X} + (\nabla \mathcal{X}) \cdot U \right)_{t(t, r_0)}
\]
\[
\left( \frac{\partial \chi}{\partial t} + \nabla \cdot (U \chi) \right) \bigg|_{r(t,r_0)} = 0
\]

Thus, (6.2) follows immediately. \(\square\)

The following result establishes exactly how the Biot-Savart kernel enters into the analysis of the Navier-Stokes equation.

**Lemma 6.2** Assume that \(U(r,t)\) is the velocity field and \(\chi(r,t)\) is the vorticity associated with an incompressible fluid in \(\mathbb{R}^2\) satisfying:

\[
\frac{\partial}{\partial t} \chi(r,t) = \nu \Delta \chi(r,t) - \nabla \cdot (U(r,t) \chi(r,t))
\]

\[
\chi(r,t) = \text{curl } U(r,t) = \frac{\partial U_2}{\partial r_1} - \frac{\partial U_1}{\partial r_2}, \quad \nabla \cdot U \equiv 0.
\]

Then, the velocity can be written explicitly in terms of the vorticity as follows:

\[
U(r,t) = \int K(r - q) \chi(q,t) dq
\]

(6.4)

where \(K(\cdot)\) is the Biot-Savart kernel which for \(r = (r_1, r_2) \in \mathbb{R}^2\) is given by:

\[
K(r) = \nabla^\perp \left( \frac{1}{2\pi} \ln(|r|) \right) = \frac{1}{2\pi |r|^2} (-r_2, r_1)
\]

(6.5)

and \(\nabla^\perp = (-\frac{\partial}{\partial r_2}, \frac{\partial}{\partial r_1})\) and \(|r|^2 = r_1^2 + r_2^2\).

**Proof:** Note that from the incompressibility condition, it follows that \(U(r,t)\) is a divergence free vector field in two dimensions. Consequently, from [Bar11], there
must exist a differentiable $\Psi : \mathbb{R}^2 \times [0, \infty) \longrightarrow \mathbb{R}^2$ so that

$$U(r, t) = \left( \frac{\partial \Psi}{\partial r_2}, -\frac{\partial \Psi}{\partial r_1} \right).$$

Since $\mathcal{X}(r, t) = \text{curl } U(r, t)$, this implies

$$\mathcal{X}(r, t) = -\Delta \Psi$$

By standard Green’s function results, such as [Eva10], $\Psi(\cdot) = G \ast \mathcal{X}(\cdot)$ where $G(r) = -\frac{1}{2\pi} \ln |r|$ for $r \in \mathbb{R}^2$ and $\ast$ denotes convolution. Consequently,

$$U(r, t) = \int K(r - q) \mathcal{X}(q, t) dq. \quad (6.6)$$

\[\square\]

### 6.2 Stochastics

Let $(S, \rho)$ be a separable metric space with metric bounded by 1. Recall the definition of the Wasserstein distance is the following.

$$W_1(\mu, \nu) := \left( \inf_{Q \in C(\mu, \nu)} \int g(r, q) Q(dr, dq) \right) \quad (6.7)$$

where $C(\mu, \nu)$ is the set of joint distributions of the probability measures $\mu, \nu \in \mathcal{P}_1(S)$ (the set of probability measures on $S$).

We begin with the following important theorem and its proof.
Theorem 6.3 \textbf{Kantorovich-Rubinstein Theorem}

For $\mu, \nu \in \mathcal{P}_1(S)$

$$\gamma_f(\mu, \nu) := \sup_{\|f\|_{L, \infty} \leq 1} \left| \int f(r)(\mu - \nu)(dr) \right| \quad (6.8)$$

then, $\gamma_f(\mu, \nu) = W_1(\mu, \nu)$.

\textit{Proof:} The following arguments are those of [Dud02] and [Pan08]. The result is first shown in the case where the supremum in the definition of $\gamma_f$ is taken over $\|f\|_L \leq 1$. Use $\gamma$ to denote (6.8) in this case. This is the original statement of the Kantorovich-Rubinstein Theorem. The result is proved by a sequence of intermediate lemmas.

Note that the sets in the definitions of $\gamma$ and $\gamma_f$ are invariant under the multiplication of $-1$. Consequently, one can remove the absolute value in their definitions.

