ALTERNATING LONGITUDINAL WEDGED COULOMB FORCES MINIMIZE
TRANSVERSE TUBE VIBRATIONS THROUGH NON-LINEAR COUPLING

by

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ALTERNATING LONGITUDINAL WEDGED COULOMB FORCES MINIMIZE TRANSVERSE TUBE VIBRATIONS THROUGH NON-LINEAR COUPLING

Abstract

by

TRIVIKRAM S. BELAGOD

The damping force and the self-excited force, which are a part of Heat Exchanger tube vibrations, act in the same (transverse) direction. The wedging process introduces alternating longitudinal coulomb forces that act at double the frequency of transverse vibrations and is defined by the wave equation. The transverse vibrations and the alternating longitudinal coulomb forces are coupled and act orthogonal to each other. Physical observations show that the transverse vibrations cannot exist without longitudinal vibrations. The governing constitutive equations for coupling can be shown theoretically through material non-linearities by considering higher order terms for the elastic energy, and geometric non-linearities by considering non-linear strain displacement relations. This non-linear constitutive equation when used in the equation of motion for transverse vibrations, the Gol’dberg tensorial result emerges. Energy reorganization due to this coupling results in reduced transverse vibration amplitudes. A simple experimental setup simulating this wedging process validates that transverse vibrations cannot occur without longitudinal vibrations.
Chapter 1 INTRODUCTION

The heat exchanger is the primary equipment used for energy transfer in different areas of engineering. The power generation industry uses the heat exchanger to extract the heat generated through burning of coal, fossil fuel or from the nuclear reactor core to be used to generate electricity. The most commonly used type of heat exchanger is the Shell and Tube type heat exchanger. The shell and tube heat exchanger consists of a tube bundle enclosed inside a shell. Heat transfer occurs across the tube walls between the fluid in the shell and the fluid in the tubes. The lower temperature tube side fluid is heated by the hot shell side fluid or the higher temperature tube side fluid heats the lower temperature shell side fluid. The Moisture Separator Re-heater (MSR) in the nuclear industry is a cross flow type shell and tube heat exchanger. The clean high pressure steam from the steam generator passes through the MSR for further re-heating and moisture removal of the shell side re-heat steam used to drive the turbines generating electricity. A simple diagram of the mechanism is shown in Figure 1.1. The tubes inside the MSR are long and have to be supported at regular intervals so as to withstand the severe operating conditions of high temperature and pressure. Typically the MSR contains 20-22 supports for the tube bundle. It is crucial to decide the tube spacing between support plates as the surface area of tubes is important for cost effective heat transfer. Long tube spans lead to lower natural frequencies causing concern for vibration amplitudes. Very short tube spans can lead to low heat transfer and higher cost. Flow induced vibration, caused due to the shell side fluid flowing over the tubes, becomes a very important phenomena in the MSR. As the fluid passes over the tube, vortex shedding, turbulent buffeting, whirling and blockage beat occurs across the tube.
Figure 1.1 A typical Shell and Tube type Heat Exchanger

Figure 1.2 Schematic of a section of the tube bundle with support plates
The dynamic force exerted on the tubes due to vortex shedding across the tube section results in oscillating the tubes in a frequency matching the vortex generation especially in a stable environment. If this frequency of vortex shedding reaches the resonant frequency then the tubes can fail by either striking adjacent tubes due to high amplitudes or by fatigue failure over a period of time. Steps have to be taken to make sure these vibrations are under critical levels with optimal heat transfer.

![Diagram of vortex shedding across the tube section](image)

**Figure 1.3 Vortex shedding across the tube section**

Many types of damping in the transverse direction cause energy dissipation for transverse vibrations. This is not sufficient to reduce vibration amplitudes during flow induced vibration for vertical MSR’s. The natural frequency of the system can be increased to reduce vibration amplitudes by increasing the stiffness of the tubes. This can be done by installing tubes with large thickness or by increasing the number of support plates. Either of these will reduce heat transfer in the MSR.
The wedging process helps eliminate these drawbacks. The wedging mechanism involves a structural modification in the design of the MSR where alternate support plates of the tube bundle are modified by cutting them across the width at the center into two halves. A piece of similar material and width but a small length is inserted between the two halves. This causes the two halves of the support plate to be pushed apart by a distance equal to the length of the small piece. The centers of the holes in adjacent support plates will move apart by the same length. This exerts a wedging force on the tubes, at the contact points with the support plate, in the direction of the transverse vibration.

Figure 1.4 Support plates before and after wedging
Figure 1.5 Schematic of the exaggerated effect of wedging

Before Wedging: Tube center and support plate hole centers are concentric

After Wedging: Tube center and support plate hole centers have moved by a distance equal to the wedge piece

Figure 1.6 Shift in the tube and support plate hole centers after wedging

Support plate cut at the center and moved apart by inserting an extra piece

Force exerted on the tubes

Support plate hole

Tube center shifted

Support plate hole center shifted

Tube center and support plate hole centers are concentric

Wedge Force
The wedging force helps to generate alternating longitudinal coulomb forces. As steady state is reached by the shell side fluid, the work done by the longitudinal vibrations and the transverse vibrations balances the work done by the forces exerted by the shell side fluid on the tube.

Figure 1.7 Alternating longitudinal coulomb forces generated by wedging

Figure 1.8 3D view of the alternating longitudinal coulomb forces
The above diagram shows how the contact region between the tube and tube support plate vary as the span lengths of the tubes vary. Longer tube spans result in smaller longitudinal vibration amplitudes leading to the points of contact shifting by a small amount on the tube support plate surface. This results in reduced wedging effects and higher transverse vibrations amplitudes. The shorter tube spans experience increased longitudinal vibration amplitudes and as a result the points of contact shift by much larger amounts on the tube support plate surface leading to increased wedging effects. This results in greater reduction of transverse tube vibration amplitudes. The longitudinal alternating coulomb forces increase with higher wedging. The work done by these alternating longitudinal coulomb forces reduces the energy available for transverse
vibrations resulting in reduced amplitudes. A simple experimental setup is used to validate the proposed theory. A stainless steel tube comparable to that used in the MSR is supported at various points to obtain 3 spans, 1 fixed-pinned and 2 pinned-pinned. The tube is struck at the center of one of the pinned-pinned tube spans and the response is measured on the other pinned-pinned tube span. The experimental results are analyzed and compared to validate the concept of energy re-organization between the longitudinal and transverse vibrations.
Chapter 2 MODE SHAPES AND NATURAL FREQUENCIES FOR THE TRANSVERSE VIBRATION EQUATION

Vibration analysis involves determining the natural frequencies and mode shapes of the vibrating system initially. The natural frequencies and mode shapes form the eigen values and eigen vectors of the governing differential equation of motion for the transverse vibration of the tube. The experimental setup consists of a hollow cylindrical tube of outer diameter 0.5 inches, thickness 0.02 inches and length 54 inches. One end of the tube is fixed and the other end is pinned and 2 continuous pinned supports in between. Two pinned supports between the 2 ends divide the beam into 3 spans. One span is fixed-pinned of length 3” and 2 pinned-pinned spans of length 25” each as shown in figure 3.1. The Receptance method is used to find the natural frequencies and the mode shapes of the beam\(^3\).

![Figure 2.1 Tube used for experimental setup](image-url)
2.1 THE RECEPTANCE METHOD

The concept of Receptance can be explained using the simple example of a spring shown in Figure 2.2.

![Figure 2.2 Harmonic force applied to a spring fixed at one end](image)

The equation of motion for the spring\[^3\]

\[
kx = Fe^{i\omega t} \tag{2.1}
\]

\[
x = Xe^{i\omega t} \tag{2.2}
\]

\[
kXe^{i\omega t} = Fe^{i\omega t} \tag{2.3}
\]

\[
\frac{X}{F} = \frac{1}{k} = \alpha \tag{2.4}
\]

where \(\alpha\) is the Receptance for the spring\[^3\]. Receptance is defined in simple terms as the dynamic response of a system to a unit dynamic load applied to it\[^3\]. Receptance is also the reciprocal of the dynamic stiffness and called the mechanical admittance or the dynamic flexibility of the system. The concept can be further extended to a spring mass system represented in Figure 2.3\[^3\].
The diagram shows a spring-mass system with spring stiffness ‘k’ and mass ‘M’. The spring is connected to a wall at one end and to the mass at the other. A harmonically varying force $F e^{jwt}$ is applied on the mass and $X e^{jwt}$ is the response of the system. To analyze this system using the Receptance method we split the spring and the mass into 2 sub-systems B and C as shown[3].

Figure 2.3 Spring-Mass system

Figure 2.4 Receptance applied to the Spring-Mass system
The sub-systems $B$ and $C$ are connected through equilibrium and compatibility conditions for the displacement and force\textsuperscript{[3]}. At any time $t$, the force applied to the system is balanced between the inertia force of the mass and force stored in the spring due to its stiffness. Both the spring and the mass experience the same displacements. Using these conditions the Receptance of the system can be evaluated. The original system denoted by ‘A’ is a combination of the 2 sub-systems\textsuperscript{[3]}.

\begin{equation}
F = F_B + F_C 
\tag{2.5}
\end{equation}

\begin{equation}
X_B = X_C = X 
\tag{2.6}
\end{equation}

Equation of motion for the mass\textsuperscript{[3]}:

\begin{equation}
M\ddot{x}_C = F_C e^{i\omega t} 
\tag{2.7}
\end{equation}
Where
\[ x_c = X_c e^{i\omega t} \]  \hspace{1cm} (2.8)
\[ \dot{x}_c = -X_c \omega^2 e^{i\omega t} \]  \hspace{1cm} (2.9)

Substituting in the equation of motion for the mass:
\[ -MX_c \omega^2 e^{i\omega t} = F_c e^{i\omega t} \]  \hspace{1cm} (2.10)

Gives the receptance \( \gamma \)
\[ \frac{X_c}{F_c} = - \frac{1}{M \omega^2} = \gamma \]  \hspace{1cm} (2.11)

Equation of motion for the spring \(^3\):
\[ k x_B = F_B e^{i\omega t} \]  \hspace{1cm} (2.12)

Where
\[ x_B = X_B e^{i\omega t} \]  \hspace{1cm} (2.13)

Substituting in the equation of motion for the spring:
\[ kX_B e^{i\omega t} = F_B e^{i\omega t} \]  \hspace{1cm} (2.14)

Gives the receptance \( \beta \)
\[ \frac{X_B}{F_B} = \frac{1}{k} = \beta \]  \hspace{1cm} (2.15)

Rewriting the force equilibrium equation (2.5) in terms of the receptances:
\[ \frac{X}{\alpha} = \frac{X_B}{\beta} + \frac{X_c}{\gamma} \]  \hspace{1cm} (2.16)

where
\[ \beta = \frac{1}{k} \quad \text{and} \quad \gamma = - \frac{1}{M \omega^2} \]  \hspace{1cm} (2.17)

Substituting for \( \beta \) and \( \gamma \) gives the overall receptance of the system:
\[ \alpha = \frac{1}{k - M \omega^2} \]  \hspace{1cm} (2.18)

The Receptance ‘\( \alpha \)’ approaches infinity as the applied frequency gets closer to the natural frequency causing resonance\(^3\).
\[ k - M \omega^2 = 0 \]  \hspace{1cm} (2.19)
We apply the Receptance method to find the natural frequencies of the cylindrical tube with multiple boundary conditions.

