STRUCTURED MAINTENANCE POLICIES ON
INTERIOR SAMPLE PATHS

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To my family.
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Structured Maintenance Policies on Interior Sample Paths

Abstract

by

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This dissertation has three chapters related to the maintenance optimization.

In the first chapter, we examine the problem of adaptively scheduling perfect inspections and preventive replacement for a multi-state, Markovian deterioration system with silent failures, such that total expected discounted cost is minimized. We model this problem as a partially observed Markov decision process for which three actions - replace, do nothing, perfectly inspect - are available at each decision epoch, and establish structural properties of the optimal policy for certain non-extreme sample paths.

The second chapter also considers discrete-time, Markovian deterioration system. However, while in the first chapter, we consider maintenance action with perfect outcome, that is, as a result of replacement, the system transits to “as
good as new state”, in the second chapter, we explore optimal policy structure for such a system in the case of stochastic repair. Particularly, we consider three types of stochastic repair. Problem 1 considers replacement of the existing system by a refurbished one, the deterioration state of which is determined upon inspection following the replacement. The distribution of the initial state of the refurbished system does not depend on the deterioration state of the existing system. That is, Problem 1 considers state independent, stochastic repair with observable repair outcome. Problem 2 considers state dependent, stochastic repair with observable repair outcome. Problem 3 considers state dependent, stochastic repair with unobservable repair outcome. For all three problems we investigate the structure of optimal maintenance policy along non-extreme sample paths.

In the third chapter, we consider discrete-time, nonstationary Markovian deterioration system with the same set of actions as in the first chapter. We investigate the structure of the optimal maintenance policy for such a system by minimizing the expected total discounted cost over an infinite horizon.
Chapter 1

Structured Maintenance Policies for Systems with Perfect Repair and Silent Failures

1.1 Introduction

Consider a production system that degrades over time due to usage and age. Because the operating and replacement costs for such systems tend to increase in the deterioration level, properly timed preventive maintenance actions can reduce total cost. Furthermore, a priori statements about the structure of such preventive maintenance policies can both accelerate their computation and ease their implementation. In this spirit, we establish structural properties of the optimal
policy that minimizes the total expected discounted cost over the infinite horizon for a system that undergoes discrete-time, finite-state Markovian deterioration with three actions available in each decision epoch: replace, produce without inspection (i.e., “no action”), and produce with a costly, perfect inspection. As a result of the replace action, the system returns to an “as good as new” state. The other two actions do not alter the deterioration state, however the inspections reveal the current deterioration state with certainty.

Other authors who consider this problem ([18], [17], [20], [22]) differ in their choice of state definition, but establish the same structural properties of the optimal policy along ordered subsets of their respective state spaces. We extend these results by introducing a new state definition and establishing the same structural properties for ordered subsets of states not previously considered.

The rest of the chapter is organized as follows. In Section 1.2, we formulate a partially observed Markov decision process (POMDP) model of the problem and articulate our contribution to the literature in detail. Section 1.3 provides preliminary results for Sections 1.4 and 1.5, in which we establish structural properties of the optimal policy. Lastly, in Section 1.6 we shortly outline the results of this chapter and our directions for future research.
1.2 Model Formulation and Literature Review

Let the underlying discrete-time Markov chain consist of deterioration states 0 through N, where state 0 is the “new” state and state N is the most deteriorated state. The system deteriorates in each period according to a known transition probability matrix, $P$, with elements $p_{ij}$ for $i, j = 0, \ldots, N$. At any given time period, we define the state of the process to be $(\pi, k)$, meaning that $k$ periods ago, the process was in information state $\pi$, and since then no information about the deterioration state has been obtained and no actions have been taken. The information state $\pi$ has components $[\pi_0, \pi_1, \ldots, \pi_N]$, where $\pi_i$ is the probability that the system is in deterioration state $i$.

In any given state $(\pi, k)$, three actions are available: replacement, no action and inspection. Upon replacement, the process reverts to state $(e_0, 0)$ at the beginning of the next period, where $e_i$ is the $N + 1$ dimensional row-vector with a 1 in the $i$th position and 0 everywhere else. The expected replacement cost incurred is

$$\pi P^k C = \sum_{i=0}^{N} \pi_i \sum_{j=0}^{N} p_{ij}^k C_j,$$

where $C_j$ is the cost of replacing the system in deterioration state $j$ and $p_{ij}^k$ is the $(i, j)$th element of the $k$-step transition matrix $P^k$. Similarly, the expected cost associated with taking no action is

$$\pi P^k L = \sum_{i=0}^{N} \pi_i \sum_{j=0}^{N} p_{ij}^k L_j,$$
where $L_j$ is the per-period operating cost in deterioration state $j$. As a result of this action, the process transitions to state $(\pi, k + 1)$. Lastly, the expected cost of inspection is

$$\pi P^k L + M,$$

where $M$ is the cost of inspection. Upon inspection, which we assume takes one full time period, the process enters state $(e_j, 0)$ with probability $\sum_{i=0}^{N} \pi_i p_{ij}$.

As a result, if $V_n(\pi, k)$ is the minimum total expected discounted cost-to-go with $n$ periods remaining starting in state $(\pi, k)$, then

$$V_n(\pi, k) = \min \begin{cases} 
\pi P^k C + \alpha V_{n-1}(e_0, 0), \\
\pi P^k L + \alpha V_{n-1}(\pi, k + 1), \\
\pi P^k L + M + \alpha\pi P^{k+1} V_{n-1}(e, 0)
\end{cases}.$$  \hspace{1cm} (1.1)

where $\alpha$ is a discount factor, $0 < \alpha < 1$, and $V_n(e, 0) \equiv [V_n(e_0, 0), \ldots, V_n(e_N, 0)]^T$.

It is well known (White [22]) that as $n$ approaches infinity, $V_n(\pi, k)$ converges to

$$V(\pi, k) = \min \begin{cases} 
\pi P^k C + \alpha V(e_0, 0), \\
\pi P^k L + \alpha V(\pi, k + 1), \\
\pi P^k L + M + \alpha\pi P^{k+1} V(e, 0)
\end{cases}.$$  \hspace{1cm} (1.2)

Note that because state $(\pi, k)$ is equivalent to state $(\pi P^k, 0)$, we could redefine the state of the process to be $\pi$ alone without affecting the model. In their analysis of this problem, Ross [20] and White [22] both take this approach. Ross [20] considers the special case in which there are only two deterioration states. He assumes that the cost of production increases in the deterioration level, but that
the costs of inspection and replacement are independent of the underlying deterioration state. Under these conditions, he shows that if the transition probability matrix is upper-triangular (i.e., the system cannot improve on its own), then the optimal policy consists of at most four regions on the interval \( \pi_1 \in [0, 1] \) in the following order: no action, inspect, no action, replace. We refer to this type of structure as a monotonic, at-most-four-region (AM4R) policy.

White [22] considers the general, \( N \) state case. He assumes that both replacement costs and production costs, as well as their differences, increase in the deterioration state. He establishes the monotonic, AM4R policy structure along straight lines of information states in increasing stochastic order, under the condition that \( P \) exhibits increasing failure rate (IFR), i.e., its rows are in increasing stochastic order.

Rosenfield [18] also considers the general, \( N \) state case, but defines the state of the process to be \((i, k)\), meaning that \( k \) periods ago the system was known to be in deterioration state \( i \) and since then no actions have been taken. The state \((i, k)\) corresponds to our state \((e_i, k)\) and to White’s state \(e_i P^k\). Rosenfield assumes that \( P \) is upper-triangular, has only one recurrent state (namely \( N \)) and is totally positive of order two (\( TP_2 \)), i.e., all of its \(2 \times 2\) determinants are nonnegative. Using the same cost assumptions as in White [22], Rosenfield establishes the monotonic, AM4R structure in \( k \) for every deterioration state \( i \). In a subsequent paper [17], he eliminates the assumptions that \( P \) is \( TP_2 \) and upper-triangular, and instead
assumes that $P$ is IFR and satisfies a new, fairly complex condition. Under these assumptions on the transition probability matrix, Rosenfield [17] proves that the optimal policy is monotone with respect to replacement in $k$ for all $i$.

Essentially, in both papers, Rosenfield establishes the structure of the optimal policy along sample paths emanating from extreme points of the state space

$$\Omega \equiv \{ \pi : 0 \leq \pi_i \leq 1, \sum_{i=0}^{N} \pi_i = 1 \},$$

namely $e_i$, $i = 0, 1, \ldots, N$. By a sample path, we mean the sequence of information states that the process would occupy over time if the decision maker takes no action. For example, the sample path emanating from information state $\pi$ is \{ $\pi, \pi P, \pi P^2, \ldots$ \}, or equivalently, \{( $\pi, 0), (\pi, 1), (\pi, 2), \ldots$ \}. Policy structure along a sample path, as opposed to along a line of information states, has intuitive appeal since such results are inherently time-based.

Although he does not explicitly treat the same model of interest here, Lovejoy [9] emphasizes that Rosenfield’s sample-path-based approach to establishing policy structure may be an interesting direction for future research. Toward that end, we extend Rosenfield’s results to sample paths that emanate from non-extreme information states in $\Omega$. That is, we characterize subsets of $\Omega$ such that sample paths emanating from information states in these subsets exhibit the same structure established in Rosenfield [18] and [17].

More specifically, in Section 1.5 we show that the optimal policy exhibits the monotonic, AM4R structure along certain non-extreme sample paths under a
subset of the assumptions used in Rosenfield [18]. Definition 1 provides a more formal characterization of a sample path with such policy structure:

**Definition 1.** Consider a sample path of states \((\pi, k), k = 0, 1, \ldots\). A monotonic, at most four-region policy along this sample path consists of three numbers \(k_1^*(\pi) \leq k_2^*(\pi) \leq k^*(\pi)\), and prescribes replacement in state \((\pi, k)\) if \(k \geq k^*(\pi)\), inspection if \(k_1^*(\pi) \leq k \leq k_2^*(\pi)\) and no action otherwise.

However, before presenting this result, we establish (in Section 1.4) the existence of a replacement control limit, \(k^*(\pi)\), for certain non-extreme sample paths under a subset of the less restrictive conditions used in Rosenfield [17].

There are some recent research results that consider POMDP and find some structural results for the optimal policy. In [5], the system with only two states, “as good as new” and bad, is considered. There are two actions, do nothing and replace, available. The true state of the system is unknown but could be probabilistically inferred through the quality of output. For such a system the conditions under which the optimal policy is monotone with respect to replacement are established.

In [8], authors consider a discrete-time Markov deterioration system with general \(N\) deterioration states for which at each decision epoch three actions available replace, do nothing, inspect. While replace and inspect actions are perfect and exactly the same as we consider in our paper, do nothing is different and means
that operated system is imperfectly monitored. Cost of replacement and inspection are constant, cost of do nothing depends on deterioration state of the system. Conditions under which the optimal policy has $AM4R$ policy structure are established. However, structure of the optimal policy is not time-based as we consider in this chapter.

Admittedly, because the deterioration state is known with certainty following replacement or inspection, the system will only occupy an information state on a non-extreme sample path if it starts in such a state at time zero. That is, after the first replacement or inspection action, which results in an extreme point information state, the system can never return to a non-extreme sample path. However, establishing structure along non-extreme sample paths is worthwhile not only for instances of this situation, but perhaps more importantly, to gain insights that might be useful in establishing similar results for systems with imperfect information due to obvious failures, imperfect replacements and/or imperfect inspections for which few results exist (Lovejoy [9]).

### 1.3 Preliminary Definitions and Results

In this section, we present preliminary results needed to establish the structure of the optimal policy along non-extreme sample paths in Sections 1.4 and 1.5. We begin with several definitions:
Definition 2. Information state $\pi'$ is stochastically larger than information state $\pi$, $\pi \prec_{st} \pi'$, if and only if $\sum_{i=m}^{N} \pi_i \leq \sum_{i=m}^{N} \pi'_i$ for all $m = 0, 1, 2, \ldots, N$.

Definition 3. Information state $\pi'$ is larger in the sense of likelihood ratio than information state $\pi$, $\pi \prec_{lr} \pi'$, if and only if $\pi_i \pi'_j - \pi_j \pi'_i \geq 0$ for all $j \geq i$.

Definition 4. A stochastic matrix $P$ is IFR if

$$\sum_{j=m}^{N} p_{i_2 j} \leq \sum_{j=m}^{N} p_{i_1 j}$$

for all $i_2 \geq i_1$ and $m = 0, 1, \ldots, N$.

Definition 5. A stochastic matrix $P$ is totally positive of order two (TP$_2$) if

$$\begin{vmatrix} p_{i_1 j_1} & p_{i_1 j_2} \\ p_{i_2 j_1} & p_{i_2 j_2} \end{vmatrix} \geq 0$$

for all $i_2 \geq i_1$ and $j_2 \geq j_1$.

Our two main results depend on the monotonicity of the value function along sample paths with stochastically or likelihood-ratio ordered information states, i.e., $\pi P^k$ increasing (in the sense of $\prec_{st}$ or $\prec_{lr}$) in $k$. It is imperative, therefore, that we be able to characterize information states that spawn such sample paths. Toward that end, part (i) of Proposition 1.3.3 establishes that any sample path that starts in increasing stochastic order, i.e., emanates from an information state $\pi \in \Omega_{\pi \prec_{st} \pi P} \equiv \{\pi \in \Omega : \pi \prec_{st} \pi P\}$, remains in increasing stochastic order as long as $P$ is IFR. Part (ii) of Proposition 1.3.3 establishes the analogous result.
for sample paths emanating from $\pi \in \Omega_{\pi<_{\ell r}\pi P} \equiv \{\pi \in \Omega : \pi <_{\ell r} \pi P\}$ and $TP_2$ matrices. As a consequence of the former result, given that $P$ is $IFR$, the set $\Omega_{\pi<_{st}\pi P}$ (respectively, $\Omega_{\pi<_{\ell r}\pi P}$) actually contains all of the information states that reside on sample paths whose information states are in increasing stochastic (likelihood ratio) order. It is precisely the sample paths in these two subsets of $\Omega$, therefore, to which the subsequent results on policy structure in Sections 1.4 and 1.5 apply. Before presenting Proposition 1.3.3, we state several well-known (cf. Rosenfield [18], White [22], Ohnishi [13]) results via Propositions 1.3.1 and 1.3.2.

**Proposition 1.3.1.**

(i) If $P$ is $IFR$, then $P^k$ is $IFR$ for any $k = 0, 1, \ldots$.

(ii) If $P$ is $TP_2$, then $P$ is $IFR$.

**Proposition 1.3.2.**

(i) If $P$ is $IFR$ and $\pi <_{st} \pi'$, then $\pi P^k <_{st} \pi' P^k$ for $k = 0, 1, \ldots$.

(ii) If $P$ is $TP_2$ and $\pi <_{\ell r} \pi'$, then $\pi P^k <_{\ell r} \pi' P^k$ for $k = 0, 1, \ldots$.

**Proposition 1.3.3.**

(i) If $P$ is $IFR$, then for any $\pi \in \Omega_{\pi<_{st}\pi P}$,

$$\pi <_{st} \pi P <_{st} \pi P^2 <_{st} \ldots .$$

Furthermore, if $P$ has a single recurrent class, then

$$\pi P^k <_{st} \Pi \equiv \lim_{n \to \infty} \pi P^n$$
for all $k$.

(ii) If $P$ is $TP_2$, then for any $\pi \in \Omega_{\pi \prec_{\ell r} P}$,

$$
\pi \prec_{\ell r} P \prec_{\ell r} \pi P \prec_{\ell r} \pi P^2 \prec_{\ell r} \ldots
$$

Furthermore, if $P$ has a single recurrent class, then

$$
\pi P^k \prec_{\ell r} \Pi
$$

for all $k$.

Proof.

(i) Since $\pi \prec_{st} P$ for $\pi \in \Omega_{\pi \prec_{st} P}$ by definition, applying Proposition 1.3.2 (i)

repeatedly to both sides of this inequality yields

$$
\pi \prec_{st} P \prec_{st} \pi P \prec_{st} \pi P^2 \prec_{st} \ldots
$$

(1.3)

The existence of the steady-state probability distribution $\Pi$ is ensured by the condition that the Markov chain has a single recurrent class. Hence $\pi P^k \prec \pi P^\infty$ for all $k$ and $\pi P^\infty = \Pi$.

(ii) This result is established similarly to part (i) by applying Proposition 1.3.2 (ii).

Note that Proposition 1.3.3 also indicates that if $P$ is $IFR$, then $\pi \prec_{st} \Pi$ for all $\pi \in \Omega_{\pi \prec_{st} P}$, or in other words, all information states on $\prec_{st}$-increasing sample
paths are less deteriorated than the steady-state distribution in the sense of the \( \prec_{\text{st}} \)-ordering. This deduction holds for \( \pi \in \Omega_{\pi \prec_{\ell_r} \pi P} \) as well, since as a consequence of Proposition 1.3.1 part \((ii)\), \( \Omega_{\pi \prec_{\ell_r} \pi P} \subset \Omega_{\pi \prec_{\text{st}} \pi P} \).

