WATER AND FAT IMAGE RECONSTRUCTION IN MAGNETIC RESONANCE IMAGING

by

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(date) __________ June 27, 2011

*We also certify that written approval has been obtained for any proprietary material contained therein.
To my wife and daughter

To all my family
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Water and Fat Image Reconstruction in Magnetic Resonance Imaging

Abstract

by

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The reconstruction of water and fat images based on chemical shift-encoded MRI has been under intensive investigation ever since 1980’s, due to its significance in biomedical and clinical researches. Various approaches have been proposed, but robust reconstruction with respect to high and ultra-high MRI remains challenging. Major challenges include: (1) Ill-posedness arising from the non-linear MR signal model. Multiple solutions may exist at pixels, whereas improper selection of solutions leads to severe estimation errors. (2) Large field inhomogeneity variation in MR field complicates reconstruction. Approaches validated on low field inhomogeneities may disfunction in the presence of large field inhomogeneity variation. (3) Critical demanding of biomedical and clinical applications on accuracy and robustness. Estimates must be at minimal error rate to be acceptable.

Since estimation of MR field map is crucial to water and fat image reconstruction, this dissertation mainly focuses on robust recovery of field map. The approach developed with this thesis research consists of a Markov random field (MRF) based energy model for casting the estimation of field map to extensively studied MRF energy optimization, and a novel Iterated Conditional Modes (ICM) algorithm providing high-performance MRF energy optimization. Compared to MRF energy model seen
elsewhere, the proposed NLR-MRF model is characteristic of non-linear least residual data cost terms and background masking accounted for improvement of accuracy and efficiency. The major components of the novel ICM algorithm are the stability tracking (ST) and median initialization algorithms. The stability tracking algorithm dynamically keeps track of iterative stability at each pixel to avoid redundant computations. It is demonstrated that with an optimal configuration, the ST algorithm can substantially speed up the ICM iterative computations with accuracy compromise. Median based algorithms are developed to address the high sensitivity of ICM to initialization. By assuming the quasi-unimodality of MR field inhomogeneity, the median-based initialization algorithm identifies all unambiguous pixels in the whole MR field, estimates the median and uniformly or block-wisely sets the median as the initial guess. Experimental validation on synthetic and a large group of in vivo 7 Tesla mouse datasets demonstrate the robustness of the proposed approach.
Chapter 1

Introduction

1.1 Problem Formulation

This dissertation addresses the problem of reconstructing water and fat images based on chemical shift-encoded MRI, where water and fat images represent the water and fat components in the scanned object and are of significance in biomedical and clinical research [3]. For \textit{in vivo} hydrogen (\textsuperscript{1}H) MR, signals can be reasonably considered to be generated by the components of water and fat [4]. Meanwhile, protons within the water molecules process at a slightly faster frequency than those in the fat molecules due to their different chemical environments. This frequency difference is referred to as the water-fat chemical shift, approximately 3.5 ppm and propositional to the \(B_0\) field strength (e.g., about 1050 Hz at 7.0 Tesla) [3].

The chemical shift-encoded MRI uses two phases to generate the water and fat images [5]. First, a sequence of image data is acquired with different echo time (TE) shifts. Then some post-processing techniques are applied to decompose the water and fat components and yield water-only and fat-only images. Assume that totally \(N\) MR images have been acquired at echo time shifts \(t_1, t_2, \cdots, t_N\). Then the standard signal
Figure 1.1 *Images in chemical shift-encoded MRI (N = 3).* (a), (b), (c): acquisitions $S_1$, $S_2$ and $S_3$. (d): water image $W$. (e): fat image $F$. (f): field map image $\Psi$.

Model of each acquisition at a pixel $p$ is [6]

\[
\begin{pmatrix}
    s_1(p) \\
    s_2(p) \\
    \vdots \\
    s_N(p)
\end{pmatrix}
= 
\begin{pmatrix}
    e^{i2\pi \psi(p)t_1} & e^{i2\pi(\Delta f + \psi(p))t_1} \\
    e^{i2\pi \psi(p)t_2} & e^{i2\pi(\Delta f + \psi(p))t_2} \\
    \vdots & \vdots \\
    e^{i2\pi \psi(p)t_N} & e^{i2\pi(\Delta f + \psi(p))t_N}
\end{pmatrix}
\begin{pmatrix}
    W(p) \\
    F(p)
\end{pmatrix}
+ \varepsilon
\]  

(1.1)

where, $s_n$ is the acquired signal, $W$ and $F$ separately the amplitude of the water and fat signals, $\Delta f$ the water-fat chemical shift, $\psi$ the $B_0$ field inhomogeneity (generally referred to as field map value), and $\varepsilon$ the white additive Gaussian noise. With respect
to this signal model (1.1), the reconstruction of water and fat images is essentially to estimate the pixel-wise $\psi$, $W$ and $P$. Figure 1.1 illustrates how images of acquisition, water, fat and field map look like. Note that if the field map value $\psi$ is available somehow, then $W$ and $F$ can be directly estimated via

$$\rho = (\psi^t \psi)^{-1} \psi^t S = \psi^+ S$$  \hspace{1cm} (1.2)$$

with $\psi^+ = (\psi^t \psi)^{-1} \psi^t$ the Moore-Penrose pseudo-inverse of $\psi$. In fact, existing reconstruction approaches are mostly concentrated on recovery of the MR field map.

Robust recovery of field map based on the non-linear signal model (1.1), nevertheless, is challenging due to its inherent ill-posedness. The signal model is likely to contain multiple solutions with respect to the phasor $e^{i2\pi\psi}$ at pixels, whereas incorrect selection of phasor solutions (thus field map values) results in severe water-fat swapping errors. Based on the local smoothness of field inhomogeneity, a large number of approaches, such as iterative least-square estimation [7], region growing [8, 9], region merging [2], and Markov random field (MRF) energy optimization [10, 11], have been proposed to overcome the ill-posedness. However, these approaches validated in low/mederate field MR becomes more error-prone in the presence of large variation of field inhomogeneities and inferior SNR which is normal for high and ultra-high field MR ($\geq 7$ Tesla).

### 1.2 Background for MR Water-Fat Image Reconstruction

Reconstruction of water and fat images intends to decompose the water and fat components. Water-only and fat-only images are of considerable significance for biomedical research and clinical diagnosis of diseases [3, 4, 12]. First, with its relatively short $T_1$ relaxation time to water, fat often appears hyperintense and bright...
in important imaging pulse sequences such as fast spin-echo (FSE), spoiled gradient echo (SPGR), and steady-state free precision (SSFP). Without suppression, fat signals may obscure the underlying pathology such as edema and inflammation, and result in aggregated motion-related artifacts and enhancing tumors [3, 12]. Therefore, water-only images with fat suppression have been widely used in clinical applications to improve visualization of abnormalities. On the other hand, the fat components in themselves may be of primary interest for clinical diagnosis of fat related diseases [3]. For example, direct visualization of fat is desirable for early detection of fatty tumor related pathologies. Quantitative measurement of fat plays a key role in evaluation of severity of fatty infiltrative diseases such as bone marrow, hepatic steatosis, breast, muscle, brain, liver, and heart [4].

As listed in Table 1.1 [3], a large number of MR imaging and processing techniques have been proposed ever since 1980’s to address the water and fat image reconstruction problem. Amongst these techniques, the family of chemical shift selective imaging techniques using fat saturation or water excitation [13–17] has been very popular. For fat saturation, prior to the excitation of water for imaging, a frequency selective radio-frequency pulse and a spoiler gradient pulse are conjunctively used to sequentially excite and saturate the fat magnetization [12]. For water excitation, a frequency selective radio-frequency pulse is used to directly excite the water magnetization with no impact on the fat magnetization among the longitudinal axis. The chemical shift selective imaging techniques are fast and applicable to most of MR pulse sequences. But they are disadvantageous of sensitivity to $B_0/B_1$ inhomogeneities and lack of sequence efficiency.

In another kind of MR water imaging technique STIR (Short-Tau Inversion Re-
Table 1.1 Techniques for MR fat suppression and water-fat separation

<table>
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<tr>
<th>Method</th>
<th>Advantages</th>
<th>Disadvantages</th>
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| Chemically selective fat suppression pulses, fat saturation | ▶ Versatile  
▶ Relatively fast  
▶ applicable to most pulse sequences | ▶ Sensitive to B0 and B1 inhomogeneities;  
▶ Low sequence efficiency |
| Spatial spectral pulses, water excitation   | ▶ Insensitive to B1 inhomogeneities  
▶ Versatile  
▶ Relatively fast  
▶ Practical to most pulse sequences except FSE | ▶ Sensitive to B0 inhomogeneities  
▶ Low sequence efficiency  
▶ Longer excitation pulses |
| Short-Tau Inversion Recovery (STIR)         | ▶ Robust to B0 and B1 inhomogeneities  
▶ Reliable fat suppression | ▶ Mixed contrast  
▶ Inherent T1 weighting  
▶ Only works with protein density and T2W  
▶ Low SNR efficiency  
▶ Suppresses short T1 species and enhancing tissue after contrast |
| Chemical-shift encoded water-fat imaging   | ▶ Robust fat saturation  
▶ Provides fat/water images  
▶ Allow recombined images  
▶ Corrects for chemical shift  
▶ Universal compatibility  
▶ Quantitative applications  
▶ High SNR efficiency | ▶ Long scan time  
▶ More complex reconstruction |

covency), fat suppression is achieved by signal nulling process of first flipping the longitudinal magnetization and then relaxing and exciting the longitudinal magnetization of water [12]. STIR is capable of producing uniform fat suppression even in the presence of strong $B_0$ and $B_1$ inhomogeneities. Its utility is seriously constrained by the intrinsic $T_1$-weighting arising from $T_2$-weighted and proton density-weighted imaging.

A third family of water-fat imaging techniques is the chemical shift-encoded imaging originally published in 1984 by Dixon [18] and extended later by other researchers
In this family of imaging techniques, the MR acquisitions explicitly encode the water-fat shift into the signal with the use of echo time (TE) shift. Post-processing techniques help to decompose water and fat components. Compared to the aforementioned fat suppression methods, the chemical shift-encoded imaging is robust to $B_0$ and $B_1$ inhomogeneities, has high SNR efficiency and universal compatibility. Additionally, it is unique in yielding the field map image which has important applications such as correction of fast acquisitions (e.g., EPI and spiral trajectories) and automated shimming [4, 20, 21]. Disadvantages of chemical shift-encoded imaging include long scan time and requiring sophisticated postprocessing methods to reconstruct the water and fat images. A couple of postprocessing techniques developed most recently will be reviewed in Chapter 2.

1.3 Overview of Contributions

The goal of this dissertation is to develop a novel water-fat image reconstruction approach with high accuracy, efficiency, and robustness. Main components in the new approach are:

- The NLR-MRF energy model. The Markov Random Field (MRF) energy model is applied to cast the recovery of field map to classical MRF energy optimization. Compared to other forms of MRF models widely used in fields of computer vision and image processing [22, 23], the NLR-MRF model is characteristic of background masking which is important to guarantee the overall efficiency and accuracy and the non-linear least residual-based data cost terms which complicate the optimization process.
• Fast Iterated Conditional Modes (ICM) with stability tracking algorithm. The ICM algorithm [24] is chosen for the NLR-MRF energy optimization due to its success in various image processing fields. Since it is of low efficiency for high-volume data computation, the stability tracking algorithm has been developed. The stability tracking algorithm constantly keeps track of stability at each pixel so that time-costly minimization computations on stable pixels can be saved. It is demonstrated with a large number of tests that stability tracking could substantially improve efficiency and degrade no accuracy with respect to the conventional ICM.

• Median value initialization algorithms for ICM. One major disadvantage of ICM is its high sensitivity to the initial guess. Based on statistical similarity between the whole field map and the unambiguous field map, median initialization algorithms tries to estimate field inhomogeneities on the unambiguous field map and then set the initial guess on the whole image grid to the median value. Depending on the variation of field inhomogeneity, uniform and block-wise median initialization can be separately applied to provide reliable initial guesses.

• Performance assessment of the proposed approach on ultra-high field MR datasets. The proposed approach, along with other approaches, has been assessed on a large group of in vivo mouse datasets acquired with 7 Tesla MRI. Experimental assessment has demonstrated the high accuracy, efficiency and robustness of the proposed approach.

1.4 Organization of Dissertation

The remainder of the dissertation is organized as follows:
• Chapter 2 reviews fundamentals of MRI to make better understanding of the MR water and fat image reconstruction. Overview of the state-of-the-art water-fat decomposition approaches are also provided. Moreover, fundamental notions related to MRF optimization techniques are introduced.

• Chapter 3 introduces the NLR-MRF energy model to cast the recovery of MR field map to MRF optimization. Two background masking algorithms have been developed to support this special forms of energy model: one is based on morphological operations, and the other base on more advanced graph cut techniques.

• Chapter 4 introduces the ICM with stability tracking approach to performing efficient optimization computation on the NLR-MRF model. The stability tracking algorithm, along with analysis on its accuracy and efficiency are discussed in detail. Moreover, uniform and block-wise initialization algorithms are developed to provide reliable initial guesses to ICM-ST.

• Chapter 5 provides experimental assessments on the ICM-ST approach. Both simulated and in vivo datasets on 7 Tesla mouse datesets are used to compare ICM-ST with other water-fat reconstruction approaches.

• Chapter 6 concludes this dissertation with the summary and further research plans with respect to the water and fat image reconstruction problem.
Chapter 2

Preliminaries

In this chapter, a brief overview of fundamental concepts in MRI such as field inhomogeneity and chemical shift is first provided. Next, the chemical shift-encoded methods and signal models used for water-fat image reconstruction are reviewed. Then previously proposed approaches for water-fat image reconstruction are briefly covered. Finally, Markov random field (MRF) energy model and various optimization techniques are discussed.

2.1 MRI Background

Magnetic Resonance Imaging (MRI) is one of the most important modalities in medical imaging [5]. Hydrogens ($^1H$) that distribute throughout the human body in the form of water, fat and other chemical components form the basis for MRI signals. Each hydrogen can be visualized as a spin which always spins around an axis and yields a tiny magnetic moment in the same direction as the angular momentum. Exerting external magnetic fields onto these spins generates the signals which are detected and measured by MR imaging devices and further transformed, via Fourier transforms, into the final image data which exposes internal physical and chemical characteristics of the scanned object.

Next, fundamental notations of MR main field, frequency field, gradient field,
field inhomogeneity and chemical shift related to the formation of MR signals are to be introduced. In-depth material can be consulted in [5, 25, 26].

2.1.1 Magnetic Fields

Basically, there are three types of external magnetic fields applied in MRI: 1) \( B_0 \), the main field, 2) \( B_1 \), the radio frequency (RF) field, and 3) \( B_G \), the gradient field.

2.1.1.1 Main Field

The static main field \( B_0 \) is applied to line up spins within the object to be imaged. Due to thermal random motion, directions of spins inside the to-be-imaged object are completely random in the absence of external magnetic field and no net magnetic field exists around the object. However, when the main field is introduced, magnetic moments generated with proton spins are lined up in two directions: oriented parallel (with lower energy state) or anti parallel (with higher energy state) to the field. Slightly more atoms will stay in the oriented parallel direction and thus generate a net magnetic field moment \( \mathbf{M} \) which points along the direction of \( B_0 \) (termed as \( z \) or longitudinal direction) and is propositional to the spin density. In general, \( \mathbf{M} \) is factorized as

\[
\mathbf{M} = M_x \mathbf{i} + M_y \mathbf{j} + M_z \mathbf{k}
\]

(2.1)

where \( \mathbf{i} \), \( \mathbf{j} \) and \( \mathbf{k} \) are the unit vector along the \( x \), \( y \) and \( z \) axis, respectively.

The \( B_0 \) field also cause spins to resonate (or process) about the longitudinal direction. The resonance frequency \( \omega \) can be simply described by the Larmor equation

\[
\omega = \gamma \cdot B_0
\]

(2.2)

where \( \gamma \) is the gyromagnetic ratio (for hydrogen, \( \gamma / 2\pi = 42.58 \text{ MHz/T} \)).
The strength of the $B_0$ field, measured with unit of Tesla, is a critical parameter for MRI. Generally, high-field MRI is desirable because it lead to better signal-to-noise ratio (SNR) and higher spectral resolution than low-field MRI. With the development of MR technologies over the last few decades, high-field (e.g., 3.0 Tesla) and ultra-high field ($\geq 7.0$ Tesla) MRI become more and more popular in biomedical studies and clinical diagnosis.