\textbf{Lemma 6.4} Define the following term for $\mu, \nu \in \mathcal{P}_1(S)$:

$$m_\rho(\mu, \nu) := \sup \left\{ \int f(x)d\mu(x) + \int g(y)d\nu(y) : f, g \in C(S, \mathbb{R}), f(x) + g(y) < \rho(x, y) \right\} \quad (6.9)$$

Then, $m_\rho(\mu, \nu) = \gamma(\mu, \nu)$

\textit{Proof:} Let $f \in C_L(S, \mathbb{R})$ satisfy $\|f\|_L \leq 1$, and pick $\varepsilon > 0$. Define $g(y) = -f(y) - \varepsilon$.

$$f(x) + g(y) = f(x) - f(y) - \varepsilon \leq \rho(x, y) - \varepsilon \quad (6.10)$$

Thus, $f$ and $g$ satisfy the conditions for $m_\rho(\cdot, \cdot)$, and have the following relation.

$$\int f(x)d\mu(dx) + \int g(y)d\nu(dy) = \int f(x)d\mu(dx) - \int f(y)d\nu(dy) - \varepsilon$$

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This implies that

\[ \int f(x)(\mu(dx) - \nu(dx)) \]

\[ \leq \sup\left\{ \int f(x)\mu(dx) + \int g(y)\nu(dy) : f(x) + g(y) < \rho(x, y) \right\} + \varepsilon \]

and, so

\[ \gamma(\mu, \nu) \leq m_\rho(\mu, \nu). \]

Now, for \( f, g \in C(S, \mathbb{R}) \) such that \( f(x) + g(y) < \rho(x, y) \). Define

\[ e(x) = \inf_y (\rho(x, y) - g(y)) = -\sup_y (g(y) - \rho(x, y)) \]

This implies

\[ f(x) \leq e(x) \leq -g(x) \]

and

\[ \int f(x)\mu(dx) + \int g(y)\nu(dy) \leq \int e(x)\mu(dx) - \int e(y)\nu(dy). \]

\( e(\cdot) \) also satisfies the following.

\[ e(x) - e(z) = -\sup_y (g(y) - \rho(x, y)) + \sup_y (g(y) - \rho(z, y)) \leq \sup_y (\rho(x, y) - \rho(z, y)) \leq \rho(x, z) \]

Thus, \( \|e\|_L \leq 1 \) and the other inequality is established.

**Lemma 6.5** If \((S, \rho)\) is a compact metric space and \( \mu, \nu \in P_1(S) \), then \( W_1(\mu, \nu) = m_\rho(\mu, \nu) \).
Proof: Denote $V = C(S \times S; \mathbb{R})$ equipped with the infinity norm, $\| \cdot \|_\infty$, and

$$U = \{ f \in V : f(x, y) < \rho(x, y) \}.$$

Note that $U$ is convex and open because $S \times S$ is compact. Define the subspace $E$ of $V$ as

$$E = \{ \phi \in V : \phi(x, y) = f(x) + g(y) \text{ where } f, g \in C(S; \mathbb{R}) \}.$$

This implies that

$$U \cap E = \{ f(x) + g(y) < \rho(x, y) \}.$$

One can define a linear functional on $E$, $r$, by

$$r(\phi) = \int f(x)\mu(dx) + \int g(y)\nu(dy) \text{ where } \phi(x, y) = f(x) + g(y).$$

So, by the Hahn-Banach Theorem, one can extend $r$ to $\tilde{r} : V \rightarrow \mathbb{R}$ such that $\tilde{r}|_E = r$ and

$$\sup_{U} \tilde{r}(\phi) = \sup_{U \cap E} r(\phi) = m_\rho(\mu, \nu)$$

Note that if $a(x, y) \geq 0$ and $c \geq 0$, then $\rho(x, y) - ca(x, y) - \varepsilon < \rho(x, y) \in U$. Thus, for arbitrary $c \geq 0$:

$$\tilde{r}(\rho - ca - \varepsilon) = \tilde{r}(\rho) - c\tilde{r}(a) - \rho(\varepsilon) \leq \sup_{U} \tilde{r} < \infty$$

Yet, the above is true only if $\tilde{r}(a) \geq 0$. So, $\tilde{r}$ is a positive linear functional on the compact space $S \times S$. By the Riesz Representation Theorem, there exists a Borel
measure, \( \eta \in M_f(S \times S) \), such that

\[
\tilde{r}(f) = \int f(x, y) \eta(dx, dy).
\]

Recall that \( \tilde{r}|_E = r \) and, one has that

\[
\int (f(x) + g(y)) \eta(dx, dy) = \int f(x) \mu(dx) + \int g(y) \nu(dy).
\]

Consequently, \( \eta \in C(\mu, \nu) \), and

\[
m_\rho(\mu, \nu) = \sup \tilde{r}(\phi) = \sup_{f(x, y) < \rho(x, y)} \int f(x, y) \eta(dx, dy) = \int \rho(x, y) \eta(dx, dy) \geq W_1(\mu, \nu).
\]

The other inequality, \( m_\rho \leq W_1 \), follows immediately as for any \( \lambda \in C(\mu, \nu) \)

\[
\int f(x) \mu(dx) + \int g(y) \nu(dy) = \int (f(x) + g(y)) \lambda(dx, dy) \leq \int \rho(x, y) d\lambda(dx, dy).
\]

Furthermore, it is clear from the last inequality that \( \gamma(\mu, \nu) \leq W_1(\mu, \nu) \).