In general, \( \Omega_{\pi \prec_{\text{st}} \pi P} \) can be identified by solving the system of inequalities
\[
\sum_{i=m}^{N} \pi_i \leq \sum_{i=m}^{N} (\pi P)_i, \quad m = 0, 1, \ldots, N - 1,
\]
plus the normalizing equation
\[
\sum_{i=0}^{N} \pi_i = 1.
\]
Similarly, \( \Omega_{\pi \prec_{\ell_r} \pi P} \) can be identified by solving the system of inequalities
\[
\begin{vmatrix}
\pi_{i_1} & \pi_{i_2} \\
(\pi P)_{i_1} & (\pi P)_{i_2}
\end{vmatrix} \geq 0
\]
for all \( i_1 \leq i_2, \ i_1, i_2 = 0, 1, \ldots, N, \) and \( \sum_{i=0}^{N} \pi_i = 1. \) For a more intuitive understanding of these ideas, consider a numerical example with \( N = 2 \) and IFR transition probability matrix
\[
P = \begin{pmatrix}
0.9 & 0.09 & 0.01 \\
0.025 & 0.8 & 0.175 \\
0 & 0.05 & 0.95
\end{pmatrix}. \tag{1.4}
\]

The simplexes in Figure 1.1 represent the set of all information states, \( \Omega. \) The shaded portion of part (a) of Figure 1.1 depicts \( \Omega_{\pi \prec_{\text{st}} \pi P} \) for the Markov chain given by equation (1.4); part (b) depicts \( \Omega_{\pi \prec_{\ell_r} \pi P}. \) The steady-state probability distribution for this Markov chain is \( \Pi = [0.052, 0.208, 0.740]. \) The relationship
\( \Omega_{\pi \prec_{\ell_r} \pi P} \subset \Omega_{\pi \prec_{\text{st}} \pi P} \) is clear, however, in contrast to the boundaries of \( \Omega_{\pi \prec_{\text{st}} \pi P}, \) the
boundaries of $\Omega_{\pi, \prec_{\pi} P}$ and $\Omega_{\pi, \prec_{\ell r} P}$ are nonlinear. The region above and to the right of the dashed line in Figure 1.1 parts (a) and (b) is the set of information states that are stochastically smaller than the steady-state probability distribution, $\Pi$. As stated previously, $\Omega_{\pi, \prec_{\pi} P}$ and $\Omega_{\pi, \prec_{\ell r} P}$ are subsets of this region, meaning that all information states on $\prec_{\pi}$- or $\prec_{\ell r}$-increasing sample paths are less deteriorated than $\Pi$.

When we describe a function to be $\prec_{\pi}$-nonincreasing (or $\prec_{\pi}$-nondecreasing) in $\pi$, we mean that the function is nonincreasing (or nondecreasing) along any set of $\pi$ in increasing stochastic order (we use $\prec_{\ell r}$-nonincreasing and $\prec_{\ell r}$-nondecreasing similarly). In Section 1.4, to establish conditions under which the value function is $\prec_{\pi}$-nonincreasing in $\pi$ and nondecreasing in $k$, we rely on Lemma 1.3.5, which establishes the monotonicity of a particular function that appears repeatedly in the value function. Lemma 1.3.5 requires the well-known (cf. Lemma 4.7.2 in Puterman [16]) result presented in Proposition 1.3.4, namely that the product of
an IFR transition probability matrix and a vector with nonincreasing components also has nonincreasing components.

**Proposition 1.3.4.** If P is IFR and $F_j$ is nonincreasing in $j$, $j = 0, \ldots, N$, then $(PF)_i = \sum_{j=0}^{N} p_{ij} F_j$ is also nonincreasing in $i$.

**Lemma 1.3.5.** If P is IFR and $F_j$ is nonincreasing in $j$, then $\pi P^k F$ is

(i) $\prec_{st}$-nonincreasing in $\pi$ and

(ii) nonincreasing in $k$ for $\pi \in \Omega_{\pi \prec_{st} \pi P}$.

**Proof.** First, note that by Proposition 1.3.1 part (i), $P^k$ is also IFR.

(i) Consider the sum $a_i = \sum_{j=0}^{N} p_{ij}^k F_j$, which enables us to express $\pi P^k F$ as

$$
\pi P^k F = \sum_{i=0}^{N} \pi_i \sum_{j=0}^{N} p_{ij}^k F_j
$$

$$
= \sum_{i=0}^{N} \pi_i a_i
$$

$$
= \sum_{j=0}^{N} \pi_j \sum_{i=0}^{j} (a_i - a_{i-1})
$$

$$
= \sum_{i=0}^{N} (a_i - a_{i-1}) \sum_{j=i}^{N} \pi_j
$$

where $a_{-1} = 0$. For two stochastically ordered information states $\pi \prec_{st} \pi'$,
equation (1.5) yields

$$\pi P^k F = \sum_{i=0}^{N} (a_i - a_{i-1}) \sum_{j=i}^{N} \pi_i$$

$$\geq \sum_{i=0}^{N} (a_i - a_{i-1}) \sum_{j=i}^{N} \pi'_i$$

$$= \sum_{i=0}^{N} \pi'_i a_i$$

$$= \pi' P^k F,$$

where inequality (1.6) follows from the fact that $a_i - a_{i-1} \leq 0$ and Proposition 1.3.4.

(ii) By Proposition 1.3.3 part (i), $\pi P^k \prec_{st} \pi P^{k+1}$. Hence, by Proposition 1.3.4,

$$\pi P^k F \geq \pi P^{k+1} F.$$

\[\square\]

1.4 A Replacement Control Limit Result

In this section, we establish that the optimal policy is monotone with respect to replacement along any $\prec_{st}$-increasing sample path, i.e., for any stochastically increasing sample path $\{(\pi, 0), (\pi, 1), \ldots, (\pi, k), \ldots\}$ there exists a $k^*(\pi)$ such that in all states $(\pi, k)$ for which $k \geq k^*(\pi)$, replacement is optimal. We establish this property of the optimal policy under the following set of assumptions:

A1. $C_j$ and $L_j$ are nondecreasing in $j$;
2. $C_j - L_j$ is nonincreasing in $j$;

3. there is a number $\delta > 0$ and an integer $k$ such that $\min_{0 \leq i \leq N} p_i^k \geq \delta$;

4. $P$ is IFR.

Assumption A1 states that as the system deteriorates it becomes more costly to operate and replace. According to Assumption A2, operating costs increase faster than replacement costs. Assumption A3 ensures the existence of the steady-state probability distribution, $\Pi$. Assumption A4 implies that the more the system deteriorates, the more likely it is to deteriorate further. To prove the existence of replacement control limits along the extreme sample paths, Rosenfield [17] imposes the same four assumptions, plus a fifth assumption that, as we emphasize following Remark 1, holds for all $\prec_{st}$-increasing sample paths.

As stated in Section 1.3, the results concerning policy structure depend on the monotonicity of the value function. Part (i) of Lemma 1.4.1 establishes that under the optimal policy, given a fixed time $k$ since the last inspection or replacement, the system is more costly to operate in stochastically larger (i.e., more deteriorated) states. Part (ii) of Lemma 1.4.1 argues that as a consequence, since the information states on sample paths in the region $\Omega_{\pi \prec_{st} \pi} P$ stochastically increase over time, the cost function is also nondecreasing in time when the process begins in an information state in this region.

**Lemma 1.4.1.** Under Assumption A1 and Assumption A4, $V(\pi, k)$ is
(i) \( \prec_{st} \)-nondecreasing in \( \pi \) for all \( k \);

(ii) nondecreasing in \( k \) for all \( \pi \in \Omega_{\pi \prec_{st} \pi P} \).

Proof. Since \( V_n(\pi, k) \) converges to \( V(\pi, k) \) as \( n \to \infty \) (White [22]), we proceed by induction on \( n \).

(i) Consider two information states \( \pi \prec_{st} \pi' \). For \( n = 1 \)

\[
V_1(\pi, k) = \min\{\pi P^k C, \pi^P L\} \\
\leq \min\{\pi' P^k C, \pi' P^k L\} \quad \text{(1.7)} \\
= V_1(\pi', k),
\]

where inequality (1.7) follows from Lemma 1.3.5 part (i). Assuming that

\[
V_2(\pi, k) \leq V_2(\pi', k), \ldots, V_{n-1}(\pi, k) \leq V_{n-1}(\pi', k),
\]

we need only prove that \( V_n(\pi, k) \) is \( \prec_{st} \)-nondecreasing in \( \pi \). By the induction assumption and Lemma 1.3.5 part (i) again,

\[
V_n(\pi, k) = \min \left\{ \begin{array}{l}
\pi P^k C + \alpha V_{n-1}(e_0, 0), \\
\pi P^k L + \alpha V_{n-1}(\pi, k + 1), \\
\pi P^k L + M + \alpha \pi P^{k+1} V_{n-1}(e, 0)
\end{array} \right\} \\
\leq \min \left\{ \begin{array}{l}
\pi' P^k C + \alpha V_{n-1}(e_0, 0), \\
\pi' P^k L + \alpha V_{n-1}(\pi', k + 1), \\
\pi' P^k L + M + \alpha \pi' P^{k+1} V_{n-1}(e, 0)
\end{array} \right\} \\
= V_n(\pi', k).
\]
(ii) For \( n = 1 \) and \( \pi \in \Omega_{\pi \prec_{st} \pi P} \),

\[
V_1(\pi, k) = \min \{\pi P^k C, \pi P^k L\} \\
\leq \min \{\pi P^{k+1} C, \pi P^{k+1} L\} \\
= V_1(\pi, k + 1),
\]

where inequality (1.8) follows from Lemma 1.3.5 part (ii). Assuming that \( V_2(\pi, k), \ldots, V_{n-1}(\pi, k) \) are nondecreasing in \( k \) for \( \pi \in \Omega_{\pi \prec_{st} \pi P} \), we need only prove that \( V_n(\pi, k) \) is nondecreasing in \( k \) for \( \pi \in \Omega_{\pi \prec_{st} \pi P} \). By Lemma 1.3.5 part (ii) and the fact that part (i) of this lemma implies \( V_{n-1}(e_0, 0) \leq V_{n-1}(e_1, 0) \leq \ldots \leq V_{n-1}(e_N, 0) \),

\[
V_n(\pi, k) \leq \min \left\{ \begin{array}{l}
\pi P^{k+1} C + \alpha V_{n-1}(e_0, 0), \\
\pi P^{k+1} L + \alpha V_{n-1}(\pi, k + 1), \\
\pi P^{k+1} L + M + \alpha \pi P^{k+2} V_{n-1}(e, 0)
\end{array} \right\}
\]

for \( \pi \in \Omega_{\pi \prec_{st} \pi P} \). By the induction assumption, \( V_{n-1}(\pi, k + 1) \leq V_{n-1}(\pi, k + 2) \), in which case

\[
V_n(\pi, k) \leq \min \left\{ \begin{array}{l}
\pi P^{k+1} C + \alpha V_{n-1}(e_0, 0), \\
\pi P^{k+1} L + \alpha V_{n-1}(\pi, k + 2), \\
\pi P^{k+1} L + M + \alpha \pi P^{k+2} V_{n-1}(e, 0)
\end{array} \right\}
\]

\[= V_n(\pi, k + 1). \]
Next we present Theorem 1.4.2, analogous to Theorem 1 in Rosenfield [17] in which he proves that the optimal policy is monotone with respect to replacement along the extreme sample paths of $\Omega$; we prove that this structure also holds for any sample path that originates from an information state in the set $\Omega_{\pi \prec_{st} \pi P}$. Furthermore, the point at which the optimal action switches to replacement is $\prec_{st}$-nonincreasing in $\pi$, meaning that the more deteriorated the initial state of the system, the sooner replacement becomes optimal.

First, consider the two functions

$$F(\pi, k) \equiv \pi P^k [C - L] + \alpha [V(e_0, 0) - V(\pi, k + 1)], \text{ and}$$

$$G(\pi, k) \equiv \pi P^k [C - L] + \alpha [V(e_0, 0) - \pi P^{k+1} V(e, 0)] - M. \quad (1.10)$$

Here, $F(\pi, k)$ denotes the cost of replacement minus the cost of no action in state $(\pi, k)$, and $G(\pi, k)$ denotes the cost of replacement minus the cost of inspection in state $(\pi, k)$. The functions $F(\pi, k)$ and $G(\pi, k)$ converge to $F^*$ and $G^*$, respectively, as $k \to \infty$ for all $\pi \in \Omega$, where

$$F^* = \lim_{k \to \infty} F(\pi, k) = \Pi (C - L) + \alpha [V(e_0, 0) - V(\Pi, \cdot)], \text{ and} \quad (1.11)$$

$$G^* = \lim_{k \to \infty} G(\pi, k) = \Pi (C - L) - M + \alpha V(e_0, 0) - \alpha \Pi V(e, 0). \quad (1.12)$$

(Rosenfield [17]). Here, $F^*$ represents the difference between the cost of replacement and the cost taking no action in information state $\Pi$; similarly $G^*$ represents the difference between the cost of replacement and the cost of inspection in information state $\Pi$. Since $\Pi P^k = \Pi$ for all $k = 0, 1, \ldots$, we use $V(\Pi, \cdot)$ here instead
of $V(\Pi, \infty)$.

**Theorem 1.4.2.** If $\Pi(C - L) \leq 0$ and Assumptions A1-A4 hold, then for all $\pi \in \Omega_{\pi \prec st \pi_P}$ there exists a critical number $k^*(\pi)$, $\prec_{st}$-nonincreasing in $\pi$, such that in state $(\pi, k)$ replacement is optimal if $k \geq k^*(\pi)$, and either inspection or no action is optimal otherwise.

**Proof.** By Assumption A2 and Lemma 1.3.5, $\pi P^k[C - L]$ is $\prec_{st}$-nonincreasing in $\pi$ and nonincreasing in $k$ for $\pi \in \Omega_{\pi \prec st \pi_P}$. Furthermore, by Lemma 1.4.1, $[V(e_0, 0) - V(\pi, k + 1)]$ is $\prec_{st}$-nonincreasing in $\pi$ and nonincreasing in $k$ for $\pi \in \Omega_{\pi \prec st \pi_P}$. Therefore, $F(\pi, k)$ is $\prec_{st}$-nonincreasing in $\pi$ and nonincreasing in $k$ for $\pi \in \Omega_{\pi \prec st \pi_P}$, as the sum of these two terms. Similarly, $G(\pi, k)$ is $\prec_{st}$-nonincreasing in $\pi$ and nonincreasing in $k$ for $\pi \in \Omega_{\pi \prec st \pi_P}$.

Rosenfield [17] argues that under the (sufficient) condition,

$$\Pi(C - L) \leq 0,$$

it is optimal to replace in state $\Pi$, in which case $F^* < 0$ and $G^* < 0$. Therefore, due to the monotonicity of $F(\pi, k)$ in $k$, for all $\pi \in \Omega_{\pi \prec st \pi_P}$ there exists a $\hat{k}(\pi) = \min \{k : F(\pi, k) < 0\}$ which implies that for $k \geq \hat{k}(\pi)$, no action is more costly than replacement. Similarly, there exists a $\hat{k}(\pi) = \min \{k : G(\pi, k) < 0\}$, which implies that for $k \geq \hat{k}(\pi)$, inspection is more costly than replacement. Hence, there exists a $k^*(\pi) = \max\{\hat{k}(\pi), \hat{k}(\pi)\}$, such that for $k \geq k^*(\pi)$ both $F(\pi, k)$ and $G(\pi, k)$ are negative, meaning replacement is optimal.
Finally, that this replacement critical number, \( k^*(\pi) \), is \( \prec_{st} \)-nonincreasing in \( \pi \) follows directly from the monotonicity of \( F(\pi, k) \) and \( G(\pi, k) \) in \( \pi \).

\( \square \)

**Remark 1.**

(a) Theorem 1.4.2 also holds, with \( k^*(\pi) = 0 \), for all \( \pi \succ_{st} \Pi \) (White [22]).

That is, it is optimal to replace in any information state that is “more deteriorated” than \( \Pi \).

(b) Theorem 1.4.2 also holds for all information states \( e_iP^k \), \( i = 0, 1, \ldots, k = 0, 1, \ldots \) (Rosenfield [17]) as long as, in addition to Assumptions A1-A4, the following holds: for any vector \( F \) with nonincreasing components \( F_j, j = 0, \ldots, N \),

\[
e_iPF \leq \max\{e_iF, \Pi F\}, \quad i = 0, 1, \ldots, N. \quad (1.13)
\]

To understand why a non-extreme sample path version of condition (1.13) is not needed in Theorem 1.4.2, note that since \( \pi \prec_{st} \Pi \) for all \( \pi \in \Omega_{\pi \prec_{st} \pi P} \) (Proposition 1.3.3 part (i)), \( \pi F \geq \Pi F \) for all \( \pi \in \Omega_{\pi \prec_{st} \pi P} \) for any nonincreasing vector \( F \) (by Lemma 1.4.1). That is, \( \max\{\pi F, \Pi F\} = \pi F \) for information states in this region. Applying Lemma 1.4.1 again yields, \( \pi F \geq \pi PF \geq \Pi F \) for all \( \pi \in \Omega_{\pi \prec_{st} \pi P} \). Clearly then, any information state \( \pi \in \Omega_{\pi \prec_{st} \pi P} \) satisfies the non-extreme sample path version of condition (1.13) (where \( e_i \) is replaced by \( \pi \)), which is why it is not necessary to invoke this rather nonintuitive condition in our proof.
of Theorem 1.4.2.

To conclude this section, we provide the following Corollary to Theorem 1.4.2, which establishes that if the system cannot improve on its own, i.e. $P$ is upper-triangular, then a replacement control limit exists along every sample path.

**Corollary 1.4.3.** If $C_N - L_N \leq 0$, Assumptions A1-A4 hold and $P$ is uppertriangular, then for all $\pi \in \Omega$ there exists a critical number $k^*(\pi)$, $\prec_{st}$-nonincreasing in $\pi$, such that in states $(\pi, k)$ replacement is optimal if $k \geq k^*(\pi)$, and either inspection or no action is optimal otherwise.

