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Magnetic field issues in magnetic resonance imaging

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Case Western Reserve University, 1993
MAGNETIC FIELD ISSUES IN
MAGNETIC RESONANCE IMAGING

by
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Submitted in partial fulfillment of the requirements
for the Degree of Doctor of Philosophy

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*We also certify that written approval has been obtained for any proprietary material contained therein.
MAGNETIC FIELD ISSUES IN MAGNETIC RESONANCE IMAGING

Abstract
by
LABROS S. PETROPOULOS

Advances in Magnetic Resonance Imaging depend on the capability of the available hardware. More homogenous main magnets and faster gradient coils will help reduce field inhomogeneity effects, will help us acquire the data faster and will improve the quality of the images. Using variational methods to minimize the total energy of the system subject to a specified set of constraints for different geometries, we obtain the optimal inductance for the coils, and thus, we can generate high strength gradient coils with faster rise times. Specifically, for the main magnet configuration, using derivative constraints, we can create a static magnetic field with reduced levels of inhomogeneity over a prescribed imaging volume. In the gradient coil, the entire design for the axial elliptical coil, and the mathematical foundation for the transverse elliptical coil have been presented. Also, there is a growing interest for designing gradient coils with reduced length. The design of a self-shielded cylindrical gradient coil with a restricted length has been presented. In order to generate gradient coils adequate for head imaging without including the human shoulders in the design, asymmetric cylindrical coils where the gradient center is shifted axially towards the end of a finite cylinder have been introduced and theoretical as well as experimental results have been presented. In order to eliminate eddy current effects in the design of the non-shielded asymmetric gradient coils, the self-shielded asymmetric cylindrical gradient coil geometry has been introduced. Apart from the elimination of eddy current effects, the potential reduction to the total torque of the transverse coil system can be up to 75%, depending on the radii of the inner and outer coils. Continuing the development of novel geometries for the gradient coils, the complete set of self-shielded cylindrical gradient coils, which are designed such that the \( z \) component of the magnetic field varies linearly along the three traditional gradient axes, has been presented. This particular set of gradient coils is important when imaging of the human wrist is performed.
In order to understand the behavior of the rf field inside a dielectric object, a mathematical model is briefly presented. We have explained why the rf magnetic field is almost linear inside the human body at the frequency of 64 MHz. Heating effects are also present when the rf field penetrates the dielectric object. Depending on the applied imaging sequence and the values of conductivity $\sigma$ and permittivity $\varepsilon$ of the dielectric object, the heating effects can be significant. Heating experiments on a head-like phantom using multi-echo spin echo sequence have been performed, and agreement with a mathematical model has been confirmed. Since the behavior of the rf field inside a dielectric object is indicative of the values of $\sigma$ and $\varepsilon$ of this object, we were able to use this information and extract $\sigma$ and $\varepsilon$ using the rf profile. Specifically, taking advantage of the relationship between the magnetization and the amplitude of the rf field for small tip angles, the values of $\sigma$ and $\varepsilon$ were extracted \textit{in vitro}. Furthermore, using least square methods in a multilayer planar theoretical model, we were able to extract the values of $\sigma$ and $\varepsilon$ for leg and head-like models. Good agreement between the prescribed and the extracted values of $\sigma$ and $\varepsilon$ was found, with the noise levels up to 3% of the rf profile's amplitude.

Although specific methods can provide an indication of the rf behavior inside a loosely dielectric object, finite element methodology is the ultimate approach for modeling the human torso and generating an accurate picture for the shape of the rf field inside this dielectric object. For this purpose we have developed a 3D finite element model, using the Coulomb gauge condition as a constraint. Agreement with the heterogeneous multilayer planar model has been established, while agreement with theoretical results from the spherical model and experimental results from the cylindrical model at 170 MHz is very good and provides an encouraging sign for using this finite element approach for modeling the rf inside the human body.
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Chapter 1

Introduction

During the past few years, in the field of Magnetic Resonance Imaging (MRI) there has been a continuing thrust to develop new techniques to obtain better images and reduce the imaging time. Diffusion/perfusion imaging, echo-planar imaging, as well as imaging of the heart, neck and lower parts of the brain are the frontiers in research in the field of MRI. These areas of imaging require localized high-strength low-inductance gradient coils with improved linearity and uniformity over a prescribed “diameter spherical volume” (DSV). Furthermore, the application of the radiofrequency (rf) pulse from a transmitter coil to a human body (dielectric object) can induce heat to areas with high conductivity. The presence of conductivity $\sigma$ and permit-
tivity \( \varepsilon \) distorts the rf field profile inside the dielectric material and creates
artifacts in the image. These artifacts can be eliminated with the design of
improved rf pulses that account for these effects, generating a nearly flat rf
profile [13].

There are three focuses of this thesis: 1) the design of actively shielded
magnets with improved homogeneity. 2) the development of novel gradient
coil geometries which are compatible with new localized and fast MRI se-
quENCES and 3) the investigation of the rf penetration and power deposition
inside the human body at high fields. Particularly, we present the mathemat-
ical analysis and the design for solenoidal type magnets. Both infinite and
finite length, minimum inductance gradient coils combined with improved
linearity and uniformity over a specific imaging volume are considered. Fur-
thermore, we present the analysis of the distortion of the rf magnetic field
profile and its effects inside a dielectric heterogeneous object, and the de-
velopment of the 3D finite element approach which will help in the detailed
description of the rf behavior inside an arbitrarily shaped heterogeneous ob-
ject.
The main body of this dissertation is divided into three major parts. The first is related to main magnet design using derivative constraints, the second corresponds to infinite and finite-length low-inductance gradient coils of novel geometries and the third describes the rf effects inside a heterogeneous dielectric object.

Chapter 2 of this thesis discusses the basic concepts and principles of MRI and the role of main magnets, gradient coils and rf in the formation of an image.

Following the presentation for the general concepts of MRI, we extend our discussion to the design of coils which produce various configurations of magnetic fields. In chapter 3, we present the basic methodology of generating magnetic fields which are constant inside the imaging volume or varying linearly along one of the three dimensions. Specifically, we review the Helmholtz coil pair design, a prototype configuration for main magnet design, and the Maxwell pair coil design, a prototype for the design of gradient coils.

Chapter 4 gives a detailed description of the method for designing main magnets and gradient coils. The theory is based on an approach first intro-
duced by Turner [13],[1] for the design of magnetic coils that produce a static magnetic field, varying linearly in space. This method can also be referred to as an “inverse approach”: imposing magnetic field constraints on cylindrical or planar geometries in order to generate the current patterns which are necessary for the desired field specifications.

In chapter 3, we extend Turner’s method to design main solenoid magnets using derivative constraints where the higher-order even-order derivatives are set small. In this chapter, we derive the expression for the magnetic field as well as the current density and the stored magnetic energy based on the continuous current distributions.

Cylindrical and planar geometries are commonly used for the design of main magnets and gradient coils. We begin the second part of the dissertation by introducing elliptical geometry for the design of novel gradient coils. A new set of functions are used for the analysis for the design of the complete set of gradient coils. A brief description has been previously introduced (Petropoulos [36, 37]). In chapter 6, we present the entire analysis and the results for the whole set of gradient coils and the creation of the
entire computer library for Mathieu functions of every kind. Section 6.3 contains the theoretical background for the design of the transverse elliptical coil. Since the elliptical geometry does not possess any symmetry in the transverse direction, we must analyze both the $x$ and $y$ transverse elliptical coils separately.

Up to this point all the different geometrical shapes previously introduced correspond to an unconstrained length for the coils. In section 7.2, we present a detailed description for the finite-size self-shielded, symmetric, axial, cylindrical gradient coil. In section 7.3, we concentrate on the theory and results for the actively shielded transverse cylindrical coils where only the length of the inner cylinder is constrained.

MRI researchers have focused their attention on the design of faster sequences that require faster gradient rise times and better slew rates. The larger dimensions of the whole-body gradient coils corresponds to higher inductance and limits the echo times of the rf sequences. The introduction of smaller and shorter gradient coils helps the development of new sequences in the areas of cardiac, pulmonary, and echo planar imaging and the progres-
sion of diffusion/perfusion studies in the brain. In order to perform imaging to the human head, we must consider a gradient coil with dimensions wide enough to include the shoulders of the person. In chapter 8, we introduce the concept of the asymmetric cylindrical gradient coil where the center of the gradient field, associated with the finite size of the coil, such that we exclude the human shoulders while the quality of the field remains the same. The entire theoretical analysis of the axial gradient coil is presented in section 8.2. Section 8.2.3 contains the results which are produced from the previous design. Furthermore, sections 8.3-8.3.3, contain the theoretical background and the results for the transverse asymmetric and finite cylindrical gradient coil.

The reduction of the eddy current effects caused by the magnet shield and generated from the gradient coils is possible with the introduction of a second cylinder outside the main gradient coil (inner cylinder). The presence of the second cylinder reduces the magnetic field close to the magnet shield by five orders of magnitude relative to its original value. Hence, this eliminates the eddy currents. This system of gradient coils is often called "self-shielded" or "actively shielded" coils. In chapter 9, we present the theoretical background
and the results of the "self-shielded" asymmetric coil set (including axial and transverse coil) where the inner cylinder is constrained to a finite length.

All the prior designs for the gradient coils referred to an axial component of the magnetic field that varies linearly along each of the three cartesian axes, producing the set of three gradient coils. The following chapter of this dissertation presents a different gradient coil design where the magnetic gradient field is directed along either one of the two transverse axes (in this thesis we choose the \( x \) direction) and is free to vary linearly along all the three cartesian axes. Because of the uniqueness for the design of each gradient coil, we will present each coil design separately. Up to this point we have presented a variety of novel gradient coils which share a common property. The component of the magnetic field which is responsible to generate the gradient field the gradient field is chosen to coincide with the direction of the magnetic field of the main magnet. For some specific applications such as the imaging of the human wrist, specific coils are required. In section 10.2, we derive the formulas and illustrate the results for a "self-shielded" cylindrical gradient coil with the length of the cylinder restricted and the magnetic
field in the $x$ direction changes linearly along the $x$ direction (the main magnet direction). In section 10.3, we introduce the theoretical development and generate the results for a finite cylindrical coil where the magnetic field in the $x$ direction varies linearly along the $y$ direction. In the final section of the second part of the thesis: we present the theoretical development and the results for a finite cylindrical coil where the $x$ component of the magnetic field varies linearly along the $z$ direction of the cylinder.

Chapter 11 is the beginning of the third part of the thesis. It introduces the concepts of the rf penetration and power deposition and answers the question of why the rf magnetic field profile for the human body is nearly flat and thus makes possible the application of MRI at high fields.

Besides its traditional properties, MRI data contain additional information that can assist us in determining the values of the dielectric properties ($\sigma$ and $\epsilon$) of a dielectric object. Since the shape of the rf field profile is strongly dependent on the values of conductivity $\sigma$ and permittivity $\epsilon$, it is possible to work backwards and take advantage of the information of the collecting data to extract the values of $\sigma$ and $\epsilon$ for the given object. In chapter 12,
we describe the theoretical background, state the experimental procedure
and present the results for extracting the values of $\sigma$ and $\varepsilon$ \textit{in vitro}. Additional computer simulations for complicated heterogeneous models are also presented.

In general, the shape of the human body does not possess any particular symmetry. A finite element approach can better describe the behavior of the rf field inside an arbitrary shaped 3D heterogeneous object. In chapter 13, we introduce the concept of the finite element method, starting with the 1D finite element approach and concentrating on the 3D finite element method. The derivation of the relevant equations as well as computer simulations and comparisons between the finite element and the soluble heterogeneous model are also presented.

The final chapter of this dissertation, states the conclusions drawn from the preceding research and discusses the potential future directions which are related to this dissertation.
Chapter 2

Background of MRI

Over the last two decades, Magnetic Resonance Imaging has emerged as a noninvasive tool for detection of tissue abnormalities in the field of radiological science. The core of MRI relies on the fact that the protons in the nucleus of an atom have spin and an associated magnetization. The application of a strong magnetic field can alter the spin state of the atom. The application of an additional weak rf field perpendicular to the strong magnetic field tips the spins to a different plane and can generate a signal characteristic for the individual atom.

The existence of the magnetic moment \( \mu \) for some nuclei was demonstrated first by the Stern-Gerlach experiment [38, 39], where a beam consist-
ing of silver atoms passed through a gradient magnetic field applied perpendicular to the direction of the magnetic field. The gradient of the field exerts a force on the atoms and causes a displacement of the beam by an amount that is proportional to the average magnetization of the system.