2.1.1.2 Radio Frequency Field

The radio frequency field $B_1$ is a short-lived (normally turned on for a few microseconds or milliseconds) alternating electromagnetic field perpendicular to the $B_0$ field [5]. The effect of applying the $B_1$ to tip the net magnetization vector $\mathbf{M}$ off the longitudinal direction, or briefly, excitation. The tip angle of $\mathbf{M}$ is determined by the strength of the $B_1$ field as well as the length of time the $B_1$ field is applied. Typically, the angle of 90 degrees is desirable in which case the net magnetization $\mathbf{M}$ in (2.1) is completely in the transverse $x$-$y$ plane.

When the $B_1$ field is turned off, the net magnetization $\mathbf{M}$ will quickly restore to its equilibrium in the longitudinal plane. This restoration process, called free precession or relaxation, can be characterized with the recovery of the longitudinal magnetization $M_z$ and the destruction of the transverse magnetization $M_{xy}$. More specifically, the longitudinal relaxation process is subject to

$$M_z(t) = M_0(1 - e^{-t/T_1}) \quad (2.3)$$

where, $M_0$ is the thermal equilibrium value for $\mathbf{M}$ in the presence of $B_0$ only, and $T_1$ is a time-related constant. The transverse relaxation process is

$$M_{xy} = M_0e^{-t/T_2} \quad (2.4)$$
where, $M_{xy} = M_x \vec{i} + M_y \vec{j}$, and $T_2$ is also a time-related constant.

Note that the vector $\mathbf{M}$ always precesses at the Larmor frequency in the whole relaxation process. This precession, based on Faraday’s Law of electromagnetic induction, causes voltage signals that are detected by the RF receiver. These voltage signals, after deliberate processing of demodulation, form the raw data which further evolves into MR images with signal localization of gradient field and Fourier transform.

2.1.1.3 Gradient Field

The gradient field $B_G$ is a special kind of inhomogeneous field intended for spatial localization of signals from different parts of the scanned object. One critical feature of $B_G$ is that its $z$ component $B_{G,z}$ varies linearly along a specific direction called the gradient direction [5]. The general form of $B_{G,z}$ is

$$B_{G,z} = G_x x + G_y y + G_z z$$  \hspace{1cm} (2.5)

where $G_x x$, $G_y y$ and $G_z z$ are the gradient fields generated by the $x$-, $y$- and $z$-gradient coils, respectively. The overall magnetic field in the presence of main field $B_0$ and the gradient field $B_{G,z}$ is

$$B = (B_0 + B_{G,z}) \vec{k} = (B_0 + G_x x + G_y y + G_z z) \vec{k}$$  \hspace{1cm} (2.6)

By spatially varying the $B_{G,z}$, sampled signals on the region of interest (ROI) are collected and two-dimensional Fourier transform are subsequently applied on them to reconstruct the final image.

2.1.2 Field Inhomogeneity

MR field inhomogeneity refers to the undesirable phenomenon that the strength of external magnetic fields ($B_0$, $B_1$ or $B_G$) varies rather than being uniform across the
imaging region. Sources of field inhomogeneity can be categorized into two groups [5, 27, 28]. The first group is related to imperfection of MRI devices, including $B_0$ field inhomogeneity, bandwidth filtering of the data, eddy currents driven by the gradient fields, and RF transmission/reception inhomogeneity. This group of inhomogeneities can be minimized by shimming techniques [5, 27, 29], special imaging sequences and different sets of coils, or device calibration using phantoms or mathematical models [27, 30–32]. The second group is concerned with properties of the imaged object, such as its shape, position, orientation, magnetic permeability, and dielectrics [27]. Since this group of field inhomogeneities is coherent, they are much more difficult to eliminate than the first group.

Two fundamental properties concerning MR field inhomogeneities are noteworthy in the context of water-fat image reconstruction. First, the map of field inhomogeneities, or field map, varies spatially smoothly across the whole image. This local smoothness feature has been extensively exploited in field inhomogeneity correction [27] and estimation researches [2, 7–11]. Statistical models such as Hidden Markov random field (MRF) have been widely used to formulate this local smoothness. The second feature is that the positive correlativity of field inhomogeneities with the main field strength. As the strength of main field $B_0$ increases, the resonance frequency and the strength of RF field to excite resonances increase. As a result, the RF field becomes less likely to be quasi-static and the field inhomogeneities increase.

2.1.3 Water-Fat Chemical Shift

As described above, field inhomogeneities exist in MR acquisitions due to distinction of chemical environment of protons and other factors. Let $B_{\text{eff}}$ denote the
Figure 2.1 *Simplified water-fat chemical shift*. The chemical shift of water (H$_2$O) protons is nearly 4.7 ppm, and that of CH$_2$ protons which is the dominant chemical moiety in fatty acid is about 1.3 ppm. So the water-fat chemical shift is approximately 3.5 ppm. (Courtesy of Dr. Cassidy et al [1]).

effective field strength at a proton. Then the effective precession frequency $\omega_{eff}$ at this proton can be described by refining the Larmor equation (2.2)

$$\omega_{eff} = \gamma \cdot B_{eff}$$  

(2.7)

From (2.7), one can see that protons in different chemical species may have different precession frequency. This frequency difference is referred to as chemical shift [5].

For *in vivo* hydrogen MRI, protons in water and fat molecules are the two major contributions to resonance signals. Protons in water precess in a slightly faster frequency than those in fat [33, 34]. As illustrated in Figure 2.1, the water-fat chemical shift $\Delta \delta \approx 3.5$ ppm. Alternatively, the chemical shift measured in terms of resonance frequency is

$$\Delta f = \gamma \cdot B_0 \cdot \Delta \delta \cdot 10^{-6}$$  

(2.8)

where $\gamma$ is the Larmor frequency. For example, the chemical shift $\Delta f \approx 1050$ Hz at
7 Tesla MRI.

While water-fat chemical shift is the main source of misregistration artefact of image data in conventional hydrogen MRI [4, 5], it has become a useful tool for diagnosis of diseases such as fatty liver and bone marrow with rapid development and application of chemical shift imaging techniques. Major water-fat imaging techniques, such as chemically selective fat suppression/water excitation, spatial-spectral pulses, STIR, SSFP methods, and chemical shift-encoded $N$-point methods, are all based on the chemical shift between water and fat [3].

2.1.4 Noise in MRI

Random noise is an important factor to contaminate MR acquisitions. Noise in MR images is primarily caused by the thermal motion of electrons within the imaged object’s conducting issue, and the RF receiver coil [5, 35, 36]. Through proper design of the RF coil, the object to be imaged turns to be the major source of noise.

Statistically, the $k$-space complex MR data can be modeled by a zero-mean Gaussian random distribution. Since the inverse Fourier transform is linear and orthogonal, the noise in the MR image data is still be Gaussian [37, 38]. Moreover, in magnitude MR images, the noise conforms to the Rician distribution and degenerates to the Rayleigh distribution in the background region of the image [35, 37, 39, 40].

The signal-to-noise (SNR), as the ubiquitous index to characterize MRI acquisition quality, is impacted by a large group of imaging parameters such as the $B_0$ field strength, RF coil selection, proton density, relaxation characteristics, tissue electrical properties, pixel dimensions, and total data acquisition time [25, 41]. Formally, SNR
can be expressed as [25]

\[ \text{SNR} \propto \Delta x \Delta y \Delta z \sqrt{T_{\text{acq, total}}} \]  

(2.9)

Herein, \( \Delta x \), \( \Delta y \) and \( \Delta z \) are the three pixel dimensions and the product \( \Delta x \Delta y \Delta z \) is the voxel volume, and \( T_{\text{acq, total}} \) is the total amount of time for the data acquisition window to be open to sample data. One can readily see from (2.9) that the increase of spatial resolution leads to decrease of SNR. Additionally, SNR is propositional to the \( B_0 \) field strength as well as the square root of the total acquisition time although this is not made explicit in Formula (2.9).

### 2.2 Chemical Shift-Encoded Methods and Signal Models

The chemical shift-encoded imaging for water and fat image reconstruction exploits the water-fat chemical shift by acquiring signals at slightly different echo times (TE) [3]. Advantages of chemical shift-encoded methods include the availability of both water and fat images, high SNR efficiency, compatibility with a wide spectrum of pulse sequences such as fast spin echo (FSE) [42–46], steady-state free precession (SSFP) [47, 48], and spoiled gradient echo (SPGR) [19, 49, 50]. Disadvantage of chemical shift-encoded methods, nonetheless, include relatively long scan times and requiring sophisticated reconstruction techniques to decompose the water and fat components.

In the original chemical shift-encoded imaging proposed by Dixon [18], two images were acquired and post-processing techniques are simply arithmetic operations such as summation and subtraction. After that, substantial extension based on more acquisitions and more sophisticated post-processing techniques has been made by
Glover [6], Ma [19] and other researchers. Assume that water and fat are the only two components to produce signals and effects of multi-peak fat and $T_2^*$ are ignorable. Then the signal intensity at each pixel $p$ of each acquisition can be formulated as [12]

$$s_n(p) = (W(p) + F(p)e^{2\pi \Delta f t_n}) \cdot e^{2\pi \psi(p)t_n} \cdot e^{i\phi_0(p)}$$

for $n = 1, 2, \cdots, N$ (2.10)

where $N$ is the number of acquisitions, $s_n(p)$ the signal intensity, $W(p)$ and $F(p)$ non-negative reals representing the magnetization magnitudes of water and fat, respectively, $\Delta f$ (in Hz) the water-fat chemical shift, $\psi(p)$ (in Hz) the field inhomogeneity, $\{t_n\}_{n=1}^{N}$ the TE shifts, and $\phi_0(p)$ the initial phase of water and fat signals. With respect to the standard signal model (2.10), reconstruction of water and fat images is essentially resolving the pixel-wise $W(p)$ and $F(p)$. Note that the water-fat chemical shift is known a priori and the echo time shifts $\{t_n\}_{n=1}^{N}$ are imaging parameters and given during the image acquiring. Therefore, only the four terms of $W, F, \psi$ and $\phi_0$ are unknown in the model (2.10).

### 2.2.1 Two-point Methods

In Dixon’s original two-point method [18], the $B_0$ field was supposed to be homogeneous all over the FOV, that is, $\psi(p) = 0$ at each pixel $p$. Due to this, only two images (i.e., $N = 2$) are needed to acquire to resolve $W$ and $F$: the in-phase image at the spin echo (i.e., $t_1 = 0$) and the out-of-phase one with the TE shift:

$$t_2 = -\frac{1}{2\Delta f}$$

(2.11)

The signal model (2.10) is simplified as

$$s_0(p) = (W(p) + F(p)) \cdot e^{i\phi_0(p)}$$

(2.12)

$$s_1(p) = (W(p) - F(p)) \cdot e^{i\phi_0(p)}$$

(2.13)
Thus $W$ and $F$ can be directly calculated via

$$W(p) = \frac{|s_0(p) + s_1(p)|}{2} \quad (2.14)$$

$$F(p) = \frac{|s_0(p) - s_1(p)|}{2} \quad (2.15)$$

As discussed in Section 2.1.2, MR field inhomogeneities is not ignorable in practical applications. Therefore, the twp equation (2.14) and (2.15) often result in incomplete separation of water and fat (that is, water-only and fat-only images contain admixture of both water and fat [12]). To avoid water-fat swapping errors, more advanced two-point methods [51–56] had been proposed using varieties of phase unwrapping, region growing, or allowing flexible echo times.

### 2.2.2 Three-point Methods

In three-point methods first proposed in [6, 57, 58], a third image (i.e., $N = 3$) was acquired to overcome the limitation of two-point methods. The echo time shifts $\{t_n\}_{n=1}^3$ were evenly set to

$$t_1 = -\frac{1}{2\Delta f}, t_2 = 0, t_3 = \frac{1}{2\Delta f} \quad (2.16)$$

resulting in relative water-fat phase shifts of $\{-\pi, 0, \pi\}$, or alternatively set to

$$t_1 = 0, t_2 = -\frac{1}{2\Delta f}, t_3 = \frac{1}{\Delta f} \quad (2.17)$$

resulting in relative water-fat phase shifts of $\{0, \pi, 2\pi\}$.

In the case of the phase shift scheme (2.16), the signal model (2.10) becomes

$$s_1 = (W - F) \cdot e^{i\pi\psi/\Delta f} \cdot e^{i\phi_0} \quad (2.18)$$

$$s_2 = (W + F) \cdot e^{i\phi_0} \quad (2.19)$$

$$s_3 = (W - F) \cdot e^{-i\pi\psi/\Delta f} \cdot e^{i\phi_0} \quad (2.20)$$
In the sequel, $p$ will be neglected for notational simplicity. Then the pixel-wise estimator of $\psi$ can be obtained from (2.18) and (2.20):

$$\hat{\psi} = 0.5 \cdot \arg(s_1 \cdot s_3^*) \quad (2.21)$$

where arg stands for the phase operation on complex numbers and $*$ is the complex conjugate operation. Likewise, for the phase shift scheme (2.17), the signal model (2.10) becomes

$$s_1 = (W - F) \cdot e^{i\pi \psi / \Delta f} \cdot e^{i\phi_0} \quad (2.22)$$

$$s_2 = (W + F) \cdot e^{i\phi_0} \quad (2.23)$$

$$s_3 = (W + F) \cdot e^{i2\pi \psi / \Delta f} \cdot e^{i\phi_0} \quad (2.24)$$

Accordingly, the pixel-wise estimator of $\psi$ is

$$\hat{\psi} = 0.5 \cdot \arg(s_3 \cdot s_2^*) \quad (2.25)$$

Now with the estimation of $\psi$, $W$ and $F$ can be resolved in the same manner as the two-point method.

Note that due to the multi-valuedness of the phase operator arg and the presence of the coefficient 0.5, (2.21) and (2.25) always provide two possible values for $\hat{\psi}$ within the fundamental range $(-\pi, \pi]$. In other words, both estimators are not single-valued and one of them must be correctly determined as the estimate of $\psi$ to avoid the solution interchange of water and fat. As indicated in [12], $\psi$ can be uniquely determined only in the case that

$$|\psi| < \Delta f / 2 \quad (2.26)$$
which is impractical for in vivo acquisitions. So it is crucial for three-point based methods to develop some reliable solution selecting strategies. In [6], an phase-unwrapping based method was developed and much more complicated algorithms had developed in other water-fat reconstruction approaches.

2.2.3 Advanced Signal Models

Although the standard signal model (2.10) had long since been used for the reconstruction of water and fat, more advanced signal models have been attracting more and more attentions [59–66]. Next, three types of advanced signal models are to be briefly reviewed.

2.2.3.1 Multiple-Component Model

While water and fat are the two most important components in MR, investigation on signals arising from other spectral components such as silicone also has many important applications [60, 61, 67–69]. Now assume that there are totally $M$ types of chemical components contributing to the acquired signals. Then the pixel-wise signal model becomes

$$s_n = \left( \sum_{m=1}^{M} \rho_m e^{i2\pi \Delta f_m t_n} \right) \cdot e^{i2\pi \psi t_n} \cdot e^{i\phi_0}, \text{ for } n = 1, 2, \cdots, N \quad (2.27)$$

where $\{\rho_m\}_{m=1}^{M}$ represent the real-valued magnitude of spectral components, and $\{\Delta f_m\}_{m=1}^{M}$ the chemical shift among components (in Hz). The TE shifts $\{t_n\}_{n=1}^{N}$ are normally evenly spaced [61] with $2N - 2 \geq M$. 

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Figure 2.2 Multiple peaks of fat protons. The MR spectrum is from a fatty human liver. The chemical shift of protons in CH=CH, CH$_2$-COOR, CH$_2$, and CH$_3$ are 5.3, 2.1, 1.3, 0.9 ppm, respectively. (Courtesy of Dr. Cassidy et al [1]).

2.2.3.2 Multi-peak Model

The standard signal model (2.10) assumes the single peak of fat. However, as shown in Figure 2.2, multiple lipid resonance signals with different $T_1$ and $T_2$ relaxation times exist in fat-containing tissues and give rise to separate resonance peaks in the hydrogen spectra [59, 65, 70]. To overcome the estimation bias caused by this single-peak simplification, a more sophisticated multi-peak fat model

$$s_n = (W + F \sum_{m=1}^{M} r_m e^{i2\pi \Delta f_m t_n}) e^{i2\pi \psi t_n}, \text{ for } n = 1, 2, \cdots, N$$

(2.28)

can be used [64–66, 70]. Herein, $\{\Delta f_m\}_{m=1}^{M}$ are fat peaks and $\{r_m(p)\}_{m=1}^{M}$ are normalized coefficients, that is,

$$r_1(p) + r_2(p) + \cdots + r_M(p) = 1$$

(2.29)

In this model, $M$, depending on applications, generally varies from 3 to 6, and values of the fat peak $\{r_m(p)\}_{m=1}^{M}$ can be assumed known a priori [59].
2.2.3.3 $T_2^*$-decay Model

By assuming that there is at most mild signal loss resulted from $T_2^*$ decay, the $T_2^*$ decay is not considered in the standard model (2.10). However, for applications such as hepatic iron overload, significant signal dephasing and $T_2^*$ decay may occur and compromise the reconstruction of fat [59, 62, 63, 71–74] and $T_2^*$ decay must be considered.