**Proof.** Since matrix $P$ is uppertriangular, the condition $\Pi(C - L) \leq 0$ of Theorem 1.4.2 is now equivalent to $C_N - L_N \leq 0$. Now, to prove that results of Theorem 1.4.2 hold, it is sufficient to show that under these conditions, $\Omega_{\pi \prec_{st} \pi P} = \Omega$, or that

$$\sum_{j=m}^{N} \pi_j \leq \sum_{j=m}^{N} \sum_{i=0}^{N} \pi_i p_{ij} \text{ for all } m = 0, 1, ..., N.$$
for all \( \pi \). Manipulating the right hand side of this inequality yields

\[
\sum_{j=m}^{N} \sum_{i=0}^{N} \pi_i p_{ij} = \pi_0 \sum_{j=m}^{N} p_{0j} + \pi_1 \sum_{j=m}^{N} p_{1j} + \ldots + \pi_N \sum_{j=m}^{N} p_{Nj} \\
= \pi_0 \sum_{j=m}^{N} p_{0j} + \pi_1 \sum_{j=m}^{N} p_{1j} + \ldots + \pi_{m-1} \sum_{j=m}^{N} p_{m-1,j} + \pi_m + \ldots + \pi_N
\]

(1.14)

\[
\geq \pi_m + \ldots + \pi_N \\
= \sum_{j=m}^{N} \pi_j,
\]

where equation (1.14) follows from the fact that

\[
\sum_{j=m+1}^{N} p_{m+1,j} = \sum_{j=m+2}^{N} p_{m+2,j} = \ldots = p_{NN} = 1
\]

for \( P \) uppertriangular.

\[\Box\]

### 1.5 Monotonic, At Most Four Region Policy Structure

In this section, we strengthen our assumption regarding \( P \) and show that the optimal policy has an appealing monotonic, AM4R structure along \( \prec_{\ell_r} \)-increasing sample paths. As a corollary to this result, we show that the more deteriorated the initial condition of the system, the sooner inspection becomes optimal. We retain assumptions \( A1, A2, \) and \( A3 \) of Section 1.4, but replace \( A4 \) with a stronger assumption, namely:
A4′. $P$ is totally positive of order two ($TP_2$).

Theorem 1.5.4 proves the existence of the critical numbers $k_1^*(\pi)$ and $k_2^*(\pi)$ that define the optimal inspection region along any $\prec_{\ell r}$-increasing sample path, which ensures that the optimal policy exhibits the monotone, AM4R structure along every $\prec_{\ell r}$-increasing sample path. This result relies on the three technical lemmas with proofs presented in Appendix.

**Lemma 1.5.1.** If $P$ satisfies A4′, and $F_i$ has the crossing property in $i$, then $\pi P^k F$ has the crossing property in both $\prec_{\ell r}$-increasing $\pi$ and in $k$ for all $\pi \in \Omega_{\pi \prec_{\ell r} \pi P}$.

**Lemma 1.5.2.** For all $\pi \in \Omega_{\pi \prec_{\ell r} \pi P}$

$$\pi P^k V(e, m) \leq V(\pi, k + m).$$

**Lemma 1.5.3.** For all $f \geq 1$ under Assumptions A1, A2, A3 and A4′,

$$H_f^k(\pi, k) \equiv V(\pi, k) - \pi P^k L - \ldots - \alpha^\ell \pi P^{k+\ell} L - \alpha^\ell M - \alpha^{\ell+1} \pi P^{k+\ell+1} V(e, 0) + f M$$

(1.15)

crosses zero at most once in $\pi \in \Omega_{\pi \prec_{\ell r} \pi P}$, and if it does, it does so from above.

**Theorem 1.5.4.** If $\Pi(C - L)$ and Assumptions A1 – A4′ hold, then a monotone, AM4R policy is optimal along every sample path emanating from an information state $\pi \in \Omega_{\pi \prec_{\ell r} \pi P}$.

**Proof.** By Proposition 1.3.1 part (i), Theorem 1.4.2 holds and we are guaranteed the existence of a finite critical number $k^*(\pi)$ for all $\pi \in \Omega_{\pi \prec_{\ell r} \pi P}$. 

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Let $k^*_2(\pi)$ be as defined in Definition 1, so that in state $(\pi, k)$ for $k^*_2(\pi) < k < k^*(\pi)$, no action is optimal. We need to show that we can separate the interval $[0, k^*_2(\pi)]$ into two subintervals $[0, k^*_1(\pi) - 1]$ and $[k^*_1(\pi), k^*_2(\pi)]$ such that in state $(\pi, k)$ if $k \in [0, k^*_1(\pi) - 1]$ no action is optimal, and if $k \in [k^*_1(\pi), k^*_2(\pi)]$ inspection is optimal. Assume without loss of generality that $k^*_2(\pi) \geq 1$; otherwise $k^*_2(\pi) = 0$ and there is nothing left to prove.

Let $J(\pi, k)$ be the difference between the cost of inspection and the cost of no action in state $(\pi, k)$

$$J(\pi, k) \equiv \alpha \left[ \frac{M}{\alpha} + \pi P^k V(e, 0) - V(\pi, k + 1) \right].$$

By construction, the optimal policy in state $(\pi, k + 1)$ prescribes no action for some number of periods and then inspection, which allows us to write $J(\pi, k)$ for $k \in [0, k^*_2(\pi) - 1]$ as

$$J(\pi, k) \geq \alpha \left[ \frac{M}{\alpha} + \pi P^{k+1} V(e, 0) - \min_{a \in A} V(\pi, k + 1, \alpha, a) \right]$$

$$= \alpha \max_{a \in A} \left\{ \frac{M}{\alpha} + \pi P^{k+1} V(e, 0) - V(\pi, k + 1, \alpha, a) \right\},$$

where $V(\pi, k + 1, \alpha, a)$ is the total expected discounted cost of using action $a$ starting in state $(\pi, k + 1)$, and $A$ is the set of actions that require no action for
some number of periods and then inspection. Hence, for $k \in [0, k_2^*(\pi) - 1]$, 

$$J(\pi, k) = \alpha \max_{\ell \leq k^*(e_0)} \left\{ \frac{M}{\alpha} + \pi P^{k+1} V(e, 0) - \pi P^{k+1} L - \alpha \pi P^{k+2} L - \ldots - \alpha^\ell \pi P^{k+\ell} L \\
- \alpha^\ell M - \alpha^{\ell+1} \pi P^{k+\ell+2} V(e, 0) \right\}$$

$$= \alpha \max_{\ell \leq k^*(e_0)} \left\{ \sum_{i=0}^N \sum_{j=0}^N \pi_{ij} H^{k+1}_{1/\alpha} (e_j, 0) \right\}$$

$$= \alpha \max_{\ell \leq k^*(e_0)} \left\{ \pi P^{k+1} H^{\ell-1}_{1/\alpha} (e, 0) \right\},$$

where $k^*(e_0)$ is the repair control limit if the system starts from information state $e_0$. Note that $k^*(e_0)$ is the largest among all $k^*(\pi), \pi \in \Omega$. By Lemma A.1 and Lemma A.3, $\pi P^{k+1} H^{\ell-1}_{1/\alpha} (e, 0)$ satisfies the crossing property in $\prec_{tr}$-increasing $\pi$ for any $k \in [0, k_2^*(\pi) - 1]$, and in $k$ for $\pi \in \Omega_{\pi \prec_{tr} \pi_P}$, hence, so does $J(\pi, k)$ in $k \in [0, k_2^*(\pi) - 1]$. As a result, we can divide $[0, k_2^*(\pi)]$ into two subintervals $[0, k_1^*(\pi) - 1]$ and $[k_1^*(\pi), k_2^*(\pi)]$, such that $J(\pi, k) \geq 0$ in $[0, k_1^*(\pi) - 1]$ and $J(\pi, k) \leq 0$ in $[k_1^*(\pi), k_2^*(\pi)]$, with $J(\pi, k_1^*(\pi)) < 0$. Thus, in $[0, k_1^*(\pi) - 1]$, no action is optimal and in $[k_1^*(\pi), k_2^*(\pi)]$ inspection is optimal.

In Corollary 1.5.5 we prove that the more deteriorated the initial state of the system, the sooner we inspect.

**Corollary 1.5.5.** If $\Pi(C - L)$ and Assumptions A1 – A4' hold, then for $\pi \in \Omega_{\pi \prec_{tr} \pi_P}$, $k_1^*(\pi)$ is $\prec_{tr}$-nonincreasing in $\pi$.

**Proof.** We proceed by contradiction, i.e. by assuming $k_1^*(\pi) < k_1^*(\pi')$ for $\pi \prec_{tr} \pi'$. By Theorem 1.5.4, we can construct $k_1^*(\pi)$ such that inspection is strictly optimal
in state \((\pi, k)\) for \(k \geq k_1^*(\pi)\), so that
\[
0 \geq J(\pi, k_1^*(\pi)) \geq \alpha \max_{\ell \leq k_1^*(\pi)} \{ \pi P^{k_1^*(\pi)+1} H_{1/\alpha}^\ell (e, 0) \}.
\]

Now, we claim that in state \((\pi', k_1^*(\pi)+1)\), inspection is suboptimal and, hence, since \(J(\pi', k_1^*(\pi)) \geq 0\),
\[
\alpha \max_{\ell \leq k_1^*} \{ \pi' P^{k_1^*(\pi)+1} H_{1/\alpha}^\ell (e, 0) \} \geq 0.
\] (1.16)

Since by Proposition 1.3.2 \((ii)\), \(\pi P^{k} \prec_{\ell r} \pi' P^{k}\) for \(\pi \prec_{\ell r} \pi'\) and \(H_{1/\alpha}^\ell (e, 0)\) satisfies the crossing property, by Lemma A.1 and Lemma A.3, we have
\[
\pi P^{k_1^*(\pi)+1} H_{1/\alpha}^\ell (e, 0) \geq \pi' P^{k_1^*(\pi)+1} H_{1/\alpha}^\ell (e, 0).
\]
Taking the maximum of both sides of the last inequality on \(\ell \in [k_1^*, k_2^*]\), we obtain
\[
0 > \max_\ell \{ \pi P^{k_1^*(\pi)+1} H_{1/\alpha}^\ell (e, 0) \}
\geq \max_\ell \{ \pi' P^{k_1^*(\pi)+1} H_{1/\alpha}^\ell (e, 0) \},
\]
which contradicts inequality (1.16) and, hence, \(k_1^*(\pi') < k_1^*(\pi)\).

To establish the same results as in Theorem 1.5.4 and Corollary 1.5.5 for the extreme sample paths of \(\Omega\), Rosenfield [18] imposes the additional assumption that \(P\) be uppertriangular. As White [22] points out, uppertriangularity of \(P\) is sufficient for condition (1.13) to hold. Although Theorem 1.5.4 and Corollary 1.5.5 do not require the uppertriangularity of \(P\) or condition (1.13) (because we restrict our attention to the subset \(\Omega_{\pi \prec_{\ell r} \pi P}\) for which condition (1.13) holds automatically), we do, like Rosenfield [18], need the stronger TP2 condition on \(P\),
whereas White [22], in establishing the AM4R structure along straight lines, does not.

1.6 Conclusion

In this chapter, we extend structural properties of the optimal policy for a classic, partially observed maintenance optimization problem in an intuitive way. More specifically, we demonstrate that these properties hold along certain time-based sample paths not previously considered. Not only can these results be exploited computationally in solving higher order instances of this problem, but they also serve as a bridge toward deriving similar results in more complex settings involving imperfect information. As an initial step in this direction, future work could consider the same framework considered here, but with obvious failures and investigate how the necessary conditions can be strengthened to establish the same structured policy results along ordered sample paths. For this problem, in which observing no failure in a particular period yields imperfect information regarding the underlying deterioration state, structural results have been examined along straight lines of information states (Ohnishi [13], Maillart [10]), but not sample paths.
Chapter 2

Structured Maintenance Policies
for System with Stochastic
Repair and Silent Failures

2.1 Introduction

Since maintenance is responsible for the system’s reliability and performance, it has a major impact on delivery, quality, and cost of the final product, and optimized, can provide a success in the market competition. Therefore, over the past few decades, there has been considerable research activity in the modeling and optimization of the system maintenance. The main objectives of maintenance are to maintain and retain the system in good working condition, as well as detect and
prevent failure. Maintenance involves repairs or replacement, and is performed to keep the system in acceptable operable condition. Most of the existing literature, however, assumes that repairs and replacement are perfect, that is, afterwards, the system is “as good as new.” In many practical situations, repair may result in a wide variety of outcomes, including the extreme - perfect or minimal repairs, intermediate ones, as well as destructive ones. Such outcomes are possible due to imperfectness of diagnosis, the replacement parts, or the ability of the repair person. We refer to repair that could with some probability result in any possible outcome listed above as a stochastic repair.

Examples where stochastic repair is possible include but are not limited to repair of the manufacturing equipment, cure of diseases, repair of software, etc. With regards to software, since it significantly impacts critical applications in all areas of human activity such as aviation, nuclear power generation, mass distraction weapon control, medicine, etc., it is extremely important to certify software integrity and reliability, and to provide its accurate functioning. To present a realistic software maintenance and reliability model, we clearly need to adopt the assumption of possibility of stochastic repair outcome.

In this chapter, we consider a production system that undergoes deterioration according to the finite multi-state discrete-time Markov chain. The states of the system’s deterioration are not directly observable by a decision maker. At each system’s deterioration state, three actions are available to the decision maker:
stochastic repair, do nothing, or inspect. As a result of stochastic repair, the system with some nonnegative probability occupies new deterioration state. If do nothing is chosen, then the system deteriorates according to a known transition probability matrix. If the inspection is chosen as the next action, the real deterioration state of the system is perfectly established.

While most of the existing literature considers perfect repair, little has been written about maintenance policy that considers stochastic repair. In this paper, we would like to establish conditions under which the maintenance policy in case of stochastic repair has an appealing, easy to implement structure.

The rest of the chapter is as follows. In Section 2.2, we give the model formulation and review the existing literature relevant to our research. In Sections 2.3, 2.4, 2.5 and , we establish the main results of the paper. Section 2.6 summarizes the chapter findings.

2.2 Model Formulation and Literature Review

In Chapter 2, we consider three problems that deal with stochastic replacement/repair. Problem 1 considers state independent, stochastic repair with observable repair outcome. In case of Problem 1, we assume that the existing system is replaced by the refurbished one, deterioration state of which is not known beforehand and is not necessarily “as good as new”. Problem 2 considers state dependent, stochastic repair with observable repair outcome. Finally, Problem
3 considers state dependent, stochastic repair with unobservable repair outcome. In case of Problems 2 and 3, we assume that the existing system undergoes the repair the result of which depends on the system’s state prior to repair. For all three problems, we investigate the structure of the optimal maintenance policy by minimizing the expected total discounted cost over an infinite horizon.

We consider the state of the system to be \((\pi, k)\) as defined in Chapter 1.

First, consider Problem 1. In any given state \((\pi, k)\), there are three actions available to the decision maker: stochastic replacement, do nothing, inspection. Upon stochastic replacement, the systems with probability \(r_i\) enters the state \((e_i, 0), i = 0, 1, \ldots, N\), where \(r_i \geq 0, \sum_{i=0}^{N} r_i = 1\). This case corresponds to a situation when the existing system is replaced by a randomly chosen refurbished one. Deterioration state of the refurbished system is known only after inspection that we assume is combined with stochastic replacement.

Stochastic replacement takes one period and its expected cost in state \((\pi, k)\), as well as the expected cost of no action and perfect inspection are the same as in Chapter 1. The finite and infinite-horizon minimum total expected discounted cost-to-go \(V_n(\pi, k)\) and \(V(\pi, k)\), respectively, are

\[
V_n(\pi, k) = \min \left\{ \begin{array}{l}
\pi P^k C + \alpha r V_{n-1}(e, 0), \\
\pi P^k L + \alpha V_{n-1}(\pi, k + 1), \\
\pi P^k L + M + \alpha \pi P^{k+1} V_{n-1}(e, 0)
\end{array} \right\}, \quad (2.1)
\]
where \( r = [r_0, \ldots, r_N] \) and \( \alpha \) is a discount factor, \( 0 < \alpha < 1 \).

Next, for Problem 2 in any given state \((\pi, k)\), there are three actions available to the decision maker: state dependent, stochastic repair with observable outcome; do nothing; or inspection. As a result of stochastic repair, the system transits to state \((e_i, 0)\), with probability \((\pi P_k R)_i, i = 0, 1, \ldots, N\). Here, \( R \) is stochastic matrix \( i \)th elements \( r_{ij}, i, j = 0, 1, \ldots, N \). We refer to matrix \( R \) as a repair matrix. The minimum total expected discounted cost-to-go with \( n \) periods remaining \( O_n(\pi, k) \) and the optimal total expected discounted cost over an infinite horizon \( O(\pi, k) \) can be written for Problem 2 as follows

\[
O_n(\pi, k) = \min \left\{ \begin{array}{l}
\pi P_k C + \alpha \pi P_k R O_{n-1}(e, 0), \\
\pi P_k L + \alpha O_{n-1}(\pi, k + 1), \\
\pi P_k L + M + \alpha \pi P^{k+1} O_{n-1}(e, 0)
\end{array} \right\}, \quad (2.3)
\]

and

\[
O(\pi, k) = \min \left\{ \begin{array}{l}
\pi P_k C + \alpha \pi P_k R O(e, 0), \\
\pi P_k L + \alpha O(\pi, k + 1), \\
\pi P_k L + M + \alpha \pi P^{k+1} O(e, 0)
\end{array} \right\}, \quad (2.4)
\]

respectively.

Finally, Problem 3 considers state dependent, stochastic repair with unobservable repair outcome. Such repair in state \((\pi, k)\) brings the system to the new
state \((\pi P^k R, 0)\), such that new state \((\pi P^k R, 0)\) is possibly less deteriorated than previous state \((\pi, k)\). The other two actions are the same as in previous problems: do nothing and inspection. For Problem 3, the minimum total expected discounted cost-to-go with \(n\) periods remaining \(U_n(\pi, k)\) and the optimal total expected discounted cost over an infinite horizon \(U(\pi, k)\) are given by the following expressions:

\[
U_n(\pi, k) = \min \left\{ \begin{array}{l}
\pi P^k C + \alpha U_{n-1}(\pi P^k R, 0), \\
\pi P^k L + \alpha U_{n-1}(\pi, k + 1), \\
\pi P^k L + M + \alpha \pi P^{k+1} U_{n-1}(e, 0)
\end{array} \right\}, \quad (2.5)
\]

and

\[
U(\pi, k) = \min \left\{ \begin{array}{l}
\pi P^k C + \alpha U(\pi P^k R, 0), \\
\pi P^k L + \alpha U(\pi, k + 1), \\
\pi P^k L + M + \alpha \pi P^{k+1} U(e, 0)
\end{array} \right\}, \quad (2.6)
\]

respectively.

