INFERENCE FOR THE LÉVY MODELS AND THEIR APPLICATIONS IN MEDICINE AND STATISTICAL PHYSICS

by

Alexandra Piryatinska

Submitted in partial fulfillment of the requirements For the degree of Doctor of Philosophy

Dissertation Advisor: Dr. Wojbor A. Woyczynski

Department of Statistics

CASE WESTERN RESERVE UNIVERSITY

January 2005
CASE WESTERN RESERVE UNIVERSITY

SCHOOL OF GRADUATE STUDIES

We hereby approve the dissertation of

ALEXANDRA PIRYATINSKA

candidate for the Doctor of Philosophy degree *

Committee Chair: ________________________________
Dr. Wojbor A. Woyczynski
Dissertation Advisor
Professor,
Department of Statistics

Committee: ________________________________
Dr. Jiayang Sun
Professor,
Department of Statistics

Committee: ________________________________
Dr. Nidhan Choudhuri
Assistant Professor,
Department of Statistics

Committee: ________________________________
Dr. Ken Loparo
Professor,
Department of Electrical Engineering and Computer Science

January 2005
# Table of Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Table of Contents</td>
<td>iii</td>
</tr>
<tr>
<td></td>
<td>List of Figures</td>
<td>vi</td>
</tr>
<tr>
<td></td>
<td>Acknowledgement</td>
<td>ix</td>
</tr>
<tr>
<td></td>
<td>Abstract</td>
<td>x</td>
</tr>
<tr>
<td>1</td>
<td>$\alpha$-Stable Distributions and Lévy Processes. Smoothly Truncated</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Lévy Distributions and Flights</td>
<td>4</td>
</tr>
<tr>
<td>1.1</td>
<td>Introduction</td>
<td>4</td>
</tr>
<tr>
<td>1.2</td>
<td>$\alpha$-Stable Distributions</td>
<td>6</td>
</tr>
<tr>
<td>1.2.1</td>
<td>Fractional moments of the $\alpha$-stable distributions and log-moments</td>
<td>11</td>
</tr>
<tr>
<td>1.2.2</td>
<td>Simulation of $\alpha$-stable random variables</td>
<td>15</td>
</tr>
<tr>
<td>1.2.3</td>
<td>Lévy stable processes</td>
<td>17</td>
</tr>
<tr>
<td>1.3</td>
<td>Smoothly Truncated Lévy Distributions and Flights</td>
<td>18</td>
</tr>
<tr>
<td>1.4</td>
<td>Fractional- and Integer-Order Moments of One-Sided $STL_\alpha$ Distributions, $0 &lt; \alpha &lt; 1$</td>
<td>24</td>
</tr>
<tr>
<td>1.4.1</td>
<td>Fractional-order moments</td>
<td>24</td>
</tr>
<tr>
<td>1.4.2</td>
<td>Integer-order moments</td>
<td>27</td>
</tr>
<tr>
<td>1.5</td>
<td>Fractional- and Integer-Order Moments for Symmetric $STL_\alpha$ distribu-</td>
<td></td>
</tr>
<tr>
<td>1.5.1</td>
<td>Fractional-order moments</td>
<td>28</td>
</tr>
<tr>
<td>1.5.2</td>
<td>Integer-order moments</td>
<td>36</td>
</tr>
<tr>
<td>1.6</td>
<td>Multiscaling of the Smoothly Truncated Lévy Flight</td>
<td>37</td>
</tr>
<tr>
<td>1.6.1</td>
<td>One-sided $STL_\alpha$ flights, $0 &lt; \alpha &lt; 1.$</td>
<td>38</td>
</tr>
<tr>
<td>1.6.2</td>
<td>Symmetric $STL_\alpha$ flights, $0 &lt; \alpha &lt; 2.$</td>
<td>39</td>
</tr>
<tr>
<td>1.7</td>
<td>Monte Carlo Method for Simulation of $STL_\alpha$ Flights</td>
<td>41</td>
</tr>
<tr>
<td></td>
<td>Bibliography</td>
<td>43</td>
</tr>
<tr>
<td>2</td>
<td>Estimation of the Parameters of $\alpha$-Stable Distributions and Smoothly</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Truncated Lévy Distributions</td>
<td>47</td>
</tr>
<tr>
<td>2.1</td>
<td>Introduction</td>
<td>47</td>
</tr>
<tr>
<td>2.2</td>
<td>Estimation of the Parameters of $\alpha$-Stable Distributions</td>
<td>50</td>
</tr>
<tr>
<td>2.3</td>
<td>Estimation of the Parameters of the Smoothly Truncated Lévy Distribu-</td>
<td></td>
</tr>
<tr>
<td></td>
<td>tions</td>
<td>61</td>
</tr>
</tbody>
</table>
4.9.4 Anomalous random walks: the stationary case . . . . . . . . . . . . . . . . . . 166
4.10 The Langevin equation . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 170
4.11 Conclusions . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 173
List of Figures

1.1 Function $f(y; \alpha = 1/2)$. .................................................. 30
1.2 Function $g(x; \alpha = 1/2)$. .................................................. 32
1.3 The multiscaling exponents $\tau(\rho)$ for one-sided $STL_\alpha(a, 1, 10^{-5})$ flights, for time-scale parameter values $a = 0.004, 10, \text{ and } 800, \text{ and the index } \alpha = 1/2 \text{ (top) and } \alpha = 3/4 \text{ (bottom)}$. .......................... 40
1.4 The multiscaling exponents $\tau(\rho)$ for symmetric $STL_\alpha(a, 1/2, 10^{-2})$ flights, for time-scale parameter values $a = 0.004, \text{ and } 800, \text{ and the index } \alpha = 3/2 \text{ (top) and } \alpha = 7/4 \text{ (bottom)}$. .................................................. 42
2.1 Boxplot of the moment estimators for an $\alpha$-stable distribution. Left: estimation of $\alpha = 1.5$; right: estimation of $\sigma = 3$. ............. 53
2.2 Quadratic estimator of $\alpha$. .................................................. 55
2.3 U-statistics estimator of $\alpha$. ............................................. 57
2.4 Fractional moment estimator of $\alpha$. ..................................... 59
2.5 Comparison of the moment, quadratic and fractional moment estimators, with sample size $n=4000$ (left), $n=10000$ (right). .......................... 60
2.6 MLE of the parameters of the IG distributions. ......................... 63
2.7 Estimates of the parameters of the $STL_{1/2}(0.004, 1, 0.0001)$ (top left), $STL_{1/2}(0.004, 1, 0.1)$ (top right), and $STL_{0.75}(0.004, 1, 0.01)$ distributions. .................................................. 65
2.8 Moment estimators of the parameters for $STL_{1.5}(0.004, 1/2, 0.1)$ (top left), $STL_{1.5}(0.004, 1/2, 0.01)$ (top right), $STL_{1.2}(0.004, 1/2, 0.01)$ (bottom left) and $STL_{0.75}(0.004, 1/2, 0.01)$ (bottom right). .................................................. 67
2.9 The characteristic functions of the $STL_\alpha$ distributions $\varphi(u; \alpha, a, \lambda)$. Top left: $\varphi(u, a)$; top right: $\varphi(u, \alpha)$; bottom: $\varphi(u, \lambda)$. .......................... 71
2.10 MLE estimators for the $STL_{1.2}(0.004, 1/2, 0.001)$ (top left), $STL_{1.2}(0.004, 1/2, 0.01)$ (top right), $STL_{1.5}(0.004, 1/2, 0.01)$ (middle left), $STL_{1.5}(0.004, 1/2, 0.1)$ (middle right), and $STL_{0.75}(0.004, 1/2, 0.01)$ (bottom). .................................................. 72
3.1 Top: mixed frequency sleep stages; bottom: low voltage irregular(active) sleep stage. .................................................. 79
<table>
<thead>
<tr>
<th>Section</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.2</td>
<td>Top: high voltage irregular (quiet) sleep; bottom: trace alternate (quiet) sleep</td>
</tr>
<tr>
<td>3.3</td>
<td>Quiet (top), and active (bottom) EEG-sleep patterns, for the fullterm neonate</td>
</tr>
<tr>
<td>3.4</td>
<td>ACF plots (top), and periodograms (bottom), for the active (left), and quiet sleep stages (right)</td>
</tr>
<tr>
<td>3.5</td>
<td>ACF for the quiet sleep patterns, calculated for every 15 seconds of observations</td>
</tr>
<tr>
<td>3.6</td>
<td>ACF for the quiet sleep patterns, calculated by removing of 15 seconds observations</td>
</tr>
<tr>
<td>3.7</td>
<td>ACF for the active sleep patterns, calculated by removing consecutive 15 second observations</td>
</tr>
<tr>
<td>3.8</td>
<td>ACF for the quiet sleep patterns, calculated by removing consecutive 15 second observations</td>
</tr>
<tr>
<td>3.9</td>
<td>Sleep stages for EEG88 (fullterm neonate); normalized power values for delta, theta, beta, and alpha spectral bands. Minute-by-minute calculations and their smoothing</td>
</tr>
<tr>
<td>3.10</td>
<td>Sleep stages for EEG40 (preterm neonate), normalized power values for delta, theta, beta, and alpha spectral bands. Minute-by-minute calculations and their smoothing</td>
</tr>
<tr>
<td>3.11</td>
<td>EEG 88 sleep stages. Separation into the clusters by the spectral band method</td>
</tr>
<tr>
<td>3.12</td>
<td>EEG 40 sleep stages, separation into the clusters by the spectral band method</td>
</tr>
<tr>
<td>3.13</td>
<td>Sleep stages for EEG88. Spectral characteristics of the EEG data calculated minute-by-minute and their smoothing</td>
</tr>
<tr>
<td>3.14</td>
<td>EEG 88 sleep stages. Separation into the cluster by the spectral and nonlinear characteristics method</td>
</tr>
<tr>
<td>3.15</td>
<td>Fullterm group: ANOVA of the sleep stages for the spectral bands characteristics</td>
</tr>
<tr>
<td>3.16</td>
<td>Preterm group: ANOVA of sleep stages for the spectral bands characteristics</td>
</tr>
<tr>
<td>3.17</td>
<td>Fullterm group: ANOVA of the sleep stages for the spectral and nonlinear characteristics</td>
</tr>
<tr>
<td>3.18</td>
<td>Preterm group: ANOVA of the sleep stages for the spectral and nonlinear characteristics</td>
</tr>
<tr>
<td>3.19</td>
<td>Comparison of fullterm and preterm groups. ANOVA for the spectral characteristics of the active sleep stages</td>
</tr>
<tr>
<td>3.20</td>
<td>Comparison of fullterm and preterm groups. ANOVA for the spectral characteristics of the quiet sleep</td>
</tr>
<tr>
<td>3.21</td>
<td>Fractional dimension D, memory parameter d, and D+d+1/2 = D+H for the fullterm neonate</td>
</tr>
</tbody>
</table>
3.22 Fractional dimension D, memory parameter d, and D+d+1/2 = D+H, for the preterm neonate. 123
3.23 Density estimates for EEG 88. 125
3.24 Density estimates for EEG 40. 126

4.1 The Hankel loop, a path of integration in (2.12). 139
4.2 The Mittag-Leffler function \(E_{1/2}(-s)\) (solid line) and its asymptotic behavior \(E_{1/2}(-s) \sim 1/\sqrt{\pi s}, \ (s \to \infty)\) (dashed line). Notice the accuracy approximation of \(E_{1/2}(-s)\) by \(1/\sqrt{\pi s}\) for \(s > 3\). 140
4.3 Plots of p.d.f.s \(g_\beta(\theta)\) of the inverse random time for different values of \(\beta\); the values \(g_\beta(\theta = 0)\) decrease as \(\beta\) increases. In contrast to the corresponding p.d.f.s \(f_\beta(t)\) of the random time depicted in Fig. A.1, \(g_\beta(\theta)\) decay exponentially as \(\tau \to \infty\), ensuring that their moments are finite, (2.7). As \(\beta \to 1\), we have \(g_\beta(\theta) \to \delta(\theta - 1)\). 141
4.4 Plot of the relative variance of the fractional anomalous drift process as a function of the order \(\beta\) of the fractional derivative in equation (5.3). 149
4.5 The solution of the subdiffusion equation (6.2), for \(\beta = 2/3\). 151
4.6 Plot of function \(k(\tau; \beta = 1/2)\), drawn in loglog scale (solid line). The dashed lines show the subdiffusive (\(\sqrt{\tau}\)) and linear (\(\tau\)) asymptotics of this function. 156
4.7 Plots of functions \(R(0; t)\), for different values of \(\beta\), describing the probabilities of the event that the fractional wandering particle has not moved up to time \(t\). 165
4.8 A schematic illustration of different random times studied in the renewal theory. 167
ACKNOWLEDGEMENTS

I would like to thank Professor Wojbor A. Woyczynski for his guidance, for involving me in his own research projects, for his support and friendship. Dr. Alexander Saichev kindly introduced me to many subtleties of anomalous diffusion. I’m grateful for his acceptance of my participation in the subdiffusion and superdiffusion project. Dr. Gyorgy Terdik spent many hours with me explaining various facets of the theory of stochastic processes and fields. I much appreciate a chance to collaborate with him on the problem of multiscaling for truncated Lévy flights, and on long-range dependence issues. Professor Jiayang Sun provided a kind commentary on EEG-sleep analysis and I owe her a solid introduction to computational statistics. I thank her for her help. Anatoly Zlotnik, an undergraduate Case student, helped me with MATLAB coding for EEG-sleep analysis and I’m grateful for his assistance. Finally, I would like to express my gratitude to Professors Ken Loparo, Mark Scher and Dr. Mark Johnson. They granted me access to the EEG data and explained the major issues of the EEG studies in pediatric neuroscience. Chapter 3 of this dissertation would have been impossible without their generosity. I would like as well to thank Dr. Nidhan Choudhuri who kindly agrees to be a member of the committee.
Lévy processes, that is stochastic processes with time-homogeneous and independent increments, found numerous applications in the physical sciences, economics and engineering.

In this dissertation we study specific theoretical issues related to the multiscaling properties of some special classes of Lévy processes and to the kinetic equations describing time-evolution of statistical mechanical systems driven by certain Lévy processes displaying, perhaps limited, fractal behavior.

To be able to apply these models to real data we also develop statistical parametric estimation procedures for them. These theoretical tools are then utilized in analysis of EEG recordings for fullterm and preterm neonates. The issues of sleep stage separations and long-memory property have been also investigated for this data set.

Dissertation Advisor: Wojbor A. Woyczynski
Introduction

Lévy processes, that is stochastic processes with time-homogeneous and independent increments, found numerous applications in the physical sciences, economics and engineering.

In this dissertation we study specific theoretical issues related to the multiscaling properties of some special classes of Lévy processes and to the kinetic equations describing time-evolution of statistical mechanical systems driven by certain Lévy processes displaying, perhaps limited, fractal behavior.

To be able to apply these models to real data we also develop statistical parametric estimation procedures for them. These theoretical tools are then utilized in analysis of EEG recordings for fullterm and preterm neonates. The issues of sleep stage separation and long memory property have been also investigated for this data set.

In Chapter 1, we consider $\alpha$-stable distributions and processes, and Smoothly Truncated Lévy ($STL_\alpha$) distributions and flights. We study behavior of the fractional and integer absolute moments for the Smoothly Truncated Lévy distributions. The multiscaling property of the Smoothly Truncated Lévy flights is investigated. We perform simulations of $STL_\alpha$ distributions and estimate their moments to illustrate the multiscaling property.

Part of the results of Chapter 1 were obtained in collaboration with Dr. Gy. Terdik and Dr. W.A. Woyczynski and will appear in a joint paper entitled ” Multi-Scaling Properties of Time-Scaled Truncated Lévy Flights”.

In Chapter 2, we review the problem of parametric estimation for the $\alpha$-stable distributions and develop estimation methods for parameters of the Smoothly Truncated Lévy distributions. In the one-sided case, we estimate parameters by using
properties studied in Chapter 1, and construct moment estimators of the parameters. In the symmetric case, we provide an algorithm for Numerical Maximum Likelihood estimators. Effectiveness of these estimators is then verified on simulated samples.

In Chapter 3, we study EEG-sleep patterns of neonates. These EEG data were collected by Dr. M. Scher of the Department of Pediatric Neurology of the Case School of Medicine and made available to us courtesy of Dr. K. Loparo of the Case School of Engineering, Department of Electrical Engineering and Computer Science.

We consider the EEG signal as a piecewise stationary time series. Several EEG spectral and non-linear dynamics characteristics are estimated. The non-parametric change-point detection algorithm and cluster analysis were applied to these characteristics to obtain the sleep stage separation. To compare spectral characteristics in different sleep stages, analysis of variance for the fullterm and preterm neonate EEG signals is performed. We also compare groups of fullterm and preterm neonates by using the above characteristics. We use fractional dimension and nonparametric change-point detection algorithm to separate the signal into quasi-stationary segments and estimate its long-memory parameter. We find the presence of the long-memory property for the active sleep stage signal. For two EEG recordings, one of them of a fullterm neonate and one of a preterm neonate, we study amplitude distributions. We find that Smoothly Truncated Lévy distributions, which were investigated in Chapters 1 and 2, provide a reasonable fit for the amplitude distribution of the fullterm neonate EEG signal observed during quasi-stationary segments. This is not the case for the preterm neonate.

In Chapter 4, kinetic equations for anomalous diffusion processes are considered. Their one-point probability density functions (p.d.f.) are exact solutions of fractional diffusion equations. The models describe asymptotic behavior of jump processes with random jump sizes and random interjump time intervals with infinite means which do not satisfy the Law of Large Numbers. In the case when these intervals have a fractional exponential p.d.f., the fractional Kolmogorov-Feller equation for the corresponding anomalous diffusion is provided and methods of finding its solutions are discussed. Some statistical properties of solutions of the related Langevin equation
are studied. The subdiffusive case is considered. Part of the results of Chapter 4 were obtained in collaboration with Dr. A.I. Saichev and Dr. W.A. Woyczynski and will be published in a joint paper entitled "Models of Anomalous Diffusion: The Subdiffusive Case," to appear in *Physica A: Statistical Mechanics.*
1.1 Introduction

The theory of $\alpha$-stable distributions was developed in the 1930s by Paul Lévy and Alexander Khinchin. The interest in them increased in the last 20 years due to their applications in finance, physics, telecommunication, insurance, and sociology, see, e.g., [B-NMR01], [Shlesinger88].

The problem with application of these models is that their distributions do not have finite moments of order $\geq \alpha$, whereas in practice, real data always have finite moments of all orders. In this context different ways of truncation have been suggested.

Truncated Lévy flights were introduced by Mantegna and Stanley [MantegnaStanley94] as a model for random phenomena which exhibit at small scales properties similar to those of self-similar Lévy processes (see, e.g., [Sato99]) but have distributions which at large scales have exponential tails and thus have finite moments of any order. Mantegna and Stanley pointed out the usefulness of truncated Lévy flights in modelling a broad spectrum of random physical phenomena ranging from turbulence to financial markets. Smoothly Truncated Lévy Flights, introduced
by Koponen [Koponen95], built on Mantegna and Stanley’s ideas but stressed the advantage of a nice analytic form.

In Section 1.2, we begin with main definitions and well known facts about $\alpha$-stable distributions and stable Lévy processes. Different parameterizations of $\alpha$-stable distributions are considered. Fractional absolute moments are calculated. This forms a background for our investigation of properties of smoothly truncated Lévy ($STL_\alpha$) distributions and $STL_\alpha$ flights.

In Section 1.3, $STL_\alpha$ distributions are defined. It turns out that in the one-sided case, $\alpha = 1/2$, we have an explicit formula for the density, which turns out to be that of the well known Inverse Gaussian distribution. In Sato’s book, p. 233, the Inverse Gaussian (IG) distribution is considered as an example of the truncated distribution obtained via the density truncation known as the Essher transform. If $f(x)$ is the density of an $\alpha$-stable distribution then $c f(x) e^{-\lambda x}$ is Esscher transform of the density. Surprisingly, the IG distribution is also an example of the distribution where truncation can be performed in the Lévy measure. More generally, we have shown that for any one-sided $\alpha$-stable distribution the truncation of the density and truncation of the Lévy measure are equivalent operations. This result was also obtained independently by Rosinski in a recent manuscript.

In Section 1.4, we study fractional- and integer-order moments of one-sided $STL_\alpha$ distributions. In the case $\alpha = 1/2$, fractional moments of order $\rho < \alpha$ are calculated and cumulants are found. Knowledge of cumulants allows us to find the integer moments. For the one-sided case, with $0 < \alpha < 1$, the asymptotics of the moments, for $a \to 0$, and $a \to \infty$, are found, and log-absolute moments’ behavior is described.

In Section 1.4, we study symmetric $STL_\alpha$, $0 < \alpha < 2$, and their fractional- and integer-order moments. The asymptotics of upper and lower bounds for the fractional absolute moments are found. This allows us to find the asymptotic behavior for the logarithm of the absolute moments. Integer-order cumulants are calculated.

In Section 1.6, we take a closer look at the scaling properties of $STL_\alpha$ flights following on the footsteps of Nakao [Nakao2000]. The typical approach, used commonly in, e.g., intermittency studies in hydrodynamics (see, Frisch [Frisch95]), calls for an
investigation of the evolution of the moments (partition functions) of \( STL_\alpha \) flights as functions of time and scale, displayed in the log-log coordinates. In contrast to Nakao’s work, we study the behavior of both fractional- and integer-order moments \( E(\left|X_\alpha(t)\right|^\rho) \), for both small, and large values of the scaling parameter \( \alpha \). In the former case, for a fixed \( t \), we obtain the behavior close to that of the \( \alpha \)-stable distribution, and for the latter, close to that of the Gaussian distribution.

More precisely, we will consider an \( STL_\alpha \) flight \( X(t) \), with discrete time parameter \( t = 0, 1, 2, 3, \ldots \). To emphasize the dependence of \( STL_\alpha \) on the scaling parameter \( \alpha \) we will write \( X(t) = X_\alpha(t) \) and calculate the form of the scaling exponent \( \tau_\alpha(\rho) \) so that

\[
E(\left|X_\alpha(t)\right|^\rho) \propto (at)^{\tau_\alpha(\rho)},
\]

which gives the following linear dependence in the log-log scales:

\[
\log E(\left|X_\alpha(t)\right|^\rho) \approx \tau_\alpha(\rho) \log(at) + \text{const}.
\]

The behavior of \( \tau(\rho) \) is different for small and large values of the scaling parameter \( \alpha \). We perform simulations of \( STL_\alpha \) distributions by using Rosinski’s series representation. The simulations are performed to investigate multiscaling properties.

### 1.2 \( \alpha \)-Stable Distributions

Let us start from the definitions of \( \alpha \)-stable distributions. There are several approaches possible. We are going to follow [SamorodnitskiTaqqu94] and choose two definitions. At first, we consider the following definition:

**Definition 1** A probability distribution is said to be stable if it has a domain of attraction, i.e. if there is a sequence of i.i.d. (independent, identically distributed) random variables \( Y_1, Y_2, \ldots \) on \( \mathbb{R}^1 \), and a sequence of positive numbers \( \{d_n\} \), and real numbers \( \{a_n\} \), such that

\[
d_n^{-1}(Y_1 + Y_2 + \cdots + Y_n) + a_n \Rightarrow X, \quad \text{as} \quad n \to \infty, \quad \text{(1.1)}
\]
(here ⇒ means the convergence in distribution). The non-degenerate limit laws $X$ in 1.1 are called stable laws.

The distribution of $Y_i$ is then said to be in the Domain of Attraction (DOA) of the distribution $X$.

**Definition 2** A random variable $X$ is said to have a strictly stable distribution if $a_n = 0$.

The second, equivalent, definition of stable random variables is formulated in terms of the characteristic functions:

**Definition 3** A random variable $X$ is said to have a stable distribution if there are parameters $0 < \alpha \leq 2$, $\sigma \geq 0$, $-1 \leq \beta \leq 1$, and $b$ (real), such that its characteristic function has the following form:

$$
\varphi(u) = E \exp iuX = \begin{cases} 
\exp \left\{-\sigma|u|^\alpha (1 - i\beta(\text{sign } u) \tan \frac{\alpha \pi}{2}) +ibu\right\}, & \text{if } \alpha \neq 1, \\
\exp \left\{-\sigma|u| (1 + i\frac{2\beta}{\pi}(\text{sign } u) \ln |u|) +ibu\right\}, & \text{if } \alpha = 1.
\end{cases}
$$

(1.2)

The parameter $\alpha$ is called the index of stability, or the tail parameter, and

$$
\text{sign } u = \begin{cases} 
1, & \text{if } u > 0; \\
0, & \text{if } u = 0; \\
-1, & \text{if } u < 0.
\end{cases}
$$

The parameters $\sigma$, $\beta$ and $b$ are called, respectively, the scale, skewness and location parameters. In this case we will write $X \sim S_\alpha(\sigma, \beta, b)$. Equivalence of the above definitions is a fundamental, and nontrivial theorem of probability theory (see, [SamorodnitskiTaqqu94])

For $\alpha \neq 1$, a stable distribution is strictly stable if and only if $b = 0$. For $\alpha = 1$, a stable distribution is strictly stable if and only if $\beta = 0$ (see, e.g., [SamorodnitskiTaqqu94]).

Let us recall the definition of the cumulant function:
Definition 4 The function
\[ \psi_X(u) = \log \varphi(u). \] (1.3)
is called the cumulant function of the random variable \( X \) with characteristic function \( \varphi(u) \).

Notice, that the cumulant function can be defined only in the case when \( \log \varphi(u) \) exists.

The derivatives of order \( k \) of the characteristic function, evaluated at \( u = 0 \), give the \( k \)-order moments \( E X^k \). Similarly, the \( k \)-order derivatives of the cumulant function, evaluated at \( u = 0 \), are called cumulants of order \( k \), and denoted \( \text{cum}_k(X) \).

Thus all the integer-order moments can be expressed in terms of cumulants:
\[ E X = \text{cum}_1(X), \]
\[ E X^2 = \text{cum}_2(X) + [\text{cum}_1(X)]^2, \] (1.4)
\[ E X^3 = \text{cum}_3(X) + 3\text{cum}_1(X)\text{cum}_2(X) + [\text{cum}_1(X)]^3, \ldots. \]

The general correspondence is as follows:
\[ E(X_1 \cdot X_2 \cdot \ldots \cdot X_n) = \sum_{\mathcal{L} \in \mathcal{P}_{(1,\ldots,n)}} \prod_{K_j \in \mathcal{L}} \text{Cum}(X_{K_j}), \]
where the summation is over all partitions \( \mathcal{L} = \{K_1, K_2, \ldots, K_k\} \) of the sequence \( 1, \ldots, n \), \( X_1, \ldots, X_n \) are random variables from the distribution with the cumulative function \( \psi_X(u) \), and \( \mathcal{P}_{(1,\ldots,n)} \) are permutations of the sequence \( 1, \ldots, n \).

The cumulant function of a stable distribution can be written in terms of the Lévy-Khinchin representation as follows: if \( \alpha = 2 \), then
\[ \psi(u) = iMu - \sigma^2 u^2, \] (1.5)
if \( \alpha < 2 \), then
\[ \psi(u) = iMu + P \int_0^\infty \xi(u,x)x^{-(\alpha+1)}\,dx + Q \int_{-\infty}^0 \xi(u,x)|x|^{-(\alpha+1)}\,dx. \] (1.6)
Here $M$ is real, $\sigma \geq 0$, $P$, $Q$ are non-negative numbers, and

$$\xi(u, x) = e^{iux} - 1 - iuxI_{[-1,1]}(x),$$

where $I_A(x)$ is the indicator function of the set $A$. In the case $\alpha < 2$, when $X$ is non-degenerate ($P + Q > 0$), one has the relation (see, formula 1.2)

$$\beta = \frac{P - Q}{P + Q}.$$

The measure

$$L(dx) = \frac{P}{x^{\alpha+1}}I_{(0,\infty)}(x)dx + \frac{Q}{|x|^\alpha}I_{(-\infty,0)}(x)dx$$

is called the \textit{Lévy measure}. This representation of the characteristic function can be used to prove the equivalence between the Definitions 1 and 3, see e.g. [Zolotarev86].

Note, that the formula (1.2) is not the only possible way to write the characteristic function of a stable distributions. In the literature, different ways to parameterize the characteristic function or, equivalently, the cumulant function of $\alpha$-stable distributions can be found, see, e.g. [Zolotarev86]. For example, for the purpose of parametric estimation Zolotarev [Zolotarev86] uses the following parameterization of the cumulant function of strictly stable random variables:

$$\psi_u = -\exp \left\{ \nu^{-1/2}(\log |u| + \tau - i\frac{\pi}{2}\theta \text{sign } t + C(\nu^{-1/2} - 1) \right\}, \quad (1.7)$$

where

$$C = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \ln n \right) \approx 0.57721$$

is the Euler constant. The parameters $\nu$, $\theta$, and $\tau$, vary within the following bounds:

$$\nu \geq 1/4, \quad |\theta| \leq \min(1, 2\sqrt{\nu} - 1), \quad |\tau| < \infty,$$

and are connected with the parameters $\alpha$, $\beta$, and $\sigma$, by the relations

$$\alpha = 1/\sqrt{\nu}, \quad (1.8)$$
\[
\beta = \theta \max(1, 1/(2\sqrt{\nu} - 1)) \text{sign}(1 - 1/\nu),
\]
(1.9)

\[
b = \begin{cases} 
0, & \text{if } \nu \neq 1, \\
\frac{\pi}{2} \tan(\pi \theta/2), & \text{if } \nu = 1.
\end{cases}
\]
(1.10)

\[
\sigma = \begin{cases} 
\exp(\tau/\sqrt{\nu} - C(1 - 1/\sqrt{\nu})) & \text{if } \nu \neq 1, \\
\exp(\tau + \log \cos(\pi \theta/2) - \log(\pi/2)) & \text{if } \nu = 1,
\end{cases}
\]
(1.11)

Also, later on, the following parameterization will be useful for us:

\[
\psi(u) = \begin{cases} 
a \Gamma(-\alpha) \left[ p(-iu)^\alpha + q(iu)^\alpha \right] + iub, & \text{for } \alpha \neq 1, \\
ap(iu) \log(iu) + q(-iu) \log(-iu) + iub, & \text{for } \alpha = 1,
\end{cases}
\]
(1.12)

where \( p + q = 1 \).

Indeed,

\[
\psi(u) = a \Gamma(-\alpha) \left[ p(-iu)^\alpha + q(iu)^\alpha \right] + iub
= a \Gamma(-\alpha) |u|^\alpha \left[ p \exp(-i\alpha \pi/2 \text{sign } u) + q \exp(i\alpha \pi/2 \text{sign } u) \right] + iub
= a \Gamma(-\alpha) \cos(\alpha \pi/2) |u|^\alpha \left[ 1 - i(p - q) \tan(\alpha \pi/2) \text{sign } u \right] + iub
= -\sigma |u|^\alpha \left[ 1 - i\beta \tan(\alpha \pi/2) \text{sign } u \right] + iub,
\]

and the correspondence between the parameters as follows:

\[
\sigma = -a \Gamma(-\alpha) \cos(\alpha \pi/2),
\]
(1.13)

\[
\beta = p - q.
\]
(1.14)

If \( \psi(u) = \psi_\alpha(u) \) is the cumulant function of an \( \alpha \)-stable random variable then

\[
\psi_1(u) = \lim_{\alpha \to 1+} \psi_\alpha(u).
\]

Therefore

\[
\psi_1(u) = \lim_{\alpha \to 1+} (a \Gamma(-\alpha) \left[ p(-iu)^\alpha + q(iu)^\alpha \right] + iub)
= iub + a \lim_{\alpha \to 1+} \frac{\Gamma(2 - \alpha)}{\alpha(\alpha - 1)} \left[ p(-iu)^\alpha + q(iu)^\alpha \right]
= iub + a \lim_{\alpha \to 1+} \frac{\Gamma(2 - \alpha)}{\alpha} \lim_{\alpha \to 1+} \left[ \frac{p(-iu)^\alpha}{\alpha - 1} + \frac{q(iu)^\alpha}{\alpha - 1} \right]
= iub + a[p(-iu) \log(-iu) + q(iu) \log(iu)].
\]
The probability densities of $\alpha$-stable random variables exist and are continuous (or even analytic) functions, but only in the cases listed below they can be written by using the elementary functions.

1. $S_2(\sigma, 0, b)$ is the Gaussian distribution with mean $b$ and variance $2\sigma^2$ and pdf (probability density function)

   $$f(x) = \frac{1}{\sqrt{4\sigma^2}} \exp \left( -\frac{(x - b)^2}{4\sigma^2} \right);$$

2. $S_1(\sigma, 0, b)$ is the Cauchy distribution, whose density

   $$f(x) = \frac{\sigma}{\pi((x - b)^2 + \sigma^2)};$$

3. $S_{1/2}(\sigma, 0, b)$ is the Lévy distribution, whose density

   $$f(x) = \left( \frac{\sigma}{2\pi} \right)^{1/2} \frac{\exp\left( -\sigma/(2(x - b))\right)}{(x - b)^{3/2}}.$$  \hspace{1cm} (1.15)

For some other cases expressions of the density function can be found in terms of special functions. For example, for $\alpha = 1/3$ (see [Zolotarev86]), the density function is

$$f(x) = \frac{1}{\sqrt{3}} x^{-7/2} \text{Ai}\left( \frac{1}{\sqrt{3}x} \right), \quad x > 0,$$  \hspace{1cm} (1.16)

where $\text{Ai}(x)$ is the so-called Airy function (see, for example [AbramowitzStegun92]).

In most cases, closed form expressions of the densities of stable distributions are unknown.

1.2.1 Fractional moments of the $\alpha$-stable distributions and log-moments

Fractional moments. It is well known that, for $\alpha$-stable distributions, the moments of order greater or equal to $\alpha$ do not exist. Therefore, only the absolute moments of the fractional order $\rho < \alpha$ are considered.
We will calculate the fractional absolute moments in terms of the characteristic function (1.12) relying on the following result: Let $X$ be a random variable with a characteristic function $\varphi(x)$. Then

$$E |X|^\rho = c(\rho) \int_{-\infty}^{\infty} [1 - \text{Re} \varphi_X(u)] \frac{du}{|u|^{1+\rho}}, \quad (1.17)$$

where

$$c(\rho) = \frac{\Gamma (1 + \rho) \sin (\rho\pi/2)}{\pi},$$

see [BahrEssen65].

We assume that the location parameter $b = 0$, and $\rho < \alpha$, and rewrite the formula (1.17) as follows:

$$E |X|^\rho = 2c(\rho) \text{Re} \int_0^{\infty} [1 - \varphi (r)] \frac{dr}{r^{1+\rho}}$$

$$= 2c(\rho) \text{Re} \int_0^{\infty} [1 - \exp \left(\Gamma (-\alpha) a u^{\alpha} [q(i)^{\alpha} + p (-i)^{\alpha}] \right) \frac{du}{u^{1+\rho}}$$

$$= \frac{2c(\rho)}{\alpha} \text{Re} \left[\left[\left[-a\Gamma (-\alpha) (q(i)^{\alpha} + p (-i)^{\alpha})\right]^{\rho/\alpha} \Gamma (-\rho/\alpha) \right] \right]$$

$$= \frac{2c(\rho)}{\rho} \text{Re} \left[\left[-a\Gamma (-\alpha) (q(i)^{\alpha} + p (-i)^{\alpha})\right]^{\rho/\alpha} \Gamma \left(1 - \rho/\alpha\right) \right]$$

$$= \frac{2}{\pi} \Gamma (\rho) \sin (\rho\pi/2) \left[a - \Gamma (-\alpha) \cos (\alpha\pi/2)\right]^{\rho/\alpha}$$

$$\times \Gamma \left(1 - \rho/\alpha\right) \text{Re} \left[\left(1 + i (q - p) \tan (\alpha\pi/2)\right)^{\rho/\alpha}\right]$$

$$= \frac{-a\Gamma (-\alpha) \cos (\alpha\pi/2)^{\rho/\alpha} \Gamma \left(1 - \rho/\alpha\right)}{\Gamma \left(1 - \rho/\alpha\right) \cos (\rho\pi/2)}$$

$$\times \text{Re} \left[\left(1 + i (q - p) \tan (\alpha\pi/2)\right)^{\rho/\alpha}\right].$$

Note that, for $\alpha \neq 1$, and $\rho \neq 1$, $-\Gamma (-\alpha) \cos (\alpha\pi/2) > 0$, and $\Gamma \left(1 - \rho/\alpha\right) \cos (\rho\pi/2) > 0$, for all $0 < \alpha < 2$, and $0 < \rho < 2$, respectively. In particular, in the symmetric case
\( p = q = 1/2, \)

\[
E|X|^\rho = \frac{\Gamma(1 - \rho/\alpha) \left[ -a \Gamma(-\alpha) \cos(\alpha \pi/2) \right]^{\rho/\alpha}}{\Gamma(1 - \rho) \cos(\rho \pi/2)} \tag{1.19}
\]

\[
= \Gamma\left(\frac{1 - \rho}{2}\right) \Gamma\left(\frac{1 + \rho}{2}\right) \frac{\Gamma(1 - \rho/\alpha) \left[ -a \Gamma(-\alpha) \cos(\alpha \pi/2) \right]^{\rho/\alpha}}{\Gamma(1 - \rho) \pi} \]

\[
= \frac{2^\rho \Gamma\left(\frac{1+\rho}{2}\right) \Gamma\left(1 - \rho/\alpha\right)}{\sqrt{\pi} \Gamma(1 - \rho/2)} \left[ -a \Gamma(-\alpha) \cos(\alpha \pi/2) \right]^{\rho/\alpha},
\]

see Shanbhag-Sreehari [ShanbhagSreehari]. For the one-sided case, \( 0 < \alpha < 1, \) and \( p = 1, \) we have

\[
E|X|^\rho = (-a \Gamma(-\alpha))^{\rho/\alpha} \frac{\Gamma(1 - \rho/\alpha)}{\Gamma(1 - \rho)}.
\]

This result is well known.

**Remark.** For the sake of completeness, we provide the following formulas from the
gamma-function calculus which were used in the above calculations:

\[ \Gamma (1 + \alpha) = \alpha \Gamma (\alpha), \]
\[ \Gamma (1 - \alpha) = -\alpha \Gamma (-\alpha), \]
\[ \Gamma (\alpha) \Gamma (-\alpha) = -\frac{\pi}{\alpha \sin \pi \alpha}, \]
\[ \Gamma (\alpha) \Gamma (1 - \alpha) = \frac{\pi}{\sin \pi \alpha}, \]
\[ \Gamma \left( \frac{1 - \alpha}{2} \right) \Gamma \left( \frac{1 + \alpha}{2} \right) = \frac{\pi}{\cos \pi \alpha/2}, \]
\[ \Gamma (1 - \alpha) = \frac{1}{2^{\alpha} \sqrt{\pi}} \Gamma \left( \frac{2 - \alpha}{2} \right) \Gamma \left( \frac{1 - \alpha}{2} \right) \]
\[ \int_0^\infty (e^{ur} - 1) \frac{dr}{r^{1+\alpha}} = \Gamma (-\alpha) (-u)^\alpha, \]
\[ 2 \text{Re} \int_0^\infty (1 - e^{ir}) \frac{dr}{r^{1+\alpha}} = -2 \Gamma (-\alpha) \text{Re} (-i)^\alpha \]
\[ = -2 \Gamma (-\alpha) \cos (\alpha \pi/2) \]
\[ = \frac{\pi}{\Gamma (1 + \alpha) \sin (\alpha \pi/2)}, \]
\[ |x|^{\rho} = c (\rho) \int_{-\infty}^{\infty} [1 - \cos (rx)] \frac{dr}{|r|^{1+\rho}}, \quad 0 < \rho < 2. \]

Notice, that the logarithm of the absolute fractional moment of the stable distribution can be written as

\[ \log \mathbb{E}|X|^{\rho} = \rho/\alpha \log a + C, \quad (1.20) \]

where constant \( C \) does not depends of \( a \).