Let $R^* = 1/T_2^*$ denote the pixel-wise decay rate. Then the pixel-wise signal model accounting for $T_2^*$ decay becomes

$$s_n = (W + Fe^{i2\pi \Delta f t_n}) \cdot e^{i2\pi \psi t_n} \cdot e^{-R_2^n}, \text{ for } n = 1, 2, \ldots, N$$  \hspace{1cm} (2.30)

Note that in this model, water and fat components are supposed to have the same $T_2^*$ decay rate at each pixel.

2.3 Existing Three-point Reconstruction Approaches

Existing water-fat reconstruction approaches are mostly based on the three-point signal model. Instead of the signal model (2.10), a more flexible formulation introduced in [7]

$$s_n(p) = (W + Fe^{i2\pi \Delta f t_n})e^{i2\pi \psi t_n}, \text{ for } n = 1, 2, 3$$  \hspace{1cm} (2.31)

will be used. Compared to (2.10), $W$ and $F$ are supposed to be complex valued to cancel the error phase term $e^{i\phi_0}$. Observe the matrix form

$$\begin{bmatrix}
s_1 \\
s_2 \\
s_3 \\
s\end{bmatrix} = \begin{bmatrix}
e^{i2\pi \psi t_1} & e^{i2\pi (\Delta f + \psi) t_1} \\
e^{i2\pi \psi t_2} & e^{i2\pi (\Delta f + \psi) t_2} \\
e^{i2\pi \psi t_3} & e^{i2\pi (\Delta f + \psi) t_3} \\
\end{bmatrix} \cdot \begin{bmatrix}
W \\
F \\
\end{bmatrix}$$  \hspace{1cm} (2.32)
and it is apparent that provided $\psi$, $W$ and $F$ will be available according to linear least-square estimation

$$\rho = (\Psi^T \Psi)^{-1} \Psi^T \cdot S = \Psi^+ \cdot S \quad (2.33)$$

Therefore, reconstruction of water and fat is reducible to the recovery of field map $\psi$. As a matter of fact, all approaches to be reviewed, except the Directed Phase Encoding, are primarily focused on the recovery of field map.

### 2.3.1 Directed Phase Encoding

The Directed Phase Encoding (DPE) approach was proposed by Xiang and An [75]. In this approach, the three images were acquired with uniformly spaced TE shifts (i.e., $t_2 = t_1 + \Delta t$ and $t_3 = t_1 + 2\Delta t$ for some $\Delta t > 0$). With the notations

$$C = e^{i2\pi \Delta f t_1}, \quad A = e^{i2\pi \Delta f \Delta t}, \quad P_0 = e^{i2\pi \psi t_1}, \quad P_1 = e^{i2\pi \psi \Delta t}$$

$$X = W \cdot P_0 \cdot P_1, \quad Y = C \cdot A \cdot F \cdot P_0 \cdot P_1$$

We have

$$s_2 = X + Y \quad (2.36)$$

$$s_1 s_3 = X^2 + Y^2 + XY(A + \frac{1}{A})$$

Then two groups of solutions for the vector $(X, Y)$ can be obtained from (2.36) and (2.37):

$$\left(\frac{s_2 + \Delta I}{2}, \frac{s_2 - \Delta I}{2}\right), \text{ and } \left(\frac{s_2 - \Delta I}{2}, \frac{s_2 + \Delta I}{2}\right) \quad (2.38)$$

With one of the solutions selected as the estimate of $X$ and $Y$, water and fat image can be reconstructed via

$$|W| = |X|, \quad |F| = |Y| \quad (2.39)$$

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However, this ambiguity resulted from multiple solutions at each pixel raises essentially the same ambiguity as that encountered by the original three-point method in [6]. A technique of “orientation filters” was therefore developed to overcome this ambiguity.

2.3.2 IDEAL and IDEAL-RG

The IDEAL (iterative decomposition of water and fat with echo asymmetry and least squares estimation) approach [7], as hinted by its name, iteratively uses least-square estimators to independently estimate the field inhomogeneity at each pixel. To eliminate estimation bias introduced by noise, field inhomogeneity estimates are smoothed by low-pass and other filtering techniques. Zero-initialization has been recommended.

For its advantages over the original three-point method, DPE and other older approaches, IDEAL has become the most prevalent approach to reconstructing water and fat images [66, 76–83]. One advantage is that IDEAL allows flexible TE shifts. Further, with minimal modification, it can be applied to address the multi-component model (2.27).

One major challenge against IDEAL is on its initialization. The suggested zero-initialization works well only in the presence of low to moderate field inhomogeneities so that the local minima nearest to zero happen to be correct estimates that IDEAL converges to. For acquisitions with large field inhomogeneities, IDEAL turns out to be unreliable. To overcome the trouble arising from the zero-initialization, an improvement based on region-growing algorithm IDEAL-RG was proposed [8]. The IDEAL-RG approach is essentially composed of two phases [4, 8]: 1) the selection of
the seed pixel; 2) square-spiral region growing with 2D linear extrapolation. In more detail,

- Selection of seed pixel. All source images are smoothes and down-sampled to low-resolution (e.g., $32 \times 32$). IDEAL estimating is then performed on the low-resolution dataset. A binary mask image is generated from the low-resolution source images and applied to the low-resolution field map estimates to remove noisy background pixels. From the remaining pixels, the so-called median “super-pixel neighborhood” consisting of the super-pixel with the median field map value and the 14 super-pixels with field map values closest to the median are chosen. Then among all super-pixels in the median super-pixel neighborhood, the one that is spatially closest to the center-of-mass super-pixels is picked as the seed of region growing procedure.

- Square-spiral region growing. Pixels in the source images corresponding to the seed super-pixel are estimated with IDEAL initialized with the estimated value of the seed super-pixel. Next, square-spiral region-growing is performed, where the estimate of each new encountered pixel is computed by 2D linear extrapolation of field map values of neighboring visited pixels.

2.3.3 Multi-resolution Estimation using Golden Section Search (MEGSS)

The multi-resolution estimation using Golden section search (MEGSS) proposed by Lu and Hargreaves [9] combines golden section search for local minima and region growing to estimate the field map. With the assumption that the echo time shifts are uniformly spaced (i.e., $T = t_2 - t_1 = t_3 - t_2$), the full-resolution source data is first pyramidally down-sampled at rate of 0.5 to produce a source data stack. Throughout
all layers of the stack and at each pixel, the three local minima of the residual function

\[ R(\psi) = ||S - \Psi(\Psi'\Psi)^{-1}\Psi'S||^2 \]  

(2.40)
in the range \([\frac{1}{2^T}, \frac{1}{2^T}]\) are located using the golden section search method [84]. Region growing is used only on the coarsest layer. For two consecutive layers, estimation on the coarser layer is expanded to initialize the finer layer using bilinear interpolation [85]. The fitting with these initial values is evaluated at each pixel with the residual \( R(\psi) \). Initial values with poor fitting evaluation are further modified using golden section search.

### 2.3.4 VARPRO-ICM

Based on Markov random field (MRF) theory, VARPRO-ICM [10] casts the estimation of field map to minimization of a penalized maximum likelihood (PML) function

\[ R(\psi) = \sum_{p \in \mathcal{P}} ||S - \Psi(\psi_p)\Psi(\psi_p)^+S||^2 + \mu \sum_{p \in \mathcal{P}} \sum_{q \in \mathcal{N}_p} \omega_{pq} |\psi_p - \psi_q|^2 \]  

(2.41)

where, \( || \cdot || \) is the 2-norm, \( \mathcal{P} \) represents the source image grid, \( \mathcal{N}_p \) the neighborhood of \( p \), \( \omega_{pq} \) the weight between \( p \) and \( q \), \( \mu \) is the regularization factor, and \( \psi : \mathcal{P} \rightarrow \mathcal{I} = [I_{\text{min}}, I_{\text{max}}] \) (\( I_{\text{min}} \) and \( I_{\text{max}} \) are the lower and upper bounds of the field inhomogeneity). In the formula (2.41), the term \( ||S - \Psi(\psi_p)\Psi(\psi_p)^+S||^2 \) measures the fitting extent with the estimate \( \psi_p \) and is named as the data cost of \( p \). The second term \( \sum_{q \in \mathcal{N}_p} \omega_{pq} |\psi_p - \psi_q|^2 \) measures the local field map smoothness around \( p \) with the estimate \( \psi_p \). Therefore, the PML function \( R(\psi) \) accounts for both fitting error and local smoothness of field map.
With the formula (2.41), the field map can be estimated by

$$\hat{\psi} = \arg\min_{\psi} R(\psi)$$ (2.42)

However, computation of (2.42) is intractable and efficient algorithms must be developed. VARPRO-ICM introduced the basic iterated conditional modes (ICM) algorithm [24] to compute the local energy function at each pixel. That is, at each iteration $k$,

$$\psi_p^{(k)} = \arg\min_{\psi_p} |S - \Psi(\psi_p)\Psi(\psi_p)^+S|^2 + \mu \sum_{q \in N_p} \omega_{pq} |\psi_p - \psi_q^{(k-1)}|^2$$ (2.43)

The iterative procedure terminates when the overall field map change falls below a predefined threshold $\varepsilon$:

$$\sum_{p \in P} |\psi_p^{(k)} - \psi_p^{(k-1)}| < \varepsilon$$ (2.44)

VARPRO-ICM can be easily extended to handle multi-coil MR acquisitions. Meanwhile, it owns high noise performance. However, VARPRO-ICM is much more time costly than other competing approaches. Furthermore, like IDEAL, the zero-initialization suggested in [10] is error-prone and often leads to severely erroneous estimates.

### 2.3.5 Algebraic Decomposition

The Algebraic Decomposition (AD) approach proposed by Jacob and Sutton [2] provides efficient field map recovery. By assuming the uniform spacedness of the echo time shift, the signal model 2.31 is transformed to two quadratic equations of

$$\gamma = e^{i2\pi\psi T}$$

$$s_3 - (1 + \lambda)s_2\gamma + \lambda s_1\gamma^2 = 0$$ (2.45)

$$s_1^* - (1 + \lambda)s_2\gamma + \lambda s_3\gamma^2 = 0$$ (2.46)
at each pixel, where $T = t_2 - t_1$, $\lambda = e^{i2\pi \Delta f T}$ and $*$ denotes the complex conjugate operation. Feasible solutions of equations (2.45) are termed as feasible solutions. Instead of directly working on (2.45), Sylvester matrix theory [86–89] is applied to denoise (2.45) and calculate the feasible solutions. Subsequently, region merging algorithm was proposed to pick one of the feasible solutions as the final solution at each pixel.

2.3.6 VARPRO-GraphCut

As described above, one challenge encountered by VARPRO-ICM is its sensitivity to initialization. To overcome this, Hernando et al [11] introduced a much more complicated VARPRO-GraphCut approach. This approach used the same PML residual function (2.41) as VARPRO-ICM to cast the estimation of field map to PML optimization. A graph cut-based algorithm was then applied to compute the optimization of (2.41). VARPRO-GraphCut has proven to be robust to recovery of field map with rapid and large variation [11]. However, it is disadvantageous of being time and space costly.

2.4 Markov Random Field and Energy Minimization

Markov random field (MRF) theory and techniques have been widely applied in computer vision, medical imaging and other fields for its capability of formulating spatial or contextual dependencies [23, 90, 91]. In this subsection, we briefly go over a few fundamental notions in MRF theory. More specific introduction can be referred to in [23, 90].
2.4.1 Markov Random Field Energy Model

A field $\mathcal{P}$ is a grid of one or higher dimensions. Each grid point $p$ in $\mathcal{P}$ is called a site. Sites in $\mathcal{P}$ are correlated via a neighborhood system

$$\mathcal{N} = \{N_p : p \in \mathcal{P}\}$$

where each $N_p$ is a site subset of $\mathcal{P}$ and is called the neighborhood of $p$. The neighborhood system is subject to:

i) For any site $p$, $p \notin N_p$;

ii) Symmetry: $p \in N_q \iff q \in N_p$ for any two sites $p$ and $q$.

The configuration for the neighborhood system is application specific. For example, the 4 or 8-neighborhood system (see Figure 2.3) is popular in image segmentation and restoration, while larger size of neighborhood systems may be preferred in other subsequent discussion.

Figure 2.3 Neighborhood System in Markov random field. (a) 4-neighborhood. (b) 8-neighborhood.
Each site in the field is associated with a quantity in form of scalars or vectors. The quantity at each site is called a label. We use $I$ to denote the possible label values. Any assignment $f : \mathcal{P} \rightarrow I$ of labels on all sites is called a labeling (or configuration). The label value $f(p)$ on a site $p$ is simplified as $f_p$. Let $\mathcal{F}$ denote the set of all possible labelings of $\mathcal{P}$ on the label set $\mathcal{L}$. Then the field $\mathcal{P}$ with the neighborhood system $\mathcal{N}$ is called a Markov random field (MRF) on the label sets $\mathcal{L}$ if there is a probability distribution $P(\cdot)$ on the labelings $\mathcal{F}$ such that for any labeling $f \in \mathcal{F}$ and site $p \in \mathcal{P}$,

i) Positivity: $P(f) > 0$;

ii) Markovianity: $P(f_p | f_{\mathcal{P} - \{p\}}) = P(f_p | f_{\mathcal{N}_p})$.

The central problem for an MRF energy model is the labeling problem that asks for the optimal labeling whenever an instance (or observation) of the field is provided. Within Bayesian maximum a posteriori (MAP) estimation framework, an energy function for the field with the typical form of

$$E(f) = \sum_{p \in \mathcal{P}} D_p(f_p) + \sum_{p \in \mathcal{P}} \sum_{q \in \mathcal{N}_p} \mu_{p,q} \cdot V_{pq}(f_p, f_q)$$

(2.48)

can be drawn out. Then a labeling $\hat{f}$ is optimal in the sense that it minimizes the energy function, that is, $\hat{f}$ satisfies

$$\hat{f} = \arg \min_f E(f)$$

(2.49)

In a MAP-MRF energy function like (2.48), the data cost term $E_{data}(f)$ measures the disparity between the labeling $f$ and the observed realization of the field. On the other hand, The smoothness cost term $E_{smooth}(f)$ measures the extent to which $f$
is not piecewise smooth. The weight parameter \( \mu_{p,q} \) is designed to keep balance between disparity and smoothness measurements. Inappropriate values of \( \mu_{p,q} \) result in oversmoothing or undersmoothing labeling. In subsequent discussion, we call \( \mu_{p,q} \) the balance factor.

In practice, it is usually difficult to learn the exact probability distribution of an MRF. Thereby, data and smoothness terms within the energy function formulation are not directly available. A plausible and widely adopted approach is to choose data cost and smoothness cost terms based on heuristics from prior knowledge. For instance, the Potts model

\[
E(f) = \sum_{p \in P} D_p(f_p) + \sum_{p \in P} \sum_{q \in N_p} \mu_{p,q} \cdot \delta_{pq}(f_p \neq f_q) \tag{2.50}
\]

is popular in process of image segmentation and stereo \([92, 93]\), while such an energy function as

\[
E(f) = \sum_{p \in P} ||f_p - f^0_p|| + \sum_{p \in P} \sum_{q \in N_p} \mu_{p,q} \cdot V_{pq}|f_p - f_q| \tag{2.51}
\]

is desirable for image restoration \([94]\).

### 2.4.2 Energy Optimization and Iterated Conditional Modes Algorithm

Exact minimization on MRF energy models is challenging because of its ill-posedness, ill-conditionedness and intractability in general \([22, 23]\). Instead, energy optimization algorithms such as simulated annealing \([95]\), graph cut \([96–98]\), iterated conditional modes (ICM) \([24]\), and loopy belief propagation (LBP) \([99–101]\), have been widely used for approximate estimation. A comprehensive comparison of these algorithms on image processing and computer vision problems can be referred to in
Initialize $\psi^{(0)}$

at iteration $k$

Compute the energy minimizer $\psi_p^{\text{new}}$

Update $\psi_p^{(k)} := \psi_p^{\text{new}}$

Goto next pixel: $p := \text{next}(p)$

Figure 2.4 Conventional Iterated Conditional Modes (ICM) Algorithm. In ICM, the time-costly minimizing computations are always run in the same way at each iteration and each pixel. Initialization of $\psi^{(0)}$ determines the convergence of ICM.