Lemma 6.6 If \((S, \rho)\) is separable and \( \mu \in \mathcal{P}_1(S) \), then there is a sequence \( \{\mu_n\} \subset \mathcal{P}_1 \) such that there are finite sets \( F_n \) with \( \mu_n(F_n) = 1 \) and \( W_1(\mu_n, \mu) \rightarrow 0 \) and \( \gamma(\mu_n, \mu) \rightarrow 0 \).

Proof: Since \((S, \rho)\) is separable, for \( n \geq 1 \) there exists a partition of \( S \) into \( \{S_{n,k}\}_{n \geq 1} \) such that \( diam(S_{n,k}) \leq \frac{1}{n} \). Further, we assume each set in the partition is non-empty.
as we can remove empty sets. Let \( x_{n,k} \in S_{n,k} \), and define the following.

\[
  f_{n,k}(x) = \begin{cases} 
  x_{n,j} & \text{if } x \in S_{n,j} \text{ for some } j \leq k \\
  x_{n,1} & \text{otherwise}
  \end{cases}
\]

Then, it follows that for large enough \( k \)

\[
  \int \rho(x, f_{n,k}(x)) \mu(dx) = \sum_{j \geq 1} \int_{S_{n,j}} \rho(x, f_{n,k}(x)) \mu(dx) 
  \leq \frac{1}{n} \sum_{j \leq k} \mu(S_{n,j}) + \int_{S \setminus (S_1 \cup \cdots \cup S_k)} \rho(x, x_{n,1}) \mu(dx) \leq \frac{2}{n}.
\]

Define \( \mu_n \in \mathcal{P}_1(S) \) as the measure concentrated on the sets \( \{S_{n,1}, \ldots, S_{n,k}\} \), and let \( \lambda_{n,k} \) be the push-forward measure of \( \mu \) under the map \( x \rightarrow (f_{n,k}(x), x) \) so that \( \lambda_{n,k} \in C(\mu_n, \mu) \):

\[
  \mathbb{W}_1(\mu_n, \mu) \leq \int \rho(x, y) \lambda_{n,k}(dx, dy) = \int \rho(f_{n,k}(x), x) \mu(dx) \leq \frac{2}{n}.
\]

As we remarked before, \( \gamma(\mu_n, \mu) \leq \mathbb{W}_1(\mu_n, \mu) \). Consequently the proof is complete.

Finally, we can prove the Kantorovich-Rubinstein Theorem with the above lemmas. Since \( (S, \rho) \) is a separable metric space, then for \( \mu, \nu \in \mathcal{P}_1(S) \), there exists \( \mu_n, \nu_n \) concentrated on compact sets such that \( \mu_n \rightarrow \mu, \nu_n \rightarrow \nu \) in both \( \gamma \) and \( \mathbb{W}_1 \). Since
these metrics agree on compact sets, one has that $\mathbb{W}_1(\mu_n, \nu_n) = \gamma(\mu_n, \nu_n)$. Thus,

$$\mathbb{W}_1(\mu, \nu) \leq \mathbb{W}_1(\mu, \mu_n) + \mathbb{W}_1(\mu_n, \nu_n) + \mathbb{W}_1(\nu_n, \nu)$$

$$= \mathbb{W}_1(\mu, \mu_n) + \gamma(\mu_n, \nu_n) + \mathbb{W}_1(\nu_n, \nu)$$

$$\leq \mathbb{W}_1(\mu, \mu_n) + \gamma(\mu_n, \mu) + \mathbb{W}_1(\nu, \nu) + \gamma(\nu, \nu_n) + \gamma(\mu, \nu).$$

Letting $n \to \infty$ shows that $\mathbb{W}_1(\mu, \nu) \leq \gamma(\mu, \nu)$. The other inequality was established previously, so we have equality in the case of $\gamma$.

To show the claim with $\gamma_f$, we must show that

$$\gamma_f(\mu, \nu) = \sup_{\|f\|_{L, \infty} \leq 1} \int f(q)(\mu(dq) - \nu(dq))$$

and

$$\gamma(\mu, \nu) = \sup_{\|f\|_L \leq 1} \int f(q)(\mu(dq) - \nu(dq))$$

agree. Clearly, $\gamma_f(\mu, \nu) \leq \gamma(\mu, \nu)$. Now for an arbitrary constant, $c$,

$$\gamma(\mu, \nu) = \sup_{\|f\|_L \leq 1} \left[ \int (f(q))(\mu(dq) - \nu(dq)) - c(\mu(S) - \nu(S)) \right]$$

$$= \sup_{\|f\|_L \leq 1} \int f(q)(\mu(dq) - \nu(dq)).$$

As both $\mu, \nu$ are probability measures. Consequently, one can choose without loss of generality that $f(x_0) = 0$ in the definition of $\gamma(\mu, \nu)$ where $x_0$ is a fixed origin in $S$.  