**Integer moments of the** \( \log |X| \). For an \( \alpha \)-stable distribution \( X \), the absolute moments of the integer orders grater then \( \alpha \) are infinite. However, random variables \( Y = \log |X| \) have finite moments of every order \( k \). This fact is very useful in moment estimation of the parameters of the stable distribution (see, [Zolotarev86]).

Observe that there is a direct connection between absolute moments of order \( \rho < \alpha \)
and log-moments of order $k$. Indeed,

$$E(\log |X|^k) = EY^k = (d/d\rho)^k(E|X|^\rho)|_{\rho=0},$$

see [Zolotarev86]. This fact is very easy to prove as follows: Let random variable $X$ have a probability density function $f(x)$. Let $k=1$. Then

$$\left.\frac{d}{d\rho}(E|X|^\rho)\right|_{\rho=0} = \left.\frac{d}{d\rho}\left(\int_{-\infty}^{0} (-x)^\rho f(x)dx + \int_{0}^{\infty} x^\rho f(x)dx\right)\right|_{\rho=0}$$

$$= \left.\frac{d}{d\rho}\left(-x^\rho f(x)dx + x^\rho f(x)dx\right)\right|_{\rho=0}$$

$$= \left.\frac{d}{d\rho}\left(-x^\rho \log(-x)f(x)dx + x^\rho \log(x)f(x)dx\right)\right|_{\rho=0}$$

$$= E \log |X|.$$

By induction the result can be proved for any order $k$.

For example, using Mathematica, we can calculate directly $E \log |X|$, and $E(\log |X|^2)$, for the one-sided strictly $\alpha$-stable random variable given by the cumulant function (1.12) ($p = 1$, $q = 0$)

$$E \log |X| = \frac{dE |X|^\rho}{d\rho}|_{\rho=0} = -\zeta + \frac{C}{\alpha} + \frac{\log(-a(-\Gamma(-\alpha)))}{\alpha},$$

(1.21)

$$E(\log |X|^2) \sim \zeta^2 - \frac{\pi^2}{6} + \frac{\zeta^2}{\alpha^2} - \frac{2\zeta^2}{\alpha^2} + \frac{2\zeta \log(a(-\Gamma(-\alpha)))}{\alpha^2} + \frac{\log(a(-\Gamma(-\alpha)))}{\alpha^2},$$

where $\zeta$ is the Euler constant.

### 1.2.2 Simulation of $\alpha$-stable random variables

The most popular and efficient method of simulation of $\alpha$-stable random variables is the Chambers-Mallows-Stuck method, see [CMS76] or, also, [SamorodnitskiTaqqu94], [Weron96].

For the symmetric case the algorithm depends on the following fact:
Let $U \sim \text{Uniform}(-\pi/2, \pi/2)$ and let $W \sim \text{Exp}(1)$. Assume that $U$ and $W$ independent. Then

$$X = \frac{\sin(\alpha U)}{(\cos U)^{1/\alpha}} \left( \frac{\cos(1-\alpha)U}{W} \right)^{(1-\alpha)/\alpha} \sim S_\alpha(1, 0, 0).$$

Skewed random variables, $X \sim S_\alpha(1, \beta, 0)$, for $\alpha \in (0, 2]$, and $\beta \in [-1, 1]$, can be simulated using the following algorithm:

- Generate a random variable $V \sim \text{Uniform}(-\pi/2, \pi/2)$ and an independent exponential random variable $W$ with mean 1;

- For $\alpha \neq 1$, compute

$$X = S_{\alpha, \beta} \times \frac{\sin(\alpha(V + B_{\alpha, \beta}))}{(\cos(V))^{1/\alpha}} \times \left( \frac{\cos(V - \alpha(V + B_{\alpha, \beta}))}{W} \right)^{(1-\alpha)/\alpha}, \quad (1.22)$$

where

$$B_{\alpha, \beta} = \frac{\arctan(\beta \tan \frac{\pi \alpha}{2})}{\alpha},$$

$$S_{\alpha, \beta} = \left[ 1 + \beta^2 \tan^2 \frac{\pi \alpha}{2} \right]^{1/(2\alpha)};$$

- For $\alpha = 1$, compute

$$X = \frac{2}{\pi} \left[ \left( \frac{\pi}{2 + \beta V} \right) \tan V - \beta \log \left( \frac{W \cos V}{\frac{\pi}{2} + \beta V} \right) \right]. \quad (1.23)$$

To simulate $Y \sim S_\alpha(\sigma, \beta, b)$, one can simulate $X \sim S_\alpha(1, \beta, 0)$, and then do the transformation

$$Y = \sigma X + b.$$
1.2.3 Lévy stable processes

Let us review the main definitions in the theory of Lévy processes.

**Definition 5** A stochastic process $X(t) \equiv X_t$, $t \geq 0$, $X(0) = 0$, in $\mathbb{R}$ is called a Lévy process if, for every $s, t \geq 0$, the increments $X(t + s) - X(s)$ are independent of the process $X(u)$, $0 \leq u \leq t$, and if the process is time homogeneous, i.e.

$$X(t + s) - X(t) \overset{d}{=} X(s),$$

where $\overset{d}{=}$ means the equality of probability distributions. Its characteristic function $\varphi(u; t), u \in \mathbb{R}^1, t \geq 0$, can be written in the form

$$\varphi(u; t) = \exp(t\psi(u))$$

where the characteristic exponent function

$$\psi(u) = iMu + \frac{Bu^2}{2} + \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{iux} + iux I_{|x|<1}\right) L(dx),$$

with $M \in \mathbb{R}$, $B > 0$, and $L$ being a nonnegative measure on $\mathbb{R}\setminus\{0\}$ satisfying the integrability condition $\int_{\mathbb{R}} \min(1, |x|^2) L(dx) < \infty$.

The case $L \equiv 0$ produces a Gaussian process which is a Brownian motion if $M = 0$ and $B = 1$. In the case $M = 0$, $B = 0$, and $L$ being a finite measure, Lévy process $X$ is just a *compound Poisson process* with the Lévy measure $L$ serving as the intensity of jump sizes.

Let $T$ be either $\mathbb{R}$, or $\mathbb{R}_+ = \{t; t \geq 0\}$.

**Definition 6** ([SamorodnitskiTaqqu94], p.311) A real-valued process $X(t), t \in T$, is self-similar with index $H$ if, for any $a > 0$, the finite dimensional distributions of process $X(at), t \in \mathbb{R}$, are equal to those of $a^H X(t), t \in \mathbb{R}$, i.e., if, for any $n \geq 1$, $t_1, t_2, \ldots, t_n \in T$, and any $a > 0$,

$$(X(at_1), X(at_2), \ldots, X(at_n)) \overset{d}{=} (a^H X(t_1), a^H X(t_2), \ldots, a^H X(t_n)).$$
Lévy processes with Lévy measures

\[ L(dx) = \begin{cases} 
  a q x^{-\alpha-1} dx, & \text{for } x > 0, \\
  a p |x|^{-\alpha-1} dx, & \text{for } x < 0,
\end{cases} \quad (1.28) \]

where \( p + q = 1 \), and \( a > 0 \), are self-similar with the self-similarity index \( H = 1/\alpha \) (see, for example [SamorodnitskiTaqqu94]), and for this process the characteristic exponent function is just the cumulant function of an \( \alpha \)-stable distribution, given, for example, in (1.6) for, \( 0 < \alpha < 2 \), and in (1.5), for \( \alpha = 2 \). For the Brownian motion, the self-similarity index \( H = 1/2 \).

### 1.3 Smoothly Truncated Lévy Distributions and Flights

We begin with a few definitions.

**Definition 7** A random variable \( X \) has a Smoothly Truncated Lévy (STL\( \alpha \)) distribution if its cumulant function \( \psi_X \) has the following form

\[ \psi(u) = i M u + \int_{\mathbb{R}\setminus\{0\}} (1 - e^{iux} + iuxI_{|x|<1}) L(dx), \quad (1.29) \]

where the Lévy measure

\[ L(dx) = \begin{cases} 
  a p \frac{e^{-\lambda x}}{x^{\alpha+1}} dx, & \text{for } x > 0, \\
  a q \frac{e^{-\lambda |x|}}{|x|^{\alpha+1}} dx, & \text{for } x < 0,
\end{cases} \quad (1.30) \]

where \( p + q = 1 \), and \( a > 0 \).

**Definition 8** A 1-dimensional Lévy process is called a Smoothly Truncated Lévy (STL\( \alpha \)) flight if its characteristic exponent function has no Gaussian part \( (B = 0) \), and is given by formulas (1.29), (1.30).
Proposition 9 The characteristic function of the Smoothly Truncated Levy distribution (1.29) with the Lévy measure (1.30) can be written in the following form

$$\psi_X(u) = a\lambda^\alpha [q\zeta_\alpha(-u/\lambda) + p\zeta_\alpha(u/\lambda)] + iuM,$$

(1.31)

where $a,p,q \geq 0$, $p+q=1$, $b$ is a real number, and

$$\zeta_\alpha(r) = \begin{cases} 
\frac{\Gamma(-\alpha)[(1-ir)^\alpha - 1]}{1 - (1-ir)\log(1-ir) + ir}, & \text{for } \alpha = 1; \\
\Gamma(-\alpha)((1-ir)^\alpha - 1 + i\alpha r), & \text{for } 1 < \alpha < 2.
\end{cases}$$

(1.32)

Proof. If $0 < \alpha < 1$, then

$$\psi(u) = iMu + ap\int_{-\infty}^{0} (1 - e^{iu(x|x|<1)}) e^{\lambda x} dx$$

$$+ ap\int_{0}^{\infty} (1 - e^{iu(x|x|<1)}) e^{-\lambda x} dx.$$ 

We can open parentheses and split this integral into the sum of the integrals (since each of the summand integrals is convergent) which yields

$$\psi(u) = iMu + ap\int_{0}^{\infty} (1 - e^{iu(x|x|<1)}) e^{\lambda x} dx$$

$$+ ap\int_{-\infty}^{0} (1 - e^{iu(x|x|<1)}) e^{-\lambda x} dx$$

$$= iMu + ap\left(\int_{-\infty}^{0} x^{-\alpha-1}(1 - e^{iu(x|x|<1)}) e^{-\lambda x} dx + \int_{0}^{1} iue^{-\lambda x} x^{-\alpha} dx\right)$$

$$+ ap\left(\int_{-\infty}^{0} (-x)^{-\alpha-1}(1 - e^{-iu(x|x|<1)}) e^{\lambda x} dx - \int_{-1}^{0} iue^{\lambda x} (-x)^{-\alpha} dx\right)$$

$$= iMu + ap\left(\Gamma(-\alpha)(\lambda^\alpha + (\lambda - iu)^\alpha) + \lambda^\alpha \int_{0}^{\lambda} iuy^{-\alpha} e^{-y} dy\right)$$

$$+ ap\left(\Gamma(-\alpha)(\lambda^\alpha + (\lambda + iu)^\alpha) - \lambda^\alpha \int_{0}^{\lambda} iuy^{-\alpha} e^{-y} dy\right)$$

$$= iub + a\lambda^\alpha \Gamma(-\alpha)\left[p((1-u/\lambda)^\alpha + 1) + q((1+u/\lambda)^\alpha + 1)\right],$$

where

$$b = M + \lambda^\alpha a(p - q)\int_{0}^{\lambda} y^{-\alpha} e^{-y} dy$$

$$= M + \lambda^\alpha a(p - q)(\Gamma(1 - \alpha) - \Gamma(1 - \alpha, \lambda)).$$
and $\Gamma(\alpha, x) = \int_x^\infty y^{\alpha-1}e^{-y}dy$ is the ”upper” incomplete Gamma function.

Let us consider the case $1 < \alpha < 2$. Now, the integral $\int_0^1 e^{-\lambda x}x^{-\alpha}dx$ diverges, so we can’t conduct calculations by separating the summands. In this situation we are going to rewrite the cumulant function $\psi(u)$ as follows:

$$\psi(u) = iMu + ap \int_0^\infty (1 - e^{iux} + iuxI_{x<1}) \frac{e^{-\lambda x}}{x^{\alpha+1}} dx + aq \int_{-\infty}^0 (1 - e^{iux} + iuxI_{x<1}) \frac{e^{-\lambda x}}{x^{\alpha+1}} dx$$

$$= ibu + ap \int_0^\infty (1 - e^{iux} + iux) \frac{e^{-\lambda x}}{x^{\alpha+1}} dx + aq \int_{-\infty}^0 (1 - e^{iux} + iux) \frac{e^{-\lambda x}}{x^{\alpha+1}} dx$$

$$= ibu + ap (\Gamma(-\alpha)(\lambda^\alpha + (\lambda - iu)^\alpha + \lambda^{-1}\alpha iu) + aq (\Gamma(-\alpha)(\lambda^\alpha + (\lambda + iu)^\alpha - \lambda^{-1}\alpha iu)$$

$$= iub + a\lambda^{\alpha}\Gamma(-\alpha) \left[p((1 - iu/\lambda)^\alpha + 1 + i\alpha u/\lambda) + q((1 + iu/\lambda)^\alpha + 1 - i\alpha u/\lambda)\right],$$

where $b = M - (p - q)\int_1^\infty iuxe^{-\lambda x}x^{-\alpha}dx$.

If $\psi(u) = \psi_\alpha$ is the cumulant function of the $ST L_\alpha$ distribution with $\alpha \neq 1$, then, as we observed before, the cumulant function $\psi_1$ is the limit of the cumulant functions $\psi_\alpha(u)$, as $\alpha \searrow 1$ (see, for example, [Zolotarev86]). To find

$$\lim_{\alpha \to 1} \psi_\alpha(u) = iub$$

$$+ a \lim_{\alpha \to 1} \left(\lambda^\alpha\Gamma(-\alpha) \left[p((1 - iu/\lambda)^\alpha + 1 + i\alpha u/\lambda) + q((1 + iu/\lambda)^\alpha + 1 - i\alpha u/\lambda)\right]\right),$$

let us first observe that

$$\lim_{\alpha \to 1} \left(\Gamma(-\alpha)((1 - iu/\lambda)^\alpha + 1)\right) = \lim_{\alpha \to 1} \left(\frac{\Gamma(2 - \alpha)}{\alpha(\alpha - 1)} ((1 - iu/\lambda)^\alpha + 1)\right)$$

$$= \lim_{\alpha \to 1} \left(\frac{\Gamma(2 - \alpha)}{\alpha} \frac{(1 - iu/\lambda)^\alpha + 1}{\alpha - 1}\right) = \lim_{\alpha \to 1} \frac{\Gamma(2 - \alpha)}{\alpha} \lim_{\alpha \to 1} \frac{(1 - iu/\lambda)^\alpha \log(1 - iu/\lambda)}{1}$$

$$= (1 - iu/\lambda) \log(1 - iu/\lambda).$$

Hence,

$$\psi_1(u) = \lim_{\alpha \to 1} \psi_\alpha(u) = iub + a\lambda[p(\lambda - iu/\lambda) \log(1 - iu/\lambda) + iu/\lambda]$$

$$+ q(\lambda + iu/\lambda) \log(1 + iu/\lambda) - iu/\lambda],$$

which proves the proposition.
Parameter $\lambda$ will be called the truncation parameter (because it truncates jumps of the stable distribution), and parameter $p$ will be called the skewness parameter. The case $p = 1$ corresponds to the distribution totally skewed to the positive side, and the case $p = q = 1/2$ yields a symmetric distribution. In what follows we shall only consider the centered $STL_\alpha$ distributions which means that we assume that parameter $M = 0$. Thus, later on, the Smoothly Truncated Lévy distribution is uniquely determined by parameters $\alpha, a, p$ and $\lambda$ and will be denoted $STL_\alpha(a, p, \lambda)$.

In the case $0 < \alpha < 1$, the cumulant function is given by the formula
\[
\psi_{X(1)}(u) = a\lambda^\alpha \Gamma(-\alpha) \left[ p \left(1 - i\frac{u}{\lambda}\right)^\alpha + q \left(1 + i\frac{u}{\lambda}\right)^\alpha - 1 \right],
\] (1.33)
and, if $p = 1$ the cumulant function
\[
\psi_{X(1)}(u) = a\lambda^\alpha \Gamma(-\alpha) \left[ \left(1 - i\frac{u}{\lambda}\right)^\alpha - 1 \right]
\] (1.34)
\[= a\Gamma(-\alpha) [ (\lambda - iu)^\alpha - \lambda^\alpha] \]
describes a distribution totally concentrated on the positive half-line.

In the symmetric case, and $\alpha = 1$,
\[
\psi_{X(1)}(u) = a\lambda^\alpha \Gamma(-\alpha) \left[ \frac{1}{2} \left(1 + i\frac{u}{\lambda}\right)^\alpha + \frac{1}{2} \left(1 - i\frac{u}{\lambda}\right)^\alpha - 1 \right] \] (1.35)
\[= a\Gamma(-\alpha) \left[ (\lambda^2 + u^2)^{\alpha/2} \cos(\alpha \arctan(u/\lambda)) - \lambda^\alpha \right].\]

In the symmetric case, and $\alpha = 1$,
\[
\psi_{X(1)}(u) = a \left(\frac{\lambda}{2} \log \left(1 + \frac{u^2}{\lambda^2}\right) - |u| \arctan \frac{|u|}{\lambda}\right).\] (1.36)

For an $STL_\alpha$ flight $X(t)$ with $X(1) \sim STL_\alpha(1, p, \lambda)$, the distribution of $X(a)$ coincides with the distribution of $Y(1) \sim STL_\alpha(a, p, \lambda)$. For this reason, the parameter $a$ will be called the time-scale parameter.

**One-sided $STL_\alpha$ distribution**, $0 < \alpha < 1$. Let us consider the one-sided case when $p = 1$, and $0 < \alpha < 1$. Also, assume that $M = 0$. Then the cumulant function
is of the form
\[ \psi_X(u) = \Gamma(-\alpha) a \lambda^\alpha (p(1 - iu/\lambda) - 1). \] (1.37)

Since this is a one-sided distribution, the Laplace transform of the density is available:
\[ \hat{f}(s) = \exp (\Gamma(-\alpha) a \lambda^\alpha ((1 + s/\lambda) - 1)). \] (1.38)

**Example:** Let \( \alpha = 1/2 \). In this case we can find the inverse Laplace transform by using Mathematica:
\[ f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{sx} \hat{f}(s) ds = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{sx} \exp \left( -\alpha \sqrt{\lambda} (\sqrt{1 + s/\lambda} - 1) \right) ds = ae^{-2a\sqrt{\pi}x} x^{-3/2} \exp \left( -a^2 \pi/x - x\lambda \right). \]

Here, \( \gamma \) is a vertical contour in the complex plane chosen so that all singularities of \( \hat{f}(s) \) are to the left of it.

Therefore, we conclude that, for \( \alpha = 1/2 \), the density of \( STL_\alpha \) is
\[ f(x) = C x^{-3/2} \exp \left( -a^2 \pi/x - x\lambda \right), \] (1.39)
where \( C = ae^{-2a\sqrt{\pi}x} \) is the normalizing constant. This family is the Inverse Gaussian exponential family. The distribution \( STL_{1/2}(a, 1, 0) \) has the probability density
\[ f(x) = \frac{a}{x^{3/2}} e^{-a^2 \pi/x}, \quad x > 0, \]
see [Seshadri99], and Lévy observed that it is the distribution of the time of the first crossing of a level by the Brownian motion [Bochner62]. Notice that in this situation the truncation of the Lévy measure corresponds to the same density truncation. This peculiar fact has been observed, for \( \alpha = 1/2 \), by Bochner, and the related transformations are often called subordination, or Esscher, or exponential, transformations (see, also, [Sato99], p. 233, for the general case).

For the case \( M = 0 \), a more general fact can be proved.
Proposition 10 If $g(x)$ is a probability density function of a one-sided $\alpha$-stable distribution, $0 < \alpha < 1$, then, for every fixed $\lambda$, there exists one, and only one, $STL_\alpha$ distribution with the density $f(x) = cg(x)e^{-\lambda x}$, where $c$ is a normalizing constant.

**Proof.** Let us consider an $\alpha$-stable one-sided random variable $X$ with the cumulant function

$$\psi_X(u) = a\Gamma(-\alpha)(iu)^\alpha,$$

see (1.12), with $p = 1, q = 0$. Then the Laplace transform of the density is

$$\hat{g}(x) = \exp(a\Gamma(-\alpha)(-s)^\alpha),$$

with the density itself

$$g(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{sx} \hat{g}(s) \, ds$$

$$= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{sx} \exp(a\Gamma(-\alpha)(-s)^\alpha) \, ds.$$

Now, let us fix $\lambda$ and consider the Laplace transform of the density of $STL_\alpha(a, 1, \lambda)$ (1.38). The density $f(x)$ itself can then be written in the form

$$f(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{sx} \hat{f}(s) \, ds$$

$$= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{sx} \exp\left(\Gamma(-\alpha)a(\lambda + s)^\alpha - \lambda^\alpha\right) \, ds$$

$$= \exp\left[-\Gamma(-\alpha)a\lambda^\alpha\right] \frac{e^{-\lambda x}}{2\pi i} \int_{\gamma_1-i\infty}^{\gamma_1+i\infty} e^{-(y+\lambda)x} \exp(a\Gamma(-\alpha)(-y)^\alpha) \, dy$$

$$= \exp\left[-\Gamma(-\alpha)a\lambda^\alpha\right] e^{-\lambda x} g(x)$$

$$= Ce^{-\lambda x} g(x),$$

where $\gamma_1$ is the contour in the complex plane which we can get from the contour $\gamma$ by the transformation $\gamma_1 = \gamma + \lambda$. Thus we demonstrated that $f(x) = cg(x)e^{-\lambda x}$, and the proposition is proved.

**Remark.** In practice, this proposition is of limited usefulness since, for one-sided distributions, the density is known only for $\alpha = 1/2$. 
1.4 Fractional- and Integer-Order Moments of One-Sided $STL_\alpha$ Distributions, $0 < \alpha < 1$

1.4.1 Fractional-order moments

For the one-sided $STL_\alpha (a, 1, \lambda)$ distribution with $0 < \alpha < 1$, the cumulant function is of the form

$$\psi_{X_a} (u) = a \lambda^\alpha \Gamma (-\alpha) \left[ (1 - i\frac{u}{\lambda})^\alpha - 1 \right]. \quad (1.40)$$

As $\lambda \to 0$, the distribution $STL_\alpha (a, 0, \lambda)$ clearly converges to the $\alpha$-stable distribution $STL_\alpha (a, 1, 0)$, for which the fractional moments of order $\rho < \alpha$ are well known and of the form

$$E(|X_a|^\rho) = (-a \Gamma (-\alpha))^{\rho/\alpha} \frac{\Gamma(1 - \rho/\alpha)}{\Gamma(1 - \rho)}, \quad 0 < \rho < \alpha, \quad (1.41)$$

see, e.g., [Wolfe75], or [Sato99], p.162.

In this subsection we address the question of dependence of the fractional moments of $STL_\alpha (a, 1, \lambda)$ on scaling parameter $a$, for a fixed $\lambda$. Recall that random variables involved here are positive and the absolute moments and moments of the same order coincide.

Since the $STL_\alpha (a, 1, \lambda)$ distribution is concentrated on the positive halfline, it is more convenient to use its Laplace transform instead of the Fourier transform. The former is given by the formula

$$\phi_{X_a} (u) = E(e^{uX_a}) = \varphi_{X_a} (-iu) = \exp (a \lambda^\alpha \Gamma (-\alpha) [(1 + u/\lambda)^\alpha - 1]),$$

so that

$$E(|X_a|^\rho) = \frac{\rho}{\Gamma(1 - \rho)} \int_0^\infty [1 - \exp (a \lambda^\alpha \Gamma (-\alpha) [(1 + u/\lambda)^\alpha - 1])] \frac{du}{u^{1+\rho}},$$

see [Wolfe75]. Integrating by parts, and changing variables, we get

$$E(|X_a|^\rho) = \frac{a \lambda^{-\rho} \Gamma (1 - \alpha)}{\alpha \Gamma (1 - \rho)} \int_1^\infty \frac{\exp (a \lambda^\alpha \Gamma (-\alpha) [u - 1])}{(u^{1/\alpha} - 1)^\rho} du, \quad 0 < \rho < \alpha. \quad (1.42)$$
Fractional-order moments in the case $\alpha = 1/2$. Observe that only in the particular case $\alpha = 1/2$ the above formulas lead to explicit expressions. Indeed, in this case,

$$E(|X_a|^\rho) = \frac{2a\lambda^{1/2-\rho}\sqrt{\pi}}{\Gamma(1-\rho)} \int_1^\infty \exp\left(-2a\sqrt{\pi}\lambda \frac{u}{(u^2-1)^{\rho}}\right) du, \quad 0 < \rho < 1/2,$$

and the fractional moments of $STL_{1/2}(a, 1, \lambda)$ distribution, are

$$E(|X_a|^\rho) = 2a^{1/2+\rho} \left(\frac{\lambda}{\pi}\right)^{1/4-\rho/2} \exp\left(2a\sqrt{\pi}\lambda\right) K_{1/2-\rho}\left(2a\sqrt{\pi}\lambda\right), \quad 0 < \rho < 1/2,$$

where $K_\nu$ is the modified Bessel function of the second kind, see [PBM86], p. 323, 2.3.5.4. Since,

$$K_\nu(z) \sim \frac{1}{2} \Gamma(\nu) \left(\frac{z}{2}\right)^{-\nu}, \quad z \to 0,$$

see [AbramowitzStegun92], 9.6.9, we obtain the following asymptotics for the moments of $STL_\alpha$:

$$E(|X_a|^\rho) \sim 2a^{1/2+\rho} \left(\frac{\lambda}{\pi}\right)^{1/4-\rho/2} \frac{1}{2} \Gamma(1/2 - \rho) \left(a\sqrt{\pi}\lambda\right)^{\rho-1/2}$$

$$= (-a\Gamma(-1/2))^{2\rho} \frac{\Gamma(1-2\rho)}{\Gamma(1-\rho)}, \quad a \to 0,$$

so, for a fixed $\lambda$, at small-time scales $a$, the moments structure of the $STL_{1/2}$ distribution is the same as that of the $1/2$-stable distribution, see (1.41).

Since

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} (1 + O(1/z)), \quad z \to \infty,$$

see Bateman-Erdélyi [EMOT54], Vol. 2., p.32, we have, for large time-scales $a \to \infty$, the asymptotics

$$E(|X_a|^\rho) \sim a^\rho (\pi/\lambda)^{\rho/2}, \quad a \to \infty, \quad 0 < \rho < 1/2,$$

which gives the following logarithmic asymptotics for the fractional moments:

$$\log E(|X|^\rho) \sim \rho \log a + \text{const.}, \quad a \to \infty.$$
Fractional-order moments for general $\alpha$, $0 < \alpha < 1$. We shall return now to the general case of $0 < \alpha < 1$. Formula (1.42) can be rewritten in the form
\[
E \left( |X_a|^{\rho} \right) = \frac{\Gamma (1 - \alpha) a^{\alpha - \rho}}{\alpha \Gamma (1 - \rho)} \int_1^\infty \frac{\exp \left( a^{\alpha} \Gamma (-\alpha) [u - 1] \right)}{(u - 1)^\rho} \left[ \frac{u - 1}{u^{1/\alpha} - 1} \right]^\rho du.
\]
Function $f(u) := (u - 1) / (u^{1/\alpha} - 1)$ has the limit $\alpha$ for $u \to 1$, and the limit 0 at $u \to \infty$, with the Taylor series expansion around $u = 1$ of the form
\[
f(u) = \alpha - \frac{1 - \alpha}{2} (u - 1) + o([u - 1]), \quad (1.44)
\]
and the expansion around $u = \infty$ of the form
\[
f(u) = \frac{1}{u^{1/\alpha - 1}} - \frac{1 - \alpha}{\alpha u^{1/\alpha}} + o \left( \frac{1}{u^{1/\alpha}} \right). \quad (1.45)
\]
Therefore,
1. For small $a$, or, equivalently, large $u / (a^{\alpha} \Gamma (-\alpha))$, expansion (1.44) applied to the integrand in
\[
E \left( |X|^{\rho} \right) = a^{\rho} \frac{(-\Gamma (-\alpha))^{\rho}}{\Gamma (1 - \rho) \lambda^{(\alpha - 1)\rho}} \int_0^\infty \frac{\exp (-u)}{u^\rho} \left[ \frac{u / (a^{\alpha} \Gamma (-\alpha))}{(u / (a^{\alpha} \Gamma (-\alpha)) + 1)^{1/\alpha} - 1} \right]^\rho du,
\]
results in asymptotics
\[
E \left( |X|^{\rho} \right) \sim \frac{a^{\rho/\alpha} (-\Gamma (-\alpha))^{\rho/\alpha}}{\Gamma (1 - \rho) \lambda^{(\alpha - 1)\rho}} \frac{(a^{\alpha} \Gamma (-\alpha))^{1/\alpha - 1}}{u^{1/\alpha - 1}} + O \left( \frac{1}{u^{1/\alpha - 1}} \right)^\rho du
\]
\[
\sim a^{\rho/\alpha} \frac{(-\Gamma (-\alpha))^{\rho/\alpha}}{\Gamma (1 - \rho)} \Gamma (1 - \rho/\alpha), \quad a \to 0, \quad 0 < \rho < \alpha
\]
which again matches the moment behavior for the $\alpha$–stable case (1.41).

2. For large $a$, or, equivalently, small $u / (a^{\alpha} \Gamma (-\alpha))$, expansion (1.44) applies to yield asymptotics
\[
E \left( |X|^{\rho} \right) \sim a^{\rho} (\Gamma (1 - \alpha))^{\rho} \lambda^{(\alpha - 1)\rho}, \quad a \to \infty, \quad 0 < \rho < \alpha.
\]
In conclusion, we obtain the following

**Theorem 11** For each $\alpha$, $0 < \alpha < 1$, and each $\rho < \alpha$,

$$\log E(|X|^\rho) \sim \begin{cases} \frac{\rho}{\alpha} \log a + c_1, & \text{as } a \to 0; \\ \rho \log a + c_2, & \text{as } a \to \infty, \end{cases} \quad (1.46)$$

where constants $c_1 = \log \left( -\Gamma(-\alpha) \frac{\rho}{\alpha} \Gamma(1 - \frac{\rho}{\alpha}) / \Gamma(1 - \rho) \right)$, and $c_2 = \rho \log(\lambda^{\alpha-1} \Gamma(1-\alpha))$.

### 1.4.2 Integer-order moments

For $\lambda > 0$, the $STL_{\alpha}(a, 1, \lambda)$ distribution has finite moments of any positive order $\rho$, the fact crucial for their application as models of the physical and economical phenomena (see, e.g., Mantegna and Stanley [MantegnaStanley94]). In this subsection we will show how to explicitly calculate all the positive integer moments of $STL_{\alpha}(a, 1, \lambda)$ by using the theory of cumulants. This method will be applied in the case of $\rho = 1, 2, 3$.

In view of (1.40), the cumulants of order $k = 1, 2, \ldots$, of $STL_{\alpha}(a, 1, \lambda)$ are given by the expression

$$\text{cum}_k (X_a) = a^{\alpha-k} \Gamma(k - \alpha), \quad k = 1, 2, \ldots.$$ 

For a random quantity $Z$ the integer-order moments of $Z$ can be expressed in terms of its cumulants as follows:

$$E(Z) = \text{cum}_1 (Z),$$

$$E(Z^2) = \text{cum}_2 (Z) + \left[ \text{cum}_1 (Z) \right]^2,$$

$$E(Z^3) = \text{cum}_3 (Z) + 3 \text{cum}_1 (Z) \text{cum}_2 (Z) + \left[ \text{cum}_1 (Z) \right]^3,$$

$$\cdots \cdots \cdots$$

$$E(Z^k) = \text{cum}_k (Z) + \cdots + \left[ \text{cum}_1 (Z) \right]^k,$$

and so on.
For a fixed $k$, and large $a$, the term $[\text{cum}_1(X_a)]^k$ dominates. Hence we conclude that

$$\log \mathbb{E}(|X_a|^k) \sim k \log a + c, \text{ as } a \to \infty,$$

where $c$ is a constant independent of $a$. As we will see in Section 1.6 this is the first evidence of the multiscaling property of $STL_\alpha(a, 1, \lambda)$ flights, and reflects its large time-scale Gaussian (with drift) behavior.

For a fixed $k$, and as $a \to 0$, the dominant term is $\text{cum}_k(X_a)$. Therefore,

$$\log \mathbb{E}(|X_a|^k) \sim \log a + c, \text{ as } a \to 0. \tag{1.48}$$

Remark 1: At this time we are not providing an evaluation of theoretical fractional moments of order $\rho, \alpha \leq \rho < \infty$. However, the method used Subsection 1.4.1 can be extended to provide information about any fractional nonnegative moment of order $\rho$, producing

$$\log \mathbb{E}(|X|^\rho) \sim \begin{cases} \min(\rho/\alpha, 1) \log a + c_1, & \text{as } a \to 0; \\ \rho \log a + c_2, & \text{as } a \to \infty, \end{cases}$$

which is also supported by our simulations in Section 1.6. There, we will simulate the $STL_\alpha$ distribution and calculate the behavior of $\log \mathbb{E}(|X_a|^\rho)$ via a sampling procedure. In the special case $\alpha = 1/2$, formula (1.43) remains valid for any positive $\rho$ giving us complete information about the behavior of moments.

### 1.5 Fractional- and Integer-Order Moments for Symmetric $STL_\alpha$ distributions, $0 < \alpha < 2$

#### 1.5.1 Fractional-order moments

As see in (1.35) and (1.36), in the case of symmetric $STL_\alpha(a, 1/2, \lambda)$ distribution the characteristic function $\varphi_X(u) = \exp(\psi_X(u))$ is real-valued, with the cumulant
function of the explicit form
\[
\psi_{X_a}(u) = \begin{cases} 
   a \Gamma(-\alpha) \lambda^\alpha \left[ \left(1 + \frac{u^2}{\lambda^2}\right)^{\alpha/2} \cos(\alpha \arctan(u/\lambda)) - 1 \right], & \text{for } \alpha \neq 1; \\
   a \left(\frac{1}{2} \lambda \log(1 + \frac{u^2}{\lambda^2}) - |u| \arctan(|u|/\lambda)\right), & \text{for } \alpha = 1.
\end{cases}
\]  

(1.49)

In spite of this fact, it is hard to replicate for them direct calculations of asymptotics of fractional moments used in Section 1.4 in the case of one-sided \(STL_\alpha\) distributions. So, to investigate the behavior of absolute moments of \(STL_\alpha(a, 1/2, \lambda)\) we will have to be satisfied with just finding asymptotics for their upper and lower bounds. The integer-order moment calculations are similar to those in Subsection 1.4.2.

Let us begin by recalling that for any random quantity \(Z\), with characteristic function \(\varphi_Z(u)\), fractional moments are given by the formula

\[
E(|Z|^\rho) = c(\rho) \int_{-\infty}^{\infty} [1 - \text{Re} \varphi_Z(u)] \frac{du}{|u|^{1+\rho}},
\]  

(1.50)

where

\[
c(\rho) = \frac{\Gamma(1+\rho) \sin(\rho \pi/2)}{\pi},
\]

see [BahrEssen65].

**The case** \(0 < \alpha < 2, \alpha \neq 1\). Denote \(\theta_y = \arctan y\) and substitute the characteristic function \(\varphi(u) = \exp(\psi(u))\), with \(\psi(u)\) given by the first formula in (1.49), into formula (1.50). In view of the symmetry of the distribution, for \(\rho < \alpha\),

\[
E(|X|^\rho) = 2c(\rho) \int_0^{\infty} \left[1 - \exp\left(-a\lambda^\alpha \Gamma(-\alpha) \left[1 - (1 + u/\lambda)^{\alpha/2} \cos(\alpha \theta_{|u|/\lambda})\right]\right]\right] \frac{du}{u^{1+\rho}}.
\]  

(1.51)

Integration by parts, and change of variable \(y = u/\lambda\), yields the following expression:

\[
E(|X|^\rho) = \frac{2c(\rho)}{\rho} a \lambda^{a-\rho} \int_0^{\infty} \exp \left[-a\lambda^\alpha \Gamma(-\alpha) (1 - (1 + y^2) \cos(\alpha \theta_y))\right] \left(1 + y^2\right)^{\alpha/2-1} \left(1 - \frac{1}{y} \sin(\alpha \theta_y) - \cos(\alpha \theta_y)\right) \frac{dy}{y^{\rho-1}}.
\]  

(1.52)
Figure 1.1: Function $f(y; \alpha = 1/2)$.

Since the asymptotics of this integral, both, for small and large time-scale $a$, is difficult to evaluate, we begin by finding upper and lower bounds for the integrands.