In 1946 Bloch et al. [10], by applying a continuous rf field to a sample in the presence of a strong magnetic field, was able to induce torque on the spins, and orient them in a plane perpendicular to their initial equilibrium position. Once the rf is turned off, the precessing magnetization induced a signal (EMF) to a pick-up coil. Following Bloch’s observation, Hahn in his 1950 article [11], modified Bloch’s approach with the application of a strong rf pulse for a short time interval. Turning off the rf pulse the system precesses freely towards its equilibrium position. This precession changes the amount of the net magnetization in the plane perpendicular to the axis of rotation and thus alters the flux that passes through a pick-up coil which is designed to respond to a frequency characteristic to a specific kind of nuclei. With the application of the rf, the reconstructed image was a single dot, since all the spins in the object were forced to rotate with the same frequency. Although
the basic theory of MRI was already known, no significant advances, relative to the field of medicine, had been made for a considerable amount of time. In 1973 the first signs started to appear that MRI might be applicable to medicine. Lauterbur [12] introduced relatively weak static linear magnetic fields, also referred to as “gradient fields”, where spatial information was assigned to each individual spin. The presence of the gradient field makes each spin rotate at a different rate, depending on its position relative to the origin. Thus the measured data contains the additional spatial information and when reconstructed, creates a picture which is almost the same as the object under examination. The combination of the gradient fields with the development of new technology for designing high strength superconducting magnets with reduced field variations, boosted the clinical application of MRI to the level that it is today.

In this chapter, we will discuss the basic principles of MRI, starting from the Bloch equations and ending with the discrete Fourier transform formulation.

For any nucleus with an intrinsic angular momentum, “spin”, $\vec{s} = hI$,
there is an associated magnetic moment, \( \vec{\mu} \), defined as

\[
\vec{\mu} = \gamma \hbar I
\]  

(2.1)

where \( \gamma \) is called the gyromagnetic ratio and for hydrogen atoms is \( 42.56 \text{MHz/Tesla} \).

The application of a static magnetic field \( \vec{H}_o = H_o \hat{z} \) provides each spin of the object with an additional energy of \( -\vec{\mu} \cdot \vec{H}_o \). The interaction between the spins and the magnetic field results in a split of the energy levels of the atom to \( 2I + 1 \) non-degenerate levels, each associated with energy

\[
E_{m_s} = -\gamma \hbar m_s H_o, \text{with } m_s = -I_z, ..., +I_z
\]  

(2.2)

Since only transitions with \( \Delta m_z = \pm 1 \) are allowed, the transition energy between two levels can be presented as

\[
\Delta E = \hbar \omega_L
\]  

(2.3)

Combining equations (2.2),(2.3), we obtain the expression for the Larmor frequency \( \omega_L \)

\[
\omega_L = \gamma H_o
\]  

(2.4)
Since Plank constant $h$ does not appear in equation (2.1), this indicates that the same result can be derived classically; the spinning top is such an example.

Our next step is to consider the macroscopic system. We define the bulk magnetization $\vec{M}$ as the sum of all individual magnetic moments in a given volume of a sample. Because our interest involves hydrogen atoms, the bulk magnetization can be considered as the number of protons in the given volume. Since there is a one-to-one correspondence between the proton and its spin state, we can define $\vec{M}$ as the quantity that represents the spin density of the system.

In a free magnetic field environment, the magnetic moments $\vec{\mu}$ of the individual nuclei are randomly oriented, such that the average bulk magnetization $\vec{M}$ in all three directions is zero. With the application of a constant magnetic field $\vec{H}$, we induce energy to the system which causes the magnetization to precess around the magnetic field at a constant rate, as shown in figure (2.1). The rate of the rotation of $\vec{M}$ is given by the Larmor equation. If there is no interaction between the object and the surrounding environment,
Figure 2.1: Rotation of magnetization around the static magnetic field $\vec{H}_o$. The direction of the Larmor frequency coincides with that of the static field.

the bulk magnetization $\vec{M}$ satisfies the following relation

$$\frac{d\vec{M}}{dt} = \gamma (\vec{M} \times \vec{H}_o)$$  \hspace{1cm} (2.5)

Since the magnetization rotates around $\vec{H}_o$ at a constant rate, it is more convenient to transform equation (2.5) to a rotating frame of reference with angular velocity $\vec{\omega}_r$. The expression for $\vec{M}$ in the rotating frame is modified
as:

$$\frac{d\vec{M}'}{dt} = \gamma \left( \vec{M}' \times \left( \vec{H}_o + \frac{\vec{w}_r}{\gamma} \right) \right)$$

(2.6)

where "the prime" indicates that the quantity is expressed in terms of the rotating coordinate system. If we choose \( \vec{w}_r \) such that, \( \vec{w}_r = -\gamma \vec{H}_o \), \( \vec{M}' \) is time independent in the rotating frame. Since \( \gamma \vec{H}_o \) is the Larmor frequency, equation (2.6) indicates that \( \vec{M}' \) is time independent in the rotating frame, if its frequency is equal in magnitude and opposite in direction to the Larmor frequency (eq. (2.4)). Then we say that we are on "RESONANCE".

Let us investigate further the behavior of the magnetic moment of the nuclei in the presence of an applied magnetic field. As we have mentioned above, this interaction between a strong static magnetic field and a non-magnetized material causes splitting of its energy states and depends on the spin of the nuclei. If we assume that the nuclei in question are protons, the interaction of the proton with the magnetic field will cause each energy level to split into two, corresponding to the two spin states of the proton. From the point of view of quantum mechanics, the process of magnetization of a
non-magnetized object can be viewed as a transition from the upper spin state to the lower one. When this process is completed there is a net energy differential. The system was in equilibrium before the whole process started and the total energy was conserved. Thus the presence of another system is necessary in order to accept this extra energy. This system, also referred to as the lattice, will accumulate the energy as heat. This procedure will continue until there is a thermal equilibrium between the spin system and the lattice. Thus, there is an additional interaction between the spin and the lattice system.

The approach to thermal equilibrium depends on the microscopic structure of the material. To describe this relaxation mechanism towards equilibrium, we introduce an additional variable $T_1$ which is called “the spin-lattice relaxation time” and is a quantity characteristic for each material. Besides $T_1$, there is an additional interaction between the spins.

The second phenomenon appears when the spins are in the $xy$ plane. They interact with the local fields $\Delta B_o$ and the effective field is $B_o + \Delta B_o$. This will force the neighboring spins to rotate at different frequencies. After a while in
the local region the spins will be uniformly distributed from $0 \to 2\pi$ and the net signal to the coil will be zero. To characterize this type of interaction, we introduce a quantity $T_2$ which is called “spin-spin relaxation time” and is also characteristic for the material. Unlike the spin-lattice relaxation mechanism, the spin-spin interaction is an irreversible mechanism. In general, $T_2$ has a smaller value than $T_1$ and only for water are they approximately the same.

The role of $T_1$ and $T_2$ for the design of an MRI experiment is very important, since different tissues in the human body have different values of $T_1$ and $T_2$ depending on the water content and chemical enviroment of the tissue. Thus, it is possible to generate an image based not only on high S/N ratio but also to high contrast between the tissues.

With the introduction of $T_1$ and $T_2$ relaxation mechanisms the Bloch equation (2.6) can be rewritten as:

$$\frac{d\vec{M}'}{dt} = \gamma \vec{M}' \times \left( \vec{H}_o + \frac{\vec{B}_0}{\gamma} \right) - \frac{M_x'^2 + M_y'^2}{T_2} - \frac{M_z' - M_0'}{T_1} \hat{z}$$ \hspace{1cm} (2.7)

The assumption in equation (2.7) is that the $T_2$ dephasing of the spins has an exponential form. This approximation is valid only for liquids. For
solid materials with a very short $T_2$, we can not consider an exponential-like behavior, and it is very difficult to evaluate $T_2$ with the available imaging methods and equipment.

In the laboratory frame, equation (2.7) can be written as

$$\frac{d\vec{M}}{dt} = \gamma (\vec{M} \times \vec{H}_o) - \frac{M_x \dot{z} + M_y \dot{y}}{T_2} - \frac{M_x - M_o}{T_1} \dot{z}$$  \hspace{1cm} (2.8)

The above equation (2.8) can be decomposed into a set of three differential equations, representing the three components of the bulk magnetization

$$\frac{dM_x}{dt} = \gamma M_y H_o - \frac{M_x}{T_2}$$  
$$\frac{dM_y}{dt} = -\gamma M_x H_o - \frac{M_y}{T_2}$$  
$$\frac{dM_z}{dt} = -\frac{M_x - M_o}{T_1}$$  \hspace{1cm} (2.9)

Combining the first two equation from (2.9) by replacing $M_+ = M_x + iM_y$, we obtain a differential equation for $M_+$. The solution of this differential equation can be done in two steps. The first one is the solution of the homogeneous part and the second one is the solution of the inhomogeneous part. Thus the solution for $M_+$ is

$$M_+(t) = M_x^0 e^{(-\omega_T - \frac{1}{T_1})t}$$  \hspace{1cm} (2.10)
with $M_x^o = M_x^o + iM_y^o$ and $\omega_L = \gamma H_o$.

The time evolution of $M_x$ can be shown for the last differential equation in (2.9) that takes the form

$$M_x(t) = M_x^o e^{-\frac{t}{T_2}} + M_o \left( 1 - e^{-\frac{t}{T_1}} \right)$$

(2.11)

with initial conditions $M_x(0) = M_x^o \neq M_o$, $M_x^o = M_y^o = 0$

A further investigation of equation (2.7) implies that at resonance and without taking into account the $T_1$ and $T_2$ effects the bulk magnetization remains time independent in the rotating frame. Thus, no signal is detected and no imaging is possible. To image an object, we have to alter the state of the spins and measure their reaction. We can achieve this with the application of a small time dependent magnetic field

$$\vec{H}_1(t) = 2H_1 \cos(\omega_{rf} t) \hat{z}$$

(2.12)

where arbitrarily the field is chosen to lie along the $\hat{z}$ axis. This field is often called “rf field” because the frequency of rotation lies in the radiofrequency region. We can decompose the expression for $\vec{H}_1(t)$ into two circular
polarizations

\[ \tilde{H}_R(t) = H_1 (\cos(\omega_r t) \hat{x} + \sin(\omega_r t) \hat{y}) \]

\[ \tilde{H}_L(t) = H_1 (\cos(\omega_r t) \hat{x} - \sin(\omega_r t) \hat{y}) \]  

(2.13)

Depending on the direction of \( \omega_r \) with respect to \( \omega_L \), the decomposed components of the alternating rf field \( \tilde{H}_R(t) \), \( \tilde{H}_L(t) \) are precessing in phase and out of phase with the magnetic moments, depending on whether we choose a counterclockwise or clockwise reference system, respectively. The effect of \( \tilde{H}_1(t) \) is to tip the spins in a direction which is perpendicular to the plane defined by the directions of the static magnetic field and the rf field. Turning off the rf field, the spins undergo two motions. The first motion is characterized by the \( T_1 \) relaxation time with the magnetic moments relaxing towards the initial value of the bulk magnetization \( \tilde{M}_0 = M_0 \hat{z} \). The second motion is the dephasing of the transverse components of the magnetization \( M_x \) and \( M_y \) due to the spin-spin interaction mechanism. With the rf field present, the Bloch equations are modified to

\[ \frac{d\tilde{M}'}{dt} = \gamma\tilde{M}' \times \gamma\tilde{H}_{eff} - \frac{M_x' \hat{z} + M_y' \hat{y}}{T_2} - \frac{M_x' - M_o'}{T_1} \hat{z} \]  

(2.14)
where \( \tilde{H}_{\text{eff}} \) is defined as

\[
\tilde{H}_{\text{eff}} = \left( H_o - \frac{\omega_r}{\gamma} \right) \hat{z} + \tilde{H}_1
\]

If no rf field is applied and we are off-resonance (the frequency of the rotating frame is different from the Larmor frequency), the bulk magnetization \( \tilde{M} \) precesses around the \( \tilde{H}_{\text{eff}} \) at an angle

\[
\theta = \arctan \frac{\omega_r}{\gamma H_o}
\]  

as illustrated in Figure (2.2).

The set of differential equations (2.9) with the rf field \( \tilde{H}_1 = \tilde{H}_L = H_1 (\cos (\omega_r f t) \hat{x} - \sin (\omega_r f t) \hat{y}) \) present can be modified as follows

\[
\frac{dM_x}{dt} = \gamma (M_y h_o + M_z H_1 \sin (\omega_r f t)) - \frac{M_x}{T_2}
\]

\[
\frac{dM_y}{dt} = -\gamma (M_x h_o - M_z H_1 \cos (\omega_r f t)) - \frac{M_y}{T_2}
\]

\[
\frac{dM_z}{dt} = -\gamma (M_z H_1 \sin (\omega_r f t) + M_y H_1 \cos (\omega_r f t)) - \frac{M_z - M_o}{T_1}
\]  

with \( \bar{h}_o = H_o + \frac{\omega_r}{\gamma} \).