### 2.2 CLOSED FORM OF RECEPTANCES FOR A BEAM

![Figure 2.6 Force balance on an elemental length $\delta x$ of the beam](image)

The equation of motion of beam flexure

\[
\frac{\partial^2 v}{\partial t^2} + \frac{E I}{A \rho} \frac{\partial^4 v}{\partial x^4} = \frac{w}{A \rho}
\]  

(2.21)

where $v$ is the deflection in the transverse direction, $E$ is the Young’s modulus of the material of the tube, $I$ is the area moment of inertia, $A$ is the area of cross section, $\rho$ is the density of the material of the tube, and $w$ is the uniformly distributed load on the beam.

For this equation small deflections are assumed along with the $z$-$y$ planes remaining plane and don’t deform. In the real case of the MSR there is two phase fluid flowing inside and outside the tube making the equation of motion more complicated than the PDE shown above. For the experiment done here this equation is adequate.
Assuming a solution of the variable separable form for the homogenous equation\textsuperscript{[3]}:

\[ v(x, t) = X(x)T(t) \]  \hspace{1cm} (2.22)

\[ T(t) = e^{i\omega t} \]  \hspace{1cm} (2.23)

\[ v(x, t) = X(x)e^{i\omega t} \]  \hspace{1cm} (2.24)

Substituting for \( v \) in the homogenous equation of motion of the beam gives:

\[ \frac{d^4 X}{dx^4} - \frac{\omega^2 A \rho}{E I} X = 0 \]  \hspace{1cm} (2.25)

The solution to the 4\textsuperscript{th} order equation will contain 4 constants satisfying the boundary conditions. The general form of the solution, function \( X(x) \) is\textsuperscript{[3]}:

\[ X(x) = A \cos \lambda x + B \sin \lambda x + C \cosh \lambda x + D \sinh \lambda x \]  \hspace{1cm} (2.26)

\[ \lambda^4 = \frac{\omega^2 A \rho}{E I} \]  \hspace{1cm} (2.27)

The constants \( A, B, C, \) and \( D \) are determined by end conditions\textsuperscript{[3]}. Solutions can be found for the system when a harmonic force, \( F e^{i\omega t} \) and a couple, \( H e^{i\omega t} \) is applied to it. The functions relating the displacement, \( v \) and the rotation, \( \theta = \frac{\partial v}{\partial x} \) to the applied loads are the receptances and can be written as\textsuperscript{[3]}:

\[ [\psi] = \begin{bmatrix} \alpha_{xh} & \alpha_{xh'} \\ \alpha_{x'h} & \alpha_{x'h'} \end{bmatrix} \begin{bmatrix} F e^{i\omega t} \\ H e^{i\omega t} \end{bmatrix} \]  \hspace{1cm} (2.28)

Consider the end span of the beam which is pinned-pinned, with a moment \( H e^{i\omega t} \) applied at one end.
The boundary conditions for the beam are:

\[ v = \frac{\partial^2 v}{\partial x^2} = 0 \text{ at } x = 0 \quad (2.29) \]

\[ v = 0 \text{ and } \frac{\partial^2 v}{\partial x^2} = H e^{i\omega t} \text{ at } x = l \quad (2.30) \]

The boundary conditions are used to obtain the constants \( A, B, C \) and \( D \) for the solution \( X(x) \)\(^3\). The final solution for the displacement and rotation at any section \( x \) of the beam can be represented by the receptance form\(^3\):

\[ v = \alpha_{x,t} H e^{i\omega t} \quad (2.31) \]

\[ \theta = \alpha_{x',t'} H e^{i\omega t} \quad (2.32) \]

Where\(^3\):

\[ \alpha_{x,t} = \left\{ \frac{(F_7 - F_8) \sin \lambda x - (F_7 + F_8) \sinh \lambda x}{-4EI\lambda^2 F_1} \right\} \quad (2.33) \]

\[ \alpha_{x',t'} = \left\{ \frac{(F_7 - F_8) \cos \lambda x - (F_7 + F_8) \cosh \lambda x}{-4EI\lambda F_1} \right\} \quad (2.34) \]

and\(^3\)

\[ F_1 = \sin \lambda l \sinh \lambda l \quad (2.35) \]
\[ F_7 = \sin \lambda l + \sinh \lambda l \]  
(2.36)

\[ F_8 = \sin \lambda l - \sinh \lambda l \]  
(2.37)

### 2.3 RECEPTANCE METHOD APPLIED TO THE EXPERIMENTAL SETUP

The applied frequency of the harmonic moment \( M_3 \) at node 3 is varied to match with the natural frequency of the tube with multiple supports. The coordinate connecting the subsystems B and C is the rotation \( \theta_1 \) at the support 1, and the rotation \( \theta_2 \) at the support 2 connects the 2 sub-systems C and D.

Consider an \( n \) degree of freedom system. Let the responses be given by the set of generalized coordinates \( q_1, q_2, q_3, \ldots q_n \) \(^{[3]}\). The kinetic energy of the \( i^{th} \) particle in a system containing \( n \) particles is given by\(^{[3]}\):

\[ T = \frac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) \]  
(2.38)
Where $\dot{x}_i, \dot{y}_i, \dot{z}_i$ are the velocities of the particle in the $x$, $y$ and $z$ directions. Using the Taylor series expansion for the velocities $\dot{x}_i$ in terms of the generalized coordinates $q_1, q_2, \ldots, q_n$ \[^3\]:

$$
\dot{x}_i = \frac{\partial x_i}{\partial q_1} \dot{q}_1 + \frac{\partial x_i}{\partial q_2} \dot{q}_2 + \cdots + \frac{\partial x_i}{\partial q_n} \dot{q}_n
$$

(2.39)

And similarly for $\dot{y}_i$ and $\dot{z}_i$ the kinetic energy term reduces to\[^3\]:

$$
2T = a_{11} \dot{q}_1^2 + a_{22} \dot{q}_2^2 + \cdots + a_{12} \dot{q}_1 \dot{q}_2 + a_{13} \dot{q}_1 \dot{q}_3 + \cdots
$$

(2.40)

The constants $a_{11}, a_{22}, a_{33}, \ldots$ are the coefficients of inertia\[^3\].

Using the Taylor series expansion for the potential energy in terms of the generalized coordinates and neglecting the higher order terms and equating the first order terms to zero on the account of equilibrium and minimum potential energy condition we have\[^3\]:

$$
2V = c_{11} q_1^2 + c_{22} q_2^2 + \cdots + c_{12} q_1 q_2 + c_{13} q_1 q_3 + \cdots
$$

(2.41)

The constants $c_{11}, c_{22}, \ldots$ are the coefficients of stability\[^3\]. Let $Q_1, Q_2, \ldots, Q_n$ be the set of generalized forces provided by the externally applied loads on the system associated with the generalized coordinates $q_1, q_2, q_3, \ldots, q_n$ where\[^3\]

$$
Q_r = \phi_r e^{i\omega t}
$$

(2.42)

Substituting for $T$ and $V$ in the Lagrange equation\[^3\]:

$$
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_r} \right) + \frac{\partial V}{\partial q_r} = Q_r
$$

(2.43)

We obtain a set of simultaneous second order ordinary differential equations in $q_r$. In matrix form it is given by\[^4\]:

$$
\begin{bmatrix}
  a_{11} & \ldots & a_{1n} \\
  \ldots & \ldots & \ldots \\
  a_{n1} & \ldots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
  \dot{q}_1 \\
  \vdots \\
  \dot{q}_n
\end{bmatrix}
+ \begin{bmatrix}
  c_{11} & \ldots & c_{1n} \\
  \ldots & \ldots & \ldots \\
  c_{n1} & \ldots & c_{nn}
\end{bmatrix}
\begin{bmatrix}
  q_1 \\
  \vdots \\
  q_n
\end{bmatrix}
= \begin{bmatrix}
  Q_1 \\
  \vdots \\
  Q_n
\end{bmatrix}
$$

(2.44)
For a harmonic loading the response can be assumed to be harmonic and of the variable separable form:\[4]\):

\[ q_r = \psi_r e^{i\omega t} \quad (2.45) \]

We obtain the equations in matrix form:

\[
\begin{bmatrix}
(c_{11} - \omega^2 a_{11}) & \cdots & (c_{1n} - \omega^2 a_{1n}) \\
\vdots & \ddots & \vdots \\
(c_{n1} - \omega^2 a_{n1}) & \cdots & (c_{nn} - \omega^2 a_{nn})
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
\vdots \\
\psi_n
\end{bmatrix} =
\begin{bmatrix}
\phi_1 \\
\vdots \\
\phi_n
\end{bmatrix}
\]

\[ Z\Psi = \Phi \quad (2.46) \]

\[ Z \text{ is the dynamic stiffness matrix}[^4]. \text{ The inverse of this matrix is the dynamic flexibility matrix or the receptance matrix}[^4]. \text{ Solving for } \Psi \text{ in the equation above we obtain the general form}[^4]:

\[ \Psi = Z^{-1}\Phi \quad (2.48) \]

\[ Z^{-1} = [\alpha] \quad (2.49) \]

\[ \psi_r = (-1)^{r+s} \frac{\Delta_{rs}}{\Delta} \phi_s \quad (2.50) \]

Where \(\psi_r\) is the response amplitude and \(\phi_s\) is the generalized force (excitation) amplitude.

\(\Delta_{rs}\) is the minor of the element of \(\Delta\) that lies in the \(r^{th}\) column and \(s^{th}\) row and \(\Delta\) is the determinant of the \([C - \omega^2 A]\) matrix[^4]. The coefficient of \(\phi_s\) is the receptance given by[^4]:

\[ \alpha_{rs} = (-1)^{r+s} \frac{\Delta_{rs}}{\Delta} \quad (2.51) \]

The general form is given by[^4]:

\[
\begin{bmatrix}
\psi_1 \\
\vdots \\
\psi_n
\end{bmatrix} =
\begin{bmatrix}
\alpha_{11} & \cdots & \alpha_{1n} \\
\vdots & \ddots & \vdots \\
\alpha_{n1} & \cdots & \alpha_{nn}
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\vdots \\
\phi_n
\end{bmatrix}
\]

\[ (2.52) \]
The receptance matrix represented by the $\alpha$'s have already been evaluated for beams with various end conditions and can be used to obtain the elemental receptance matrices for the different beam spans of the experimental setup\textsuperscript{[3]}. Using equation (2.52) with value of $n$ equal to 3 for the 3 rotations at the supports $\theta_1, \theta_2, \theta_3$, they are related to the end moment $M_3$ through the global receptance matrix:

$$
\begin{bmatrix}
\theta_1 \\
\theta_2 \\
\theta_3
\end{bmatrix} =
\begin{bmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} \\
\alpha_{31} & \alpha_{32} & \alpha_{33}
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
M_3
\end{bmatrix}
$$

(2.53)

The global receptance matrix can be assembled by considering the elemental receptance matrices for each span separately as shown below.