**Lemma 12** Let $\alpha \in (0, 2) \setminus \{1\}$, and $y \geq 0$. Then the following inequalities hold:

$$
\Gamma(-\alpha)(\alpha - 1) \leq \frac{\Gamma(-\alpha) \left( 1 - (1 + y^2)^{\alpha/2} \cos(\alpha \theta_y) \right)}{(1 + y^2)^{\alpha/2} - 1} \leq -\Gamma(-\alpha) \cos \alpha \pi/2, \quad (1.53)
$$

$$
\Gamma(-\alpha)(\alpha - 1) \leq \Gamma(-\alpha) \left( \frac{1}{y} \sin(\alpha \theta_y) - \cos(\alpha \theta_y) \right) \leq -\Gamma(-\alpha) \cos(\alpha \pi/2). \quad (1.54)
$$

**Proof.** If $\alpha \in (0, 1)$, then $\Gamma(-\alpha) < 0$, and $\cos(\alpha \pi/2) > 0$. Therefore, the inequality (1.53) can be written as

$$
1 - \alpha \leq \frac{(1 + y^2)^{\alpha/2} \cos(\alpha \theta_y) - 1}{(1 + y^2)^{\alpha/2} - 1} \leq \cos \alpha \pi/2. \quad (1.55)
$$

Let us consider the function

$$
f(y) = \frac{(1 + y^2)^{\alpha/2} \cos(\alpha \theta_y) - 1}{(1 + y^2)^{\alpha/2} - 1}, \quad y \geq 0. \quad (1.56)
$$

This function is monotone and increasing. A plot of this function, for $\alpha = 1/2$, is presented in Figure 1.1. Observe that

$$
\lim_{y \to 0^+} f(y) = \lim_{y \to 0^+} \left( \frac{(1 + y^2)^{\alpha/2} \cos(\alpha \theta_y) - 1}{(1 + y^2)^{\alpha/2} - 1} \right) = \frac{\alpha(1 + y^2)^{\alpha - 1}}{\alpha(1 + y^2)^{\alpha - 1}} \left( \cos(\alpha \theta_y) - \frac{\sin(\alpha \theta_y)}{y} \right) = 1 - \alpha.
$$
On the other hand, 

\[ \lim_{y \to \infty} f(y) = \lim_{y \to \infty} \frac{(1 + y^2)^{\alpha/2} \cos(\alpha \theta y) - 1}{(1 + y^2)^{\alpha/2} - 1} \]

\[ = \lim_{y \to \infty} \frac{\alpha(1 + y^2)^{\alpha/2 - 1}}{\alpha(1 + y^2)^{\alpha/2} - 1} \left( \cos(\alpha \theta y) - \frac{\sin(\alpha \theta y)}{y} \right) = \cos(\alpha \pi / 2). \]

In the case \( 1 < \alpha < 2, \Gamma(-\alpha) > 0, \) and \( \cos(\alpha \pi / 2) < 0. \) Therefore, the inequality (1.53) can be written as follows:

\[ \alpha - 1 < \frac{1 - (1 + y^2)^{\alpha/2} \cos(\alpha \theta y)}{(1 + y^2)^{\alpha/2} - 1} \leq -\cos \alpha \pi / 2. \quad (1.57) \]

This function is increasing and the limits are:

\[ \lim_{y \to 0^+} f(y) = \lim_{y \to 0^+} \frac{1 - (1 + y^2)^{\alpha/2} \cos(\alpha \theta y)}{(1 + y^2)^{\alpha/2} - 1} = \alpha - 1, \]

\[ \lim_{y \to \infty} f(y) = \lim_{y \to \infty} \frac{1 - (1 + y^2)^{\alpha/2} \cos(\alpha \theta y)}{(1 + y^2)^{\alpha/2} - 1} = -\cos(\alpha \pi / 2). \]

Let us consider inequality (1.58). In the case \( 0 < \alpha < 1, \) it yields the following inequality:

\[ 1 - \alpha \leq \cos(\alpha \theta y) - \frac{1}{y} \sin(\alpha \theta y) \leq \cos(\alpha \pi / 2). \quad (1.58) \]

Let us consider the function

\[ g(x) := \cos(\alpha \theta y) - \frac{1}{y} \sin(\alpha \theta y). \]

This function is increasing. A plot of this function, for \( \alpha = 1/2, \) is presented in Figure 1.2. Let us find its limits at zero and infinity:

\[ \lim_{y \to 0^+} \left( \cos(\alpha \theta y) - \frac{\sin(\alpha \theta y)}{y} \right) = 1 - \alpha. \]

\[ \lim_{y \to \infty} \left( \cos(\alpha \theta y) - \frac{\sin(\alpha \theta y)}{y} \right) = \cos(\alpha \pi / 2). \]
In the case $1 < \alpha < 2$, we have the following inequality:

$$\alpha - 1 \leq \cos(\alpha \theta_y) - \frac{1}{y} \sin(\alpha \theta_y) \leq -\cos(\alpha \pi/2).$$

The proof is the same as that of (1.58).

The upper bounds. Applying inequalities (1.53) and (1.58) to formula (1.52) we have

$$E |X|^\rho \leq \frac{2c(\rho)}{\rho} a\lambda^{\alpha-\rho} \int_0^\infty \exp \left[-a\lambda^\alpha \Gamma(-\alpha)(\alpha - 1)((1 + y^2) - 1)\right]$$

$$\times (1 + y^2)^{\alpha/2-1}(-\Gamma(-\alpha) \cos(\alpha \pi/2)) \frac{dy}{y^{\rho-1}}.$$  \hfill (1.59)

Denote

$$A = -2c(\rho)a\lambda^{\alpha-\rho}\Gamma(-\alpha) \cos(\alpha \pi/2)/\rho,$$

$$b = a\lambda^\alpha \Gamma(-\alpha)(\alpha - 1).$$

Observe that, for $0 < \alpha < 1$ and for $1 < \alpha < 2$, constant $b$ is positive. Thus we get

$$E |X|^\rho \leq A \int_0^\infty \exp \left[-b((1 + y^2)^{\alpha/2} - 1)\right] (1 + y^2)^{\alpha/2-1} \frac{dy}{y^{\rho-1}}$$

$$= \frac{A}{\alpha} \int_1^\infty \exp(-b(v - 1))(v^{2/\alpha-1})^{-\rho/2}dv$$

$$= \frac{A}{\alpha} \int_1^\infty \frac{\exp(-b(v - 1))}{(v - 1)^{\rho/2}} \left(\frac{v - 1}{v^{2/\alpha-1}}\right)^{\rho/2}dv.$$  \hfill (1.60)
Now, function
\[ f(y) = \frac{y - 1}{y^{2/\alpha} - 1} \]
can be expanded in a power series around 1 and infinity similarly to the procedure used in Section 1.4, the only difference being that, \( 1/\alpha \) is replaced by \( 2/\alpha \).

If \( a \to 0 \), or, equivalently, \( b \to 0 \), then

\[
E|X|^\rho \leq \frac{A}{\alpha} \int_1^{\infty} \frac{\exp(-b(v-1))}{(v-1)^{\rho/2}} \left( \frac{v - 1}{v^{2/\alpha - 1}} \right)^{\rho/2} dv
\]

\[
= \frac{A}{b\alpha} \int_0^{\infty} \frac{\exp(-u)}{(u/b)^{\rho/\alpha}} \left( \frac{u/b}{(u/b + 1)^{2/\alpha} - 1} \right)^{\rho/2} du
\]

\[
= \frac{A}{b\alpha} \int_0^{\infty} \frac{\exp(-u)}{(u/b)^{\rho/\alpha}} \left( \frac{b^{2/\alpha - 1}}{u^{2/\alpha - 1} - 1} + o \left( \frac{1}{(u/b)^{2/\alpha}} \right) \right) du
\]

\[
\sim \frac{A}{b\alpha} b^{\rho/\alpha} \Gamma(1 - \rho/\alpha)
\]

\[
= a^{\rho/\alpha} 2 \sin(\rho\pi/2) \cos'(\alpha\pi/2) \left( \frac{\Gamma(-\alpha)(\alpha - 1)^{\rho/\alpha} \Gamma(\rho) \Gamma(1 - \rho/\alpha)}{\pi\alpha(1 - \alpha)} \right).
\]

\[ (1.61) \]

**Lemma 13** Let \( c_a \leq b_a \sim d_a \quad (c_a \geq b_a \sim d_a) \quad a \to h \quad \text{were } h = \pm \infty. \) Then \( \forall \epsilon > 1 \exists a_{\epsilon} : \forall a < a_{\epsilon} (a > a_{\epsilon}) c_a \leq d_a \cdot \epsilon \quad (c_a \geq d_a \cdot \epsilon), \) and therefore

\[
\log c_a \leq \log d_a + \log \epsilon \quad (\log c_a \geq \log d_a + \log \epsilon) \quad \text{as } a \to h
\]

The proof of the lemma follows directly from the definition of asymptotics.

Therefore, by Lemma 13, we obtain the following

**Theorem 14** There exists a positive constant \( c_1 \) such that, for sufficiently small \( a \), we have the following inequality:

\[
\log(E|X|^\rho) \leq \frac{\rho}{\alpha} \log a + c_1.
\]

If \( a \to \infty \), or equivalently, \( b \to \infty \) and \( u/b \to 0 \), then
\[ E|X|^\rho \leq \frac{A}{b\alpha} \int_0^\infty \exp(-u) \left( \frac{u/b}{(u/b + 1)^{2/\alpha} - 1} \right)^{\rho/2} d\rho \]
\[ \sim \frac{A}{b\alpha} \int_0^\infty \exp(-u) \left( \frac{\alpha}{2} \right)^{\rho/2} d\rho \]
\[ = \frac{A}{b\alpha} \left( \frac{\alpha}{2} \right)^{\rho/2} \Gamma(1 - \rho/2) \]
\[ = a^{\rho/2} \frac{2 \sin(\pi/2) \cos(\alpha\pi/2)}{\pi\alpha(1 - \alpha)} \Gamma(\alpha - 2) \Gamma(2 - \alpha) \Gamma(1 - \rho/2), \]
\[ (1.62) \]

Therefore, we obtain the following

**Theorem 15** There exists a positive constant \( c_2 \) such that, for sufficiently large \( a \), we have the following inequality:

\[ \log(E|X|^\rho) \leq \frac{\rho}{2} \log a + c_2. \]

**The lower bounds.** Here we proceed in a manner analogous to the estimates used for upper bounds but employing the complementary inequalities in (1.53) and (1.58).

\[ E|X|^\rho \geq \frac{2c(\rho)}{\rho} a\lambda^{\alpha - \rho} \int_0^\infty \exp \left[ a\lambda^\alpha \Gamma(-\alpha) \cos(\alpha\pi/2)(1 + y^2) - 1 \right] dy \frac{(1 + y^2)^{\alpha/2 - 1} (\Gamma(-\alpha)(\alpha - 1))}{y^{\rho-1}}. \]
\[ (1.63) \]

Denote

\[ A_1 = 2c(\rho) a\lambda^{\alpha - \rho} \Gamma(-\alpha)(\alpha - 1)/\rho, \]

\[ b_1 = -a\lambda^\alpha \Gamma(-\alpha) \cos(\alpha\pi/2). \]

Then the above inequality can be written as follows:

\[ E|X|^\rho \geq A_1 \int_0^\infty \exp \left[ -b_1((1 + y^2)^{\alpha/2} - 1) \right] (1 + y^2)^{\alpha/2 - 1} \frac{dy}{y^{\rho-1}}. \]
\[ (1.64) \]
Note, that the right-hand sides of the (1.64) and (1.60) differ only by a multiplicative constant. Therefore we can use the same calculations as before to get the following results:

\[
E|X|^{\rho} \geq A_1 \int_0^\infty \frac{\exp(-b_1(v-1))}{(v-1)^{\rho/2}} \left( \frac{v-1}{v^{2/\alpha-1}} \right)^{\rho/2} dv
\]

\[
\sim \frac{A_1}{b_1 \alpha} b_1^{\rho/\alpha} \Gamma(1-\rho/\alpha)
\]

\[
= a^{\rho/\alpha} \frac{2\sin(\rho \pi/2)(1-\alpha)}{\pi \alpha \cos(\alpha \pi/2)} (-\Gamma(-\alpha) \cos(\alpha \pi/2))^{\rho/\alpha} \Gamma(\rho) \Gamma(1-\rho/\alpha),
\]

as \(a \to 0\). By lemma 13, we obtain the following

**Theorem 16** There exists a constant \(c_3\) such that for sufficiently small \(a\) the following inequality holds:

\[
\log(E|X|^{\rho}) \geq \frac{\rho}{\alpha} \log a + c_3.
\]

When \(a \to \infty\),

\[
E|X|^{\rho} \geq A_1 \int_0^\infty \frac{\exp(-b_1(v-1))}{(v-1)^{\rho/2}} \left( \frac{v-1}{v^{2/\alpha-1}} \right)^{\rho/2} dv
\]

\[
\sim \frac{A}{b \alpha} \left( \frac{\alpha \beta}{2} \right)^{\rho/2} \Gamma(1-\rho/2)
\]

\[
= a^{\rho/2} \frac{2\sin(\pi/2)(1-\alpha)}{\pi \alpha \cos(\alpha \pi/2)} \left( \frac{\lambda^{\alpha-2} \Gamma(1-\alpha) \cos(\alpha \pi/2)}{2} \right)^{\rho/2} \Gamma(\rho) \Gamma(1-\rho/2).
\]

Therefore, we obtain the following

**Theorem 17** There exist a constant \(c_4\) such that, for large enough \(a\),

\[
\log(E|X|^{\rho}) \geq \frac{\rho}{2} \log a + c_4.
\]

Thus the asymptotic dependence of the log-fractional absolute moments on \(\log a\) is linear, both, for small and large time-scales \(a\). As we shall see in Section 1.6, for time-scales \(a \to 0\), a symmetric \(STL_\alpha\) flight converges to the Lévy \(\alpha\)-stable flight, and,
for time-scales \( a \to \infty \), it converges to the Brownian motion. This result gives the logarithmic asymptotics indicated above. The above discussion can be summarized as follows:

\[
\log(\mathbb{E} |X|^\rho) \sim \begin{cases} 
(\rho/\alpha) \log a + C_1, & \text{for } a \to 0; \\
(\rho/2) \log a + C_2, & \text{for } a \to \infty;
\end{cases}
\]  

for some constants \( C_1, C_2 \).

**The case \( \alpha = 1 \).** Note that the cumulant function \( \psi(u) = \psi_\alpha(u) \) in (1.49), for \( \alpha = 1 \), is the limit of the cumulant functions \( \psi_\alpha(u) \) as \( \alpha \searrow 1 \), and the same is true for the corresponding characteristic functions. Therefore, for any \( \rho < 1 \), the integrand in (1.51) for \( \alpha = 1 \) is the bounded limit of the integrands for \( \alpha \searrow 1 \). As a result, for \( \alpha = 1 \), we get the asymptotics

\[
\log(\mathbb{E} |X|^\rho) \sim \begin{cases} 
\rho \log a + C_3, & \text{for } a \to 0; \\
(\rho/2) \log a + C_4, & \text{for } a \to \infty;
\end{cases}
\]  

for some constants \( C_3, C_4 \).

### 1.5.2 Integer-order moments

For \( STL_\alpha(a, 1/2, \lambda) \), \( \alpha \neq 1 \), the cumulant technique used in Subsection 1.5.1 permits only evaluation of absolute moments of even order \( \rho = 2n \). Other moments can be obtained via the Central Limit Theorem as is done in Section 4. The calculations analogous to those of Subsection 2.2 give the following logarithmic asymptotics:

For a fixed \( k \), and large \( a \),

\[
\log \mathbb{E} \left( |X_a|^k \right) \sim \frac{k}{2} \log a + c, \quad \text{as } a \to \infty,
\]  

where \( c \) is a constant independent of \( a \). As we will see in Section 1.6 this is the first evidence of the multiscaling property of \( STL_\alpha(a, 1, \lambda) \) flights, and reflects its large time-scale Gaussian (no drift) behavior.

For example, if \( 2n = 4 \), then

\[
\mathbb{E} (X^4) = \text{cum}_4(X) + 3 [\text{cum}_2(X)]^2.
\]
Since, for any symmetric distribution $STL_\alpha(a, 1/2, \lambda)$,
\[
    \text{cum}_{2k}(X) = a\lambda^{\alpha-2k} \Gamma(2k - \alpha),
\]
we have the asymptotics $E(X^4) \sim a^2$, as $a \to \infty$.

For small $a$, we have that
\[
    \log E(|X_a|^k) \sim \log a + c, \quad \text{as } a \to 0.
\]

## 1.6 Multiscaling of the Smoothly Truncated Lévy Flight

Consider now the $STL_\alpha$ flight $X(t) = X_a(t)$ determined by cumulant function
\[
    \psi_{X_a(t)}(u|\lambda) = t a \lambda^\alpha \Gamma(-\alpha) \left[ p \left( 1 + \frac{i u}{\lambda} \right)^\alpha + q \left( 1 - \frac{i u}{\lambda} \right)^\alpha - 1 \right],
\]
see (1.35).

Our basic contention, grounded in moment calculations from Sections 1.4 and 1.5, is that two different rescalings of $STL_\alpha$ flights give two different behaviors for large and small $t$'s: the rescaling
\[
    a^{-1/\alpha}X_a(t) \Rightarrow S_\alpha(t), \quad a \to 0,
\]
where the convergence $\Rightarrow$ is that of one-point distributions, gives the behavior of the $\alpha$-stable process $S_\alpha(t)$, while the rescaling
\[
    a^{-1/2}(X_a(t) - E(X_a(t))) \Rightarrow B(t), \quad a \to \infty,
\]
gives the behavior of the Brownian motion process $B(t)$.

We will argue this multiscaling statement by looking at the asymptotics of the logarithmic partition functions
\[
    \log E(|a^{-1/\alpha}X_a(t)|^\rho) = -\frac{\rho}{\alpha} \log a + \log E(|X_a(t)|^\rho), \quad a \to 0,
\]
\[
    = -\frac{\rho}{\alpha} \log a + \frac{\rho}{\alpha} \log(at) + c = \frac{\rho}{\alpha} \log t + c,
\]
for \( \rho < \alpha \), where the behavior of the last term is known from Sections 1.4 and 1.4, and

\[
\log \mathbb{E} \left( |a^{-1/2}(X_a(t) - \mathbb{E}(X_a(t)))|^\rho \right) = \frac{\rho}{2} \log t + c, \quad a \to \infty, \tag{1.71}
\]

in view the general Central Limit Theorem. Indeed, the limit distribution of

\[
a^{-1/2}(X_a(t) - \mathbb{E}(X_a(t)))
\]

is Gaussian with mean zero and variance \( \text{Var}(X_1(t)) \propto t \) so that its \( \rho \)-th central moments grow like \( t^{\rho/2} \).

Remark: Since the characteristic functions of the processes in question are analytic the convergence of the corresponding logarithmic partition functions implies the convergence of processes in (1.68) and (1.69). A different approach to this problem, in a more general context of tempered stable processes has been announced by Rosinski [Rosinski02]. It is our understanding that the full version of his results will be submitted for publication soon, see [Rosinski04].

### 1.6.1 One-sided \( STL_\alpha \) flights, \( 0 < \alpha < 1 \).

For convenience let us introduce the multiscaling exponent \( \tau(\rho) \) such that the partition function

\[
\mathbb{E}(X_\rho^\alpha(t)) \propto t^{\tau(\rho)}.
\]

Results of Section 1.4 and the comments above immediately give that in the case of one-sided \( STL_\alpha \) flights, \( 0 < \alpha < 1 \),

\[
\log \mathbb{E} \left( |X_a(t)|^\rho \right) \sim \tau(\rho) \log(at) + \text{const},
\]

where, for \( a \to 0 \),

\[
\tau(\rho) = \frac{\rho}{\alpha}, \quad \text{for} \quad \rho < \alpha,
\]

and, for \( a \to \infty \),

\[
\tau(\rho) = \rho, \quad \text{for} \quad \rho < \alpha, \quad \text{and} \quad \rho = 1, 2, \ldots
\]
The simulations provided below give information about this asymptotic behavior.

In Figure 1.6.1 we show the multiscaling exponents $\tau(\rho)$ for $STL_\alpha(a, 1, 10^{-5})$ flights, for time-scale parameter values $a = 0.004$, and 800, and the index $\alpha = 1/2$ (top) and $\alpha = 3/4$ (bottom).

The “solid” lines represent the limiting Gaussian (with drift) case; the “dashed” lines correspond to the $\alpha$-stable processes. The “triangle-down” data points are obtained, for $a = 800$, using theoretical values of the moments. The ”triangle-up” data points are obtained by simulation of $STL_\alpha(800, 1, 10^{-5})$ flights (the nontrivial simulation procedure is described in the Appendix) and estimation of $\tau(\rho)$. The two “squares” points are obtained, for $a = 0.004$, by calculating the exact values of the moments of integer order. The “asterisk” data points are obtained by simulation of $STL_\alpha(0.004, 1, 10^{-5})$ flights and estimation of $\tau(\rho)$.

Additionally, in the case $\alpha = 1/2$ (top), we are able to find numerically values of all fractional moments and use this information to calculate $\tau(\rho)$. The “dash-point-dash” line corresponds to $a = 800$, and “dotted” line corresponds to $a = 0.004$.

For small time-scale parameter value $a = 0.004$, the behavior of the multiscaling exponent $\tau(\rho)$, $\rho < \alpha$ is close to that of the $\alpha$-stable process; the smaller $\rho$ the better the approximation.

If $a = 800$, for the range of $\rho$ covered by our estimations, the situation is close to that for a Gaussian process with drift.

Remark. Note that in the case $\alpha = 1/2$ the probability density is explicitly known and we have available exact numerical evaluation of the fractional and integer order moments. So our estimations of $\tau(\rho)$ are more accurate here. We used them as a validation tool for our simulations.

1.6.2 Symmetric $STL_\alpha$ flights, $0 < \alpha < 2$.

Results of Section 1.5 and comments at the beginning of this section give that in the case of symmetric $STL_\alpha$ flights, $0 < \alpha < 2$,

$$\log E (|X_\alpha (t)|^\rho) \sim \tau(\rho) \log(\alpha t) + \text{const},$$
Figure 1.3: The multiscaling exponents $\tau(\rho)$ for one-sided $STL_\alpha(a, 1, 10^{-5})$ flights, for time-scale parameter values $a = 0.004, 10, 800$, and the index $\alpha = 1/2$ (top) and $\alpha = 3/4$ (bottom).
where, for $a \to 0$,

$$
\tau(\rho) = \frac{\rho}{\alpha}, \quad \text{for } \rho < \alpha,
$$

and, for $a \to \infty$,

$$
\tau(\rho) = \frac{\rho}{2}, \quad \text{for } \rho < \alpha, \quad \text{and } \rho = 1, 2, \ldots
$$

The simulations provided below give information about this asymptotic behavior for intermediate values of $a$ and values of $\rho$ not accounted for above.

In Fig. 1.4 we show the multiscaling exponents $\tau(\rho)$ for symmetric $STL_\alpha(a, 1/2, 10^{-2})$ flights, for time-scale parameter values $a = 0.004$, and 800, and the index $\alpha = 3/2$ (top) and $\alpha = 7/4$ (bottom).

The the usage of special lines in this Figure 1.4 is identical to that of Figure 1.6.1 which was explained in Subsection 1.6.1.

For small time-scale parameter value $a = 0.004$, the behavior of the multiscaling exponent $\tau(\rho)$, $\rho < \alpha$ is close to that of the $\alpha$-stable process; the smaller $\rho$ the better the approximation.

If $a = 800$, for the range of $\rho$ covered by our estimations, the situation is close to that for a Gaussian process with no drift.

### 1.7 Monte Carlo Method for Simulation of $STL_\alpha$ Flights

For our simulations of STL random variables we have used the series representations obtained in Rosinski [Rosinski02]. In the one-sided $STL_\alpha(a, 1, \lambda)$ case, with $0 < \alpha < 1$, the expansion is as follows:

$$
S_0 = \sum_{j=1}^{\infty} \left( \left( \frac{\alpha \gamma_j}{a} \right)^{-1/\alpha} \wedge \frac{e_j u_j^{1/\alpha}}{\lambda} \right),
$$

where $\{u_j\}$ is an sequence of independent and identically distributed (iid) uniform random variables on the interval $[0, 1]$, $\{e_j\}$ and $\{e'_j\}$ sequences of iid exponential
Figure 1.4: The multiscaling exponents $\tau(\rho)$ for symmetric $STL_\alpha(a, 1/2, 10^{-2})$ flights, for time-scale parameter values $a = 0.004$, and 800, and the index $\alpha = 3/2$ (top) and $\alpha = 7/4$ (bottom).
random variables with parameter 1, and $\gamma_j = e'_1 + \cdots + e'_j$. Additionally, the sequences $\{u_j\}$, $\{e_j\}$ and $\{e'_j\}$ must be independent of each other. The wedge $\wedge$ stands for the operation of taking the minimum of two real numbers.

In the symmetric $STL_\alpha(a, 1/2, \lambda)$ case the expansion is

$$S_1 = \sum_{j=1}^{\infty} \left( v_j \left( \frac{\alpha \gamma_j}{a} \right)^{-1/\alpha} \wedge \frac{e_j u_j^{1/\alpha}}{\lambda} \right),$$

where the additional sequence $\{v_j\}$ sequence appearing here consists of independent random variables taking values $\pm 1$ with equal probabilities $1/2$.

The $STL_\alpha(a, p, \lambda)$ flight at an integer time $t = 2, 3, \ldots$, is simulated as a sum of iid copies of the same flight at time 1.

In the one sided-case $p = 0$, we choose $\lambda = 10^{-5}$, and $a = 0.004$ or $= 800$. In the symmetric case, we choose $\lambda = 10^{-2}$, and $a = 0.004$ or $= 800$.

To obtain the multiscaling exponent $\tau(\rho)$, we take into account the fact that, for small and large $a$ the relationship between $\log \mathbb{E}( |X_a(t)|^\rho)$ and $\log a$, is nearly linear, choose times exponentially $t = 1, 2, 2^2, \ldots, 2^5$, and estimate the slope via linear regression.

**Bibliography**


Chapter 2

Estimation of the Parameters of \( \alpha \)-Stable Distributions and Smoothly Truncated Lévy Distributions

2.1 Introduction

Probability models using \( \alpha \)-stable distributions and smoothly truncated Lévy distributions arise in various fields, including medicine, statistical physics, finance, hydrology, and computer science. To apply them in practical problems we have to be able to obtain information about parameters of these distributions.

In this Chapter we begin with a review of the problem of parametric estimation for the \( \alpha \)-stable distributions, but our main goal is to develop methods of estimation of parameters for the Smoothly Truncated Lévy distributions.

Since \( \alpha \)-stable distributions do not have finite moments and only in few cases have explicit expression for the density function, most of the standard methods of estimations such as the method of moments and the maximum likelihood method are not applicable directly.

The problem of estimation of the stability index \( \alpha \) was addressed first, in Fama and Roll’s work (1968) [FamaRoll68, FamaRoll71], where they used sample quantiles to construct estimators for the symmetric case. Later Press [Press72] constructed
estimators of the tail thickness based on the structure of the characteristic function. But the properties of this estimator depend in essential way on the choice of fixed argument values of the characteristic function.

Another method popular in applications is the Hill’s method [Hill75]. This method was constructed to investigate the special case of heavy-tailed distributions with Pareto-type tails. Hill constructed a simple estimator for the tail index by maximizing the conditional likelihood function. In applications, many authors use this estimator for the general class of stable distributions. The statistical software ”R” uses this method as a ”build-in” function for estimation of the index parameter $\alpha$.

Another important approach is due to Zolotarev (see[Zolotarev86]). It uses the moments properties of the logarithm of the absolute value of the stable random variable.

Meerschaert and Sheffler constructed quadratic estimators for the parameter $\alpha$ (see [MeerschaertScheffler98]). The scale invariant corrections of the quadratic estimators were introduced in [BianchiMeerschaert].

Recently, Fan [Fan] used the method of U-statistics for estimation of the tail index $\alpha$ and the scale parameter $\sigma$.

A numerical algorithm implementing Maximum Likelihood Estimation method for all parameters of the $\alpha$-stable distributions was constructed by Nolan [Nolan], who also provided program STABLE for an implementation of this algorithm.

The class of $STL_\alpha$ distributions is an intermediate class of distributions between $\alpha$-stable and normal distributions. The $\alpha$-stable distributions do not have finite moments of order greater or equal to $\alpha$, but for real data we can calculate empirical moments of any order. Therefore, $STL_\alpha$ distributions are a reasonable model for fitting of real data having an empirical density function similar around zero to an $\alpha$-stable density but rapidly decaying away from zero. To apply this model one has to be able to estimate parameters of such distributions.

Notice that if an $STL_\alpha$ distribution is close to the normal distribution then utilization of this model may be unnecessary. We can just assume normality for such data and work with the normal model. Moreover, in such a case it is practically
impossible to estimate all parameters $\alpha, a, \lambda, p$.

Thus our main goal is parameter estimation for $STL_\alpha$ distributions which are close to $\alpha$-stable distributions.

In Section 2.2 we recall some methods of estimation of the tail and scale parameters of the $\alpha$-stable distributions. We give a description of the algorithms due to Fama and Roll, and to Hill. These estimators are not suitable for us because they use the tail behavior. We would like to estimate parameters from the behavior of the density around zero to be able to use it in the $STL_\alpha$ case. Therefore, we consider the method of moments of Zolotarev, the quadratic estimation of Meerschaert and Sheffler and its modification, and the U-statistics approach. These methods use logmoments and fractional moments and therefore are applicable for our purpose.

We also construct estimators of the index parameter which use the properties of the fractional moments. To decide which estimator is best for our applications we performed simulations of $\alpha$-stable distribution using the Chambers-Mellows algorithm and estimated the index and scale parameters. Based on these simulations, we decided to use the Zolotarev estimators in the remainder of this chapter.

Section 2.3 is devoted to the problem of parameter estimation for $STL_\alpha$ distributions. First, we find the maximum likelihood estimators for the one-sided $STL_{1/2}(a, 1, \lambda)$ distributions, where we have an explicit expression for the density function.

Second, we consider one-sided $STL_\alpha$ distributions. In this case we use the limit results from Chapter 1 and estimate the index and scale parameters from the corresponding limiting stable distribution. To estimate the truncation parameter we use the method of moments. Expressions for the cumulants derived in Chapter 1 are used to get moment estimators.

In the symmetric case we provide an algorithm for the Numerical Maximum Likelihood estimation. As the starting point, the Zolotarev estimators for the index and scale parameters are used. The moment method is used to find the starting point for the truncation parameter in the Numerical MLE procedure.
2.2 Estimation of the Parameters of $\alpha$-Stable Distributions

Let $X \sim S_\alpha(\sigma, \beta, 0)$ be an $\alpha$-stable random variable. We assume that the location parameter is zero. We will consider different ways to estimate $\alpha$ and $\sigma$, and perform simulations to compare different estimation methods.

1. Fama-Roll Method. These estimators were constructed for the symmetric stable law with $1 < \alpha \leq 2$. The following estimator of $\sigma$, the scale parameter, is proposed

\[ \hat{\sigma} = \frac{\hat{x}_{.72} - \hat{x}_{.28}}{1.654}, \quad (2.1) \]

where $x_p$ is $p$-th percentile.

McCulloch (1986) ([McCulloch86]) noticed that Fama and Roll based their estimator of $\sigma$ on the fortuitous observation that $(\hat{x}_{.72} - \hat{x}_{.28})/\sigma$ lies within 0.4% of 1.654 for all $1 < \alpha \leq 2$, when $\beta = 0$. This enables them to estimate $\sigma$ by (2.1) with less then 0.4% asymptotic bias, without knowing $\alpha$.

Then $\alpha$ can be estimated from the tail behavior of the distribution. Fama and Roll take $\hat{\alpha}$ satisfying

\[ S_\hat{\alpha} \left( \frac{\hat{x}_f - \hat{x}_{1-f}}{2\hat{\sigma}} \right) = f. \quad (2.2) \]

They find that $f = 0.95, 0.96, 0.97$ works best. Tabulated values of $S_\alpha$ can be found in [SamTaqqu94], p.598.

2. Hill estimator. Consider i.i.d. random variables $X_1, X_2, \ldots$, with common distribution $F$, with regularly varying tail

\[ P(X_1 > x) = x^{-\alpha} L(x), \quad x \to \infty, \quad \alpha > 0, \quad (2.3) \]

where $L$ is a slowly varying function, and $\alpha$ is the index of regular variation. Let us consider the order statistics in decreasing order

\[ X_{(1)} \geq X_{(2)} \geq \cdots X_{(n)}. \]
Then the Hill estimator $H_{k,n}$ based on $(k + 1)$st upper order statistics, and a sample of size $n$, is

$$H_{k,n} = \frac{1}{k} \sum_{i=1}^{k} \log \frac{X_{(i)}}{X_{(k+1)}}. \quad (2.4)$$

Concerning the asymptotic behavior of this estimator, it is known that $H_{k,n}$ is consistent in sense that

$$H_{k,n} \xrightarrow{P} 1/\alpha, \quad n \to \infty, \ k \to \infty, \ n/k \to \infty.$$

Unfortunately it is hard to choose the right value of $k$. Therefore, in practice, it is advisable to construct a plot of points $\{(k, H_{k,n}), 1 \leq k \leq n - 1\}$, called the Hill plot, and infer the value $\gamma$ from the stable region in the graph. This is sometimes difficult since the plot may be volatile or may not spend a large portion of the display space in the neighborhood of $\gamma$. In fact, it is becoming increasingly clear that the traditional Hill plot is most effective only when the underlying distribution is Pareto or very close to Pareto. For the Pareto distribution

$$1 - F(x) = \left(\frac{x}{\sigma}\right)^{-\alpha}, \quad x > \sigma > 0.$$ 

3. Method of moments. This method was introduced by Zolotarev (see, [Zolotarev86]). Let $Y, Y_1, \ldots, Y_n$ be i.i.d. strictly stable random variables given by cumulant function (1.7). Since the moments of order greater then or equal to $\alpha$ do not exist, the collection of independent (within each collection) random variables

$$U_i = \text{sign} Y_i, \quad V_i = \log |Y_1|, \ i = 1, \ldots, n, \quad (2.5)$$

is considered. For these random variables all the moments exist and can be calculated. It was shown that the following equalities hold [Zolotarev86]:

$$\nu = \frac{6}{\pi} Var V - \frac{3}{2} Var U + 1, \quad \theta = EU, \quad \tau = EV, \quad (2.6)$$

where $\nu = \frac{1}{\alpha^2}$. The relationships between parameters $\beta, \sigma$ and $\theta, \tau$ are given by (1.9), (1.11).
Using the method of moment we can find the following estimator:

\[ \hat{\nu} = \frac{6}{\pi} S^2_V - \frac{3}{2} S^2_U + 1, \]  

(2.7)

where

\[ S^2_U = \frac{\sum_{i=1}^{n} (U_i - \bar{U})}{n - 1} \xrightarrow{P} E U^2 - (EU^2) = \text{Var} U, \]  

(2.8)

and

\[ S^2_V = \frac{\sum_{i=1}^{n} (V_i - \bar{V})}{n - 1} \xrightarrow{P} E V^2 - (EV^2) = \text{Var} V. \]

Also \( S^2_U \xrightarrow{P} \text{Var} U \) and \( E\hat{\nu} = \nu \). Therefore this estimator is consistent and unbiased.

Using (1.8) we get

\[ \hat{\alpha} = 1/\sqrt{\hat{\nu}}. \]

(2.9)

Now let us consider estimation of the scale parameter \( \sigma \). Using (2.6) and (1.11), one can conclude that

\[ \hat{\sigma} = 1/2 \exp(\hat{\tau}/\sqrt{\hat{\nu}} - C(1 - 1/\sqrt{\hat{\nu}})), \quad \text{if} \ \nu \neq 1, \]  

(2.10)

where

\[ \hat{\tau} = \frac{1}{n} \sum_{i=1}^{n} V_i. \]  

(2.11)

The symmetric case, with \( \alpha = 1 \), corresponds to the Cauchy distribution with a known density function, and we can estimate its parameters by using the maximum likelihood estimators (MLE).

To verify the above estimation procedures, we simulate the i.i.d symmetric \( S_{1.5}(3, 0, 0) \) random variables, for \( n = 1000, 4000, 10000 \), and estimate \( \alpha \) and \( \sigma \) using the method of moments 500 times. Let us mention that in simulations some \( x_i = 0 \). Since in this method we have to find \( \log |x_i| \), which is \( -\infty \) at zero, we just exclude these points from the calculations. The boxplots for these estimators are shown in Figure 2.1. The left picture corresponds to the estimator of \( \alpha \) and the right picture corresponds to \( \sigma \). One can see that when sample size increases the variance decreases and we have
Figure 2.1: Boxplot of the moment estimators for an $\alpha$-stable distribution. Left: estimation of $\alpha = 1.5$; right: estimation of $\sigma = 3$. 
fewer outliers and they are less spread out. Also for the case $\alpha = 1.5$, we calculate the mean, the standard deviation of the estimators. The results are presented in the table below:

<table>
<thead>
<tr>
<th>$\alpha$ sample size</th>
<th>mean</th>
<th>std</th>
<th>min value</th>
<th>max value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>1.5162</td>
<td>0.1223</td>
<td>1.2326</td>
<td>1.9651</td>
</tr>
<tr>
<td>4000</td>
<td>1.5027</td>
<td>0.0573</td>
<td>1.3648</td>
<td>1.7637</td>
</tr>
<tr>
<td>10000</td>
<td>1.5026</td>
<td>0.0363</td>
<td>1.4058</td>
<td>1.6179</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\sigma$ sample size</th>
<th>mean</th>
<th>std</th>
<th>min value</th>
<th>max value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>2.7356</td>
<td>0.6382</td>
<td>1.6054</td>
<td>5.7163</td>
</tr>
<tr>
<td>4000</td>
<td>2.6253</td>
<td>0.2687</td>
<td>2.0571</td>
<td>3.9646</td>
</tr>
<tr>
<td>10000</td>
<td>2.6147</td>
<td>0.1704</td>
<td>2.1605</td>
<td>3.2171</td>
</tr>
</tbody>
</table>

4. **Quadratic estimator.** The quadratic estimator was suggested in [MeerschaertScheffler98], and its scale-invariant correction was introduced in [BianchiMeerschaert].

Let $X, X_1, \ldots, X_n$ be i.i.d. random variables which belong to the domain of attraction of a non-degenerate random variable $Y$ which has a stable distribution.