Since, it is very difficult to solve analytically the above set of the differential equations, it is wise to move in the rotating frame of reference with
some additional simplifications. Assuming that we are on resonance \( \tilde{h}_o = 0 \), with the rf field \( \tilde{H}_1 \) directed in the \( \hat{x} \) direction of the rotating frame and not considering the relaxation mechanisms for the moment, the set of differential
equations (2.16) in the rotational frame of reference have the form

\[
\begin{align*}
\frac{dM'_x}{dt} & = 0 \\
\frac{dM'_y}{dt} & = \gamma M'_x H_1 \\
\frac{dM'_z}{dt} & = -\gamma M'_y H_1
\end{align*}
\] (2.17)

The solution of the equations (2.17) is

\[
\begin{align*}
M'_x(t) & = M'_x(0) \\
M'_y(t) & = M'_y(0) \cos(\omega_1 t) - M'_z \sin(\omega_1 t) \\
M'_z(t) & = M'_y(0) \sin(\omega_1 t) + M'_z \cos(\omega_1 t)
\end{align*}
\] (2.18)

Under the previous assumptions equation (2.18) indicates the bulk magnetization is rotated around the \( \hat{z} \) direction (of the rotating frame) with angular frequency

\[
\omega_1 = -\gamma H_1
\] (2.19)

Thus the application of the rf field \( H_1 \) tilts the bulk magnetization, originally oriented in the \( \hat{z} \), by angle \( \theta \) as shown in Figure (2.3). The relation between this angle and the magnitude of the rf field is
[\theta = \gamma \int_0^\tau H_1 \, d\tau]

(2.20)

where \( \tau \) is the duration of the rf pulse.

In this discussion, we considered that the rf field is applied along the \( \hat{z} \) direction of the rotating frame, but similar results can be obtained by the application of the rf field in the other transverse direction of the rotating frame. For the remaining discussion in this chapter, we will consider the rotating frame as the principal reference frame (leaving the primes understood) unless stated otherwise.

Let us summarize, what we have presented so far. MRI is based on the intrinsic property of the nuclei, hence the Nuclear part of the imaging. When a strong magnetic field is applied all the spins tend to align with the direction of the field (chosen to be the \( z \) direction). The net magnetization (bulk magnetization) is then precessing around this field with frequency described by the Larmor equation. This effect does not alter the state of the spins and no significant effect is observed. The application of a time varying magnetic field (rf field) applied along a prescribed direction tilts the magnetization to
a direction perpendicular to the plane defined by the static field and the rf field. When the rf field is turned off relaxation phenomena occurs, hence the Magnetic part of the imaging. The first has its origin in the interaction of the spins with the surrounding environment, and it is a reversible process. The second is originated by the interaction between the spins, and it is an irreversible process. The behavior of the bulk magnetization is described by the set of differential equations, known as Bloch equations. Since it is difficult to solve these equations exactly, a transformation to a rotating frame of reference is considered. If the frequency of the rotating frame coincides with the Larmor frequency, hence the Resonance part of the imaging, the Bloch equation is simplified and a solution is feasible with additional assumptions.

Now, we proceed further, investigating the detection and the data manipulation mechanisms. For a receiver coil, an rf resonator is constructed, designed to respond only at the Larmor frequency. Since the pick up is coupled with the sample, any change of the magnetic flux passing through the object will induce an electromotive force (EMF). The fluctuating voltage
that is generated in the coil is

\[ V(t) = -\frac{d}{dt} \int_V \vec{M}(\vec{x}, t) \cdot \vec{B}_c(\vec{x}) \, d^3x \]  \hspace{1cm} (2.21)

where \( V \) is the volume of the imaging region and \( \vec{B}_c(\vec{x}) \) the sensitivity of the receiver coil. The receiver coil is designed to generate a \( \vec{B}_c(\vec{x}) \) with the largest possible component in the transverse direction. Thus, the contribution of the longitudinal component of the magnetization is almost negligible. Along with the actual signal, high frequency noise is present. A low pass filter is inserted between the pickup coil and the data reconstruction system allowing frequencies at the range of few KHz.

As stated in equations (2.10),(2.11), the application of the static field \( H_0 \) causes each spin in a paramagnetic material to rotate around the axis of the static field with the frequency defined by the Larmor equation. The additional application of the rf field tips these spins at the transverse plane. Therefore, it is impossible to distinguish two points separated by a distance in a volume according to their position, because all the spins will precess with the same frequency and thus, we obtain the same signal at the same position.
for both of these points and for all the other points inside the imaging volume.

The separation of the spatial location of two spins with a different position inside an imaging volume, can be accomplished by the introduction of a set of three independent coils which vary in space and time. This three set of coils are called gradient coils and each one is named individually in exact agreement with the three axes of a cartesian coordinate system. Each of these coil sets is designed to create a magnetic field with its $z$ component varying linearly along each one of the three primary axes, depending on the gradient set. Although the other two components of the magnetic field are not taken into account for the design of the gradient coils, they can produce significant artifacts depending on the choice of the gradient strength [26].

With the introduction of the gradient field in the $z$ direction, the expression of the total magnetic field has the form

$$\vec{H} = (\vec{H}_o + \vec{G} \cdot \vec{r}) \hat{z}$$

(2.22)

Thus in the Larmor equation, the angular frequency can be modified to
contain the spatial dependence and can be written as

\[ \vec{J}(x) = \gamma \left( H_o + \vec{G} \cdot \vec{x} \right) \vec{z} \]  

(2.23)

Then the expression of the bulk magnetization is modified as

\[ \vec{M}(x, t) = M_0 e^{i(\omega t + \frac{1}{\gamma} G_z t)} \]  

(2.24)

With the presence of gradient coils, a complete MRI system is created. During the application of the rf pulse, a z-gradient coil is switched on, in order to select the slice of the object being imaged. When the rf pulse and the slice select gradient are turned off, a gradient coil in the transverse direction, say \( \hat{x} \), is switched on. Since at this moment, this gradient field is the only magnetic field present in this reference frame, the excited spins precess in the transverse plane with angular frequency \( \omega = \gamma G_z x \), with \( G_z \) the field strength. Thus, spins at different locations inside the object precess with different frequencies. This mechanism induces a signal in the receiver that is now position dependent and can be written as

\[ S(t) = \frac{1}{\gamma G_z} \int M(\omega) e^{-\gamma G_z x t} d\omega \]  

(2.25)
or

$$S(k_x) = \int M(x) e^{-i2\pi k_x x} dx$$  \hspace{1cm} (2.26)$$

with $k_x = \gamma G_x t$.

The interpretation of equation (2.26) is that the signal is the Fourier transform of the spin density (bulk magnetization). Thus, by performing an inverse Fourier transform to the signal, we get a spatial dependent magnetization and hence the image. The previous equation (2.26) corresponds to the imaging procedure in one dimension (1D). To expand the method into two dimensions (2D), the excitation of the spins in the other transverse direction is required. The application of the $y$-gradient coil (phase encoding) is then applied. The presence of this additional gradient field introduces an additional phase to the spins which is equal to $k_y y = \gamma G_y y \tau$, where $\tau$ is the time duration of the second gradient field. Then the modified signal is

$$S(k_x, k_y) = \int M(x, y) e^{-i2\pi(k_x x + k_y y)} dx dy$$  \hspace{1cm} (2.27)$$

The image can be recovered by performing a double Fourier transform of the signal (2.27). We can proceed further by performing a 3D imaging pro-
cedure, introducing an additional phase factor \( k_z z = \gamma G_z z \tau_z \) in the slice select direction. The duration \( \tau_z \) defines the total volume of the object to be imaged.

The role of the individual gradients as described above can be easily modified, depending on the design of the sequence and the direction of the slice which is needed to be excited.

The preceding interpretation of the Fourier transform assumes that the sampling time interval (\( \Delta t \)) is infinitesimal and the number of sampling points \( N \) is infinitely large, in order to approach the continuous case. In fact, the entire experiment is confined in a prescribed total time for the data acquisition. Specifically, there is a critical frequency \( f_c \) which is the largest frequency component that can be sampled. From the sampling theorem, \( f_c = \gamma G_z L / 2 \), where \( G_z \) is the maximum strength of the gradient in the read direction and \( L \) represents the entire field-of-view (FOV). According to Nyquist relationship, in order to avoid aliasing ("overlapping" of two points
positioned at different locations), the sampling rate has to be such that

\[ \Delta t = \frac{1}{2f_c} = \frac{1}{2\pi c L} \]

The discrete analog for the continuous Fourier transform from equation (2.26) is

\[ S\left(\frac{n}{L}\right) = \sum_{m=-\frac{N}{2}}^{\frac{N}{2}-1} \rho\left(\frac{mL}{N}\right) e^{-i2\pi nm} \quad (2.28) \]

and the inverse discrete Fourier transform of the signal is

\[ \rho\left(\frac{mL}{N}\right) = \frac{1}{N} \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} S\left(\frac{n}{L}\right) e^{i2\pi mn} \quad (2.29) \]

where \( \rho(x) \) is another representation for the bulk magnetization and \( n \) is the total number of points that ought to be collected.

There are additional concepts in MRI that have equal importance with the concepts which are presented in this chapter. The main focus of this dissertation is on the continuous improvement and development of coils for MRI and the understanding of the rf behavior. For the reader who is interested for in depth analysis, additional reading is available in Schumacher [24], Abraham [23] and Slichter [3].
Figure 2.3: Rotation of the bulk magnetization towards the positive $\hat{y}$-axis, under the application of an rf field in the $\hat{z}$ direction. In this case, the angle of rotation is $\alpha$. 
Chapter 3

Theoretical Background for MRI Coil Design

3.1 Introduction

There exists an ongoing interest for the improvement and/or the introduction of novel techniques for the design of main magnet and gradient coils. Until 1980, because of the lack of superconducting technology, resistive magnets were used as main magnets in MRI. In general, there were low field magnets (up to 0.15 Tesla(T)) with high levels of inhomogeneity over the imaging volume. Although low cost maintenance magnets, they are very susceptible to voltage fluctuations which create an additional increase in their inhomogeneity levels. After 1984, superconducting magnets have been used as the
main magnet in MRI. Even though their operational cost is larger than the
previous magnet technology, they can be ramped up to 25 T. are very low
inhomogeneity magnets, and are not sensitive to voltage fluctuations.

Up to now, the design of both the superconducting and resistive magnets
is based on the iterative technique of the Helmholtz pair (see section 3.2)
configuration, where by adding more coils, the higher order derivatives of
the magnetic field are eliminated and the homogeneity is improved. The dis-
advantage of this procedure is that it requires an initial guess of the position
of the coils that is very close to their final position, in order to get the de-
sired homogeneity of the magnetic field. Present works by Thompson [5, 4],
[22, 21], suggest an alternative way for designing main magnets. According
to this method, the values of the magnetic field at certain positions of the
Diameter Spherical Volume (DSV) are given as constraints and the distribu-
tion of the current density is generated. The advantage of this method over
the prior one is that no preceding knowledge is needed for the initial position
of the coils.

Besides the design of the main magnet in MRI, the design of high per-
formance gradient coils is equally important. Recall we have noted that, although the basic principal of NMR was known since 1946, the suggestion of linearly varied static magnetic fields (gradient fields) by P. Lauterbur in 1973 [42] revolutionized the concept of NMR with the addition of the spatial variation to the phase of the magnetization. This idea opened a new horizon to the field of NMR. In addition to its spectroscopic capabilities, imaging in vivo was also made available. The first set of gradient coils \((x, y, z)\) was based on the idea of Maxwell pair (see section 3.2) where pairs of coil loops with opposite current are placed in a specified position and produce a static linearly varying magnetic field. This method of design of the gradient coils was used until 1988. Their design was based on the same method that is used for the Helmholtz pair for the production of main magnets with more relaxed field specifications. In 1988 Turner [43, 1] suggested a novel technique for designing gradient coils. This technique is based on the Lagrange multiplier technique and the desired current density is obtained by extremizing a functional that contains the stored magnetic energy and the value of the magnetic field at certain points of the DSV as constraints.
In this chapter, we will discuss the concepts of a Helmholtz pair, a Maxwell pair, the inductance of the coils and the field linearity. The primary focus of this chapter is an extensive presentation of the procedure that is used for the design of high performance gradient coils using the magnetic field or derivative of the magnetic field as constraints. The theory of minimization using the derivative of the magnetic field at the origin as constraints in the minimization function will be discussed in the following chapter 5.

3.2 Theoretical Background for the Gradient Coil Design

Any line-current segment \( l \, dl \), positioned at \( \vec{r}_1 \) with respect to the origin of an orthogonal coordinate system, generates a magnetic field at position \( \vec{r}_2 \), \( d\vec{B}(\vec{r}_2) \), which is given by Biot-Savart law

\[
    d\vec{B}(\vec{r}_2) = \frac{\mu_0 \, l \, dl \times (\vec{r}_2 - \vec{r}_1)}{4\pi \, |\vec{r}_2 - \vec{r}_1|^3} \quad (3.1)
\]

For a wire loop lying on the \( (\rho, \phi) \) plane of a cylindrical coordinate system, the expression of the \( z \) component of the magnetic field \( \vec{B} \), along the
$z$ direction can be derived from the equation (3.1) and can be shown to be

$$B_z(z) = \frac{\mu_0 I}{2} \frac{a^2}{(z^2 + a^2)^{3/2}}$$  \hspace{1cm} (3.2)$$

where $a$ is the radius of the loop.