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Yet, a function with Lipschitz constant bounded by 1 and 0 at the origin, \( x_0 \), satisfies:

\[
\sup_{\|f\|_{L^1} \leq 1} f(r) = \rho(r, x_0) \leq 1
\]

as \( \rho \) is a metric bounded by 1. This implies that \( \|f\|_{\infty} \leq 1 \) if \( \|f\|_{L^1} \leq 1 \) and \( f(x_0) = 0 \). Consequently, we obtain that \( \gamma_f(\mu, \nu) = \gamma(\mu, \nu) = W_1(\mu, \nu) \) for all \( \mu, \nu \in \mathcal{P}_1(S) \).

**Theorem 6.7** Assume \((S, \varrho)\) is a complete, separable metric space with a bounded metric and countably dense set, \( T \). The set of discrete, finite, Borel measures:

\[
\mathcal{M}_{f,d} := \left\{ \mu := \sum_{i=1}^{N} a_i \delta_{t_i}, N \in \mathbb{N}, t_i \in T \text{ and } a_i \in [0, \infty) \cap \mathbb{Q} \; i = 1, \ldots, N \right\}
\]  

is dense in \((\mathcal{M}_f(S), \gamma_f)\). Similarly, the set of discrete, finite, Borel, signed measures:

\[
\mathcal{M}_{f,s,d} := \left\{ \mu := \sum_{i=1}^{N} a_i \delta_{t_i}, N \in \mathbb{N}, t_i \in T \text{ and } a_i \in \mathbb{Q} \; i = 1, \ldots, N \right\}  
\]  

is dense in \((\mathcal{M}_f(S), \gamma_f)\).

**Proof:** We show the claim for the measures as the argument for the other case is analogous. Let \( \varepsilon > 0 \) and \( \mu \in \mathcal{M}_f(S) \). We may assume that \( \mu \) is not the 0 measure. Since \((S, \varrho)\) is separable, there is a countable set, \( \{x_n\}_{n \in \mathbb{N}} \subset T \), such that \( S = \bigcup_{n \in \mathbb{N}} B_\varepsilon(x_n) \) where \( B_\varepsilon(x_n) \) is the ball of radius \( \varepsilon \) centered at \( x_n \). Define \( \bar{B}_n := B_\varepsilon(x_n) \setminus \bigcup_{i=1}^{n-1} B_\varepsilon(x_i) \) which partitions \( S \) into disjoint sets. This allows one to create the well-defined map, \( f : S \rightarrow S \), which sends each element in \( \bar{B}_n \) to \( x_n \).

Now, define the measure, \( \nu \in \mathcal{M}_f(S) \), as the push-forward of the measure, \( \mu \),
under $f$. Consequently, by writing $a_i := \mu(\tilde{B}_i) \in \mathbb{R}$, we have that

$$
\nu(\cdot) = \mu(f^{-1}(\cdot)) = \sum_{i=1}^{\infty} a_i \delta_{x_i}(\cdot).
$$

Since $\mu$ is a finite measure, we must have that $a := \sum_{i=1}^{\infty} a_i < \infty$, and $\nu(S) = \mu(S) = a$.

By the Kantorovich-Rubinstein Theorem:

$$
\gamma_f(\mu, \nu) = a \gamma_f\left(\frac{\mu}{a}, \frac{\nu}{a}\right) = a \mathbb{W}_1\left(\frac{\mu}{a}, \frac{\nu}{a}\right).
$$

Note that the product measure, $\lambda = \frac{\mu}{a} \times \frac{\nu}{a} \in C\left(\frac{\mu}{a}, \frac{\nu}{a}\right)$. Consequently, by Fubini-Tonelli,

$$
a \mathbb{W}_1\left(\frac{\mu}{a}, \frac{\nu}{a}\right) \leq a \int \varrho(x, y) \lambda(dx, dy) = \sum_{i=1}^{\infty} a_i \int_{\tilde{B}_i} \varrho(x, x_i) \mu(dx) \leq a \varepsilon \mu(S).
$$

Since $\varepsilon$ was arbitrary, it follows that one can approximate $\mu$ by infinite sums of point masses with arbitrary weights.

Note that for $\beta > 0$, one can pick a discrete measure $\eta$ such that $\gamma_f(\nu, \eta) < \beta$.

Choose $M \in \mathbb{N}$ so that $\sum_{i=M+1}^{\infty} a_i < \beta$. Define $\eta \in \mathbb{M}_f$ by $\eta = \sum_{i=1}^{M} a_i \delta_{x_i}$

$$
\gamma_f(\nu, \eta) = \sup_{\|g\|_{L, \infty} \leq 1} \left| \int g(r)(\nu(dr) - \eta(dr)) \right| = \sup_{\|g\|_{L, \infty} \leq 1} \sum_{i=M+1}^{\infty} a_i \varrho(x_i) \leq \sum_{i=M+1}^{\infty} a_i < \beta.
$$