[102]. Herein, we will briefly review the ICM algorithm which our water-fat reconstruction is primarily based on.

The conventional ICM is by first Besag [24] and has been widely used in varieties of image reconstruction problems [10, 103–108]. As illustrated by figure 2.4, ICM is a greedy algorithm and always calculate the local MRF energy functions in the same way at each iteration and each pixel. With respect to the energy function (2.48), the pixel-wise minimizer $\hat{f}_p$ for ICM is

$$\hat{f}_p = \arg \min_{f_p \in \mathcal{L}} E_p(f) = \arg \min_{f_p \in \mathcal{L}} \left[ D_p(f_p) + \sum_{q \in \mathcal{N}_p} \mu_{pq} \cdot V_{pq}(f_p, f_q) \right]$$

(2.52)

where $p \in \mathcal{P}$ and $\mathcal{L}$ is the label set. After being initialized with some $\psi^{(0)}$, ICM
Figure 2.5 *Prior and posterior neighbors in ICM*. Highlighted is the $5 \times 5$ neighborhood of the pixel $p$. – represents prior neighbor, and + represents posterior neighbor.

iteratively calculate the minimizer at iteration $k (\geq 1)$ in the manner that

$$ f_p^{(k)} = \arg \min_{f_p \in \mathcal{I}} \left[ D_p(f_p) + \sum_{q \in \mathcal{N}_p^-} \mu_{pq} \cdot V_{pq}(f_p, f_q^{(k-1)}) \right] $$

(2.53)

For convenience, we call those pixels which are the neighbors of $p$ and computed before $p$ the prior neighbors of $p$, and those neighboring to but computed after $p$ the posterior neighbors, as illustrated in Figure 4.3. Later, $\mathcal{N}_p^-$ and $\mathcal{N}_p^+$ are used to separately denote the set of prior and posterior pixels. Then a popular alternative of (2.53), called asynchronous ICM, is

$$ f_p^{(k)} = \arg \min_{f_p \in \mathcal{I}} \left[ D_p(f_p) + \sum_{q \in \mathcal{N}_p^-} \mu_{pq} \cdot V_{pq}(f_p, f_q^{(k)}) + \sum_{q \in \mathcal{N}_p^+} \mu_{pq} \cdot V_{pq}(f_p, f_q^{(k-1)}) \right] $$

(2.54)

Correspondingly, the formula (2.53) is called synchronous.

Note that the data cost term $D_p(\psi_p)$ in (2.53) and (2.54) can be pre-computed before iterations start. With this, the computational time cost of ICM at each pixel $p \in \mathcal{P}$ and each iteration $k$ is $\Theta(|\mathcal{L}| \times |\mathcal{N}_p|)$. Therefore, the overall time cost of ICM
on $n$ iterations is

$$O(|I| \times |N_p| \times |P| \times n)$$ (2.55)

where all pixels are supposed to have the same size of neighborhood. As we will see in subsequent discussion, this time cost is intolerably expensive in the case of large label set, neighborhood or slow convergence.

Another major disadvantage of ICM is on its high sensitivity to the initial guess $\psi^{(0)}$ [102]. Inappropriate initialization is prone to result in severely wrong estimation. Although a few initialization schemes, such as Winner-Take-All [24, 102], All-Zero [10] and Combined-Local-Minima [109], have been proposed, but how to properly initialize ICM remains to be challenging.
Chapter 3

NLR-MRF Energy Model

As described in previous chapter, reconstruction of water and fat images in magnetic resonance imaging (MRI) boils down to the recovery of field map. A crucial feature on MR field map is of its local smoothness (see Figure 3.1(a)), that is, field inhomogeneity at one pixel are highly related to its neighboring pixels. This feature of local smoothness enable the use of smoothing techniques such as phase unwrapping [6, 51], region growing [8, 9, 46, 53, 110], region merging [2], and low-pass filtering [7]. One commonality of these estimating techniques is that pixel-wise field inhomogeneity calculation is separated from correction. Recently, Markov random field (MRF) in which the two estimating criteria of least residual and local smoothness are combined, has become popular to model the local smoothness of field map and used by VARPRO-ICM [10], Graph-Cut [11], and JIGSAW [111]. MRF-based approaches are more likely to provide correct estimates for field map with large and rapid variation despite they might be more time costly [11].

Another important feature related to the recovery of field map is the relatively high contrastedness between object and noise (air) regions (see Figure 3.1(b)) in MR acquisitions [5, 25]. This feature greatly facilitates accurately masking background off from the whole image which turns to be challenging in general imaging processing applications [97, 112]. The background masking in MR acquisitions have been implicitly or explicitly exploited in approaches such as [8, 9] to exclude estimation bias.
In this chapter, a novel MRF energy model, termed as NLR-MRF (abbreviated for \(l\)-normed non-linear least residual MRF) is developed. Compared to energy model used by VARPRO-ICM, Graph-Cut and JIGSAW, NLR-MRF is integrated with background masking so that accuracy and especially efficiency can be substantially improved. Moreover, compared to general MRF formulation used in image processing, NLR-MRF has a much more complicated data cost term so that it is even harder for MRF optimization algorithms to yield the correct estimates.

3.1 The Non-linear Least Residual MRF Model

3.1.1 The Model Formulation

Let \(\mathcal{P}\) denote the image grid of a field map, \(\mathcal{F}\) the foreground object region and \(\mathcal{B}\) the background noise region (see Figure 3.3). Mathematically,

\[
\mathcal{P} = \mathcal{F} \cup \mathcal{B}, \quad \text{and} \quad \emptyset = \mathcal{F} \cap \mathcal{B}
\]  

(3.1)
Let $\psi_{\text{max}}$ and $\psi_{\text{min}}$ be some upper and lower bound of the field inhomogeneities on $\mathcal{P}$, separately, and $\mathbf{I} = [\psi_{\text{min}}, \psi_{\text{max}}]$ the range of the whole field map. Then the $l$-normed non-linear least residual MRF (NLR-MRF) energy functional is defined as:

$$E(\psi) = \sum_{p \in \mathcal{F}} D_p^l(\psi_p) + \sum_{p \in \mathcal{F}} \mu_p \sum_{q \in \mathcal{N}_p \cap \mathcal{F}} \omega_{pq} |\psi_p - \psi_q|^l$$

(3.2)

where, $\psi : \mathcal{P} \rightarrow \mathbf{I}$ is a field map configuration, $l \geq 1$ is an integer regulating the order of norm, $\mathcal{N}_p$ the neighborhood of pixel $p$, $W_p = (\omega_{pq})_{q \in \mathcal{N}_p}$ the weight matrix, $\mu_p$ the regularization factor, and

$$D_p^l(\psi_p) = ||S - \Psi(\psi_p)\Psi(\psi_p)^+ S||^l$$

(3.3)

is the $l$-normed non-linear least residual at $p$ and measures. Recall that

$$\Psi = \begin{pmatrix}
e^{i2\pi \psi t_1} & e^{i2\pi (\Delta f + \psi)t_1} \\
e^{i2\pi \psi f t_2} & e^{i2\pi (\Delta f + \psi)t_2} \\
e^{i2\pi \psi f t_3} & e^{i2\pi (\Delta f + \psi)t_3}
\end{pmatrix}$$

(3.4)

is the coefficient matrix of the three-point signal model (2.32). As with general MRF energy models, the data cost $E_{\text{data}}$ in (3.2) measures the overall fitting of the configuration $\psi$ with the acquired data, the smoothness cost $E_{\text{smooth}}$ measures the overall smoothness of $p$, and the estimate $\hat{\psi}$ for the field map $\psi$ corresponds to the minimal solution of the energy functional (3.2), that is,

$$\hat{\psi} = \arg \min_{\psi} E(\psi) = \arg \min_{\psi} \left[ \sum_{p \in \mathcal{F}} D_p^l(\psi_p) + \sum_{p \in \mathcal{F}} \mu_p \sum_{q \in \mathcal{N}_p \cap \mathcal{F}} \omega_{pq} |\psi_p - \psi_q|^l \right]$$

(3.5)

### 3.1.2 NLR-MRF versus PML Model

Compared to the penalized maximum likelihood (PML) models

$$E(\psi) = \sum_{p \in \mathcal{P}} ||S - \Psi(\psi_p)\Psi(\psi_p)^+ S||^l + \sum_{p \in \mathcal{P}} \mu_p \sum_{q \in \mathcal{N}_p} \omega_{pq} |\psi_p - \psi_q|^l$$

(3.6)
used by approaches such as VARPRO-ICM [10], Graph-Cut [11], and JIGSAW [111]. The NLR-MRF model is characteristic of its background masking. Exclusion of air regions can improve both accuracy and efficiency of subsequent estimation. Note that the signals of the air regions are generated completely by noise and the signal model (2.32) is only fit for foreground pixels. Therefore, using (2.32) to recover field map on air regions are not reliable and of little use in light of reconstructing water and fat images. As we can see in next chapter, noise regions may severely bias the estimation of unambiguous field map. Moreover, by excluding air regions which generally occupy a big chunk of area in the field of view (FOV), much less number of pixels is to be computed and both computation time and space can be saved. As a matter of fact, according to our test on mouse datasets, background masking can averagely save half of the total time cost.

Meanwhile, the NLR-MRF energy model allows flexible option of norm order $l$ while the PML model (3.6) always assumes 2-norm. As commented in [22], with $l < 2$, the resulting smoothness term stresses more on piecewise instead of total smoothness and can better model the discontinuous nature of images. Throughout all of our experiments, $l$ has always been set as 1.

3.1.3 Data Cost: Periodic and Non-convex

Another characteristics of the NLR-MRF energy model is on the data cost term’s non-convexity and periodicity in case of uniform spaced echo time shifts.

To simplify the signal model and maximize noise performance, uniform spaced echo time shifts have been generally adopted in most of Dixon signal models [113]. Based on the assumption of symmetry of echo time shifts, it has been shown in [9]
Figure 3.2 *Periodicity and non-convexity of pixel-wise data cost function.* One typical plot of $D^1_p(\psi_p)$ which is periodic and non-convex. The fundamental period is coincident to $1/T = 3153$ with $T = 3.17e^{-4}$. There exists two local minima (highlighted by arrows) within one period.

that the pixel-wise data cost function $D^1_p(\psi_p)$ is periodic with fundamental period of $1/T$, where $T = t_2 - t_1 (= t_3 - t_2)$. In fact, note that the coefficient matrix $\Psi$ can be rewritten as

$$
\Psi = \begin{pmatrix}
e^{i2\pi\psi t_1} & 0 & 0 \\
0 & e^{i2\pi\psi t_2} & 0 \\
0 & 0 & e^{i2\pi\psi t_3}
\end{pmatrix}
\Phi(\psi_p)
\times
\begin{pmatrix}
e^{i2\pi\Delta f t_1} \\
1 \\
e^{i2\pi\Delta f t_2} \\
1 \\
e^{i2\pi\Delta f t_3} \\
1
\end{pmatrix}
\Lambda (3.7)
$$

Then $\Psi^+(\psi_p) = \Lambda^+ \cdot \Phi(-\psi_p)$ and thus

$$
D^1_p(\psi_p) = ||\Psi(\psi_p) \cdot \Psi^+(\psi_p) \cdot S - S||
= ||(\Lambda\Lambda^+ - I) \cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & e^{i2\pi\psi T} & 0 \\
0 & 0 & e^{i2\pi\psi 2T}
\end{pmatrix} \cdot S||
(3.8)
$$

is periodic of $1/T$.

Meanwhile, the pixel-wise data cost function $D^1_p(\psi_p)$ may not be convex and have more than one (however, no more than according to our observation) local minimum
within a single period (see Figure 3.2). Note that global minimum in one period is not necessarily the optimal estimate \cite{9, 102}. Consequently, when gradient-based algorithms such as graph cut and ICM are applied to compute the minimization of the NLR-MRF model, the initialization becomes critical in that the algorithms will generally converge to the local minimum closest to the initial guess. We will see in the discussion of next chapter how this nonconvexity feature complicate the initialization of the ICM computation.

### 3.2 Background Masking Algorithms

Background masking algorithms for the NLR-MRF energy model must be highly accurate so that no region of interest (ROI) is excluded. Next, we will introduce two automated masking algorithms, one is based on intensity histogram and morphological operations while the other one is adapted from a well-known foreground extraction utility developed for color images.

#### 3.2.1 Histogram-based Masking Algorithm

Histogram-based background masking techniques have been used by IDEAL-RG \cite{8}, MEGSS \cite{9} and some other approaches to remove air regions in MR acquisitions. Our masking method is composed of a series of morphological operations meant to minimize true-negative masking error. First, pick one of the acquisitions as the template image. Then dilate the template image. Next, based on the histogram of the dilated image, a threshold is generated \cite{114} with which a binary image is yielded. Then holes in the binary image are filled and erosion is performed on the hole-filled image. Finally, perform closing operation on the eroded image and output the binary
Figure 3.3 *Histogram-based background masking.* (a) Template image. (b) Dilation. (c) Binary image with a threshold generated from intensity histogram. (d) Fill hole. (e) Binary mask with erosion.

The histogram-based masking algorithm has desirable performance on MR acquisitions of high signal-to-noise ratio (SNR). However, low SNR may compromise its accuracy by classifying air pixels into foreground, as shown in Figure 3.4. Next, we introduce the adapted GrabCut masking which is capable of providing accurate mask for acquisitions with different levels of SNR.

### 3.2.2 GrabCut-based Masking Algorithm

The original GrabCut developed by Rother, Kolmogorov, and Blake [112] is a robust interactive foreground extraction method on general color image foreground extraction. For the three images of a three-point MR dataset, we can take each of them as an RGB channel of color images. Meanwhile, the whole image is directly set as the initial foreground which is to be refined with subsequent GrabCut computations to save the human interaction in the original GrabCut. Such an initialization for
GrabCut is feasible due to the relatively higher contrastedness of MR acquisitions to general images.

Due to the use of more advanced graph cut and other algorithms [112] than the simple histogram-based thresholding, the GrabCut-based method owns desirable accuracy. Therefore, although the histogram-based method is much more efficient, the GrabCut-based method appears to be preferable.

### 3.3 NLR-MRF on Advanced Signal Models

The NLR-MRF energy model (3.2) is based on the standard three-point signal model (2.31). This formulation, however, can be easily extended to formulate field inhomogeneities in advanced multi-peak fat and $T_2^*$ decay signal models.
Consider the multi-peak fat model ([11, 49]):

\[ s_n(p) = (W(p) + F(p) \sum_{m=1}^{M} r_m(p) e^{i2\pi \Delta f_m t_n}) e^{i2\pi \psi(p)t_n} \]  (3.9)

where, \( M \) is the number of fat peaks, \( n = 1, 2, \cdots, N \), \( r_1(p) + r_2(p) + \cdots + r_M(p) = 1 \) for each individual pixel \( p \), and all frequency shifts \( \{\Delta f_m\}^M_{m=1} \) of fat peaks are known. Additionally, the relative fractions \( \{r_m\}^M_{m=1} \) can be measured from separate calibration [49] so that we can assume that they are also known. Under these assumptions, the multi-peak model (3.9) contains the same three types of unknown parameters as the standard model (1.1) and thus \( N = 3 \) acquisitions are sufficient to perform the water-fat image reconstruction. By replacing \( \Psi \) in data cost (3.3) with the new coefficient matrix

\[
\tilde{\Psi} = \begin{pmatrix}
    e^{i2\pi \psi t_1} & e^{i2\pi \psi t_1} & \sum_{m=1}^{M} r_m e^{i2\pi \Delta f_m t_1} \\
    e^{i2\pi \psi t_2} & e^{i2\pi \psi t_2} & \sum_{m=1}^{M} r_m e^{i2\pi \Delta f_m t_2} \\
    e^{i2\pi \psi t_3} & e^{i2\pi \psi t_3} & \sum_{m=1}^{M} r_m e^{i2\pi \Delta f_m t_3}
\end{pmatrix} \tag{3.10}
\]

the NLR-MRF energy model is still available to cast the water-fat image reconstruction to MRF energy optimization.