### 3.2.1 Generalized Helmholtz Coils

The Helmholtz pair, shown in figure 3.1, consists of two sets of coils with the same number $N$ of turns each. These coils share the same axis, same radius, same direction of the current and are placed in equal distance from the point halfway from their common axis.

Assuming that the origin of the coordinate system is located on the geometric center of one of the coils, the resulting $z$ component of the magnetic field $\vec{B}_z$ along the $z$ direction is an extension of equation (3.2) and for the set of these two coils is

$$B_z(z) = \frac{N \mu_0 I a^2}{2} \left[ \frac{1}{(z^2 + a^2)^{3/2}} + \frac{1}{((2b - z)^2 + a^2)^{3/2}} \right]$$  \hspace{1cm} (3.3)$$

where $2b$ is the the axial distance between the two coils. The position of the Helmholtz coil in the $z$ direction is chosen to eliminate the second derivative
Figure 3.1: Helmholtz pair coil configuration.

of the magnetic field at the midpoint of their common axis with respect to the \( z \)-variable. Before reducing we have,

\[
\left. \frac{d^2 B_z}{dz^2} \right|_{z=b} = -\frac{3}{2} \frac{N \mu_0 I a^2}{b} \left[ \frac{a^2 + b^2 - 5b^2 + b^2 + a^2 - 5b^2}{(b^2 + a^2)^{7/2}} \right]
\]  

(3.4)

Setting equation (3.1) to zero, we obtain the following relationship between
the separation of the coils $b$ and the radius of the coils $a$.

$$2b = a$$  \hspace{1cm} (3.5)

The above relation is only valid for two coils with the condition that the thickness of the wire for each coil is infinitely small and that the wires are packed very close together. Also, this expression is true only for the $z$ dependence of the $z$ component of the magnetic field. The general expression of the magnetic field involving the radial dependence is in terms of elliptic integrals of the first and second kind and can be found in Stratton [29]. Additional elimination of higher order derivatives for the magnetic field in the radial and the axial direction is not an easy task, using analytical methods.

Over the years, numerical minimization routines were used to eliminate up to the 12th derivative of the magnetic field in all three directions. The number of derivatives that can be nullified depends on the the number of coils that are used. Since the sets of coils are placed symmetrically around the midpoint of their axial distance, and the current density is also symmetric around this point, all odd derivatives of the $z$ component of the magnetic field are
cancelled out at the midpoint.

In conclusion, the property that the generalized Helmholtz coil provides is that it is possible to produce magnetic fields which are very homogeneous over a defined spherical volume, and deviate only by a few parts-per-million (ppm).

This principle is widely used today for the design of MRI main magnets.

### 3.2.2 Generalized Maxwell Coils

The basic principle for the design of gradient coils is the Maxwell pair configuration. The theory for the Maxwell coil is similar to the theory of the Helmholtz coil. The difference is only on the current direction. In the Helmholtz coil case the current density is symmetric around their midpoint, but in the Maxwell case it is antisymmetric. Thus, the expression for the $z$ component of the magnetic field is

$$B_z(z) = \frac{N \mu_0 I a^2}{2} \left[ \frac{1}{(z^2 + a^2)^{3/2}} - \frac{1}{((2b - z)^2 + a^2)^{3/2}} \right]$$  (3.6)

At the middle point of their separation ($z = b$), all the even derivatives vanish due to the antisymmetric nature of the current. The basic concept of the Maxwell pair design is that it eliminates the third derivative of the
\( z \) component of the magnetic field along the \( z \) axis. This can be done by finding the relation between the separation of the coils and their radii. The expression for the third derivative of the gradient field at the origin has the uncombined preliminary form

\[
\frac{d^3 B_z}{dz^3} \bigg|_{z=a} = -15 \frac{\mu I a^4}{2} \left[ -b \left(b^2 + a^2\right) - 2b \left(b^2 + a^2\right) + 7b^3 \right] - b \left(b^2 + a^2\right) - 2b \left(b^2 + a^2\right) + 7b^3 \right] f \left( \left(b^2 + a^2\right)^{3/2} \right)
\] (3.7)

Equation (3.7) vanishes when

\[
2b = \sqrt{3}a
\] (3.8)

The elimination of additional derivatives for the Maxwell pair design and hence the improvement of the magnetic field homogeneity, requires the introduction of extra coils that obey the same current configuration. Although, for the Helmholtz pair the stored magnetic energy is not a concern since the ramping of the magnet occurs over a long period of time, for the Maxwell pair the stored magnetic energy is an important factor since we want to switch these coils on and off as fast as we can. Methods that minimize the stored energy have been developed over the years [43, 1, 54, 55, 36] and play a major
role in the design of high performance gradient coils.

Until now, the only gradient coil that has been mentioned corresponds to the magnetic field which changes linearly along the axial (z) direction. The configuration of the Maxwell pair design is unable to generate magnetic fields whose z components vary linearly along the other two directions (x or y). The Golay pair of coils, given by Siebold [33], is the appropriate shape for the creation of such gradient fields. Figure 3.2 shows a Golay (saddle) pair schematic which generates a gradient field in the x-direction. Adjusting the dimensions between the coils, we should be able to improve the quality of the magnetic field over a certain region. There is, however, just so much that we are able to do with only one pair of saddle coils for the magnetic field homogeneity. For further improvement of the the linearity and uniformity of the coil, additional golay pairs have to be considered. This is not an easy task, since the current pattern is a vector addition of two different directions, the axial and the azimuthal. Thus, the upfront prediction of the correct position of the current patterns is very difficult. For more complicated gradient requirements this is almost impossible and very time consuming.
Figure 3.2: Golay pair of coils for creating an \( z \)-gradient field.

3.3 Quality Characteristics

3.3.1 Energy, Inductance and Image Quality

Unlike the ramping of main magnets that can take a considerable amount of time, the application of gradient coils during an MRI experiment is driven by a gradient amplifier which generates a pulse with a certain time duration.
The duration of time that the gradients are turned on almost defines the total time of the MRI experiment. Since there exist prescribed limits of the largest power that the amplifier can dump into the system, the inductance is a major factor of how fast we can achieve the highest gradient strength. Especially for fast MRI sequences, we need gradient coils that can be switched on and off quite fast, in order to decrease the echo time ($TE$) and thus to improve the signal-to-noise ratio by diminishing the $T_2^*$ dephasing effects. One way to accomplish this is the reduction of the magnetic energy which the amplifier puts into the gradient system which corresponds to the reduction of the total inductance of the coils.

For a given gradient coil system, the time which is necessary to get from zero gradient strength to the prescribed gradient strength is called "rise time". The rise time of gradient coils $t_r$ is defined in terms of the parameters of the system as

$$t_r = \frac{LI^2}{P} \quad (3.9)$$

where $L$ is the total inductance of the coil, $I$ is the maximum current needed
(assuming linear current ramping), and $P$ is the maximum power that the gradient amplifier can deliver. Since the maximum power and the maximum current are fixed for an amplifier, equation (3.9) indicates that there is one-to-one correspondence between the rise time and the inductance of the coil. Furthermore, the relationship between the stored energy and the inductance of the system is

$$W = \frac{1}{2} L I^2$$  \hspace{1cm} (3.10)

Combining equations (3.9), (3.10), we obtain the relation between the rise and the energy as

$$t_r = \frac{2W}{P}$$  \hspace{1cm} (3.11)

As we mentioned previously in chapter 2, there are two dephasing mechanisms which appear after the application of an rf pulse. Although the $T_1$ relaxation mechanism plays an important role to the design of the experiment, the $T_2^*$ dephasing mechanism is the major process which determines the signal-to-noise (S/N) ratio in the MRI image. A lot of tissues in the human body have a very short value for $T_2^*$ and imaging them is very difficult.
unless the total repetition time \( (T_R) \) of an MRI sequence is reduced so that the spin-spin relaxation effects for these tissues are insignificant. The reduction of the repetition time can be achieved by decreasing the waiting time where the three gradients are switched on and off. Equation (3.9) indicates that the decrease in the total inductance of a gradient coil by a factor of two results in the reduction of the rise time by the same amount, assuming that all the other parameters of the system remain unchanged. A typical gradient coil has inductance of 57.2 \( \mu H \). For a gradient amplifier that provides 100 A at 100 V, the rise time is 572 \( \mu sec \). The values of \( T_2^* \) for some of the tissues in the human body are smaller than 2 \( msec \), and by the time that imaging is performed the signal from these tissues is reduced to approximately 1/3 of its original strength. Thus important information about these tissues have been lost and can not be recovered, unless the time between the end of the rf pulse and the beginning of collection of the data from the read gradient is decreased to a level insensitive to \( T_2^* \) effects. Equation (3.9) indicates that the decrease of the inductance of the gradients by a factor of two results in the reduction of \( t_r \) by the same amount, assuming that all the other param-
eters of the system remain the same. For a traditional 2D sequence, with no rephasing lobes or spoiling gradients present, a factor of two reduction in the rise time for each of the three gradients results in the reduction of the echo time of at least 3msec. Since the $T_2^*$ does not interfere with the signal, we can further reduce the imaging time by decreasing the number of points which are collected. Another advantage that short rise times of gradient coils provides us, is that in the case where $T_1^*$ is considerably larger than the gradient ramping time, we can collect more points in the time interval that the gradients are on and improve our S/N ratio. Also, the resolution of the image can be improved by decreasing the time that the slice select gradient is on, thus reducing the thickness of the slice that is imaged.

In conclusion of this section, there exists a vast field of applications for fast gradient coils, including Echo Planar Imaging (EPI), diffusion/perfusion imaging, cardiac imaging, snapshot imaging, etc.
3.4 Inhomogeneity effects

The design of gradient coils must satisfy prescribed specifications inside the volume of interest. Besides the high magnetic field strength and low inductance coils, major consideration is given in the improvement of the linearity and the uniformity of field generated by the coils. The linearity of the coils depends on the contribution of high order field derivatives along the gradient axis. Although the loops in the gradient coil configuration are positioned to eliminate the third order derivative of the field, higher order terms are still present. At the origin, their contribution to the gradient field is almost negligible. When we approach the boundaries of the DSV, the effect from the higher derivatives becomes more significant. New methods for designing gradient coils have been introduced [43, 1, 54, 55] [36, 87, 20] in order to improve the linearity of the coils. Higher order derivatives can have devastating effects on the image quality. With these derivatives present, the points that are further away from the center of the coil precess faster or slower than is expected. This depends on the sign of the immediate ranking order derivative.
Thus, when the data are reconstructed, these points are placed at higher or lower position than the true one and the image is deformed [56].

In early stages of MRI, the acquired images suffered from additional phase effects. These effects have their origin in the other two associated components of the gradient field. Norris and Hutchinson [26] explain the effects of the concomitant magnetic field gradients on images.

Although our focus is to generate a gradient field along the axis of interest, the Maxwell equations indicate that other components of the magnetic field are present. Assuming that we are able to generate a gradient field of strength $G$ (say in the $z$ direction), then the magnetic field ought to have the form

$$B_z(z) = xG\hat{z}$$

(3.12)

According to Maxwell equations, in free space the magnetic field must satisfy the conditions $\vec{\nabla} \times \vec{B} = 0$ and $\vec{\nabla} \cdot \vec{B} = 0$. Even though the expression of the magnetic field (3.12) meets the divergence condition, the curl condition is not satisfied. To generate a gradient field which is consistent with both of
these restrictions, we alter the expression (3.12) as

$$B_z(x) = rGz + zG\hat{z}$$

(3.13)

Then the total magnetic field present, after the rf is turned off, is

$$|\vec{B}| = \sqrt{(B_o + rG)^2 + z^2G^2}$$

(3.14)

with $B_o$ is the value of the main magnet field. Assuming that $G << B_o$,
equation (3.11) is modified as

$$|\vec{B}| = B_o + rG + \frac{z^2G^2}{2B_o}$$

(3.15)

Equation (3.15) indicates that a low main magnet field can alter the homogeneity of
the total field. An extra frequency appears in the Larmor

equation. Assuming that the gradient strength is time independent and the
application of the gradient lasts time $t_G$, the additional phase shift is

$$\phi_e = \frac{\gamma z^2G^2t_G}{2B_o}$$

(3.16)

Equation (3.16) implies that the extra phase error is inversely proportional to
the strength of the main magnet. Increasing the main magnet strength by a
factor of 3 will decrease the phase shift by the same amount. Thus, at higher field strengths, the contribution of concomitant gradients is insignificant.