Figure 2.9 Simply supported beam loaded with end moments

Consider the beam element with rotation $\theta_1 = \psi_1 e^{i\omega t}$ and loading $\phi_1 e^{i\omega t} = M_1 e^{i\omega t}$ at support 1 and rotation $\theta_2 = \psi_2 e^{i\omega t}$ and loading $\phi_2 e^{i\omega t} = M_2 e^{i\omega t}$.

$$
\psi_1 = \alpha_{11} M_1 + \alpha_{12} M_2 
$$

(2.54)

$$
\psi_2 = \alpha_{21} M_1 + \alpha_{22} M_2 
$$

(2.55)

$$
\begin{bmatrix}
\psi_1 \\
\psi_2
\end{bmatrix} =
\begin{bmatrix}
\alpha_{11} & \alpha_{12} \\
\alpha_{21} & \alpha_{22}
\end{bmatrix}
\begin{bmatrix}
M_1 \\
M_2
\end{bmatrix}
$$

(2.56)

The compatibility conditions at the 2 supports for the experimental setup are:
\[ \theta_{1B} = \theta_{1C} = \theta_1 \]  \hspace{1cm} (2.57)
\[ \theta_{2C} = \theta_{2D} = \theta_2 \]  \hspace{1cm} (2.58)
\[ \theta_{3D} = \theta_3 \]  \hspace{1cm} (2.59)

The force balance at these supports are:
\[ M_{1B} + M_{1C} = 0 \]  \hspace{1cm} (2.60)
\[ M_{2C} + M_{2D} = 0 \]  \hspace{1cm} (2.61)
\[ M_{3D} = M_3 \neq 0 \]  \hspace{1cm} (2.62)

Let the receptance for the system B, fixed-pinned span be represented by \( \beta \). The relation between the response and loading at support 1 is given by:
\[
\begin{bmatrix} \theta_{0B} \\ \theta_{1B} \end{bmatrix} = \begin{bmatrix} \beta_{00} & \beta_{01} \\ \beta_{10} & \beta_{11} \end{bmatrix} \begin{bmatrix} M_0 \\ M_{1B} \end{bmatrix} \]  \hspace{1cm} (2.63)

The values for \( \beta_{00}, \beta_{01} \) and \( \beta_{11} \) equal zero since they all represent the response at the fixed end for a loading either at \( x = 0 \) or \( x = l \). The response equation for the fixed-pinned beam reduces to:
\[ \theta_{1B} = \theta_1 = \beta_{11} M_{1B} \]  \hspace{1cm} (2.64)

For the pinned-pinned span between supports 1 and 2, with rotations \( \theta_{1C} = \theta_1 \) and \( \theta_{2C} = \theta_2 \) and moments \( M_{1C} \) and \( M_{2C} \), using compatibility conditions we obtain:
\[
\begin{bmatrix} \theta_{1C} \\ \theta_{2C} \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} M_{1C} \\ M_{2C} \end{bmatrix} \]  \hspace{1cm} (2.65)
\[ \theta_{1C} = \theta_1 = \gamma_{11} M_{1C} + \gamma_{12} M_{2C} \]  \hspace{1cm} (2.66)
\[ \theta_{2C} = \theta_2 = \gamma_{21} M_{1C} + \gamma_{22} M_{2C} \]  \hspace{1cm} (2.67)

For the pinned-pinned span between supports 2 and 3, with rotations \( \theta_{2D} = \theta_2 \) and \( \theta_{3D} = \theta_3 \) and moments \( M_{2D} \) and \( M_{3D} \), using compatibility conditions we obtain:
For the assumption that a moment $M_3$ is applied at the support 3 the receptances $\alpha_{13}, \alpha_{23}, \alpha_{33}$ can be found. Each of these receptances are given by the forms:

$$\alpha_{13} = \frac{\theta_1}{M_3}$$  \hspace{1cm} (2.71)

$$\alpha_{23} = \frac{\theta_2}{M_3}$$  \hspace{1cm} (2.72)

$$\alpha_{33} = \frac{\theta_3}{M_3}$$  \hspace{1cm} (2.73)

The natural frequencies of the system can be found by considering any one of the receptance values and equating its denominator to zero. Eliminating $\theta_1$ from the first 2 equations we obtain $M_{1C}$ in the form:

$$M_{1C} = -\frac{\gamma_{12}}{\beta_{11} + \gamma_{11}} M_{2C}$$  \hspace{1cm} (2.74)

Eliminating $\theta_2$ from the next two equations and substituting for $M_{1C}$ we obtain $M_{2D}$ in the form:

$$M_{2D} = \frac{-\delta_{23}}{\left(\frac{-\gamma_{12}}{\beta_{11} + \gamma_{11}} + \gamma_{22} + \delta_{22}\right)} M_{3D}$$  \hspace{1cm} (2.75)

Substituting for $M_{2D}$ in the equation for $\theta_3$ we obtain the direct receptance:

$$\theta_3 = \left[\frac{-\delta_{23}^2}{\left(\frac{-\gamma_{12}}{\beta_{11} + \gamma_{11}} + \gamma_{22} + \delta_{22}\right)} + \delta_{33}\right] M_{3D}$$  \hspace{1cm} (2.76)

where
\[
\alpha_{33} = \left[\frac{-\delta_{23}^2}{\left(-\frac{y_{12}^2}{\beta_{11} + y_{11}} + y_{22} + \delta_{22}\right)} + \delta_{33}\right]
\]  
(2.77)

The receptances \(\alpha_{13}, \alpha_{23}\) are found by the similar method and are given by:

\[
\alpha_{13} = y_{11} \left[\frac{(\delta_{22} + y_{22})(\alpha_{33} - \delta_{33}) + \delta_{23}}{\delta_{32}y_{21}}\right] - \frac{y_{12}}{\delta_{32}}(\alpha_{33} - \delta_{33})
\]  
(2.78)

\[
\alpha_{23} = \frac{\delta_{22}}{\delta_{32}}(\alpha_{33} - \delta_{33}) + \delta_{23}
\]  
(2.79)

The global receptance matrix can be assembled by deriving the components as shown so far by considering moments \(M_1\) and \(M_2\) applied individually at supports 1 and 2. The derived receptance \(\alpha_{33}\) will reach infinity when the forcing frequency matches the resonant frequency. The denominator of the receptance is the frequency equation of the system ‘A’. The natural frequencies are calculated by equating the denominator of \(\alpha_{33}\) to zero and are shown in the graph below.
Figure 2.10 Receptance graph showing the natural frequencies of the system.
When the frequency equation equals zero the curve crosses the $\lambda$ axis. These points correspond to the natural frequencies of the system. The $\lambda$ value is related to the frequency as:

$$\omega = \lambda^2 \sqrt{\frac{E I}{A \rho}}$$

(2.80)

where $\omega = 2\pi f$ is the angular frequency, $E$ is the Young’s modulus of the tube material, $I$ is the area moment of inertia of the tube, $A$ is the area of cross-section of the tube, and $\rho$ is the density of the material of the tube.

The table below shows the first 8 natural frequencies:

<table>
<thead>
<tr>
<th>$\lambda$ value from graph</th>
<th>Frequency $f$ (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.135</td>
<td>95.32</td>
</tr>
<tr>
<td>0.184</td>
<td>177.08</td>
</tr>
<tr>
<td>0.260</td>
<td>353.58</td>
</tr>
<tr>
<td>0.300</td>
<td>470.74</td>
</tr>
<tr>
<td>0.385</td>
<td>775.28</td>
</tr>
<tr>
<td>0.425</td>
<td>944.75</td>
</tr>
<tr>
<td>0.510</td>
<td>1360.44</td>
</tr>
<tr>
<td>0.545</td>
<td>1553.58</td>
</tr>
</tbody>
</table>

Table 2.1 Natural frequencies of the beam

The validity of the results can be verified by comparing the Fast Fourier Transform graphs of the transient curve representing the experimental data. The graph shows that the first natural frequency, 95.3 Hz, of the system was excited during the transient vibration process.
2.4 MODE SHAPES

When a moment is applied at the end of the beam, bending moments are generated at the supports. The rotations at each section of the beam are related to these moments through the receptances of the beam section as shown below\[3\].

\[
\begin{bmatrix}
\theta_1 \\
\theta_2
\end{bmatrix} =
\begin{bmatrix}
\alpha_{11} & \alpha_{12} \\
\alpha_{21} & \alpha_{22}
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2
\end{bmatrix} e^{i\omega t} \tag{2.81}
\]

\[
\begin{bmatrix}
\psi_1 \\
\psi_2
\end{bmatrix} =
\begin{bmatrix}
\alpha_{11} & \alpha_{12} \\
\alpha_{21} & \alpha_{22}
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2
\end{bmatrix} \tag{2.82}
\]

where \(\psi_1\) and \(\psi_2\) are the amplitudes of the response \(\theta_1\) and \(\theta_2\), \(\phi_1\) and \(\phi_2\) are the amplitudes of the loading\[3\], \(M_1\) and \(M_2\) and \(\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}\) are the receptances of a simply supported beam.

Solving the above 2 simultaneous equations for \(\psi_2\) and \(\phi_2\) we have\[1\]:

\[
\begin{bmatrix}
\psi_2 \\
\phi_2
\end{bmatrix} =
\begin{bmatrix}
\frac{\alpha_{22}}{\alpha_{21}} & -\alpha_{21} + \frac{\alpha_{11}}{\alpha_{21}} \\
\frac{1}{\alpha_{21}} & \frac{\alpha_{11}}{\alpha_{21}}
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
\phi_1
\end{bmatrix} \tag{2.83}
\]

The rotation and loading at one end of the beam section is related to the known rotation and loading at the other end. The 4 X 4 matrix that was derived is called the ‘Transfer Matrix’\[1\]. This relation is used to determine the loading and rotation at each support of the beam. The fixed end is assumed to take a unit moment and along with the condition of zero rotation, forms the initial conditions for the beam. Using these values, moment and rotation at support 1 and subsequently for supports 2 and 3 are obtained. Since the Transfer Matrix is defined for pinned-pinned beams, the zero rotation at the fixed end automatically takes care of the result for the rotation and loading at the pinned end of the fixed-pinned beam. The table below shows the values of the moments and rotations calculated for a unit moment and 0 rotation applied at the fixed end.
Table 2.2 Moments and rotations at the supports of the beam

<table>
<thead>
<tr>
<th>Support</th>
<th>Moment</th>
<th>Rotation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed end</td>
<td>1 (assumed)</td>
<td>0 (boundary condition)</td>
</tr>
<tr>
<td>Pinned support “1”</td>
<td>-1.999</td>
<td>-0.00005744</td>
</tr>
<tr>
<td>Pinned Support “2”</td>
<td>-1.6805</td>
<td>0.000117</td>
</tr>
<tr>
<td>Pinned Support “3”</td>
<td>-0.0131311</td>
<td>0.00019</td>
</tr>
</tbody>
</table>

The deflection of the beam sections are related to moments at the supports through the receptances as\[^3\]:

\[
v = \beta_{x0'} M_1 + \beta_{xl'} M_2 \tag{2.84}\]

where \(v\) is the deflections at any section ‘\(x\)’ of the beam, \(\beta_{x0'}\) is the receptance relating the displacement at any \(x\) due to the moment \(M_1\) and \(\beta_{xl'}\) is the receptance relating the displacement at any ‘\(x\)’ due to the moment \(M_2\)[\(^3\)]. By substituting the appropriate functions and load values the mode shapes across the length of the beam can be obtained. The first 4 mode shapes are shown in the graphs below.