Since in reality, there are a lot of cases of stochastic repair, it is undoubtedly important to consider maintenance policies that take such repair into consideration, which, as noted earlier in this chapter, existing literature does not.

Minimal imperfect repair is considered by Brown and Proschan [2], but within the context of completely different problem. Minimal imperfect repair is such a repair that with probability \(p\) brings the system to “as good as new” state and with probability \(1 - p\) system is in “as good as old” one. Pham and Wang [15] propose classification of maintenance models according to the degree of operating
conditions of an item after repair.

White [22] and Rosenfield [17] consider a model similar to the one described here but with perfect repair. White [22] establishes conditions under which optimal maintenance policy along straight lines of information states in increasing stochastic order has the monotonic structure that consists of at most four regions. Rosenfield [17] establishes the monotonic, AM4R policy structure for every sample path originated from the information state $e_i, i = 0, 1, \ldots, N$.

Smallwood and Sondik [21] formulate the problem as a finite-state, discrete-time partially observable Markov process. With four actions available at each decision epoch - replace only; manufacture (do nothing); inspect and replace; examine product - they show that with the finite number of decision epochs remaining, the cost function is piecewise-linear and convex. They modeled the result of imperfect repair using the repair matrix.

Ohnishi et al. [14] consider replacement problem with minimal repair. They consider the system that deteriorates according to a discrete-time Markov decision process with deterioration levels $0, 1, \ldots$. The system is imperfectly monitored and at each decision epoch could be in two observable conditions: available (A) and unavailable (F). Accordingly, the state of the system is represented as a pair $(\pi, A)$ or $(\pi, F)$. In any state $(\pi, A)$, do nothing and preventive replacement actions are available, while in state $(\pi, F)$, minimal repair or failure replacement could be performed. As a result of minimal repair, the system from state $(\pi, F)$ transits to
state \( (\pi, A) \). Ohnishi et al. establish conditions under which the optimal policy is monotone with respect to failure replacement along any likelihood ratio increasing states, and find the special type of optimal policy being \((i, I)\) type, \(i \leq I\). The \((i, I)\) policy performs minimal repair for the failure up to deterioration level \(i\); a failure replacement at the first failure after the deterioration level exceeds level \(i\); and a preventive replacement is performed at the first transition to the level which is larger than level \(I\) without failure at any level between levels \(i\) and \(I\) (Ohnishi [14]). Both replacement, preventive and failure, are perfect.

Ivy et al. [7] consider multi-state deteriorating system with imperfect observations probabilistically related to the real system’s deterioration state with this set of actions: do nothing, repair, replace. While replacement results in a good as new system, repair depends on the current deterioration state of the system and if repair action \(a_l\) is chosen, \(l = 1, \ldots, N - 2\), at state \(j\), then the new state of the system is \(\text{max}(0, j - l)\). That is, the result of repair action is not stochastic.

In this chapter, we would like to consider stochastic, imperfect repair problem and establish conditions under which the maintenance policy for such a problem has monotone, at-most-four region structure.
2.3 Problem 1: State Independent, Stochastic Repair with Observable Repair Outcome

First, we establish that the optimal policy is monotone with respect to repair along any \( \prec_{st} \)-increasing sample path, i.e., for any stochastically increasing sample path \( \{(\pi, 0), (\pi, 1), \ldots, (\pi, k), \ldots\} \) there exists a \( k^*(\pi) \), such that in all states \( (\pi, k) \) for which \( k \geq k^*(\pi) \), repair is optimal. We establish this property of the optimal policy under the following set of assumptions:

A1. \( C_j \) and \( L_j \) are nondecreasing in \( j \);

A2. \( C_j - L_j \) is nonincreasing in \( j \);

A3. there exists a unique steady-state probability distribution \( \Pi \), such that \( \Pi = \Pi P \);

A4. \( P \) is IFR.

The results concerning the policy structure depend on the monotonicity of the value function, that we establish in Lemma 2.3.1, which is analogous to Lemma 1.4.1.

Lemma 2.3.1. Under Assumption A1 and Assumption A4, \( V(\pi, k) \) given by equation (2.2) is

(i) \( \prec_{st} \)-nondecreasing in \( \pi \) for all \( k \);
(ii) nondecreasing in $k$ for all $\pi \in \Omega_{\pi \succ_{st} \pi P}$.

Next, we present Theorem 2.3.2, analogous to Theorem 1.4.2. In this theorem, we prove that the optimal maintenance policy is monotone with respect to stochastic repair with observable outcome along any sample path that originates from an information state in the set $\Omega_{\pi \succ_{st} \pi P}$. Furthermore, in this theorem we establish that the more the system deteriorated initially, the sooner repair becomes optimal. To proof this theorem, we need to use next two functions

$$F(\pi, k) \equiv \pi P^k [C - L] + \alpha [rV(e, 0) - V(\pi, k + 1)], \quad \text{and} \quad (2.7)$$

$$G(\pi, k) \equiv \pi P^k [C - L] + \alpha [rV(e, 0) - \pi P^{k+1} V(e, 0)] - M. \quad (2.8)$$

The functions $F(\pi, k)$ and $G(\pi, k)$ converge to $F^*$ and $G^*$, respectively, as $k \to \infty$ for all $\pi \in \Omega$, where

$$F^* = \lim_{k \to \infty} F(\pi, k) = \Pi(C - L) + \alpha [rV(e, 0) - V(\Pi, \cdot)], \quad (2.9)$$

$$G^* = \lim_{k \to \infty} G(\pi, k) = \Pi(C - L) + \alpha [rV(e, 0) - \Pi V(e, 0)] - M. \quad (2.10)$$

Here, the interpretation of functions $F(\pi, k), G(\pi, k), F^*$, and $G^*$ is analogous to one in Chapter 1.

**Theorem 2.3.2.** If $\Pi(C - L) \leq 0$ and Assumptions A1-A4 hold, then for all $\pi \in \Omega_{\pi \prec_{st} \pi P}$ there exists a critical number $k^*(\pi)$, $\prec_{st}$-nonincreasing in $\pi$, such that in state $(\pi, k)$ repair is optimal if $k \geq k^*(\pi)$, and either inspection or no action is optimal otherwise.
We omit the proof since it is similar to the proof of Theorem 1.4.2. Next, we provide the following corollary to Theorem 2.3.2, which establishes that if the system cannot improve on its own, i.e., if \( P \) is upper triangular, then a repair control limit exists along every sample path.

**Corollary 2.3.3.** If \( \Pi(C - L) \leq 0 \), Assumptions A1-A4 hold and \( P \) is upper triangular, then for all \( \pi \in \Omega \) there exists a critical number \( k^*(\pi) \), \( \prec_{st} \)-nonincreasing in \( \pi \), such that in states \((\pi, k)\) repair is optimal if \( k \geq k^*(\pi) \), and either inspection or no action is optimal otherwise.

**Proof.** The proof is similar to the proof of Corollary 1.4.3 in Chapter 1. \( \square \)

Next, if we strengthen our assumption regarding transition probability matrix \( P \), then we could show that the optimal policy has an appealing monotone, AM4R structure along \( \prec_{\ell r} \)-increasing sample paths. As a corollary to this result, we show that the more deteriorated the initial condition of the system, the sooner inspection becomes optimal. We retain assumptions A1, A2, and A3, but replace A4 with a stronger assumption, namely:

\( A4'. \) \( P \) is totally positive of order two \( (TP_2) \).

Note that Lemma 2.3.4 is true under less restrictive conditions, that is, under condition that \( P \) is IFR and \( \pi \in \Omega_{\pi \prec_{st} \pi P} \).

**Lemma 2.3.4.** If \( P \) is \( TP_2 \) and \( \pi \in \Omega_{\pi \prec_{\ell r} \pi P} \), then

\[
\pi P^k V(e, m) \leq V(\pi, k + m).
\]
Lemma 2.3.5. For all $f \geq 1$ under Assumptions A1, A2, A3 and A4', function

$$H^J_f(\pi, k) \equiv V(\pi, k) - \pi P^k L - \ldots - \alpha^\ell \pi P^{k+\ell} L - \alpha^\ell M - \alpha^{\ell+1} \pi P^{k+\ell+1} V(e, 0) + f M$$

(2.11)

crosses zero at most once in $\pi \in \Omega_{\pi \prec \ell r}$, and if it does, it does so from above.

Proof. Since Do Nothing and Inspect components of function $V(\pi, k)$ are the same as components of function $V(\pi, k)$ of Chapter 1, we consider only Repair component in the proof of this lemma.

For $\ell = 0$,

$$H^J_0(\pi, k) = V(\pi, k) - \pi P^k L - M - \alpha \pi P^{k+1} V(e, 0) + f M$$

$$= \min \left\{ \begin{array}{l}
\pi P^k C + \alpha r V(e, 0) - \pi P^k L - M - \alpha \pi P^{k+1} V(e, 0) + f M, \\
\pi P^k L + \alpha V(\pi, k + 1) - \pi P^k L - M - \alpha \pi P^{k+1} V(e, 0) + f M, \\
\pi P^k L + M + \alpha \pi P^{k+1} V(e, 0) - \pi P^k L - M - \alpha \pi P^{k+1} V(e, 0) + f M \\
\end{array} \right\}$$

$$= \min \left\{ \begin{array}{l}
\pi P^k (C - L) + \alpha \left[ r V(e, 0) - \pi P^{k+1} V(e, 0) \right] + M(f - 1), \\
\alpha \left[ V(\pi, k + 1) - \pi P^{k+1} V(e, 0) \right] + M(f - 1), \\
f M \\
\end{array} \right\}.$$
Now, assume that $H^1_f(\pi, k)$ through $H^{\ell-1}_f(\pi, k)$ satisfy the crossing property in $\prec_{\ell_r}$-increasing $\pi$ and consider $H^\ell_f(\pi, k)$,

$$H^\ell_f(\pi, k) = V(\pi, k) - \sum_{i=0}^{\ell} \alpha^i \pi P^{k+i}L - \alpha^\ell M - \alpha^{\ell+1} \pi P^{k+\ell+1}V(e, 0) + fM$$

Again, consider only the first component of the above expression, which equals to

$$a^\ell_f(\pi, k) = \pi P^k(C-L) + \alpha rV(e, 0) - \sum_{i=1}^{\ell} \alpha^i \pi P^{k+i}L - \alpha^\ell M - \alpha^{\ell+1} \pi P^{k+\ell+1}V(e, 0) + fM$$

The function $\pi P^k(C-L)$ is $\prec_{\ell_r}$-nonincreasing in $\pi$ by Assumption A2 and Lemma 1.3.5. The terms $\alpha rV(e, 0), \alpha^\ell M$ and $fM$ are constants. The functions $\alpha^i \pi P^{k+i}L$, $i = 1, 2, \ldots, \ell$ are $\prec_{\ell_r}$-nondecreasing in $\pi$ by Assumption A1 and Lemma 1.3.5 and, hence, the functions $-\alpha^i \pi P^{k+i}L$, $i = 1, 2, \ldots$ are $\prec_{\ell_r}$-nonincreasing in $\pi$. The function $V(e_i, 0)$ is nondecreasing in $e_i$, so $-\pi P^{k+\ell+1}V(e, 0)$ is $\prec_{\ell_r}$-nonincreasing in $\pi$ by Lemma 1.3.5. Since all the terms in function $a^\ell_f(\pi, k)$ are nonincreasing, then function $a^\ell_f(\pi, k)$ is nonincreasing in $\pi$, and hence, satisfies crossing property
Theorem 2.3.6 proves the existence of the critical numbers $k_1^*(\pi)$ and $k_2^*(\pi)$ that define the optimal inspection region along any $\prec_{\ell r}$-increasing sample path, which ensures that the optimal policy exhibits the monotone, AM4R structure along every $\prec_{\ell r}$-increasing sample path. This result relies on the previous technical lemmas, that provide a basis for the existence of inspection control limit. We omit the proof of this theorem since it is analogous to the proof of Theorem 1.5.4.

Theorem 2.3.6. If $\Pi(C - L)$ and Assumptions A1 – A4' hold, then a monotone, AM4R policy is optimal along every sample path emanating from an information state $\pi \in \Omega_{\pi \prec_{\ell r} \pi P}$.

2.4 Problem 2: State Dependent, Stochastic Repair with Observable Repair Outcome

In this section, we consider state dependent, stochastic repair with observable repair outcome and establish structural properties of the optimal maintenance policy. The finite and infinite horizon cost functions in state $(\pi, k)$ are given in Section 2.2 by equations (2.3) and (2.4), respectively. First, in Lemma 2.4.1, we prove monotonicity of the cost function. Second, in Theorem 2.4.3, we establish
monotonicity of the optimal policy with respect to repair along any \(\prec_{st}\)-increasing sample path under the following set of assumptions:

1. \(C_j\) and \(L_j\) are nondecreasing in \(j\);
2. \(C_j - L_j\) is nonincreasing in \(j\);
3. \(P\) and \(R\) are IFR and have unique steady-state probability distributions, 
   \(\Pi_P\) and \(\Pi_R\), respectively, such that \(\Pi_R \prec_{st} \Pi_P\).

Here, \(\Pi_P\) and \(\Pi_R\) are steady-state probability distributions of matrices \(P\) and \(R\), respectively.

**Lemma 2.4.1.** Under Assumptions A1-A3, \(O(\pi, k)\) given by equation (2.4) is nondecreasing in \(\prec_{st}\)-increasing \(\pi\) and nonincreasing in \(k\).

**Proof.** First, let’s prove the property of \(O_n(\pi, k)\) in \(\pi \in \Omega_{\pi \prec_{st} \pi_P}\). Consider two information states \(\pi \prec_{st} \hat{\pi}\), then for \(n = 1\) by Proposition 1.3.2,

\[
O_1(\pi, k) = \min \{\pi P^k C, \pi P^k L\} \\
\leq \min \{\hat{\pi} P^k C, \hat{\pi} P^k L\} \\
= O_1(\hat{\pi}, k).
\]

Assuming that \(O_{n-1}(\pi, k) \leq O_{n-1}(\hat{\pi}, k)\) for \(\prec_{st}\)-increasing \(\pi\), we prove that \(O_n(\pi, k)\)
is also nondecreasing in \( \prec_{st} \)-increasing \( \pi \)

\[
O_n(\pi, k) \leq \min \begin{cases}
\hat{\pi} P^k C + \alpha \pi P^k R O_{n-1}(e, 0), \\
\hat{\pi} P^k L + \alpha O_{n-1}(\pi, k + 1), \\
\hat{\pi} P^k L + M + \alpha P^{k+1} O_{n-1}(e, 0)
\end{cases}
\]

\[
\leq \min \begin{cases}
\hat{\pi} P^k C + \alpha \hat{\pi} P^k R O_{n-1}(e, 0), \\
\hat{\pi} P^k L + \alpha O_{n-1}(\hat{\pi}, k + 1), \\
\hat{\pi} P^k L + M + \alpha \hat{\pi} P^{k+1} O_{n-1}(e, 0)
\end{cases}
\]

\[
= O_n(\hat{\pi}, k),
\]

which is true by the induction assumption on \( n \).