Even though we want high gradient strengths for applications to fast imaging, their effects can produce significant phase errors which can make their application difficult. Thus, there is a trade-off between gradient strength and linearity and uniformity conditions. Thus more elastic limits in the design for gradient coils must be imposed.

In later chapters, we will present methods for designing high performance gradient coils which attempt to reduce the defects that we described in this chapter.
Chapter 4

Target Field Method

4.1 Introduction

Since the first introduction of the gradient field in MRI by Lauterbur [42], there is an interest for a continuous improvement in existing methods, the development of new ones and the extension to different geometrical shapes.

As we said at the beginning, the basic methodology for the design of gradient coils was based on the Maxwell pair configuration. Initially, the wire loops were placed at prescribed positions on a cylindrical surface. Using Biot-Savart law, the strength of the magnetic field, its linearity, and uniformity in a DSV are evaluated. If the field does not satisfy the present conditions, we alter the positions of the loops, introduce additional loops and increase
the current value. Using minimization techniques, we obtain the positions of the loops which are in agreement with the prescribed conditions. Although this method appear to be simple, there are a lot of disadvantages associated with its application.

The first disadvantage is that prior knowledge is needed for the positions of the loops. This guess must be close to the final solution. If the initial guess is not well established, the majority of the methods fail to produce the correct results, because they diverge during the process.

The second disadvantage is the time duration for the application of this approach. Since these methods are iteration techniques, considerable CPU time has to be consumed. Thus, in order to make this method competitive with the other existing techniques, more powerful computers have to be used. This makes these techniques less attractive for original research and development.

The third disadvantage involves the design of the transverse gradient coils. Based on the Maxwell pair design, the current density necessary for generating an axial gradient field can have only one component which is directed
azimuthally and varies along the axial direction of the cylindrical geometry. Thus, the configuration of gradient coils can be easily implemented, because the shape of the wire loops can have circular shape. Therefore, the Biot-Savart law is adequate to evaluate the magnetic field in the volume of interest.

Moving towards the design of transverse gradient coils, the situation changes drastically. The current density, essential to generate magnetic fields in either two transverse directions of the cylinder, must be composed of two components. Although the boundary conditions for the current density are well established by Maxwell's equations, it is difficult to construct current patterns which lie on the surface of the cylinder [27], because the current patterns are composed of considerably small line segments which are restricted to lie on the surface of the cylinder. This must be done numerically which is not a trivial task. Furthermore, it is well known [43, 1], [56, 55] that the shape of current patterns necessary for the creation of transverse gradient coils diverges from the conventional circular shape. Most of these methods [51, 48, 27] are incapable of generating current patterns which are closer
to the actual shape of the current distribution. Also, the basic focus of these methods is to generate gradient fields consistent with prescribed specifications. Little or no specific concern has been taken for minimizing the energy of the system.

In his papers Turner [13, 1] suggested a different approach for the design of gradient coils. According to his method, the values of the magnetic field at specified points in the DSV are preassigned. Furthermore, the current density is restricted to lie on the surface of the cylinder. Also, we take advantage of the fact that the relations between the magnetic field and the stored energy with the current density are linear and quadratic, respectively. Forming a functional $F$ which contains the energy and the field points as Lagrange multipliers, we minimize it with respect to the current density. Thus, we obtain a linear equation for the current density which contains the Lagrange multipliers and the description of the geometry of the system. By inverting this equation, we obtain the expression for the continuous current distribution. Using the continuity equation, we can proceed and discretize the continuous current density.
Gradient coils create eddy currents, because they change the magnetic flux on the magnet shield when they are turned on and off. The outcome of eddy current effects can be devastating, since eddy currents generate fields which change their shape in time and space, without following a known pattern [56, 49]. Thus, it is difficult to correct their effects on the MRI image.

One way to eliminate eddy current effects is the introduction of a second coil outside of the primary gradient coil. This coil operates as a shield for the magnetic field of the inner coil and nulls the magnetic field outside from the combined gradient coil system. The combined gradient coil system is called a "self-shielded" gradient coil or "actively-shielded" gradient coil. The details for an actively-shielded coil has been discussed previously [56], and we will not attempt to describe them again.

Even though the entire methodology for the design of gradient coils has been described previously [43, 1], [56], we will also provide an extensive overview, because it is essential for this dissertation.
4.2 Theoretical Development

As a starting point, let us consider that the current density distribution is restricted to lie on the surface of a particular geometric shape. In order to evaluate analytically the expressions of the magnetic field and the stored energy, we need to relate the shape of the coil with a known reference frame. Then we are able to define a complete set of three orthogonal vectors \( \hat{a}_1, \hat{a}_2, \hat{a}_3 \) forming the basis of the frame and the associate set of three independent variables \( \zeta_1, \zeta_2, \zeta_3 \). The restriction on the current density can be implemented as follows:

\[
\vec{J}(\vec{r}) = [j_{\alpha_1}(\zeta_1, \zeta_2) \hat{a}_1 + j_{\alpha_2}(\zeta_1, \zeta_2) \hat{a}_2] \frac{\delta(\zeta_3 - \zeta_c)}{f_3} \tag{4.1}
\]

where \( f_3 \) is the metric which depends on the geometry of the system and can be represented with the 3D Jacobian of the system with respect to the cartesian system. The delta function term enforces the restriction that the current distribution lies on the constant surface \( \zeta_3 = \zeta_c \).

The basic idea behind this method is to generate a current density distribution capable of producing magnetic fields which satisfy prescribed re-
quirements and boundary conditions. Turner [43, 1] demanded that the
\( \dot{\alpha}_1 \)-component of the magnetic field \( \vec{B}(\vec{r}) \) takes prescribed values at certain
points inside the volume of interest. This can be expressed in a mathematical
form as:

\[
B_{\dot{\alpha}_1}(\vec{r}_j) = H_{\alpha_1 \cdot \vec{r}_j} \quad \text{with } i = 1, 2, 3 \text{ and } j = 1, ..., N
\]  

(4.2)

where \( N \) represents the number of the constraint points. \( B_{\dot{\alpha}_1 \cdot \vec{r}_j} \) is the
prescribed value of the magnetic field at the point \( \vec{r}_j \).

In free-space, Maxwell equations describe the behavior of the magnetic
field. Thus, the magnetic field must satisfy the relations \( \vec{\nabla} \times \vec{B} = 0 \), \( \vec{\nabla} \cdot \vec{B} = 0 \).
Thus the choice of the constraint points 4.2 must be consistent with these
two relations.

Our next step is to relate the current density with known quantities in
electromagnetism. Thus, we need the expression of the Green function, char-
acteristic for the geometrical shape and the reference system. A system with
separable coordinates facilitates the construction of a Green function in this
reference frame. A representative form of the Green function for such a
coordinate system \((\zeta_1, \zeta_2, \zeta_3)\) is

\[
G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} = \int \int dk_1 dk_2 \Omega(k_1, k_2, \zeta_1, \zeta_2) \Psi_1(k_1, k_2, \zeta_3<) \Psi_2(k_1, k_2, \zeta_3>) \tag{4.3}
\]

where \(k_1, k_2\) are the two orthogonal directions in the Fourier space, corresponding to the directions \(\hat{a}_1, \hat{a}_2\), respectively. \(\Omega(k_1, k_2, \zeta_1, \zeta_2)\) contains all the information of the system and can be analyzed in a set of orthogonal functions which are characteristic for the system. \(\Psi_1(k_1, k_2, \zeta_3<)\) and \(\Psi_2(k_1, k_2, \zeta_3>)\) define the point in space relative to the source. The lesser \((\zeta_3<)\) and greater \((\zeta_3>)\) expressions can be interchanged as a constant or independent variable, depending on the position of the source (current) relative to the point of interest.

The knowledge of the Green function can help to evaluate the expression for the vector potential

\[
\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{f}(\vec{r}')}{|\vec{r} - \vec{r}'|} \, d^3x'
\tag{4.4}
\]
The magnetic field can be evaluated as $\vec{B} = \nabla \times \vec{A}$ and the stored magnetic energy $W_m$ is

$$W = \frac{1}{2} \int_V \vec{A} \cdot \vec{j} \, d^4 r.$$  \hfill (4.5)

Since we have expressed the Green function in the Fourier domain, it is necessary to do the same with the current density. The expressions of the Fourier transform of the two components are

$$j_{\alpha_1}(k_1, k_2) = \iiint d\zeta_1 d\zeta_2 \Omega(k_1, k_2, \zeta_1, \zeta_2) j_{\alpha_1}(\zeta_1, \zeta_2)$$ \hfill (4.6)

$$j_{\alpha_2}(\zeta_1, \zeta_2) = \iiint dk_1 dk_2 \Omega^*(k_1, k_2, \zeta_1, \zeta_2) j_{\alpha_1}(k_1, k_2)$$ \hfill (4.7)

Similar expressions can be obtained for the $j_{\alpha_2}$ component. In our discussion earlier in this section, the total current density is considered as a vector superposition of two components. In order to proceed further with this theoretical development, we have to find a relation between these two components. The reason is that we will be able to express the magnetic field and the stored energy in terms of only one component. Since there is no free-charge present, the current density $\vec{j}$ must satisfy the continuity equation $\nabla \cdot \vec{j} = 0$. Applying this condition to equation 4.1, we obtain the relation for the two components,
which is
\[ \frac{1}{f_1} \frac{\partial j_{\dot{\alpha}_1}(\zeta_1, \zeta_2)}{\partial \zeta_1} + \frac{1}{f_2} \frac{\partial j_{\dot{\alpha}_2}(\zeta_1, \zeta_2)}{\partial \zeta_2} = 0 \]
or
\[ \frac{1}{f_1} \frac{\partial j_{\dot{\alpha}_1}(k_1, k_2)}{\partial \zeta_1} + \frac{1}{f_2} \frac{\partial j_{\dot{\alpha}_2}(k_1, k_2)}{\partial \zeta_2} = 0 \] (4.8)
where \( f_1, f_2 \) represent the weighting factors in the two directions \( \dot{\alpha}_1, \dot{\alpha}_2 \) respectively. Using equation 4.8, we can obtain a relationship between the two Fourier components of the current density
\[ j_{\dot{\alpha}_2}(k_1, k_2) = -\frac{k_1 f_2}{k_2 f_1} j_{\dot{\alpha}_1}(k_1, k_2) \] (4.9)
Thus, we succeeded to express the magnetic field and the stored energy of \( j_{\dot{\alpha}_1}(k_1, k_2) \).

We now proceed with the mathematical development of the methodology. We construct the functional \( \mathcal{E} \) which contains the magnetic energy \( W_m \) and the \( z \) component of the magnetic field in terms of Lagrange multipliers. The only independent variable in this functional is the Fourier component of the current. The mathematical formalism for \( \mathcal{E} \) is
\[ E(j_{\dot{\alpha}_1}(k_1, k_2)) = W - \sum_{j=1}^{\infty} \lambda_j (B_{\dot{\alpha}_1}(\vec{r}_j) - B_{\dot{\alpha}_2}(\vec{r}_j)) \] (4.10)
Thus, $\mathcal{E}$ is a quadratic function of the current density. By extremizing $\mathcal{E}$ with respect to $j_{\alpha_1}(k_1, k_2)$, we obtain an equation for the current density. We have previously mentioned this procedure as a minimization mechanism, but in this current paragraph we refer to it as an extremization technique. We will show that these two views coincide. If we differentiate $\mathcal{E}$ twice with respect to $j_{\alpha_1}(k_1, k_2)$, the only surviving term is the integral of the magnetic energy independent of $j_{\alpha_1}(k_1, k_2)$. Because we deal with a classical system, this integral is always positive, which means that the second derivative is always positive. Thus, the extremization of the energy functional is eventually a minimization procedure.

Returning to the discussion for the $j_{\alpha_1}(k_1, k_2)$, we have argued that it is possible to obtain a linear equation for this current density. The mathematical expression is

$$j_{\alpha_1}(k_1, k_2) = \sum_{j=1}^{\nu} \lambda_j \Omega(\bar{r}_j) \quad (4.11)$$

where $\lambda_j$ are the Lagrange multipliers and $\Omega(\bar{r}_j)$ contains all the information of the geometry of the system.
The following step is to replace the Lagrange multipliers. This can be done with the help of equation (4.2). According to this equation, \( \lambda_j \) must satisfy the following relation

\[
B_{\theta,SC} (\vec{r}_k) = \sum_{j=1}^{N} C_{kj} \lambda_j \text{ with } i = 1, 2, 3 \text{ and } k = 1, \ldots, N \quad (4.12)
\]

\( C_{kj} \) is an \( M \times N \) matrix, where \( M \) represents the number of points where the constraints are evaluated and \( N \) defines the number of constraints.

Equation (4.12) contains a set of linear equations which contain \( \lambda_j \). Solving for \( \lambda_j \), we can replace their values in the equation (4.11). Thus, we get an expression for \( j_{\hat{a}_i}(k_1, k_2) \). Working backwards, and with the help of equations (4.7),(4.9), we get the expressions of both the components of the current density in the real space.