Figure 2.11 Mode shape of the beam corresponding to the first natural frequency
Figure 2.12 Mode shape of the beam corresponding to the second natural frequency

Figure 2.13 Mode shape of the beam corresponding to the third natural frequency
Figure 2.14 Mode shape of the beam corresponding to the fourth natural frequency
Chapter 3 WEDGED AND UN-WEDGED VIBRATION EXPERIMENT

3.1 DESIGN OF THE EXPERIMENT

The Moisture Separator Re-heater is a cross flow Shell and Tube heat exchanger where the reheat steam flows perpendicular to the tubes carrying the heating steam. The reheat steam absorbs heat from the heating steam across the tube wall which is then used to increase the efficiency of the turbine and the overall plant efficiency. The tubes experience a force due to the flow which sets up transverse tube vibrations due to flow induced vibration. These vibrations have to be kept under critical limits to prevent damage to the tubes. The tube bundle in the MSR is a collection of U-tubes fixed at the 2 ends. Each U-tube has three regions, an upper straight piece, the U bend, and the lower straight piece. The straight piece of the U-tube is a fixed-pinned beam with numerous pinned supports in between due to the many tube support plates.

![U-tube used in the Moisture Separator Re-heater](image)

Figure 3.1 U-tube used in the Moisture Separator Re-heater
Normal method of reducing transverse vibrations is to increase the natural frequencies of the tubes by increasing the number of support plates, thickening the tubes or increasing the diameter of the tubes which is uneconomical and impractical for a Moisture Separator Re-heater. Wedging forms a suitable alternative to provide damping in these tubes. In this mechanism alternate support plates are selected and are cut at the center of the plate across its width. To this gap a piece of same material is inserted which causes the two pieces of the support plate to be pulled apart. This wedging force causes longitudinal alternating coulomb forces which dampens longitudinal vibrations. This causes more energy to be removed through longitudinal vibrations leading to reduced transverse vibration amplitudes.

Figure 3.2  3D view showing the wedged support plates
There exists a shifting point contact at all times between the tube and tube support plate. Alternating longitudinal coulomb forces are generated at this region. These forces increase with wedging. As such more energy is needed to setup longitudinal vibrations when the tubes are set in motion. This results in energy re-organization between the longitudinal and transverse vibrations. The remaining lower energy for transverse vibrations cause reduced vibration amplitudes.

Figure 3.3 Alternating Longitudinal coulomb forces generated at the support region
3.2 EXPERIMENTAL SETUP

A 304 grade Stainless Steel tube of outer diameter 0.5” and inner diameter 0.46” is selected. Metal plates are used for supporting the tube. One end of the tube is fixed and the other end is pinned. In between 2 pinned supports are inserted with 2 spans of 25” from the pinned end and a third span of 3” at the fixed end. A wooden frame is built around the tube and support system to hold them together. The pinned supports 2 and 4 are fixed with respect to the wooden frame. The pinned support 3 can be displaced in the z-direction to simulate the wedging effect. The fixed end of the tube is bolted to the wooden frame and the wooden frame is clamped to a rigid surface to make sure only the tube vibrates when excited. The experiments were conducted for 2 arrangements, horizontal and vertical.
3.3 STABILITY OF THE SETUP

The setup was placed on a heavy and sturdy table and a few discrepancies such as the table wobbling were overcome by placing weights around the setup to make the table sturdier. For the vertical arrangement the wooden frame was clamped rigidly to a concrete column of the building with the fixed end resting on the floor.
3.4 MEASUREMENTS AND INSTRUMENTATION

The analysis is done by studying the response of the tube when loaded. The loading is given in the form of an impulse or an instantaneous force at the mid-point of one span. The response to this impulse is measured at the mid-point of the adjacent span. The Linear Variable Differential Transformer (LVDT) was used to measure the amplitude variation in the beam. It consists of a cylinder with a set of primary and secondary coils. A metal pin acting as a magnetic core moves through the center of the cylinder. The movement of the metal pin in the electric field generated by the coils causes a variation in the voltage output. This variation in the voltage is calibrated to read linear displacement of the magnetic core. The data recorded by the LVDT is read using a computer via the Optim Megadac data acquisition system and the TCS data acquisition software.

3.5 EXPERIMENTAL PROCEDURE

The whole setup was clamped to a sturdy table and the instruments were connected for data collection. The knife edge of the hammer was used to strike the tube at the midpoint of one span and the response was measured at the midpoint of the adjacent span. After a few trials wedging was done by inserting metal plates of certain thickness between the movable support plate and the wooden frame, thereby displacing it in the z-direction and shifting its center by a distance equal to the thickness of the metal plate which measured about 1/10th of an inch. The tube was struck again at the same point with the wedging and the response was measured. The same procedure was repeated for the vertical arrangement.
Chapter 4 EXPERIMENTAL RESULTS

The data recorded during the experiments show a transient decaying curve, represented by Figures 4.1 through 4.4. The curve shows the transverse vibration amplitude variation of the tube with respect to time. A Fast Fourier Transform was performed on the raw data to extract the frequency content of the curves. The FFT graphs show that the first natural frequency, which matches with the theoretical value, was excited during the loading process for both the wedged and un-wedged case, whether horizontal or vertical. The same natural frequency value for all cases suggests that the stiffness of the tube system has not changed and that the damping that is obtained for the wedged case comes in during the dynamic process of vibration.
Figure 4.1 Time-Amplitude response of Horizontal Un-wedged tube
Figure 4.2 Time-Amplitude response of Horizontal Wedged tube
Figure 4.3 Time-Amplitude response of Vertical Un-wedged tube
Figure 4.4 Time-Amplitude response of Vertical Wedged tube
4.1  HORIZONTAL WEDGED AND UN-WEDGED TUBE

The curves for the horizontal wedged and un-wedged case show a steep drop in amplitude for the first few cycles and then a more gradual decay in the vibration. The weight of the tube acts in the direction of the applied force on the tube causing the negative part of the curve to deflect slightly more than the positive side. The weight of the tube causes it to rest on the support plate providing a wedging effect in that direction. As a result of this the un-wedged case shows two regions just like the wedged case, one with rapid decay and the other with a more gradual decay. In the wedged case, the self weight of the tube adds to the wedging effect causing a rapid decay of the amplitude in the first few cycles. The subsequent few cycles show diminished vibration amplitudes as compared to the same region for the un-wedged case.

4.2  VERTICAL WEDGED AND UNWEDGED CASE

The curves for the un-wedged case indicate a gradual and uniform decay of vibration amplitudes. The self weight of the tube does not affect the amplitude readings since it does not act in the direction of transverse vibrations. The wedged case shows two regions of amplitude decay, an initial rapid decay and a gradual decay for the subsequent cycles. In both cases the initial rapid decay indicates a form of energy removal for transverse vibrations.

4.3  DAMPING ANALYSIS

As the vibration mechanism begins, higher transverse vibration amplitudes result in higher magnitude of longitudinal alternating coulomb forces. As such longitudinal vibrations utilize more energy provided by the loading on the tube. As the curve decays transverse vibrations become more dominant. But the energy available for transverse
vibrations is much lower compared to the un-wedged case resulting in reduced vibration amplitudes. The alternating longitudinal coulomb forces increase with wedging which causes energy re-organization thereby reducing transverse vibration amplitudes.

4.4 LOGARITHMIC DECREMENT AND COULOMB DAMPING

The logarithmic decrement is a measure of the amount of damping in a system for the transient part of the vibration for one complete cycle of oscillation. It is calculated using the formula:

$$\delta = \frac{1}{n} \ln \left( \frac{x_0}{x_n} \right)$$  \hspace{1cm} (4.1)

where $x_0$ is the greater of the two amplitudes and $x_n$ is the amplitude $n$ periods away.

Coulomb damping arises when the tube slides on the surface of the support plate. At the contact points a friction force is generated due to the wedging in the tube system. This is the alternating longitudinal coulomb force that resist the longitudinal vibrations. The amplitude reduction due to coulomb friction is given by $^{[11]}$:

$$\Delta = \frac{4 \mu N}{k}$$  \hspace{1cm} (4.2)

Where $\Delta$ is the change in amplitude for one complete cycle, $\mu$ is the friction coefficient, $N$ is the reaction due to the wedging force, $k$ is the axial stiffness of the tube. The coulomb friction damping varies linearly as compared to the logarithmic decrement which varies exponentially for a transient curve as shown by equations 4.1 and 4.2 and further illustrated by the 2 regions in the experimental results in Figure 4.5.
Figure 4.5 Logarithmic decrement for a transient vibration curve
Figure 4.6 Logarithmic decrement for Horizontal Un-wedged curve
Figure 4.7 Coulomb Damping for Horizontal Wedged curve
Figure 4.8 Logarithmic decrement for Vertical Un-wedged curve
Figure 4.9 Coulomb and Logarithmic damping for Vertical Wedged curve
<table>
<thead>
<tr>
<th>CASE</th>
<th>Friction coefficient for the first half of the curve</th>
<th>Logarithmic decrement for second half of vibration curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizontal Un-Wedged</td>
<td>0.231</td>
<td>0.0671</td>
</tr>
<tr>
<td>Horizontal Wedged</td>
<td>0.277</td>
<td>0.0602</td>
</tr>
<tr>
<td>Vertical Un-Wedged</td>
<td>-</td>
<td>0.0727</td>
</tr>
<tr>
<td>Vertical Wedged</td>
<td>0.213</td>
<td>0.07167</td>
</tr>
</tbody>
</table>

Table 4.1 Coulomb Friction coefficient and Logarithmic decrement values for the 2 characteristic regions of the curve

### 4.4.1 HORIZONTAL WEDGED AND UNWEDGED CASE

The horizontal un-wedged tube shows a rapid decay as compared to that of the vertical tube for the initial part because the self weight of the tube simulates a wedging effect. The higher amplitude values at the start of measurement are associated with higher longitudinal coulomb forces. This causes a large portion of the energy to be used by alternating longitudinal coulomb forces to resist the longitudinal vibrations. The subsequent cycles of vibration have lower remaining energy thus resulting in reduced transverse vibration amplitudes. The horizontal wedged tube has a logarithmic decrement value for the second half of the vibrations comparable to that of the un-wedged case.