Now consider the \( T^*_2 \) decay signal model

\[ s_n(p) = (W(p) + F(p)e^{i2\pi \Delta f t_n}) e^{i2\pi \psi(p)t_n} e^{-R^*_2(p)t_n} \]  (3.11)

where, \( R^*_2 = 1/T^*_2 \) is the decay rate and \( n = 1, 2, \cdots, N \). Again, by substituting \( \Psi \) in (3.3) with the \( T^*_2 \) decay coefficient matrix

\[
\tilde{\Psi} = \begin{pmatrix}
    e^{i2\pi (\psi+iR^*_2)t_1} & e^{i2\pi (\Delta f+\psi+iR^*_2)t_1} \\
    e^{i2\pi (\psi+iR^*_2)t_2} & e^{i2\pi (\Delta f+\psi+iR^*_2)t_2} \\
    \cdots & \cdots \\
    e^{i2\pi (\psi+iR^*_2)t_N} & e^{i2\pi (\Delta f+\psi+iR^*_2)t_N}
\end{pmatrix} \tag{3.12}
\]

we still have the NRL-MRF energy model. Although \( N \) is larger than 3 and the unknown \( \psi \) and \( R^*_2 \) are both contained in the matrix, estimation of \( \psi \) and \( R^*_2 \) can be decoupled because of their uncorrelatedness in the Cramer-Rao bound analysis [115].
3.4 Discussion

The exact recovery of field inhomogeneities on the standard three-point and advanced signal models is essentially an ill-posed problem. In this chapter, we proposed the NLR-MRF energy model to cast the estimation of field inhomogeneities to MRF energy optimization. One advantage of MRF-based reconstructing approaches over others like IDEAL and MEGGS is that MRF energy model elegantly joins both characteristics of local minimization (via data cost terms) and local smoothness (via smoothness cost terms) of the optimal estimates within one formulation. Meanwhile, estimates yielded by MRF optimization techniques, such as Graph Cut, ICM, Simulated Annealing, or Loopy Belief Propagation, can provide relatively accurate estimates to the optimal solutions.

One disadvantage of the NLR-MRF energy model is concerned with the configuration of such parameters as the neighborhood system, regularization factor, and discretization of the effective field inhomogeneity range. As a matter of fact, configuration of these parameters are the optimization algorithm and data specific. For example, the Graph Cut-based approach has a very different configuration from the VARPO-ICM approach. Therefore, an important task of developing a new optimization approach for the NLR-MRF energy model is to validate the effectiveness of the parameter setup.

Meanwhile, the NLR-MRF model may be a crude model for the recovery of MR field inhomogeneities. The data cost and smoothness cost terms may not be adequate to reflect importance features of field inhomogeneities like variation speed. In the case of rapid variation field inhomogeneities, lack of characterizing variation speed may result in extreme difficulty of closely approximate the optimal solutions.
Chapter 4

Energy Optimization with ICM-ST

With the NLR-MRF energy model proposed in Chapter 3, the recovery of field map is cast to MRF energy minimization. As we have seen, global exact optimal estimation on the NLR-MRF energy model is ill-posed and intractable and in practice energy optimization algorithms such as ICM [24], graph cut-based [96–98], and LBP-based [99–101], are applied to provide approximate estimates. Subsequently, we will adopt the ICM algorithm to handle optimization of the NLR-MRF energy model for its demonstrated effectiveness in medical and generic image processing [10, 103–108].

While it has been generally considered as efficient when compared to other optimization approaches such as graph cut-based or LBP-based [102], the ICM algorithm is still time costly in the case of large label sets, neighborhood system or large number of iterations. Meanwhile, ICM is also constrained by its sensitivity to initialization: inappropriate initial guesses leads to total wrong estimation.

In this chapter, we introduce an improved ICM method, termed ICM-ST, to provide efficient and accurate computation on the NLR-MRF energy model. Compared to the conventional ICM algorithm, ICM-ST is substantially more efficient by keeping track of estimate stability across iterations. Meanwhile, ICM-ST always produces the same estimates as the conventional ICM at each iteration. The accuracy advantage of ICM-ST over conventional ICM is resulted from the use of a new initialization
method which takes advantage of the median of the unambiguous part of the whole MR field map.

4.1 Stability Tracking on ICM

The basic idea of stability tracking for the ICM computation is to always keep track of variation of label value at each pixel so that the time costly synchronous minimization

$$\psi_p^{(k)} = \arg \min_{\psi_p \in I} \left[ D_p(\psi_p) + \sum_{q \in N_p} \mu_{pq} \cdot V_{pq}(\psi_p, \psi_q^{(k-1)}) \right]$$ (4.1)

or asynchronous minimization

$$\psi_p^{(k)} = \arg \min_{\psi_p \in I} \left[ D_p(\psi_p) + \sum_{q \in N_p^e} \mu_{pq} \cdot V_{pq}(\psi_p, \psi_q^{(k)}) + \sum_{q \in N_p^o} \mu_{pq} \cdot V_{pq}(\psi_p, \psi_q^{(k-1)}) \right]$$ (4.2)

at a pixel $p$ can be saved whenever label values of $p$’s prior neighbors at last and current iteration and $p$’s posterior neighbors at current iteration have not been changed. Next, we will focus on stability tracking on synchronous (4.1) since the discussion on (4.2) is completely the same.

4.1.1 Stability of Labeling

Let $f_p^{(k)}$ denote the estimate produced with (4.1) at the pixel $p$ and iteration $k$ (in case that $k = 0$, $f_p^{(0)}$ represents the initial guess). Then $p$ is called stable at iteration $k$ if the estimates at $p$ are the very same for the consecutive iteration $k$ and $k - 1$. Formally,

**Definition 4.1.1.** A pixel $p$ is stable at iteration $k$ ($k \geq 1$), if $\psi_p^{(k)} = \psi_p^{(k-1)}$. 
It can be readily observed that

**Proposition 4.1.1.** A pixel $p$ is stable at iteration $k$ if all its neighbors are stable at iteration $k - 1$.

**Proof.** According to the formula (4.1), at iteration $k - 1$ and $k$, we have

$$
\psi_p^{(k-1)} = \arg \min_{\psi_p} \left[ D_p(\psi_p) + \sum_{q \in N_p} \mu_{p,q} V_{pq}(\psi_p, \psi_q^{(k-2)}) \right] \tag{4.3}
$$

$$
\psi_p^{(k)} = \arg \min_{\psi_p} \left[ D_p(\psi_p) + \sum_{q \in N_p} \mu_{p,q} V_{pq}(\psi_p, \psi_q^{(k-1)}) \right] \tag{4.4}
$$

Since any neighboring site $q$ of $p$ is stable at iteration $k - 1$ and $k$, $\psi_q^{(k-1)} = \psi_q^{(k-2)}$. Therefore, $\psi_p^{(k)} = \psi_p^{(k-1)}$. That is, $p$ is stable at iteration $k$. "

For the asynchronous iteration (4.2), Proposition 4.1.1 with slight modification is still valid:

**Proposition 4.1.2.** A pixel $p$ is stable at iteration $k$ if all its prior neighbors are stable at iteration $k$ and all its posterior neighbors are stable at iteration $k - 1$. "

**Definition 4.1.2.** The **cumulative stable number** of a pixel $p \in P$ at iteration $k$ is $n$, if $p$ remains stable for the latest consecutive $n$ iterations, i.e.,

$$
\psi_p^{(k)} = \psi_p^{(k-1)} = \cdots = \psi_p^{(k-n)} \neq \psi_p^{(k-n-1)}. \tag{4.5}
$$

We use $\text{cus}(p, k)$ to denote the cumulative stable number.

**Definition 4.1.3.** A site $p$ is **cumulatively stable** at iteration $k$ with respect to a threshold integer $\text{cus}_{\text{thr}}(\geq 1)$, if

$$
\text{cus}(p, k) \geq \text{cus}_{\text{thr}}. \tag{4.6}
$$
4.1.2 The Stability Tracking Algorithm

From Property 4.1.1 and 4.1.2, it can be inferred that the computation of (4.1) and (4.2) at a pixel \( p \) is redundant if all its neighbors have been known to be stable. Based on this observation, we have the Stability Tracking algorithm \( ST(k) \) as illustrated by Figure 4.1.

With the \( ST(k) \) algorithm, to compute the estimate of a pixel \( p \) at iteration \( k \), the \( ST(k) \) algorithm first checks its stability by looking up the CUS table. If \( p \) is cumulatively stable (that is, \( CUS(p, k - 1) \geq \text{cus}_\text{thr} \)), no further computation is needed for \( p \) and \( ST(k) \) jumps to the next pixel. Otherwise, depending on which
updating scheme is used, formula (4.1) or (4.2) is calculated at $p$ to produce an estimate. With this fresh estimate, stability of $p$ within the current iteration is verified. If $p$ is stable for the current iteration, the $CUS$ number of $p$ increments by $\beta$. Otherwise, the $CUS$ number is decreased with the ratio of $\alpha$; the $CUS$ number of each neighbor of $p$ also needs to be decreased in the same way. The presence of update operation in Figure 4.1(a) is for asynchronous minimization of formula (4.1), and after removing it (see Figure 4.1(b)), the synchronous minimization of formula (4.2) is computed.

In the ST($k$) algorithm, the two-dimensional table $CUS$ is for tracking and looking up stability at each pixels. The threshold integer $cus\_thr$ is preset for identifying cumulative stability property at each iteration. As we will see, $CUS(p,k)$ is always no greater than $cus(p,k)$, so $CUS(p,k) \geq cus\_thr$ at line 2 implies that $p$ is cumulatively stable with respect to $cus\_thr$. Initialization of $CUS$ before starting the first iteration ST(1) is:

$$CUS(p,0) = 0, \text{ for each } p \in \mathcal{P}$$

(4.7)

implying that no pixel is stable.

The most part of the ST($k$) algorithm, as illustrated in Figure 4.1, is for maintaining the $CUS$ table. However, compared to the time cost of minimizer computation on formula (4.1) or (4.2), the overhead $\mathcal{O}(|N_p| \times |\mathcal{P}|)$ of maintaining the $CUS$ structure is minor and ignorable.

The update policy of the $CUS$ structure adopted by the ST($k$) algorithm is known as Additive-Increase Multiplicative-Decrease (AIMD) in networking research for congestion control [116, 117]. Such an AIMD policy is critical to guarantee the accuracy of the ST algorithm. The AIMD parameters $\alpha$ should be constrained in the unit
interval $[0, 1)$ and $\beta$ no less than 1. Note the decreasing of CUS number is necessary when instability is detected at a pixel. As shown in Figure 4.2, a pixel may become instable so that the minimizer needs to be re-computed after being stable within a large number of iterations. Furthermore, this group of figure also illustrates how fast the instability ratio $r_{ins}$ becomes so that the ST($k$) can substantially decrease the entire time cost.

As discussed in the next section, an optimal configuration in terms of both efficiency and accuracy for ICM-ST is:

$$\alpha = 0, \quad \beta = 1, \quad cus_{thr} = 2$$  \hspace{1cm} (4.8)
Algorithm 1 ICM with Stability Tracking (ICM-ST)

1: generate the background mask.
2: pre-compute all pixels’ data costs.
3: generate the initial guess of field map.
4: repeat the ST\((k)\) algorithm until the instable ratio \(r_{ins}(k)\) falls below the error threshold \(\varepsilon\), i.e.
   \[ r_{ins}(k) < \varepsilon \] (4.9)
5: reconstruct water and fat images via \(\rho = \Psi^+ \cdot S\).

4.2 Performance Analysis of ICM-ST

With the ST\((k)\) algorithm, the main components of ICM-ST algorithm is illustrated in Algorithm 1. The background masking of Step 1 can be performed with the GrabCut-based or histogram-based masking algorithm introduced in the previous chapter. To process Step 2, the field map interval \(I = [\psi_{min}, \psi_{max}]\) can be evenly discretized to \(I_m\), where

\[
I_m = \{\psi_{min} + \frac{k}{m}(\psi_{max} - \psi_{min}) : k = 0, 1, 2, \cdots, m\} \tag{4.10}
\]

and the size of \(m\) can be determined by following the method given in[10]. For Step 3, we will develop a median-based algorithm in Section 4.3. In step 4, \(r_{ins}(k)\) represents the instable ratio at iteration \(k\), namely,

\[
r_{ins}(k) = \frac{|\{p \in \mathcal{P} : cus(p, k) < cus\_thr\}|}{|\mathcal{P}|} \tag{4.11}
\]

where \(\{p \in \mathcal{P} : cus(p, k) < cus\_thr\}\) is the collection of all cumulatively instable pixels at iteration \(k\) with respect to the threshold \(cus\_thr\).

Next, we present theoretical analysis on the accuracy and efficiency of ICM-ST by comparing it to the conventional ICM. We will see how the stability tracking mechanism is able to substantially speed up the conventional ICM iterative computations with no compromise of accuracy.
4.2.1 Accuracy of Stability Tracking

We show that for both synchronous (4.1) and asynchronous (4.2), the parameter configuration
\[ \alpha = 0, \beta = 1, \text{cus}_\text{thr} \geq 2 \]  
makes the stability tracking algorithm produce the very same estimate as the conventional ICM iteration at each iteration and each pixel.

First of all, from the ST(k) Algorithm, it can be readily observed that for both synchronous and asynchronous updating

\[ \text{Proposition 4.2.1.} \text{ At any pixel } p \text{ and iteration } k \geq 0, \]

\[ CUS(p, k) \leq \text{cus}(p, k). \]  

\[ \text{Proposition 4.2.2.} \text{ Given a pixel } p \text{ with } CUS(p, n) = t, \]

(1) \[ \text{cus}(q, n) \geq t - 1 \text{ for any } q \in N_p^-; \]
(2) \( \text{cus}(q, n) \geq t \) for any \( q \in \mathbb{N}^+_p \).

**Proof.** For (1), assume that there exists some \( q \in \mathbb{N}^-_p \) with \( \text{cus}(q, n) \leq t - 2 \). Then there is an index \( k \) such that \( n - t + 2 \leq k \leq n \) and \( q \) is not stable at iteration \( k \). That is \( CUS(p, k) \leq 1 \). Then \( CUS(p, n) \leq CUS(p, k) + (n - k) \leq 1 + t - 2 = t - 1 \). Thus \( CUS(p, n) \leq t - 1 \), contradicted to the assumption that \( CUS(p, n) = t \).

Similarly, we can prove (2). \( \Box \)

Let \( \psi(t, k) \) denote the label value yielded by the ST\( (k) \) iteration with the threshold value \( \text{cus\_thr} = t \). Let \( \hat{\psi}(k) \) denote the value yielded by the \( k \)-th ordinary ICM iteration. Moreover, \( \psi_p(t, k) \) and \( \hat{\psi}_p(k) \) denote the values of \( \psi(t, k) \) and \( \hat{\psi}(k) \) at pixel \( p \), respectively. Then for both updating policies, under the configuration of (4.12), we have

**Theorem 4.2.3.** Assume \( \alpha = 0, \beta = 1 \), and \( \text{cus\_thr} = t \geq 2 \). Then for each iteration index \( k \geq 0 \),

\[
\psi(t, k) = \hat{\psi}(k) \tag{4.14}
\]

**Proof.** We will present an inductive proof for asynchronous updating (4.1.2), and a similar argument can be applied to show synchronous updating (4.1.1).

First of all, note that \( \psi_p(t, 0) \) and \( \hat{\psi}_p(0) \) are both equal to the initial guess at \( p \). Therefore, the equation (4.14) holds in the case of \( k = 0 \).

Now assume that the equation \( \psi(t, k) = \hat{\psi}(k) \) holds for all \( k \leq n \). We need to prove \( \psi(t, n + 1) = \hat{\psi}(n + 1) \). Assume that the iterating order on all pixels is \( p_1 \to p_2 \to p_3 \to \cdots \to p_{|\mathbb{P}|} \). Then we show by induction that

\[
\psi_{p_j}(t, n + 1) = \hat{\psi}_{p_j}(n + 1) \tag{4.15}
\]
always holds for \( j = 1, 2, \ldots, |\mathcal{P}| \). For notational simplicity, we use \( \psi_{q,n}^{\text{cur}} \) and \( \hat{\psi}_{q,n}^{\text{cur}} \) to denote the iterated term \( \psi_{q,n}^{\text{cur}} \) in formula (4.1.2) for the \((n+1)\)-th iteration, respectively. Then according to formula (4.1.2), to prove (4.15), we only need to verify

\[
\psi_{q,n}^{\text{cur}} = \hat{\psi}_{q,n}^{\text{cur}}
\]  

for each neighbor \( q \) of \( p_j \).