The entire methodology presented in this section is not limited to the type of functions described here. There is a whole variety of functions which can replace the stored magnetic energy and another variety of constraints which involve magnetic fields, or derivatives of them, or any polynomial containing powers of the magnetic field and its derivatives.
In chapter 5, we will discuss the application of this method to the main magnet design, where the derivatives of the field replace the magnetic field constraints.
Part I

Main Magnet
Chapter 5

Solenoidal Main Magnet Coil

5.1 Introduction

Recalling from section 1.2, there exist different implementations of the general target field approach. An alternative technique is the replacement of the values of the magnetic field with the values of its derivatives at the origin. This can be understood as follows. If we consider a Taylor expansion of the magnetic field around the origin, the effect of the higher order derivatives becomes significant when we move away from the origin. In order to compensate for this effect, we have to set these derivatives to zero. If we wish to extend the region of the homogeneity of the magnetic field, additional numbers of the field derivatives must be vanished. This procedure cannot
last forever. Because the elimination of each additional derivative adds to the energy of the system. Thus, after a certain point the system becomes energetically unfavorable.

There exist many publications suggesting different methods for the creation of a constant magnetic field with high degree of homogeneity [4, 21, 22, 17, 18]. In this chapter, we will present a different method for the design of main magnets. Even though the primary focus of this dissertation is on the design of gradient coils and the investigation of rf effects, we will discuss the basics for the design of main magnets and we will not attempt to get into details. Our purpose is to introduce the idea of derivative constraints, and show that design of main magnets is feasible using Turner’s approach. Specifically, we will describe the entire theoretical background behind the non-shielded and self-shielded solenoidal magnet designs. We will also present results from the non-shielded case.
5.2 Derivative Constraints Methodology

Let us consider a cylindrical geometry \((\rho, \phi, z)\). The current density must lie on the surface of the cylinder with radius \(a\), and thus, its general expression (4.1) is modified as follows

\[
\tilde{j}(\phi, z) = \left[ j_{x\phi}(\phi, z) \hat{\alpha}_\rho + j_{xz}(\phi, z) \hat{\alpha}_z \right] \delta(\rho - a)
\]  

(5.1)

where \(a\) is the radius of the cylinder and \(\hat{\alpha}_\rho, \hat{\alpha}_z\) are the unit vectors on the azimuthal and axial direction, respectively. In order to find the expression of the vector potential, the free-space green function is needed for a cylindrical coordinate system. This expression can be obtained from Jackson [44] and is

\[
G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i m (\phi - \phi')} e^{ik(z - z')} I_m(k\rho_<) K_m(k\rho_>)
\]  

(5.2)

where \(I_m(k\rho_<)\) and \(K_m(k\rho_>)\) are the modified Bessel functions of the first and the second kind, respectively.
Then combining equations (5.2) and (4.4), the three components of the vector potential in the cylindrical coordinates are

\[
A_\rho(\vec{r}) = -\frac{i\mu_0}{4\pi} \sum_{m=-\infty}^{\infty} \epsilon^{im\phi} \int_{-\infty}^{\infty} dk \, e^{ikz} j^\rho_\phi(m,k) \\
\left[ I_{m-1}(k\rho) K_{m-1}(k\rho) - I_{m+1}(k\rho) K_{m+1}(k\rho) \right] (5.3)
\]

\[
A_\phi(\vec{r}) = \frac{\mu_0}{4\pi} \sum_{m=-\infty}^{\infty} \epsilon^{im\phi} \int_{-\infty}^{\infty} dk \, e^{ikz} j^\phi_\phi(m,k) \\
\left[ I_{m-1}(k\rho) K_{m-1}(k\rho) + I_{m+1}(k\rho) K_{m+1}(k\rho) \right] (5.4)
\]

\[
A_z(\vec{r}) = \frac{1}{4\pi} \sum_{m=-\infty}^{\infty} \epsilon^{im\phi} \int_{-\infty}^{\infty} dk \, e^{ikz} j^z_\phi(m,k) \\
\left[ \frac{m}{k} I_{m}(k\rho) K_{m}(k\rho) \right] (5.5)
\]

where we have made the Fourier transforms

\[
j^\rho_\phi(m,k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \, e^{-im\phi} \int_{-\infty}^{\infty} dz \, e^{-ikz} j^\rho_\phi(\phi,z) (5.6)
\]

\[
j^z_\phi(m,k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \, e^{-im\phi} \int_{-\infty}^{\infty} dz \, e^{-ikz} j^z_\phi(\phi,z) (5.7)
\]

with

\[
j^\rho_\phi(\phi,z) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \epsilon^{im\phi} \int_{-\infty}^{\infty} dk \, e^{ikz} j^\rho_\phi(m,k)
\]

\[
j^z_\phi(\phi,z) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \epsilon^{im\phi} \int_{-\infty}^{\infty} dk \, e^{ikz} j^z_\phi(m,k)
\]
Since there is no free-charge on the system, the continuity equation for the current density is modified to $\nabla \cdot \vec{J} = 0$. Considering the Fourier transform for both the components of the current density, the continuity equation takes the form

$$\frac{m}{u} j^\prime_\phi(m, k) - k j_z(m, k) = 0.$$  \hspace{1cm} (5.8)

or

$$j^\prime_\phi(m, k) = \frac{m}{ku} j_z(m, k)$$ \hspace{1cm} (5.9)

Equation (5.9) is valid, except in the case when $m = 0, k = 0$. In this situation, from equation (5.8), we conclude that the $\phi$ component of the current density should vanish in order to total current density to satisfy the continuity equation. This is exactly the current configuration for the design of coil with azimuthal symmetry.

We now proceed to calculate the three components of the magnetic field. Using the relation between the vector potential and the magnetic field, the expression of the three components of the magnetic field in cylindrical coor-
dinates are

\[ B_\varphi(\vec{r}) = \frac{\mu_0 a}{2\pi} \sum_{m=\pm\infty} e^{im\phi} \int_{-\infty}^{\infty} dk \, e^{ikz} j^m_\varphi(m,k) \varphi_{\varphi}(m,k) \]  \hspace{1cm} (5.10)\]

\[ B_\rho(\vec{r}) = \frac{\mu_0 a}{2\pi} \sum_{m=\pm\infty} e^{im\phi} \int_{-\infty}^{\infty} dk \, e^{ikz} \frac{j^m_\varphi(m,k)}{\rho} \varphi_{\varphi}(m,k) \]  \hspace{1cm} (5.11)\]

\[ B_z(\vec{r}) = \frac{\mu_0 a}{2\pi} \sum_{m=\pm\infty} e^{im\phi} \int_{-\infty}^{\infty} dk \, e^{ikz} \frac{j^m_\varphi(m,k)}{k} \varphi_{\varphi}(m,k) \]  \hspace{1cm} (5.12)\]

where \( I'_m(k\rho_<) \) represents the derivative of the Bessel function with respect to its argument and \( \Theta(x-a) \) is the step function.

In order to apply Turner's technique, we need to evaluate the stored magnetic energy. Combining equation (5.2) and (4.5), the expression of the energy in cylindrical coordinates is

\[ W = -\frac{\mu_0}{2} \sum_{m=\pm\infty} \int_{-\infty}^{\infty} dk \left[ a^2 I'_m(ka) K'_m(ka) j^m_\varphi(m,k) \right]^2 \]  \hspace{1cm} (5.13)\]
Thus, we have developed the basic theory for the design of any cylindrical magnetic coil.

5.2.1 Non-Shielded Main Magnetic Field

In earlier chapter, we have mentioned that the design of the magnetic field for main magnets is based on the Helmholtz pair configuration. We intend to generate magnetic fields which are constant inside the desired imaging region and symmetric around the geometric center of the cylinder. We can produce such fields by eliminating the higher order derivatives of the magnetic field.

The conditions which we describe above reflect the behavior of the current density. The requirements that the current density must be azimuthally oriented and is dependent only on the position $z$, combined with the demand that the magnetic field is constant (independent of $\phi, z$), leads to the conclusion that $m = 0$. Recall also that the field is symmetric around the origin of the cylinder, which means that the field is an even function of $z$. Therefore, we must replace the complex exponential $e^{ikz}$ with the $\cos(kz)$ term.

Under these conditions the expression of the current density in the Fourier
domain is transformed to

\[
\vec{j}_{\phi}(m.k) = j_{\phi}^{*}(0.k)\hat{v}_{\phi} = j_{\phi}^{*}(k)\hat{v}_{\phi}\delta(\rho - a)
\]  

(5.14)

with

\[
j_{\phi}^{**}(k) = j_{\phi}^{*}(k)
\]  

(5.15)

and

\[
j_{\phi}^{*}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz \cos(kz) j_{\phi}^{*}(k)z
\]  

(5.16)

Under these assumptions, equations (5.12) and (5.13) can be modified to describe the magnetic field and stored energy for the region inside the cylinder (\(\rho < a\)), as

\[
B_z(\rho, z) = \frac{\mu_0 a}{2\pi} \int_{-\infty}^{\infty} dk \cos(kz) j_{\phi}^{*}(k)k I_0(k\rho) K_1(ka)
\]  

(5.17)

\[
W = \frac{\mu_0 a^2}{2} \int_{-\infty}^{\infty} dk \, I_1(ka) K_1(ka) j_{\phi}^{**}(k)
\]  

(5.18)

where, we have replaced \(I_0(k\rho) = I_1(k\rho), K_0(k\rho) = -K_1(k\rho)\).

Since the current density is symmetric around the origin, odd derivatives of the magnetic field are by definition equal to zero. Then, the expression of
any even order derivative of the magnetic field at the origin can be expressed in the following form

\[ B^{(2n)}_z = \frac{\partial^{(2n)} B_z}{\partial z^{(2n)}} = \frac{\mu_0 m}{2\pi} \int_{-\infty}^{\infty} dk \left(-1\right)^n k^{2n+1} \cos(kz) j_\phi(k) K_1(ka) \]  

(5.19)

where \( n = 0 \) represents the actual value of the magnetic field \( B_0 \), and \( (2n) \) indicates the order of the derivative of the magnetic field with respect to \( z \) at \( \rho = 0, \phi = 0 \). Thus, the constraints which are imposed for this problem have the form

\[ B^\rho = 0, \quad B^{(2n)}_z = 0 \text{ for } n = 1, \ldots, N \]  

(5.20)

where \( N \) is the total number of constraints which coincides with the number of even derivatives.

Therefore, the expression for the functional \( \mathcal{E} \) has the form

\[ \mathcal{E}(j_\phi(k)) = W_m - \sum_{n=0}^{N} \lambda_n \left( B^{(2n)}_z - B^{(2n)\text{ic}}_z \right) \bigg|_{z=0, \rho=0} \]  

(5.21)

Minimizing \( \mathcal{E} \) with respect to \( j_\phi(k) \), we obtain an expression for the current
density

\[ j_\phi^s(k) = \frac{1}{2\pi a l_m'(ka)} \sum_{n=0}^{N} \lambda_n (-1)^n k^{2n+1} \]  

(5.22)

The determination of Lagrange multipliers \( \lambda_n \) can be done through the constraint relation (5.20), as

\[ \sum_{m=0}^{N} C_{nm} \lambda_m = B_{zsc}^{(2n)} \]

where

\[ C_{nm} = \frac{\mu_0}{4\pi^2} \int_{-\infty}^{\infty} dk (-1)^{m+n} k^{2n+2m+3} \frac{K_m(ka)}{I_m(ka)} \bigg|_{\tau=0, \varphi=0} \]  

(5.23)

Evaluating the coefficients (5.23), we get the values for the \( \lambda_n \). Substituting \( \lambda_n \) into equation (5.22), we obtain the expression for the Fourier component of the current density. Thus, performing the inverse Fourier transform on the \( j_\phi^s(k) \), we acquire the expression for the continuous current distribution \( j_\phi^s(z) \).

In the remainder of this section, we will present two characteristic examples demonstrating the efficiency of this method. The geometry of the cylinder for the main magnet design is shown in Figure 5.1.
Figure 5.1: Cylindrical geometry for the main magnet design.

The radius of the cylinder of an infinite length is chosen to be $a = 0.5 \, m$.