### 4.4.2 VERTICAL WEDGED AND UN-WEDGED CASE

The vertical arrangement does not experience the self weight effects of the tube. The vertical un-wedged tube shows a gradual decay of vibration amplitudes with viscous damping dominating. The vertical wedged case shows a rapid decay for the initial part of the curve due to the wedging force generating coulomb friction force at the support plate region. In this region coulomb damping dominates over viscous damping. This is
indicated by a linear decay in the amplitude of the curve for this region. As the energy is removed by the alternating longitudinal coulomb forces it causes a gradual decay of transverse vibration amplitudes for the subsequent cycles of vibration where viscous damping is more dominant.

4.5 DAMPING IN TERMS OF ENERGY

Finite element analysis was done using Abaqus to obtain the static deflection of the tube for a unit load applied to it at the center of the pinned-pinned span\textsuperscript{9}.

\begin{equation}
    k = \frac{F}{v_{\text{max}}} \tag{4.3}
\end{equation}

Figure 4.10 Finite Element result of the beam for static deflection due to a unit load

The value of deflection obtained where the unit load is applied is used to determine an equivalent single degree of freedom stiffness of the tube considered.

Where \( k \) is the equivalent stiffness of the beam, \( v_{\text{max}} \) is the deflection due to a unit applied load \( F \). The stiffness of the tube does not change for the wedged or the un-wedged case.
The energy stored in a spring is represented by:

\[ E = \frac{1}{2} k x^2 \]  

(4.4)

Where \( k \) is the stiffness of the spring, \( x \) is the deflection in the spring due an applied force and \( E \) is the energy stored in the spring. When compared with the beam, \( k \) forms the equivalent stiffness of the beam and \( x(t) \) the transverse vibration amplitude obtained from the experimental results. The graphs, 4.11, 4.12 and 4.13 show the energy distribution during the transient vibration of the beam. Further explanation of these curves are given below.

4.5.1 HORIZONTAL WEDGED AND UN-WEDGED CASE

The curves for both wedged and un-wedged case can be divided into 2 regions, first region consisting of the first 5 cycles of oscillation and the second region constituting the remaining part of the transient curve. These 2 regions were elucidated in detail in section 4.4 as the Coulomb dominant region and Logarithmic Decrement dominant region. The un-wedged case shows lower, if not comparable, total energy as compared to the wedged case during the first 5 cycles of oscillation. In the second region for the subsequent few cycles the energy available for the un-wedged case is higher than that compared with the wedged case. This can be explained for the wedged case by the fact that the magnitude of the longitudinal coulomb forces generated due to wedging, which resist the longitudinal vibration, are much higher than the un-wedged case indicating higher energy removal during this time. As the transverse vibration amplitudes reduce the magnitude of the longitudinal coulomb forces also reduce but the amount of energy remaining in the system for transverse vibration is much lesser leading to reduced transverse vibration amplitudes.
4.5.2 VERTICAL WEDGED AND UN-WEDGED CASE

The wedged case is divided into 2 regions, one region constituting the first 0.15 seconds, which is Coulomb damping dominant, and the second region constituting the subsequent cycles, which is Logarithmic Decrement dominant, whereas for the un-wedged case both regions are Logarithmic Decrement dominant. A comparison of the second region for the wedged and un-wedged cases indicate that the energy available for the wedged case is much less than that available for the un-wedged case due to the energy removal from the system through longitudinal vibrations which are resisted by the large magnitude alternating longitudinal coulomb forces. This is seen by the rapid decay of the wedged vibration curve and a gradual decay of the un-wedged vibration curve.
Energy available for the first few cycles

Energy available for un-wedged case is more than wedged case

Figure 4.11 Energy distribution comparison between the Horizontal Un-wedged and Wedged tubes
Figure 4.12 Energy distribution of Vertical Un-wedged tubes
Figure 4.13 Energy distribution of Vertical Wedged tubes
Chapter 5 NON-LINEAR LONGITUDINAL-TRANSVERSE COUPLING TENSORIAL RESULT FOR THE ISOTROPIC SOLID

5.1 ELASTICITY, ISOTROPY AND DEFORMATION IN MATERIALS

An elastic material is one in which the deformation is independent of the path by which the body deforms from an initial state to a final state\[^{[2]}\]. Forces acting on an elastic body will deform it, and once these forces are removed the body will return to its initial state. Stresses arise in the body due to the deformation or the strain induced due to the applied loads on the body. This stress is independent of strain history or the rate of strain of the body. To solve the problem of equilibrium of a deformed body we need along with the equations of motion, six equilibrium equations, constitutive equations relating the stress and strain and the strain-displacement relations. The stress strain relations for a material have to be determined experimentally but some theoretical insight can be obtained by using the conservation of energy principle and the following assumptions\[^{[8][10][2][5][6]}\].

1. The deformation process is adiabatic, i.e., no heat (energy) loss occurs during the process.
2. The material is perfectly elastic and independent of the manner of deformation and the body returns to its initial state when the forces are removed.
3. The role of dissipative forces is negligible or in other words it is a conservative process.

Under these conditions the work done to deform a material of volume dV is given by\[^{[10]}\]:

\[
\delta I = W(\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{13}, \epsilon_{23})dV
\]

(5.1)
where \( W(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}) \) is the strain energy density function that depends on the strain components. These strain components can be expressed in terms of principal strains \((\varepsilon_1, \varepsilon_2, \varepsilon_3)\) and the corresponding direction cosines \((\theta, \phi, \psi)\) of the three mutually perpendicular principal axes with respect to the reference axes \(X_1, X_2\) and \(X_3\).

\[
\delta I = W(\varepsilon_1, \varepsilon_2, \varepsilon_3, \theta, \phi, \psi) dV \quad (5.2)
\]

The dependence on the direction cosines make the material anisotropic where the body represents different properties in different directions, such as responses to forces and work done to deform it. A material that does not show this dependence where the response is the same for a given force in all directions is called an isotropic material\(^{[10]}\).

In a more general sense the strain energy density function is dependent on the principal strains and in turn the strain invariants:

\[
\delta I = W(\varepsilon_1, \varepsilon_2, \varepsilon_3) dV \quad (5.3a)
\]

\[
\delta I = W(I_1, I_2, I_3) dV \quad (5.3b)
\]

From the first law of thermodynamics we have\(^{[2]}:\)

\[
\delta W_e + \delta \overline{Q} = \delta U + \delta T \quad (5.4)
\]

Where \(\delta W_e\) is the external work done, \(\delta \overline{Q}\) is the heat added to the system, \(\delta U\) is the internal strain energy and \(\delta T\) is the change in the kinetic energy in the system. The conservation of kinetic energy law states\(^{[2]}:\)

\[
\delta W_e + \delta W_i = \delta T \quad (5.5)
\]

Where \(\delta W_e\) is the external work done, \(\delta W_i\) is the work done due to internal forces and \(\delta T\) is the change in the kinetic energy of the system. Substituting for \(\delta W_e\) in the second equation and assuming no heat is added to the system, \(\delta \overline{Q} = 0\)^{[2]},

\[
\delta W_i = -\delta U \quad (5.6)
\]
The variation in the external work can be represented by the variation in the work done by external forces $F_i$, and due to the surface stresses $\sigma_i^{[2]}$:

$$\delta W_e = \int_V F_i \delta u_i dV + \int_S \sigma_{ij} n_i \delta u_i dS$$

(5.7)

Using Gauss’ theorem$^{[2]}$ and $\sigma_{ij} = \sigma_{ji}^{[2]}$:

$$\delta W_e = \int_V \sigma_{ji} \frac{\partial (\delta u_i)}{\partial x_j} dV$$

(5.8)

The displacement gradient tensor $\frac{\partial (\delta u_i)}{\partial x_j}$ can be represented by its symmetric and skew symmetric parts$^{[2]}$:

$$\frac{\partial (\delta u_i)}{\partial x_j} = \delta \varepsilon_{ij} + \delta \omega_{ij}$$

(5.9)

Substituting in equation (5.8) we get:

$$\delta W_e = \int_V \left( \sigma_{ji} \delta \varepsilon_{ij} + \sigma_{ji} \delta \omega_{ij} \right) dV$$

(5.10)

The skew symmetric part is zero and the equation reduces to:

$$\delta W_e = \int_V \sigma_{ij} \delta \varepsilon_{ij} dV$$

(5.11)

From the equation of the first law of thermodynamics and with the assumption of no temperature change in the body we have$^{[3]}$:

$$\delta W_e = \delta U$$

(5.12)

$$\delta U = \int_V \sigma_{ij} \delta \varepsilon_{ij} dV$$

(5.13)

$$\delta U = \int_V \delta W dV$$

(5.14)
\( W \) is the strain energy density or the strain energy per unit volume under deformation and 
\( U \) is the total strain energy.

\[
\delta W = \sigma_{ij} \delta \epsilon_{ij} \tag{5.15}
\]

From the definition of a total derivative we have\(^2\):

\[
\delta W = \frac{\partial W}{\partial \epsilon_{ij}} \delta \epsilon_{ij} \tag{5.16}
\]

\[
\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}} \tag{5.17}
\]

The stress is the derivative of the strain energy density function and is given as\(^8\):

\[
\sigma_{ij} = \frac{\partial W(\epsilon)}{\partial \epsilon_{ij}} \tag{5.18}
\]

Using Taylor series expansion for the strain energy density function about the zero strain state\(^8\):

\[
W(\epsilon) = W(0) + \sum_{i,j} \frac{\partial W(0)}{\partial \epsilon_{ij}} \epsilon_{ij} + \sum_{i,j,k,l} \frac{1}{2!} \frac{\partial^2 W(0)}{\partial \epsilon_{ij} \partial \epsilon_{kl}} \epsilon_{ij} \epsilon_{kl} \tag{5.19}
\]

+ \cdots

The first term is zero since there is no strain energy in the material during zero strain as the body has not deformed. The second term is also zero since the derivative represents residual stress in the body

\[
\sigma_{ij,\text{residual}} = \frac{\partial W(0)}{\partial \epsilon_{ij}} = 0 \tag{5.20}
\]

Neglecting the higher order terms in the series we obtain an expression for the strain energy density of the material\(^8\):

\[
W(\epsilon) = \frac{1}{2} \frac{\partial^2 W(0)}{\partial \epsilon_{ij} \partial \epsilon_{kl}} \epsilon_{ij} \epsilon_{kl} \tag{5.21}
\]
\[ W(\epsilon) = \frac{1}{2} C_{ijkl} \epsilon_{ij} \epsilon_{kl} \]  

(5.22)

\[ C_{ijkl} = \frac{\partial^2 W(0)}{\partial \epsilon_{ij} \partial \epsilon_{kl}} \]  

(5.23)

where \( C_{ijkl} \) represents the 4\(^{th}\) order stiffness tensor and its components are the elastic moduli. This coefficient tensor consists of \( 3^4 = 81 \) components that must be calculated to solve the problem. By applying the strain tensor symmetry and the symmetry of the partial derivative to this tensor the number of components can be reduced to 21\(^{[8]}\).

\[ \epsilon_{ij} = \epsilon_{ji} \rightarrow C_{ijkl} = C_{jikl} \]  