First, consider \( j = 1 \). If \( CUS(p_1, n) < t \), the ST algorithm recomputes the label of \( p_1 \) at iteration \( n + 1 \) using formula (4.1.2). Now that \( p_1 \) is the first pixel computed at iteration \( n + 1 \), all its neighbors are posterior to it and their estimates have not been updated yet. Therefore, for each \( q \in \mathcal{N}_{p_1} \),

\[
\psi_{q,n}^{\text{cur}} = \psi_q(t, n) = \hat{\psi}_q(n) = \hat{\psi}_{q,n}^{\text{cur}}
\]

and thus \( \psi_{p_1}(t, n+1) = \hat{\psi}_{p_1}(n+1) \). In the case that \( CUS(p_1, n) = t \), the ST algorithm skips computation at \( p_1 \) and so \( \psi_{p_1}(t, n+1) = \psi_{p_1}(t, n) = \hat{\psi}_{p_1}(n) \) by induction. From Proposition 4.2.2, for any \( q \in \mathcal{N}_{p_1} \),

\[
cus(q, n) \geq CUS(p_1, n) = t
\]

Since \( t \geq 2 \), it follows that \( \psi_q(t, n) = \psi_q(t, n-1) \). With the inductive assumption,

\[
\hat{\psi}_{q,n}^{\text{cur}} = \hat{\psi}_q(n) = \psi_q(t, n) = \psi_q(t, n-1) = \hat{\psi}_q(n-1) = \hat{\psi}_{q,n-1}^{\text{cur}}
\]

Therefore, it follows from formula (4.1.2) that \( \hat{\psi}_{p_1}(n+1) = \hat{\psi}_{p_1}(n) = \psi_{p_1}(t, n+1) \).

Assume that \( \psi_{p_j}(t, n + 1) = \hat{\psi}_{p_j}(n + 1) \) holds for all \( j < m \). To finish proving equation (4.15), we only need to show that

\[
\psi_{p_m}(t, n + 1) = \hat{\psi}_{p_m}(n + 1)
\]
For this, first consider the case $CUS(p_m, n) < t$. In this case, for any $q \in N^-_{p_m}$,

$$
\psi^{\text{cur}}_{q,n} = \psi_q(t, n + 1) = \hat{\psi}_q(n + 1) = \hat{\psi}^{\text{cur}}_{q,n}
$$

(4.21)

And for any $q \in N^+_{p_m}$,

$$
\psi^{\text{cur}}_{q,n} = \psi_q(t, n) = \hat{\psi}_q(n) = \hat{\psi}^{\text{cur}}_{q,n}
$$

(4.22)

Since equation (4.21) and (4.22) say that $\psi^{\text{cur}}_{q,n} = \hat{\psi}^{\text{cur}}_{q,n}$ holds for any $q \in N_{p_m}$, it follows from formula (4.1.2) that

$$
\psi_{p_m}(t, n + 1) = \hat{\psi}_{p_m}(n + 1)
$$

(4.23)

Now consider the other case that $CUS(p_m, n) = t$. In this case, the ST algorithm skips computation of formula (4.1.2) at $p_m$. So

$$
\psi_{p_m}(t, n + 1) = \psi_{p_m}(t, n) = \hat{\psi}_{p_m}(n)
$$

(4.24)

Clearly, equation (4.19) still holds for all pixels $q \in N^+_{p_m}$. As for any $q \in N^-_{p_m}$, it follows from Lemma 2 that $ras(q, n) \geq t - 1 \geq 1$. Thus, similar to previous discussion on $p_1$, we still have the equation (4.19). Hence, with formula (4.1.2), it still holds that $\hat{\psi}_{p_m}(n + 1) = \hat{\psi}_{p_m}(n) = \psi_{p_m}(t, n + 1)$.

With Theorem 4.2.3, one can see that an optimal configuration for ST($k$) to achieve desirable efficiency and the same accuracy as ICM is:

$$
\alpha = 0, \beta = 1, cus\_thr = 2
$$

(4.25)

Meanwhile, algorithm the configuration for the ST($k$) algorithm is:

$$
\alpha = 0, \beta = 1, cus\_thr = 1
$$

(4.26)
also has desirable performance according to our test. Such a configuration is slightly faster than (4.25) which generates almost the same results as it. However, with other configurations, efficiency of stability tracking with respect to conventional ICM can not be guaranteed any more.

4.2.2 Efficiency with Stability Tracking

As we have seen in Chapter 2, the time cost of each conventional iteration is

\[ T_{ICM}(k) = \mathcal{O}(|I_m| \times |N_p| \times |P|). \]  

(4.27)

In contrast, since ST\((k)\) only compute instable pixels \(\bar{S}(k)\) and therefore, the time cost of each ST\((k)\) is

\[ T_{ST}(k) = \mathcal{O}(|I_m| \times |N_p| \times |\bar{S}(k)|). \]  

(4.28)

Therefore, the efficiency improvement of ST\((k)\) over the conventional ICM at each iteration is

\[ \frac{T_{ST}(k)}{T_{ICM}(k)} = \mathcal{O}\left(\frac{|I_m| \times |N_p| \times |P|}{|I_m| \times |N_p| \times |\bar{S}(k)|}\right) = \mathcal{O}(r_{ins}(k)) \]  

(4.29)

implying that the effectiveness of stability tracking for efficiency improvement is determined by the variation of the instability ratio \(r_{ins}(k)\).

Exponential decreasing of \(r_{ins}(k)\) has always been observed throughout our tests on more than 100 mouse datasets with different parameter configuration of \(\alpha, \beta\) and \(cus\_thr\). Typical Linear fittings of log\((r_{ins}(k))\) on six mouse datasets are illustrated in Figure 4.4, where the AIMD parameters follows the optimal configuration: \(\alpha = 0, \beta = 1, cus\_thr = 2\). Figure 4.5 directly shows the exponential decreasing of \(r_{ins}(k)\) on one data set with different configuration of \(\alpha, \beta\) and \(cus\_thr\).
Figure 4.4 Linear fittings of $\log(r_{\text{ins}}(k))$ on 6 mouse datasets. The $\log(r_{\text{ins}}(k))$ curves are in green, and the linear fitting lines are in red.

Figure 4.5 Exponential decreasing of $r_{\text{ins}}(k)$ on one mouse dataset. Exponential decreasing of $r_{\text{ins}}(k)$ can be directly observed with different configuration of $\alpha$, $\beta$ and $\text{cus}_\text{thr}$. Note that the $r_{\text{ins}}$ rapidly decreases to below 0.1 after the first few iteration so that the time cost is ignorable for subsequent iterations.
Now assume the validity of exponential decreasing on $r_{\text{ins}}(k)$, that is,

$$r_{\text{ins}}(k) = \begin{cases} 1, & \text{for } k \leq \text{cus}_\text{thr}; \\ \xi^{k-\text{cus}_\text{thr}}, & \text{for } k > \text{cus}_\text{thr}. \end{cases}$$

with respect to some base $\xi \in (0, 1)$. Then for $n(\geq \text{cur}_\text{thr} + 1)$ conventional ICM iterations, the total time cost is

$$T_{ICM} = \mathcal{O}(|I_m| \times |N_p| \times |P| \times n)$$

The total time cost of $n$ ST iterations is

$$T_{ST} = \mathcal{O}(|I_m| \times |N_p| \times |P| \times \sum_{k=1}^{n} r_{\text{ins}}(k))$$

$$\approx \mathcal{O}(|I_m| \times |N_p| \times |P| \times (\text{cur}_\text{thr} + \frac{\xi}{1-\xi})).$$

Therefore, for large number of iterations $n$, the speedup ratio $\rho$ of stability tracking over the conventional ICM is

$$\rho = \frac{T_{ICM}}{T_{ST}} \approx \frac{n}{\text{cus}_\text{thr} + \frac{\xi}{1-\xi}}$$

Particularly when $\text{cus}_\text{thr} = 2$, the speedup ratio is

$$\rho \approx \begin{cases} 1, & \text{for } n \leq \text{cus}_\text{thr}; \\ \frac{1-\xi}{\frac{1}{2}-\xi}n, & \text{for } n \gg \text{cus}_\text{thr}. \end{cases}$$

### 4.3 Median Initialization for ICM-ST

Gradient descent-based algorithms such as ICM-ST are sensitive to initial guesses and produce desirable results only when appropriately initialized [22, 24, 102]. This is especially the case when ICM-ST is applied to handle the NLR-MRF energy model (3.2) developed in Chapter 3 due to the non-convexity of data cost terms within it. As
illustrated in Figure 4.6, initialization schemes such as “Winner-Take-All” [24, 102], “Greatest-Feasible-Solution” [2] and “All-Zero” [10], result in severely erroneous field map recoveries when applied to ICM-ST.

In this section, we propose a novel initialization algorithm for ICM-ST which essentially estimates solutions on ambiguous pixels and takes their median as the initial guess of ICM-ST at each pixel. In subsequent discussion, we assume that the echo time shifts in the signal model (1.1) are uniformly spaced, that is

\[ t_2 - t_1 = t_3 - t_2 = T. \] (4.35)

### 4.3.1 Unambiguous Field Map

Unambiguous pixels refer to those whose signal equations (1.1) have unique solution of \( \psi \) in the fundamental range \( \left[ -\frac{1}{2T}, \frac{1}{2T} \right) \) (alternatively, the phasor \( e^{i2\pi\psi T} \) has the unique value). Theoretically, pixels containing both fat and water are mostly unambiguous [8]. In addition, fat pixels are highly likely to be unambiguous with respect to the single-peak model (1.1), largely due to that the olefinic fat protons resonate at a frequency close to the water resonance [2]. Subsequently, unambiguous map will refer to the field map consisting of unambiguous pixels, as illustrated in Figure 4.7(b).

Initializing ICM-ST with median of unambiguous map is motivated by three observations which have been made on our datasets. First, the whole field map statistically similar to the unambiguous map. Finite Gaussian Mixture (FGM) models have been widely used to formulate MR field inhomogeneity [27, 118, 119]. As sampled from the whole field map, the unambiguous field inhomogeneity can also be modeled with FGM. Furthermore, these two types of FGM models are statistically similar to
Figure 4.6 *Reconstructions of ICM-ST with different initialization methods.* In the first column are the correct reconstructions with the proposed initialization scheme. In the second column are the reconstructions with the “Greatest-Feasible-Solution” initialization proposed for Algebraic Decomposition. In the third column are the reconstructions with the “Winner-Take-All” initialization. In the forth column are the reconstructions with the “All-Zero” initialization.
The whole and unambiguous field inhomogeneity. (a) the “gold standard” whole field map. (b) the unambiguous field map with unambiguous pixels identified by the HR based method [2]. The unambiguous percentage is 76.8%, and the median and standard variance are separately 1909.4 and 309.9. (c) The quasi-unimodal probability density functions (PDF) of the whole field map and unambiguous map. The median and standard variance pair on the whole field map is (1930.5, 307.3), while it is (1909.4, 309.9) on the unambiguous field. (d) The difference of the “gold standard” field map in (a) with the ICM-ST reconstruction initialized with the median 1930.5.

Secondly, the probability density functions of MR field inhomogeneity have great chance of containing a single dominant mode or multiple dominant modes which cluster within a small range (e.g., in the two-sigma or three-sigma interval, as shown in Figure 4.8). We call this type of probability density functions quasi-unimodal. Although robust methods to automatically assess quasi-unimodality are not yet available, a few statistical techniques are applicable. For example, kernel density estima-
Figure 4.8 *Quasi-unimodality of field inhomogeneity probability density function.* The quasi-unimodality is roughly assessed with the two-sigma and three-sigma rules. The two-sigma intervals for (a), (b), (c) and (d) are $806 \pm 251\, \text{Hz}$, $-411 \pm 643\, \text{Hz}$, $1096 \pm 328\, \text{Hz}$ and $-344 \pm 353\, \text{Hz}$, respectively; corresponding, the two-sigma percentages are $94.3\%$, $93.7\%$, $95.1\%$ and $95.5\%$. The three-sigma intervals for (a), (b), (c) and (d) are $806 \pm 377\, \text{Hz}$, $-411 \pm 964\, \text{Hz}$, $1096 \pm 492\, \text{Hz}$ and $-344 \pm 529\, \text{Hz}$, respectively; and the three-sigma percentages separately are $97.9\%$, $99.7\%$, $99.2\%$ and $99.5\%$.

Thirdly, let $\psi$ denote the correct reconstruction of field inhomogeneity and $\tilde{\psi}$ the new inhomogeneity reconstruction by ICM-ST initialized with the median of $\psi$. Then with our tests, $\tilde{\psi}$ would be almost identical to $\psi$ as long as the latter is quasi-unimodality. Figure 4.7(d) shows an instance for the identicalness between $\psi$ and $\tilde{\psi}$. Therefore, one can assert that the median value of field map provide a good initial guess for ICM-ST iterations.
4.3.2 Estimate the Unambiguous Map

Due to noise contamination, the signal equations (1.1) cannot be directly applied to calculate and identify unambiguous pixels. Instead, techniques such as harmonic retrieval (HR) analysis used by the Algebraic Decomposition approach [2] are available for this task. With HR analysis, at each foreground pixel \( p \in \mathcal{F} \), the signal equations (1.1) is first transformed to the quadratic equations of \( \gamma \):

\[
\begin{align*}
    s_3(p) - (1 + \lambda)s_2(p)\gamma + \lambda s_1(p)\gamma^2 &= 0 \\
    s_1(p)^* - (1 + \lambda)s_2(p)^*\gamma + \lambda s_3(p)^*\gamma^2 &= 0
\end{align*}
\tag{4.36}
\]

where \( T = t_2 - t_1 \), \( \gamma = e^{2\pi i T} \), \( \lambda = e^{2\pi i \Delta f T} \), and the notation * denotes the complex conjugate operation. The solutions of (4.36) with respect to the phasor \( \gamma \) are called feasible solutions. It is easy to see that unambiguous pixels are just those with unique feasible phasor solution. For the same reason of noise corruption, straightforward calculation on (4.36) to solve phasors is not applicable. Note that the polynomial greatest common divisor (GCD) of polynomials \( P(\gamma) \) and \( Q(\gamma) \), where

\[
\begin{align*}
    P(\gamma) &= s_3(p) - (1 + \lambda)s_2(p)\gamma + \lambda s_1(p)\gamma^2 \\
    Q(\gamma) &= s_1(p)^* - (1 + \lambda)s_2(p)^*\gamma + \lambda s_3(p)^*\gamma^2
\end{align*}
\tag{4.37}
\tag{4.38}
\]

indicates the feasible solutions. Then associated techniques from Sylvester matrix analysis can be used to determine the GCD of \( P(\gamma) \) and \( Q(\gamma) \) [2, 126, 127]. In more detail, The GCD-based algorithm to estimate feasible solutions is:

(i) Construct the Sylvester matrix of \( P(\gamma) \) and \( Q(\gamma) \)

\[
M = \begin{pmatrix}
    \lambda s_1 & -(1 + \lambda)s_2 & s_3 & 0 \\
    0 & \lambda s_1 & -(1 + \lambda)s_2 & s_3 \\
    \lambda s_3^* & -(1 + \lambda)s_2^* & s_1^* & 0 \\
    0 & \lambda s_3^* & -(1 + \lambda)s_2^* & s_1^*
\end{pmatrix}
\tag{4.39}
\]
Based on the Sylvester matrix theory, when $M$ is triangulated to $\tilde{M}$ using only row matrix operations [2], the GCD of $P(\gamma)$ and $Q(\gamma)$ corresponds to the $(4 - r)$th row of $\tilde{M}$. Herein, $r$ represents the rank of GCD and will also be estimated later.

(ii) Denoise $M$. First, SVD decompose $M = U\Sigma V^*$. Then construct $\Sigma' = \Sigma$ with further modification as: $\Sigma'(4, 4) = 0$, and $\Sigma'(3, 3) = 0$ if $\Sigma(3, 3) / \Sigma(1, 1)$ is below some threshold. Now set $r$ to be

$$r = \begin{cases} 2, & \Sigma'(3, 3) = 0; \\ 1, & \Sigma'(3, 3) \neq 0. \end{cases} \quad (4.40)$$

and $M' = U\Sigma' V^*$ the denoised $S$.

(iii) LU decompose $M' = LU$. Then the $(4 - r)$th row of $U$ provides the feasible solutions of $\gamma$.

After all feasible solutions have been calculated, the unambiguous pixels along with their field inhomogeneity estimates are accordingly available. However, as observed from Figure 4.9(a), the quasi-unimodality of the unambiguous map’s PDF may not hold due to phase-wrapping occurring with a considerable number of pixels. Therefore, prior to estimating the median, phase wrapping at zero should be tested and phase unwrapping is performed whenever the test result comes out to be significant. The phase wrapping test can be performed with the statistic

$$X_0 = \frac{|\{p \in \mathcal{U} : -w \leq \hat{\psi}(p) \leq w\}|}{|\mathcal{U}|} \quad (4.41)$$

Herein, $\mathcal{U}$ denotes the unambiguous field map, $\hat{\psi}(p)$ the unambiguous solution at $p$, and $[-w, w]$ the window around 0. Given a significance level $\sigma$, $X_0 < \sigma$ implies
Figure 4.9 Phase unwrapping for median estimation. (a) The histogram of the unambiguous solutions which are directly obtained with the HR-based method. Phase wrapping apparently occurred at zero. The window size $w = 105$ is the width of 5 discretization steps, $\varepsilon = 0.10$ and $X_0 = 0.028$. Since $X_0 < \varepsilon$, the test confirms the occurrence of phase wrapping. (b) The phase wrapped histogram of (a). Before phase unwrapping, the median is 864.2; after phase wrapped, the median is 1318.6. (c) A histogram of unambiguous solutions. There is no significant phase wrapping occurring. With $w = 105$, the statistic $X_0 = 0.141 > \varepsilon$ and so the test confirms no occurrence of phase wrapping.

significant occurrence of phase wrapping. Throughout our tests, $\sigma$ has been fixed as 0.10 and $w$ the width of 5 discretization steps, as shown in Figure 4.9.