Then the quadratic estimator of $1/\alpha = \gamma$ is given by the formula:

$$\hat{\gamma}([X_i]_{i=1,...,n}) = \frac{C + \log(\sum_{i=1}^{n} (X_i - \bar{X}_n)^2)}{2(C + \log n)}.$$  \hspace{1cm} (2.12)

This is a consistent estimator. The quadratic estimator (2.12) is shift-invariant but not scale invariant. The median adjusted quadratic estimator, which is both scale and shift-invariant, as well as consistent was given in [BianchiMeerschaert]. The result is as follows:

**Theorem 18** [BianchiMeerschaert] Suppose $X \in DOA(\alpha)$, for some $0 < \alpha < 2$, and define

$$\hat{\phi} = \hat{\gamma}([X_{0,i}]_{i=1,...,n}),$$  \hspace{1cm} (2.13)

where $[X_{0,i}] = [(X_i - \tilde{M}_n)/M_n]$, $\tilde{M}_n$ is the sample median of $[X_i]$, and $M_n$ is the sample median of $[|X_i - \tilde{M}_n|]$. Then
1. $\hat{\phi}_n \xrightarrow{p} 1/\alpha$, as $n \to \infty$,

2. there exist some $\tilde{c}_n \xrightarrow{p} 0$, and some $\alpha/2$ stable random variable $Y_0$, with $E[\log Y_0] = 0$, such that, as $n \to \infty$,

$$2 \log n(\hat{\phi}_n - 1/\alpha - \tilde{c}_n) \Rightarrow \log Y_0.$$  \hspace{1cm} (2.14)

Here DOA stands for the domain of attraction. We carried out estimation of $1/\hat{\alpha} = \hat{\phi}$ based on the above Theorem. The simulation was conducted for $n = 1000, 4000, 10000$. The results are presented in Figure 2.2 and the table below:

<table>
<thead>
<tr>
<th>sample size</th>
<th>mean</th>
<th>std</th>
<th>min value</th>
<th>max value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>1.4273</td>
<td>0.1405</td>
<td>0.7855</td>
<td>1.6552</td>
</tr>
<tr>
<td>4000</td>
<td>1.4452</td>
<td>0.1086</td>
<td>0.9415</td>
<td>1.6332</td>
</tr>
<tr>
<td>10000</td>
<td>1.4363</td>
<td>0.1140</td>
<td>0.9239</td>
<td>1.6124</td>
</tr>
</tbody>
</table>

5. **Estimation of parameter $\alpha$ by using $U$-statistic.** This estimator was introduced in [Fan2004] and it is based on the following property of $\alpha$-stable distributions:

$$X_1 + X_2 \overset{d}{=} 2^{1/\alpha} X_1.$$
Therefore
\[ E \log |X_1 + X_2| = \frac{1}{\alpha} \log 2 + E \log |X_1|, \]
and then
\[ \frac{1}{\alpha} = \frac{E \log |X_1 + X_2| - E \log |X_1|}{\log 2}. \]
Furthermore,
\[ \log \sigma = E(\log |X|) - C(1/\alpha - 1). \]
Let kernel \( h \) be a real function defined by the formula
\[ h(x_1, x_2) = \left( \log |x_1 + x_2| - \frac{1}{2}(\log |x_1| + \log |x_2|) \right) / \log 2. \] (2.15)
We can define the following U-statistics based on kernel \( h \):
\[ U_n(h) = \binom{n}{2} \sum_{1 \leq i < j \leq n} h(X_i, X_j). \] (2.16)
Then, if \( X_1, \ldots X_n \) are i.i.d. observations from a strictly stable distribution, then the estimator
\[ \frac{1}{\alpha} = U_n. \]
is asymptotically normal in view of the properties of U-statistics. More precisely, we have

**Theorem 19** [Fan2004] Suppose \( X_1, \ldots X_n \) are i.i.d. observations from a strictly \( \alpha \)-stable distributed population, and \( U_n(h) \) is defined by (2.16). Then \( U_n(h) \) is an unbiased estimator of \( 1/\alpha \), and
\[ \frac{\sqrt{n}}{2\sqrt{\zeta_1}} \left( U_n(h) - \frac{1}{\alpha} \right) \overset{d}{\to} Z, \quad \text{as} \quad n \to \infty, \]
where \( Z \) is a random variable with the standard normal distribution, and
\[ \zeta_1 = \text{Var}(E(h(X_1, X_2)|X_1)). \]
This estimator is inefficient in the computational sense. It is hard to perform for large sample sizes, but, for small sample size, this estimator works well and can be useful in applications. It gives us an opportunity to construct confidence intervals. To implement this estimation procedure we performed 100 simulations, for \( n = 1000, 2000, 3000 \). The results, for \( \alpha = 1.5 \), are presented in Figure 2.3 and the table below:

<table>
<thead>
<tr>
<th>sample size</th>
<th>mean</th>
<th>std</th>
<th>min value</th>
<th>max value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>1.5089</td>
<td>0.1235</td>
<td>1.1963</td>
<td>1.9587</td>
</tr>
<tr>
<td>2000</td>
<td>1.5037</td>
<td>0.0640</td>
<td>1.3153</td>
<td>1.8641</td>
</tr>
<tr>
<td>3000</td>
<td>1.5034</td>
<td>0.0401</td>
<td>1.3865</td>
<td>1.6690</td>
</tr>
</tbody>
</table>

To estimate the scale parameter \( \sigma \), by using \( U \)-statistics one can define the following kernel

\[
h_\alpha(x_1, x_2) = (1 - C/\log 2) \frac{\log |x_1| + \log |x_2|}{2} - C(h(x_1, x_2) - 1),
\]

where \( h(x_1, x_2) \) is the kernel used for the estimation of \( \alpha \) and given by (2.15).
Then the $U$-statistics
\[
U_n(h_\sigma) = \left( \frac{n}{2} \right) \sum_{1 \leq i < j \leq n} h_\sigma(X_i, X_j)
\]
(2.17)
is an unbiased estimator of $\log \sigma$. Note that $h_\sigma(X_1, X_2)$ has finite moments and, therefore, the estimator $U_n(h_\sigma)$ asymptotically normal.

6. **Estimation by using fractional moments.** Here we assume that we have the $\alpha$-stable variable $X$ with parameterizations (1.12) of the characteristic function, i.e. $X \sim S_\alpha(a, p, 0)$, with parameters $a$, $p$, $q = 1 - p$ and $b = 0$.

We can estimate parameter $\alpha$ by using the following property of the $\alpha$-stable distribution:
\[
\log \mathbb{E}|X|^{\rho} = \left( \frac{\rho}{\alpha} \right) \log a + \text{const},
\]
(2.18)
see for example [Zolotarev86].

If $a$ is fixed and $X_1, X_2 \sim S_\alpha(a, p, 0)$ then $X_1 + X_2 \sim S_\alpha(2a, p, 0)$. So, from the samples with parameter $a$ we can get sample from the distributions with parameters $2a, 4a, \ldots, 2^k a$, $k \in \mathbb{N}$. By using this property we can get estimators of $\alpha$ from the fractional sample moments as estimators of fractional moments, and by fitting the linear regression
\[
\log \mathbb{E}|X|^{\rho} = b \log ta + \text{const} = b \log t + c_1,
\]
where $c_1 = \text{const} + \log a$, $t = 1, 2, \ldots, 2^k$.

The estimation algorithm is as follows:

- Take a sample $x_1^1, \ldots, x_n^1$. Assume that $n = 2^k$, $k > 4$.
- Consider the samples $x_i^t = \sum_{m=(i-1)+1}^{ti} x_m^1$, $t = 2, 4, 8$.
- Choose $\rho$ (small, for example $\rho = 0.1$, because $\rho$ should be less then $\alpha$), and find sample moments $m_t = t/n \sum_{m=1}^{n/t} (x_m^t)^{\rho}$.
- Fit the linear regression $\log m_t = b \log t + C$, and get the least-square estimator $\hat{b}$ of $b$. 
• Find estimator $\hat{\alpha} = \rho/\hat{b}$ of $\alpha$.

We run the simulations for the symmetric $S_{1.5}(0,1,0)$, for $n = 4000, 10000$. We tried this method also for $n = 1000$, but the error turned out to be large and we decided not to present this result. The results, for $n = 4000, 10000$, are presented in Figure 2.4 and the table below:

<table>
<thead>
<tr>
<th>sample size</th>
<th>mean</th>
<th>std</th>
<th>min value</th>
<th>max value</th>
</tr>
</thead>
<tbody>
<tr>
<td>4000</td>
<td>1.5112</td>
<td>0.0963</td>
<td>1.2795</td>
<td>1.8511</td>
</tr>
<tr>
<td>10000</td>
<td>1.5029</td>
<td>0.0708</td>
<td>1.3152</td>
<td>1.7082</td>
</tr>
</tbody>
</table>

Our conclusion is that this method does not work well for a small sample size. For sample sizes $n = 4000$, and larger, it works well.

**Comparison of the estimators.** In comparison we exclude the method of $U$—*statistics* estimation since it is convivial only for small sample sizes, and compare simulations and estimations of parameter $\alpha$ using the method of moments, quadratic estimation and the suggested above method of fractional moments.

The comparison results are presented in Figure 2.5. The figure on the left corresponds to the sample of size $n = 4000$, and the one on the right to $n = 10000$. The order of the box plots is as follows: 1 - moment estimator, 2 - quadratic estimator, 3 - fractional moment estimator.

We can see that the Zolotarev moment estimator is better than the other two. Therefore, in the next section, we are going to use it exclusively.
Figure 2.5: Comparison of the moment, quadratic and fractional moment estimators, with sample size $n=4000$ (left), $n=10000$ (right).
2.3 Estimation of the Parameters of the Smoothly Truncated Lévy Distributions

2.3.1 Estimation of the parameters for one-sided \( STL_{1/2}(a, 1, \lambda) \) distribution

In this subsection we consider one-sided \( STL_{1/2}(a, 1, \lambda) \) distributions which have an explicit expression for the density. They are known as Inverse Gaussian distributions. In this case the maximum likelihood estimation can be carried out.

Let us assume that the location parameter \( b = 0 \). Then the density has the following form:

\[
g_{\lambda} = e^{c\sqrt{2\lambda}} c e^{\frac{-c^2}{2x} - \lambda x} \frac{1}{x^{3/2} \sqrt{2\pi}},
\]

where \( c = a \sqrt{2\pi} \).

Let \( x_1, \ldots, x_n \) be a sample from the independent identically distributed \( (i.i.d) \) random variables \( X_1, \ldots, X_n \) such that \( X_i \sim STL_{1/2}(a, 1, \lambda) \). Then the likelihood function has the following form:

\[
L(c, \lambda) = \prod_{i=1}^{n} g(x_i|c, \lambda) = \prod_{i=1}^{n} c \exp \left( -\frac{c^2}{2x_i} - \lambda x_i + c\sqrt{2\lambda} \right) \frac{1}{x_i^{3/2} \sqrt{2\pi}}.
\]

To maximize the likelihood function it suffices to maximize the loglikelihood function \( l = \log L \), which has the form

\[
l(c, \lambda) = \log L = n \log c - \frac{c^2}{2} \sum_{i=1}^{n} \frac{1}{x_i} - \lambda \sum_{i=1}^{n} x_i + nc\sqrt{2\lambda} - \frac{3}{2} \log \left( \sum_{i=1}^{n} \frac{1}{x_i} \right) + \log \sqrt{2\pi}.
\]

To find the extremal points we solve the following system of the equations:

\[
\begin{align*}
\frac{\partial l}{\partial c} &= n - c \sum_{i=1}^{n} \frac{1}{x_i} + n \sqrt{2\lambda} = 0, \\
\frac{\partial l}{\partial \lambda} &= - \sum_{i=1}^{n} x_i + \frac{nc}{\sqrt{2\lambda}} = 0,
\end{align*}
\]

which gives the MLE estimators

\[
\begin{align*}
\hat{c} &= \sqrt{\frac{\sum_{i=1}^{n} \frac{n}{x_i}}{n}}, \\
\hat{\lambda} &= \frac{1}{2} \frac{\sum_{i=1}^{n} \frac{2}{x_i} - \frac{n}{x}}{n}.
\end{align*}
\]
Here $\bar{x} = (1/n) \sum_{i=1}^{n} x_i$ is the usual sample mean. It is easy to check that this is the global maximum. Then we substitute $\hat{c} = \hat{a}\sqrt{2\pi}$ and obtain the MLE of the parameters $a$ and $\lambda$:

$$\hat{a} = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{n}{\sum_{i=1}^{n} \frac{1}{x_i} - \frac{n}{\bar{x}}}}$$

$$\hat{\lambda} = \frac{1}{2} \frac{n}{\sum_{i=1}^{n} \frac{x_i}{x_i} - \frac{n}{\bar{x}}}.$$ 

**Simulations.** We simulated 500 samples of $n = 2^{12} = 4096$ random numbers from $STL_{1/2}(0.004, 1, 0.0001)$ and estimated parameters $a$ and $\lambda$. We repeated this procedure for $STL_{1/2}(0.004, 1, 0.1)$. The results are shown in Figure 2.6 and the table below:

<table>
<thead>
<tr>
<th>parameter</th>
<th>mean</th>
<th>std</th>
<th>min value</th>
<th>max value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a=0.004$</td>
<td>0.0040</td>
<td>$10^{-5}$</td>
<td>0.0039</td>
<td>0.0042</td>
</tr>
<tr>
<td>$\lambda = 0.0001$</td>
<td>0.0016</td>
<td>0.0051</td>
<td>0</td>
<td>0.0903</td>
</tr>
<tr>
<td>$a=0.004$</td>
<td>0.0040</td>
<td>$10^{-5}$</td>
<td>0.0039</td>
<td>0.0041</td>
</tr>
<tr>
<td>$\lambda = 0.1$</td>
<td>0.1207</td>
<td>0.0560</td>
<td>0.0261</td>
<td>0.5003</td>
</tr>
</tbody>
</table>

We can see that this method works well for parameter $a$. For very small $\lambda$, the method does not work well, but in this case we could use the $\alpha$-stable model. For $\lambda = 0.1$, MLE estimation works well.

### 2.3.2 Estimation of the parameters for one sided $STL_{\alpha}(a, 1, \lambda)$ distributions, moment estimators

Let us consider now one-sided $STL_{\alpha}(a, 1, \lambda)$ distributions. As we mentioned in the introduction, the problem of parameter estimation for such distributions is practically solvable only in the case of distribution close to $\alpha$-stable distributions. It means that the product of the parameters $a\lambda$ should be small enough. This conclusion follows from the previous chapter.

In that case we can use the corresponding limiting stable distribution for the estimation of the index parameter $\alpha$ and the scale parameter $a$ from the previous
Figure 2.6: MLE of the parameters of the IG distributions.
section. Let us emphasize that we must not use estimation which is based on the properties of the tail. Since we have shown that, for the $STL_\alpha$, the limits (as $a \to 0$) of the logarithm of the fractional moments behave as in the $\alpha$-stable case, Zolotarev moment estimators, quadratic, and fractional moment estimators are all suitable in this case. Based on the simulations of the previous section we decided to use the method of moments of Zolotarev. The estimator for $\alpha$ is given by (2.9). Notice that in this case the estimator of $\alpha$ is biased. By using relationship between scale parameter $\sigma$ and $a$ (see 1.13) we can find that

$$\hat{a} = \exp \left[ \hat{\alpha} (\bar{V} + C) - C \right] - \Gamma(-\hat{\alpha}),$$

(2.19)

where $C$ is the Euler constant.

To find the truncation parameter we can use the first moment of the $X_i$. Notice that

$$E_X = \text{cum}_1 = a \lambda^{\alpha-1} \Gamma(1 - \alpha).$$

Therefore

$$\hat{\lambda} = \left( \frac{\bar{x}}{\hat{a} \Gamma(1 - \hat{\alpha})} \right)^{1/(\hat{\alpha}-1)}.$$

So, putting everything together, we have

$$\hat{\alpha} = \frac{1}{\sqrt{\frac{6}{\pi^2} S^2_V + 1}}.$$

(2.20)

Here, $S^2_V$ is given by (2.8),

$$\hat{a} = \frac{\exp \left[ \hat{\alpha} (\bar{V} + C) - C \right]}{-\Gamma(-\hat{\alpha})},$$

(2.21)

$$\hat{\lambda} = \left( \frac{\bar{x}}{\hat{a} \Gamma(1 - \hat{\alpha})} \right)^{1/(\hat{\alpha}-1)}.$$

(2.22)

**Simulations.** First, we simulate 500 samples of $n = 2^{12} = 4096$ pseudo-random numbers from $STL_{1/2}(0.004, 1, 0.0001)$, $STL_{1/2}(0.004, 1, 0.1)$, and 100 samples from
We can see that, for very small $\lambda$, the estimates of $\alpha$ and $a$ are very good, but estimates of $\lambda$ have large errors.
2.3.3 Estimators of the parameters of the symmetric \( STL_\alpha(a, 1, \lambda) \) distributions; moment estimations and numerical MLE

Moment estimators of the parameters of symmetric \( STL_\alpha \) distributions. Let us consider a sample from the symmetric \( STL_\alpha(a, 1/2, \lambda) \). In this case we can also use the limiting \( \alpha \)-stable distribution for the estimation of parameters \( \alpha \) and \( a \). Then the estimator of \( \alpha \) is given by (2.20). The estimation of parameter \( a \) using the method of moments, and the relationship between \( a \) and \( \sigma \), gives

\[
\hat{a} = \exp \left[ \hat{\alpha} \left( \bar{V} + C \right) - C \right] \frac{\cos(\hat{\alpha} \pi/2) \Gamma(-\hat{\alpha})}{-\cos(\hat{\alpha} \pi/2) \Gamma(-\hat{\alpha})},
\]

(2.23)

where \( \bar{V} = \frac{\sum_{i=1}^{n} \log|x_i|}{n} \).

To find the moment estimator of \( \lambda \) in the symmetric case of two-sided distribution we have to use the second cumulant since the first cumulant is zero, therefore

\[
\hat{\lambda} = \left( \frac{\text{Var}(x)}{\hat{a} \Gamma(2 - \hat{\alpha})} \right)^{1/(\hat{a} - 2)}.
\]

Simulations. Consider the two-sided case. We simulate 100 random samples of size \( n = 1000, 2000, 4000 \) from \( STL_{1.5}(0.004, 1/2, 0.1) \), and of size \( n = 2^{12} \) from \( STL_{1.2}(0.004, 1/2, 0.01) \), and \( STL_{1.5}(0.004, 1/2, 0.01) \). Parameter \( a \) is small so that the distribution is close to \( \alpha \)-stable. We calculate mean and standard deviation of these estimators. The results are presented in the tables below

\[
\alpha = 1.5
\]

<table>
<thead>
<tr>
<th>sample size</th>
<th>mean</th>
<th>std</th>
<th>min value</th>
<th>max value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>1.5300</td>
<td>0.1252</td>
<td>1.2108</td>
<td>2.1477</td>
</tr>
<tr>
<td>2000</td>
<td>1.5161</td>
<td>0.0770</td>
<td>1.3244</td>
<td>1.7371</td>
</tr>
<tr>
<td>4000</td>
<td>1.5226</td>
<td>0.0590</td>
<td>1.3786</td>
<td>1.6568</td>
</tr>
</tbody>
</table>
Figure 2.8: Moment estimators of the parameters for ST L
1.5 (0.004, 1/2, 0.01) (top left), ST L
1.5 (0.004, 1/2, 0.01) (top right), ST L
1.2 (0.004, 1/2, 0.01) (bottom left) and ST L
0.75 (0.004, 1/2, 0.01) (bottom right).
We can see that the estimators of $\alpha$, $a$, and $\lambda$ are biased. Simulations show that these estimators do not give satisfactory results. In the case when $a$ and $\alpha$ are very small, these estimates are not bad, but estimation of the truncation parameter needs to be improved.

Numerical Maximum Likelihood Estimators (MLE) for Symmetric $STL_\alpha(a, 1, \lambda)$ distributions. In the case of symmetric $STL_\alpha(a, 1/2, \lambda)$ distribution we develop below an algorithm for parameter estimation via Numerical MLE. Assume that $\alpha \neq 1$. 

<table>
<thead>
<tr>
<th>sample size</th>
<th>mean</th>
<th>std</th>
<th>min value</th>
<th>max value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>0.036</td>
<td>0.0015</td>
<td>0.0003</td>
<td>0.0100</td>
</tr>
<tr>
<td>2000</td>
<td>0.037</td>
<td>0.0011</td>
<td>0.0013</td>
<td>0.0072</td>
</tr>
<tr>
<td>4000</td>
<td>0.0036</td>
<td>0.0007</td>
<td>0.0021</td>
<td>0.0057</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>sample size</th>
<th>mean</th>
<th>std</th>
<th>min value</th>
<th>max value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>0.3218</td>
<td>0.2927</td>
<td>0.0005</td>
<td>1.3731</td>
</tr>
<tr>
<td>2000</td>
<td>0.2353</td>
<td>0.1881</td>
<td>0.0003</td>
<td>0.7714</td>
</tr>
<tr>
<td>4000</td>
<td>0.1628</td>
<td>0.1186</td>
<td>0.0011</td>
<td>0.5799</td>
</tr>
</tbody>
</table>

For the sample of size $n = 2^{12}$, the results are as follows:

<table>
<thead>
<tr>
<th>sample size</th>
<th>mean</th>
<th>std</th>
<th>min value</th>
<th>max value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 1.5$</td>
<td>1.4757</td>
<td>0.0330</td>
<td>1.3790</td>
<td>1.5629</td>
</tr>
<tr>
<td>$a = 0.004$</td>
<td>0.0043</td>
<td>0.0006</td>
<td>0.0030</td>
<td>0.0063</td>
</tr>
<tr>
<td>$\lambda = 0.01$</td>
<td>0.0924</td>
<td>0.1086</td>
<td>0</td>
<td>0.4800</td>
</tr>
<tr>
<td>$\alpha = 1.2$</td>
<td>1.1943</td>
<td>0.0261</td>
<td>1.2108</td>
<td>1.2548</td>
</tr>
<tr>
<td>$a = 0.004$</td>
<td>0.0042</td>
<td>0.0004</td>
<td>0.0031</td>
<td>0.0053</td>
</tr>
<tr>
<td>$\lambda = 0.01$</td>
<td>0.0652</td>
<td>0.0794</td>
<td>0</td>
<td>0.3320</td>
</tr>
<tr>
<td>$\alpha = 1.2$</td>
<td>1.1943</td>
<td>0.0261</td>
<td>1.2108</td>
<td>1.2548</td>
</tr>
<tr>
<td>$a = 0.004$</td>
<td>0.0042</td>
<td>0.0004</td>
<td>0.0032</td>
<td>0.0053</td>
</tr>
<tr>
<td>$\lambda = 0.01$</td>
<td>0.0652</td>
<td>0.0794</td>
<td>0</td>
<td>0.3320</td>
</tr>
<tr>
<td>$\alpha = 0.75$</td>
<td>0.7554</td>
<td>0.0147</td>
<td>0.7215</td>
<td>0.7951</td>
</tr>
<tr>
<td>$a = 0.004$</td>
<td>0.0039</td>
<td>0.0004</td>
<td>0.0032</td>
<td>0.0047</td>
</tr>
<tr>
<td>$\lambda = 0.01$</td>
<td>0.0330</td>
<td>0.0302</td>
<td>0.0011</td>
<td>0.1405</td>
</tr>
</tbody>
</table>
Let $f(x|\alpha, a, \lambda)$ be the density of $ST L_\alpha(a, 1/2, \lambda)$. The log-likelihood function for an i.i.d. $ST L_\alpha$ sample $X_1, \ldots, X_n$ is given by

$$l(\alpha, a, \lambda) = \sum_{i=1}^{n} \log(f(X_i|\alpha, a, \lambda)).$$

We do not have an explicit formula for the density, but we know the exact expression for the characteristic function $\varphi_{X_\alpha}(u) = \exp(\psi_{X_\alpha}(u))$ where the cumulant function

$$\psi_{X_\alpha}(u) = a \Gamma(-\alpha) \lambda^\alpha \left[ \left(1 + \frac{u^2}{\lambda^2}\right)^{\alpha/2} \cos(\alpha \arctan(u/\lambda)) - 1 \right], \quad \text{for } \alpha \neq 1. \tag{2.24}$$

The density function is the Fourier transform of the characteristic function:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixu} \varphi(u) du.$$ 

Therefore, if we can perform the numerical Fourier transform efficiently then we can find the numerical MLE. The problem is to approximate efficiently the integral

$$f(x) = \int_{-c}^{c} e^{-ixu} \varphi(u) du, \tag{2.25}$$

where $c$ is unknown. In the case of fixed $c$ the following lemma provides the connection between numerical calculation of the integral (2.3.3) by the Midpoint Rule and the Discrete Fourier Transform (DFT) (see [Menn2004], p.35, and also [MennRachev2004]). Recall that the Midpoint Rule calls for a partition of the interval $[-c, c]$ by the grid $-c = c_0 < c_1 < \cdots < c_n = c$ and the following calculation:

$$\int_{-c}^{c} g(x) dx = \sum_{k=1}^{n} \int_{c_{k-1}}^{c_k} g(x) dx \approx \sum_{k=1}^{n} g(x_k)(c_k - c_{k-1}), \quad \text{with } x_k := \frac{c_k - c_{k-1}}{2}.$$

**Lemma 20** [MennRachev2004] Let $N = 2^n \in \mathbb{N}$ be a natural number, $c \in \mathbb{R}_+$ a positive real number, and $h := \frac{2c}{N}$. Furthermore, define an equidistant grid over the interval $[-c, c]$ via $t_j := -a + jh \ (j = 0, \ldots, N)$. The midpoints are given by
$t_j^* := 0.5(t_j + t_{j+1})$
\((j = 0, \ldots, N - 1)\). Define the vector \(y \in \mathbb{C}^N\), and \(C \in \mathbb{C}^N\), as

\[
y_j := (-1)^j \varphi(t_j^*), \quad j = 0, \ldots, N - 1, \quad \text{(2.26)}
\]

\[
C_k := (-1)^k i \frac{2c}{N} e^{-i \frac{k\pi}{N}}, \quad k = 0, \ldots, N - 1. \quad \text{(2.27)}
\]

For

\[
x_k := -\frac{N\pi}{2c} + \frac{\pi}{a} k, \quad k = 0, \ldots, N - 1, \quad \text{(2.28)}
\]

we have

\[
\sum_{j=0}^{N-1} \varphi(t_j^*) e^{-it_j^*x_k} (t_{j+1} - t_j) = C_k \cdot DFT(y)_k. \quad \text{(2.29)}
\]

Thus, the application of the "midpoint rule" in the evaluation of the integral (2.3.3) can be realized with the help of the Fast Fourier Transform algorithm.

The next question is how to select the cut-off point \(c\).

In Figure 2.9 we plot the characteristic functions \(\varphi(u, \alpha, a, \lambda)\) of symmetric \(STL_{\alpha}(a, 1/2, \lambda)\) distributions showing the dependence on a single parameter with two remaining parameters fixed. In the top-left plot we show \(\varphi(u, a)\); in top-right, \(\varphi(u, \alpha)\); at the bottom: \(\varphi(u, \lambda)\). We can see that if \(a\) and \(\alpha\) are fixed then a change of the parameter \(\lambda\) does not change the speed of convergence of the characteristic function to zero as \(u \to \infty\). Since in the case close to an \(\alpha\)-stable distribution we can estimate \(\alpha\) and \(a\) sufficiently well, the cut-off point \(c\) can be selected by using \(\alpha_0\) and \(a_0\) as preliminary estimators of \(\alpha\) and \(a\) from the limiting \(\alpha\)-stable distribution. For the preliminary estimator of the parameter \(\lambda\) we will use the estimator \(\lambda_0\) given by the cumulants, since the large error in this parameter estimates does not influence our selection of the cut-off point. Then we can find numerically a \(c\) such that \(\varphi(c, \hat{\alpha}_0, \hat{a}_0, \hat{\lambda}_0) < 10^{-8}\). For this \(c\), using the above described method, the numerical values of the density function with given parameters \(\alpha, a, \lambda\) can be found on the grid (2.28) \(x_k, k = 0, \ldots, J\). If \(z_1, \ldots, z_n\) are our sample values then values of the density function at these points can be approximated by using cubic splines. The numerical
Figure 2.9: The characteristic functions of the $STL_\alpha$ distributions $\varphi(u;\alpha, a, \lambda)$. Top left: $\varphi(u, a)$; top right: $\varphi(u, \alpha)$; bottom: $\varphi(u, \lambda)$.

MLE was performed in MATLAB with starting points $\hat{\alpha}_0, \hat{a}_0, \hat{\lambda}_0$. The code is given in the appendix.

Simulations Since as we have seen above, the moment estimators are biased, we are going to use the numerical MLE to estimate parameters $\alpha$, $a$, and $\lambda$, and use the moment estimates as starting points.

We simulated 100 samples for $STL_{1.2}(0.004, 1/2, 0.001)$, $STL_{1.2}(0.004, 1/2, 0.01)$, $STL_{1.5}(0.004, 1/2, 0.01)$, and $STL_{1.5}(0.004, 1/2, 0.1)$, with sample size $n = 2^{12}$, and found numerical MLE for these samples. The boxplot of these estimates is presented in Figure 2.10. The mean and the standard deviations for these simulations are presented in the table below:
Figure 2.10: MLE estimators for the $STL_{1.2}(0.004, 1/2, 0.001)$ (top left), $STL_{1.2}(0.004, 1/2, 0.01)$ (top right), $STL_{1.5}(0.004, 1/2, 0.01)$ (middle left), $STL_{1.5}(0.004, 1/2, 0.1)$ (middle right), and $STL_{0.75}(0.004, 1/2, 0.01)$ (bottom).
<table>
<thead>
<tr>
<th>sample size</th>
<th>mean</th>
<th>std</th>
<th>min value</th>
<th>max value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 1.2$</td>
<td>1.2034</td>
<td>0.0283</td>
<td>1.1406</td>
<td>1.3068</td>
</tr>
<tr>
<td>$a = 0.004$</td>
<td>0.0040</td>
<td>3.7825e-04</td>
<td>0.0032</td>
<td>0.0052</td>
</tr>
<tr>
<td>$\lambda = 0.001$</td>
<td>0.0360</td>
<td>0.0597</td>
<td>2.0633e-06</td>
<td>0.3571</td>
</tr>
<tr>
<td>$\alpha = 1.2$</td>
<td>1.2021</td>
<td>0.0385</td>
<td>1.1067</td>
<td>1.3034</td>
</tr>
<tr>
<td>$a = 0.004$</td>
<td>0.0040</td>
<td>3.0005</td>
<td>0.0027</td>
<td>0.0053</td>
</tr>
<tr>
<td>$\lambda = 0.01$</td>
<td>0.0913</td>
<td>0.0866</td>
<td>0.0027</td>
<td>0.3319</td>
</tr>
<tr>
<td>$\alpha = 1.5$</td>
<td>1.4913</td>
<td>0.0531</td>
<td>1.3917</td>
<td>1.6625</td>
</tr>
<tr>
<td>$a = 0.004$</td>
<td>0.0041</td>
<td>0.0009</td>
<td>0.0020</td>
<td>0.0057</td>
</tr>
<tr>
<td>$\lambda = 0.01$</td>
<td>0.1014</td>
<td>0.0889</td>
<td>0.0027</td>
<td>0.4002</td>
</tr>
<tr>
<td>$\alpha = 1.5$</td>
<td>1.4871</td>
<td>0.0263</td>
<td>1.4072</td>
<td>1.5581</td>
</tr>
<tr>
<td>$a = 0.004$</td>
<td>0.0041</td>
<td>0.0005</td>
<td>0.0031</td>
<td>0.0056</td>
</tr>
<tr>
<td>$\lambda = 0.1$</td>
<td>0.1693</td>
<td>0.1527</td>
<td>0.0031</td>
<td>0.8639</td>
</tr>
<tr>
<td>$\alpha = 0.75$</td>
<td>0.7932</td>
<td>0.0153</td>
<td>0.7576</td>
<td>0.8347</td>
</tr>
<tr>
<td>$a = 0.004$</td>
<td>0.0039</td>
<td>0.0003</td>
<td>0.0032</td>
<td>0.0047</td>
</tr>
<tr>
<td>$\lambda = 0.01$</td>
<td>0.0330</td>
<td>0.0302</td>
<td>0.0011</td>
<td>0.1425</td>
</tr>
</tbody>
</table>

We can see that the estimates of $\alpha$ and $a$ are good. Estimates for $\lambda$ are not as satisfactory, but still much better than those we obtained by the method of moments.

**Appendix**

**MATLAB code for the MLE of $STL_{\alpha}(a, 1/2, \lambda)$**

```matlab
% MLE for the symmetric STL_{alpha} (a,1/2, lambda) distributions:
% m=15

function[est]=mleSTL(XX,m)
[alpha, a, lambda]=TSestsym(XX);
[UR]=URange(alpha,a,lambda);
x0=[alpha a lambda];
est=fminsearch('LLfun',x0,[],XX,UR,m);

% moment estimators for STL_{alpha} (a,1/2, lambda) distributions
% starting point for the MLE
function[alpha,a,lambda]=TSestsym(X);
```
U=sign(X);
V=log(abs(X(X>0|X<0)));
nu=6/pi^2*var(V)-3/2*var(U)+1;
alpha=1/sqrt(nu)
M=mean(V);
C=0.577215665; %Euler constant
a=exp(alpha*(M+C)-C)/(-cos(-alpha*pi/2)*Gamma(-alpha));
b=1/(alpha-2);
lambda=(var(X)/(a*Gamma(2-alpha)))^b;
estsym=[alpha a lambda];

% calculate cutting point c
function [UR]=URange(alpha,a,lambda)
s0=10^-5;
u0=10^-8;
utop=abs(fzero('CharFun',s0,[],alpha,a,lambda,u0));
UR=[-floor(nutop)-1 floor(nutop)+1];

% Characteristic function of STL
function [Phi]=CharFun(u,alpha,a,lambda,u0)
Phi=exp(a*gamma(-alpha)*(lambda^alpha).*((1+u.^2/(lambda^2)).^(alpha/2).*...cos(alpha*atan(u/lambda))-1)-u0);

% numerical calculation of -loglikelihood function
function[LLf]=LLfun(par, X,UR,m)
alpha=par(1);
a=par(2);
lambda=par(3);
if nargin<4 m=15; end
\[ y, x \] = STLpd(\alpha, a, \lambda, UR, m);
xx = sort(X);
Y = spline(x, y, xx);
LLf = -sum(log(Y));

**Bibliography**

[BianchiMeerschaert] Bianchi, K., Meerschaert M. Scale and shift invariant estimators for the heavy tail index \( \alpha \), see (http://unr.edu/homepage/mcubed/KBest.pdf).


Chapter 3

Statistical Analysis of EEG-Sleep Patterns of Neonates

3.1 Introduction

In this Chapter we analyze the EEG-sleep signal of neonates. Methodologies to analyze EEG-sleep could be useful for the assessment of functional brain maturation of infants at risk for developmental disabilities (see, [SSB96], [WhitneyThoman]), as well as specific clinical syndromes, such as sudden infant death syndrome (see, e.g., [GAH95]). The maturity level of brain in a human neonate at a given corrected age can vary widely and be dependent on circumstances such as premature birth, mothers’ drugs addiction, etc.

A noninvasive, reliable quantitative evaluation of brain dysmaturity is thus important in developing an understanding of the interplay between genetic endowment and environmental influences on neonatal brain development. Different measures of maturation based on the statistical analysis of temporal patterns of neonatal EEG-sleep were developed (see, e.g., [Scher97a, Scher97b]). Ultradian rhythm of the sleep (change between sleep stages) is one of the important characteristic for the dysmaturity measures.

Neurophysiologists have traditionally identified, by visual analysis of the EEG of
fullterm neonates4 distinct encephalogram patterns during sleep (see, [S-PWB92], [Scher99]):

- **Mixed frequency active** (MFA) sleep usually begins as a sleep cycle following falling asleep. This sleep stage is characterized by stationary and continuous signals with frequencies spread across the human EEG spectrum (with most of the signal power between 0.5 and 10 Hz). The energy is dominant in the theta band with intermittent delta waveforms (see, Figure 3.1 (top));

- **Low voltage irregular** (LVI) active sleep is characterized by relatively lower amplitude signals at broadband higher frequency distributions (see, Figure 3.1 (bottom));

- **High voltage slow** (HVS) quiet sleep appears following mixed frequency active sleep, and is a brief state, characterized by a shift in the frequency distribution to the higher power in the lower (delta) frequencies (see, Figure 3.2 (top));

- **Trace alternate** (TA) quiet sleep is characterized by non-stationary signals of alternating broad band bursts of the activity with intermittent epochs of relative EEG quiescence, comprised over lower amplitude and lower frequencies (see, Figure 3.2) (bottom).

Notice that both undefined sleep stages, and the awake state, can be present in the EEG signal.

Reorganization of sleep is known to occur around the 36th week of post-conceptional age, after which time, the ultradian sleep rhythm of the term infant becomes better recognized. The difference in EEG-sleep organization between pre- and fullterm cohorts at matched post-conceptional term ages were studied in, e.g., [SSDAG92], [SJSCDSHB03].

In this Chapter we consider the EEG-sleep patterns of the full- and preterm neonates from the Pittsburgh data set made available to us by Dr. Mark Scher. The massive data were collected for hundreds of patients, each recording containing 24 channel information measured at 64 HZ for several (6-8) hours.
Figure 3.1: Top: mixed frequency sleep stages; bottom: low voltage irregular (active) sleep stage.
Figure 3.2: Top: high voltage irregular (quiet) sleep; bottom: trace alternate (quiet) sleep.
The first goal of our studies is to develop an automatic procedure for scoring of sleep stages. Manual scoring of these data was provided by Dr. Scher. We also study different spectral EEG characteristics, and measures from nonlinear dynamical system theory, to analyze the EEG-sleep signals for the full- and preterm infants.

Methods of automatic detection of the sleep stages for adults were developed in, e.g., [GasBar-on], [ShimadaShiina00] [BrodskyDarkovski93], [KRDF01]. In [D-HKGKLHKPRDSTD04] various methods of automatic detection of sleep stages are compared. For neonates, methods of automatic detection of sleep stages using discrete wavelet transform were developed in [TLJS01].

The EEG signals are traditionally analyzed by using the spectral characteristics of the signals such as power spectrum, delta, theta power etc. Methods of nonlinear dynamics are also used in their analysis. According to the current understanding neurophysiological mechanisms generating the EEG, the signal stems from a highly nonlinear system. The studies [RKS95], and [PDK92], provide evidence for nonlinearity in EEG signals, but not for low dimensional deterministic dynamics.

The ability of several spectral characteristics (e.g.,delta, theta power) and nonlinear dynamics characteristics (e.g., correlation dimension, largest Lyapunov exponent) to discriminate between different sleep stages were compared in [FRMS96]. It was shown that the combination of spectral and nonlinear dynamics characteristics yielded better overall discrimination of sleep stages then spectral characteristics alone.

The EEG signal is non-stationary. A single set of model parameters (e.g. power spectrum) therefore cannot represent the EEG-signal adequately. Nonstationary phenomena are present in the EEG usually in the form of transient events, such as sharp waves, spikes or spike-wave discharges which are characteristic for the epileptic EEG, or as alteration of relatively homogeneous intervals (segments) with different statistical features (e.g., with different amplitude or variance) [LopesdaSilva78].

Note, that most of the spectral and non-linear characteristics of the stochastic processes which are used to describe EEG-signal assume its stationarity. Therefore, the most popular approach to analysis of the EEG as an inherently non-stationary signal is to consider it as a piecewise stationary process, i.e. as a process, "glued"
from random stationary processes with different probabilistic characteristics. Thus, the first step in EEG analysis should involve separation of the EEG into stationary segments by estimating the transition points which in the established statistical terminology are called "change-points". When this problem is solved, mathematical models can be fitted for each of the segments. A method for segmentation of the EEG, based on nonparametric statistical analysis, was proposed in [BDKS99].

In Section 3.2 we describe various characteristics and methods for analysis of the EEG-sleep patterns.

In Section 3.3 we provide statistical analysis of channel $C_5 - F_1$ of the EEG-sleep signal. In our model we assume that the signal is a piecewise stationary time series $X_i$ without trend. We provide detailed analysis for two EEG recordings. The first is EEG 88 of a healthy fullterm female neonate, 39 weeks post-conceptual age, and 39 weeks gestation age. The second is EEG 40 of a healthy preterm male neonate, with matching post-conceptual age and 30 weeks gestation age. The frequency of the recording is 64 Hz. To validate our finding we are also providing ANOVA and MANOVA analysis of different EEG measures for groups of 6 fullterm and 6 preterm neonates.

In particular, In Subsection 3.3.1 we compare signals from active and quiet sleep stages (scored manually) for the fullterm and preterm neonate EEG signals. We also study the issue of stationarity within one sleep stage.

Subsection 3.3.2 is devoted to methods of sleep stage separation. We calculate the spectral EEG characteristics such as Delta, Theta, Beta, and Alpha powers minute-by-minute and perform a change-point detection following [BDKS99], [BrodskyDarkovski93]. Then we smooth the spectral mean values of the spectral characteristics of the spectral bands within homogeneous segments. Smooth data for the Delta, Theta, Beta, and Alpha powers are considered as multidimensional variables and we apply cluster analysis to separate them into different groups. The same procedure is applied to other spectral and non-linear characteristics.

In Subsection 3.3.3 we perform ANOVA analysis of different spectral characteristics of the EEG signal to investigate their behavior in different sleep stages.
In Subsection 3.3.4 we perform ANOVA and MANOVA analysis of different spectral and non-linear characteristics to investigate differences between fullterm and preterm cohorts.