Finally, note that $\eta$ can be approximated elements in $\mathbb{M}_{f,d}$. For $\varepsilon > 0$, pick $\theta_i \in \mathbb{Q} \cap [0, \infty)$ such that $|a_i - \theta_i| < \varepsilon 2^{-i}$. Then, by defining $\psi := \sum_{i=1}^{M} \theta_i \delta_{x_i}$, one has the
following.
\[
\gamma_f(\psi, \eta) = \sup_{\|g\|_{L, \infty} \leq 1} \left| \int g(r)(\psi(dr) - \eta(dr)) \right|
\]

\[
= \sup_{\|g\|_{L, \infty} \leq 1} \sum_{i=1}^{M} (a_i - \theta_i) f(x_i)
\]

\[
\leq \sum_{i=1}^{M} |a_i - \theta_i|
\]

\[
< \varepsilon
\]

\[\Box\]

**Proposition 6.8** Let \(\mu, \nu \in M_f(S)\) with total masses \(m\) and \(n\) respectively. The following metric is equivalent to the Kantorovich-Rubinstein distance, \(\gamma_f,:\)

\[
\tilde{\gamma}_f(\mu, \nu) := (n \wedge m)W_1(\frac{\mu}{m}, \frac{\nu}{n}) + |m - n|
\]  \(\text{(6.12)}\)

where \(W_1\) is the Wasserstein distance.

**Proof**: We follow the proof given in [Kot08]. Note that by the Kantorovich-Rubinstein Theorem, we have already established the result in the case where \(m = n = 1\). Similarly, if either of \(\mu\) or \(\nu\) is the zero measure, we have equality by the results from Chapter 2. We first prove the case where \(m = n > 0\). By normalizing these measures, we have that by the Kantorovich-Rubinstein Theorem:

\[
\gamma_f(\mu, \nu) = m\gamma_f\left(\frac{\mu}{m}, \frac{\nu}{m}\right) = mW_1\left(\frac{\mu}{m}, \frac{\nu}{m}\right)
\]

which is (6.12).
Now, for the general case, suppose that $m > n > 0$.

\[
\gamma_f(\mu, \nu) = \sup_{\|f\|_{L, \infty} \leq 1} \int f(q)(\mu(dq) - \nu(dq))
\]

\[
= \sup_{\|f\|_{L, \infty} \leq 1} \left[ \int f(q) \left( \frac{n}{m} \mu(dq) - \nu(dq) \right) + \int f(q) \frac{m - n}{m} \mu(dq) \right]
\]

\[
\leq \sup_{\|f\|_{L, \infty} \leq 1} \int f(q) \left( \frac{n}{m} \mu(dq) - \nu(dq) \right) + \sup_{\|f\|_{L, \infty} \leq 1} \int f(q) \frac{m - n}{m} \mu(dq)
\]

\[
= \gamma_f \left( \frac{n}{m}, \nu \right) + \int 1(q) \frac{m - n}{m} \mu(dq)
\]

\[
= n \gamma_f \left( \frac{\mu}{m}, \frac{\nu}{n} \right) + m - n
\]

\[
= n \mathcal{W}_1 \left( \frac{\mu}{m}, \frac{\nu}{n} \right) + m - n
\]

where the last equality follows from the previous case. Consequently, we have shown that $\gamma_f(\mu, \nu) \leq \tilde{\gamma}_f(\mu, \nu)$. We need to establish the other inequality to show that these metrics are equivalent.

Now, assume that $\frac{n}{m} \mu \neq \nu$, so that $c := \gamma_f \left( \frac{n}{m} \mu, \nu \right) > 0$. Choose $f_i$ with $\|f_i\|_{L, \infty} \leq 1$ such that

\[
c = \lim_{i \to \infty} \int f_i(q) \left( \frac{n}{m} \mu(dq) - \nu(dq) \right) = \lim_{i \to \infty} \left[ \int f_i(q) \frac{n}{m} \mu(dq) - \int f_i(q) \nu(dq) \right].
\]

Both integrals are bounded sequences, $\alpha_i$ and $\beta_i$. Consequently, by compactness we may assume, without loss of generality that these sequences converge to $\alpha, \beta \in \mathbb{R}$, respectively. Consider the case where $-\beta < c$. This must imply that $\alpha > 0$, and
hence $\alpha_i > 0$ for large enough $i$. This implies

$$\int f_i(q) \frac{m - n}{n} \mu(dq) = \alpha_i \frac{m}{n} \frac{m - n}{m} \rightarrow \alpha \frac{m}{n} \frac{m - n}{m} = \alpha \left( \frac{m}{n} - 1 \right) > 0.$$  

This implies that

$$\gamma_f\left( \frac{n}{m} \mu, \nu \right) = c = \lim_{i \rightarrow \infty} \int f_i(q) \left( \frac{n}{m} \mu(dq) - \nu(dq) \right)$$

$$\leq \lim_{i \rightarrow \infty} \int f_i(q) \left( \frac{n}{m} \mu(dq) - \nu(dq) \right) + \lim_{i \rightarrow \infty} \int f_i(q) \frac{m - n}{m} \mu(dq)$$

$$= \lim_{i \rightarrow \infty} \int f_i(q) \mu(dq) - \nu(dq) \leq \gamma_f(\mu, \nu).$$