(5.24)

\[ \epsilon_{kl} = \epsilon_{lk} \rightarrow C_{ijkl} = C_{ijlk} \]  

(5.25)

\[ \frac{\partial^2 W}{\partial \epsilon_{ij} \partial \epsilon_{kl}} = \frac{\partial^2 W}{\partial \epsilon_{kl} \partial \epsilon_{ij}} \rightarrow C_{ijkl} = C_{klij} \]  

(5.26)

Considering the definition of the stress in terms of the strain energy density function\(^{[8]}\):

\[ \sigma_{ij} = \frac{\partial W(\epsilon)}{\partial \epsilon_{ij}} \]  

(5.27)

\[ \sigma_{ij} = \frac{\partial}{\partial \epsilon_{ij}} \left\{ \frac{1}{2} C_{pqrs} \epsilon_{pq} \epsilon_{rs} \right\} \]  

(5.28)

\[ \sigma_{ij} = \frac{1}{2} C_{pqrs} \left( \frac{\partial \epsilon_{pq}}{\partial \epsilon_{ij}} \epsilon_{rs} + \frac{\partial \epsilon_{rs}}{\partial \epsilon_{ij}} \epsilon_{pq} \right) \]  

(5.29)

\[ \sigma_{ij} = \frac{1}{2} C_{pqrs} (\delta_{lp} \delta_{jq} \epsilon_{rs} + \delta_{lr} \delta_{js} \epsilon_{pq}) \]  

(5.30)

\[ \sigma_{ij} = \frac{1}{2} (C_{ijrs} \epsilon_{rs} + C_{pqij} \epsilon_{pq}) \]  

(5.31)

The summation rule is applied to the repeating indices on both terms of the right hand side of the above equations. The stress term reduces to\(^{[8]}\):

\[ \sigma_{ij} = C_{ijkl} \epsilon_{kl} \]  

(5.32)
This forms the constitutive equation for the linear elastic solid under the assumptions mentioned above. The stress components are the linear functions of the infinitesimal strain components. This stiffness tensor is positive definite.

\[ \mathbf{e} : \mathbf{C} : \mathbf{e} \geq 0 \]  
\[ C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \geq 0 \]

This suggests that the tensor can be inverted giving \[^8\]:

\[ \varepsilon_{ij} = S_{ijkl} \sigma_{kl} \]

where the 4\(^{th}\) order tensor, \(S\) is the compliance tensor. Consider 2 sets of basis vectors \(\{e_i\}\) and \(\{e'_i\}\) where\[^8\]:

\[ \varepsilon_{ij} = S_{ijkl} \sigma_{kl} \]  
\[ \varepsilon'_{ij} = S'_{ijkl} \sigma'_{kl} \]

in the \(\{e_i\}\) basis and\[^8\]

\[ \varepsilon'_{ij} = S'_{ijkl} \sigma'_{kl} \]

in the \(\{e'_i\}\) basis. The stresses and the compliance tensors in the two different bases are related to each other through the transformation law for 2\(^{nd}\) and 4\(^{th}\) order tensors by the orthogonal matrices \([Q]\)^\[^8\]:

\[ \sigma'_{ij} = Q_{ij} Q_{jq} \sigma_{pq} \]
\[ S'_{ijkl} = Q_{ij} Q_{jq} Q_{kr} Q_{ls} S_{pqrs} \]

For a particular set of orthogonal matrices \([Q]\) the compliance tensors are the same in both the primed and the unprimed bases.

\[ S'_{ijkl} = S_{ijkl} \]  
\[ S_{ijkl} = Q_{ij} Q_{jq} Q_{kr} Q_{ls} S_{pqrs} \]

\([Q]\) being the symmetry transformation, for the same material properties in the two bases the above expression suggests material isotropy\[^8\]. A material is thus said to be elastically
isotropic when the elastic moduli are invariant with respect to all orthogonal transformations. To summarize the theoretical explanation of the constitutive relations, for a perfectly elastic isotropic material under adiabatic conditions and a conservative deformation process, the stress is a function of the strain induced in the material. The stress term derived so far has been for the linear case where only up to the quadratic terms in the strain energy density function were considered. To develop the non-linear theory, higher order terms have to be considered and for the subsequent analysis the third degree terms in the strain energy density function will be considered.

5.2 NON-LINEARITY IN ISOTROPIC SOLIDS

Consider the following plane in an isotropic solid:

![Diagram showing deformation in an isotropic solid](image)

**Figure 5.1 Deformation in an isotropic solid**

Two points $A_1$ and $B_1$ in the $xy$ plane of a semi infinite case of an isotropic solid are considered. The 2 points are connected through inter-atomic bonding. As the isotropic
solid deforms during transverse vibrations, the 2 points will shift from their original position as shown in the diagram. The points will displace in the y direction through a distance \(dy\) and in the x direction through a distance \(dx\). The coupling between the two displacements, which are mutually perpendicular, can be expressed by considering higher order terms in the strain energy density function and strain-displacement relations. The Taylor’s series expansion of the strain energy about zero strain is considered to derive the strain energy density function up to the third approximation[^8].

\[
W(\epsilon) = W(0) + \sum_{i,j} \frac{\partial W(0)}{\partial \epsilon_{ij}} \epsilon_{ij} + \sum_{i,j,k,l} \frac{1}{2!} \frac{\partial^2 W(0)}{\partial \epsilon_{ij} \partial \epsilon_{kl}} \epsilon_{ij} \epsilon_{kl} \\
+ \sum_{i,j,k,l,p,q} \frac{1}{3!} \frac{\partial^3 W(0)}{\partial \epsilon_{ij} \partial \epsilon_{kl} \partial \epsilon_{pq}} \epsilon_{ij} \epsilon_{kl} \epsilon_{pq}
\]

(5.42)

From previous assumptions the first and the second terms disappear because there is no strain energy for zero strain in the system and the residual stress is zero in the second term. Considering the third term alone results in the linear stress-strain relation. The linear strain term and the cubic expression (fourth term) in the Taylor’s series expansion will account for the material and physical non-linearity for the vibration coupling. The expression (5.42) simplifies to[^5][^6]:

\[
W(\epsilon) = \mu \epsilon_{ik}^2 + \left(\frac{K}{2} - \frac{\mu}{3}\right) \epsilon_{il}^2 + \frac{A}{3} \epsilon_{ik} \epsilon_{il} \epsilon_{kl} + B \epsilon_{ik}^2 \epsilon_{il} + \frac{C}{3} \epsilon_{il}^3
\]

(5.43)

The strain term for the vibration coupling problem includes the geometric non-linearity and is given by[^5]:

\[
\epsilon_{ij} = \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_l} \frac{\partial u_l}{\partial x_j} \right]
\]

(5.44)
where $u_l$ is the displacement vector along the 3 reference axes $x$, $y$ and $z$. Substituting this form for $e_{ij}$ in the strain energy density function expression and retaining terms up to and including the third order, we have:

$$W(\varepsilon) = \frac{1}{4} \mu \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right)^2 + \left( \frac{1}{2} K - \frac{1}{3} \mu \right) \left( \frac{\partial u_i}{\partial x_i} \right)^2$$

$$+ \left( \mu + \frac{1}{4} A \right) \frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_i} \frac{\partial u_l}{\partial x_l}$$

$$+ \left( \frac{1}{2} B + \frac{1}{2} K - \frac{1}{3} \mu \right) \frac{\partial u_i}{\partial x_i} \left( \frac{\partial u_l}{\partial x_l} \right)^2$$

$$+ \frac{1}{12} A \frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_i} \frac{\partial u_l}{\partial x_l} + \frac{1}{2} B \frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_i} \frac{\partial u_l}{\partial x_l}$$

$$+ \frac{1}{3} C \left( \frac{\partial u_l}{\partial x_i} \right)^3$$

(5.45)

The equation of motion for an isotropic solid is:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ik}}{\partial x_k}$$

(5.46)

The term on the right hand side indicates force per unit volume. The Landau stress can be expressed as the derivative of the strain energy density function with respect to the strain component:

$$\sigma_{ik} = \frac{\partial W(\varepsilon)}{\partial (\partial u_i/\partial x_k)}$$

(5.47)

Considering the case of plane waves propagating along the $OX$ axis only,

$$u(x, t) = \| u_x(x, t) + \| u_y(x, t) + \| u_z(x, t)$$

(5.48)

And using the above relations, the equation of motion (5.47) reduces to:
\[ \rho_o \frac{\partial^2 u_x}{\partial t^2} - \alpha \frac{\partial^2 u_x}{\partial x^2} = \beta \frac{\partial^2 u_x}{\partial x^2} \frac{\partial u_x}{\partial x} + \gamma \left( \frac{\partial^2 u_y}{\partial x^2} \frac{\partial u_y}{\partial x} + \frac{\partial^2 u_x}{\partial x^2} \frac{\partial u_x}{\partial x} \right) \] (5.49)

for the longitudinal direction\(^6\), and:

\[ \rho_o \frac{\partial^2 u_y}{\partial t^2} - \mu \frac{\partial^2 u_y}{\partial x^2} = \gamma \left( \frac{\partial^2 u_y}{\partial x^2} \frac{\partial u_x}{\partial x} + \frac{\partial^2 u_x}{\partial x^2} \frac{\partial u_y}{\partial x} \right) \] (5.50)

for the transverse direction\(^6\). The square terms in the above equations depend on all components of the displacement vector and explain the coupling between the transverse and longitudinal directions. This coupling indicates that when the beam vibrates in the transverse direction a longitudinal displacement is also induced. This causes energy reorganization between the transverse and longitudinal vibrations. With an increase in the wedging force the friction developed causes the longitudinal vibrations to utilize more energy to overcome the alternating longitudinal coulomb forces. The higher transverse vibration amplitude causes higher coulomb forces resulting in the longitudinal vibrations utilizing more energy as compared to the un-wedged case.

5.3 LONGITUDINAL AND TRANSVERSE VIBRATION COUPLING IN BEAMS

From Figure 5.1 the section of the isotropic solid can be considered as a beam which when vibrates, the \(z\)-\(y\) planes do not deform and the material is perfectly elastic. As the beam vibrates in the transverse direction longitudinal vibration occurs simultaneously. The behavior of the planes in a beam and the particles in that plane, comparable to the isotropic solid, is depicted in Figure 5.2.
Since the beam vibration is comparable to that of the isotropic solid, longitudinal and transverse vibration coupling is said to exist in beams. This causes energy re-organization in beam vibrations leading to damped transverse vibration amplitudes.
Chapter 6 NON-LINEARITY AND ANALYTICAL RESULTS
FOR A VIBRATING STRING

The equation of motion for the isotropic solid\(^6\) with the second degree terms in \(\partial u_i / \partial x_k\) accounts for the coupling between the longitudinal and transverse vibrations. This is extended to the case of a vibrating string\(^7\).

6.1 EQUATION OF MOTION FOR A STRING WITH COUPLING

Consider an isotropic solid whose \(z\)-\(y\) plane does not deform but can move in the \(x\), \(y\) or \(z\) directions. Consider a point at the center of the plane of this isotropic solid and assume a string passing through this point.