Subsection 3.3.5 is devoted to a study of the long-memory property for fullterm and preterm neonate EEG patterns. We use fractional dimension, a local nonlinear measure, for separation of our signal into quasistationary segments. Then, for every segment, we estimate the memory parameter by using the local Whittle estimator. We found presence of long-memory property within the active sleep segments. The self-similarity property is also investigated. Our conclusion is that the EEG signal is not self-similar.

Finally, in Subsection 3.3.6, we study the amplitude distributions for the quasistationary segments obtained in Subsection 3.3.5. We discover that, for fullterm babies, the model of Smoothly Truncated Lévy distribution, studied in Chapters 1, 2 and 4, is appropriate. The MLE for the parameters of these distributions described in Chapter 2 is also performed.

### 3.2 Preliminaries

In this Section we introduce our notation, recall the main definitions, and describe the methods which we are going to use to analyze the EEG data of neonates.

We consider our data \( x_1, x_2, \ldots \), to be a trajectory of a stochastic processes (time series) \( X_t, t = 1, 2, \ldots \), such that \( X_0 = 0 \).

Denote by \( \gamma(t, s) = \text{cov}(X_t, X_s) \) the autocovariance function of this process. Denote by \( r(k) = \gamma(k)/\gamma(0) \) the autocorrelation function of the process \( X \).

Let us recall that process \( X_t \) is said to be weakly stationary if its mean is constant and its autocovariance function \( \gamma(t, s) = \gamma(t - s) \).

For most of the characteristics used here, the stationarity assumption is essential. Although our data represent a nonstationary signal, they can be split into pieces which can be reasonably assumed to be stationary in the week sense.
Denote by \( f(\lambda) \) the spectral density, corresponding to the autocovariance function \( \gamma(t) \), i.e.,
\[
f(\lambda) = \sum_{h=-\infty}^{\infty} \gamma(h)e^{-i2\pi\lambda h},
\]
and by \( I_n \) the periodogram of \((x_t)\), i.e.,
\[
I_n(\lambda) = \frac{1}{2\pi n} \left| \sum_{t=1}^{n} x_t e^{-i2\pi\lambda t} \right|^2, \quad 0 < \lambda < 1.
\]

Recall that the periodogram is an unbiased and consistent estimator of the spectrum density of the time series.

Let \( \hat{\gamma}_X(h) = \frac{1}{n} \sum_{t=1}^{n-|h|} (x_{t+|h|} - \bar{x})(x_t - \bar{x}) \) be the sample autocovariance function. Then the connection between the periodogram and the sample autocovariance function is as follows
\[
I_n(\omega_k) = \sum_{|h|<n} \hat{\gamma}_X(h)e^{-i\omega_k},
\]
where \( \omega_k = 2\pi k/n \).

Recall that process \( X = \{X_t, t \in \mathbb{R}\} \) is said to be self-similar with index \( H \), if \( \{X_{at}, t \in \mathbb{R}\} \) and \( \{a^H X_t, t \in \mathbb{R}\} \) have identical finite-dimensional distributions, for all \( a > 0 \).

### 3.2.1 Spectral characteristics of EEG signals

In this section we are utilizing the measures described in [FRMS96]. The calculation of the spectral characteristics of the EEG signal is based on the Fast Fourier Transform (FFT). All spectral characteristics rely on estimations of the power spectrum density \( f(\lambda) \) via periodograms.
The relative delta power (DEL) is obtained from the normalized spectral densities by integration between 0.5 Hz and 3.5 Hz:

\[ \text{DEL} = \int_{0.5}^{3.5} f(\lambda) d\lambda. \]

The spectral edge (EDGE) is defined as the frequency below which 90% of the total power is accumulated (cut-off frequency = 45 Hz):

\[ \int_{0.5}^{\text{EDGE}} f(\lambda) d\lambda = 0.9 \int_{0.5}^{45} f(\lambda) d\lambda. \]

The first spectral moment as a characteristic of EEG was first proposed in [Hjorth70]. The first spectral moment is defined by the formula

\[ M_1 = \int \lambda f(\lambda) d\lambda. \]

The use of the spectral entropy (SEN) was suggested in [ISSTUIKH91]. The spectral entropy

\[ \text{SEN} = -\int f(\lambda) \log(f(\lambda))/N, \]

where \( N \) is the number of discrete frequencies.

The entropy of amplitudes (ENA) is calculated from the normalized amplitude distribution \( P_i \) of the time series:

\[ \text{ENA} = -\sum P_i \log P_i / \log(N), \]

where \( N \) is the number of the boxes. Calculation of the amplitude distribution is carried out by partitioning of the amplitude domain into 100 equal-size boxes covering the interval between the maximum and the minimum amplitude of the respective EEG segments.
3.2.2 Fractional (Hausdorff) dimension, nonlinear measure

In this subsection we will employ ideas due to [ConstantineHall94] and [GneitingSchlather04].

The Hausdorff or fractal dimension is a convenient way of characterizing the smoothness of a profile, or an isotropic surface. If the surface is very smooth, in particular, if it is has a continuously varying tangent plane, then the fractal dimension $D_3 = 2$. If the surface is extremely rough and irregular, then in the limit, as roughness increases without bound, the fractal dimension approaches $D_3 = 3$. For the hypersurface in $\mathbb{R}^n$ the fractal dimension $D_n \in [n - 1, n)$. There exists a large literature devoted to the study of fractal dimension; different definitions of the fractal dimension are possible [Falconer90].

In a variety of practical problems, data are only available through one-dimensional line transect "samples" of the surface. Such transect samples are representable by curves in a plane perpendicular to that of the projection of the surface.

A capacity-based definition of the fractional dimension $D$ of the curve $C$ is given by formula
\[
2 - D = \lim_{\epsilon \to 0} \frac{\log a(\epsilon)}{(\log \epsilon)^{-1}},
\]
where $a(\epsilon)$ denotes the area of the union of all discs of radius $\epsilon$ centered at points on a small ("homogeneous") section of the curve.

The two dimensional fractal dimension $D$ of such curve is a number between 1 (for a smooth curve) and 2 (for a rough curve).

Methods for estimating $D$. Let $X$ denote the height of the curve above arbitrary fixed level. We observe $X$ at points on a grid. If we assume that $X$ may be modelled as a stationary Gaussian process then $D$ may be expressed in terms of the behavior near the origin of the covariance function
\[
\gamma(t) = \text{cov}\{X(t + s), X(s)\}
\]
or, equivalently, in terms of the behavior in a neighborhood of infinity of the spectral
density

\[ f(\lambda) = (2\pi)^{-1} \int_{-\infty}^{\infty} \rho(t)e^{-\lambda t}dt. \]

If \( \rho \) is sufficiently smooth on \((0, \infty)\), and if

\[ \rho(0) - \rho(t) \sim c|t|^\alpha, \quad \text{as } |t| \to \infty, \] (3.1)

then, by the Abelian and Tauberian theorems (see, for example, [Stein99], section 2.8),

\[ f(\lambda) \sim c' |\lambda|^{-(\alpha+1)}, \quad \text{as } |\lambda| \to \infty, \]

where the constant \( c' \) depends only on \( c \) and \( \alpha \). Therefore, \( \alpha \) may be estimated via either the covariance function or the spectral density.

For a stationary Gaussian trajectory the fractal dimension is related to \( \alpha \) by the simple formula

\[ D = 2 - \frac{\alpha}{2}, \] (3.2)

see, for example, [Adler81].

In general, nothing specific appears to be known about relationship between \( \alpha \) and the fractal dimension.

We will use the fractal dimension estimators via the variogram [ConstantineHall94].

Let \( X \) be the stationary stochastic process which is observed at \( n \) equally spaced "time" points. Let \( Y_j = X(\nu j/n), 0 < j \leq n - 1 \), be the observed data, a realization of the process, where \( \nu = \nu(n) \) increases, and \( \nu/n \) decreases, as \( n \to \infty \).

The variogram is defined via the formula

\[ v(t) = \mathbb{E}(X(t) - X(0))^2 = 2(\gamma(0) - \gamma(t)), \]

where \( \gamma(t) \) is the autocovariance function. The formula

\[ \hat{g}_j = \hat{v}(\nu j/n) = (n - j)^{-1} \sum_{i=0}^{n-j-1} (Y_{i+j} - Y_i)^2. \]

gives an unbiased estimator of \( g_j = v(\nu j/n) \).
If we assume that
\[ v(t) \sim \text{const}|t|^\alpha, \quad \text{as} \quad t \to 0, \]
then
\[ \hat{v}(t) = \alpha \log |t| + \text{const} + \text{error}, \quad \text{as} \quad t \to 0. \]

The variogram estimator of $D$ is then given by $\hat{D} = 2 - \hat{\alpha}/2$, where $\hat{\alpha}$ is the slope in a log-log plot of $\hat{v}(t)$ versus $t$.

### 3.2.3 Long-range dependence and estimations of the long memory parameter

Long-range dependence has been observed in many real-life time series, e.g., in network traffic and finance. It is characterized, in the finite variance case, by slowly decaying covariances. The spectral density of such process tends to infinity as frequency tends to zero [TaqqueTeverovsky98].

We assume that our process $X_t$ has mean $\mu$ and variance $\sigma^2$, and is weakly stationary with autocorrelation function $r(k)$, $k \geq 0$. If the autocorrelation function has the following form,
\[ r(k) \sim k^{(1-2H)}L(k), \quad k \to \infty, \tag{3.3} \]
where $1/2 < H < 1$, and $L(k)$ is a slowly varying function at infinity, then process $X$ is said to have long-range dependence property. Parameter $H$ is called the Hurst parameter. Notice that if parameter $H \in (0, 1/2)$ then the process does not have the long memory property. One also uses the the memory parameter $d$ which for the finite variance process, is defined as follows:
\[ d = H - 1/2. \]

For the infinite variance, $\alpha$-stable process
\[ d = H - 1/\alpha. \]
Note that if the process is self-similar then the relationship between the memory parameter \( H \) and the fractional dimension \( D \) he in n-dimensional space is as follows:

\[
D + H = n + 1.
\]

Several methods of estimation of the parameters \( H \) and \( d \) are known in the literature, see, e.g., [TaqqueTeverovsky98]. In this chapter we will use the local Whittle estimation method [Robinson95].

**Local Whittle estimation.** This estimator is a semi-parametric estimator, which specifies the parametric form of the spectral density when \( \nu \) is close to zero, namely, it assumes that

\[
f(\nu) \sim G(d)|\nu|^{-2d}, \quad \text{as} \quad \nu \to 0,
\]

where \( G(d) \) is a bounded function of \( d \). The estimate for \( d \) is obtained by minimization of

\[
R(d) = \log \left( \frac{1}{M} \sum_{j=1}^{M} I(\nu_j) \right) - 2d \frac{1}{M} \sum_{j=1}^{M} \log(\nu_i).
\]

Note that only frequencies up to \( 2\pi M/N \) are included and that the estimator is sensitive to how we choose the cut-off point \( M \). For different \( M \) we will get different estimates.

### 3.2.4 Change point detection

We will use a non-parametric method for change-point detection suggested in [BrodskyDarkovski93]). It was applied to EEG signals in [BDKS99].

This method will be used to find the change-points of the distribution of the diagnostic (test) sequences obtained from the original EEG signals. The change of the distribution is characterized by the change of the mean.

The description of the method follows bellow.

Let \( x_1, \ldots, x_n \) be a realization of the diagnostic sequence. To detect the change-
points we consider the following family of statistics

\[ Y_N(n, \delta) = \left( 1 - \frac{n}{N} \right) \frac{n}{N} \delta \left[ \frac{1}{n} \sum_{k=1}^{n} x_k - \frac{1}{N-n} \sum_{k=n+1}^{N} x_k \right], \tag{3.6} \]

where \( 0 \leq \delta \leq 1, 1 \leq n \leq N - 1. \)

This family of statistics is a generalized variant of the Kolmogorov-Smirnov statistics, which is used for testing coincidence or difference of distribution functions of two samples (with fixed \( n \)).

We calculate the difference between the arithmetic means of the first \( n \) samples and an arithmetic mean of the last \( N-n \) samples, times a factor depending on \( \delta \). Then, the detection point \( n^* \) maximizes the differences over \( n \).

Any change-point estimation method can be characterized by the false alarm probability (i.e., the probability of a positive decision about the presence of change points when no change occurred), by the probability of false tranquility (i.e., the probability of the absence of designated change-points, when there actually was a change), and by the estimation error (in time) for a change-point location.

For the above-defined class of statistics these values are functions of the parameter \( \delta \) (this is true for any given threshold). It can be shown that, under weak mathematical assumptions (see [BrodskyDarkovski93]), the above-defined family of statistics gives, asymptotically (as \( N \to \infty \)), an optimum estimator for the change-point. An important property of this statistics is that the choice of \( \delta = 0 \) provides the minimum for false alarm probability. On the other hand, \( \delta = 1 \) corresponds to the minimum of the false tranquility probability, and the choice \( \delta = 0.5 \) guarantees the minimal estimation error.

We suppose that the diagnostic sequence \( X^N = \{x^N(k)\}_{k=1}^{N} \) formed from the initial sample is presented as a sum of a step function of time \( f(\cdot) \) and a centered random sequence \( \xi(\cdot) \). Namely, \( x^N_k = f(k/N) + \xi(k), k = 1, 2, \ldots, N, \text{E}\xi(k) \equiv 0. \)

Times of jumps \( \theta_i, 0 < \theta_i < 1, \) of function \( f(\cdot) \) are normalized change-points, i.e., change points \( n^*_i = \lfloor \theta_i N \rfloor. \)

Concerning the sequence \( \xi = \{\xi(n)\} \), we assume that it satisfies the mixing con-
dition, that its moment generation function is finite in some neighborhood of zero, and that there exists the limit
\[
\lim_{n \to \infty} \frac{1}{n} E \left( \sum_{k=1}^{n} \xi(k) \right)^2 = \sigma^2. \tag{3.7}
\]
Limit (3.7) exists if the sequence \( \xi \) is ‘glued’ from a finite number of stationary random sequences with integrable correlation functions. These conditions are fulfilled for our data set.

To compute the threshold let us suppose that the homogeneity (no change-points) hypothesis is true. Then using the limit theorem it can be shown that, under weak conditions, for example, stationarity, we have
\[
P\{ \max_{1 \leq n \leq N-1} \sqrt{N}|Y_N(n, 0)| > C \} \to f(C), \quad \text{as} \quad N \to \infty,
\]
where
\[
f(C) = 2 \sum_{k=1}^{\infty} (-1)^{k+1} \exp(-2k^2(C/\sigma)^2).
\]
For a given false alarm probability \( P_{fa} \), constant \( C \) can be calculated from the following equation
\[
F(C) = P_{fa}.
\]

**Our algorithm of change point detection** is as follows:

**Step (a) Calculating the diagnostic sequence.** From the original data set construct the diagnostic sequence.

**Step (b) Checking the homogeneity hypothesis.** We compute the value \( \max\{|Y_N(n, \delta = 0.7)|, 1 \leq n \leq N - 1\} = \eta_N \), and the threshold \( C \) with fixed false alarm probability \( P_{fa} = 0.05 \). If \( \eta_N \leq C \), then the homogeneity hypothesis is accepted and the procedure is completed; in the opposite case we go to the next step.

**Step (c) Preliminary estimation of the change-point.** The global maximum of the statistics \( |Y_N(n, \delta = 0.7)| \), call it \( n_1 \), is assumed to be the estimate of the first found change point. Then we form two new samples:
\[
Z_1 : 1 \leq n \leq n_1 - [\epsilon N], \quad \text{and} \quad Z_1 : n_1 + [\epsilon N] \leq n \leq N.
\]
Here $\epsilon$ is the number which is computed from the size of the sample, and the steepness of the statistics maximum. It gives a preliminary estimate of the confidence interval for the change-point.

Then, each of the new samples $Z_1$ and $Z_2$, is checked for homogeneity (step(b)), and if the homogeneity is not observed then we move on to step (c). The procedure is repeated until we obtain statistically homogeneous segments. As a result of this effort we obtain a set of preliminary estimates of ordered change points, with $k$ being the preliminary estimate of the number of change points.

**Step (d) Rejection of doubtful change-points.** The following subsamples are formed ($s = 2, \ldots, k - 1$):

\[
X_1 : 1 \leq n \leq n_1 + \frac{1}{2}(n_2 - n_1), \\
X_s : n_{s-1} + \frac{1}{2}(n_s - n_{s-1}) \leq n \leq n_s + \frac{1}{2}(n_{s+1} - n_s), \quad s = 2, \ldots, k - 1, \\
X_k : n_{k-1} + \frac{1}{2}(n_k - n_{k-1}) \leq n \leq N,
\]

Hence, inside each subsample $X_i$ there is a single preliminary change point estimate $n_i$. Each sample is then analyzed by the procedure of step (c), but with lower false alarm probability. If the homogeneity hypothesis is accepted for the sample, then the corresponding change point is rejected.

**Step (e) Final estimation of change point.** For each sample $X_i$ (of the volume $N_i$ remaining after step (d)) the statistic $Y_{N_i}(n, 1/2)$ is computed. The maximum point of the modulus for this statistic is assumed to be the final estimate of the $i$–th change point. Then the confidence interval can be computed from the statistic $Y_{N_i}(n, 0)$.

### 3.3 Analysis of EEG-Sleep Patterns

We study fullterm babies of 38-42 weeks conceptional age, and premature babies at the matching post-conceptional age. For a detailed study we choose two babies, one of them fullterm, and the other preterm.
Recording EEG 88 is a recording of a female fullterm baby, 40 weeks conceptual age (CA), 3 days after its birth. Recording EEG 40 is a recording of a preterm baby of the matching post-conceptual age. For a detailed study we select channel C5-Fp1, our motivation being that this brain channel gives good information about overall brain activity. The central part of the brain develops faster then the front part.

Similar results were obtained by running our experiments for different channels.

3.3.1 Quiet and active sleep stages, stationarity of the EEG recording within each sleep stage

It is obvious that EEG recording is a nonstationary time series. But most of the techniques of time series analysis, and the most popular techniques of the nonlinear dynamical systems theory, such as fractional dimension, correlation dimension and so on, are well established only for stationary time series. To take advantage of the stationarity assumption we have to divide our data into pieces which are close to stationary and only then perform our statistical analysis. The weak stationarity of the individual pieces has to be verified.

The natural first step is to separate different sleep stages. We will consider division into the active (MFS + LVI) sleep, quiet (HVI+TA) sleep and indeterminate (ID) sleeps for the babies of 38 weeks CA. Our goal is to reproduce manual scoring carried out by Dr. M. Scher. His scoring is done minute-by-minute by visual analysis of the EEG patterns. Then we could analyze differences between preterm babies and fullterm babies at the same CA. Notice that, sometimes, one minute comprises more than one sleep state. In such cases Dr. Scher’s scoring chooses the state which is dominant during that minute.

To check the stationarity, we take two samples. Each of them is 3 minutes long taken at 64HZ, which gives us 11520 observations. The first sample, according to Dr. Scher classification, corresponds to quiet sleep, while the second – to active sleep. Both of them are taken from recording EEG88. This two time series are presented in Figure 3.3.1.

The autocorrelation functions (ACF) of these patterns is presented in Figure 3.4
Figure 3.3: Quiet (top), and active (bottom) EEG-sleep patterns, for the fullterm neonate (top). It is clear that ACF in quiet sleep state decays faster. The periodograms for these two recordings are shown in Figure 3.4. We can see that the spectrum behavior around zero is singular around zero. These two observations suggest that long-range dependence structure for active sleep stage may be worth investigating. We will do it later on.

Next we will examine each piece to see if we can consider it as stationary?

**Verification of stationarity:** Let us start with the quiet sleep. We divide data into 15 second pieces and plot the ACF for every 15 seconds of the recording. The results are presented in Figure 3.5. Every row contains analysis of 1 minute of the signal, so some variability can be observed, but it is not large. To assess this variability quantitatively we plot ACF for 3 minutes of quiet sleep observations. From this time series we remove the first 15 seconds of observations and calculate ACF for the remaining data. We display this ACF on the same plot, and then, again, we remove the first 15 seconds of the observations from the remaining time series and find the new ACF, and display it on the same graph. We repeat the same procedure one more time. The result of this procedure is presented in Fig. 3.6. We can see that
Figure 3.4: ACF plots (top), and periodograms (bottom), for the active (left), and quiet sleep stages (right).
Figure 3.5: ACF for the quiet sleep patterns, calculated for every 15 seconds of observations.
Figure 3.6: ACF for the quiet sleep patterns, calculated by removing of 15 seconds observations.
different plots almost coincide. Therefore we can assume stationarity within the first minute of this segment.

Now we will repeat the same exploratory analysis for the active sleep record. The results are presented in the Fig. 3.7-3.8. The conclusion is that we can also assume stationarity for the active sleep pattern within the first minute of this segment.

3.3.2 Sleep stages separation

The basis for our work here is the visual minute-by-minute scoring of stages performed by Dr. M Scher. Five different sleep stages were separated by him: Trace Alternate
Figure 3.8: ACF for the quiet sleep patterns, calculated by removing consecutive 15 second observations.
(TA) quiet sleep stage, High Voltage Slow (HVS) quiet sleep stage, Indeterminate Sleep (IS), Low Voltage Irregular (LVI) active sleep, Mixed Frequency Active (MFA) sleep stage. Our limited goal is separation of active (TA+HVS) and quiet (LVI+MFA) sleep stages, and finding which stages can be recognized automatically. From the perspective of neuroscience the periodicity in changes of sleep stages reflects the level of brain maturation.

We conducted two sleep-stage separation experiments using cluster analysis and change-point detection methods. The first method relies on the distribution of power over four fundamental frequency bands, delta (0.5-3.5 Hz), theta (4-7.5 Hz), beta (8-12.5 Hz), alpha (13-17 Hz), and the second on other spectral characteristics such as the first spectral moment, the spectral edge, the spectral entropy, and other nonlinear dynamics characteristics such as the amplitude entropy and the fractional dimension.

The experiments were conducted for the recording EEG 88 (180 minutes at 64 HZ, for a fullterm baby) and EEG 40 (211 minutes at 64 HZ, for a premature baby). The detailed implementation of the two approaches, and the results obtained, are described below.

The spectral band method.

In the first step a testing sequence was constructed by calculating, minute-by-minute, 4-dimensional vectors with coordinates representing percentage of power concentrated in each of the fundamental bands. As a result we obtained a 4-dimensional test time series; 180 observations for the fullterm baby and 211 observations for the preterm baby.

Each of the four coordinate time series was now smoothed by using a change point detection for the mean described in Subsection 3.2.4. The final step was to use the cluster analysis ("kmean" function in S-plus) to collapse these 4-dimensional vectors into one of the several categories (sleep stages). Fig. 3.9 [Fig.3.10] show the results of the above smoothing procedure for recordings EEG 88 (fullterm) [EEG 40 (preterm)] for each of four coordinate of the test time series, and overlays them with the results of Dr. Scher’s manual scoring.

Fig. 3.11-3.11 show the results of the clustering analysis with several choices of
Figure 3.9: Sleep stages for EEG88 (fullterm neonate); normalized power values for delta, theta, beta, and alpha spectral bands. Minute-by-minute calculations and their smoothing.

the number of clusters: 5 clusters representing stages TA, HVS, IS, LVI, MFA, 3 clusters representing active (TA+HVS), indeterminate (IS) and quiet (LVI+MFA) sleep stages, 4 clusters representing TA, HVA, LVI, MFA, and 2 clusters representing active and quiet sleep stages. Again these results are overlayed with manual classification.

Our conclusions are as follows: For the first 120 minutes of recording EEG 88 (fullterm baby) the 5-cluster analysis reproduced manual scoring very well for the TA stage and the main tendencies were well represented for all categories. Similar behavior has been observed for 3-, 4-, and 2-cluster analysis.

For the first 120 minutes of the recording EEG 40 (premature baby) the 3-, and, especially, 2-cluster analysis give the best results.
Figure 3.10: Sleep stages for EEG40 (preterm neonate), normalized power values for delta, theta, beta, and alpha spectral bands. Minute-by-minute calculations and their smoothing.
Figure 3.11: EEG 88 sleep stages. Separation into the clusters by the spectral band method.
Figure 3.12: EEG 40 sleep stages, separation into the clusters by the spectral band method.
Figure 3.13: Sleep stages for EEG88. Spectral characteristics of the EEG data calculated minute-by-minute and their smoothing.
Figure 3.14: EEG 88 sleep stages. Separation into the cluster by the spectral and nonlinear characteristics method.
Spectral and nonlinear characteristics method.

The procedure here was similar to the one used above by the spectral band method, but the test time series was obtained by minute-by-minute calculation of the following six characteristics: the first spectral moment, the spectral edge, the spectral entropy, the amplitude entropy, and the fractional dimension.

That gave us initially a 6-dimensional test time series, which was smoothed out, coordinatewise via the same change-point detection method. After comparison with the manual score Fig. 3.13 we decided to restrict the cluster operation to 3-D time series including the spectral moment, the spectral entropy, and the fractional dimension coordinates, choosing a 2- and 3-cluster analysis. The outcome is shown in Fig. 3.14.

Our conclusion is that there is significant correlation between the manual scoring and our automatized approach.

In the next section we will use ANOVA on a larger sample of neonates to quantitatively assess the ability of various characteristics used above to emulate manually scored sleep stages. This will validate the use of the above procedure for automatic scoring and suggest ways to improve it.

3.3.3 ANOVA in sleep stages separations

As we have seen before, delta and theta power spectral band transitions correlated well with sleep stages. To verify if this statement can be extended to a broader class of neonates, additional 6 fullterm and 6 preterm recordings have been examined.

For every EEG recording in each group we calculate the percentage of the power in the delta, theta, beta, and alpha power bands and average them within each sleep stage in the group. Then we perform ANOVA and MANOVA, separately for the fullterm and preterm group. Given the conclusions of the previous Subsection we concentrate here on the first two hours of the EEG recordings.

The results of ANOVA for the fullterm group are presented in Fig.3.15 and the S-plus ANOVA’s output:
Figure 3.15: Fullterm group: ANOVA of the sleep stages for the spectral bands characteristics.
Response: Delta

<table>
<thead>
<tr>
<th>Df</th>
<th>Sum of Sq</th>
<th>Mean Sq</th>
<th>F Value</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>states</td>
<td>3</td>
<td>9.67277</td>
<td>3.224258</td>
<td>167.1842</td>
</tr>
<tr>
<td>Residuals</td>
<td>587</td>
<td>11.32068</td>
<td>0.019286</td>
<td></td>
</tr>
</tbody>
</table>

Response: Theta

<table>
<thead>
<tr>
<th>Df</th>
<th>Sum of Sq</th>
<th>Mean Sq</th>
<th>F Value</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>states</td>
<td>3</td>
<td>0.1144465</td>
<td>0.03814885</td>
<td>114.4168</td>
</tr>
<tr>
<td>Residuals</td>
<td>587</td>
<td>0.1957174</td>
<td>0.00033342</td>
<td></td>
</tr>
</tbody>
</table>

Response: Beta

<table>
<thead>
<tr>
<th>Df</th>
<th>Sum of Sq</th>
<th>Mean Sq</th>
<th>F Value</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>states</td>
<td>3</td>
<td>0.000280745</td>
<td>0.00009358177</td>
<td>7.588176</td>
</tr>
<tr>
<td>Residuals</td>
<td>587</td>
<td>0.007239223</td>
<td>0.0001233258</td>
<td></td>
</tr>
</tbody>
</table>

Response: Alpha

<table>
<thead>
<tr>
<th>Df</th>
<th>Sum of Sq</th>
<th>Mean Sq</th>
<th>F Value</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>states</td>
<td>3</td>
<td>0.00219392</td>
<td>0.0007313057</td>
<td>19.24188</td>
</tr>
<tr>
<td>Residuals</td>
<td>587</td>
<td>0.02230949</td>
<td>0.000380059</td>
<td></td>
</tr>
</tbody>
</table>

We can see that the p-values are 0, so that the means for different bands and corresponding sleep stages are different. The boxplot indicates good separation between active ("LVI", "MFA") and quiet ("HNI", "TA") sleep stages. For the beta and alpha spectral bands the separation is not so clear. Our conclusion is as follows: to separate quiet and active sleep stages we should use delta and theta spectral bands and perform for them the change-point detection and cluster analysis described in the previous Chapter. Observe that residuals have a skewed distribution. For the delta band it looks like a bimodal distribution. The issue of how to fix this by using an elementary transformation could be a topic for further research. The distribution of the residuals for the ANOVA with factors beta and alpha are skewed and we are not going to use them in sleep stage separations. They do not give good result.
The results of ANOVA for the preterm group are presented in Fig.3.16 and the S-plus ANOVA’s output:

Response: DeltaP

<table>
<thead>
<tr>
<th></th>
<th>Df</th>
<th>Sum of Sq</th>
<th>Mean Sq</th>
<th>F Value</th>
<th>Pr(F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>states</td>
<td>3</td>
<td>9.42403</td>
<td>3.141342</td>
<td>98.47848</td>
<td>0</td>
</tr>
<tr>
<td>Residuals</td>
<td>644</td>
<td>20.54280</td>
<td>0.031899</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Response: ThetaP

<table>
<thead>
<tr>
<th></th>
<th>Df</th>
<th>Sum of Sq</th>
<th>Mean Sq</th>
<th>F Value</th>
<th>Pr(F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>states</td>
<td>3</td>
<td>0.0368549</td>
<td>0.01228496</td>
<td>69.27689</td>
<td>0</td>
</tr>
<tr>
<td>Residuals</td>
<td>644</td>
<td>0.1142013</td>
<td>0.00017733</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Response: BetaP

<table>
<thead>
<tr>
<th></th>
<th>Df</th>
<th>Sum of Sq</th>
<th>Mean Sq</th>
<th>F Value</th>
<th>Pr(F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>states</td>
<td>3</td>
<td>0.0000397515</td>
<td>0.00001325051</td>
<td>9.803036</td>
<td>2.4914e-006</td>
</tr>
<tr>
<td>Residuals</td>
<td>644</td>
<td>0.0008704783</td>
<td>0.00000135167</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Response: AlphaP

<table>
<thead>
<tr>
<th></th>
<th>Df</th>
<th>Sum of Sq</th>
<th>Mean Sq</th>
<th>F Value</th>
<th>Pr(F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>states</td>
<td>3</td>
<td>0.000420747</td>
<td>0.0001402488</td>
<td>21.4299</td>
<td>3.060885e-013</td>
</tr>
<tr>
<td>Residuals</td>
<td>644</td>
<td>0.004214685</td>
<td>0.0000065445</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We can see that all p-values are zero, so that the mean values of the power spectrum bands are different for different sleep stages. Here the delta and theta bands also give information useful in separation of active and quiet sleep. But separation is less sharp than for fullterm neonates. For these recordings the automatic separation of sleep stages can lead to more errors. The residuals are similar to those of fullterm neonate recordings.

The ANOVA for the other spectral characteristics and nonlinear characteristics of the EEG signal gives the following results:
Figure 3.16: Preterm group: ANOVA of sleep stages for the spectral bands characteristics.
The box-plots for the factors and residuals for fullterm group are presented in Fig. 3.17. They suggest that we should choose spectral moments and spectral entropy for separation of active and quiet sleep stages. Also the fractional dimension separates the trace alternate sleep stage well. For spectral moments and spectral entropy characteristics residuals are close to normal and the use of ANOVA is appropriate.
Figure 3.17: Fullterm group: ANOVA of the sleep stages for the spectral and nonlinear characteristics.
The ANOVA for the spectral and nonlinear characteristics of the preterm group gave the following results:

**Response: SpMomentP**

<table>
<thead>
<tr>
<th></th>
<th>Df</th>
<th>Sum of Sq</th>
<th>Mean Sq</th>
<th>F Value</th>
<th>Pr(F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>states</td>
<td>3</td>
<td>29.62161</td>
<td>9.873871</td>
<td>73.94013</td>
<td>0</td>
</tr>
<tr>
<td>Residuals</td>
<td>644</td>
<td>85.99894</td>
<td>0.133539</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Response: EDGEP**

<table>
<thead>
<tr>
<th></th>
<th>Df</th>
<th>Sum of Sq</th>
<th>Mean Sq</th>
<th>F Value</th>
<th>Pr(F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>states</td>
<td>3</td>
<td>5.808</td>
<td>1.935847</td>
<td>1.019827</td>
<td>0.3832826</td>
</tr>
<tr>
<td>Residuals</td>
<td>644</td>
<td>1222.448</td>
<td>1.898211</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Response: SpEntrP**

<table>
<thead>
<tr>
<th></th>
<th>Df</th>
<th>Sum of Sq</th>
<th>Mean Sq</th>
<th>F Value</th>
<th>Pr(F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>states</td>
<td>3</td>
<td>4.433234</td>
<td>1.477745</td>
<td>99.60109</td>
<td>0</td>
</tr>
<tr>
<td>Residuals</td>
<td>644</td>
<td>9.554791</td>
<td>0.014837</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Response: AmplEntrP**

<table>
<thead>
<tr>
<th></th>
<th>Df</th>
<th>Sum of Sq</th>
<th>Mean Sq</th>
<th>F Value</th>
<th>Pr(F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>states</td>
<td>3</td>
<td>0.130591</td>
<td>0.04353038</td>
<td>15.52959</td>
<td>9.183218e-010</td>
</tr>
<tr>
<td>Residuals</td>
<td>644</td>
<td>1.805171</td>
<td>0.00280306</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Response: FracdimP**

<table>
<thead>
<tr>
<th></th>
<th>Df</th>
<th>Sum of Sq</th>
<th>Mean Sq</th>
<th>F Value</th>
<th>Pr(F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>states</td>
<td>3</td>
<td>1.314007</td>
<td>0.4380022</td>
<td>68.03036</td>
<td>0</td>
</tr>
<tr>
<td>Residuals</td>
<td>644</td>
<td>4.146288</td>
<td>0.0064383</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The box-plots for the factors and residuals for the preterm group are presented in Fig. 3.18. We can see that, for this group, the use of the spectral moments and the fractional dimension for the sleep stage separations is fully justified.
Figure 3.18: Preterm group: ANOVA of the sleep stages for the spectral and nonlinear characteristics.
3.3.4 MANOVA for full term group and preterm group

We consider again two groups: 6 fullterm neonates and 6 preterm neonates. We would like to discern differences between them in the corresponding sleep stages. For this purpose we conduct minute-by-minute calculations of spectral moments, edge, spectral entropy, amplitude entropy, fractional dimension for the first two hours of observations. In every group we separate observations into active and quiet sleep stages. Then we perform the MANOVA for active and quiet sleep stage. Below we reproduce an S-plus output of MANOVA for both cases. We can see that, for both sleep stages, the p-values are zero. Therefore we can conclude that our groups are significantly different.

Active sleep stage

<table>
<thead>
<tr>
<th>Df</th>
<th>Wilks Lambda approx. F num df</th>
<th>den df</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>babyA</td>
<td>1</td>
<td>0.6075</td>
<td>90.1754</td>
</tr>
<tr>
<td>Residuals</td>
<td>702</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Quiet sleep stage

<table>
<thead>
<tr>
<th>Df</th>
<th>Wilks Lambda approx. F num df</th>
<th>den df</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>babyQ</td>
<td>1</td>
<td>0.4453</td>
<td>138.7924</td>
</tr>
<tr>
<td>Residuals</td>
<td>561</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Next we use ANOVA to compare fullterm and preterm group in every characteristic. The summary is as follows:

Active sleep

Response: Spectral Moment

<table>
<thead>
<tr>
<th>Df</th>
<th>Sum of Sq</th>
<th>Mean Sq</th>
<th>F Value</th>
<th>Pr(F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>babyA</td>
<td>1</td>
<td>32.9748</td>
<td>32.97481</td>
<td>188.6194</td>
</tr>
<tr>
<td>Residuals</td>
<td>702</td>
<td>122.7250</td>
<td>0.17482</td>
<td></td>
</tr>
</tbody>
</table>

Response: EDGE

<table>
<thead>
<tr>
<th>Df</th>
<th>Sum of Sq</th>
<th>Mean Sq</th>
<th>F Value</th>
<th>Pr(F)</th>
</tr>
</thead>
</table>
### Response: Spectral Entropy

<table>
<thead>
<tr>
<th>Df</th>
<th>Sum of Sq</th>
<th>Mean Sq</th>
<th>F Value</th>
<th>Pr(F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>babyA</td>
<td>1</td>
<td>1.44623</td>
<td>1.44623</td>
<td>95.33633</td>
</tr>
<tr>
<td>Residuals</td>
<td>702</td>
<td>10.64918</td>
<td>0.01517</td>
<td></td>
</tr>
</tbody>
</table>

### Response: Amplitude Entropy

<table>
<thead>
<tr>
<th>Df</th>
<th>Sum of Sq</th>
<th>Mean Sq</th>
<th>F Value</th>
<th>Pr(F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>babyA</td>
<td>1</td>
<td>0.001030</td>
<td>0.00102975</td>
<td>0.2851646</td>
</tr>
<tr>
<td>Residuals</td>
<td>702</td>
<td>2.534977</td>
<td>0.00361107</td>
<td></td>
</tr>
</tbody>
</table>

### Response: Fractional dimension

<table>
<thead>
<tr>
<th>Df</th>
<th>Sum of Sq</th>
<th>Mean Sq</th>
<th>F Value</th>
<th>Pr(F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>babyA</td>
<td>1</td>
<td>2.844659</td>
<td>2.844659</td>
<td>268.8293</td>
</tr>
<tr>
<td>Residuals</td>
<td>702</td>
<td>7.428320</td>
<td>0.010582</td>
<td></td>
</tr>
</tbody>
</table>

Quiet sleep

### Response: Spectral Moment

<table>
<thead>
<tr>
<th>Df</th>
<th>Sum of Sq</th>
<th>Mean Sq</th>
<th>F Value</th>
<th>Pr(F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>babyQ</td>
<td>1</td>
<td>48.00952</td>
<td>48.00952</td>
<td>297.3754</td>
</tr>
<tr>
<td>Residuals</td>
<td>561</td>
<td>90.57016</td>
<td>0.16144</td>
<td></td>
</tr>
</tbody>
</table>

### Response: EDGE

<table>
<thead>
<tr>
<th>Df</th>
<th>Sum of Sq</th>
<th>Mean Sq</th>
<th>F Value</th>
<th>Pr(F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>babyQ</td>
<td>1</td>
<td>399.0136</td>
<td>399.0136</td>
<td>229.5507</td>
</tr>
<tr>
<td>Residuals</td>
<td>561</td>
<td>975.1512</td>
<td>1.7382</td>
<td></td>
</tr>
</tbody>
</table>

### Response: Spectral Entropy
We see that, in active sleep, the amplitude entropy is not a significant variable in comparison of the two groups. All other variables are significant. The box-plots for the factors and histograms of the residuals and QQ-plots of the residuals for active and quiet sleep stages are presented in Fig. 3.19-3.20. We can see that the spectral moments and the spectral entropy for the active sleep stage are close to normal, and also give good separation between groups. All others residuals for the active sleep stages have skewed distributions. For the quiet sleep, spectral moment residuals are more or less normal; all other are skewed. How to transform these data to get a normal distribution will be the subject of further investigations. At present, our conclusion is that the use of spectral moments and spectral entropy for comparison of the two groups in the active sleep is justified. The same conclusion applies for discriminant analysis.