Next, we prove monotonicity of \( O(\pi, k) \) in \( k \). For \( n = 1 \) by Proposition 1.3.2,

\[
O_1(\pi, k) = \min\{\pi P^k C, \pi P^k L\}
\]

\[
\leq \min\{\pi P^{k+1} C, \pi P^{k+1} L\}
\]

\[
= O_1(\pi, k + 1).
\]

Assume that functions \( O_2(\pi, k), \ldots, O_{n-1}(\pi, k) \) are nondecreasing in \( k \) for \( \pi \in \Omega_{\pi \prec_{st} \pi P} \) and let’s prove this for \( O_n(\pi, k) \). Taking into account that \( \pi P^k F \leq \pi P^{k+1} F \) for \( \pi \in \Omega_{\pi \prec_{st} \pi P} \) and \( F \) nondecreasing and, since \( O_n(e_0, 0) \leq O_n(e_1, 0) \ldots \leq O_n(e_N, 0) \) for any \( n = 1, 2, \ldots \), then by Lemma 1.3.5 part (ii), \( \pi P^{k+1} O_{n-1}(e, 0) \leq \)

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\( \pi^{k+2} O_{n-1}(e, 0) \), and we can receive
\[
O_n(\pi, k) = \min \left\{ \begin{array}{l}
\pi P^k C + \alpha \pi P^k RO_{n-1}(e, 0), \\
\pi P^k L + \alpha O_{n-1}(\pi, k + 1), \\
\pi P^k L + M + \alpha \pi P^{k+1} O_{n-1}(e, 0)
\end{array} \right\}
\]
\[
\leq \min \left\{ \begin{array}{l}
\pi P^{k+1} C + \alpha \pi P^k RO_{n-1}(e, 0), \\
\pi P^{k+1} L + \alpha O_{n-1}(\pi, k + 1), \\
\pi P^{k+1} L + M + \alpha \pi P^{k+1} O_{n-1}(e, 0)
\end{array} \right\}
\]
\[
= O_n(\pi, k + 1).
\]

Redefine two functions
\[
F(\pi, k) \equiv \pi P^k [C - L] + \alpha [\pi P^k RO(e, 0) - O(\pi, k + 1)], \quad (2.12)
\]
\[
G(\pi, k) \equiv \pi P^k [C - L] + \alpha [\pi P^k RO(e, 0) - \pi P^{k+1} O(e, 0)] - M. \quad (2.13)
\]

By Assumption A2 and Lemma 1.3.5, \( \pi P^k [C - L] \) is \( \prec_{st} \)-decreasing in \( \pi \). Since \( O(e_i, 0) \) is non-decreasing in \( i \), then by Lemma 1.3.5, \( \pi P^k RO(e, 0) - \pi P^{k+1} O(e, 0) \) is \( \prec_{st} \)-decreasing in \( \pi \). Hence, \( G(\pi, k) \) is \( \prec_{st} \)-decreasing in \( \pi \). As \( k \to \infty \), the functions \( F(\pi, k) \) and \( G(\pi, k) \) converge to \( F^* \) and \( G^* \), respectively, for all \( \pi \in \Omega \),
where

\[
F^* = \lim_{k \to \infty} F(\pi, k) = \Pi_P(C - L) + \alpha[\Pi_P R O(e, 0) - O(\Pi_P, \cdot)],
\]

(2.14)

\[
G^* = \lim_{k \to \infty} G(\pi, k) = \Pi_P(C - L) - M + \alpha[\Pi_P R O(e, 0) - \Pi_P O(e, 0)].
\]

(2.15)

Note, information state \( \Pi_P R \) is likelihood ratio smaller than \( \Pi_P \), but likelihood ratio larger than \( \Pi_R \). The following lemma provide a basis for monotonicity of function \( F(\pi, k) \):

**Lemma 2.4.2.** For all \( \pi \in \Omega_{\pi < \ast \pi_P} \) under Assumptions A2 and A3,

\[
\pi P^k O(e, m) \leq O(\pi, k + m).
\]

**Proof.** The fact that \( \min \{a, b, c\} \) is less than or equal any of \( a, b \) or \( c \), yields

\[
\sum_{j=0}^{N} [\pi P^k]_j \min \{a_j, b_j, c_j\} \leq \min \left\{ \sum_{j=0}^{N} [\pi P^k]_j a_j, \sum_{j=0}^{N} [\pi P^k]_j b_j, \sum_{j=0}^{N} [\pi P^k]_j c_j \right\}.
\]

Let

\[
a_n(e_j, m) = e_j P^m C + \alpha e_j P^m R O_{n-1}(e, 0),
\]

\[
b_n(e_j, m) = e_j P^m L + \alpha O_{n-1}(e_j, m + 1),
\]

\[
c_n(e_j, m) = e_j P^m L + M + \alpha e_j P^{m+1} O_{n-1}(e, 0).
\]

Here, \( a_n(e_j, m) \), \( b_n(e_j, m) \) and \( c_n(e_j, m) \) represent the total expected discounted cost-to-go with \( n \) periods left starting from state \( (e_j, m) \) under the actions repair, no action and inspection, respectively.
For $n = 1$, $a_1(e_j, m) = e_j P^m C$, $b_1(e_j, m) = e_j P^m L$, and $c_1(e_j, m) = e_j P^m L + M$, in which case

$$\sum_{j=0}^{N} [\pi P^k]_j O_1(e_j, m) = \sum_{j=0}^{N} [\pi P^k]_j \min \{a_1(e_j, m), b_1(e_j, m), c_1(e_j, m)\}$$

$$\leq \min\left\{\sum_{j=0}^{N} [\pi P^k]_j e_j P^m C, \sum_{j=0}^{N} [\pi P^k]_j e_j P^m L, \sum_{j=0}^{N} [\pi P^k]_j (e_j P^m L + M)\right\}$$

$$= \min\left\{\pi P^k P^m C, \pi P^k P^m L, \pi P^k P^m L + \sum_{j=0}^{N} [\pi P^k]_j M\right\}$$

$$= \min\{\pi P^k+m C, \pi P^k+m L, \pi P^k+m L + M\}$$

$$= O_1(\pi, k + m).$$

That is, we have

$$\pi P^k O_1(e, m) \leq O_1(\pi, k + m).$$

Now, assume that this result holds for $\pi P^k O_2(e, m)$ through $\pi P^k O_{n-1}(e, m)$ and
consider \( \pi^{P}kO_{n}(e, m) \):

\[
\pi^{P}kO_{n}(e, m) = \sum_{j=0}^{N} [\pi^{P}]_{j} \min \{a_{n}(e_{j}, m), b_{n}(e_{j}, m), c_{n}(e_{j}, m)\}
\]

\[
\leq \min \left\{ \sum_{j=0}^{N} [\pi^{P}]_{j} (e_{j}P^{m}C + \alpha e_{j}P^{m}RO_{n-1}(e, 0)), \sum_{j=0}^{N} [\pi^{P}]_{j} (e_{j}P^{m}L + \alpha O_{n-1}(e_{j}, m+1)), \sum_{j=0}^{N} [\pi^{P}]_{j} (e_{j}P^{m}L + M + \alpha e_{j}P^{m+1}O_{n-1}(e, 0)) \right\}
\]

\[
= \min \left\{ \pi^{P}k^{+m}C + \alpha \pi^{P}k^{+m}RO_{n-1}(e, 0), \pi^{P}k^{+m}L + \alpha \pi^{P}kO_{n-1}(e, m+1), \pi^{P}k^{+m}L + M + \alpha \pi^{P}k^{+m+1}O_{n-1}(e, 0) \right\}
\]

Since \( \pi^{P}kO_{n-1}(e, m+1) \leq O_{n-1}(\pi, k + m + 1) \) by the induction assumption, we have

\[
\pi^{P}kO_{n}(e, m) \leq O(\pi, k + m).
\]

Therefore, we finally have that

\[
\pi^{P}kO(e, m) \leq O(\pi, k + m).
\]

\[ \square \]

Theorem 2.4.3 represents the main result achieved for Problem 2. According to this theorem the optimal policy is monotone with respect to repair along stochastically increasing sample path.
Theorem 2.4.3. If \( \Pi_P(C - L) \leq 0 \) and Assumptions A1-A4 hold, then for all \( \pi \in \Omega_{\pi \prec_{st} \pi P} \) there exists a critical number \( k^*(\pi) \), \( \prec_{st} \)-nonincreasing in \( \pi \), such that in state \((\pi, k)\) replacement is optimal if \( k \geq k^*(\pi) \), and either inspection or no action is optimal otherwise.

Proof. By Assumption A2 and Lemma 1.3.5, \( \pi P^k[C - L] \) is \( \prec_{st} \)-nonincreasing in \( \pi \) and nonincreasing in \( k \) for \( \pi \in \Omega_{\pi \prec_{st} \pi P} \).

Furthermore, if \( \Pi_R \prec_{st} \pi P^k \), then \( \pi P^k R \prec_{st} \pi P^k \) and, hence,

\[
\pi P^k RO(e, 0) - O(\pi, k + 1) \leq \pi P^k O(e, 0) - O(\pi, k + 1) \leq 0,
\]

by Lemma 2.4.2 for all \( \pi \in \Omega_{\pi \prec_{st} \pi P} \). Similarly, \( \pi P^k RO(e, 0) - \pi P^{k+1} RO(e, 0) \leq 0 \).

Since \( \Pi_P R \prec_{st} \Pi_P \), then by by Lemma 2.4.2, \( \Pi_P RO(e, 0) - O(\Pi_P, \cdot) \leq 0 \) in equation (2.14), and by Lemma 1.3.5 and Lemma 2.4.1, \( \Pi_P RO(e, 0) - \Pi_P O(e, 0) \leq 0 \) in equation (2.15).

If \( \Pi_P(C - L) \leq 0 \), then it is optimal to replace in state \( \Pi_P \), in which case \( F^* < 0 \) and \( G^* < 0 \). Therefore, since \( F(\pi, k) \) became negative for some \( k \), for all \( \pi \in \Omega_{\pi \prec_{st} \pi P} \) there exists a \( \hat{k}(\pi) = \min \{k : F(\pi, k) < 0\} \) which implies that for \( k \geq \hat{k}(\pi) \), no action is more costly than repair. Similarly, there exists a \( \hat{k}(\pi) = \min \{k : G(\pi, k) < 0\} \), which implies that for \( k \geq \hat{k}(\pi) \), inspection is more costly than repair. Hence, there exists a \( k^*(\pi) = \max\{\hat{k}(\pi), \hat{k}(\pi)\} \), such that for \( k \geq k^*(\pi) \) both \( F(\pi, k) \) and \( G(\pi, k) \) are negative, meaning replacement is optimal.

Finally, replacement critical number, \( k^*(\pi) \), monotonicity in \( \prec_{st} \)-nonincreasing \( \pi \), follows directly from the \( \prec_{st} \)-monotonicity of \( F(\pi, k) \) and \( G(\pi, k) \) in \( \pi \). \( \square \)
Intuitively, in state \((\pi, k)\) such that \(\pi P^k \prec_{st} \Pi_R\) repair is not optimal since it worsens the current state instead of improving it. Also, if \(R\) is lower triangular matrix, then \(\Pi_R = [0, \ldots, 1]\), and we could disregard requirement for state \((\pi, k)\) to be more deteriorated than \(\Pi_R\) to start repair.

### 2.5 Problem 3: State Dependent, Stochastic Repair with Unobservable Repair Outcome

In this section, we consider state dependent, stochastic repair with unobservable repair outcome. For this case, the cost functions for finite and infinite horizons are given by equations (2.5) and (2.6), respectively.

Define \(\Omega_{\pi \prec_{st} \pi R} \equiv \{\pi : \pi \prec_{st} \pi R\}\). See Fig. 2.1 that illustrates regions \(\Omega_{\pi \prec_{st} \pi R}\) and \(\Omega_{\pi \prec_{st} \pi P}\).

Consider the current state of the system being \((\pi, k)\), such that \(\pi P^k \in \Omega_{\pi \prec_{st} \pi P}\) and \(\pi P^k \notin \Omega_{\pi \prec_{st} \pi R}\). If in state \((\pi, k)\) we perform repair, then the system transits to a new state \((\pi P^k R, 0)\), such that \(\pi P^k R \prec_{st} \pi P^k\) (Fig. 2.2). Repair improves
the current state of the system if $\Pi_R \prec_{st} \Pi_P$. By specifying some characteristics of repair matrix, we could achieve the various degree of system’s improvement.

First in this section, we establish the conditions under which the cost function given by equation (2.6) is monotone function of $\prec_{st}$-increasing $\pi$ and $k$. Next, in Theorem 2.5.2, we establish conditions under which the optimal policy is monotone with respect to repair along any $\prec_{st}$-increasing sample path.

We receive the mentioned above results under the following set of assumptions:

A1. $C_j$ and $L_j$ are nondecreasing in $j$;

A2. $C_j - L_j$ is nonincreasing in $j$;

A3. $P$ is IFR and upper triangular (UT);

A4. $R$ is IFR and there exists unique steady-state probability distribution $\Pi_R$;

A5. $\Pi_R \prec_{st} \Pi_P$.

Note that in Chapter 1, we proved that for UT transition-probability matrix $\Omega \equiv \Omega_{\pi \prec_{st} \pi_P}$ and any sample path that starts in $\pi \in \Omega$ is stochastically increasing.
in $\pi$ and $k$. In Lemma 2.5.1, we use this fact to establish monotonicity of the cost function.

**Lemma 2.5.1.** Under Assumptions A1, A3-A5, the cost function $U(\pi, k)$ given by equation (2.6) is nondecreasing in $\pi \in \Omega$ and nonincreasing in $k$.

**Proof.** First, let’s prove the property of $U_n(\pi, k)$ in $\pi \in \Omega$. Consider two information states $\pi \prec_{st} \hat{\pi}$, then for $n = 1$ by Proposition 1.3.1 part (i),

$$U_1(\pi, k) = \min \{ \pi P^k C, \pi P^k L, \pi P^k L + M \}$$

$$\leq \min \{ \hat{\pi} P^k C, \hat{\pi} P^k L, \hat{\pi} P^k L + M \}$$

$$= U_1(\hat{\pi}, k).$$

Assuming that $U_{n-1}(\pi, k) \leq U_{n-1}(\hat{\pi}, k)$ for $\pi \in \Omega$, let’s prove that $U_n(\pi, k)$ is also nondecreasing in $\pi$,

$$U_n(\pi, k) = \min \left\{ \begin{array}{l} \pi P^k C + \alpha U_{n-1}(\pi P^k R, 0), \\
\pi P^k L + \alpha U_{n-1}(\pi, k + 1), \\
\pi P^k L + M + \alpha \pi P^{k+1} U_{n-1}(e, 0) \end{array} \right\}$$

$$\leq \min \left\{ \begin{array}{l} \hat{\pi} P^k C + \alpha U_{n-1}(\hat{\pi} P^k R, 0), \\
\hat{\pi} P^k L + \alpha U_{n-1}(\hat{\pi}, k + 1), \\
\hat{\pi} P^k L + M + \alpha \hat{\pi} P^{k+1} U_{n-1}(e, 0) \end{array} \right\}$$

$$= U_n(\hat{\pi}, k),$$

by the induction assumption on $n$. 

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Let’s prove property in $k$. For $n = 1$ by Proposition 1.3.1 part (i),

$$U_1(\pi, k) = \min\{\pi P^k C, \pi P^k L, \pi P^k L + M\}$$

$$\leq \min\{\pi P^{k+1} C, \pi P^{k+1} L, \pi P^{k+1} L + M\}$$

$$= U_1(\pi, k + 1).$$

Assume $U_2(\pi, k), \ldots, U_{n-1}(\pi, k)$ are nondecreasing in $k$ for $\prec_{s.t}$-increasing $\pi$ and let’s prove this for $U_n(\pi, k)$. Taking in account that $\pi P^k F \leq \pi P^{k+1} F$ for $\pi$ and $F$ nondecreasing and, since $U_n(e_0, 0) \leq U_n(e_1, 0) \ldots \leq U_n(e_N, 0)$ for any $n = 1, 2, \ldots$, then $\pi P^{k+1} U_{n-1}(e, 0) \leq \pi P^{k+2} U_{n-1}(e, 0)$, and we can receive

$$U_n(\pi, k) = \min \begin{cases} 
\pi P^k C + \alpha U_{n-1}(\pi P^k R, 0), \\
\pi P^k L + \alpha U_{n-1}(\pi, k + 1), \\
\pi P^k L + M + \alpha \pi P^{k+1} U_{n-1}(e, 0) 
\end{cases}$$

$$\leq \min \begin{cases} 
\pi P^{k+1} C + \alpha U_{n-1}(\pi P^{k+1} R, 0), \\
\pi P^{k+1} L + \alpha U_{n-1}(\pi, k + 1), \\
\pi P^{k+1} L + M + \alpha \pi P^{k+1} U_{n-1}(e, 0) 
\end{cases}$$

$$\leq \min \begin{cases} 
\pi P^{k+1} C + \alpha U_{n-1}(\pi P^{k+1} R, 0), \\
\pi P^{k+1} L + \alpha U_{n-1}(\pi, k + 2), \\
\pi P^{k+1} L + M + \alpha \pi P^{k+2} U_{n-1}(e, 0) 
\end{cases}$$

$$= U_n(\pi, k + 1).$$

\[\square\]
Here, we also use the two functions

\[ F(\pi, k) \equiv \pi P^k [C - L] + \alpha [U(\pi P^k R, 0) - U(\pi, k + 1)], \]

\[ G(\pi, k) \equiv \pi P^k [C - L] + \alpha [U(\pi P^k R, 0) - \pi P^{k+1} U(e, 0)] - M. \]

Since we consider transition probability matrix \( P \) to be UT, the unique steady-state probability distribution is \( \Pi_P = [0, \ldots, 1] \). As \( k \to \infty \), functions \( F(\pi, k) \) and \( G(\pi, k) \) converge to \( F^* \) and \( G^* \), respectively, for all \( \pi \in \Omega \):

\[ F^* = \lim_{k \to \infty} F(\pi, k) = (C_N - L_N) + \alpha [U(R_N, 0) - U(e_N, 0)], \]

\[ G^* = \lim_{k \to \infty} G(\pi, k) = (C_N - L_N) - M + \alpha [U(R_N, 0) - U(e_N, 0)], \]

where \( R_N \) is the \( N \)th row of matrix \( R \). Furthermore, since \( R_N \prec_{st} \Pi_P \), then \( U(R_N, 0) - U(e_N, 0) \leq 0 \). This fact along with condition that \( C_N - L_N \leq 0 \), provide that both functions \( F^* \) and \( G^* \) are negative and, hence, functions \( F(\pi, k) \) and \( G(\pi, k) \) have crossing property and they cross zero from above. This result guarantees the monotonicity of the optimal policy with respect to repair. We state this result in Theorem 2.5.2.

**Theorem 2.5.2.** If \( (C_N - L_N) \leq 0 \) and Assumptions A1-A5 hold, then for all \( \pi \in \Omega \) there exists a critical number \( k^*(\pi) \), such that in state \( (\pi, k) \) repair is optimal if \( k \geq k^*(\pi) \), and either inspection or no action is optimal otherwise.
2.6 Conclusion

This chapter discusses a structure of the optimal inspection and repair policy of a discrete-time, partially observable Markov deterioration system with stochastic repair. We consider three types of stochastic repair:

1. state independent, stochastic repair with observable repair outcome;

2. state dependent, stochastic repair with observable repair outcome;

3. state dependent, stochastic repair with unobservable repair outcome.

In case of state independent stochastic repair with observable repair outcome, we show that the optimal policy is a monotone, at-most-four-region policy under the same set of conditions as we consider in Chapter 1.

For state dependent, stochastic repair with observable repair outcome and state dependent, stochastic repair with unobservable repair outcome, we prove that the optimal maintenance policy is monotone with respect to repair.

Clearly, we could consider this chapter’s three problems as a generalization of the Chapter 1. For example, if in Problem 1 vector \( r = [r_0, r_1, \ldots, r_N] = [1, 0, \ldots, 0] \), then the repair outcome is perfect as modeled in Chapter 1.