For the first example, we have chosen six constraints which are shown in Table 5.1. The first constraint sets the strength of the magnetic field at $1.5 \, T$. The rest of the constraints set the even derivatives of the magnetic field equal to zero, in a consecutive order. Therefore, the lowest derivative which contributes to the inhomogeneity of the magnetic field is the $12^{th}$ order.
<table>
<thead>
<tr>
<th>$n$</th>
<th>$\rho_i$</th>
<th>$z_i$</th>
<th>$B_{z=a}(2n)$</th>
</tr>
</thead>
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<td>1.5000000004</td>
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</tr>
<tr>
<td>5</td>
<td>0.0</td>
<td>0.000</td>
<td>0.0000000000</td>
</tr>
</tbody>
</table>

Table 5.1: Constraint set used for the design of a non-shielded main-magnet coil. Values for $\rho$ and $z$ are in m, values for $B_{z=a}(2n)$ are in T.

derivative. Furthermore, there exists an one-to-one correspondence between
the order of the field derivative present and the level of the inhomogeneity of
the magnetic field. Assuming only the $z$ component of the field, the level of
the inhomogeneity of the magnetic field is proportional to $(zB_a(0))^{12}$. Thus,
at distance $z = 0.25 \text{ m (50 cmDSV)}$, the difference from the actual value of
the magnetic field at $1.5 T$ is on the order of $8 \text{ ppm}$. The shape of the current
distribution necessary to create these characteristics is shown in Figure 5.2.
We notice that the shape of the current density includes three peaks which
contain most of the current. Knowing the current distribution, we are able
to evaluate the resulting magnetic field. In Figure 5.3, we show a plot of the
divergence of the field from its $1.5 \text{ T}$ value. At the point of $z = 0.23 \text{ m}$, the
Figure 5.2: Current distribution $j_d(z)$ for the non-shielded main-magnet, using six sets of constraints.

Theoretically estimated inhomogeneity is 7 ppm. The actual inhomogeneity is larger, is due to the contribution of the higher field derivatives.

To increase the homogeneity region of the magnetic field, we must increase the number of constraints. In the second example, we have chosen seven constraints overall, and the lowest field derivative present is the 14th one. In this case, we have reduced the inhomogeneity of the magnetic field (same
magnet strength) for the 50 cm DSV imaging region to 1 ppm. The plot of the resulted current distribution is shown in Figure 5.4. We denote that the number of highest current peaks has been increased to four. The explanation of this phenomenon is that we need additional current strength (additional degree of freedom) in order to null the 12th field derivative. Additionally, Figure 5.5 displays the variation of the magnetic field from its actual value of 1.5 T. We notice that the homogeneous region is extended further compared to the previous case.

Although from an engineering perspective, the discrete current distribution is inefficient for the actual design of the main magnet, it gives a good indication for the number of coils, their position, and the current value which are necessary to create the desired homogeneity of the magnetic field. Therefore, we can use this information for the initial step in the iteration routine, which will save valuable computing time and will decrease the chances of the divergence for the minimization routines, since this guess is very close to the final answer.

So far, we have dealt with the non-shielded magnet designs. Besides
Figure 5.3: Plot of the homogeneity of the static field $\frac{B_z - B_o}{B_o}$, for the main magnet, using six constraint points.

The concern for the homogeneity of the magnetic field, another design requirement is the decrease of the distance where the value of $B_o$ drops to 5 Gauss ($G$). The reason is that magnetic fields of this strength can be hazardous to the human health (phosphines, nausea) and create defects in cardiac pacemakers or other electronic devices inserted in the human body.

For the non-shielded case, the magnetic field drops slowly as we move away
Figure 5.4: Current distribution $j_\phi(z)$ for the non-shielded main-magnet, using seven sets of constraints.

from the center. In order to decrease the distance of the 5 (G) line, two designs must be considered. The first is to shield the magnet with iron. Although the iron-shield reduces the 5 (G) line, because it traps the flux of the magnetic field lines, it is not desired due to the large weight (close to 100 tons) of the entire system. The most reasonable approach is the design of an “actively-shielded” magnet.
Figure 5.5: Plot of the homogeneity of the static field (see Figure 5.3) for the main magnet, using seven constraint points.

5.3 Actively-Shielded Main Magnets

The roles of the actively-shielded magnet design are to reduce significantly the distance of the 5 (G) line from the center of the magnet, to retain the homogeneity of the magnetic field inside the DSV and to generate a design with the minimum possible weight. In order to achieve these three goals, a second cylinder with radius b must be introduced outside the primary coil.
The current of the second cylinder has opposite sign from the primary cylinder and is sufficiently smaller in magnitude. Thus, it affects the magnitude of the field outside the DSV region, while there is no significant effect on the homogeneity of the main magnetic field.

With the introduction of the second coil, the formulas which we have presented in the section 5.2 must change in order to account for the second coil. Then, the expression of the $z$ component of the magnetic field in the three regions ($\rho < a$, $a < \rho < b$, $b < \rho$) are

$$B_z = \frac{\mu_0}{2\pi} \int_{-\infty}^{\infty} dk \cos(kz) k \left[ a j_\alpha(k) K_1(ka) I_0(k\rho) ight.$$  
$$+ b j_\alpha(k) K_1(kb) I_0(k\rho) \right] \text{ for } \rho < a \tag{5.24}$$

$$B_z = \frac{\mu_0}{2\pi} \int_{-\infty}^{\infty} dk \cos(kz) k \left[ -a j'_\alpha(k) K_0(k\rho) I_1(ka) ight.$$  
$$+ b j'_\alpha(k) K_1(kb) I_0(k\rho) \right] \text{ for } a < \rho < b \tag{5.25}$$

$$B_z = \frac{\mu_0}{2\pi} \int_{-\infty}^{\infty} dk \cos(kz) k \left[ -a j'_\alpha(k) K_0(k\rho) I_1(ka) ight.$$  
$$- b j'_\alpha(k) K_0(k\rho) I_1(kb) \right] \text{ for } b < \rho \tag{5.26}$$

where $j_\alpha(k)$, $j'_\alpha(k)$ are the Fourier transforms of the current densities for the
inner coil (radius $a$) and the outer coil (radius $b$), respectively. The behavior of the current distribution of the outer coil $j^b_\phi(k)$ is identical to that of the inner coil. It is assumed to be directed azimuthally and vary only along the $z$ direction. The representation of the Fourier transforms are analogous to the Fourier transform formulas of the inner coil. The expression of the magnetic energy of the system is

$$W = -\frac{\mu_0}{2} \sum_{m=\infty}^{\infty} \int_{-\infty}^{\infty} dk \left[ a^2 I'_m(ku) K'_m(ku) |j^a_\phi(m,k)|^2 ight. \\
\left. + b^2 I'_m(kb) K'_m(kb) |j^b_\phi(m,k)|^2 \\
+ ab I'_m(ku) K'_m(kb) \left(j^a_\phi(m,k) j^{*b*}_\phi(m,k) + j^{*a*}_\phi(m,k) j^b_\phi(m,k) \right) \right] (5.27)$$

We now proceed to calculate the derivatives of the the magnetic field in the two regions $\rho < a, b < \rho$. Assuming again, that the magnetic field is symmetric around the origin and is independent on the azimuthal angle $\phi$, which equates the conjugate components of the current density for the inner and the outer coil. The analogous relation for the magnetic field derivatives at the origin is now
for $\rho < a$ and $n = 1, \ldots, N$

$$\frac{\partial^{(2n)} B_z}{\partial z^{(2n)}} = \frac{\mu_0}{2\pi} \int_{-\infty}^{\infty} dk (-1)^n k^{2n+1}$$

$$[a j_0^*(k) K_1(\rho a) + b j_0^*(k) K_1(\rho b)]$$  \hspace{1cm} (5.28)

for $b < \rho$ and $m = N + 1, \ldots, N_{\text{tot}}$

$$\frac{\partial^{(2m)} B_z}{\partial z^{(2m)}} = \frac{\mu_0}{2\pi} \int_{-\infty}^{\infty} dk (-1)^m k^{2m+1}$$

$$[-a j_0^*(k) I_1(\rho a) K_0(\rho c) - b j_0^*(k) I_1(\rho b) K_0(\rho c)]$$  \hspace{1cm} (5.29)

where $N_{\text{tot}}$ represents the total number of constraints.

Again, we construct the functional $\mathcal{E}$ as

$$\mathcal{E}(j_0^*(k), j_0^*(k)) = W_m \left[ \sum_{n=0}^{N} \lambda_n \left( B_z^{(2n)} - B_z^{(2n)} \right) + \sum_{m=N+1}^{N_{\text{tot}}} \lambda_m \left( B_z^{(2m)} - B_z^{(2m)} \right) \right]_{x=0, \rho=0, c}$$  \hspace{1cm} (5.30)

where $\lambda_n, \lambda_m$ represent the Lagrange multipliers for the inside and the outside constraints respectively. Also, $\rho_c$ is the point where the 5 G condition is imposed. $\mathcal{E}$ is a function of both $j_0^*(k), j_0^*(k)$. Therefore, we must minimize $\mathcal{E}$ with respect to these two quantities. Doing so, we obtain a matrix equation
for these two components. The solution of the matrix equation results to the following expressions

\begin{align}
J_2^a(k) &= \frac{1}{2\pi a I_n' (ka)} \sum_{n=0}^{N} \lambda_n (-1)^n k^{2n+1} \\
J_2^b(k) &= -\frac{1}{2\pi b} \sum_{n=N+1}^{\infty} \lambda_n (-1)^n k^{2n+1} \frac{K_n(k\rho_c)}{K_1(kb)}
\end{align} (5.31)

The expression of the magnetic field in the two regions is

\begin{align}
B_z^{(2a)} &= \frac{\mu_0}{12\pi^2} \int_{\rho}^{\infty} dk \left[ \sum_{n=0}^{N} \lambda_n (-1)^{i+n} k^{(2n+2i-N+1)} \frac{K_n(k\rho_c)}{I_n(ka)} \\
&\quad - \sum_{n=N+1}^{\infty} \lambda_n (-1)^{i+n-N-1} k^{(2n+2i-N)} K_n(k\rho_c) \right] \tag{5.33}
\end{align}

for \( \rho < a \)

\begin{align}
B_z^{(2b)} &= -\frac{\mu_0}{12\pi^2} \int_{-\rho}^{-\infty} dk \left[ \sum_{n=0}^{N} \lambda_n (-1)^{i+n-N-1} k^{(2n+2i-N)} K_n(k\rho_c) \\
&\quad + \sum_{n=N+1}^{\infty} \lambda_n (-1)^{i+n-2N-2} k^{(2n+2i-4N-2)} K_n^2(k\rho_c) \frac{I_1(kb)}{K_1(kb)} \right] \tag{5.34}
\end{align}

for \( b < \rho \)

The associated magnetic energy is

\begin{align}
W_m &= \frac{\mu_0}{4\pi^2} \int_{-\infty}^{\infty} dk \left\{ \frac{K_n(ka)}{I_n(ka)} \left[ \sum_{n=0}^{N} \lambda_n (-1)^n k^{2n+1} \right]^2 \right\}
\end{align}
\[ + K_0(k\rho_c) \left( \frac{I_1(kb)}{K_1(kb)} \left[ \sum_{n=N+1}^{N_{tot}} \lambda_n(-1)^{i+n-N-1}k^{2n+2i-N} \right]^2 \right. \\
\left. + \left[ \sum_{n=0}^{N} \lambda_n(-1)^{n}k^{2n+1} \sum_{n=N+1}^{N_{tot}} \lambda_n(-1)^{i+n-N}k^{2n+2i-N} \right] \right) \] (5.35)

The determination of the Lagrange multipliers \( \lambda_n \) can be done with the help of the constraint equations (5.28),(5.29).

Therefore, when the values of the Lagrange multipliers are determined, we can evaluate both Fourier components of the current density and in succession the magnetic field and the stored energy. Since the stored energy is not a major concern for the design of main magnets, an additional elimination of higher order derivatives is possible.

We now proceed with the design of a self-shielded magnet set. In this design, we employ all the restrictions for the current densities which have been mentioned above. Furthermore, the radius of the inner cylinder is chosen to be \( a = 0.5 \text{ m} \) while the radius of the outer cylinder is \( b = 0.625 \text{ m} \). Our main objective is to generate a homogeneous magnetic field with less than 4 ppm variation over a 40 DSV imaging region which drops to 5G at 3 m from the center of the coils. In order to satisfy all the requirements, a total
<table>
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<th>$B_{z_n}(2n)$</th>
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Table 5.2: Constraint set used for the design of a self-shielded main-magnet coil. Values for $\rho$ and $z$ are in m, while values for $B_{z_n}(2n)$ are in T.

of 11 constraint points were considered, as shown in Table 5.2. The first seven constraints establish the strength of the magnetic field of 0.5 T and the number of even field derivatives that must be eliminated (up to the $12^{th}$ one). The four remaining constraints are responsible for the behavior of the magnetic field for the region outside both coils. Specifically, the first of four constraints demands that the value of the field is 5 G at distance $\rho_c = 2.5 \text{ m}$ from the center of the coils. The remaining three constraints eliminate the contribution of the field derivatives (up to the $6^{th}$) at this point.