3.3.5 Long-range dependence and fractional dimension in EEG patterns

In this section we would like to investigate the issues of long-range dependence in the EEG data sets. As before, we choose EEG 88 and EEG 40 for this analysis.
Figure 3.19: Comparison of fullterm and preterm groups. ANOVA for the spectral characteristics of the active sleep stages.
Figure 3.20: Comparison of fullterm and preterm groups. ANOVA for the spectral characteristics of the quiet sleep.
At first we would like to separate our data into stationary increments and estimate parameter \( d \) \((H = d + 1/2)\) of long-range dependence for each of them. We assume that our patterns have finite variance. Since \( d \) and \( H \) are global parameters we would like to have the longest possible quasi-stationary pieces. On the other hand, the fractional dimension \( D \) is a local parameter which indicates roughness of the time series and can be estimated for short intervals. So, in this case, we divided our EEG into 10 second intervals and estimated parameter \( D \) in each of them. Then we applied our change-point detection method to this statistics and obtained intervals which we assume to be quasi-stationary. One of our reasons to calculate the fractional dimension is that in the case of self-similar processes \( D + H = constant \). A violation of this equality would prove that our data are not self-similar.

To estimate the memory coefficient \( d \) we used the local Whittle method. Other methods were also explored but, in my opinion, the local Whittle method gives better (more robust) results.

In Fig. 3.21-3.22 plots of fractional dimension \( D \), together with its smoothing by means of quasi-stationary intervals (top) are shown, for EEG88 and EEG 40 recordings, respectively. The middle pictures show estimates for the long-memory parameter \( d \), for the corresponding EEG recordings (”solid line”), and manually scored sleep stages (”dashed line” ). Here we used the following coding for the sleep stages:”MFA” =0.4, ”LVI” = 0.2, ”IS” =0, ”HVS” =-0.2, ”TA”=-0.4. We observe that, especially for the fullterm neonate (EEG 88), during the quiet sleep (HVS,TA) the process mostly does not display long-range dependence, but that, for active sleep, the long-range dependence effect is present. The bottom pictures show the sums of the smoothed values of fractional dimensions \( D \) and memory parameters \( H = d + 1/2 \) (in the case of finite variance). Clearly these sums are not constants in either case indicating that our processes are not self-similar.

3.3.6 Distributions of the amplitude

In these Section we study the amplitude distribution of the EEG patterns. We consider the channel C3-Fp1 from two recordings: EEG 88 (fullterm neonate) and EEG
Figure 3.21: Fractional dimension $D$, memory parameter $d$, and $D + d + 1/2 = D + H$ for the fullterm neonate
Figure 3.22: Fractional dimension $D$, memory parameter $d$, and $D+d+1/2 = D+H$, for the preterm neonate.
40 (preterm neonate). The problem of distributional description is meaningful if our patterns are homogeneous. Therefore we use the result from the previous section which provided separation of the recording into quasi-stationary pieces using the tool of the fractional dimension.

We estimate the empirical p.d.f. by using MATLAB function ”ksdensity” with a gaussian kernel and find that our distribution is bell-shaped. In this context we decide to test the hypothesis that the amplitude on quasi-stationary intervals have $STL_\alpha$ distributions.

We use the results from Chapter 2 and find the MLE of $STL_\alpha$ distributions for the given segments. The Kolmogorov-Smirnov test is performed for the corresponding segments. Figure 3.23 shows the kernel estimate of the sample density (solid line), the normal density for the corresponding estimated mean and standard deviation (“dash points”) and the estimated $STL_\alpha$ densities for one of the segments of the EEG 88 (fullterm neonate). In that case the estimated parameters are as follows $\alpha = 1.1181$, $a = 262.8396$, $\lambda = 0.0079$. The result of the Kolmogorov-Smirnov test is to accept the null hypothesis that our sample is from the $STL_{1.12}(262.84, 1/2, 0.0079)$ distribution; $p-value \approx 1$.

For the EEG 88 recording of the fullterm infant, our separation into the quasi-stationary segments gave to us 53 segments during 183 minute recording. Out of these 53 segments only one segment has a normal distribution, while 50 have $STL_\alpha$ distributions with $p-value > 0.1$.

We performed the same segmentation and estimation for the parameters of the distributions of the EEG 40 recording of the preterm baby (211 minutes). We obtained 63 quasi-homogeneous segments and none of them had a normal distribution; 7 had $STL_\alpha$ distributions according to the Kolmogorov-Smirnov test. All others had neither normal nor $STL_\alpha$ distribution. An attempt to check goodness-of-fit of $\alpha$-stable distributions was not successful either. The situation is illustrated in Fig.3.24. The empirical distribution is between the normal and the $STL_\alpha$ distribution.

The conclusion of this subsection is that, for the fullterm baby recording EEG 88
Figure 3.23: Density estimates for EEG 88.
Figure 3.24: Density estimates for EEG 40.
displayed $STL_\alpha$ distribution of amplitude within the quasi-stationary segments. For the preterm infant recording EEG 40 is not appropriate.

**Bibliography**


Chapter 4

Models of Anomalous Diffusion: The Subdiffusive Case

4.1 Introduction

Over the last several decades, beginning with the work of Montroll and his collaborators, see, Montroll and Weiss (1965), the physical community showed a steady interest in the anomalous diffusions, a somewhat vague term that describes diffusive behavior in absence of the second moments, with scaling different than that of the classical Gaussian diffusion; see, e.g., Shlesinger (1988), Saichev and Zaslavsky (1997), and Metzler and Klafter (2000). More recently, this model found applications in several other areas, including econophysics and mathematical finance; see e.g., Scales et al. (2000) and Mainardi et al. (2000). The standard Lévy processes and their calculus have an enormous mathematical literature including comprehensive monographs by Kwapien and Woyczynski (1992), Bertoin (1996), and Sato (1999), and an edited volume of papers devoted to recent results in the theory and applications of Lévy processes, see Barndorff-Nielsen, Mikosch and Resnick, Eds., (2001), which includes an elementary survey of Lévy processes in the physical sciences written by one of the authors of this article, see Woyczynski (2001).

Despite all these efforts the deeper understanding of dynamical features of anomalous diffusions that would satisfy both the mathematical and the physical audiences is still lacking. Moreover, although it is well known that one-point probability den-
sity functions (p.d.f.) of such processes can be described by the diffusion equations involving derivatives of fractional orders, their physically and computationally useful theory is still only in their beginning stage, especially in the case of nonlinear fractional phenomena, see, e.g., Biler, Funaki and Woyczynski (1999), and Biler, Karch and Woyczynski (2001).

The goal of the present Chapter is to provide a lucid and rigorous analysis of anomalous diffusions which provides some new insights into the dynamics of their statistical properties and serve as a basis for further developments both in the mathematical and the physical literatures.

To motivate our work on anomalous diffusion it is worth recalling a few elementary facts regarding the classical Brownian motion diffusion (the Wiener process) \( B(t), t \geq 0 \), which can be defined as a Gaussian process with zero mean and the covariance function \( \langle B(t)B(s) \rangle = \min(t, s) \). The physicist usually thinks about it as a limit (in distribution) of a process of independent Gaussian (say) jumps taken at smaller and smaller time intervals of (nonrandom) length \( 1/n \). In other words,

\[
B(t) = \lim_{n \to \infty} \sum_{k=1}^{\lfloor nt \rfloor} B_k \left( \frac{1}{n} \right)
\]

where \( \lfloor a \rfloor \) denotes the integer part of number \( a \), and \( B_1(1/n), B_2(1/n), \ldots \), are independent copies of the random variable \( B(1/n) \).

The key observation is that this mental picture can be extended to the situation when the fixed inter-jump times \( \Delta t = 1/n \) are replaced by random inter-jump times of an accelerated Poissonian process \( N(nt), t \geq 0 \), independent of \( B(t) \). Here, \( N(t) \) is the standard Poisson process with mean \( \langle N(t) \rangle = t \), and variance \( \langle (N(t) - t)^2 \rangle = t \). Since the mean \( \langle N(nt) \rangle = nt \) is finite, in view of the Law of Large Numbers,

\[
\frac{N(nt)}{nt} \to 1, \quad \text{as} \quad t \to \infty,
\]

so that the Brownian diffusion \( B(t) \) can also be viewed as a limit of the process of
random Gaussian jumps at random Poissonian jump times:

\[
\lim_{n \to \infty} B\left(\frac{N(nt)}{n}\right) =_{d} \lim_{n \to \infty} \sum_{k=1}^{N(nt)} B_k \left(\frac{1}{n}\right) =_{d} \lim_{n \to \infty} \sum_{k=1}^{\lfloor tn \rfloor} B_k \left(\frac{1}{n}\right) =_{d} B(t) \quad (1.3)
\]

It is a worthwhile exercise to see directly that the process

\[
Z(t) = \lim_{n \to \infty} B\left(\frac{N(nt)}{n}\right)
\]

represents a Brownian motion diffusion. Indeed, \(Z(t)\) is clearly a process with independent and time-homogeneous (stationary) increments and its one-point cumulative distribution function (c.d.f.)

\[
P(Z(t) \leq z) = \lim_{n \to \infty} \sum_{k=0}^{\infty} P(B(k/n) \leq z) \cdot P(N(nt) = k)
\]

\[
= \lim_{n \to \infty} \int_{-\infty}^{z} \sum_{k=0}^{\infty} \frac{e^{-x^2/(2k/n)}}{\sqrt{2\pi k/n}} e^{-nt} \frac{(nt)^k}{k!} \frac{1}{\sqrt{2\pi t}} dx = \int_{-\infty}^{z} \frac{e^{-x^2/(2t)}}{\sqrt{2\pi t}} dx
\]

because the Poissonian distribution in question has mean \(nt\) and standard deviation \(\sqrt{nt}\). Asymptotically, for large \(n\), it is concentrated around the value \(k = nt\) and acts, in the limit, as the Dirac delta centered at \(k/n = t\).

The jump-times process \(T(n), n = 1, 2, \ldots\), the random time of the \(n\)-th jump and the Poissonian process \(N(t)\) are mutually inverse random function as

\[
T(n) = \min\{t : N(t) \geq n\},
\]

and the inter-jump random time intervals \(T(n) - T(n-1), n = 1, 2, \ldots\), are independent and have identical exponential p.d.f. \(e^{-t}, t \geq 0\).

The main thrust of this Chapter is to replace the Brownian motion \(B(t)\) by a general Lévy diffusion \(X(t)\), the random times \(T(n)\) by an increasing infinite mean continuous parameter process \(T(\theta)\) (sometimes called the subordinator) with the inverse random time \(\Theta(t)\) replacing the Poisson process \(N(t)\), and consider the composite process \(X(\Theta(t))\) which we call here anomalous diffusion. It is the infinite mean of the
random time $T(\theta)$ which makes the diffusion $X(\Theta(t))$ anomalous and non-Gaussian since the above Law of Large Numbers argument no longer applies.

The Chapter is organized as follows: Sections 2 and 3 introduce the concepts, and study statistical properties of fractional random time and inverse random time. These concepts are based on the principles of infinite divisibility and self-similarity under rescaling. The role of the Mittag-Leffler functions as generalizations of the exponential distribution is here explored. Section 4 investigates anomalous diffusions evolving in such inverse fractional random time and it is followed by a study of the special case of fractional drift processes in Section 5. Section 6 discusses the fractional subdiffusion in the case when the usual Brownian diffusion is run in the inverse fractional random time; the multidimensional case is discussed in Section 7. Then, Section 8 introduces the concept of tempered anomalous subdiffusion which has small jumps similar to those of the fractional anomalous diffusion while preserving the important finiteness of all moments that is characteristic of the Brownian motion.

Finally, Sections 9 and 10 show that anomalous diffusions studied in sections 2-8 can be interpreted as limit cases of pure jump processes which we call *anomalous random walks*. We first look at their statistical properties, related Kolmogorov-Feller equations, and then at the Langevin-type equations driven by them.

Since our development depends on mathematical tools of the theory of infinitely divisible distributions and processes, we provide compact appendices to explain their essential features.

4.2 Random time and its inverse

The basic element in the construction of our model of anomalous diffusion is the concept of random time $T(\theta)$, which is a stochastic process depending on a non-negative parameter $\theta$. We will impose physically justifiable requirements that the random time be nonnegative, nondecreasing as a function of parameter $\theta$, and that its increments be independent and time-homogeneous, with infinitely divisible and selfsimilar probability distributions.
More precisely, to satisfy the time-homogeneity and infinite divisibility assumptions, we will suppose that \( T(\theta) \) is a Lévy infinitely divisible process (see Appendix A) with \( T(\theta = 0) = 0 \), nondecreasing trajectories, and such that the p.d.f. of \( T(\theta) \) is equal to \( f(t; \theta) \) (in brief, \( T(\theta) \sim_d f(t, \theta) \)). In this context, denoting \( f(t; 1) = f(t) \), we have that, for every \( n = 2, 3, \ldots \),

\[
T(1) =_d T_1(1/n) + \ldots + T_n(1/n),
\]

where the random quantities on the right-hand side are independent copies of \( T(1/n) \) and \( =_d \) denotes the equality of probability distributions. In terms of the "mother" Laplace transform

\[
\hat{f}(s) = \int_0^\infty e^{-st} f(t) \, dt
\]

the above infinite divisibility condition can be written in the form

\[
\hat{f}(s; 1/n) = \int_0^\infty e^{-ts} f(t, 1/n) \, dt = \hat{f}^{1/n}(s).
\]

More generally, we thus obtain that, for any \( \theta > 0 \),

\[
T(\theta) =_d \lim_{n \to \infty} \sum_{k=1}^{[\theta n]} T_k(1/n), \tag{2.1}
\]

where \( [x] \) denotes the integer part of number \( x \).

The monotonicity of the trajectories of random time \( T(\theta) \) permits us to introduce the *inverse random time*

\[
\Theta(t) = \min\{\theta : T(\theta) \geq t\}.
\]

The p.d.f. \( g(\theta; t) \) of the inverse random time \( \Theta(t) \), which is a probability density with respect to the variable \( t \) whereas \( \theta \) plays the role of a parameter, can be calculated from the p.d.f. \( f(t; \theta) \) of the random time \( T(\theta) \), which is a probability density with respect to the variable \( \theta \) with \( t \) playing the role of a parameter. Indeed, observing the equivalence

\[
T(\theta) < t \quad \iff \quad \Theta(t) \geq \theta,
\]
we get the relationship

\[ F(t; \theta) = P(T(\theta) < t) = P(\Theta(t) \geq \theta) = \int_{\theta}^{\infty} g(\theta'; t) \, d\theta'. \]

which gives the following formula for the p.d.f. \( g(\theta; t) \) of \( \Theta(t) \) in terms of the p.d.f. \( f(t; \theta) \):

\[
g(\theta; t) = -\frac{\partial F(t; \theta)}{\partial \theta} = -\frac{\partial}{\partial \theta} \int_{-\infty}^{t} f(t'; \theta) \, dt'.
\] (2.2)

To satisfy the selfsimilarity requirement we will additionally assume that the random time \( T(\theta) \) is a \( \beta \)-stable process with the selfsimilarity parameter \( \beta, 0 < \beta < 1 \), and p.d.f. \( f_{\beta}(t; \theta) \) with the Laplace transform

\[
\hat{f}_{\beta}(s; \theta) = e^{-\theta s^\beta}
\]

(see Appendix A, formula (A.7)). The selfsimilarity of \( T(\theta) \) implies the following scaling property of the random time:

\[
f_{\beta}(t; \theta) = \frac{1}{\theta^{1/\beta}} f_{\beta}\left(\frac{t}{\theta^{1/\beta}}\right),
\] (2.3)

where, following the convention established above, \( f_{\beta}(t) \equiv f_{\beta}(t, 1) \). Substituting (2.3) into (2.2) immediately yields the following self similarity property for the p.d.f. \( g_{\beta}(\theta; t) \) of the inverse random time \( \Theta(t) \):

\[
g_{\beta}(\theta; t) = \frac{1}{t^{\beta}} g_{\beta}\left(\frac{\theta}{t^{\beta}}\right),
\] (2.4)

where

\[
g_{\beta}(\theta) = \frac{1}{\beta \theta^{1+1/\beta}} f_{\beta}\left(\frac{1}{\theta^{1/\beta}}\right).
\] (2.5)

The above formula for \( g_{\beta}(\theta) \), and the quoted above form (A.7) of the Laplace transform of the \( \beta \)-stable p.d.f. \( f_{\beta}(t) \), imply together the following useful equality:

\[
\exp(-\theta s^\beta) = \beta \int_{0}^{\infty} e^{-st} \frac{\theta}{t^{\beta+1}} g_{\beta}\left(\frac{\theta}{t^{\beta}}\right) \, dt.
\] (2.6)
Moments $\langle T^\nu \rangle$ of the $\beta$-stable random time with p.d.f. $f_\beta(t)$ are infinite for $\nu \geq \beta$. However, moments $\langle \Theta^\nu \rangle$ of the inverse random time $\Theta$ with p.d.f. $g_\beta(\theta)$ are finite for all $\nu > 0$, and we can find them by multiplying both sides of (2.6) by $\theta^{\nu-1}$, and integrating the resulting equality with respect to $\theta$. This procedure gives

$$
\frac{\Gamma(\nu)}{s^{\nu\beta}} = \beta(\Theta^\nu) \int_0^\infty e^{-st^{\nu\beta-1}} dt = \beta(\Theta^\nu) \frac{\Gamma(\nu\beta)}{s^{\nu\beta}},
$$

so that

$$
\langle \Theta^\nu \rangle = \int_0^\infty \theta^\nu g_\beta(\theta) d\theta = \frac{\Gamma(\nu)}{\beta \Gamma(\nu/\beta)} = \frac{\Gamma(\nu+1)}{\Gamma(\nu/\beta+1)}.
$$

Consequently, the Laplace transform $\hat{g}_\beta(s)$ of the p.d.f. $g_\beta(\theta)$ has the Taylor expansion

$$
\hat{g}_\beta(s) = \langle e^{-s\Theta} \rangle = \sum_{n=0}^\infty \frac{(-s)^n}{\Gamma(n\beta + 1)} = E_\beta(-s),
$$

where

$$
E_\beta(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\beta + 1)}
$$

is the Mittag-Leffler function with parameter $\beta$ (see, e.g., Saichev and Woyczynski (1997), Section 6.10). As a result, the Laplace transform $\hat{g}_\beta(s; t)$ of p.d.f. $g_\beta(\theta; t)$ (2.4) can be written in the form

$$
\hat{g}_\beta(s; t) = E_\beta(-s t^{\beta}),
$$

and it is easy to see (see, again, Saichev and Woyczynski (1997), Section 6.10)) that $\hat{g}_\beta(s; t)$ from (2.10) is a causal solution of the nonhomogeneous linear fractional differential equation

$$
\frac{d^\beta \hat{g}_\beta(s; t)}{dt^\beta} + s \hat{g}_\beta(s; t) = \frac{t^{-\beta}}{\Gamma(1 - \beta)} \chi(t),
$$

where $d^\beta / dT^\beta$ is the Riemann-Liouville fractional differentiation operator defined via the formula

$$
\frac{d^\beta u(t)}{dt^\beta} = \frac{1}{\Gamma(-\beta)} \int_0^t (t-s)^{-\beta-1} u(s) ds.
$$
Notation $\chi(t)$ stands here for the Heaviside unit step function.

Remark: A couple of observations are in order here. We use the term "fractional differential equation" although fractional differentiation is really a nonlocal integral operator; this usage is, however, traditional. In this context, instead of solving an initial value problem it is more natural to consider the inhomogeneous equation (2.11) whose source term on the right-hand side automatically enforces the causality condition and guarantees the uniqueness of the solutions. Notice that, as $\beta \to 1^-$, the source term converges weakly to the Dirac delta $\delta(t)$. Thus, in the limit case $\beta = 1$, equation (2.11) becomes a classical differential equation $\dot{g}_\beta(s; t) + s\dot{g}_\beta(s; t) = \delta(t)$, which is equivalent to the intitial value problem $\dot{g} + sg = 0$, $g(0) = 1$.

For the benefit of the reader we would also like to cite the following standard references on fractional calculus: Samko, Gilbas and Marichev (1993), Gorenflo and Mainardi (1997), Podlubny (1999), Butzer and Westphal (2000), and West, Bologna and Grigolini (2003).

Recall that the Mittag-Leffler function $E_\beta(z)$ has an integral representation

$$E_\beta(z) = \frac{1}{2\pi i} \int_{\mathcal{H}} \frac{e^y y^{\beta - 1}}{y^{\beta} - z} \, dy,$$

where the integration is performed along the Hankel loop depicted in Fig.4.1. The loop envelops the negative part of the real axis and all singular points of the integrand.


If $z = -s$ is a negative real number, then the integral in (2.12) is reduced to the
definite integral

$$E_\beta(-s) = \frac{s}{\pi} \sin(\pi\beta) \int_0^\infty \frac{x^{\beta-1} e^{-x} dx}{x^{2\beta} + s^2 + 2sx\beta \cos(\pi\beta)}, \quad (s > 0, \quad 0 < \beta < 1). \quad (2.13)$$

As $s \to \infty$, the terms in the denominator of the integrand are dominated by $s^2$ which gives the asymptotic formula

$$E_\beta(-s) \sim \frac{1}{s \Gamma(1-\beta)}, \quad (s \to \infty). \quad (2.14)$$

Of course, for $\beta = 1/2$, integral (2.13) boils down to the familiar complementary error function:

$$E_{1/2}(-s) = \exp(s^2) \operatorname{erfc}(s),$$

where

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-z^2} dz.$$  

The plot of $E_{1/2}(-s)$, indicating its asymptotics

$$E_{1/2}(-s) \sim 1/\sqrt{\pi} s, \quad (s \to \infty),$$

is shown in Fig.4.2.

Finally, note that in the limiting case $\beta = 1$, $E_1(-s)$ is simply the exponential function $e^{-s}$, with asymptotics dramatically different than the asymptotics of $E_\beta(-s)$, for $0 < \beta < 1$.  

Figure 4.1: The Hankel loop, a path of integration in (2.12).
Figure 4.2: The Mittag-Leffler function $E_{1/2}(-s)$ (solid line) and its asymptotic behavior $E_{1/2}(-s) \sim 1/\sqrt{\pi s}$, $(s \to \infty)$ (dashed line). Notice the accuracy approximation of $E_{1/2}(-s)$ by $1/\sqrt{\pi s}$ for $s > 3$.

4.3 Distribution of the inverse random time

In this section we provide detailed information about p.d.f. $g_\beta(\theta)$ of the inverse random time $\Theta(1)$. Observe that, in view of (2.5), $g_\beta(\theta)$ is the p.d.f. of the random variable $T^{-\beta}$, where $T$ has the $\beta$-stable p.d.f. $f_\beta(t)$. In particular, in view of the asymptotic relation

$$f_\beta(t) \sim \frac{\beta}{\Gamma(1-\beta)t^{1+\beta}} \quad (t \to \infty),$$

see (A8),

$$g_\beta(\theta = 0) = \frac{1}{\Gamma(1-\beta)}.$$

According to (2.8-9), the characteristic function (Fourier transform) $\tilde{g}_\beta(u) = \langle e^{iu\Theta(1)} \rangle$ of p.d.f. $g_\beta(\theta)$ is of the form

$$\tilde{g}_\beta(u) = \hat{g}_\beta(s = -iu) = E_\beta(iu),$$
so that the function $g_{\beta}(\theta)$ itself is equal to the inverse Fourier transform

$$g_{\beta}(\theta) = \frac{1}{2\pi} \int E_{\beta}(iu) e^{-iu\theta} \, du.$$  \hfill (3.1)

Using this equality and the integral representation (2.12) of the Mittag-Leffler function it is easy to show that

$$g_{\beta}(\theta) = \frac{1}{\pi \beta} \Im e^{i\pi \beta} \int_{0}^{\infty} \exp \left( -x^{1/\beta} - x\theta e^{i\pi \beta} \right) \, dx, \quad \text{for } 0 < \beta \leq 1/2, \hfill (3.2)$$

and

$$g_{\beta}(\theta) = \frac{1}{\pi \beta} \Re \int_{0}^{\infty} \exp \left( ix\theta + x^{1/\beta} e^{-i\pi/2\beta} \right) \, dx, \quad \text{for } 1/2 \leq \beta \leq 1. \hfill (3.3)$$

Plots of p.d.f.s $g_{\beta}(\theta)$, for different values of $\beta$, are shown in Fig.4.3.

For some special values of parameter $\beta$, the integral (3.2) can be explored analytically in more detail. For example, consider $\beta = 1/n$, $n = 2, 3, \ldots$, and introduce an auxiliary complex function

$$w_{\beta}(\theta) = \frac{1}{\pi \beta} e^{i\pi \beta} \int_{0}^{\infty} \exp \left( -x^{1/\beta} - x\theta e^{i\pi \beta} \right) \, dx.$$ 

Clearly

$$\Im w_{\beta}(\theta) = g_{\beta}(\theta), \quad 0 < \beta \leq 1/2.$$
Finding the \((n - 1)\)-st derivative of \(w_{1/n}(\theta)\) with respect to \(\theta\) we obtain that

\[
\frac{d^{n-1}w_{1/n}(\theta)}{d\theta^{n-1}} = (-1)^n \frac{n}{\pi} \int_0^\infty x^{n-1} \exp \left( -x^n - x\theta e^{i\pi/n} \right) \, dx.
\]

Integrating by parts we get the equation

\[
\frac{d^{n-1}w_{1/n}(\theta)}{d\theta^{n-1}} + \frac{(-1)^n}{n} \theta w_{1/n}(\theta) = \frac{(-1)^n}{\pi}, \quad \theta > 0.
\]

Separating its imaginary part we arrive at the following differential equation for p.d.f. \(g_{1/n}(\theta)\):

\[
\frac{d^{n-1}g_{1/n}(\theta)}{d\theta^{n-1}} + \frac{(-1)^n}{n} \theta g_{1/n}(\theta) = 0.
\]

The change of variables \(z = \theta / \sqrt{n}\) reduces the above equation to the generalized Airy equations

\[
g^{(n-1)}(z) + (-1)^n z g(z) = 0, \quad z > 0, \ n = 2, 3, \ldots.
\]

The initial conditions for these equations can be determined by evaluation, for \(\theta = 0\), of the integral (3.2) and its derivatives with respect to \(\theta\). For \(n = 2\), the Airy equation has a solution \(g(z) = e^{-z^2/2}\). As a result, we obtain that

\[
g_{1/2}(\theta) = \frac{1}{\sqrt{\pi}} \exp \left( -\frac{\theta^2}{4} \right), \quad \theta \geq 0. \tag{3.4}
\]

In other words, the inverse random time, for \(\beta = 1/2\), has a one-sided Gaussian distribution. As a matter of fact, the corresponding random time’s \(1/2\)-stable density (in our case \(\lambda = 1/\sqrt{2}\))

\[
f_{1/2}(t) = \frac{\lambda}{\sqrt{2\pi}} t^{-3/2} e^{-\lambda^2/2t}, \tag{3.5}
\]

is traditionally called the inverse Gaussian distribution (see, e.g., Seshadri (1999)) and its history goes back to 1915 when Schrödinger and Smoluchowski independently derived it as the p.d.f. of the first passage time of level \(\lambda\) for the standard Brownian motion.
In the case $\beta = 1/3$,

$$g_{1/3}(\theta) = \sqrt[3]{3} \text{Ai} \left( \frac{\theta}{\sqrt[3]{3}} \right), \quad (3.6)$$

where $\text{Ai}(z)$ is the well-known Airy function, that is, the vanishing at infinity solution of the equation $g''(z) - zg(z) = 0$ (see Abramowitz and Stegun (1964), and Mathematica, for an efficient numerical and symbolic implementation). Taking into account relations (2.5) one obtains

$$f_{1/3}(t) = \frac{1}{\sqrt{3}} e^{-7/2} \text{Ai} \left( \frac{1}{\sqrt[3]{3t}} \right). \quad (3.7)$$

### 4.4 1-D anomalous diffusion

In this section we will study statistical properties of the anomalous diffusion

$$\mathcal{X}(t) = X(\Theta(t)), \quad (4.1)$$

where $X(\theta), \theta \geq 0$, is a random function (stochastic process) and $\Theta(t)$ is the inverse random time introduced in Section 2. The rationale for introduction of such anomalous diffusion was given in the Introduction and an additional physical justification can be found in Section 8 which discusses anomalous random walks.

The above transformation is known as the subordination and it was introduced in Bochner (1955). Both, Bertoin (1996), and Sato (1999) devote chapters to this concept and the related concept of the subordinator, i.e., $\Theta(t)$ in (4.1). Mathematical, and physical citations on subordination, closely related to the subject matter of this paper are Mainardi et al. (2001), Mainardi et al. (2003), Wyss and Wyss (2001), Barkai (2002) and Stanislavski (2000, 2003).

We shall assume that processes $X(\theta)$ and $\Theta(t)$ are statistically independent and that they have the one-point p.d.f.s $w(x; \theta)$ and $g(\theta; t)$, respectively. Then the p.d.f. $f(x; t)$ of the composite random function $\mathcal{X}(t)$, obtained with the help of the total probability formula, is given by the expression

$$f(x; t) = \int_0^\infty w(x; \theta) g(\theta; t) \, d\theta, \quad (4.2)$$
and the mean square of process $X(t)$ is equal to
\[
\langle X^2(t) \rangle = \int_0^\infty \langle X^2(\theta) \rangle g(\theta; t) d\theta .
\] (4.3)

Also, the characteristic function $\tilde{f}(u; t) = \langle \exp(iuX) \rangle$ can be written in the form
\[
\tilde{f}(u; t) = \int_0^\infty \tilde{w}(u; \theta) g(\theta; t) d\theta ,
\] (4.4)
where $\tilde{w}(u; \theta)$ is the characteristic function of the random function $X(\theta)$.

Let us now assume that $X(\theta), X(0) = 0$, is a Lévy process (see Appendix B), that is a process with time-homogeneous (stationary) and independent increments and with an infinitely-divisible one point p.d.f. with characteristic function
\[
\tilde{w}(u; \theta) = \tilde{w}^{\gamma \theta}(u) = \exp[\gamma \theta \Psi(u)],
\]
see (B.2), where $\Psi(u)$ is the logarithm of the characteristic function of the underlying "mother" infinitely divisible p.d.f. Substituting this expression into (4.4) gives
\[
\tilde{f}(u; t) = \int_0^\infty e^{\gamma \Psi(u) \theta} g(\theta; t) d\theta = \hat{g}(-\gamma \Psi(u); t).
\] (4.5)

Furthermore, we shall assume that $T(\theta)$ is a $\beta$-stable random time introduced in Section 2 so that the inverse random time $\Theta(t)$ has the p.d.f. $g_\beta(\theta; t)$ given by the formula (2.4). Then, taking into account (2.10), the formula (4.5) yields
\[
\tilde{f}(u; t) = E_\beta(\gamma t^\beta \Psi(u)) .
\] (4.6)

In turn, it follows from (2.11) that the characteristic function of the anomalous diffusion $X(t)$ is a solution of the fractional differential equation
\[
\frac{d^\beta \tilde{f}(u; t)}{dt^\beta} = \gamma \Psi(u) \tilde{f}(u; t) + \frac{t^{-\beta}}{\Gamma(1 - \beta)} \chi(t) .
\] (4.7)

In the special case when $X(t)$ is a symmetric $\alpha$-stable process, $0 < \alpha \leq 2$, with the logarithm of the mother characteristic function $\Psi(u) = -\sigma^\alpha |u|^\alpha$, taking the inverse
Fourier transform, (4.7) yields the following fractional diffusion equation for the p.d.f. \( f(x; t) \) of the process \( X(t) \):
\[
\frac{\partial^\beta f}{\partial \tau^\beta} = \sigma^\alpha \frac{\partial^\alpha f}{\partial |x|^\alpha} + \frac{\tau^{-\beta}}{\Gamma(1 - \beta)} \chi(\tau) \delta(x) \quad (\tau = \gamma^{1/\beta} t)
\]
the fractional derivative with respect to spatial variable \( x \) is understood here and thereafter in the Weyl sense, see Appendix B. The solution of this equation can be found by substituting into (4.2) a selfsimilar p.d.f. \( w(x; \theta) = f_\alpha(x; \theta) \) of a symmetric \( \alpha \)-stable p.d.f., see (B7), and the \( \beta \)-stable p.d.f \( g(\theta; t) = g_\beta(\theta; t) \) given in (2.4). As a result we obtain

\[
f(x; t) = \frac{1}{\sigma \tau^\mu} f_{\alpha, \beta} \left( \frac{x}{\sigma \tau^\mu} \right),
\]
where
\[
f_{\alpha, \beta}(y) = \int_0^\infty g_\beta(z) f_\alpha \left( \frac{y}{z^{1/\alpha}} \right) \frac{dz}{z^{1/\alpha}}, \quad \mu = \frac{\beta}{\alpha}.
\]

Let us find the moments of the p.d.f. (4.9). The self-similarity of \( f(x; t) \) and (4.10) imply that
\[
\langle |X(\tau)|^\kappa \rangle = \langle |X|^\kappa \rangle_\alpha \langle \Theta^{\kappa/\alpha} \rangle_\beta \sigma^\kappa \tau^{\kappa/\alpha},
\]
where the angled brackets’ subscripts mean averaging, respectively, with help of the symmetric stable p.d.f. \( f_\alpha(x) \), and the p.d.f. \( g_\beta(\tau) \). In view of (A16) and (2.7), we get
\[
\langle |X|^\kappa \rangle_\alpha = \frac{2}{\pi} \Gamma(\kappa) \Gamma \left( 1 - \frac{\kappa}{\alpha} \right) \sin \left( \frac{\pi \kappa}{2} \right), \quad \langle \Theta^{\kappa/\alpha} \rangle_\beta = \frac{\Gamma \left( \frac{\kappa}{\alpha} + 1 \right)}{\Gamma \left( \frac{\kappa}{\alpha} \right)}.
\]
Substituting these expressions into (4.11), and utilizing the symmetrization formula of the theory of the Gamma function, we obtain finally that
\[
\langle |X(\tau)|^\kappa \rangle = \frac{2}{\beta^\kappa \sin \left( \frac{\pi \kappa}{\alpha} \right)} \frac{\Gamma(\kappa)}{\Gamma \left( \frac{\kappa/\alpha}{\alpha} \right)} \sigma^\kappa \tau^{\kappa/\alpha} \quad (\kappa < \alpha).
\]
Similarly, for \( \alpha > 1 \), we can find the value of the p.d.f. \( f(x; t) \) of the anomalous diffusion \( X(t) \), see (4.9), at \( x = 0 \):
\[
\sigma f(0; t) = \frac{1}{\tau^\mu} f_\alpha(0) \langle \Theta^{-1/\alpha} \rangle_\beta = \frac{\tau^{-\mu}}{\alpha \Gamma \left( 1 - \frac{\beta}{\alpha} \right) \sin \left( \frac{\pi \beta}{\alpha} \right)} \quad (\alpha > 1).
\]
Its asymptotics, for \( x/\sigma \tau^\mu \to \infty \), is described by the formula

\[
\sigma f(x; t) \sim \frac{1}{\pi} \frac{\tau^\beta}{|x|^\alpha} \frac{\Gamma(1 + \alpha)}{\Gamma(1 + \beta)} \sin \left( \frac{\pi \alpha}{2} \right) \quad (|x| \gg \sigma \tau^\mu).
\]

Observe that, for \( \alpha < 1 \) (\( 0 < \beta < 1 \)), the value of the p.d.f. \( f(x; t) \) diverges to \( \infty \) as \( x \to 0 \).

For certain values of \( \alpha \) and \( \beta \), explicit solutions of equation (4.8) can be found, see, e.g., Saichev and Zaslavsky (1997). We shall list them below.

In the case \( \alpha = \beta \), equation (4.8) has the solution of the form

\[
f(x; t) = \frac{1}{\pi |y|^{\alpha + 1}} \frac{\sin \left( \frac{\pi \alpha}{2} \right)}{|y|^{-\alpha} + |y|^{-\beta} + 2 \cos \left( \frac{\pi \beta}{2} \right)} \quad (y = \frac{x}{\sigma \tau}).
\]

For \( \beta \to 1 \), the above solution converges, as expected, to the Cauchy p.d.f.

For \( \alpha = 1, \beta = 1/2 \), the p.d.f. \( f(x; t) \) can be expressed in terms of the integral exponential function \( \text{Ei}(z) \) since

\[
f_{\alpha,\beta}(y) = -\frac{1}{2\pi \sqrt{\pi}} \exp \left( \frac{y^2}{4} \right) \text{Ei} \left( -\frac{y^2}{4} \right).
\]

So far we have only considered symmetric anomalous diffusions and the corresponding equation (4.8) for the p.d.f. \( f(x; t) \) contained only symmetric fractional derivatives with respect to \( x \). However, modeling of various physical phenomena, such as the accumulation processes, require consideration of processes \( X(\theta) \) which are nondecreasing and have nonnegative values. In such cases, the p.d.f.s \( f(x; t) \) of the anomalous diffusions \( \chi(t) \) satisfy an analogue of the fractional differential equation (4.8) in which the symmetric fractional derivative in \( x \) is replaced by a one-sided fractional derivative, either right-sided or left-sided. Here, a good example is the equation

\[
\frac{\sqrt{\partial f}}{\partial \sqrt{\tau}} = \frac{\sqrt{\partial f}}{\partial \sqrt{x}} + \frac{1}{\sqrt{\pi \tau}} \chi(\tau) \delta(x).
\]

Its solution is also described by the general formulas (4.9-10). Substituting into (4.10)
\( f_{\alpha=1/2}(x) \) given by (A.10), and \( g_{\beta=1/2}(\tau) \) given by (3.4), we obtain in this case that
\[
f(x; t) = \frac{1}{\pi} \sqrt{\frac{t}{x + t}} \chi(x), \quad x > 0, t > 0.
\]

Remark: It is worth mentioning that the case of coupled anomalous diffusion \( \mathcal{X}(t) = X(\Theta(t)) \), where the processes \( X(\theta), \theta \geq 0 \), and \( \Theta(t) \) are not statistically independent, has been recently discussed by Meerschaert, Benson, Scheffler and Becker-Kern (2002).

### 4.5 Anomalous drift process

In this section we will discuss statistical properties of the fractional pure drift process, a simple example of the anomalous diffusion introduced in Section 4 which corresponds to the case \( X(\theta) = \gamma \theta, \gamma > 0 \). In this case,
\[
\mathcal{X}(t) = \gamma \Theta(t).
\] (5.1)

The simplicity of this case will permit us to elucidate the analysis of the general case, and the results obtained herein will prove useful in Section 6 where anomalous subdiffusion processes are going to be introduced.