Let the string be fixed at one end and consider two points on the string, \(X_1\) and \(X_2\), separated by a small distance. When the string vibrates these 2 points move to new positions, \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\).

Figure 6.1 String assumed to pass through the center of a plane in an Isotropic solid
At time $t = 0$:

$$x_i(t = 0) = X_i \tag{6.1}$$

Let the string oscillate, $t \neq 0$, thereby displacing the point to the new position with displacements $U_i, V_i$ and $W_i$ in the $x,y$ and $z$ directions respectively. The 3 components of the new length $ds$ is:

$$dx = [X_2 + U_2(X)] - [X_1 + U_1(X)]$$

$$dx = dX - dU = dX - U_X dX = dX(1 - U_X) \tag{6.2}$$

$$dy = V_2 - V_1 = dV = V_X dX \tag{6.3}$$

$$dz = W_2 - W_1 = dW = W_X dX \tag{6.4}$$

Using the above relations (6.2) to (6.4) we have:

$$ds^2 = (dX + U_X dX)^2 + (V_X dX)^2 + (W_X dX)^2 \tag{6.5}$$

$$ds = [(dX + U_X dX)^2 + (V_X dX)^2 + (W_X dX)^2]^{1/2} \tag{6.6}$$

Assuming that the deformation is small:

$$\frac{ds - dX}{dX} \ll 1 \tag{6.7}$$

$$ds = dX[1 + 2U_X + U_X^2 + V_X^2 + W_X^2]^{1/2} \tag{6.8}$$

$$\varepsilon = 2U_X + U_X^2 + V_X^2 + W_X^2 \tag{6.9}$$
Applying binomial series expansion\cite{7} to $(1 + \varepsilon)^{1/2}$ and neglecting terms higher than second order:

\[(1 + \varepsilon)^{1/2} = 1 + U_x + \frac{1}{2} U_x^2 + \frac{1}{2} V_x^2 + \frac{1}{2} W_x^2 \]  

\[ds = dX\left(1 + U_x + \frac{1}{2} U_x^2 + \frac{1}{2} V_x^2 + \frac{1}{2} W_x^2\right)\]  

The tension in the string at any instant is given by:

\[T(X, t) = T_0 + EA\left\{\frac{ds - dX}{dX}\right\}\]  

\[T(X, t) = T_0 + EA\left(U_x + \frac{1}{2} U_x^2 + \frac{1}{2} V_x^2 + \frac{1}{2} W_x^2\right)\]  

Figure 6.3 Components of the tension in the string

Consider the motion in the \(x\) direction alone and let \(m\) be the mass per unit length of the string. The tension in the string can be resolved into its respective components along the \(x\) and \(y\) directions. The equation of motion in the \(x\) direction alone is given by:

\[mdX\ U_{tt} = T(1 + U_x)\frac{dX}{ds}\]  

\[m\ U_{tt} = \frac{\partial}{\partial X} \left[T(1 + U_x)\frac{dX}{ds}\right]\]
\[
\frac{dX}{ds} = \left( 1 + U_X + \frac{1}{2} U_X^2 + \frac{1}{2} V_X^2 + \frac{1}{2} W_X^2 \right)^{-1} 
\]

Applying the binomial expansion\(^7\) to the right hand side of the above relation and omitting higher order terms and substituting for \(T\), the equation of motion reduces to\(^7\):

\[
U_{tt} - c_1^2 U_{XX} - \frac{1}{2} \left( c_1^2 - c_0^2 \right) \frac{\partial}{\partial X} [V_X^2 + W_X^2] = 0 
\tag{6.18}
\]

The equation of motion for the \(y\) direction is obtained in a similar manner as\(^7\):

\[
m V_{tt} = \frac{\partial}{\partial X} \left[ T \frac{\partial V}{\partial s} \right] 
\tag{6.19}
\]

\[
V_{tt} - c_0^2 V_{XX} - (c_1^2 - c_0^2) \frac{\partial}{\partial X} \left[ V_X \left( U_X + \frac{1}{2} V_X^2 + \frac{1}{2} W_X^2 \right) \right] = 0 
\tag{6.20}
\]

\[
c_0 = \sqrt{\frac{T_0}{m}}; \quad c_1 = \sqrt{\frac{EA}{m}} 
\tag{6.21}
\]

The last term in the equations of motion can be considered as the forcing terms. Considering the static case and making time constant, we assume a practical condition:

\[
\frac{c_1^2}{c_0^2} = \frac{EA}{T_0} \gg 1 
\tag{6.22}
\]

In the case of metal strings this ratio lies between 400-1000\(^7\). As such neglecting the \(c_0\) term, the equation of motion for the longitudinal direction reduces to:

\[
U_{tt} - c_1^2 U_{XX} - \frac{1}{2} c_1^2 \frac{\partial}{\partial X} [V_X^2 + W_X^2] = 0 
\tag{6.23}
\]

The \(U_{tt}\) term is neglected due to the fact that time is constant and steady state motion is assumed. The equation left behind to solve for is:

\[
U_{XX} = -\frac{1}{2} \frac{\partial}{\partial X} [V_X^2 + W_X^2] 
\tag{6.24}
\]

Integrating this equation twice with respect to \(x\) and using the boundary conditions:
\[ U(0, t) = U(l, t) = 0 \]  \hspace{1cm} \text{(6.25)}

\[
U = -\frac{1}{2} \int_{0}^{\chi} (V_{x}^{2} + W_{x}^{2})dX + \frac{X}{2l} \int_{0}^{l} (V_{x}^{2} + W_{x}^{2})dX \hspace{1cm} \text{(6.26)}
\]

This solution relates the longitudinal displacement to the transverse displacement and thus shows the coupling between the 2 vibrations. This analogy for the string as a part of an isotropic solid can be extended to the case of a beam.

### 6.2 LONGITUDINAL AND TRANSVERSE COUPLING IN BEAMS

The mechanism in Figure 6.5 demonstrates that the transverse and longitudinal vibrations in a beam are coupled as explained below. For the mode of vibration shown in Figure 6.5 the points below the neutral axis are pulled apart setting them in tension and points above the neutral axis are pushed towards each other setting them in compression. The material below the neutral axis can be compared to a collection of strings oscillating together. The same mechanism is valid when the beam reaches the other extreme position with the portion above the neutral axis experiencing tension.
Just as in the case of the beam behaving similar to that of the isotropic solid, the comparison to the string further establishes the fact that longitudinal and transverse vibrations in beams are coupled. This coupling results in energy re-organization for the case of wedging leading to damped transverse vibration amplitudes.
Chapter 7 COMPARISONS OF EXPERIMENTAL AND ANALYTICAL RESULTS

The equations of motion for the isotropic solid in the longitudinal and the transverse directions were derived by considering higher order terms in the Taylor’s series expansion of the strain energy as a function of the total strain. These equations are:

\[ \rho_o \frac{\partial^2 u_x}{\partial t^2} - \alpha \frac{\partial^2 u_x}{\partial x^2} = \beta \frac{\partial^2 u_x}{\partial x^2} \frac{\partial u_x}{\partial x} + \gamma \left( \frac{\partial^2 u_y}{\partial x^2} \frac{\partial u_y}{\partial x} + \frac{\partial^2 u_z}{\partial x^2} \frac{\partial u_z}{\partial x} \right) \]  

\[ \rho_o \frac{\partial^2 u_y}{\partial t^2} - \mu \frac{\partial^2 u_y}{\partial x^2} = \gamma \left( \frac{\partial^2 u_x}{\partial x^2} \frac{\partial u_x}{\partial x} + \frac{\partial^2 u_x}{\partial x^2} \frac{\partial u_y}{\partial x} \right) \]  

(7.1)  

(7.2)

The right hand side of the two equations, can be considered as forcing functions. The forcing function for equation (7.1) contains \( x, y \) and \( z \) direction displacements while equation (7.2) has \( x \) and \( y \) displacements. The interaction of the displacements in both transverse and longitudinal directions establishes the coupling that exists between them.

The equations of motion showing interactions between the longitudinal and transverse vibrations in a string were derived by considering a plane in the isotropic solid and a point in that plane through which the string passes. These equations are:

\[ U_{tt} - c_1^2 U_{xx} - \frac{1}{2} (c_1^2 - c_0^2) \frac{\partial}{\partial x} [V_x^2 + W_x^2] = 0 \]  

\[ V_{tt} - c_0^2 V_{xx} - (c_1^2 - c_0^2) \frac{\partial}{\partial x} \left[ V_x \left( U_x + \frac{1}{2} V_x^2 + \frac{1}{2} W_x^2 \right) \right] = 0 \]  

(7.3)  

(7.4)

for the longitudinal and transverse directions respectively. The solution to the longitudinal equation of motion is given by:
This solution also shows the longitudinal and transverse displacement interactions explaining the coupling of the two vibrations. The beam is considered as a part of this semi-infinite isotropic solid vibrating in the transverse direction. The beam consists of planes like the isotropic solid that do not deform but displace in the $x$, $y$ and $z$ directions. The displacement of the plane during vibration results in one part of it being in tension while the other part is in compression. The part in tension is approximated to a collection of strings which accounts for the coupling between the longitudinal and transverse vibrations. This coupling results in energy re-organization between the longitudinal and transverse vibrations. The wedging performed on the tube generates alternating longitudinal coulomb forces during the dynamic process. This wedge force increases with the increase in wedging and it removes part of the energy of the transverse vibrations helping to control the amplitudes.