The different repair outcomes could be modeled for Problems 2 and 3 as well. Recall, that in Problem 3, if repair is done in state \((\pi, k)\) then the system transits to a new state \((\pi P^k R, 0)\). By manipulating the repair matrix \(R\) we could achieve different repair outcomes. For example, if we consider matrix \(R\) with first column...
all ones and zero everywhere else, then the repair is perfect and the results received in Chapter 1 are applicable.
Chapter 3

Inspection and Replacement

Scheduling under Age-Dependent Markovian Deterioration

3.1 Introduction

We consider optimal maintenance policy structure for a production system that undergoes age-dependent Markovian deterioration. At each decision epoch, three maintenance actions are available to a decision maker: replace, produce without inspection, or produce with a costly, perfect inspection. Each action takes one period, with the replacement action making the system “as good as new” and the inspection action revealing the current deterioration state of the system.
with certainty. While producing, the system deteriorates according to a known, age-dependent discrete time Markov chain. The action costs depend on the deterioration level of the system and its age. We model the problem as a discrete-time, finite-state, partially observable Markov decision process (POMDP).

Most POMDP maintenance models of this type of system ([11], [17], [18], [20], [22]) assume that the transition probability matrix describing the deterioration process is known and age-independent, and establish conditions under which the optimal policy has a distinctive structure. In this paper, we establish conditions under which the optimal replacement and inspection policy has an appealing structure when the deterioration process is age-dependent.

The paper is organized as follows. In Section 3.2, we present a formal description of our model and provide an overview of the existing literature. Section 3.3 provides preliminary definitions and results for further sections. Section 3.4 establishes properties of the cost function. Results with regard to policy structure along straight lines of ordered process states are given in Section 3.5. In Section 3.6, we present the main results of the paper - the structural properties of the optimal policy along certain age- and deterioration level-based subsets of the process states. Finally, in Section 3.7, we summarize our findings and outline future research directions.
3.2 Model Formulation and Literature Review

Let the underlying discrete-time Markov chain consist of deterioration states 0 through \( N \), where state 0 is the “good as new” state and state \( N \) is the most deteriorated state. A system of age \( t \) deteriorates according to the known transition probability matrix \( P_t = (p_{ij}(t)) \), \( t = 1, 2, \ldots \). At any given time, we define the state of the process to be \( (\pi, t) \), meaning that the system is currently in state \( \pi \) and is \( t \) time units old. Here, the information state \( \pi \) has components \([\pi_0, \pi_1, \ldots, \pi_N]\), where \( \pi_i \) is the probability that the system is in deterioration state \( i \). The information state space is defined as \( \Omega \equiv \{ \pi : 0 \leq \pi_i \leq 1, \sum_{i=0}^{N} \pi_i = 1 \} \).

In any given state \( (\pi, t) \), three actions are available: replace, produce without inspection, or produce with a costly, perfect inspection. We refer to these actions as replacement (RP), no action (DN), and inspection (IN), respectively. Upon replacement, the process reverts to state \((e_0, 0)\) at the beginning of the next period, where \( e_i \) is the row-vector with 1 in the \( i \)th position and 0 everywhere else. The expected replacement cost incurred is

\[
\pi C(t) = \sum_{i=0}^{N} \pi_i C_i(t),
\]

where \( C_i(t) \) is the cost of replacing a system of age \( t \) in deterioration state \( i \).

Similarly, the expected cost associated with taking no action is

\[
\pi L(t) = \sum_{i=0}^{N} \pi_i L_i(t),
\]

where \( L_i(t) \) is the per-period operating cost for a system of age \( t \) in deterioration...
state \( i \). As a result of this action, the process transitions to state \((\pi P, t + 1)\).

Lastly, the expected cost of inspection is

\[
\pi L(t) + M,
\]

where \( M \) is the cost of inspection. Upon inspection, which we assume takes one full period, the process enters state \((e, t + 1)\) with probability \((\pi P)\), \( j = 0, 1, \ldots, N \).

As a result, if \( V_n(\pi, t) \) is the minimum total expected discounted cost-to-go with \( n \) periods remaining starting in state \((\pi, t)\), we have:

\[
V_n(\pi, t) = \min \left\{ \begin{array}{l}
RP_n(\pi, t) \equiv \pi C(t) + \alpha V_{n-1}(e_0, 0), \\
DN_n(\pi, t) \equiv \pi L(t) + \alpha V_{n-1}(\pi P, t + 1), \\
IN_n(\pi, t) \equiv \pi L(t) + M + \alpha \pi P V_{n-1}(e, t + 1)
\end{array} \right\},
\]

where \( \alpha \) is a discount factor, \( 0 < \alpha < 1 \), and

\[
V_{n-1}(e, t + 1) \equiv [V_{n-1}(e_0, t + 1), V_{n-1}(e_1, t + 1), \ldots, V_n(e_N, t + 1)]^T.
\]

As \( n \to \infty \), \( V_n(\pi, t) \) converges to

\[
V(\pi, t) = \min \left\{ \begin{array}{l}
RP(\pi, t) \equiv \pi C(t) + \alpha V(e_0, 0), \\
DN(\pi, t) \equiv \pi L(t) + \alpha V(\pi P, t + 1), \\
IN(\pi, t) \equiv \pi L(t) + M + \alpha \pi P V(e, t + 1)
\end{array} \right\},
\]

where the proof of convergence \( V_n(\pi, t) \) to \( V(\pi, t) \) as \( n \to \infty \) is given in Appendix.

The majority of the existing literature that considers the structure of the optimal replacement and inspection policy for this partially observable deteriorating
system assumes a stationary environment, i.e., that the system’s deterioration process stays the same over time and is not influenced by its age. For such POMDPs (Maillart and Zheltova [11], Rosenfield [17] and [18], Ross [20], White [22]) the same structural properties of the optimal policy have been established along different ordered subsets of the state space. That is, under reasonable conditions the optimal policy consists of at most four regions along ordered subsets of the state space in the order: no action, inspection, no action, replacement. We refer to this type of structure as a monotone, at-most-four-region (AM4R) policy.

In contrast, Benyamini and Yechiali [1] consider a fully observable case with an age-dependent deterioration process. They consider two models. In the first model, at each state \((e_i, t), i = 0, 1, \ldots, N\), only two actions are available: replacement or no action. As a result of replacement, the system enters state \((e_0, 0)\). The action costs depend on the system’s state and age. For this model, Benyamini and Yechiali find conditions under which the optimal policy is an age-dependent control limit policy (CLP) with respect to replacement. For the age-dependent CLP, there exists state-age pair \((i^*(t), t^*(i))\) such that in any state \((e_i, t)\) the system is replaced, when \(i \geq i^*(t)\) for any fixed \(t\) or when the age of the system \(t \geq t^*(i)\) for fixed \(e_i\).

The second model in [1] allows three actions in each state \((e_i, t)\): replace, repair to a better state \((e_j, t)\) such that \(j < i\), or no action. Benyamini and Yechiali establish conditions under which the optimal policy is a 3-way CLP. The
3-way CLP policy is defined as a combination of age-dependent CLPs for every pair of possible actions, namely, no action and repair, no action and replacement, and repair and replacement.

Makis and Jardine [12] consider a system with catastrophic failures that has discrete deterioration states. The system deteriorates according to a continuous probability distribution. Although the deterioration process is age-independent, the failure rate of the system depends on its age and deterioration state. Free, perfect observations are available at equidistant discrete time points 0, Δ, 2Δ, . . . .

Makis and Jardine consider two models. In the first model, there are two actions available at each decision epoch: preventive replacement of the existing system a time units from now (0 ≤ a ≤ Δ) or no action. In the second model, the set of actions consists of immediate replacement and no action. The action costs in both models are independent of the system’s state and remain constant over time. Both models are formulated as partially observable semi-Markov decision processes. Conditions that ensure optimal control limit policies with respect to preventive replacement and immediate replacement are established for each model.

Ghasemi et al. [4] also consider a state and age-dependent failure rate. They consider a system with catastrophic failures and free, imperfect information available at each decision epoch. With two actions available at each decision epoch, immediate replacement and no action, and the same cost structure as in [12], the problem in [4] is modeled as a discrete-time POMDP. Conditions under which the
optimal policy is monotone with respect to immediate replacement are established.

This paper extends the existing results for the age-independent POMDPs in [11] and [22] by allowing the transition probabilities and action costs to depend explicitly on both the deterioration state of the system and its age. Though we differ from [1], [4], [12] by considering age-dependent partially observable deterioration process and actions available at each decision epoch, we pursue the same objective: to find conditions under which the optimal maintenance policy has an appealing, easy to implement structure.

### 3.3 Preliminary Definitions and Results

In this section, we present preliminary results needed to establish the properties of the cost function in Section 3.4 and the structure of the optimal policy in Sections 3.5 and 3.6. We begin with several definitions:

Proposition 3.3.1 is well-known (see, for example, Ohnishi [13]) and establishes the hierarchy of the partial orderings for the information states of the process.

**Proposition 3.3.1.** If $\pi \prec_{tr} \hat{\pi}$ then $\pi \prec_{st} \hat{\pi}$.

Parts (i), (ii), and (iii) of Proposition 3.3.2 can be found in Rosenfield [18], Derman [3], and Ohnishi [13], respectively. Part (i) establishes relation between $TP_2$ and $IFR$ matrices, while parts (ii) and (iii) establish the preservation of the matrix properties through multiplication.
Proposition 3.3.2.

(i) If $P$ is $TP_2$, then $P$ is IFR.

(ii) If $P$ is IFR, then $P^n$ is IFR for any $n = 0, 1, \ldots$.

(iii) If $P$ and $Q$ are $TP_2$, then $PQ$ is $TP_2$.

Part (i) of Proposition 3.3.3 is due to White [22] and establishes that given an age-independent IFR transition probability matrix, an initially more deteriorated system, in the sense of stochastic ordering, stays this way over time. Part (ii) of Proposition 3.3.3 is due to Ohnishi [13] and establishes the analogous result for the likelihood ratio ordering and a $TP_2$ transition probability matrix.

Proposition 3.3.3.

(i) If $P$ is IFR and $\pi \prec_{st} \hat{\pi}$, then $\pi P^n \prec_{st} \hat{\pi} P^n$ for $n = 0, 1, \ldots$.

(ii) If $P$ is $TP_2$ and $\pi \prec_{tr} \hat{\pi}$, then $\pi P^n \prec_{tr} \hat{\pi} P^n$ for $n = 0, 1, \ldots$.

Proposition 3.3.4 implies that if rows of the transition probability matrices are in $\prec_{st}$- or $\prec_{tr}$-increasing order as the system’s age progresses, then starting from the same information state, after one more period an older system occupies a more deteriorated state in the sense of the stochastic or likelihood ratio ordering, respectively. Let $[P_t]_i$ is an $i$th row of the transition probability matrix $P_t$.

Proposition 3.3.4.
(i) If $P_t$ is IFR and $[P_t]_i \prec_{st} [P_{t+1}]_i$, $t = 1, 2, \ldots$, then $\pi P_t \prec_{st} \pi P_{t+1}$, for $\pi \in \Omega$ and $t = 1, 2, \ldots$.

(ii) If $P_t$ is TP$_2$ and $[P_t]_N \prec_{\ell r} [P_{t+1}]_0$, $t = 1, 2, \ldots$, then $\pi P_t \prec_{\ell r} \pi P_{t+1}$, for $\pi \in \Omega$ and $t = 1, 2, \ldots$.

Proof.

(i) Indeed, consider

$$\sum_{i=m}^{N} [\pi P_t]_i = \sum_{i=m}^{N} \sum_{j=0}^{N} \pi_j p_t(j, i)$$

$$= \pi_0 \sum_{j=m}^{N} p_t(0, j) + \pi_1 \sum_{j=m}^{N} p_t(1, j) + \ldots + \pi_N \sum_{j=m}^{N} p_t(N, j)$$

$$\leq \pi_0 \sum_{j=m}^{N} p_{t+1}(0, j) + \pi_1 \sum_{j=m}^{N} p_{t+1}(1, j) + \ldots + \pi_N \sum_{j=m}^{N} p_{t+1}(N, j)$$

$$= \sum_{i=m}^{N} [\pi P_{t+1}]_i.$$ 

(ii) Note that since $P_t$ is TP$_2$ for $t = 1, 2, \ldots$, then $[P_t]_0 \prec_{\ell r} [P_t]_1 \prec_{\ell r} \ldots \prec_{\ell r} [P_t]_N$. 

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Consider

\[
\begin{align*}
\sum_{j=0}^{N} \pi_j p_t(j, k) \sum_{j=0}^{N} \pi_j p_t(j, z) &= \\
&= \sum_{j=0}^{N} \pi_j \pi_j [p_t(j, k) p_t(j, z) - p_t(j, k) p_t(j, z)] \\
&+ \sum_{i=0}^{N-1} \sum_{j=i+1}^{N} \pi_i \pi_j [p_t(i, k) p_t(i, z) - p_t(i, k) p_t(i, z)] \\
&+ \sum_{i=0}^{N-1} \sum_{j=i+1}^{N} \pi_i \pi_j [p_t(j, k) p_t(j, z) - p_t(j, k) p_t(j, z)]
\end{align*}
\]

(3.3)

\[
\begin{align*}
\sum_{j=0}^{N} \pi_j p_t(j, k) \sum_{j=0}^{N} \pi_j p_{t+1}(j, k) - \sum_{j=0}^{N} \pi_j p_t(j, z) \sum_{j=0}^{N} \pi_j p_{t+1}(j, z) &= \\
&= \sum_{j=0}^{N} \pi_j \pi_j \ [p_t(j, k) p_{t+1}(j, z) - p_{t+1}(j, k) p_t(j, z)] \\
&+ \sum_{i=0}^{N-1} \sum_{j=i+1}^{N} \pi_i \pi_j \ [p_t(i, k) p_{t+1}(i, z) - p_{t+1}(i, k) p_t(i, z)] \\
&+ \sum_{i=0}^{N-1} \sum_{j=i+1}^{N} \pi_i \pi_j \ [p_t(j, k) p_{t+1}(j, z) - p_{t+1}(j, k) p_t(j, z)]
\end{align*}
\]

(3.4)

\[
\begin{align*}
\sum_{j=0}^{N} \pi_j p_t(j, k) \sum_{j=0}^{N} \pi_j p_{t+1}(j, k) - \sum_{j=0}^{N} \pi_j p_t(j, z) \sum_{j=0}^{N} \pi_j p_{t+1}(j, z) + \sum_{i=0}^{N-1} \sum_{j=i+1}^{N} \pi_i \pi_j \ [p_t(i, k) p_{t+1}(i, z) - p_{t+1}(i, k) p_t(i, z)]
\end{align*}
\]

(3.5)

where (3.3), (3.4) and (3.5) are nonnegative if \([P_t]_j \prec_{\ell r} [P_{t+1}]_j, [P_t]_i \prec_{\ell r} [P_t]_i\) and \([P_t]_j \prec_{\ell r} [P_t]_i\) for \(i \leq j\), respectively.

\(\square\)

We define a sample path as the sequence of information states that the process would occupy over time if the decision maker takes no action. That is, the sample
path emanating from state \((\pi, t)\) is \(\{\pi, \pi P_t, \pi P_{t+1}, \ldots\}\) (Fig 3.1). If we consider strongly ergodic, non-stationary Markov chains (see Isaacson and Madsen [6]), then starting from any state \((\pi, t)\) such a chain reaches a unique steady-state probability distribution \(\Pi\), i.e., the sample path originating from state \((\pi, t)\) is \(\{\pi, \pi P_t, \pi P_{t+1}, \ldots, \Pi\}\) for any \(\pi \in \Omega\) and \(t = 1, 2, \ldots\). Let \(\mathcal{A}\) be the class of sequences of stochastic matrices ([6]) such that any sequence of transition probability matrices \(\{P_t, t = 1, 2, \ldots\}\) that belongs to \(\mathcal{A}\) defines a strongly ergodic, non-stationary Markov chain.

In Proposition 3.3.5, we establish conditions under which for each \(t = 1, 2, \ldots\), the sample path originating from any information state \((\pi, t)\) is in increasing stochastic or likelihood ratio order if \(\pi\) is in \(\Omega_{st}(t) \equiv \{\pi \in \Omega : \pi \prec_{st} \pi P_t\}\) or \(\Omega_{lr}(t) \equiv \{\pi \in \Omega : \pi \prec_{lr} \pi P_t\}\), respectively.
Proposition 3.3.5.

(i) If $P_t$ is IFR and $[P_t]_i \prec_{st} [P_{t+1}]_i$ for all $t = 1, 2, \ldots$ and $i = 0, 1, \ldots, N$, then for any $\pi \in \Omega_{st}(t)$, $\pi \prec_{st} \pi P_t \prec_{st} \pi P_t P_{t+1} \prec_{st} \ldots$.

Furthermore, if $\{P_t, t = 1, 2, \ldots\} \in A$, then $\pi \prec_{st} \pi P_t \prec_{st} \pi P_t P_{t+1} \prec_{st} \ldots \prec_{st} \Pi$;

(ii) If $P_t$ is TP and $[P_t]_N \prec_{tr} [P_{t+1}]_0$ for all $t = 1, 2, \ldots$, then for any $\pi \in \Omega_{tr}(t)$, $\pi \prec_{tr} \pi P_t \prec_{tr} \pi P_t P_{t+1} \prec_{tr} \ldots$.

Furthermore, if $\{P_t, t = 1, 2, \ldots\} \in A$, then $\pi \prec_{tr} \pi P_t \prec_{tr} \pi P_t P_{t+1} \prec_{tr} \ldots \prec_{tr} \Pi$.

Proof.