For the drift process the logarithm of the characteristic function of the underlying "mother" infinitely divisible distribution has a degenerate form \( \Psi(u) = iu \), and equation (4.7) assumes the form
\[
\frac{\partial^\beta \tilde{f}}{\partial \tau^\beta} = iu \tilde{f} + \frac{\tau^{-\beta}}{\Gamma(1 - \beta)} \chi(\tau) \quad (\tau = \gamma^{1/\beta} t).
\] (5.2)

Taking the inverse Fourier transform in variable \( u \), we obtain the following equation for the p.d.f. \( f(x; t) \) of the anomalous drift process \( \mathcal{X}(t) \):
\[
\frac{\partial^\beta f}{\partial \tau^\beta} = \frac{\partial f}{\partial x} + \frac{\tau^{-\beta}}{\Gamma(1 - \beta)} \chi(\tau) \delta(x).
\] (5.3)
Substituting $\Psi(u) = iu$ into (4.6), applying the inverse Fourier transform, and taking into account formula (3.1), we arrive at the following solution of equation (5.3)

$$f(x; t) = \frac{1}{\tau^\beta} g_\beta \left( \frac{x}{\tau^\beta} \right). \quad (5.4)$$

For different values of $\beta$, the shapes of $f(x; t)$, as a function of $x$, are the same as the shapes of the plots of $g_\beta(\theta)$ shown in Fig. 3. As $\beta \to 1$, the p.d.f. (5.4) weakly converges to the Dirac delta:

$$\lim_{\beta \to 1^-} f(x; t) = \delta(x - \tau).$$

This behavior can be viewed as a sort of Law of Large Numbers for $\beta \to 1$.

The properties of the function $g_\beta(\theta)$ immediately give the following rates of growth of the moments of the fractional anomalous drift process (5.1):

$$\langle X^n(t) \rangle = \langle \Theta^n \rangle \tau^{n\beta}, \quad \langle \Theta^n \rangle = \frac{n!}{\Gamma(n\beta + 1)}.$$  

In particular, for $\beta \to 1_-$, the relative variance of the fractional drift,

$$\frac{\sigma_X^2(t)}{\langle X(t) \rangle^2} = 2 \frac{\Gamma^2(\beta + 1)}{\Gamma(2\beta + 1)} - 1 \quad (5.5)$$

converges to zero confirming the Law of Large Numbers behavior as $\beta < 1$ approaches 1. A plot of the relative variance (5.5) as a function of $\beta$ is shown in Fig. 4.4.

### 4.6 Fractional anomalous subdiffusion

In this section the underlying Lévy infinitely divisible process $X(\theta)$ will be taken to be the Brownian motion (Wiener) process $B(\theta)$ with the logarithm of the characteristic function of the "mother" infinitely divisible distribution of the form $\Psi(u) = -u^2$. Note that with this definition $\langle B^2(t) \rangle = 2t$ (and not $t$ as in the standard definition).

The random time $T(\theta)$ will be assumed to be a $\beta$-stable process with p.d.f. (2.3). The corresponding anomalous diffusion

$$X(t) = \sqrt{\gamma} B(\Theta(t)) \quad (6.1)$$
will be called fractional anomalous subdiffusion.

Our first task is to find an equation for the one-point p.d.f. \( f(x; t) \) of \( \mathcal{X}(t) \). We will start with an equation for its characteristic function. The latter is obtained immediately from equation (4.7) by substituting \( \Psi(u) = -u^2 \) which yields

\[
\frac{d\tilde{f}}{d\tau^\beta} + u^2 \tilde{f} = \frac{\tau^{-\beta}}{\Gamma(1 - \beta)} \chi(\tau) \quad (\tau = \gamma^{1/\beta} t).
\]

Taking the inverse Fourier transform in variable \( u \) we obtain a subdiffusion equation for the p.d.f. \( f(x; t) \) of the process (6.1):

\[
\frac{\partial^\beta f}{\partial \tau^\beta} = \frac{\partial^2 f}{\partial x^2} + \frac{\tau^{-\beta}}{\Gamma(1 - \beta)} \chi(\tau) \delta(x). \quad (6.2)
\]

This equation can be solved with the help of the solution (5.4) of the fractional drift equation (5.3).

Indeed, let us split the characteristic function of the fractional drift into the even and odd (in Fourier variable \( u \)) components:

\[
\tilde{f} = \tilde{f}_{\text{even}} + \tilde{f}_{\text{odd}}
\]
and substitute it into equation (5.2). As a result we obtain two equations:

\[
\frac{\partial^{\beta} \tilde{f}_{\text{even}}}{\partial \tau^{\beta}} = iu \tilde{f}_{\text{odd}} + \frac{\tau^{-\beta}}{\Gamma(1-\beta)} \chi(\tau), \quad \text{and} \quad \frac{\partial^{\beta} \tilde{f}_{\text{odd}}}{\partial \tau^{\beta}} = iu \tilde{f}_{\text{even}}.
\]

Applying fractional derivative of order $\beta$ with respect to $\tau$ to the first equation, taking into account the standard fractional calculus formula

\[
\frac{\partial^{\delta} \tau^{\delta} \chi(\tau)}{\partial \tau^{\beta}} = \frac{\Gamma(\delta + 1)}{\Gamma(1 + \delta - \beta)} \tau^{\delta-\beta} \chi(\tau),
\]

see, e.g., Saichev and Woyczynski (1997), and eliminating the odd component $\tilde{f}_{\text{odd}}$, we obtain a closed equation for $\tilde{f}_{\text{even}}(u; t)$:

\[
\frac{\partial^{2\beta} \tilde{f}_{\text{even}}}{\partial \tau^{2\beta}} + u^2 \tilde{f}_{\text{even}} = \frac{\tau^{-2\beta}}{\Gamma(1 - 2\beta)} \chi(\tau).
\]

As a result, the even component

\[
f_{\text{even}}(x; t) = \frac{1}{2} [f(x; t) + f(-x; t)] = \frac{1}{2} f(|x|; t)
\]

(6.3)

of the p.d.f. $f(x; t)$ of the fractional drift satisfies equation

\[
\frac{\partial^{2\beta} f}{\partial x^{2\beta}} = \frac{\partial^2 f}{\partial x^2} + \frac{\tau^{-2\beta}}{\Gamma(1 - 2\beta)} \chi(\tau) \delta(x).
\]

A substitution $\beta \mapsto \beta/2$ reduces this equation to the equation (6.2). Thus, the sought solution of the fractional subdiffusion equation (6.2) can be obtained from (6.3) and (5.4) by replacing $\beta$ by $\beta/2$ which gives

\[
f(x; t) = \frac{1}{2^{\tau^{\beta/2}} \tau^{\beta/2}} g_{\beta/2} \left( \frac{|x|}{\tau^{\beta/2}} \right)
\]

(6.4)

In particular, for $\beta = 2/3$, recalling (3.7), we obtain that

\[
f(x; t) = \frac{1}{2} \sqrt{\frac{9}{\tau}} \frac{\sqrt{3}}{\tau} \text{Ai} \left( \frac{|x|}{\sqrt{3\tau}} \right), \quad \beta = \frac{2}{3}.
\]
Figure 4.5: The solution of the subdiffusion equation (6.2), for $\beta = 2/3$.

A plot of this solution, for $\tau = 1$, is given in Fig.4.5.

The sub-Fickian behavior of the anomalous diffusion (6.1), justifying the term *subdiffusion*, is obtained by substituting $\alpha = 2$ into the general formula (4.12), and finding the limit as $\kappa \to 2\_$, which gives

$$\langle X^2(t) \rangle = \frac{2}{\Gamma(\beta + 1)} \tau^\beta. \quad (6.5)$$

In other words, the variance of $X^2(t)$ grows sublinearly as $t$ increases.

The solution (6.4) of the subdiffusive equation (6.2) can be used immediately to calculate, for instance, the p.d.f. $f(t; \ell)$ of the first crossing time $T$ of the level $\ell$ by a subdiffusion process (6.1) starting at 0:

$$T = \min\{t : X(t) \geq \ell\}.$$ 

Indeed, in view of the well known formula,

$$P(T > t) = \int_{-\infty}^\ell f(x; t|\ell) \, dx, \quad (6.6)$$

where $f(x; t|\ell)$ is a solution of the equation (6.2) satisfying the boundary condition

$$f(x = \ell; t) = 0.$$
Solving the above boundary-value problem by the standard reflection method we obtain
\[ f(x; \tau|\ell) = \frac{1}{2^{\tau/\beta}} \left[ g_{\beta/2} \left( \frac{|x|}{\tau^{\beta/2}} \right) - g_{\beta/2} \left( \frac{|x - 2\ell|}{\tau^{\beta/2}} \right) \right]. \tag{6.7} \]
Substituting this expression in (6.6) we get
\[ P(T > t) = \int_{0}^{t/\ell^{\beta/2}} g_{\beta/2}(z) \, dz, \tag{6.8} \]
which, together with (2.5), gives
\[ f(\tau; \ell) = -\frac{dP(T > t)}{dt} = \frac{1}{2^{\tau/\ell^{\beta/2}}} f_{\beta/2} \left( \frac{\sqrt{\tau}}{\ell^{\beta/2}} \right). \tag{6.9} \]

### 4.7 Multi-dimensional fractional subdiffusion

The above analysis of the one-dimensional subdiffusion can be easily extended to the multidimensional case. Let us consider, for example, a \( d \)-dimensional vector random process
\[ \mathbf{X}(t) = \{X_1(t), X_2(t), \ldots, X_d(t)\}, \tag{7.1} \]
with components which are independent Wiener processes of random argument \( \Theta(t) \):
\[ X_k(t) = \sqrt{\gamma} B_k(\Theta(t)) \quad (k = 1, 2, \ldots, d). \]

The \( d \)-dimensional characteristic function of the random vector \( \mathbf{X}(t) \)
\[ \hat{f}_d(\mathbf{u}; t) = \langle e^{i\mathbf{u} \cdot \mathbf{X}(t)} \rangle = \int_{0}^{\infty} \tilde{w}_d(\mathbf{u}; \theta) g(\theta; t) \, d\theta, \tag{7.2} \]
where
\[ \tilde{w}_d(\mathbf{u}; \theta) = e^{-\gamma(\mathbf{u} \cdot \mathbf{u}) \theta} \]
is the \( d \)-dimensional characteristic function of the vector Gaussian diffusion process \( \mathbf{X}(\theta) = \sqrt{\gamma} \mathbf{B}(\theta) \).
If $T(\theta)$ is a $\beta$-stable random time, then an evaluation of the integral in (7.2) gives $\tilde{f}_d$ in terms of the Mittag-Leffler function:

$$\tilde{f}_d(u; t) = E_\beta(-\tau^\beta (u \cdot u)), \quad (\tau = \gamma^{1/\beta} t).$$

Also, the characteristic function $\tilde{f}_d$ satisfies equation

$$\frac{d^\beta \tilde{f}_d}{d\tau^\beta} + (u \cdot u) \tilde{f}_d = \frac{\tau^{-\beta}}{\Gamma(1 - \beta)} \chi(\tau).$$

This equation, in turn, is equivalent to the $d$-dimensional subdiffusion equation for the p.d.f. $f_d(x; t)$ of the random vector $X(t)$ (7.1):

$$\frac{\partial^\beta f_d}{\partial \tau^\beta} = \Delta f_d + \frac{\tau^{-\beta}}{\Gamma(1 - \beta)} \chi(\tau) \delta(x), \quad (7.3)$$

where $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_d^2$ is the usual $d$-dimensional Laplacian.

A solution of this equation can be found substituting into (4.2) the p.d.f.

$$w_d(x; \theta) = \left(\frac{1}{2\pi \theta}\right)^{d/2} \exp\left(-\frac{r^2}{4\theta}\right), \quad r = \sqrt{x_1^2 + \cdots + x_d^2},$$

of random process $X(\theta)$, and the p.d.f. $g_\beta(\theta; t)$ given in (2.4). As a result, we obtain a self-similar solution

$$f_d(x; t) = \frac{1}{\tau^{n\beta/2}} h_d\left(\frac{r}{\tau^{\beta/2}}\right), \quad \tau = \gamma^{1/\beta} t, \quad (7.4)$$

where

$$h_d(y) = \frac{1}{(4\pi)^{d/2}} \int_0^\infty g_\beta(z) \exp\left(-\frac{y^2}{4z}\right) \frac{dz}{z^{d/2}}.$$ 

In the 3-dimensional space, a comparison of (7.4) and (6.4) gives

$$h_3(y) = -\frac{1}{4\pi y} \frac{d g_{3/2}(y)}{dy}. \quad (7.5)$$

In particular, for $\beta = 2/3$,

$$h_3(y) = -\frac{3^{1/3}}{4\pi y} \text{Ai}'\left(\frac{y}{3^{1/3}}\right),$$

where $\text{Ai}'(\theta)$ is the derivative of the Airy function introduced in (3.5).
4.8 Tempered anomalous subdiffusion

Introduced in Section 6 fractional subdiffusion \( \mathcal{X}(t) = \sqrt{\gamma} B(\Theta(t)), \ t \geq 0 \), where \( B \) is the Brownian motion process and random time \( \Theta(t) \) has the inverse \( \beta \)-stable p.d.f., \( 0 \leq \beta < 1 \), owed its subdiffusive, for all times \( t \), behavior \( \langle \mathcal{X}^2(t) \rangle = \frac{2}{\Gamma(\beta+1)} \tau^\beta \tau = \gamma^{1/\beta} t \), see (6.5), to the fact that, for the \( \beta \)-stable random time \( T(\theta) \), the mean, and thus the second moment as well, were infinite. From the physical perspective it is desirable to also have a model that circumvents the infinite-moment difficulty while preserving the subdiffusive behavior, at least for small times.

For this purpose we will consider in this section an anomalous diffusion of the form
\[
\mathcal{X}(t) = \sqrt{\gamma} B(\Theta(t)),
\] (8.1)
where \( \Theta(t) \) is the inverse tempered \( \beta \)-stable random time, with the tempered \( \beta \)-stable random time \( T(\theta) \) defined by the Laplace transform
\[
\hat{f}_{\beta,\text{tmp}}(s; \delta) = \langle e^{-sT(\theta)} \rangle = e^{\alpha \Phi(s)} = \exp \left( \theta \left( \delta^\beta - (s + \delta)^\beta \right) \right), \quad 0 \leq \beta < 1,
\] (8.2)
where
\[
\Phi(s) = \int_0^\infty (e^{-sz} - 1) \phi(z) \, dz, \quad \phi(z) = \frac{\beta}{\Gamma(1-\beta)} z^{-\beta-1} e^{-\beta z} \chi(z),
\]
for some \( \delta > 0 \). The form of the Lévy intensity function \( \phi(z) \) guarantees its \( \beta \)-stable behavior for small jump sizes \( z \); the tempering influence of the exponential term gives the existence of all moments of \( T(\theta) \). In that sense tempered stable processes occupy an intermediate place between pure stable processes and the classical Brownian motion diffusion processes. More information about properties of tempered stable processes is provided in Appendices A and B.

Using formulas (4.3) and (2.2) for the p.d.f. of the inverse random time, we can express the mean square of the process \( \mathcal{X}(t) \) (8.1) via the cumulative distribution function (c.d.f.) \( F_{\beta,\text{tmp}}(t; \theta) \) of the random time \( T(\theta) \):
\[
(\sigma^2)(t) = \langle \mathcal{X}^2(t) \rangle = \langle \langle W^2(\Theta(t)) \rangle_B \rangle_\theta = \langle 2\Theta(t) \rangle = 2\gamma \int_0^\infty F_{\beta,\text{tmp}}(t; \theta) \, d\theta.
\]
The last expression was obtained by integration by parts and application of the fact that, for each \( t \), the values of \( F_{\beta,\text{tmp}}(t; \theta) \to 0 \) as \( \theta \to \infty \).

The Laplace image of \( \langle X^2(t) \rangle \) in variable \( t \) is

\[
(\hat{\sigma}^2)(s) = \int_0^\infty e^{-st} \langle X^2(t) \rangle \, dt = \frac{2\gamma}{s} \int_0^\infty (\hat{f}_{\beta,\text{tmp}}(s))^\theta \, d\theta = -\frac{2\gamma}{s \ln \hat{f}_{\beta,\text{tmp}}(s)}, \quad (8.3)
\]

where \( \hat{f}_{\beta,\text{tmp}}(s) = \hat{f}_{\beta,\text{tmp}}(s;1) \) is the mother Laplace transform of the tempered \( \beta \)-stable random time \( T(\theta) \). Substituting into (8.3) the Laplace image of \( f_\beta(s;\delta) \) from (8.2), we obtain

\[
(\hat{\sigma}^2)(s) = \frac{2\gamma}{s[(s + \delta)^\beta - \delta^\beta]}.
\]

In a dimensionless variable \( p = s/\delta \) the above expression takes the form

\[
(\hat{\sigma}^2)(s) = \frac{2\gamma}{\delta^{1+\beta}} \hat{k}(p; \beta), \quad \text{where} \quad \hat{k}(p; \beta) = \frac{1}{p[(p + 1)^\beta - 1]}, \quad p = \frac{s}{\delta}.
\]

So the mean square

\[
\langle X^2(t) \rangle = \frac{2\gamma}{\delta^\beta} k(\tau; \beta), \quad \tau = \delta t,
\]

where \( k(\tau; \beta) \) is the inverse Laplace transform of \( \hat{k}(p; \beta) \). The asymptotics of the function \( k(\tau; \beta) \) is then determined by the asymptotics of \( \hat{k}(p; \beta) \) which is

\[
\hat{k}(p; \beta) \sim \frac{1}{p^{\beta+1}} \quad (p \to \infty); \quad \hat{k}(p; \beta) \sim \frac{1}{p^2} \quad (p \to 0).
\]

The corresponding asymptotics of \( k(\tau; \beta) \) is as follows:

\[
k(\tau; \beta) \sim \frac{\tau^\beta}{\Gamma(1+\beta)} \quad (\tau \to 0); \quad k(\tau; \beta) \sim \frac{\tau}{\beta} \quad (\tau \to \infty).
\]

Thus, for small \( \tau \), the mean square \( \langle X^2(t) \rangle \) grows according to the subdiffusive law \( \tau^\beta \) while, for large \( \tau \), the subdiffusive law is replaced by the classical Fickian law of the Gaussian linear diffusion.
We illustrate the above dichotomy of the behavior of $\langle X^2(t) \rangle$ in 4.6 in the case of $\beta = 1/2$, where the explicit expression

$$k(\tau; \beta = 1/2) = \sqrt{\frac{\tau}{\pi}} e^{-\tau} + \tau + \left( \frac{1}{2} + \tau \right) \text{erf} \left( \sqrt{\tau} \right),$$

is available.

4.9 Anomalous random walks

Anomalous diffusions discussed in the preceding sections can be viewed as scaling limits of anomalous random walks which change their positions by independent random jumps occurring at independent random intervals which are, as an ensemble, independent of the jumps. These anomalous random walks are not Markovian, except in the special case when the random intervals widths have an exponential p.d.f.
4.9.1 Anomalous diffusion as a limit of anomalous random walks

Anomalous random walk will be introduced here in the spirit of the so-called renewal theory. The starting point will be a sequence

$$\tau(n), \ n = 1, 2, \ldots$$

describing time intervals between consecutive events, which in our case are the jumps of the anomalous random walk $X(t), t \geq 0$, under consideration. The jump sizes $h(n), n = 1, 2, \ldots$, are themselves random quantities. So, the jumps occur at times

$$T(n) = \sum_{k=1}^{n} \tau(k), \ n = 1, 2, \ldots,$$

$$\tau(n) = T(n) - T(n - 1),$$

and the anomalous random walk itself can be thought of as a solution of the stochastic differential equation

$$\frac{dX(t)}{dt} = \sum_{k} \xi(k) \delta(t - T(k)), \quad X(t = 0) = 0. \quad (9.1)$$

It is convenient here to think about $X(t)$ as a description of the coordinate of a particle in the 1-D space which starts at the origin and then jumps distance $\xi(n)$ at times $T(n)$. However, many other interpretations are possible, including an application to modeling evolution of prices in financial markets, etc.

The general assumption is that jumps $\{\xi(n)\}$ are assumed to be independent from each other, and, as an ensemble, independent of the sequence $\tau(n), n = 1, 2, \ldots$, which are statistically independent random quantities themselves. The jumps $\xi(n)$ have identical p.d.f. $w(x)$, and the time intervals $\tau(n)$ have identical p.d.f. $f(t)$.

The time intervals $T(m)$, which, for any $n \geq m$, have the same probability distributions as the time intervals

$$T(n) - T(n - m) = \tau(n) + \tau(n - 1) + \cdots + \tau(n - m + 1), \quad (9.2)$$
have the following p.d.f., and c.d.f., respectively,

\[ f(t; m) = f(t) \ast \cdots \ast f(t), \quad \text{and} \quad F(t; m) = F(t) \ast f(t) \ast \cdots \ast f(t), \quad (9.3) \]

where

\[ F(t) = P(\tau(n) \leq t) = \langle \chi(t - \tau(n)) \rangle \]

is the common c.d.f. of the random intervals \( \tau(n) \).

The solution of equation (9.1), that is the anomalous random walk in question, is of the form

\[ X(t) = \sum_{k=1}^{M(t)} \xi(k), \quad (9.4) \]

where \( M(t) \) is the random number of jumps taken by the anomalous random walk up to time \( t \). The process \( M(t) \) is often called the counting process (or the renewal process) associated with the random walk \( X(t) \). The probability \( R(m; t) \) that the interval \((0, t]\) contains \( m \) jumps is

\[ R(m; t) = P(M(t) = m) = P(m \leq M(t) < m + 1) = P(M(t) < m + 1) - P(M(t) < m). \]

On the other hand,

\[ P(M(t) < m) = P(T(m) > t) = 1 - F(t; m), \quad (9.5) \]

so that

\[ R(m; t) = F(t; m) - F(t; m + 1), \quad \text{for} \quad m \geq 1, \quad (9.6) \]

and

\[ R(0; t) = 1 - F(t; 1). \]

According to the total probability formula, the p.d.f. of \( X(t) \) is

\[ f(x; t) = \sum_{m=0}^{\infty} R(m; t) w(x; m), \quad (9.7) \]
where probabilities $R(m; t)$ are defined by equalities (9.6), while

$$w(x; m) = \underbrace{w(x) \ast \cdots \ast w(x)}_{m\text{-times}}, \quad w(x; 0) = \delta(x).$$

Our final assumption is that the common p.d.f. $f(t)$ of the random durations $\tau(n)$ of the inter-jump time intervals is infinitely divisible. In this case there is an obvious relationship between formulas (2.2) and (9.6). Indeed, the infinitely divisible time $T(\theta)$ introduced in (2.1) coincides, for integer values $\theta = n$, with the values of the random time $T(n)$ introduced at the beginning of this subsection. Similarly, the c.d.f.s $F(t; m)$ and $F(t; m + 1)$, appearing in the right-hand side of the equality (9.6), can be viewed as the values of the c.d.f. $F(t; \theta)$ of the infinitely divisible time $T(\theta)$, for integer values of its variable $\theta$.

The inverse random time $\Theta(t)$ of Section 2 is then related to the counting process $M(t)$ via the following relations:

$$M(t) \leq \Theta(t), \quad M(t = T(m)) = \Theta(T(m)),$$

while the probability that the time interval $(0, t]$ contains $m$ jumps is

$$R(m; t) = \int_m^{m+1} g(\theta; t) \, d\theta,$$  \hspace{1cm} (9.8)

where, we recall, $g(\theta; t)$ is the p.d.f. (2.2) of the inverse random time $\Theta(t)$. Integrating both sides of (2.2) with respect to $\theta$ over the interval $[m, m + 1]$, and taking into account (9.8), brings us back to the formula (9.6).

Our final goal in this subsection is to find the asymptotic behavior of the p.d.f. $f(x; t)$ of the anomalous random walk $X(t)$ at large temporal and spatial scales. We begin by observing that if the p.d.f. $g(\theta; t)$ varies slowly (smoothly) with $\theta$, then the equality (9.8) one can be replaced by an approximate equality

$$R(m; t) \simeq g(m; t).$$
Correspondingly, the sum (9.7) takes the form

\[ f(x; t) \approx R(0; t) \delta(x) + \sum_{m=1}^{\infty} g(m; t) w(x; m). \]  

(9.9)

Now, if we assume that the p.d.f. \( w(\xi) \) of the random jump sizes is infinitely-divisible and generates a one-parameter semigroup \( w(x; \theta) \) of infinitely divisible p.d.f.s such that \( w(x; m) = w(x; \theta = m) \), and such that \( w(x; \theta) \) vary slowly (smoothly) as \( \theta \) varies over intervals of length one, then the sum in (9.9) can be approximated by the integral and we have

\[ f(x; t) \approx R(0; t) \delta(x) + \int_{1}^{\infty} g(\theta; t) w(x; \theta) d\theta. \]  

(9.10)

If the probability of the absence of the jumps,

\[ R(0; t) \approx \int_{0}^{1} g(\theta; t) d\theta, \]  

(9.11)

is small then the expression (9.10) can be replaced by

\[ f(x; t) \approx \int_{0}^{\infty} g(\theta; t) w(x; \theta) d\theta. \]  

(9.12)

We recognize the right-hand side as the familiar integral expression (4.2) for the p.d.f. of the anomalous diffusion \( X(t) \) introduced in (4.1).

Taking the Laplace transform in variable \( t \) of both sides of the expression (2.2) for the p.d.f. \( g(\theta; t) \), we obtain

\[ \hat{g}(\theta; s) = -\frac{1}{s} \frac{\partial \hat{f}(\theta)}{\partial \theta} = -\frac{1}{s} \hat{f}(\theta) \Phi(s), \quad \text{where} \quad \Phi(s) = \ln \hat{f}(s). \]  

(9.13)

Recall that if the logarithm of the Laplace transform of the p.d.f. of the inter-jump time intervals has a \( \beta \)-stable-like asymptotics

\[ \Phi(s) \sim -\frac{1}{\gamma} s^\beta \quad (\gamma > 0, \quad 0 < \beta < 1, \quad s \to \infty), \]  

(9.14)
then the mean $\langle \tau \rangle = \infty$. Replacing $\Phi(s)$ in (9.13) by its asymptotics (9.14), we obtain
the asymptotics of the Laplace image $\hat{g}(\theta; t)$:

$$\hat{g}(\theta; s) \sim \frac{1}{\gamma} s^{\beta - 1} \exp\left(-\frac{\theta}{\gamma} s^{\beta}\right) = -\frac{1}{\beta \theta} \frac{\partial}{\partial s} \exp\left(-\frac{\theta}{\gamma} s^{\beta}\right) \quad (s \to \infty).$$

Applying the inverse Laplace transform to both sides of the above equality, and taking into account that $-\frac{\partial}{\partial s} \exp(-\mu s^{\beta})$ is the Laplace transform of $\frac{t}{\mu^{1/\beta}} f_{\beta}\left(\frac{t}{\mu^{1/\beta}}\right)$, with $f_{\beta}(t)$ being the $\beta$-stable p.d.f. introduced in Section 2, we finally obtain the asymptotics

$$g(\theta; t) \sim \frac{\tau}{\beta \theta^{1+1/\beta}} f_{\beta}\left(\frac{\tau}{\theta^{1/\beta}}\right) = \frac{1}{\tau^{\beta}} g_{\beta}\left(\frac{\theta}{\tau^{\beta}}\right) \quad (\tau = \gamma^{1/\beta} t \to \infty). \quad (9.15)$$

Similarly, if the logarithm of the characteristic function of the p.d.f. $w(x; \theta)$ possesses asymptotics

$$\ln \hat{w}(u; \theta) \sim -\sigma^\alpha |u|^\alpha \theta, \quad (u \to \infty) \quad (9.16)$$

then the p.d.f. $w(x; \theta)$ converges, as $\theta \to \infty$, to a symmetric stable p.d.f. (B7) (with the substitution $\tau = \theta$, $\gamma = 1$). Substituting asymptotics (9.14), (9.16) into (9.12), we obtain the solution (4.9) of the fractional diffusion equation (4.8). Thus, the p.d.f. of the anomalous diffusion described by a fractional diffusion equation provides asymptotics of the p.d.f. of the anomalous random walk at large temporal and spatial scales.

### 4.9.2 The fractional Kolmogorov-Feller equations for anomalous random walks

In this subsection we will find an equation for the p.d.f. $f(x; t)$ of the anomalous random walk $X(t)$ which is explicitly given by formula (9.7). For this purpose let us take the Fourier transform of (9.7) with respect to $x$, which gives

$$\tilde{f}(u; t) = \sum_{m=0}^{\infty} R(m; t) \tilde{w}^m(u). \quad (9.17)$$
The above series is equal to the generating function $\mathcal{R}(t, z)$ of $R(m, t)$, evaluated at $z = \tilde{w}(u)$. Hence, in view of Appendix C, the Laplace image of series (9.17) is given by the expression

$$\hat{\tilde{f}}(u; s) = \frac{1 - \hat{f}(s)}{s[1 - \hat{f}(s) \tilde{w}(u)]}.$$  \hspace{1cm} (9.18)

This equality can be rewritten in the form

$$\frac{1}{\hat{f}(s)} \hat{\tilde{f}}(u; s) - \tilde{w}(u) \hat{\tilde{f}}(u; s) = \frac{1 - \hat{f}(s)}{s \hat{f}(s)}.$$  \hspace{1cm} (9.19)

After taking the inverse Fourier transform in space and the inverse Laplace transform in time, equation (9.19) gives an integral equation for the p.d.f. $f(x; t)$ of the anomalous random walk which is given by the expression (9.7).

As an example consider first the Poissonian case of inter-jump intervals with the exponential p.d.f. with mean $1/\nu$. Its Laplace image is

$$\hat{f}(s) = \frac{\nu}{\nu + s}.$$  \hspace{1cm} (9.20)

Substituting this expression in (9.18) we obtain equation

$$s \hat{\tilde{f}}(u; s) + \nu [1 - \tilde{w}(u)] \hat{\tilde{f}}(u; s) = 1.$$

Applying the inverse Laplace transform in $s$ to both sides of this equality, and setting $\hat{\tilde{f}}(u; t = 0) = 0$, we obtain a differential equation for the characteristic function of the anomalous random walk:

$$\frac{d\hat{f}(u; t)}{dt} + \nu [1 - \tilde{w}(u)] \hat{f}(u; t) = \delta(t - 0_+).$$

Taking the inverse Fourier transform we find that the p.d.f. $f(x; t)$ of random walk $X(t)$ satisfies the integro-differential equation

$$\frac{\partial f(x; t)}{\partial t} + \nu [f(x; t) - f(x; t) * w(x)] = \delta(t - 0_+)\delta(x),$$
which is equivalent to the following initial-value problem for the homogeneous equation
\[
\frac{\partial f(x; t)}{\partial t} + \nu [f(x; t) - f(x; t) * w(x)] = 0, \quad f(x; t = 0) = \delta(x),
\] (9.21)
alogous to the classical Kolmogorov-Feller equation.

In the case when \( \hat{f}(s) \) and \( \tilde{w}(u) \) have the small \( s \) and \( u \), respectively, asymptotics of the forms
\[
\frac{1}{f(s)} \sim 1 + \frac{s^\beta}{\gamma} \quad (s \to 0, \gamma > 0); \quad \tilde{w}(u) \sim 1 - \sigma^\alpha |u|^\alpha \quad (u \to 0),
\]
substituting them into (9.19), we find the asymptotic equation
\[
s^\beta \hat{f} + \gamma \sigma^\alpha |u|^\alpha \hat{f} = s^{\beta-1} \quad (s \to 0, \quad u \to 0)
\]
for \( \hat{f}(u; s) \), which is obviously equivalent to the fractional diffusion equation (4.8).

If the inter-jump intervals \( \tau(n) \) have the fractional exponential p.d.f. with the Laplace transform \( (1 + s^\beta)^{-1} \), \( 0 < \beta < 1 \), then (9.18) yields the equation
\[
s^\beta \hat{f} + [1 - \tilde{w}(u)] \hat{f} = s^{\beta-1} .
\]
Applying the inverse Laplace transform to both sides of this equality we obtain an equation for the characteristic function:
\[
\frac{\partial^\beta \hat{f}(u; t)}{\partial u^\beta} + [1 - \tilde{w}(u)] \hat{f}(u; t) = \frac{t^{-\beta}}{\Gamma(1 - \beta)} \chi(t) .
\] (9.22)
Now, applying the inverse Fourier transform to (9.22) we obtain the fractional Kolmogorov-Feller equation
\[
\frac{\partial^\beta f(x; t)}{\partial u^\beta} + f(x; t) - f(x; t) * w(x) = \frac{t^{-\beta}}{\Gamma(1 - \beta)} \delta(x) \chi(t) .
\] (9.23)
An anomalous random walk with the p.d.f. satisfying the above equation will be called here fractional wandering.
Comparing (9.23) with (4.7), (4.6), we find an explicit expression for the characteristic function of the fractional wandering:

\[ \tilde{f}(u; t) = E_\beta \left( [\tilde{w}(u) - 1] t^\beta \right). \]  

(9.24)

Formula (4.4) also gives an alternative form of this characteristic function

\[ \tilde{f}(u; t) = \int_0^\infty g_\beta(\theta; t) e^{\theta [\tilde{w}(u) - 1]} d\theta, \]  

(9.25)

where \( g_\beta(\theta; t) \) is given by equality (2.4).

It follows from (9.24) that the p.d.f. of the fractional wandering has the following structure:

\[ f(x; t) = R(0; t) \delta(x) + f_c(x; t), \]  

(9.26)

where \( f_c(x; t) \) is the continuous part of the p.d.f., while

\[ R(0; t) = E_\beta(-t^\beta) \]  

(9.27)

represent the probability of the event that, up to time \( t \), the particle has not moved. For \( \beta = 1 \), when the fractional Kolmogorov-Feller equation reduces to the standard Kolmogorov-Feller equation (9.21), the probability of jumps’ absence (9.27) decays exponentially to zero since

\[ R(0; t) = e^{-t}. \]

However, for \( 0 < \beta < 1 \), it decays much slower. More precisely, according to (2.14), we have the following asymptotic formula:

\[ R(0; t) \sim \frac{1}{\Gamma(1 - \beta) t^\beta} \quad (t \to \infty). \]

Functions \( R(0; t) \), for different values of \( \beta \), are plotted in Fig.4.7.

Formula (9.25) also gives an expressions for the probabilities that the fractional wandering particle makes \( m \) jumps by time \( t \):

\[ R(m; t) = \frac{t^{m\beta}}{m!} \int_0^\infty \theta^m e^{-\theta t^\beta} g_\beta(\theta) d\theta. \]
Figure 4.7: Plots of functions $R(0; t)$, for different values of $\beta$, describing the probabilities of the event that the fractional wandering particle has not moved up to time $t$.

Using the steepest descent method it is easy to show that the functions $R(m; t)$ satisfy the asymptotic relation

$$R(m; t) \sim \frac{1}{t^{\beta} g_\beta \left( \frac{m}{t^{\beta}} \right)}, \quad (t^{\beta} \to \infty).$$

### 4.9.3 Subdiffusive Fickian laws for anomalous random walk

Expressions (9.18) and (9.22) are convenient for analysis of Fickian laws of diffusion of anomalous random walks. Indeed, expansion of the right-hand sides of the equalities (9.18) and (9.22) in the Taylor series in powers of $u$ gives the Laplace transform of the moments of the process $X(t)$. If, for instance, the small $u$ asymptotics of $\tilde{w}(u)$, is of the form

$$\tilde{w}(u) \sim 1 - \frac{1}{2} \sigma^2 u^2 \quad (u \to 0),$$
then, substituting it into (9.18), and expanding the resulting expression into the Taylor series in \( u \), we obtain

\[
\hat{f}(u; s) \sim \frac{1}{s} - \frac{1}{2} \sigma^2 u^2 \frac{\hat{f}(s)}{s[1 - \hat{f}(s)]}.
\]

The first summand on the right-hand side is responsible for the normalization condition. The second summand gives rise to the Fickian diffusion law for the anomalous random walk. To see this, observe that the Laplace image of the mean square of the anomalous random walk

\[
\hat{\sigma}^2(s) = \int_0^\infty \langle X^2(t) \rangle e^{-st} dt = \sigma^2 \frac{\hat{f}(s)}{s[1 - \hat{f}(s)]}.
\]

Substituting the asymptotics of the Laplace image of the one-sided \( \beta \)-stable p.d.f., \( 0 < \beta < 1 \) (see Appendix A), we obtain the following subdiffusive Fickian law for the anomalous random walk:

\[
\hat{\sigma}^2(s) \sim \frac{\sigma^2}{s^{\beta+1}} \quad (s \to 0) \quad \Rightarrow \quad \langle X^2(t) \rangle \sim \frac{\sigma^2}{\Gamma(1 + \beta)} t^\beta \quad (t \to \infty).
\]

If the mean duration of the inter-jump intervals is finite \( \langle \tau \rangle < \infty \), then one obtains the Laplace image asymptotics

\[
\hat{f}(s) \to 1 - \langle \tau \rangle s \quad (s \to 0) \quad \Rightarrow \quad \hat{\sigma}^2(s) \sim \frac{D}{s^2}, \quad D = \frac{\sigma^2}{\langle \tau \rangle},
\]

which yields, asymptotically, the classical linear Fickian law of diffusion

\[
\langle X^2(t) \rangle \sim D t, \quad (t \to \infty).
\]

4.9.4 Anomalous random walks: the stationary case

In this subsection we will briefly discuss anomalous random walks in the stationary case which corresponds, intuitively speaking, to the situation when the sequence of random jump times

\[\ldots, T(-2), T(-1), T(0), T(1), T(2), \ldots\]
Figure 4.8: A schematic illustration of different random times studied in the renewal theory.

extends from $-\infty$ to $+\infty$, with independent, and identically distributed interjump time intervals

$$\tau(n) = T(n) - T(n - 1)$$

with a common p.d.f. $f(t)$.

For a certain random $n_0$, the time $t = 0$ is contained in the random interval

$$[T(n_0 - 1), T(n_0)),$$

and the p.d.f. of $T(n_0)$ is equal to the stationary p.d.f.

$$f^+(t) = \frac{d}{dt} \lim_{s \to \infty, s \in [T(n_s - 1), T(n_s))} P(T(n_s) - s \leq t) = \frac{1 - F(t)}{\langle \tau \rangle} \quad (9.28)$$

of the so-called excess life $\tau^+ = T(n_s) - s$, where $T(n_s - 1) \leq s < T(n_s)$, cf. Fig. 4.8. The term ”excess life” is borrowed from the reliability theory (see, Karlin and Taylor, p. 415).

More formally, the starting point of Subsection 9.1 will be here adjusted as follows: The stationary anomalous random walk $X_{st}(t), t \geq 0$, is defined as a solution of the stochastic differential equation

$$\frac{dX_{st}}{dt} = \sum_{k=1}^{\infty} \xi(k) \delta(t - T_k), \quad X(t = 0) = 0, \quad (9.29)$$

where

$$T(1) = \tau^+, \quad T(n) = \tau^+ + \tau(2) + \cdots + \tau(n), \quad n = 2, 3, \ldots, \quad (9.30)$$
where $\tau^+, \tau(2), \tau(3), \ldots$, are independent, $\tau^+$ has the p.d.f. $f^+(t)$ defined in (9.28), and $\tau(2), \tau(3), \ldots$, have the p.d.f. $f(t)$. In this case, for any $s > 0$, the excess life $T(n_s) - s$ has the same stationary p.d.f. $f^+(t)$, independent of time $s$. As before, the jump sizes $\xi(k)$ are independent, independent of $T(k)$’s, and have a common p.d.f. $w(x)$.