The wedged response curves shown in Figures 4.1 through 4.4 show a rapid decay in transverse vibration amplitudes during the first few cycles. This decay follows a linear behavior indicating coulomb damping. When wedged, alternating longitudinal coulomb forces generated due to the wedge force, resist the longitudinal vibrations from being excited to higher levels. There is a feedback effect from the longitudinal vibrations to control transverse vibrations due to wedging. Due to this feedback (coupling) effect, restricting longitudinal vibration amplitudes leads to reduced transverse vibration amplitudes. Work is done by the alternating longitudinal coulomb forces to resist

$$U = -\frac{1}{2} \int_0^X (V_x^2 + W_x^2) dX + \frac{X}{2l} \int_0^l (V_x^2 + W_x^2) dX$$

(7.5)
longitudinal vibrations. Thus energy is removed from the system resulting in less energy available for transverse vibrations. This leads to reduced transverse vibration amplitudes.
Chapter 8 CONCLUSIONS

A simple experiment was designed to measure the transverse vibration amplitudes of a tube with multiple supports as found in a heat exchanger. The theory explaining the non-linear coupling between longitudinal and transverse vibrations for an isotropic solid and a string was presented. Transverse vibrations give rise to longitudinal vibrations in either the wedged or the un-wedged case. Results showed that the longitudinal vibrations are coupled to the transverse vibrations in the wedged case for the vertical configuration. The longitudinal vibration has a feedback effect to control the transverse vibrations when the tube is wedged. Analysis of the experimental results for the wedged vertical tube showed 2 types of damping, coulomb damping and logarithmic decrement damping. The un-wedged vertical tube experiment showed only logarithmic decrement damping and no resistance to longitudinal vibrations and no feedback effect. Alternating longitudinal coulomb forces generated by wedging remove energy and minimize transverse tube vibrations through non-linear coupling.
APPENDIX

A.1 STRAIN ENERGY FOR THE LINEAR STRESS STRAIN RELATION

The Taylor’s series expansion of the strain energy about zero strain as a function of the total strain for an isotropic solid is given by:

\[
W(\varepsilon) = W(0) + \sum_{i,j} \frac{\partial W(0)}{\partial \varepsilon_{ij}} \varepsilon_{ij} + \sum_{i,j,k,l} \frac{1}{2!} \frac{\partial^2 W(0)}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \varepsilon_{ij} \varepsilon_{kl} + \sum_{i,j,k,l,p,q} \frac{1}{3!} \frac{\partial^3 W(0)}{\partial \varepsilon_{ij} \partial \varepsilon_{kl} \partial \varepsilon_{pq}} \varepsilon_{ij} \varepsilon_{kl} \varepsilon_{pq} + \cdots
\]

(A.1)

The total strain is expressed as its pure shear and hydrostatic parts as:

\[
\varepsilon_{ij} = \left[ \varepsilon_{ij} - \frac{1}{3} \delta_{ij} \varepsilon_{ll} \right] + \frac{1}{3} \delta_{ij} \varepsilon_{ll}
\]

(A.2)

The first term in the strain energy is zero since there is no strain in the solid. The first derivative in the second term represents the stress in the solid for zero strain and is zero due to the assumption of a perfectly elastic isotropic solid. The series condenses to:

\[
W(\varepsilon) = \sum_{i,j,k,l} \frac{1}{2!} \frac{\partial^2 W(0)}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \varepsilon_{ij} \varepsilon_{kl} + \sum_{i,j,k,l,p,q} \frac{1}{3!} \frac{\partial^3 W(0)}{\partial \varepsilon_{ij} \partial \varepsilon_{kl} \partial \varepsilon_{pq}} \varepsilon_{ij} \varepsilon_{kl} \varepsilon_{pq} + \cdots
\]

(A.3)

To explain the non-linearity in the tube vibration terms up to third order are sufficient. The case of linear stress strain relation can be obtained by considering the second term only.
\[ \epsilon_{ij}\epsilon_{kl} = \left[ \epsilon_{ij} - \frac{1}{3} \delta_{ij}\epsilon_{mm} \right] \left[ \epsilon_{kl} - \frac{1}{3} \delta_{kl}\epsilon_{nn} \right] \]

\[ \times \left[ \epsilon_{kl} - \frac{1}{3} \delta_{kl}\epsilon_{nn} \right] + \frac{1}{3} \delta_{ij}\epsilon_{mm} \]

(A.4)

\[ \epsilon_{ij}\epsilon_{kl} = \left[ \epsilon_{ij} - \frac{1}{3} \delta_{ij}\epsilon_{mm} \right] \left[ \epsilon_{kl} - \frac{1}{3} \delta_{kl}\epsilon_{nn} \right] \]

\[ + \left[ \epsilon_{ij} - \frac{1}{3} \delta_{ij}\epsilon_{mm} \right] \left( \frac{1}{3} \delta_{kl}\epsilon_{nn} \right) \]

\[ + \left( \frac{1}{3} \delta_{ij}\epsilon_{mm} \right) \left( \epsilon_{kl} - \frac{1}{3} \delta_{kl}\epsilon_{nn} \right) + \frac{1}{9} \delta_{ij}\delta_{kl}\epsilon_{mm}^2 \]

(A.5)

\[ \epsilon_{ij}\epsilon_{kl} = \left[ \epsilon_{ij} - \frac{1}{3} \delta_{ij}\epsilon_{mm} \right] \left[ \epsilon_{kl} - \frac{1}{3} \delta_{kl}\epsilon_{nn} \right] + \frac{1}{3} \epsilon_{ij}\delta_{kl}\epsilon_{nn} \]

\[ - \frac{1}{9} \delta_{ij}\delta_{kl}\epsilon_{mm}^2 + \frac{1}{3} \delta_{ij}\epsilon_{mm}\epsilon_{kl} - \frac{1}{9} \delta_{ij}\delta_{kl}\epsilon_{mm}^2 \]

\[ + \frac{1}{9} \delta_{ij}\delta_{kl}\epsilon_{mm} \]

(A.6)

\[ \epsilon_{ij}\epsilon_{kl} = \left[ \epsilon_{ij} - \frac{1}{3} \delta_{ij}\epsilon_{mm} \right] \left[ \epsilon_{kl} - \frac{1}{3} \delta_{kl}\epsilon_{nn} \right] \]

\[ + \frac{1}{3} \left( \epsilon_{ij}\delta_{kl} + \delta_{ij}\epsilon_{kl} \right)\epsilon_{mm} - \frac{1}{9} \delta_{ij}\delta_{kl}\epsilon_{mm}^2 \]

(A.7)

The form proposed by Landau and Lifshitz:

\[ W(\epsilon) = \mu \left( \epsilon_{ik} - \frac{1}{3} \delta_{ik}\epsilon_{ll} \right) + \frac{1}{2} K \epsilon_{ll}^2 \]

(A.8)

\[ K = \left( \lambda + \frac{2}{3} \mu \right) \]

(A.9)

Where \( \mu \) and \( K \) are the bulk modulus and the modulus of compression respectively.

Neglecting the first order terms in the derived equation for the strain energy it compares quite well with the Landau and Lifshitz equation. The derivative of the strain energy term with respect to the total strain gives the linear stress strain relation.
A.2 STRAIN ENERGY FOR NON-LINEAR STRESS STRAIN RELATIONS

Using the result for \( \varepsilon_{ij} \varepsilon_{kl} \) we derive the term \( \varepsilon_{ij} \varepsilon_{kl} \varepsilon_{pq} \):

\[
\varepsilon_{ij} \varepsilon_{kl} \varepsilon_{pq} = \left\{ \varepsilon_{ij} \left[ \frac{1}{3} \delta_{ij} \varepsilon_{mm} \right] \left[ \varepsilon_{kl} - \frac{1}{3} \delta_{kl} \varepsilon_{nn} \right] \right. \\
+ \frac{1}{3} \left( \varepsilon_{ij} \delta_{kl} + \delta_{ij} \varepsilon_{kl} \right) \varepsilon_{mm} \\
- \frac{1}{9} \delta_{ij} \delta_{kl} \varepsilon_{mm}^2 \left\} \left( \varepsilon_{pq} - \frac{1}{3} \delta_{pq} \varepsilon_{rr} \right) + \frac{1}{3} \delta_{pq} \varepsilon_{rr} \\
\varepsilon_{ij} \varepsilon_{kl} \varepsilon_{pq} = \left\{ \varepsilon_{ij} \left[ \frac{1}{3} \delta_{ij} \varepsilon_{mm} \right] \left[ \varepsilon_{kl} - \frac{1}{3} \delta_{kl} \varepsilon_{nn} \right] \right. \\
- \frac{1}{3} \delta_{pq} \varepsilon_{rr} \\
+ \left( \frac{1}{3} \left( \varepsilon_{ij} \delta_{kl} + \delta_{ij} \varepsilon_{kl} \right) \varepsilon_{mm} \right) \varepsilon_{pq} - \frac{1}{3} \delta_{pq} \varepsilon_{rr} \\
- \left( \frac{1}{9} \delta_{ij} \delta_{kl} \varepsilon_{mm}^2 \right) \varepsilon_{pq} - \frac{1}{3} \delta_{pq} \varepsilon_{rr} \\
+ \left( \left[ \varepsilon_{ij} - \frac{1}{3} \delta_{ij} \varepsilon_{mm} \right] \varepsilon_{kl} \\
- \frac{1}{3} \delta_{kl} \varepsilon_{nn} \right) \left( \frac{1}{3} \delta_{pq} \varepsilon_{rr} \right) \\
+ \left[ \frac{1}{3} \left( \varepsilon_{ij} \delta_{kl} + \delta_{ij} \varepsilon_{kl} \right) \varepsilon_{mm} \right] \left( \frac{1}{3} \delta_{pq} \varepsilon_{rr} \right) \\
- \left( \frac{1}{9} \delta_{ij} \delta_{kl} \varepsilon_{mm}^2 \right) \left( \frac{1}{3} \delta_{pq} \varepsilon_{rr} \right) \\
\right. \\
(A.10)
\]

Evaluating and simplifying this equation we obtain:
\[ \epsilon_{ij} \epsilon_{kl} \epsilon_{pq} = \epsilon_{ij} - \frac{1}{3} \delta_{ij} \epsilon_{mm} \left[ \epsilon_{kl} - \frac{1}{3} \delta_{kl} \epsilon_{nn} \right] \epsilon_{pq} \]

\[ - \frac{1}{3} \delta_{pq} \epsilon_{rr} \]

\[ + \frac{1}{3} \left( \delta_{kl} \epsilon_{ij} \epsilon_{pq} + \delta_{ij} \epsilon_{kl} \epsilon_{pq} + \delta_{pq} \epsilon_{ij} \epsilon_{kl} \right) \epsilon_{mm} \]

\[ - \frac{1}{9} \left( \delta_{ij} \delta_{kl} \epsilon_{pq} + \delta_{ij} \delta_{pq} \epsilon_{kl} + \delta_{pq} \delta_{kl} \epsilon_{ij} \right) \epsilon_{mm}^2 \]

\[ + \frac{1}{9} \delta_{ij} \delta_{kl} \delta_{pq} \epsilon_{mm}^3 \]

Combining the 2 results and substituting them in the strain energy Taylor’s series expansion it is comparable to the form that Landau and Lifshitz have put forth:

\[ W(\epsilon) = \mu \epsilon_{ik}^2 + \left( \frac{K}{2} - \frac{\mu}{3} \right) \epsilon_{il}^2 + \frac{A}{3} \epsilon_{ik} \epsilon_{il} \epsilon_{kl} + B \epsilon_{ik}^2 \epsilon_{il} + \frac{C}{3} \epsilon_{il}^3 \]

The non-linear stress strain relation is be obtained by differentiating the above strain energy term with respect to strain component.

**A.3 FAST FOURIER TRANSFORM CURVES FOR THE EXPERIMENTAL SETUP**

The curves shown below are the Fast Fourier Transforms of the experimental output for the setup considered in this thesis. These curves show the frequency content in the transient time-amplitude response curves. It can be seen that the amplitude peaks for all four cases at around 95 Hz which forms one of the natural frequencies of the experimental setup considered here. This value compares well with the theoretical value of first natural frequency derived from the receptance method. This proves that there is no change in the natural frequency of the system when wedging is done.
Figure A.5 Fast Fourier Transform of Horizontal Un-Wedged tube
Figure A.6: Fast Fourier Transform of Horizontal Wedged Tube
Figure A.7 Fast Fourier Transform for wedged Vertical Tube
Figure A.8 Fast Fourier Transform for un-wedged vertical tube
REFERENCES