(i) Since for $\pi \in \Omega_{st}(t)$, $\pi \prec_{st} \pi P_t$, then applying Propositions 3.3.3 (i) and 3.3.4 (i) yields

$$\pi \prec_{st} \pi P_t \prec_{st} \pi P_t^2 \prec_{st} \pi P_t P_{t+1} \prec_{st} \pi P_{t+1}.$$ 

Applying Propositions 3.3.3 (i) and 3.3.4 (i) further, we could establish that

$$\pi \prec_{st} \pi P_t \prec_{st} \pi P_t P_{t+1} \prec_{st} \ldots \prec_{st} \pi P_{t+k} \prec_{st} \ldots \prec_{st} \pi P_t \cdots P_{\infty} \equiv \Pi.$$ 

Note, that the existence of the steady-state probability distribution $\Pi$ is ensured by the condition that $P_t \in A$. 

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This result is established similarly to part (i) by applying Propositions 3.3.3 (ii) and 3.3.4 (ii).

If function is nonincreasing (or nondecreasing) along any set of $\pi$ in increasing stochastic order (we use $\prec_{\ell_{\ell}}$-nonincreasing and $\prec_{\ell_{r}}$-nondecreasing similarly) we describe this function as $\prec_{st}$-nonincreasing (or $\prec_{st}$-nondecreasing) in $\pi$.

To establish monotonicity of the cost function in stochastically nondecreasing $\pi$’s and in $t$ in Section 3.4, we use the result of Lemma 3.3.7, in which we establishes conditions that guarantee the monotonicity of a particular function that appears repeatedly in the cost function. Lemma 3.3.7 relies on a well-known Proposition 3.3.6 ([16]).

**Proposition 3.3.6.** If $P$ is IFR and $F_j$ is nondecreasing in $j$, $j = 0, \ldots, N$, then
\[
(PF)_i = \sum_{j=0}^{N} p_{ij} F_j
\]
is also nondecreasing in $i$.

**Lemma 3.3.7.** If $P_t$ is IFR and $[P_t]_i \prec_{st} [P_{t+1}]_i$ for all $t = 1, 2, \ldots$, and $i = 0, 1, \ldots, N$, and $F_j$ is nondecreasing in $j$, then $\pi P_t \cdots P_{t+k} F$ is

(i) $\prec_{st}$-nondecreasing in $\pi$ and

(ii) nondecreasing in $k$ for $\pi \in \Omega_{st}(t)$.

**Proof.** (i) Consider $\pi \prec_{st} \hat{\pi}$. Then by Propositions 3.3.3 (i) and 3.3.4 (i),
\[
\pi P_t \prec_{st} \hat{\pi} P_t, \ldots, \pi P_t \cdots P_{t+k} \prec_{st} \hat{\pi} P_t \cdots P_{t+k}
\]
and, hence, by Proposition 3.3.6, $\pi P_t \cdots P_{t+k} F \leq \hat{\pi} P_t \cdots P_{t+k} F$. 

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Let $\pi \in \Omega_{st}(t)$, then by Proposition 3.3.5 (i), $\pi \prec_{st} \pi P_t \prec_{st} \pi P_t P_{t+1} \prec_{st} \ldots \prec_{st} \pi P_t \cdots P_{t+k}$. Hence, by Lemma 3.3.7 part (i), the result follows.

\[\square\]

3.4 Properties of the Optimal Cost Function

In this section, we establish properties of the cost function $V(\pi, t)$ in $\pi$ and $t$. First, in Lemma 3.4.1, we establish concavity of the cost function in $\pi$, which we use in Section 3.5 to prove the convexity of the regions where replacement and inspection are optimal, respectively.

**Lemma 3.4.1.** The cost function $V(\pi, t)$ is piecewise-linear concave in $\pi \in \Omega$ for any fixed $t$, $t = 1, 2, \ldots$.

**Proof.** Since $V_n(\pi, t)$ converges to $V(\pi, t)$ as $n \to \infty$, it is sufficient to show that $V_n(\pi, t)$ is piecewise-linear concave for all $n = 0, 1, \ldots$. Furthermore, since $V_n(\pi, t)$ is given by equation (3.1), it is sufficient to show that $RP_n(\pi, t)$, $DN_n(\pi, t)$ and $IN_n(\pi, t)$ are piecewise-linear concave for all $n = 0, 1, \ldots$. Clearly, $RP_n(\pi, t)$ and $IN_n(\pi, t)$ are linear functions of $\pi$.

To prove that $DN_n(\pi, t)$ is piecewise-linear concave in $\pi$ for all $n = 0, 1, \ldots$, we proceed by induction on $n$. For $n = 0$, $V_0(\pi, t) = 0$. For $n = 1$, $DN_1(\pi, t) = \pi L(t)$ is linear in $\pi$. Assuming that $DN_2(\pi, t), \ldots, DN_{n-1}(\pi, t)$ are piecewise-linear concave, which implies that $V_1(\pi, t), \ldots, V_{n-1}(\pi, t)$ are piecewise-linear concave
in $\pi$, we need to prove that $DN_n(\pi, t)$ is piecewise-linear concave in $\pi$. Since

$$DN_n(\pi, t) = \pi L(t) + \alpha V_{n-1}(\pi P_t, t + 1)$$

is a sum of linear and piecewise-linear concave in $\pi$ components, then $DN_n(\pi, t)$ is piecewise-linear concave in $\pi$. \hfill \Box

Next, in Lemma 3.4.2, we prove that the cost function is $\prec_{st}$-nondecreasing in $\pi$ and nondecreasing in $t$, that is, the cost function is nondecreasing as the system deteriorates and/or ages. We establish these properties as well as the results of Section 3.5 under the following set of assumptions:

A1. $C_j(t)$ and $L_j(t)$ are nondecreasing in both $j$ and $t$;

A2. $C_j(t) - L_j(t)$ is nonincreasing in both $j$ and $t$;

A3. $[P_t]_i \prec_{st} [P_{t+1}]_i$, $t = 1, 2, \ldots$ and $i = 0, 1, \ldots, N$;

A4. $P_t$ is IFR for all $t = 1, 2, \ldots$.

Assumption A1 states that as the system deteriorates and/or ages, it becomes more costly to operate and replace. According to Assumption A2, operating costs increase faster than replacement costs as system’s deterioration and/or age progresses. Assumption A3 implies that the older the system, the more probable that it will move to more deteriorated states. Assumption A4 implies that the more the system deteriorates, the more likely it is to deteriorate further.

In Lemma 3.4.2, we prove monotonicity of the cost function in $\prec_{st}$-nondecreasing $\pi$ and $t$ in case of age-dependent deterioration under assumptions A1, A3 and A4.
For age-independent deterioration process in [11], we assume that $C_j$ and $L_j$ are nondecreasing in $j$ and $P$ is IFR to prove monotonicity of the cost function. Here, to establish the analogous properties of the cost function in Lemma 3.4.2, we extend the assumptions in [11] by taking into account age-dependency of the deterioration process. In [1], to prove the monotonicity of the cost function for the first model in case of perfectly observable age-dependent deterioration process, Benyamini and Yechiali also use assumptions $A_1$, $A_3$ and $A_4$ in addition to assumption that replacement always costs more than do nothing for one-step horizon problem. Under the similar assumptions, we extend the results of [1] to the case of partially observable age-dependent deterioration process.

**Lemma 3.4.2.** Under Assumptions $A_1$, $A_3$ and $A_4$, $V(\pi, t)$ is

(i) $\prec_{st}$-nondecreasing in $\pi$ for all fixed $t$;

(ii) nondecreasing in $t$ for all $\pi \in \Omega$.

**Proof.** Since $V_n(\pi, t)$ converges to $V(\pi, t)$ as $n \to \infty$, we proceed by induction on $n$.

(i) Consider two information states $\pi \prec_{st} \hat{\pi}$. For $n = 1$,

$$V_1(\pi, t) = \min\{\pi C(t), \pi L(t)\} \leq \min\{\hat{\pi} C(t), \hat{\pi} L(t)\} = V_1(\hat{\pi}, t),$$

which follows from Lemma 3.3.7 part (i). Assuming that

$$V_2(\pi, t) \leq V_2(\hat{\pi}, t), \ldots, V_{n-1}(\pi, t) \leq V_{n-1}(\hat{\pi}, t),$$
we need only prove that $V_n(\pi, t)$ is $\prec_{st}$-nondecreasing in $\pi$. By the induction assumption and Lemma 3.3.7 part (i) again,

$$V_n(\pi, t) = \min \begin{cases} 
\pi C(t) + \alpha V_{n-1}(e_0, 0), \\
\pi L(t) + \alpha V_{n-1}(\pi P_t, t + 1), \\
\pi L(t) + M + \alpha \pi P_t V_{n-1}(e, t + 1) 
\end{cases} \leq \min \begin{cases} 
\hat{\pi} C(t) + \alpha V_{n-1}(e_0, 0), \\
\hat{\pi} L(t) + \alpha V_{n-1}(\hat{\pi} P_t, t + 1), \\
\hat{\pi} L(t) + M + \alpha \hat{\pi} P_t V_{n-1}(e, t + 1) 
\end{cases} = V_n(\hat{\pi}, t).$$

(ii) For $n = 1$ and $\pi \in \Omega$, $V_1(\pi, t) = \min\{\pi C(t), \pi L(t)\} \leq \min\{\pi C(t+1), \pi L(t+1)\} = V_1(\pi, t+1)$. Assuming that $V_2(\pi, t), \ldots, V_{n-1}(\pi, t)$ are nondecreasing in $t$ for $\pi \in \Omega$, we need to prove that $V_n(\pi, t)$ is nondecreasing in $t$ for $\pi \in \Omega$:

$$V_n(\pi, t) = \min \begin{cases} 
\pi C(t) + \alpha V_{n-1}(e_0, 0), \\
\pi L(t) + \alpha V_{n-1}(\pi P_t, t + 1), \\
\pi L(t) + M + \alpha \pi P_t V_{n-1}(e, t + 1) 
\end{cases} \leq \min \begin{cases} 
\pi C(t+1) + \alpha V_{n-1}(e_0, 0), \\
\pi L(t+1) + \alpha V_{n-1}(\pi P_{t+1}, t + 2), \\
\pi L(t+1) + M + \alpha \pi P_{t+1} V_{n-1}(e, t + 2) 
\end{cases} = V_n(\pi, t+1),$$
for $\pi \in \Omega$, since by the induction assumption and part (i) of this lemma,

$$V_{n-1}(\pi, t + 1) \leq V_{n-1}(\pi, t + 2).$$

\[\square\]

### 3.5 Optimal Policy Structure Along Line Segments

In this section, we establish the result analogous to the result in [22]. Namely, we find the conditions under which the optimal policy has monotone, AM4R structure along certain line segments of information states. Furthermore, we prove that the more deteriorated and/or older the system, the sooner inspection and replacement become optimal along such line segments.

Consider the following functions

1. $\Delta_1(\pi, t) \equiv IN(\pi, t) - DN(\pi, t)$,
2. $\Delta_2(\pi, t) \equiv RP(\pi, t) - DN(\pi, t)$,
3. $\Delta_3(\pi, t) \equiv RP(\pi, t) - IN(\pi, t)$.

Here, $\Delta_1(\pi, t)$ denotes the cost of inspection minus the cost of no action in state $(\pi, t)$, $\Delta_2(\pi, t)$ denotes the cost of replacement minus the cost of no action in state $(\pi, t)$, and $\Delta_3(\pi, t)$ denotes the cost of replacement minus the cost of inspection in state $(\pi, t)$. 
Lemma 3.5.1. Functions $\Delta_1(\pi, t), \Delta_2(\pi, t),$ and $\Delta_3(\pi, t)$ are convex functions of $\pi \in \Omega$ for any fixed $t$.

Proof. The result follows from Lemma 3.4.1. \qed

Let $\Omega_{RP}(t)$, $\Omega_{DN}(t)$ and $\Omega_{IN}(t)$ be the set of all information states where replacement, no action, inspection are optimal, respectively, for a system of age $t$. Then Lemma 3.5.1 implies that $\Omega_{RP}(t)$ and $\Omega_{IN}(t)$ are convex subsets of $\Omega$ and, therefore, the optimal policy has at most one replacement region and at most one inspection region along any line segment of information states. In Lemma 3.5.2, we formally state the convexity of $\Omega_{RP}(t)$ and $\Omega_{IN}(t)$.

Lemma 3.5.2. For any state $(\pi, t)$ with fixed $t$, $\Omega_{RP}(t)$ and $\Omega_{IN}(t)$ are convex subsets of $\Omega$.

Before establishing next result, we define the line segment between two states $(\pi^1, t)$ and $(\pi^2, t)$, $[\pi^1, \pi^2](t)$, as follows

$$[\pi^1, \pi^2](t) \equiv \{ \pi \in \Omega : \pi = (1 - \lambda)\pi^1 + \lambda\pi^2, \lambda \in [0, 1] \}.$$ 

Since $[\pi^1, \pi^2](t) = [\pi^1, \pi^2](t + k)$, we drop the dependency on age for notational convenience.

In Theorem 3.5.3, we establish the main result of this section, namely that the optimal policy along any stochastically ordered line segment is a monotone, AM4R policy for any fixed $t$. Note, that in case of fixed $t$, Assumption A3 is irrelevant. Theorem 3.5.3 is analogous to the main result in [22].
Theorem 3.5.3. Under Assumptions A1, A2 and A4, there exists an optimal inspection and replacement policy which divides each line segments $[\pi^1, \pi^2]$, such that $\pi^1 \prec_{st} \pi^2$, into at most four segments, i.e., there exist $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq 1$ in general such that

$$[\pi^1, (1 - \lambda_1)\pi^1 + \lambda_1\pi^2] \subset \Omega_{DN}(t),$$

$$[(1 - \lambda_1)\pi^1 + \lambda_1\pi^2, (1 - \lambda_2)\pi^1 + \lambda_2\pi^2] \subset \Omega_{IN}(t),$$

$$[(1 - \lambda_2)\pi^1 + \lambda_2\pi^2, (1 - \lambda_3)\pi^1 + \lambda_3\pi^2] \subset \Omega_{DN}(t),$$

$$[(1 - \lambda_3)\pi^1 + \lambda_3\pi^2, \pi^2] \subset \Omega_{RP}(t).$$

Proof. The result follows from Lemmas 3.4.2 and 3.5.2.

In Corollary 3.5.5, we compare the optimal policies along the same line segments for two systems of ages $t$ and $t + k$, respectively, and show that for the older system, inspection and replacement become optimal earlier along the segment. To establish these results, we consider $\lambda_1$ and $\lambda_3$ as a functions of the system’s age $t$, i.e., we consider $\lambda_1(t)$ and $\lambda_3(t)$, and use Lemma 3.5.4, which implies that the more the system deteriorated and/or older, the more expensive no action and inspection become than replacement.

Lemma 3.5.4. Under Assumptions A1 – A4, functions $\Delta_2(\pi, t)$ and $\Delta_3(\pi, t)$ are

(i) $\prec_{st}$-nonincreasing in $\pi$ for any fixed $t$;

(ii) nonincreasing functions of $t$ for $\pi \in \Omega$. 

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Figure 3.2: Properties of $\lambda_1(t)$ and $\lambda_3(t)$ in $t$.

**Proof.** Since

$$\Delta_2(\pi, t) = \pi[C(t) - L(t)] + \alpha [V(e_0, 0) - V(\pi P_t, t + 1)]$$
and

$$\Delta_3(\pi, t) = \pi[C(t) - L(t)] - M + \alpha [V(e_0, 0) - \pi P_t V(e, t + 1)],$$

the results follow from Assumption A2 and Lemma 3.4.2.

Properties of $\lambda_1(t)$ and $\lambda_3(t)$ in $t$ from Corollary 3.5.5 are illustrated on Figure 3.5.

**Corollary 3.5.5.** Under Assumptions A1-A4, $\lambda_1(t)$ and $\lambda_3(t)$ are nonincreasing in $t$.

**Proof.** To prove that $\lambda_1(t)$ is nonincreasing in $t$, we need to show that if no action is optimal in state $(\pi, t + k)$, then it is optimal in state $(\pi, t)$, i.e., if $V(\pi, t + k) = DN(\pi, t + k)$, then $V(\pi, t) = DN(\pi, t)$, which is true since by Lemma 3.4.2 part (ii), $V(\pi, t) \leq V(\pi, t + k)$ for any $\pi \in \Omega$.

To prove that $\lambda_3(t) \geq \lambda_3(t + k)$, it is sufficient to show that if $\Delta_2(\pi, t) \leq 0$ and $\Delta_3(\pi, t) \leq 0$ then $\Delta_2(\pi, t + k) \leq 0$ and $\Delta_3(\pi, t + k) \leq 0$, which follows from Lemma 3.5.4 (ii).
3.6 Optimal Policy Structure Along Sample Paths

In this section, we establish the result analogous to the result in [11]. First, we establish monotonicity of the optimal policy with respect to replacement along \(\prec_{st}\)-increasing sample paths under Assumptions \(A1 - A4\). Second, we strengthen the assumptions stated in Section 3.4 and establish that under these new assumptions the optimal policy is a monotone, AM4R policy along any likelihood ratio increasing sample path.

3.6.1 Replacement Control Limit

In this section, we consider the optimal policy structure along \(\prec_{st}\)-increasing sample paths. Since we consider the age-dependent deterioration process, it is important to ensure the existence of the unique steady-state probability distribution \(\Pi\) and, hence, unique sample path starting from state \((\pi, t)\). That is, we need to assume that the underlying non-stationary Markov chain is strongly ergodic and update assumption \(A4\) to a new one:

\[ A4. \{P_t, t = 1, 2, \ldots\} \in \mathcal{A}, \text{ and } P_t \text{ is IFR for all } t. \]

We retain the same notation for updated assumption \(A4\). The part \(\{P_t, t = 1, 2, \ldots\} \in \mathcal{A}\) of Assumption \(A4\) implies that the system has a unique steady-state probability distribution \(\Pi\) and, if no action is taken, the system follows \(\{\pi, \pi P_t, \pi P_t P_{t+1}, \ldots, \Pi\}\) sample path.
In [11], we found the conditions under which the optimal policy is monotone with respect to replacement in case of age-independent deterioration. Here, to achieve the similar result we generalize some conditions. Particularly, in Assumptions $A_1$ and $A_2$, the costs are not only functions of the system’s deterioration level as considered in [11], but also are functions of the system’s age $t$.