In the stationary case the discussion of Subsections 9.1-3 has to be adjusted to reflect assumptions (9.29-30). Thus the p.d.f. and the c.d.f. of the time intervals $T(m)$ are, respectively,

$$f_{st}(t; m) = f^+(t) * f(t) \cdots * f(t), \quad \text{and} \quad F_{st}(t; m) = F^+(t) * f(t) \cdots * f(t), \quad (9.31)$$

and the solution of the equation (9.29) is of the form

$$X_{st}(t) = \sum_{k=1}^{N(t)} \xi(k), \quad (9.32)$$

where the counting process $N(t)$ represents the random number of jumps taken by the anomalous random walk $X_{st}(t)$ up to time $t$.

With $P(m; t)$ denoting the probability that the interval $(0, t]$ contains $m$ jumps of $X_{st}(t)$, the p.d.f. of $X_{st}(t)$ is

$$f_{st}(x; t) = \sum_{m=0}^{\infty} P(m; t) w(x; m), \quad (9.33)$$

with $w(x; m)$ defined, as in (9.7), as the $m$-fold convolution of $w(x)$, the common p.d.f. of $\xi_k$. However, in the present stationary case, taking first the Fourier transform of (9.33) in $x$, and then the Laplace transform in $t$, gives the following result

$$\hat{f}_{st}(u; s) = \frac{1}{s} \left[ 1 + \frac{\hat{w}(u)}{\langle \tau \rangle} - \frac{1 - \hat{f}(s)}{s} \frac{1 - \hat{f}(s)}{\hat{w}(u)} \right]. \quad (9.34)$$

It differs from the analogous expressions (9.18) and from the equation (9.19) which follows from (9.18). Formula (9.34) can also be written in the form

$$\frac{1}{f} \hat{f}_{st} \hat{w} \hat{f}_{st} = \frac{1}{s} \left[ 1 - \hat{w} + \frac{1}{\langle \tau \rangle s} (\hat{w} - 1) (1 - \hat{f}) \right], \quad (9.35)$$
analogous to (9.19), which is equivalent to the integral equation for the p.d.f. \( f(x; t) \) of the random wandering. In particular, substituting here the Laplace image (9.20) of the exponential p.d.f., we again obtain the classical Kolmogorov-Feller equation.

To use formula (9.28), the p.d.f. \( f(t) \) must have the finite mean so, in particular, that formula is not applicable to the \( \beta \)-stable inter-jump intervals with \( \beta \leq 1 \). However, it is perfectly applicable in the case of the tempered stable p.d.f. \( f_{\beta, \text{tmp}}(\tau; \delta) \) discussed in Section 8. For \( s \gg \delta \) that p.d.f. has the asymptotic behavior corresponding to the Laplace transform of one-sided \( \beta \)-stable p.d.f. (A7), while, for \( |s| \ll \delta \), it has the asymptotics

\[
\hat{f}_{\beta, \text{tmp}}(s; \delta) \sim \exp \left( -\beta \delta^{\beta-1} s + \frac{1}{2} \delta^{\beta-2} \beta(1 - \beta) s^2 \right)^{\frac{1}{2}}. \tag{9.36}
\]

Hence, in particular, in this case, both the mean and the variance of \( \tau \) are finite:

\[
\langle \tau \rangle = \beta \delta^{\beta-1}, \quad \sigma^2_\tau = \delta^{\beta-2} \beta(1 - \beta). \tag{9.37}
\]

We should emphasize here that in contrast to the expression (9.18), formula (9.34) implies the exact law of the linear diffusion. Indeed, the main asymptotics of (9.34), for \( u \to 0 \), is

\[
\tilde{\hat{f}}_\delta(u; s) \sim \frac{1}{s} + \hat{w}(u) - \frac{1}{\langle \tau \rangle} s^2 \quad (u \to 0),
\]

so that

\[
g(s) = \frac{D}{s^2} \quad \Rightarrow \quad \langle X^2(t) \rangle \equiv D t.
\]

Remark: Observe that if \( \langle \tau \rangle = \infty \) (as in the case of the \( \beta \)-stable inter-jump time intervals) then the stationary excess time \( \tau^+ \) is infinite with probability 1, since, for each \( t \geq 0 \),

\[
F^+(t) = P(\tau^+ \leq x) = \frac{1}{\langle \tau \rangle} \int_0^x (1 - F(s)) ds = 0.
\]

The same picture emerges from (9.34) as

\[
\tilde{\hat{f}}_\delta(u; s) = \frac{1}{s} \quad \Rightarrow \quad \hat{f}_{\text{st}}(x; t) \equiv \delta(x).
\]
In other words, if $\langle \tau \rangle = \infty$, then the stationary case is trivial with the wandering particle always remaining motionless ($X(t) \equiv 0$).

### 4.10 The Langevin equation

In this section we will discuss statistical properties of stationary solutions of the Langevin equation driven by anomalous random walks and anomalous diffusion processes. Initially, suppose that process $Z(t)$ satisfies the Langevin equation driven by an anomalous random walk, that is

$$\frac{dZ(t)}{dt} + \gamma Z(t) = \sum_{n=1}^{\infty} \xi(n) \delta(t - T(n)),$$

(10.1)

where $\{\xi(n)\}$ are independent Gaussian random variables with zero mean and identical variances $\sigma^2_\xi$, while $\{T(n)\}$ are the random jump times.

In the case of the stationary anomalous random walk discussed in Subsection 9.4, the random quantity

$$Z_{st} = \sum_{n=1}^{\infty} \xi(n) e^{-2\gamma T(n)}$$

(10.2)

has the statistical properties equivalent to the statistical properties of the stationary solution of the stochastic equation (10.1). Such solutions are often called the Ornstein-Uhlenbeck processes. Here $T_n$ are given by equalities (9.30) and, to avoid the trivial case, we assume that $\langle \tau^+ \rangle$ is finite. From (10.2) it is easy to see that the characteristic function of the stationary solution of the Langevin equation (10.1) is of the form

$$\left\langle \exp \left[ -\frac{1}{2} \sigma^2_\xi u^2 \sum_{n=1}^{\infty} e^{-2\gamma T(n)} \right] \right\rangle,$$

(10.3)

where the angled brackets denote averaging over the ensemble of independent random variables

$$\{\tau^+, \tau(2), \ldots, \tau(k) \ldots \}.$$  

(10.4)
In particular, it follows from (10.3) that the mean-square of the stationary solution is

$$\langle Z_{st}^2 \rangle = \sigma^2 \xi^2 \sum_{n=1}^{\infty} \langle e^{-2\gamma T(n)} \rangle.$$  \hfill (10.5)

A calculation of the means on the right-hand side, with help of the p.d.f. $f_{st,n}(t; n)$ in (9.31), gives

$$\langle e^{-2\gamma T(n)} \rangle = \frac{1 - \hat{f}(2\gamma)}{2\gamma \langle \tau \rangle} \hat{f}^n(2\gamma).$$

Substituting these expressions into (10.5) and summing the resulting geometric progression, yields

$$\langle Z_{st}^2 \rangle = \frac{\sigma^2 \xi^2}{2\gamma \langle \tau \rangle}.$$  \hfill (10.6)

In the particular case of the Ornstein-Uhlenbeck process driven by an anomalous random walk with tempered $\beta$-stable inter-jump times, in view of (9.37), we have

$$\langle Z_{st}^2 \rangle = \frac{\sigma^2 \xi^2 \delta^{1-\beta}}{2\gamma \beta}.$$  \hfill (10.8)

In the case of nonstationary anomalous random walk discussed in Subsections 9.1-3 the situation changes qualitatively. The characteristic function of the random variable $Z_{st}$ is then described by by the expression

$$\left\langle \exp \left[ -\frac{1}{2} \sigma^2 \xi^2 \sum_{m=1}^{\infty} e^{-2\gamma T(m)} \right] \right\rangle,$$  \hfill (10.7)

where the random time $T(m)$ is described by the equality given at the beginning of Subsection 9.1, and is equal to the sum of $m$ independent random variables with identical p.d.f. $f(t)$. Proceeding as in (10.5-6), it follows from (10.3) that

$$\langle Z_{st}^2 \rangle = \sigma^2 \xi^2 \frac{\hat{f}(2\gamma)}{1 - \hat{f}(2\gamma)}.$$  \hfill (10.8)

If the mean duration of the inter-jump intervals $\tau(m) = T(m) - T(m-1)$ is finite, then the following relation is true:

$$\hat{f}(s) \sim 1 - \langle \tau \rangle s \quad (s \to 0),$$
while the asymptotic behavior of the expression (10.8), for $\gamma \to 0$, is described by the formula (10.6).

If the asymptotics of the corresponding characteristic function of the interjump time intervals $\tau(m)$ is of the form

$$\hat{f}(s) \sim 1 - \varepsilon s^\beta, \quad (s \to 0, \quad 0 < \beta < 1),$$

then, for $\gamma \to 0$, the formula (10.8) implies the following fractional asymptotics:

$$\langle Z^2_{st} \rangle \sim \frac{\sigma^2}{\varepsilon (2\gamma)^\beta} \quad (0 < \beta < 1).$$

Finally, let us now explore in more detail the limiting case of a continuous infinitely-divisible time, putting

$$\sigma^2 = D \varepsilon$$

and replacing the p.d.f. in (9.3) by its continuous time analogue with the Laplace transform $\hat{f}(s)$, where $\hat{f}(s)$ is the Laplace transform of an infinitely-divisible mother p.d.f. $f(t)$. Letting $\varepsilon \to 0$ in (10.7) we see that the limit characteristic function of the corresponding Ornstein-Uhlenback process is of the form

$$\langle \exp \left[ -\frac{D}{2} u^2 \int_0^\infty e^{-2\gamma T(\theta)} d\theta \right] \rangle.$$

Here, the averaging is over the ensemble of the infinitely-divisible time $T(\theta)$, see (2.1). In particular, it follows from (10.11) that

$$\langle Z^2_{st} \rangle = D \int_0^\infty \langle e^{-2\gamma T(\theta)} \rangle d\theta = D \int_0^\infty \hat{f}(2\gamma) d\theta = \frac{D}{\ln \hat{f}(2\gamma)}.$$  

Substituting here the Laplace transform $\hat{f}_\beta(s)$ of the $\beta$-stable p.d.f. (A7) we return to the formula (10.10):

$$\langle X^2_{st} \rangle = \frac{D}{(2\gamma)^\beta}.$$  

In conclusion we shall calculate the fourth moment of the random variable $Z_{st}$ which is given by the expression

$$\langle Z^4_{st} \rangle = 3D^2 \int_0^\infty d\theta_1 \int_0^\infty d\theta_2 \langle e^{-2\gamma [T(\theta_1) + T(\theta_2)]} \rangle.$$
Recalling that $T(\theta)$ is a process with statistically independent increments, we can rewrite the last equality in the form

$$\langle Z_{st}^4 \rangle = 6D^2 \int_0^\infty d\varrho_1 \hat{f}^{\theta_1}(2\gamma) \int_0^{\theta_2} d\theta_2 \left( \frac{\hat{f}(4\gamma)}{\hat{f}(2\gamma)} \right)^{\theta_2}.$$ 

Evaluating the integrals we finally obtain that

$$\langle Z_{st}^4 \rangle = \frac{6D^2}{\ln \hat{f}(2\gamma) \ln \hat{f}(4\gamma)}.$$

In the case of a $\beta$-stable time, when $\ln \hat{f}(s)$ is given by the equality (A7), the above formula implies that

$$\langle Z_{st}^4 \rangle = \frac{6D^2}{(2\gamma)^\beta \cdot (4\gamma)^\beta}.$$ 

(10.14)

It follows from (10.13-14) that, for $\beta < 1$, the p.d.f. of the stationary solutions of the Langevin equation is not Gaussian. Indeed, the kurtosis excess

$$\kappa = \frac{\langle Z_{st}^4 \rangle - 3\langle Z_{st}^2 \rangle^2}{\langle Z_{st}^2 \rangle^2} = 3 \left[ 2^{1-\beta} - 1 \right] > 0.$$

### 4.11 Conclusions

The Chapter introduces a novel approach to modelling anomalous diffusions in the subdiffusive case. After introduction of the concept of the $\beta$-stable random time and its inverse we construct a random function $\mathcal{X}(t)$ (4.1) whose marginal p.d.f. is an exact solution of the fractional diffusion equation. In the next step the concept of the fractional advection (5.1) is introduced and explored. In particular, its detailed statistical properties are investigated and their relationship to the statistical properties of the model subdiffusion process (6.1) is discussed. We also show that $\beta$-stable p.d.f. $f_\beta(t), 0 < \beta < 1$, see (2.5), plays a key role in the analysis of the anomalous subdiffusion. Analytical properties of this p.d.f. and its probabilistic interpretation are studied. Finally, we introduce related anomalous random walks and discuss statistical properties of the stationary solutions of the Langevin equation driven by anomalous random walks and anomalous diffusions.
Appendix A:
Infinitely divisible and stable distributions

A (generalized) p.d.f. \( f(x) \) is said to be infinitely divisible if, for any \( \theta > 0 \), function \( \tilde{f}(u; \theta) = \text{def} \ f^\theta(u) \), is also the characteristic function of a random variable which we will denote by \( X(\theta) \).

Infinitely divisible p.d.f. \( f(x) \) of a random variable \( X \) can always be represented as the distribution of the sum of an arbitrary number of independent, identically distributed random variables. Indeed, for any \( n = 1, 2, \ldots \),

\[
X = X_1^{(1/n)} + X_2^{(1/n)} + \cdots + X_n^{(1/n)},
\]

where the random variables on the right-hand side are independent and each has the p.d.f. \( f(x; 1/n) \); hence the term infinitely divisible p.d.f.. The notation \( =_d \) signifies equality of the probability distributions of random variables.

The simplest nontrivial infinitely divisible distribution is the Poisson distribution with the p.d.f.

\[
f(x) = \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \delta(x - n),
\]

and the characteristic function

\[
\tilde{f}(u) = \exp \left[ \lambda(e^{iu} - 1) \right]. \tag{A.1}
\]

Parameter \( \lambda \) is assumed to be positive; it is the mean value of the Poissonian random variable. The Poissonian random variable takes only nonnegative integer values with the probability of value \( n \) being \( e^{-\lambda} \lambda^n/n! \).

If \( T \geq 0 \) is a nonnegative random variable, it is often more convenient to use the Laplace transform

\[
\tilde{f}(s) = \langle e^{-st} \rangle = \int_0^\infty e^{-st} f(t) \, dt.
\]

of its p.d.f. \( f(x) \) instead of the, complex-valued in general, characteristic function. The Poisson random variable is nonnegative and substituting \( u = is \) in (A.1) we
obtain the Laplace transform of the Poissonian random variable

\[ \hat{f}(s) = \exp \left[ \lambda(e^{-s} - 1) \right]. \]

Using Poissonian random variables as building blocks one can construct other infinitely divisible distributions which play a key role in the theory of anomalous diffusion. Indeed, let us consider a random variable

\[ X = \sum_m a_m K_m, \]

where \( \{K_m\} \), are independent random integers with the Poisson distributions with parameters \( \{\lambda_m\} \), respectively, while \( \{a_m\} \) are deterministic real-valued coefficients. Obviously, the characteristic function of \( X \) is equal to

\[ \hat{f}(u) = \prod_m \exp \left[ \lambda_m(e^{iua_m} - 1) \right] = \exp \left[ \sum_m \lambda_m (e^{iua_m} - 1) \right]. \]

Note that random variable \( X \), a mixture of the rescaled Poisson random variables, can take as its values arbitrary integer multiplicities of the constants \( a_n \).

Taking a continuum limit of the above mixtures of Poissonian distributions by putting

\[ a_m = m \Delta, \quad \lambda_m = \psi(m\Delta) \Delta, \]

for a given nonnegative function \( \psi(u) \), usually called the intensity or rate function, and letting \( \Delta \to 0 \), we arrive at an infinitely divisible random variable with the characteristic function of the form

\[ \hat{f}(u) = e^{\Psi(u)}, \quad \text{where} \quad \Psi(u) = \int \psi(z)(e^{iz} - 1) \, dz, \quad (A.2) \]

as long as the above integral is well defined for a given intensity function \( \psi(u) \). This \( \hat{f}(u) \) is the characteristic function of the random variable

\[ X =_d \lim_{\Delta \to 0} \Delta \sum_m m K_m, \quad (A.3) \]
which we will call the generalized (or compound) Poissonian random variable with the intensity function \( \psi(u) \).

If, for \( u < 0 \), the intensity function \( \psi(u) = 0 \), then the corresponding generalized Poissonian random variable \( T \) is nonnegative and, in analogy with (A.2), its Laplace transform is

\[
\hat{f}(s) = e^{\Phi(s)}, \quad \text{where} \quad \Phi(s) = \Psi(is) = \int_{0}^{\infty} \psi(z)(e^{-s} - 1) \, dz. \tag{A.4}
\]

In the particular case of the intensity function

\[
\psi(z) = \frac{\beta}{\Gamma(1 - \beta)} z^{-\beta-1} e^{-\delta z} \chi(z),
\]

where \( \chi(x) \) is the unit step function, and \( 0 < \beta < 1 \), the integral \( \Phi(s) \) in (A4) exists and can be explicitly evaluated with the help of the formula

\[
\int_{0}^{\infty} \frac{e^{-az} - e^{-bz}}{z^{\gamma+1}} \, dz = \Gamma(-\gamma) \, (a^\gamma - b^\gamma) \quad (\gamma < 1) \tag{A.5}
\]

to give

\[
\hat{f}_\beta(s; \delta) = \exp \left( \delta^\beta - (s + \delta)^\beta \right) \quad (0 < \beta < 1). \tag{A.6}
\]
an example of the Laplace transform of an infinitely divisible p.d.f. \( f_\beta(s; \delta) \). The p.d.f. described by the Laplace transform (A.6) is called the tempered \( \beta \)-stable p.d.f.

Recall that a p.d.f. \( f(x) \) is called strictly stable if the p.d.f. \( f_n(x) \) of the sum of \( n \) independent random variables, each with p.d.f. \( f(x) \), is of the form

\[
f_n(x) = \frac{1}{c_n} f \left( \frac{x}{c_n} \right),
\]

for a certain sequence of constants \( c_n \). In the language of the characteristic functions and the Laplace transforms the above condition can be written as follows:

\[
\hat{f}^n(u) = \hat{f}(c_n u), \quad \hat{f}^n(s) = \hat{f}(c_n s).
\]

It is clear that, for \( \delta = 0 \), formula (A.6) produces the Laplace transform

\[
\hat{f}_\beta(s) = e^{-s^\beta}, \quad (0 < \beta < 1) \tag{A.7}
\]
of a strictly stable distribution \( f_\beta(t) \). A random variable \( T \) with the Laplace transform (A.7) has an infinite mean. Relying on the general Tauberian theorems describing the relationship between asymptotics of p.d.f.s and their Laplace transforms on can show that

\[
\hat{f}_\beta(s) \sim 1 - s^\beta \quad (s \to 0) \quad \Leftrightarrow \quad f_\beta(t) \sim \frac{\beta}{\Gamma(1 - \beta)} t^{-\beta - 1} \quad (t \to \infty)
\]

\((0 < \beta < 1)\). (A.8)

The strictly stable p.d.f. \( f_\beta(u) \) corresponding to the Laplace transform (A.7) is of the form

\[
f_\beta(t) = \frac{1}{\pi} \int_0^\infty \exp \left[ -\cos \left( \frac{\pi \beta}{2} \right) u^\beta \right] \cos \left[ ut - u^\beta \sin \left( \frac{\pi \beta}{2} \right) \right] du . \tag{A.9}
\]

Since the absolute value of the integrand in (A.9) converges to zero exponentially as \( |u| \to \infty \), the p.d.f. \( f_\beta(t) \) is infinitely differentiable. Graphs of the stable p.d.f. \( f_\beta(t) \), for several values of parameter \( \beta \), are given in Fig.A.1.

In general, the integral in (A.9) cannot be evaluated in a closed form. However, for \( \beta = 1/2 \), and \( \beta = 1/3 \), this can be done:

\[
f_{1/2}(t) = \frac{1}{2t\sqrt{\pi t}} \exp \left( -\frac{1}{4t} \right) , \quad \text{and} \quad f_{1/3}(t) = \frac{1}{t^3 \sqrt{3t}} \text{Ai} \left( \frac{1}{\sqrt{3t}} \right) , \tag{A.10}
\]

where \( \text{Ai}(x) \) is the so-called Airy function.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figa1.png}
\caption{Plots of the stable p.d.f.s \( f_\beta(t) \), for different values of \( \beta \).}
\end{figure}
The characteristic function of a symmetric $\alpha$-stable random variable is of the form

$$\Theta_\alpha(u) = e^{-|u|^\alpha}, \quad 0 < \alpha < 2,$$

and its p.d.f. is given by the integral formula

$$f_\alpha(x) = \frac{1}{\pi} \int_0^\infty e^{-u^\alpha \cos(ux)} \, du.$$  

As in the case of (A.8), the power asymptotics of the characteristic function (A.11),

$$\Theta_\alpha(u) \sim 1 - |u|^\alpha, \quad u \to 0,$$

implies the power law of decay of the p.d.f. itself:

$$f_\alpha(x) \sim \frac{1}{\pi} \Gamma(\alpha + 1) \sin \left(\frac{\pi \alpha}{2}\right) |x|^{-\alpha - 1}, \quad (|x| \to \infty).$$  

(A.12)

Hence, the variances of all $\alpha$-stable random variables, $0 < \alpha < 2$, are infinite.

On the other hand, the exponential decay of the characteristic function (A.11), for $u \to \infty$, makes it possible to expand the p.d.f. in the Taylor series

$$f_\alpha(x) = \frac{1}{\pi \alpha} \sum_{n=0}^\infty (-1)^n \frac{\Gamma\left(\frac{2n+1}{\alpha}\right)}{(2n)!} x^{2n}.$$  

Plots of the symmetric stable p.d.f.s $f_\alpha(x)$, for different values of $\alpha$, are shown in Fig. A.2.

Although the second moment of a symmetric $\alpha$-stable distribution is infinite, moments of order $\kappa$ less than $\alpha$ are finite. We shall calculate them relying on the Parseval equality

$$\langle g(X) \rangle_\alpha = \int g(x) f_\alpha(x) \, dx = \frac{1}{2\pi} \int \tilde{g}(u) e^{-|u|^\alpha} \, du,$$

(A.13)

with $g(x) = |x|^\kappa$ and $\tilde{g}(u) = \int g(x) e^{iux} \, dx$ being the generalized Fourier transform. For $u \neq 0$, the values of the latter can be found by the summation of the divergent Fourier integral (see, for instance, Saichev and Woyczynski (1997), Chapter 8):

$$\tilde{g}(u) = -\frac{2\Gamma(1+\kappa)}{|u|^\kappa} \sin \left(\frac{\pi \kappa}{2}\right), \quad (u \neq 0).$$
Substituting this expression in (A.13), we obtain the divergent integral

\[
\int |x|^\kappa f(x) \, dx = -\frac{2\Gamma(1+\kappa)}{\pi} \sin \left( \frac{\pi \kappa}{2} \right) \int_0^\infty e^{-u^\alpha} \frac{du}{u^{\kappa+1}}.
\]  

(A.14)

**FIGURE A.2**

Plots of the symmetric \(\alpha\)-stable p.d.f.s for different values of \(\alpha\).

However, since

\[
\int |x|^\kappa f(x) \, dx = \int |x|^\kappa [f(x) - \delta(x)] \, dx, \quad \kappa > 0,
\]

the right-hand side of the equality (A.14) can be replaced by the regularized integral

\[
\int |x|^\kappa f(x) \, dx = -\frac{2\Gamma(1+\kappa)}{\pi} \sin \left( \frac{\pi \kappa}{2} \right) \int_0^\infty e^{-u^\alpha} \frac{1}{u^{\kappa+1}} \, du.
\]

(A.15)

Finally, using the formula (A.5), we obtain

\[
\langle |X|^\kappa \rangle_\alpha = \frac{2}{\pi} \Gamma(\kappa) \Gamma \left( 1 - \frac{\kappa}{\alpha} \right) \sin \left( \frac{\pi \kappa}{2} \right) \quad (\kappa < \alpha).
\]

(A.16)

We will conclude this appendix by constructing a symmetric tempered 1/2-stable random variable, that is a generalized Poissonian random variable which for "small
values” behaves like the symmetric $\alpha$-stable random variable with $\alpha = 1/2$, but which has a finite variance. For this purpose, substitute into (A.2) the intensity function

$$
\psi(z) = \frac{e^{-|z|}}{2\sqrt{2\pi}|z|\sqrt{|z|}},
$$

to obtain the logarithm of the characteristic function

$$
\Psi(u) = \sqrt{2\delta} - \sqrt{\delta + \sqrt{\delta^2 + u^2}}.
$$

(A17)

For $u \to 0$, its main asymptotics is as follows:

$$
\Psi(u) \sim -\frac{1}{4\sqrt{2}\delta^{3/2}} u^2, \quad (u \to 0).
$$

Thus, the variance of the corresponding tempered $1/2$-stable random variable $X$ can be explicitly calculated:

$$
\sigma^2_X = \frac{1}{\sqrt{8}} \delta^{-3/2} \simeq 0.354 \delta^{-3/2} < \infty.
$$

(A18)

On the other hand, for $u \to \infty$, the asymptotics of $\Psi(u)$ in (A17) is as follows:

$$
\Psi(u) \sim -\sqrt{|u|}, \quad (u \to \infty),
$$

which corresponds to the asymptotics of the $\alpha$-stable p.d.f. with index $\alpha = 1/2$.

**FIGURE A.3**

A log-log plot of the p.d.f. of the tempered $1/2$-stable random variable.
(A19) (continuous line) compared to the asymptotics of the 1/2-stable p.d.f. $f_{1/2}(x)$ (dashed line).

A log-log plot of the p.d.f. $f(x)$ with the characteristic function

$$\tilde{f}(u) = \exp \left( \sqrt{2\delta} - \sqrt{\delta + \sqrt{\delta^2 + u^2}} \right)$$

and $\delta = 10^{-3}$, is shown in Fig. A.3. The upper, dashed straight line corresponds to the asymptotic formula (A.12):

$$f_{1/2}(x) \sim \frac{1}{2\sqrt{2\pi}} |x|^{-3/2}, \quad (|x| \to \infty).$$

One can see that almost up to the value $x \approx 1/\delta$ the p.d.f. $f(x)$ remains close to the asymptotics of the stable p.d.f. Afterwards it quickly goes to zero to assure the finiteness of the variance (A.18).

**Appendix B:**

**Lévy’s infinitely-divisible processes**

Let $\tilde{f}(u)$ be the characteristic function of an infinitely divisible p.d.f. In the context of infinitely divisible processes we shall call it the *mother characteristic function*. Consider the stochastic process

$$X(t) = \lim_{\Delta t \to 0} \sum_{k=1}^{[t/\Delta t]} X_k^{(\gamma \Delta t)},$$

where the summands $X_k^{(\gamma \Delta t)}$ are statistically independent and have the same characteristic function $\tilde{f}^{\gamma \Delta t}(u)$. The characteristic function of process the $X(t)$ is

$$\tilde{f}(u; t) = \tilde{f}^{\gamma t}(u) = \exp[\gamma t \Psi(u)],$$

where $\Psi(u)$ is the logarithm of the mother characteristic function. Applying the Laplace transformation to (B.2) we obtain the following expression for the Laplace-Fourier transform of the marginal p.d.f. $f(x; t)$ of the process $X(t)$:

$$\hat{f}(u; s) = \int_0^\infty dt e^{-st} \int dx e^{iux} f(x; t) = \frac{e^{iux_0}}{s - \gamma \Psi(u)}$$

(B.3)
The equality (B.3) permits a direct derivation of an equation for the p.d.f. \( f(x; t) \). Indeed, rewrite (B.3) in the form

\[
\hat{s} \hat{f} - \gamma \hat{\Psi}(u) \hat{f} = 1
\]  

(B.4)

and assume that

\[ f(x; t = 0) \equiv 0. \]

Then the inverse Laplace-Fourier transform of the first summand on the left-hand side of the equality (B4) is the derivative:

\[
s \hat{f}(u; s) \mapsto \frac{\partial f(x; t)}{\partial t}.
\]

In what follows we will encounter expression of the form \( s^\beta \hat{f} \) to which we will formally attach the fractional derivatives

\[
s^\beta \mapsto \frac{\partial^\beta}{\partial t^\beta}.
\]

In the spatial domain one can consider so-called Weyl multiplier operators corresponding to multiplication of the Fourier transform by \( \hat{\Psi}(u) \). Let \( \hat{f}(u) \) be the characteristic function of a symmetric \( \alpha \)-stable random variable,

\[
\hat{f}(u) = \exp(\Psi(u)), \quad \text{where} \quad \Psi(u) = -\sigma^{\alpha} |u|^\alpha.
\]  

(B.5)

**Definition:** The operator in the \( x \)-space corresponding to multiplication by a fractional power of \(-iu\) in the Fourier space will be called the right fractional derivative (in the sense of Weyl). In other words,

\[
(-iu)^\alpha \mapsto \frac{\partial^\alpha}{\partial x^\alpha},
\]

or, more precisely,

\[
\left[ \frac{\partial^\alpha}{\partial x^\alpha} g(x) \right](u) := (-iu)^\alpha \hat{g}(u).
\]
The analogous operator corresponding to multiplication by a fractional power of $iu$ will be called the left fractional derivative:

$$(iu)^\alpha \mapsto \frac{\partial^\alpha}{\partial(-x)^\alpha}.$$ 

The logarithm of the mother characteristic function in (B.5) is a power of the absolute value of $u$ and the operator in the $x$-space corresponding to this multiplier will be called the symmetric fractional derivative:

$$|u|^\alpha \mapsto -\frac{\partial^\alpha}{\partial|x|^\alpha}.$$ 

Relations between these fractional differentiation operators can be easily established relying on algebraic identities

$$(\pm iu)^\alpha = |u|^\alpha \cos\left(\frac{\pi \alpha}{2}\right) \pm i s |u|^\alpha \sin\left(\frac{\pi \alpha}{2}\right), \quad s = \frac{u}{|u|},$$

and

$$|u|^\alpha = \frac{(-iu)^\alpha + (iu)^\alpha}{2 \cos(\pi \alpha/2)}.$$ 

Consequently,

$$\frac{\partial^\alpha}{\partial|x|^\alpha} = \frac{-1}{2 \cos(\pi \alpha/2)} \left[ \frac{\partial^\alpha}{\partial x^\alpha} + \frac{\partial^\alpha}{\partial(-x)^\alpha} \right].$$

If the mother characteristic function of the summands of the infinitely-divisible process (B.1) is equal to (B.5), then the Laplace-Fourier transform $\hat{f}(u; s)$ of the p.d.f. $f(x; t)$ of the infinitely-divisible process $X(t)$ satisfies equation

$$s \hat{\Theta} + \gamma \sigma^\alpha |u|^\alpha \hat{\Theta} = 1.$$ 

Applying the inverse Laplace and Fourier transformations to both sides of the above equation we get the following equation in fractional partial derivatives for $f(x; t)$:

$$\frac{\partial f}{\partial t} = \gamma \sigma^\alpha \frac{\partial^\alpha f}{\partial|x|^\alpha} + \delta(x) \delta(t). \quad (B.6)$$
This equation can be solved by finding the inverse Fourier transform of the corresponding characteristic function

\[ \hat{f}(u; t) = \exp \left( -\gamma \sigma^\alpha |u|^\alpha t \right). \]

As a result we obtain the p.d.f. of the so-called Lévy flights or \( \alpha \)-stable Lévy processes:

\[ f_\alpha(x; \tau) = \frac{1}{\sigma \tau^{1/\alpha}} f_\alpha \left( \frac{x}{\sigma \tau^{1/\alpha}} \right), \quad \tau = \gamma t. \tag{B.7} \]

As we already observed, the mean square of the Lévy flights is infinite but moments of order \( \kappa < \alpha \) are finite:

\[ \langle |X(\tau)|^\kappa \rangle = \langle |X|^\kappa \rangle \tau^{\kappa/\alpha}, \quad -1 < \kappa < \alpha, \tag{B.8} \]

where \( \langle |X|^\kappa \rangle \) is given by the expression (A.16).

Observe that Lévy flights can describe the intermediate asymptotics of stochastic processes with finite variance. To explain what we mean by this statement let us consider the following example of a tempered Lévy flight: Take

\[ \psi(z) = -\frac{|z|^{-\alpha-1}}{2\Gamma(-\alpha) \cos \left( \frac{\pi \alpha}{2} \right)} e^{-\delta |z|} \]

as the intensity function in the generalized Poissonian random variable (A.3). Substituting this intensity function into (A.2) we obtain the logarithm of the characteristic function corresponding to the tempered \( \alpha \)-stable p.d.f.:

\[ \Psi(u, \alpha, \delta) = \frac{1}{\cos \left( \frac{\pi \alpha}{2} \right)} \left[ \delta^\alpha - (\delta^2 + u^2)^{\alpha/2} \cos \left( \alpha \arctan \left( \frac{|u|}{\delta} \right) \right) \right]. \tag{B.9} \]

In particular, for \( \alpha = 1 \), we obtain

\[ \Psi(u, \alpha = 1, \delta) = -\frac{2}{\pi} \arctan \left( \frac{|u|}{\delta} \right) |u| + \frac{\delta}{\pi} \ln \left( 1 + \frac{u^2}{\delta^2} \right), \]

while, for \( \alpha = 1/2 \), we get the expression (A.17).
For \( u \to 0 \), the main asymptotics of \( \Psi(u, \alpha, \delta) \) in (B.9) is

\[
\Psi(u, \alpha, \delta) \sim -\frac{\alpha(1 - \alpha)}{2 \cos(\pi \alpha/2) \delta^{2-\alpha}} u^2, \quad (u \to 0).
\]

Consequently, in this case, the Lévy infinitely-divisible process with the characteristic function

\[
\tilde{f}(u; \tau; \alpha; \delta) = \exp(\tau \Psi(u, \alpha, \delta))
\]

has a finite mean square which grows linearly; indeed, for any \( \tau > 0 \), and small \( \delta > 0 \),

\[
\langle X(\tau) \rangle \approx \frac{\alpha(1 - \alpha)}{\cos(\pi \alpha/2) \delta^{2-\alpha}} \tau
\]  

(B10)

**FIGURE B.1**

Dependence of the moment \( \langle |X(\tau)|^{1/4} \rangle \) on \( \tau \) for the tempered Lévy flight corresponding to \( \alpha = 1/2 \). Dashed lines correspond to the law \( \tau^{1/2} \), characteristic of the Lévy flight, and \( \tau^{1/8} \), characteristic of the Wiener process. One can observe that, initially, the moment of the order 1/4 grows like that of the Lévy flight.

On the other hand, the tails of the p.d.f. of \( X(t) \) have the power asymptotics (A.12), so that the moments \( \langle |X(\tau)|^{\kappa} \rangle \) of order \( \kappa < \alpha \) depend on the time in the
way characteristic for Lévy flights. This behavior is clear from Fig. B.1 which shows dependence of $\langle |X(\tau)|^{1/4} \rangle$ on $\tau$ obtained, as in (A.15), by numerical integration of the formula

$$
\langle |X(\tau)|^{\kappa} \rangle = -\frac{2\kappa \Gamma(\kappa)}{\pi} \sin \left( \frac{\pi \kappa}{2} \right) \int_0^\infty \tilde{f}(u; \tau; \alpha; \delta) - 1 \frac{1}{u^{\kappa+1}} du.
$$

For a comprehensive analysis of subtle properties of the general tempered stable processes, which were first investigated explicitly in Mantegna and Stanley (1995), see Rosinski (2002).

**Appendix C:**

**Fractional exponential distribution**

In addition to the $\beta$-stable p.d.f. $f_\beta(\tau)$ discussed in Appendix A, the one-sided p.d.f. $\varphi_\beta(\tau)$, with an infinite mean, and the Laplace transform

$$
\hat{\varphi}(s) = \frac{1}{1 + s^\beta} \quad (0 < \beta < 1),
$$

plays an essential role in the study of anomalous diffusion. We will call it *fractional exponential distribution*. Let us explore its properties.

First, notice that the equality (C.10) is equivalent to the following fractional differential equation:

$$
\frac{d^\beta \varphi}{dt^\beta} + \varphi = \delta(t).
$$

The solution of this equation is

$$
\varphi_\beta(\tau) = -\frac{1}{\tau} D_\beta(-\tau^\beta) \chi(\tau), \quad \text{where} \quad D_\beta(z) = \beta z \frac{dE_\beta(z)}{dz},
$$

and $E_\beta(z)$ is the Mittag-Leffler function. It is easy to see that, as in (2.12), the function $D_\beta(z)$ is expressed by a contour integral

$$
D_\beta(z) = \frac{z}{2\pi i} \int_{\mathcal{H}} \frac{e^y \, dy}{y^\beta - z},
$$
where the integration is carried out along the Hankel loop depicted in Fig. ???. For the negative real values of its argument, it is easy to obtain the following integral representation for the fractional exponential p.d.f.:

\[ \varphi_{\beta}(\tau) = \frac{\sin(\pi \beta)}{\pi} \tau^{\beta-1} \int_0^\infty \frac{x^\beta e^{-x} \, dx}{x^{2\beta} + \tau^{2\beta} + 2x^\beta \tau^\beta \cos(\pi \beta)}, \quad 0 < \beta < 1. \] (C.11)

The above formula yields the following asymptotics

\[ \varphi_{\beta}(\tau) \sim \frac{\tau^{\beta-1}}{\Gamma(\beta)} \quad (\tau \to 0); \quad \varphi_{\beta}(\tau) \sim \frac{\tau^{-\beta-1}}{\Gamma(1 - \beta)} \quad (\tau \to \infty). \] (C.12)

Fractional exponential p.d.f.s are sometimes written as a mixture of standard exponential distributions. To find this mixture let us return to the integral (C.11) and introduce a new variable of integration \( \mu = \tau/x \). As a result, (C.11) assumes the form

\[ \varphi_{\beta}(\tau) = \int_0^\infty \frac{1}{\mu} \exp \left( -\frac{\tau}{\mu} \right) \xi_{\beta}(\mu) \, d\mu, \] (C.13)

where

\[ \xi_{\beta}(\mu) = \frac{1}{\pi \mu} \frac{\sin(\pi \beta)}{\mu^{\beta} + \mu^{-\beta} + 2 \cos(\pi \beta)}. \] (C.14)

The latter function can be thought of as the spectrum of the mean values of the standard exponential p.d.f.

\[ \varphi(\tau) = \frac{1}{\mu} \exp \left( -\frac{\tau}{\mu} \right). \]

For \( \beta \to 1 \), the spectrum (C.14) of the random times weakly converges to the Dirac-delta, while \( \varphi_{\beta}(t) \) approaches the standard exponential p.d.f.
FIGURE C.1
Plots of the p.d.f. $\varphi_\beta(\tau)$, for $\beta = 1/2$, in the linear (left) and the log-log (right) scales. The latter clearly shows the power asymptotics (C.12) of the fractional exponential p.d.f.

Finally, observe that, for $\beta = 1/2$, the fractional exponential p.d.f. can be explicitly written in terms of the complementary error function:

$$\varphi_{1/2}(\tau) = \sqrt{\frac{\pi}{\tau}} - \pi e^\tau \text{erfc} (\sqrt{\tau}).$$

A plot of the above p.d.f. is given in Fig. C.1.

Bibliography