Now, assume that $\beta \leq -c$. Since $c > 0$, we may assume that $\beta_i < 0$ for all $i$.

$$\gamma_f\left( \frac{n}{m} \mu, \nu \right) = c = \lim_{i \rightarrow \infty} \int f_i(q) \left( \frac{n}{m} \mu(dq) - \nu(dq) \right)$$

$$= \frac{n}{m} \lim_{i \rightarrow \infty} \int f_i(q) \left( \mu(dq) - \frac{m}{n} \nu(dq) \right)$$

$$= \frac{n}{m} \lim_{i \rightarrow \infty} \int f_i(q) \left( \mu(dq) - \nu(dq) \right) + \frac{m - n}{n} \lim_{i \rightarrow \infty} \int f_i(q) \nu(dq)$$

$$\leq \frac{n}{m} \lim_{i \rightarrow \infty} \int f_i(q) \left( \mu(dq) - \nu(dq) \right)$$

$$\leq \frac{n}{m} \gamma_f(\mu, \nu)$$

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Note that for \( m > n \),

\[
(n \land m) \gamma_f\left(\frac{\mu}{m}, \frac{\nu}{m}\right) = \gamma_f\left(\frac{n}{m} \mu, \frac{n}{m} \nu\right).
\]

The left side is symmetric with respect to \( m \) and \( n \). Consequently, in both cases we get that

\[
(n \land m) \gamma_f\left(\frac{\mu}{m}, \frac{\nu}{m}\right) \leq \gamma_f(\mu, \nu).
\]

Now, note that

\[
m - n = \int_\Omega (\mu(d\omega) - \nu(d\omega)) \leq \gamma_f(\mu, \nu).
\]

Combining these inequalities yields

\[
\gamma_f(\mu, \nu) \leq (n \land m) \gamma_f\left(\frac{\mu}{m}, \frac{\nu}{m}\right) + |m - n| \leq 2\gamma_f(\mu, \nu)
\]

Using the Kantorovich-Rubinstein Theorem yields the result.

In the study of stochastic differential equations, questions of existence and uniqueness arise naturally. The following theorems provide the necessary tools to help address these questions. A classical result is the well-known Hölder’s Inequality. We first need the following definition for an arbitrary measure space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \mu)\), we denote \( L_p(\tilde{\Omega}, \tilde{\mathcal{F}}, \mu) \) as the space of measurable real-valued functions \( f \) such that:

\[
\|f\|_p := \left\{ \int_{\tilde{\Omega}} |f|^p(\omega) \mu(d\omega) \right\}^{\frac{1}{p}} < \infty
\]

**Theorem 6.9** *Hölder’s Inequality*

Let \( p \in (1, \infty) \) and \( q := \frac{p}{p - 1} \) and \( f \in L_p(\tilde{\Omega}, \tilde{\mathcal{F}}, \mu), g \in L_q(\tilde{\Omega}, \tilde{\mathcal{F}}, \mu) \), then \( fg \in L_{pq}(\tilde{\Omega}, \tilde{\mathcal{F}}, \mu) \).
\[ L_1(\tilde{\Omega}, \tilde{\mathcal{F}}, \mu) \text{ with} \]
\[ \|fg\|_1 \leq \|f\|_p \|g\|_q \]

**Proof** For a proof of the inequality, we refer to [Fol99].

Of particular importance is the Cauchy-Schwarz Inequality which can be obtained by picking \( p = q = \frac{1}{2} \) in Hölder’s Inequality. Typically, one uses Cauchy-Schwarz in tandem with Doob’s Martingale Inequality to obtain global estimates on behavior.

Before we state Doob’s inequality, we provide the definition for a martingale in the Hilbert space setting. Let \((\mathbb{H}, \|\cdot\|_\mathbb{H})\) be a separable, real Hilbert space with scalar product \( \langle \cdot, \cdot \rangle_\mathbb{H} \) and norm \( \|\cdot\|_\mathbb{H} \). An \( \mathbb{H} \)-valued martingale is a stochastic process (i.e. a collection of \( \mathbb{H} \)-valued random variables) \( \{X_t\}_{t \geq 0} \) such that the following hold:

- \( \{X_t\} \) is adapted to the filtration \( \{\mathcal{F}_t\}_{t \geq 0} \). That is, \( X_t \) is \( \mathcal{F}_t \) measurable for all \( t \geq 0 \)
- \( X_t \) is integrable for all \( t \geq 0 \) in the sense that \( E \|X_t\|_\mathbb{H} < \infty \).
- \( E(X_t|\mathcal{F}_s) = X_s \) a.s. for \( t \geq s \geq 0 \).