Since we consider strongly ergodic non-homogeneous Markov chain, the functions $\Delta_2(\pi, t)$ and $\Delta_3(\pi, t)$ converge to $\Delta^*_2$ and $\Delta^*_3$, respectively, as $\pi \to \Pi$, where

$$\Delta^*_2 = \lim_{t \to \infty} \Delta_2(\pi, t) = \Pi[C(\Pi) - L(\Pi)] + \alpha[V(e_0, 0) - V(\Pi, \cdot)], \quad (3.9)$$

$$\Delta^*_3 = \lim_{t \to \infty} \Delta_3(\pi, t) = \Pi[C(\Pi) - L(\Pi)] - M + \alpha[V(e_0, 0) - \Pi V(e, \cdot)]. \quad (3.10)$$

Here, $\Delta^*_2$ represents the difference between the cost of replacement and the cost taking no action in steady-state probability distribution $\Pi$; similarly, $\Delta^*_3$ represents the difference between the cost of replacement and the cost of inspection in steady-state probability distribution $\Pi$; $C(\Pi) = [C_0(\Pi), \ldots, C_N(\Pi)]^T$ and $L(\Pi) = [L_0(\Pi), \ldots, L_N(\Pi)]^T$ are cost of replacement and cost of no action, respectively, in state $(\Pi, \cdot)$. Under updated assumption $A_4$, when the system reaches state $\Pi$, it means that the system start to deteriorate according to a constant transition probability matrix $P$, such that $\Pi = \Pi P$ and the actions costs do not depend on the age of the system anymore. Hence, we use $V(\Pi, \cdot)$ instead of $V(\Pi, \infty)$ and $V(e, \cdot)$ instead of $V(e, \infty)$.

In Theorem 3.6.1, we prove that the optimal policy is monotone with respect to replacement along stochastically nondecreasing sample paths, that is, there
exists a critical number $t^*(\pi, t)$ such that replacement is optimal at any state $(\pi, t)$ when $t \geq t^*(\pi, t)$. Furthermore, this critical number is $\prec_{st}$-nonincreasing in $\pi$ and nonincreasing in $t$, meaning that the more deteriorated the initial state of the system and/or the older the system initially, the sooner the replacement becomes optimal. Proof of Theorem 3.6.1 relies on the result of Lemma 3.5.4 and is similar to the proof of Theorem 1 in [11].

**Theorem 3.6.1.** If $\Pi[C(\Pi) - L(\Pi)] \leq 0$ and Assumptions A1-A4 hold, then for all $\pi \in \Omega_{st}(t)$ there exists a critical number $t^*(\pi, t)$, $\prec_{st}$-nonincreasing in $\pi$ and $t$, such that for all information states $\pi P_t \cdots P_{t+k}$ of sample path emanating from state $(\pi, t)$, the replacement is optimal if $t + k \geq t^*(\pi, t)$, and either inspection or no action is optimal otherwise.

### 3.6.2 Monotone, At Most Four Region Policy Structure

We begin this section with definition of monotone, AM4R policy along the sample path:

**Definition 6.** Consider a sample path originating from state

\[(\pi, t) : \{\pi, \pi P_t, \pi P_t P_{t+1}, \ldots\} \]  

A monotone, at most four-region policy along this sample path consists of three numbers $t^*_1(\pi, t) \leq t^*_2(\pi, t) \leq t^*(\pi, t)$, and prescribes replacement in state $\pi P_t \cdots P_{t+k}$ if $t + k \geq t^*(\pi, t)$, inspection if $t^*_1(\pi, t) \leq t + k \leq t^*_2(\pi, t)$ and no action otherwise.
In [11], we consider the age-independent deterioration and establish that the optimal policy along $\prec_{\ell r}$-increasing sample paths is a monotone, AM4R policy. To achieve this result, we assume in [11] that the transition probability matrix is $TP_2$ and consider sample paths starting in $\Omega_{\ell r}$. To receive the similar optimal policy structure in case of age-dependent deterioration, we retain assumptions $A1$, $A2$, and replace $A3$ and $A4$ with a stronger assumptions, namely:

$A3'$. $[P_t]_N \prec_{\ell r} [P_{t+1}]_0$;

$A4'$. $\{P_t, t = 1, 2, \ldots\} \in \mathcal{A}$, and $P_t$ is totally positive of order two ($TP_2$) for all $t$.

Assumption $A3'$ states that the corresponding rows of matrix $P(t)$ are $\prec_{\ell r}$-increasing with the age $t$ of the system and implies that the older the system, the more probable that it will move to more deteriorated states. Assumption $A4'$ implies that the system has a unique steady-state probability distribution $\Pi$ and, if no action is taken, the system follows $\{\pi, \pi P_t, \pi P_t P_{t+1}, \ldots, \Pi\}$ sample path. Considering transition probability matrices $P_t$ to be $TP_2$ for all $t$ implies that the more the system deteriorates the more likely it is to deteriorate further.

Under these assumptions, we show that the optimal policy has an appealing monotonic, AM4R structure along $\prec_{\ell r}$-increasing sample paths starting in $\Omega_{\ell r}(t)$. Furthermore, we show that the more deteriorated the initial condition of the system and/or the older the system, the sooner inspection becomes optimal.

In Theorem 3.6.2, we prove the existence of the critical numbers $t_1^*(\pi, t)$ and
$t_2^*(\pi, t)$. These critical numbers define the optimal inspection region along any $\prec_{t_r}$-increasing sample path, which ensures that the optimal policy exhibits a monotone, AM4R structure along every $\prec_{t_r}$-increasing sample path. This result relies on three technical lemmas presented in Appendix and is similar to the proof of Theorem 2 in [11].

**Theorem 3.6.2.** If $\Pi[C(\Pi) - L(\Pi)] \leq 0$ and Assumptions $A1 - A4'$ hold, then a monotone, AM4R policy is optimal along every sample path emanating from state $(\pi, t)$, where $\pi \in \Omega_{t_r}(t)$.

Numerical example (Fig. 3.3) illustrates the results of Theorem 3.6.2.

In Corollary 3.6.3, we establish properties of the inspection critical number $t_1^*(\pi, t)$ analogous to one that we receive in Corollary 2 in [11] for the age-independent deterioration case. Namely, in Corollary 3.6.3, we show that the more initially deteriorated and/or older the system the sooner we start the inspection, i.e., $t_1^*(\pi, t)$ is $\prec_{t_r}$-nonincreasing in $\pi$ for fixed $t$ and nonincreasing in $t$ for $\pi \in \Omega_{t_r}(t)$. Note, that the similar conclusions about the replacement critical number $t^*(\pi, t)$ are established in Theorem 3.6.1 under weaker assumptions.

**Corollary 3.6.3.** If $\Pi[C(\Pi) - L(\Pi)]$ and Assumptions $A1 - A4'$ hold, then $t_1^*(\pi, t)$ is $\prec_{t_r}$-nonincreasing in $\pi$ for fixed $t$ and nonincreasing in $t$ for $\pi \in \Omega_{t_r}(t)$. 

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Figure 3.3: Example of monotone, AM4R optimal policy.
3.7 Conclusion

We establish structural properties of the optimal policy for a not previously considered, age-dependent partially observable maintenance optimization problem with age-dependent costs. More specifically, we explore the optimal policy structure along two different ordered subsets of information states. First, we establish monotone, AM4R policy structure along straight lines of stochastically increasing information states. Second, the same policy structure is established along likelihood ratio increasing sample paths which are age-dependent sample paths reflecting further deterioration of the system. We also establish sensitivity of the inspection and replacement control limits on the age of the system.

The results in this paper are based on the previous results in [11], which establishes conditions under which the optimal policy has the same monotone, AM4R structure along sample paths of likelihood ratio increasing information states but in an age-independent setting. To establish the age-dependent result, we assume that the system deteriorates faster as it gets older and the operating and replacement costs increase with age (assumption A3) as well as deterioration level (assumptions A1 and A2). The current chapter results could be considered generalization of result of Chapter 1. In particular, if $P_t$ is the same for all $t$ then the results of current chapter could be applied for the problem in Chapter 1.
Chapter 4

Appendix

A Appendix 1

Like Rosenfield [18], we refer to a function that crosses zero at most once as a function that has the “crossing property.”

Lemma A.1. If $P$ satisfies $A4'$, and $F_i$ has the crossing property in $i$, then $\pi P^k F$ has the crossing property in both $\prec_{\ell r}$-increasing $\pi$ and in $k$ for all $\pi \in \Omega_{\pi \prec_{\ell r} \pi P}$.

Proof. Without loss of generality, we consider the case in which $F_i$ crosses zero at most once from above and assume it crosses at $j$, i.e.

$$F_0 \geq 0, F_1 \geq 0, \ldots, F_{j-1} > 0, F_j < 0, F_{j+1} \leq 0, \ldots, F_N \leq 0.$$  

Since by Proposition 1.3.2 (ii), $\pi \prec_{\ell r} \pi'$ implies $\pi P^k \prec_{\ell r} \pi' P^k$, $k = 0, 1, \ldots$, it is sufficient to show that $\pi F < 0$ implies $\pi' F \leq 0$ for any $\pi \prec_{\ell r} \pi'$.
By definition for \( \pi \prec_{\ell r} \pi' \),
\[
\frac{\pi'_0}{\pi_0} \leq \frac{\pi'_i}{\pi_i} \leq \ldots \leq \frac{\pi'_j}{\pi_j} \leq \ldots \leq \frac{\pi'_N}{\pi_N}.
\]
Let \( d_i = \frac{\pi'_i}{\pi_i}, i = 0, 1, \ldots, N \). If \( d_j = \infty \), then \( \pi_j = 0 \), and hence, since \( \pi \prec_{\ell r} \pi' \), it must be the case that
\[
\pi_{j+1} = 0, \ldots, \pi_N = 0.
\]
If \( \pi_j = 0, \ldots, \pi_N = 0 \), then \( \pi F \geq 0 \), so we need not consider \( d_j = \infty \).

If \( d_i, i = 0, \ldots, j \) are finite, then
\[
\pi' F = \pi'_0 F_0 + \ldots + \pi'_{j-1} F_{j-1} + \pi'_j F_j + \ldots + \pi'_N F_N
\]
\[
= d_0 \pi_0 F_0 + \ldots + d_{j-1} \pi_{j-1} F_{j-1} + d_j \pi_j F_j + \ldots + d_N \pi_N F_N
\]
\[
\leq d_j (\pi_0 F_0 + \ldots + \pi_{j-1} F_{j-1} + \pi_j F_j + \ldots + \pi_N F_N)
\]
\[
= d_j \pi F.
\]
Therefore, if \( \pi F < 0 \), then so is \( \pi' F \).

To show that the crossing property holds in \( k \) for \( \pi \in \Omega_{\pi \prec_{\ell r} P} \), we need to prove that \( \pi P^k F < 0 \) implies \( \pi P^{k+1} F \leq 0 \). Since for \( \pi \in \Omega_{\pi \prec_{\ell r} P} \), \( \pi P^k \prec_{\ell r} P^{k+1} \) by Proposition 1.3.3 (ii), the crossing property in \( k \) is a direct consequence of the crossing property in \( \pi \).  

\[\square\]

**Lemma A.2.** For all \( \pi \in \Omega_{\pi \prec_{\ell r} P} \)
\[
\pi P^k V(e, m) \leq V(\pi, k + m).
\]
Proof. The fact that \( \min \{a, b, c\} \) is less than or equal any of \( a \), \( b \) or \( c \), yields
\[
\sum_{j=0}^{N} [\pi P^k]_j \min \{a_j, b_j, c_j\} \leq \min \left\{ \sum_{j=0}^{N} [\pi P^k]_j a_j, \sum_{j=0}^{N} [\pi P^k]_j b_j, \sum_{j=0}^{N} [\pi P^k]_j c_j \right\}.
\]

Let
\[
a_n(e_j, m) = e_jP^mC + \alpha V_{n-1}(e_0, 0),
b_n(e_j, m) = e_jP^mL + \alpha V_{n-1}(e_j, m + 1),
c_n(e_j, m) = e_jP^mL + M + \alpha e_jP^{m+1}V_{n-1}(e, 0).
\]

Here \( a_n(e_j, m) \), \( b_n(e_j, m) \) and \( c_n(e_j, m) \) represent the total expected discounted cost-to-go with \( n \) periods left starting from state \((e_j, m)\) under the actions repair, no action and inspection, respectively.

For \( n = 1 \), \( a_1(e_j, m) = e_jP^mC \), \( b_1(e_j, m) = e_jP^mL \), and \( c_1(e_j, m) = e_jP^mL + \]
$M$, in which case

$$\sum_{j=0}^{N} [\pi P^k]_j V_1(e_j, m) = \sum_{j=0}^{N} [\pi P^k]_j \min \{a_1(e_j, m), b_1(e_j, m), c_1(e_j, m)\}$$

$$\leq \min \left\{ \sum_{j=0}^{N} [\pi P^k]_j e_j P^m C, \sum_{j=0}^{N} [\pi P^k]_j e_j P^m L, \sum_{j=0}^{N} [\pi P^k]_j (e_j P^m L + M) \right\}$$

$$= \min \left\{ \pi P^k P^m C, \pi P^k P^m L, \pi P^k P^m L + \sum_{j=0}^{N} [\pi P^k]_j M \right\}$$

$$= \min \{\pi P^{k+m} C, \pi P^{k+m} L, \pi P^{k+m} L + M\}$$

$$= V_1(\pi, k + m).$$

That is, we have

$$\pi P^k V_1(e, m) \leq V_1(\pi, k + m).$$

Now we assume that this result holds for $\pi P^k V_2(e, m)$ through $\pi P^k V_{n-1}(e, m)$ and
consider $\pi^P V_n(e, m)$:

$$\pi^P V_n(e, m) = \sum_{j=0}^{N} \left[ \pi^P \right]_j \min \{a_n(e_j, m), b_n(e_j, m), c_n(e_j, m)\}$$

$$\leq \min \left\{ \sum_{j=0}^{N} \left[ \pi^P \right]_j (e_j P^m C + \alpha V_{n-1}(e_0, 0)) , \sum_{j=0}^{N} \left[ \pi^P \right]_j (e_j P^m L + \alpha V_{n-1}(e_j, m+1)) , \sum_{j=0}^{N} \left[ \pi^P \right]_j (e_j P^m L + M + \alpha e_j P^{m+1} V_{n-1}(e_0)) \right\}$$

$$= \min \left\{ \pi^P k + m + C + \alpha V_{n-1}(e_0, 0) , \pi^P k + m + L + \alpha P^k V_{n-1}(e, m+1) , \pi^P k + m + L + M + \alpha P^k + m + 1 V_{n-1}(e, 0) \right\}$$

Since $\pi^P V_{n-1}(e, m+1) \leq V_{n-1}(\pi, k + m + 1)$ by the induction assumption, we have

$$\pi^P V_n(e, m) \leq \min \left\{ \pi^P k + m + C + \alpha V_{n-1}(e_0, 0) , \pi^P k + m + L + \alpha P^k V_{n-1}(\pi, k + m + 1) , \pi^P k + m + L + M + \alpha P^k + m + 1 V_{n-1}(e, 0) \right\}$$

$$= V_n(\pi, k + m).$$

Therefore, we finally have that $\pi^P V(e, m) \leq V(\pi, k + m)$. \hfill \Box

**Lemma A.3.** For all $f \geq 1$ under Assumptions A1, A2, A3 and A4',

$$H_f^\ell(\pi, k) \equiv V(\pi, k) - \pi^P k L - \ldots - \alpha^\ell \pi^P k + \ell L - \alpha^\ell M - \alpha^{\ell+1} \pi^P k + \ell + 1 V(e, 0) + f M$$

crosses zero at most once in $\pi \in \Omega_{\pi \prec \ell, \pi^P}$, and if it does, it does so from above.


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Appendix

Let $V_n(\pi, t, \alpha, a)$ is $n$-period cost starting in state $(\pi, t)$ using policy $a$ and a discount factor $\alpha$. Then, we could define $V(\pi, t, \alpha, a) = \lim_{n \to \infty} V_n(\pi, t, \alpha, a)$, and in Lemma C.1, we show that this limit exists. The optimal infinite-horizon cost is $V(\pi, t) = \inf_a V(\pi, t, \alpha, a)$, where $a$ is the policy that specifies an action in every state $(\pi, t)$ for every decision epoch.

**Lemma C.1.** As $n \to \infty$, $V_n(\pi, t, \alpha, a)$ converges up to $V(\pi, t, \alpha, a)$ uniformly in $\pi, t, a$ and $\alpha < 1$.

**Lemma C.2.** If $P_t, \ldots, P_{t+k}$ satisfies $A4'$, and $F_i$ has the crossing property in $i$, then $\pi P_t \cdots P_{t+k} F$ has the crossing property in $<_{\ell r}$-increasing $\pi$ and in $k$ for all $\pi \in \Omega_{\ell r}(t)$.

**Lemma C.3.** If $P_t$ is $T P_2$ and $\pi \in \Omega_{\ell r}(t)$, then for any fixed $t$, $\pi P_t V(e, t) \leq V(\pi P_t, t)$.

**Lemma C.4.** For all $f \geq 1$ under Assumptions $A1$, $A2$, $A3$ and $A4'$, function

$$H^f_\ell (\pi, t) \equiv V(\pi, t) - \pi L(t) - \ldots - \alpha^\ell \pi P_t \ldots P_{t+\ell-1} L(t + \ell) - \alpha^\ell M$$

$$- \alpha^{\ell+1} \pi P_t \ldots P_{t+\ell} V(e, t + \ell + 1) + f M$$

(C-1)

crosses zero at most once in $\pi \in \Omega_{\ell r}(t)$ and in $t$, and if it does, it does so from above.
Bibliography