Note that the two equations in (4.36) are periodic of $\frac{1}{T}$ with respect to $\psi$. Therefore, phase unwrapping, if necessary, can be simply performed by either adding $\frac{1}{T}$ to all negative inhomogeneity solutions, or alternatively, subtracting all positive inhomogeneity solutions with $\frac{1}{T}$. Then the median of the unambiguous field map can be estimated after phase unwrapping is done.

The whole median estimation procedure is summarized in Algorithm 2.

4.3.3 Block-wise Median Initialization

The uniform median initialization algorithm described above is based on the assumption that the probability density function is quasi-unimodal, characteristic of
Algorithm 2 Median Value Initialization

1: calculate feasible phasor solutions $\gamma$ on all foreground pixels, and identify the unambiguous map.
2: for each unambiguous phasor solution, calculate the field inhomogeneity value within fundamental range $[-\frac{1}{2T}, \frac{1}{2T})$.
3: test phase wrapping and perform phase unwrapping if necessary.
4: calculate the median of unambiguous field inhomogeneities and uniformly set it as the initial value of ICM-ST at each foreground pixel.

one single mode or tightly clustered modes which have been widely observed in our experiments on mouse datasets. However, field maps with large or rapid variation are likely to obviate from quasi-unimodality and thereby ICM-ST with the uniform initialization cannot converge to the correct recoveries. In this case, the whole image grid can be divided into multiple blocks such that the PDF at each block is quasi-unimodal. Then the uniform median initialization algorithm is applied to each separate block.

In the presence of large or rapid variation of field inhomogeneity, advantage of block-wise median initialization over the uniform median initialization is illustrated by Figure 4.10. In Figure 4.10(a), the correct field map varies from -877.6 Hz to 1749.9 Hz, with the global median 488.7 Hz. However, the upper part and the lower part (divided by the green line) have distinctively different distributions: the minimum, median and maximum of field map values on the upper part are $\{-877.6, -120.9, 383.6\}$ while these quantities on the lower part are $\{-205.0, 530.7, 1749.9\}$, respectively. The difference between the global median value 488.7 and the median value -120.9 on the upper part is too big for ICM-ST iterations to converge to the correct estimate at each pixel. By contrast, the block-wise initialization with the upper -120.9 and lower part 530.7 enables ICM-ST to converge correctly at each pixel(Fig. 4.10(c)). Likewise,
Figure 4.10 Improvement of ICM-ST with the block-wise initialization. (a) and (c) are field map recoveries with uniform median initialization. (b) and (d) are field map recoveries with block-wise median initialization, in which errors in (s) and (c) are corrected.

with the block-wise initialization applied for the field map in Figure 4.10(a), ICM-ST yielded the correct reconstruction shown in Figure 4.10(d).

A major challenge for block-wise initialization is to automatically determine how to divide the image grid into blocks. With our mouse datasets acquired with 7.0 Tesla MRI, block-wise initialization, if necessary, requires no more than three blocks of division. However, for field map with extremely rapid or large variation, division with more number of blocks may be indispensable, and an adaptive method for block division will be very helpful yet still to be investigated.

4.4 Configuration of Regularization Factor

The regularization factor $\mu$ in NLR-MRF energy model (3.2) regulates the relative weight of data cost to smoothness cost. Undersized regularization values often result in under-smooth recoveries, while oversized values leads to results almost identical to initial guesses. Regularization configuration usually depends on both the MRF
energy formulation and the optimization algorithm. Some configuration methods can be referred to in [10, 11, 128, 129].

The gradient-based regularization configuration scheme for the NLR-MRF model (3.2) and the ICM-ST algorithm is based on controlling the ratio of data cost gradient to smoothness cost gradient within a reasonable range. Given a field map estimate $\psi_p$ of pixel $p$, we use $\nabla D^l_p(\psi_p)$ and $\nabla S_p(\psi_p)$ to separately denote the absolute value of data cost and smoothness cost gradients, where,

$$S_p(\psi_p) = \mu_p \sum_{q \in N_p \cap \mathcal{F}} \omega_{pq} |\psi_p - \psi_q|^l$$  \hspace{1cm} (4.42)

Despite the high dimensionality of $S_p(\psi_p)$, we can use the local smoothness feature of field map to estimate its gradients at each pixel. In fact, since field map values of $p$ with its neighboring pixels should be nearly the same, with $\varrho(p)$ denoting the approximate value, we have

$$S_p(\psi_p) \approx \mu_p \sum_{q \in N_p \cap \mathcal{F}} \omega_{pq} |\psi_p - \varrho_p|^l = \mu_p \omega |\psi_p - \varrho_p|^l$$

where $\omega = \sum_{q \in N_p \cap \mathcal{F}} \omega_{pq}$ is the sum of neighboring weights and constant with respect to $p$ and $\psi$. Thus, the gradient of $S_p(\psi_p)$ is

$$\nabla S_p(\psi_p) = l\mu_p \omega |\psi_p - \varrho_p|^{l-1}$$  \hspace{1cm} (4.43)

Then the gradient ratio $\rho(\psi_p)$ of data cost to smoothness cost is

$$\rho(\psi_p) = \frac{\nabla D^l_p(\psi_p)}{\nabla S_p(\psi_p)} = \frac{\nabla D^l_p(\psi_p)}{l\mu_p \omega |\psi_p - \varrho_p|^{l-1}}$$  \hspace{1cm} (4.44)

Alternatively, the regularization factor at $p$ is

$$\mu_p = \frac{\nabla D^l_p(\psi_p)}{l \omega \rho(\psi_p) |\psi_p - \varrho_p|^{l-1}}$$  \hspace{1cm} (4.45)
Equation (4.45) provides very useful heuristic for regularization factor configuration. Generally, the gradient ratio $\rho(\psi_p)$ can be set to be some number less than 10. The term $|\psi_p - \varrho_p|$ vanishes for the 1-normed NLR-MRF models, and can be replaced by $|\psi_{\text{max}}|$ in 2-normed models. $\nabla D^t_p(\psi_p)$ can be evaluated by the mean value of all absolute gradients of the data cost $D^t_p(\psi_p)$.

4.5 Discussion

Compared to the conventional ICM optimization, the ICM-ST is characteristic of the stability tracking for high computational efficiency and median initialization to provide initial guess and improve reconstruction accuracy.

Note that the accuracy analysis of stability tracking in Section 4.2.1 does not involve the specific formulation of NLR-MRF energy model so that it can be extended to compute MRF formulation raised in other application settings. Figure 4.11 presents the application of ICM-ST in an ongoing research on diagnosis of polycystic kidney disease with CT imaging. As shown in Figure 4.11(a), cysts in the kidney image may be blurred by noise. Therefore, optimization based on ICM-ST is applied to enhance the contrast of the image to highlight the cysts based on a normal MRF energy model:

$$\hat{f} = \arg_f \min \left[ \sum_{p \in \mathcal{P}} ||f_p - f^0_p|| + \sum_{p \in \mathcal{P}} \sum_{q \in \mathcal{N}_p} \mu_{p,q} V_{pq} |f_p - f_q| \right]$$

(4.46)

Background masking is first applied to avoid unnecessary computations on air regions. However, since a label set with large large of around 800 labels and $5 \times 5$ square neighborhoods have to be used, the per-iteration time cost for conventional ICM is high up to about 6 seconds. Nevertheless, with the help of stability tracking, the per-iteration time cost decreases to below 1.5 seconds right after the first 15 iterations,
Figure 4.11 *ICM-ST for CT imaging of cystic kidney disease.* ICM-ST is applied to enhance the contrast of cystic regions (indicated by the arrow) contained in the kidney. (a) Before enhancement processing. The cystic region is unclear due to noise. (b) After enhancement processing. The cystic region is clearly visualized. (c) The decrease of \( r_{\text{ins}} \). The per-iteration time cost decrease from 6.3 seconds to below 1.5 seconds after the first 15 iterations.
Figure 4.12 *The impact of background masking on median initialization.* (a) the PDF of unambiguous field maps with and without background masking. (b) the source image. (c) and (d) are the wrong water and fat image reconstructions from ICM-ST with the non-background masked median. (e) and (f) are the correct water and fat images reconstructed by ICM-ST with the background masked median.

as shown in Figure 4.11(c).

The background masking is important for the estimation of unambiguous field and its median. Feasible solution estimates of pixels in the air region based on the quadratic equations (4.36) is not reliable and may severely bias the unambiguous field and statistics such as the PDF and median on it. As a result, the reconstruction of water and fat images may be totally wrong. Figure 4.12 demonstrates the impact
of background masking on the estimation initialization. From Figure 4.12(a) one can observe that the PDF without background masking is not quasi-unimodal. The median with no background masking is -1270, far from the median value -523 with background masking. Figure 4.12(c) and (d) are separately the total wrong water and fat images reconstructed by ICM-ST with the non-background masking median. Figure 4.12(c) and (d) present the correct reconstruction of water and fat images with the median after background masking.

An important limitation of the median initialization algorithms is on the assumption of even spacedness of TE shifts. While this has been prevalent in the research of water-fat image reconstruction, non-evenly spaced TE shifts have also been investigated [7, 44]. Therefore, to make the ICM-ST approach work in this asymmetric case, more efforts are needed to further improve the median initialization.
Chapter 5

Experimental Results

In this chapter, performance of the ICM-ST reconstruction approach will be evaluated on both synthesized and in vivo 7.0 Tesla mouse whole-body datasets. First, the efficiency of ICM-ST will be demonstrated. Then the analysis of accuracy based on comparison to IDEAL [7] and VARPRO-ICM [10] will be presented.

5.1 Experimental Setting

Major parameters for ICM-ST and VARPRO-ICM were configured as follows. The residual norm \( l = 1 \). The neighborhood system was \( 5 \times 5 \) square weighted with the inverse of Euclidean distance. Field inhomogeneity values were bounded in the range of \( \pm \frac{1}{T} \) where \( T = 0.3172 \times 10^{-3} \) is the effective time shift. Moreover, after the median \( \zeta \) of unambiguous field map had been generated with the median initialization algorithm, the bound interval was further centralized at \( \zeta \), that is, the effective lower/upper bounds of field map were:

\[
\psi_{\min} = -\frac{1}{T} - \zeta, \quad \psi_{\max} = \frac{1}{T} + \zeta.
\]  

(5.1)

Following [10], the discretization steps \( m \) was set to 300. The AIMD parameters for stability tracking were always set as:

\[
\alpha = 0, \beta = 1, \text{cus\_thr} = 2.
\]  

(5.2)
Figure 5.1 Efficiency comparison of ICM-ST and VARPRO-ICM. Time costs on 60 datasets are shown. The number of iterations for VARPRO-ICM is always 40. (a) Comparison between ICM-ST and VARPRO-CIM. (b) VARPRO-ICM vs VARPRO-ICM with background masking.

Algorithms except the GrabCut-based background masking were implemented on a desktop computer with 2.66GHz Intel Core2 CPU, 4GB RAM, MATLAB 2010a 64 bit, without using any parallel computing technique. The GrabCut-based background masking was implemented with C++.

5.2 Efficiency Assessment

The efficiency advantage of ICM-ST over VARPRO-ICM is apparent. Figure 5.1 presents the efficiency of ICM-ST and VARPRO-ICM over 60 in vivo mouse whole-body datasets. The number of iterations for both VARPRO and VARPRO with background masking was set to 40, and the loop bound for the cumulative instability ratio $r_{ins}$ of ICM-ST was 0.001 (which normally needs more than 40 iterations to attain). For ICM-ST, reconstruction time cost on datasets with moderate object region size took $\sim$105 seconds, including $\sim$23 seconds for data costs precomputation, $\sim$4 seconds for calculation of the regularization parameter $\mu$, $\sim$14 seconds for median
initialization algorithm. Downsampling the images to $256 \times 128$ could decrease the time cost on the median initialization algorithm to less than 6 seconds while not causing significant bias on the median estimates. For VARPRO-ICM, the time cost was averagely more than 1000 seconds. If we apply the background masking on VARPRO-ICM, the time cost on datasets with moderate object region size took $\sim 390$ seconds.

With respect to IDEAL, pixels in the air regions generally asked for much more iterations than those foreground pixels. When the increment step $\Delta \psi$ was set to 5 Hz ([7]), IDEAL appeared to be more time costly than ICM-ST. However, when the background masking was applied, IDEAL was faster than ICM-ST, normally converging within 40 seconds.

5.3 Accuracy Assessment

Accuracy assessment was performed on both synthetic and \textit{in vivo} datasets with quantitative comparison of ICM-ST, VARPRO-ICM and IDEAL. VARPRO-ICM was implemented completely based on [10] except that background masking was used which apparently would not degrade VARPRO-ICM. IDEAL was straightforwardly implemented according to [7] and enhancing techniques such as region growing [8] or any others proposed later was not considered.

5.3.1 Assessment on Synthetic Datasets

Following the method described in [4, 7], we generated two synthetic datasets randomly selected from a large group of whole-body mouse datasets. One was acquired on a high fat diet mouse and the other on a low fat diet mouse at 7.0 Tesla.
MR [130, 131]. ICM-ST was first applied on the datasets to generate the estimated water, fat and field map which contained no evident visual error. The synthetic water and fat images were generated by applying wavelet-denoising on the estimated water and fat images separately. The estimated field maps were smoothed by applying the Hamming filter in the Fourier domain of field maps. Then the synthetic field maps were generated by scaling the estimated field maps at different levels so as to simulate the increasing severity of field inhomogeneity measured by maximum value of inhomogeneities. The scaling step was set as 200 and 13 levels were investigated, namely,

$$\Psi_s = \Psi \times \frac{k}{\max(\text{abs}(\Psi))}, \quad k = 200, 400, 600, \ldots, 2600 \tag{5.3}$$

where $\Psi_s$ represents the scaled synthetic field map and $\Psi$ the estimated field map. The water-fat chemical shift is 1051 Hz and the echo time shifts produced fat-water phase of $\{\pi/6, 5\pi/6, 9\pi/6\}$. Now the synthetic datasets were generated by applying the synthetic water, fat and field map images, as well as the chemical shift and phases into the standard signal model Equation (2.31). Finally, complex Gaussian noise was added to each slice of the datasets with the same SNR = 20 as in [4, 7].

The relative errors (averaged on both high and low fat diet mouse synthetic datasets) of ICM-ST and IDEAL on field maps were measured and shown in Figure 5.2. The error rate is defined as

$$\frac{|\{p \in \mathcal{F} : \text{abs}(\hat{\Psi}(p) - \Psi_s(p)) \geq \frac{3}{L} \cdot \frac{2}{T}\}|}{|\mathcal{F}|} \tag{5.4}$$

where $\mathcal{F}$ is the foreground of each acquisition, and $\frac{3}{L} \cdot \frac{2}{T}$ is the length of 3 discretized steps used in ICM-ST, around 63 in our tests. Since IDEAL is also sensitive to the initial guess and the uniform zero initialization proposed in [7] is not valid, we
Figure 5.2 Relative error rates of ICM-ST and IDEAL on synthetic datasets. ICM-ST had stable performance with various field inhomogeneities.

tested IDEAL with five different initial guess for each case of reconstructions and calculated the error rate on the best result. Since the uniform initialization proposed for VARPRO-ICM [10] generated severely erroneous reconstructions in most cases, its relative errors were not drawn out. From Figure 5.2, it can be observed that the error rate of ICM-ST constantly remains low with respect to different levels of field inhomogeneities. By contrast, with the rising of field inhomogeneities, the error rate of IDEAL sharply increases.

5.3.2 Assessment on In Vivo Datasets

To assess accuracy of ICM-ST on in vivo data, 313 whole-body mouse datasets were acquired from 10 low fat diet mice and 12 high fat diet mice with high resolution shifted spin echo scans on a Bruker BioSpec 7T/30cm System. A T1-weighted rapid acquisition with relaxation enhancement (RARE) sequence with varying echo asymmetry delays \{79.3, 396.4, 713.6\} $\mu$s was used to achieve \{\pi/6, 5\pi/6, 9\pi/6\} radian shifts between fat and water. Other major parameters include: field of view (FOV)
of 102 × 40 to 102 × 50 mm, matrix size= 512 × 256, 4 averages, pulse repetition time (TR) = 1087 ms/gated, echo time (TE) = 9.1 ms.