By [MP80] we note that an \( \mathbb{H} \)-valued martingale always has a modification that is cadlag (i.e. an \( \mathbb{H} \)-valued martingale such that \( X(\cdot) = \tilde{X}(\cdot) \) a.s. and \( \tilde{X}(\cdot) \) has sample paths that are continuous from the right and limits from the left exist).

**Theorem 6.10 Doob’s Martingale Inequality**

*For any stopping time \( \tau \) and \( \mathbb{H} \)-valued martingale \( X_t \)

\[
E \sup_{0 \leq t \leq \tau} \|X_t\|_\mathbb{H}^2 \leq 4E \|X_\tau\|_\mathbb{H}^2
\]

**Proof** For a proof of the inequality, we refer to [EK05].

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Theorem 6.11  Gronwall’s Lemma Suppose $u, v, w$ are $\mathbb{R}$-valued piecewise continuous functions on $a \leq t \leq b$. Suppose $u(t)$ is nonnegative on $[a, b]$ and the following holds for $a \leq t \leq b$:

$$v(t) \leq w(t) + \int_a^t u(s)v(s)ds$$

then,

$$v(t) \leq w(t) + \int_a^t u(s)w(s) \exp \left( \int_s^t u(x)dx \right) ds$$

Proof For the proof, we refer to [Kot08].

Theorem 6.12 Contraction Mapping Principle

Let $(S, d)$ be a complete metric space. Suppose $\psi : S \rightarrow S$ satisfies the following for $x, y \in S$:

$$d(\psi(x), \psi(y)) = \delta d(x, y)$$

where $0 < \delta < 1$. Then, there exists a unique element $z \in S$ such that $\psi(z) = z$.

Proof: For a proof of the statement, see [GG99].

Definition 6.13 Itô Integrals and Quadratic Variation

The quadratic variation processes form a central role in the stochastic integration theory. To define these processes rigorously we follow [MP80] and [Kot08]. For a sequence of partitions of $[0, \infty)$, $\{t^n_0 < t^n_1 < \ldots < t^n_k < \ldots\}$ such that for all $T > 0 \max_{t^n_k \leq T} (t^n_k - t^n_{k-1}) \to 0$ as $n \to \infty$. An $\mathbb{H}$-valued stochastic process is said to be of finite quadratic variation if there exists a monotone increasing real-valued process, $<< X(\cdot) >>$, such that for every $t > 0$ we have the following:

$$\sum_{k \geq 0} \|X(t^n_k \wedge t) - X(t^n_{k-1} \wedge t)\|^2_{\mathbb{H}} \to << X >> (t) \text{ in probability, as } n \to \infty$$

(6.13)
where \( \wedge \) denotes minimum. We call \( << X >> (t) \) a quadratic variation for the process \( \{X(t)\}_{t \geq 0} \)

**Proposition 6.14** Suppose \( X(\cdot) \) is an \( \mathbb{H} \)-valued square-integrable martingale. Then, there is a unique quadratic variation \( << X >> (\cdot) \) such that (6.13) holds. If \( X(\cdot) \) is continuous, then so is its quadratic variation \( << X >> (t) \).

**Proof:** See [MP80], [Kot08].

Now, for two \( \mathbb{H} \)-valued processes with finite quadratic variation, \( X_1(\cdot) \) and \( X_2(\cdot) \), we define the mutual quadratic variation of \( X_1(\cdot) \) and \( X_2(\cdot) \) as follows:

\[
<< X_1, X_2 >> (\cdot) := \frac{1}{4} (<< X_1 + X_2 >> (\cdot) - << X_1 - X_2 >> (\cdot)) \tag{6.14}
\]

Note that this expression is well defined as the existence of the quadratic variation processes, \( << X_1 >> \) and \( << X_2 >> \), imply the existence of both \( << X_1 + X_2 >> \) and \( << X_1 - X_2 >> \). Furthermore, we note that the mutual quadratic variation defines a bilinear form by [MP80].

We recall the definition for the Itô integral driven by a continuous square integrable martingale, \( m(\cdot) = (m_1(\cdot), \ldots, m_d(\cdot))^T \), where \( d \in \mathbb{N} \) and \( T \) denotes transpose. Define \( \mathcal{M}_{d \times d} \) to be the set of \( d \times d \) matrices over \( \mathbb{R} \) and define \( L_{2,F,loc}([0, \infty) \times \Omega : \mathcal{M}_{d \times d}) \) to be the set of \( \phi(\cdot, \cdot) \) that are \( M_{d \times d} \) valued, \( F_T \)-adapted, jointly measurable in \( (t, \omega) \) with respect to \( dt \times \mathbb{P} \) (where \( dt \) is the Lebesgue measure on \( [0, \infty) \)). Also, such \( \phi \) must satisfy the following.