To evaluate the reconstructions, all the three images of estimated water, fat and field map were closely inspected with the error location and the number of pixels recorded. Four error levels were categorized based on the number and location of pixels at which water-fat swapping occurs in reconstructions, that is,

- **excellent** — no visible error
- **good** — errors occur at object boundaries, and less than 80 pixels
- **fair** — errors occur at object boundaries, and less than 160 pixels
- **poor** — errors occur inside object regions or more than 160 pixels.

Note that reconstructions of good or fair level can be corrected by either manual editing.

The overall reconstruction performance of ICM-ST on the 313 datasets is illustrated in Figure 5.3. With the uniform median initialization, 260 reconstructions were at excellent level, 21 good, 12 fair while 20 reconstructions contained severe water-fat swapping errors. Estimate errors on non-excellent, especially poor reconstructions were mainly caused by improper initialization. With the block-wise median initialization, the reconstructions had been significantly improved: 8 poor reconstructions had been shifted to excellent level; the number of fair and good reconstructions turned to 5 and 12; 288 datasets at excellent level. The field inhomogeneities mostly varied from moderate to large ranges, with length over 1400Hz in most cases. Since zero
Figure 5.3 Overall performance of ICM-ST on 313 datasets. With uniform initialization, the number of reconstructions at different level is 260: 21:12:20. With the improvement of block-wise initialization, that grows to be 286:14:5:8.

Initialization was not proper for most of the datasets, VARPRO-ICM yielded inferior reconstructions for most of datasets. IDEAL was evaluated based on the best one of 10 recoveries from IDEAL with different uniform initialization at each dataset. However, reconstructions from IDEAL on the mouse datasets were far from desirable, with severe water-fat swapping widely observed.

Figure 5.4 shows the reconstructions of ICM-ST, VARPRO-ICM and IDEAL on a low fat diet mouse dataset. The field inhomogeneities vary in the range of [-2054, -920]. Since the PDF of field inhomogeneities is apparently quasi-unimodal, the uniform initialization was able to generate a good initial guess such that ICM-ST yielded excellent water, fat and field map images (Figure 5.4(a), 5.4(b), 5.4(c)). VARPRO-ICM initialized with zero produced totally wrong reconstructions: severe water-fat swapping occurred in the most area of the image (Figure 5.4(d), 5.4(e), 5.4(f)). At best efforts, IDEAL generated reconstructions at fair level since water-fat
Figure 5.4 Reconstructions on a high fat diet mouse acquisition in presence of low field inhomogeneities. The field inhomogeneity variation is slightly larger than the chemical shift value of 1051 Hz. First row: from ICM-ST. Second Row: from VARPRO-ICM. Third row: from IDEAL.
Figure 5.5 *Reconstructions with moderate field inhomogeneities.* Reconstructions of ICM-ST and IDEAL on a high fat diet mouse acquisition. (a), (b), (c): from ICM-ST; (d), (e), (f): from IDEAL. (e): the 1-normed residual at a pixel where the IDEAL converged to wrong local minimum.

swapping was always observed in the image center and border area. A typical case of IDEAL reconstruction results is displayed in Figure 5.4(g), 5.4(h) and 5.4(i).

Figure 5.5 compares ICM-ST with IDEAL on another high fat diet mouse acquisition with the moderate field inhomogeneity. Reconstructions produced by ICM-ST with the uniform median initialization are the excellent level (see Figure 5.5(a), 5.5(b) and 5.5(c)). The field inhomogeneities vary in the interval [1250, 2647]. The best
results produced by IDEAL was at poor level (shown in Figure 5.5(d), 5.5(e) and 5.5(f)). Figure 5.5(g) illustrates the convergence of ICM-ST and IDEAL at a pixel at the center of the image grid (as indicated by the arrow in Figure 5.5(f)). At this pixel, with the initial guess of 2203, ICM converged to 2621 which is one step away from the optimal estimate 2644. Starting from this same initial guess, IDEAL converged to another different local minima 1591 leading to the water-fat swapping. With our tests, other initial guesses which could fixed the wrong convergence at this pixel yet raised estimating errors at some other pixels.

The reconstructions from ICM-ST and IDEAL on another high fat diet mouse acquisition are shown in Figure 5.6. High field inhomogeneities are present in this dataset, effectively varying in the range of [45, 2227]. Due to the large inhomogeneity variation, ICM-ST with various uniform initialization failed to yield results at excellent level. However, by dividing the image into upper and lower blocks and applying the block-wise initialization, ICM-ST yielded reconstructions at excellent level (Figure 5.6(a), 5.6(b), 5.6(c)). IDEAL still failed to produce acceptable results with various initial guesses. Figure 5.6(d), 5.6(e) and 5.6(f) illustrate one of the best results IDEAL had generated on the dataset. In this case, block-wise initialization was applied, but reconstruction errors could be clearly observed at the upper and lower part the mouse body.
Figure 5.6 *Reconstructions with moderate field inhomogeneities.* Reconstructions of ICM-ST and IDEAL on a high fat diet mouse acquisition. (a), (b), (c): from ICM-ST; (d), (e), (f): from IDEAL.
Chapter 6
Future Work and Conclusions

6.1 Future Work

6.1.1 Accurate Energy Model

With the use of NLR-MRF energy models, the recovery of field map images was casted to the optimization of MRF energy. This model, however, only contains the data cost and smoothness cost terms and does not account for other important features such as variation rate of field inhomogeneities. Meanwhile, the regularization factor in the NLR-MRF has great impact on the estimated results, and improper values result in over-smoothing or under-smoothing estimates. Generally, there is a range for the value of regularization factor to yield the visually correct reconstructions. However, for quantitative application of MR water-fat imaging, it is necessary to find out the optimal value of the regularization factor. As noted in [4], this issue could be addressed with the analysis of local impulse response (LIR) [132].

6.1.2 Automatic Block-wise Median initialization

Based on the statistical similarities between the whole and unambiguous field maps, we developed the median initialization algorithm which could provide good initial guesses for ICM-ST on acquisitions with moderate variation of field inhomogeneities. In the case of large field inhomogeneities, the block-wise initialization can be applied so that the uniform median initialization provides good initial guesses on
each block of the image grid. But at current stage, when the block-wise initialization should be applied and especially how to set up the partitioning locations are not clear to us yet.

6.1.3 Extension to Advanced Signal Models

A large number of research works on the advanced multi-peak fat and $T_2^*$ signal models demonstrates that they can improve the estimation accuracy of water and fat images than the standard three-point signal models. As illustrated in previous chapters, the ICM-ST could be theoretically extended to process these advanced signal models. However, experimental validation on a large number of datasets are desired.

6.1.4 Other Applications

The stability tracking algorithm for ICM is not specific to the computation of the NLR-MRF energy model for the water-fat image reconstruction, but a general approach for MRF energy optimization. In particular, it can be applied to image processing and computer vision problems where ICM is appropriate as the optimization method.

6.2 Conclusions

Reconstruction of water and fat images in MRI has many important applications in biomedical and clinical research. It provides better visualization of the scanned objects as well as enables quantification of fat tissues. In this thesis research, a novel approach, ICM-ST, has been developed to address the reconstruction of water and
fat images. In this approach, a non-linear least residual-based MRF energy model is used to characterize the local smoothness of MR field inhomogeneities so that the recovery of field inhomogeneities can be casted to optimization of MRF energy functions. Background masking is embedded into the NLR-MRF model to enhance the efficiency and accuracy of optimizing algorithms. Based on the conventional ICM algorithm, multiple algorithms have been developed to address the optimization on the NLR-MRF energy model. To overcome the low efficiency of ICM, the stability tracking algorithm is developed to avoid redundant computations. With the fast convergence of ICM, the stability tracking algorithm could substantially decrease the time cost with no loss of accuracy relative to ICM. The stability tracking algorithm is also applicable to ICM to optimize any other MRF energy model. To overcome the high sensitivity of ICM to initial guess, median-based algorithms based on the statistical similarity between the whole and the unambiguous field maps have been developed. The uniform median initialization has good performance on moderate to large field inhomogeneities. In the case of especially large or rapid field inhomogeneity variation, the block-wise initialization can be applied. Validation and comparative analysis on a large number of synthetic and in vivo 7T whole-body mouse datasets demonstrate high performance of the prosed ICM-ST approach.
Appendix

In this Appendix, MATLAB code for the stability tracking and median initialization algorithms are provided.

A.1 Code for Stability Tracking

Prior to the stability tracking, mask, discretization of the field inhomogeneity interval, data costs, median value, and the regularization parameter should have been computed. Meanwhile, in order to maximally exploit the performance of MATLAB, vectorization has been applied during the computation of smoothness cost.

```matlab
function Fieldmap=icm_st(dCost, init, mask, rr, Wt, mu, dfir, eps)
% dCost: the precomputed data cost of all pixels, nrow * ncol * dfir
% init: the value for uniform initialization
% mask: padded background mask with size (nrow+2*radius)-by-(ncol+2*radius)
% rr: the radius of neighborhood system
% WeightM: the weight matrix with size (2*size+1)-by-(2*size+1)
% mu: the regularization parameter
% dfir: discretized field inhomogeneity range
% eps: the convergence level, usually set to 0.001
```
dsteps = length(dfir); % discretized steps
nrow = size(mask, 1) - 2*rr; % number of rows of image
ncol = size(mask, 2) - 2*rr; % number of cols of image
totalPixels = nnz(mask); % number of foreground pixels
boundary = getBounds(mask, rr); % to adjust the weight

% median initialization
map_new = mask; map_new(rr+1:nrow+rr, rr+1:ncol+rr)=init;

map_temp=repmat(dfir, (2*rr)^2, 1);
map_temp=reshape(map_temp, 2*rr, 2*rr*(dsteps+1));
Wt_temp_ker=repmat(Wt * mu, 1, dsteps+1);

% set up AIMD parameters
stable = 0 * mask; alpha = 0; beta = 1; cus_thr = 2;

% initially, no pixel stable
rins = 1;

% ICM iterations with stability tracking
while rins > eps
    map_cur = map_new;
    for jj = rr+1: rr+nrow
for \( kk = rr+1: rr+\text{ncol} \)

% only compute on unstable foreground pixels

if \((\text{mask}(jj, kk) \neq 0 \&\& \text{stable}(jj, kk) < \text{cus}_\text{thr})\)

\[
\text{mapTemp} = \text{map_temp} - \text{repmat}(\text{map_cur}(jj-\text{radius}: jj+\text{radius}, ... \\
kk-\text{radius}: kk+\text{radius}), 1, \text{dsteps}+1);
\]

\[
\text{if boundary}(jj, kk) == (2*\text{radius})^2 \\
\quad \text{WTemp} = \text{Wt_temp_ker};
\]

else

\[
\text{boundV} = \text{WeightM} \cdot \text{boundary}(jj-\text{radius}: jj+\text{radius}, ... \\
kk-\text{radius}: k+\text{radius})*((2*\text{radius})^2/\text{boundary}(jj, kk)/2; \\
\quad \text{WTemp} = \text{repmat}(\text{boundV}, 1, \text{dsteps} + 1);
\]

end

\[
\text{tempCost} = \text{WTemp} \cdot \text{abs(}\text{mapTemp});
\]

\[
\text{temp} = \text{reshape(}\text{tempCost}, (2*\text{radius})^2, (\text{dsteps} + 1)));
\]

\[
\text{sCost} = \text{sum(}\text{temp});
\]

\[
\text{totalCost} = \text{dCost}(:, jj-\text{radius}, kk-\text{radius}) + \text{sCost}(::); \\
[\text{pholder, idx}] = \text{min(}\text{totalCost}(::));
\]

\[
\text{map}_{\text{new}}(jj, kk) = \text{dfir}(\text{idx});
\]

% multiplicative decrease if not stable

if \(\text{map}_{\text{new}}(jj, kk) \neq \text{map}_{\text{cur}}(jj, kk)\)

\[
\quad \text{map}_{\text{cur}}(jj, kk) = \text{map}_{\text{new}}(jj, kk);
\]

\[
\quad \text{stable}(jj-\text{radius}: jj+\text{radius}, kk-\text{radius}: k+\text{radius})=... 
\]
alpha * stable(jj-rr: jj+rr, kk-r: kk+rr);
else
    \% additive increase if stable
    stable(jj, kk) = min(stable(jj, kk) + beta, cus_thr);
end
end
end
end

\% update rins
stablePixels = nnz(stable == cus_thr);
rins = (totalPixels - stablePixels)/totalPixels;
end

\% this is final estimate of field map
Fieldmap = map_cur(rr+1: nrow+rr, rr+1: ncol+rr);
end

\%\% a helper function for icm-st
\% to adjust the weight matrix of each pixel
function boundary = getBounds(mask, rr)
    boundary = zeros(size(mask));
    for jj = radius+1: nrow+radius
        for k = radius+1: ncol+radius
            \%
A.2 Code for Median Value Initialization

In this implementation, feasible solutions are first calculated, then phase wrapping is test and phase unwrapping is made if phase wrapping is detected. Finally, the median of the phase unwrapped list, \( \text{medGuess} \), is returned for uniform initialization. Meanwhile, the unambiguous solution matrix, \( \text{basicFieldmap} \), can be returned for block-wise initialization.

\[%\] The median initialization for ICM-ST
\%
\% (1) first compute the unambiguous map
\% (2) make phase wrapping test and do phase unwrapping
\% if necessary

function medGuess = medianinit(image, TEs, mask, rr, dsteps, eps)
\%
\% image: the mri source images
\% TEs: the echo time shifts
\% mask: the mask with pad rr
% rr: the pad size
% dsteps: the discretizing steps for field map interval
% eps: some value less than 0.15

%% part 1: parameter initialization

T=(TEs(2)-TEs(1))*1.0e-3; ELTA=(2*pi)/(3*T);
lambda=exp(1i*DELTA*T);

nrow = size(mask, 1)-2*rr;
ncol=size(mask, 2)-2*rr;
nsamples=size(image, 3);

%% part 2: compute sylvester roots

alignedRoot=zeros(nrow, ncol);
oneRoot=zeros(nrow, ncol);

crImage=conj(image(:, :, nsamples:-1:1));

auxSyl=[lambda, -(1+lambda), 1];
xImage=zeros(size(image));

for jj = 1:3
xImage(:, :, jj)=image(:, :, jj) * auxSyl(jj);
crImage(:, :, jj) = crImage(:, :, jj) * auxSyl(jj);
end

zr = zeros(nrow, ncol, 1); Syl_row1 = cat(3, xImage, zr);
Syl_row2 = cat(3, zr, xImage);
Syl_row3 = cat(3, crImage, zr);
Syl_row4 = cat(3, zr, crImage);
Syl = cat(4, Syl_row1, Syl_row2, Syl_row3, Syl_row4);
alg_init = zeros(nrow, ncol);

for jj = 1: nrow
    for kk = 1: ncol
        if mask(jj+rr, kk+rr) ~= 0
            [U, Sigma, V] = svd(squeeze(Syl(jj, kk, :, :)).');
            Sigma(4,4) = 0;
            if Sigma(3,3) < eps * Sigma(1, 1)
                Sigma(3,3) = 0; r = 2;
            else
                r = 1;
            end
            [L, U] = lu(U * Sigma * V');
            root = roots(U(4 - r, :));
        end
    end
end
if length(root) == 1
    oneRoot(jj, kk) = 1;
    alignedRoot(jj, kk) = root;
    alg_init(jj, kk) = angle(root);
else
    alg_init(jj, kk) = max(angle(root));
end
end
end

basicFieldmap = angle(alignedRoot)/(2*pi*T);

%% phase unwrapping of basicFieldmap
n1root = nnz(oneRoot); index = 1; list = zeros(n1root, 1);

for rr = 1: nrow
    for cc = 1: ncol
        if (oneRoot(rr, cc) == 1)
            list(index) = basicFieldmap(rr, cc);
            index = index + 1;
        end
    end
end
total = nnz(list); negatives = nnz(list < 0);
positives = nnz(list > 0);

tor_coef = 5;
step = (2/T)/dsteps * tor_coef;
pValue = 0.10;

calc = (nnz(list <= step) - nnz(list < -step))/total;

if (calc < pValue)
    if (negatives > positives)
        list(list > 0) = list(list > 0) - 1/T;
    else
        list(list < 0) = list(list < 0) + 1/T;
    end
end

medGuess = median(list);
Bibliography