\[
\sum_{k,l=1}^{d} E \int_0^T \phi_{kl}(t, \cdot)^2 << m_l >> (ds) < \infty \quad \forall T > 0
\]
Given a sequence of \( \{0 = t^n_0 < \ldots < t^n_k < \ldots\} \), and \( \tilde{\phi}(t^n_k) \) \( \mathcal{F}_{t^n_k} \)-adapted \( \mathcal{M}_{d \times d} \)-valued random variables for \( i = 0, 1, \ldots \), define

\[
\phi_n(t, \omega) = \tilde{\phi}(t^n_0) + \sum_{k=1}^{\infty} \tilde{\phi}(t^n_{k-1}) 1\{t^n_{k-1} \leq t < t^n_k\}(t) \forall t \text{ a.s.}
\]

Such processes are called simple. We define the stochastic integral by:

\[
\int_0^t \phi_n(s)m(ds) := \tilde{\phi}_0 + \sum_{k=1}^{\infty} \tilde{\phi}(t^n_{k-1})(m(t^n_k \wedge t) - m(t^n_{k-1} \wedge t))
\]

By [IW89], the simple processes are dense in \( L_{2,\mathcal{F},\text{loc}}([0, \infty) \times \Omega : \mathcal{M}_{d \times d}) \) in the \( L_2 \) metric. Consequently, the definition of the stochastic integral can be extended to arbitrary \( \phi \). It can be shown that if \( \phi \in L_{2,\mathcal{F},\text{loc}}([0, \infty) \times \Omega : \mathcal{M}_{d \times d}) \), and \( \phi_n \) is a sequence of simple functions converging to \( \phi \), then the stochastic integrals converge in probability by [IW89]:

\[
\int_0^t \phi(s)m(ds) = \lim_{n \to \infty} \int_0^t \phi_n(s)m(ds).
\]

We note that \( \int_0^t \phi(s)m(ds) \) is a continuous square-integrable \( \mathbb{R}^d \)-valued martingale and has quadratic variation process given by:

\[
<< \int_0^t \phi(s)m(ds) >> = \sum_{i,j,k=1}^{d} \int_0^t \phi_{ij}(s)\phi_{ik}(s) << m_j, m_k >> (ds) \tag{6.15}
\]

where \( << \cdot, \cdot >> \) denotes the mutual quadratic variation of the components.

**Generalization of Brownian Motion differentials**

Typically, one of the main choices is \( m(t) = \beta(t) \) where \( \beta(t) \) is \( d \)-dimensional Brownian motion. However, such a choice will not be sufficient for our analysis. Denote \( \mathbb{R}^+ := \)
[0, ∞).

**Definition 6.15** Following [Wal84] and [Kot08], a process exists called **Standard Gaussian Space-Time White Noise**, \( w(dq, dt, \omega) \). It is a finitely additive random signed measure on the Borel sets of \( \mathbb{R}^d \times \mathbb{R}^+ \) of finite Lebesgue measure \( |A| \) such that the following holds:

- \( \int_0^\infty \int 1_A(q,t)w(dq,dt,\cdot) \) is a normally distributed random variable with mean 0 and variance \(|A|\)
- if \( A \cap B = \emptyset \) then \( \int_0^\infty \int 1_A(q,t)w(dq,dt,\cdot) \) and \( \int_0^\infty \int 1_B(q,t)w(dq,dt,\cdot) \) are independent

Let \( w_i(dq, dt, \cdot) \) be independent standard Gaussian White-Noise for \( i = 1, \ldots, d \). We define \( d \)-dimensional standard Gaussian White-Noise as

\[
 w(dq, dt, \cdot) = (w_1(dq, dt, \cdot), \ldots, w_d(dq, dt, \cdot))^T
\]

Finally, we provide a fundamental theorem from stochastic integration theory, Itō’s Formula. We state the theorem as in [Kot08] and [IW89]

**Theorem 6.16** **Itō’s Formula** Let \( \phi(r, t) : \mathbb{R}^{d+1} \rightarrow \mathbb{R} \) be twice continuously differentiable in space and once continuously differentiable in \( t \) such that all partial derivatives are bounded. Further, we let \( m(\cdot) \) be a continuous square-integrable \( \mathbb{R}^d \)-valued martingale and \( b(\cdot) \) a continuous process of bounded variation. Set

\[
 a(t) := b(t) + m(t)
\]
Then, $\phi(a(\cdot), t)$ is a continuous, locally square integrable semimartingale and the following holds:

$$
\phi(a(t), t) = \phi(a(0), 0) + \int_0^t \left( \frac{\partial}{\partial s} \phi \right) (a(s), s) ds
$$

$$
+ \int_0^t (\nabla \phi)(a(s), s) \cdot (b(ds) + m(ds)) + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \left( \frac{\partial^2}{\partial s_i \partial s_j} \phi \right) (a(s), s) \langle m_i, m_j \rangle (ds)
$$

(6.16)

where $\langle m_i, m_j \rangle (\cdot)$ are the mutual quadratic variations of the one-dimensional components of $m(\cdot)$. 

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References