Figure 5.6 displays both continuous current density distributions for the
inner, while Figure 5.7 illustrates the current density for the outer coil. We notice the four peaks which appear for the inner current density and indicate probable positions of the coils, are helpful for the initial guess in the discretization procedure. Furthermore, the shape of the current density for the outer coil indicates two peaks, one with a negative sign and the other with a positive sign. Thus, there is a change of the current density from the non-shielded design to the actively-shielded one. There are additionally two peaks which appear at the self-shielded design. Also the additional current density of the outer coil contributes to the homogeneity of the magnetic field, while at the same time shields the magnetic field outside. In Figure 5.8, we display the variation of the magnetic field from its prescribed value, 0.5T. We notice that the inhomogeneity of the field is less that 4 ppm inside the 40 cm DSV. At larger distances, the contribution of the 14th derivative becomes significant and the homogeneity becomes worse.

In conclusion, we have presented a design for main magnets, non-shielded and actively-shielded, using derivative constraints. Although additional discretization techniques must be employed, the behavior of the continuous
Figure 5.6: Current density distribution for the self-shielded magnet design. Seven major current peaks appear on both sides of the current pattern of the inner coil. The current distribution provides valuable information about the number of coils required, their initial position, and the sign and magnitude of the current for each coil. For additional information, we refer the reader to Thompson [5] and Pizzanefsky [1] papers which provide valuable information about the design of main magnets.
Figure 5.7: Current density distribution for the self-shielded magnet design. Two major peaks one positive and the other negative appear in the current pattern of the outer coil.
Figure 5.8: Plot of the homogeneity of the static field (see Figure 5.3) for the main magnet, using inside and outside constraint points.
Part II
Gradient Coils
Chapter 6

Elliptical Coils

6.1 Introduction

The second part of this dissertation deals with the mathematical development and novel geometries of gradient coils. As we have mentioned earlier, gradient coils are geometries which generate static magnetic fields varying linearly along one of the primary axes for a given geometry. We have also discussed the necessity of designing these gradient coils in order to obtain the lowest possible energy associated with the best linearity and uniformity of the field over the imaging volume.

Up to now, the most commonly used shape for the gradient coils is the cylindrical one [1, 27, 18, 56]. But, there are a lot of disadvantages with the
cylindrical coil design. First, they produce significant eddy current effects in the main magnet shield, since they are placed very close to it. Second, with the cylindrical coil design, in order to generate a magnetic field with the desired linearity and uniformity characteristics over a preassigned imaging volume, we create magnetic fields in areas which are not used for imaging purposes. Therefore, we dissipate a larger amount of stored energy than is usually needed and this affects the inductance of the coils. For gradient coils in free-space, the stored magnetic energy is proportional to the fifth power of the coil radius. But when the coil is placed inside the magnet, the relationship changes and the dissipated energy is scaled as $r^N$, where $r$ is the radius of the coil and $N$ is a power greater than 5.

Taking advantage of the relation between the stored energy and the geometry of the coil, there is a tendency to develop gradient coils which will occupy a smaller volume. Among these geometries are biplanar coils [54, 55, 20]. Although the stored energy is reduced, additional difficulties are present. Specifically, assuming that the planes are in the $xz$ plane and their separation is in the $y$ direction, the geometric set up of biplanar coils does not favor
the generation of high quality magnetic fields in the z direction [55]. For high quality fields, the return current paths in the biplanar coils are squeezed towards the edges of the coils which are very close to the main magnet shield. This results in a creation of eddy current effects which do not follow any symmetric pattern and it is very hard to correct. This phenomenon is also referred to as the “edge effect”. In addition, the dissipated energy of the transverse biplanar gradient coils is slightly higher, because the current patterns are confined to lie and close on each plane of the biplanar design, since no interconnection between the planes is allowed.

Elliptical coil set is a geometry which combines both advantages of the cylindrical geometry and the biplanar coils. Comparing with the cylindrical design, the shape of the elliptical coil fits closer to the shape of the human torso. Therefore, the current patterns are closer to the imaging region and it takes less effort to generate the desired magnetic field. Also, the currents for the elliptic coils are placed further away from the magnet shield and result in the reduction of coupling with the coil with the main magnet and thus, the reduction of eddy current effects. Since elliptic coils occupy less
volume, the dissipated energy is also reduced, and we can drive the coil to higher gradient strengths with the same power amplifier. Comparing with the biplanar design, elliptical coils occupy approximately the same volume and consume the same amount of the stored energy. But the elliptic geometry is a compact design which allows the current density to flow freely on its surface without any restrictions. This enable us to generate both transverse gradient coils.

In this chapter, we present the complete set of the elliptical coils. Crooks et al [19] have presented a complete set of gradient coils where one of the three axes, \( y \), is "quasi-elliptical", while the two axes, \( x \) and \( z \), maintain their original cylindrical shape. Although this idea approaches the elliptical geometry, it does not coincide with it by strict mathematical definition. Furthermore, the methodology, which is described in Crook's patent, follows the iterative approach. According to this technique, a certain number of coils with prescribed positions and current magnitude are defined and the magnetic field and the stored energy are evaluated. If the quality of the field and the magnitude of the energy do not satisfy basic gradient coil characteristics,
the parameters of the coils are altered and the same methodology is applied again. This procedure continues until we reach the point where the desired characteristics for the magnetic field are obtained. Thus, this technique does not guarantee the possible lowest energy for this particular design.

Unlike Crook's methodology, we will present an alternative approach for the design of the complete set of the elliptical coils. The technique which will be presented in this chapter is based on Turner's approach [43, 1]. We will also discuss the full mathematical development in an elliptic coordinate reference frame. For the axial gradient coil (z gradient coil), the basic methodology has been presented before [36, 37], [56]. In this chapter, we will also present the mathematical tools necessary for the development of the elliptical coils and the results from the design of the axial elliptic coil. In the last two chapters, we will also describe the analytical theory for the design of the transverse (x and y) elliptical gradient coils.
6.2 Axial Elliptic Gradient Coil

In this section, we will concentrate on the design of the axial elliptic gradient coil. Since each gradient coil possesses distinct symmetric conditions, the theory and development for the generation of the gradient fields along each one of the three axes must be dealt with separately. In the following subsection, we will discuss the theoretical development necessary for the design of the axial elliptical coil. In the following subsections, we will display the numerical methodology of evaluating the functions associated with elliptical geometry, the design procedure, the results and conclusions drawn from this design. Although parts of this section appear in McLachlan [75], we consider it necessary to mention them in this dissertation in order to provide a comfortable understanding of the following sections.

6.2.1 Theory

As a starting point, we consider the reference frame described by elliptical coordinates $(\mu, \theta, z)$ accompanied with a set of three orthonormal vectors $(\mathbf{\hat{a}}_\mu, \mathbf{\hat{a}}_\theta, \mathbf{\hat{a}}_z)$ which defines the basis of the system. This elliptic geometry is
Figure 6.1: Elliptic coil geometry and the associated elliptic coordinates.

displayed in Figure 6.1. According to this definition, the $z$ coordinate coincides with the primary axis of the main magnet, while the other two $\mu, \theta$ correspond to the radial and azimuthal directions, respectively. The relationship between the elliptical and cartesian coordinates is

$$x = \frac{a}{2} \cosh \mu \cos \theta = \frac{a}{2} \eta \xi$$  \hspace{1cm} (6.1)
\[ y = \frac{\alpha}{2} \sinh \mu \sin \theta = \frac{\alpha}{2} \sqrt{\eta^2 - 1} \sqrt{1 - \xi^2} \quad (6.2) \]

\[ z = z \quad (6.3) \]

where \( \eta = \cosh \mu \), \( \xi = \cos \theta \) and \( \alpha \) is constant, defined as

\[ \frac{\alpha}{2} = \sqrt{R_1^2 - R_2^2} \quad (6.4) \]

where \( R_1 \) and \( R_2 \) are the lengths of the semimajor and semiminor axes of the ellipse, respectively.

Recalling from chapter 3, the expression of the Green function in terms of orthogonal coordinates is adequate with the calculation of the vector potential, the magnetic field, and the dissipated energy. In elliptical coordinates, Laplace’s equation can be separated, and thus, an analytic expression for the Green function can be derived.

It is well known that the characteristic solution of Laplace’s equation in cylindrical, spherical, and cartesian coordinates are modified Bessel functions. Spherical Bessel functions of imaginary argument and trigonometric functions, respectively. For the elliptical coordinates, the solution of Laplace’s equation can be represented by a complete set of orthonormal func-
tions, which are called "Mathieu functions" [75]. Different formulas of the Green function in elliptical coordinates involving Mathieu functions combined with Newmann boundary conditions have appeared in the literature [76]-[80]. There is no publication notable to us which describes the free-space Green function in elliptical coordinates. The mathematical development of the Green function has been presented previously [36, 37], [56]. In the present section, we present the final expression of the Green function, which is

\[
G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} = 2 \int_{-\infty}^{\infty} dk e^{ikz(z-z')}
\sum_{n=0}^{\infty} \left[ ce_{2n}(\theta) ce_{2n}(\theta') \frac{Ce_{2n}(\mu_<) F_{ek_{2n}(\mu >)} \left( \frac{p_{2n}}{s_{2n}^2} \right)^2}{(p_{2n})^2} \\
+ ce_{2n+1}(\theta) ce_{2n+1}(\theta') \frac{Ce_{2n+1}(\mu_<) F_{ek_{2n+1}(\mu >)} \left( \frac{s_{2n+1}}{p_{2n+1}} \right)^2}{(p_{2n+1})^2} \\
+ se_{2n+1}(\theta) se_{2n+1}(\theta') \frac{Se_{2n+1}(\mu_<) G_{ek_{2n+1}(\mu >)} \left( \frac{s_{2n+1}}{p_{2n+1}} \right)^2}{(s_{2n+1})^2} \\
+ se_{2n+2}(\theta) se_{2n+2}(\theta') \frac{Se_{2n+2}(\mu_<) G_{ek_{2n+2}(\mu >)} \left( \frac{s_{2n+2}}{p_{2n+2}} \right)^2}{(s_{2n+2})^2} \right] (6.5)
\]

where

\[
p_{2n}' = (-1)^n ce_{2n}(0, q) ce_{2n}(\frac{\pi}{2}, q) / A_0^{(2n)}
\]
\[ p_{2n+1} = (-1)^{n+1} \text{ce}_{2n+1}(0, q) \text{ se}_{2n+1}(\frac{\pi}{2}, q) / \sqrt{q} A_{1}^{(2n+1)} \]
\[ s'_{2n+1} = (-1)^{n} \text{se}_{2n+1}(0, q) \text{ se}_{2n+1}(\frac{\pi}{2}, q) / \sqrt{q} B_{1}^{(2n+1)} \]
\[ s'_{2n+2} = (-1)^{n+1} \text{se}_{2n+2}(0, q) \text{ se}_{2n+2}(\frac{\pi}{2}, q) / \sqrt{q} B_{2}^{(2n+2)} \]

The functions which are described in the expression of the Green function are Mathieu functions. Specifically, \( ce_{r}(\theta, -q), ce_{r}(\theta', -q) \) are even Mathieu functions (as a function of \( \theta \)) expressed as series in cosines\[75\]: \( se_{r}(\theta, -q), se_{r}(\theta', -q) \) are odd Mathieu functions expressed as series in sines. \( Ce_{r}, Fe_{r}, \) are even modified Mathieu functions (as a function of \( \mu \)), and \( Se_{r} \) and \( Ge_{r} \) are odd modified Mathieu functions expressed as series of modified Bessel functions of the first and second kind, respectively\[81\]. The prime on se, implies a derivative with respect to \( \theta \). Also, \( \mu_{>}(\mu_{<}) \) is the greater (lesser) of \( \mu \) and \( \mu' \).

In exact agreement with the behavior of the current density for a cylindrical coil (chapter 5) necessary to generate an axial gradient field, the current density in the elliptical coordinates must behave the same way for the design of an axial elliptic gradient coil. Therefore, it has to vary along the \( \hat{z} \) direc-
tion and directed along the "elliptic azimuthal direction" $\hat{\alpha}_\theta$. Furthermore, the current density is restricted to lie on the surface of the ellipse $\mu = \mu_s$.

With these restrictions, the expression of the current density is

$$\vec{J} = J(z) \hat{\alpha}_\theta (\mu - \mu_s)$$

or

$$\vec{J} = J(z), \left[ -\cosh \mu \sin \theta \hat{a}_\theta + \cos \theta \sqrt{\cosh^2 \mu - 1} \hat{a}_\mu \right] \delta (\mu - \mu_s) \quad (6.6)$$

The above expression for the current density satisfies the continuity equation. Since there is only $z-$dependence on the scalar behavior of the current density, the Fourier transform of $J(z)$ is defined in a similar manner as

$$\hat{J}(k_z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dz' e^{-ik_z z'} J(z') \quad (6.7)$$

The combination of the expression of the vector potential $\hat{A}$ (4.4) and the expression of the current density in the Fourier domain (6.7) is sufficient for the derivation of the three components of $\hat{A}$ in the elliptic coordinate system.

Replacing $\cosh \mu = \eta$, $\cosh \mu_s = \eta_0$, and considering the case where $\eta < \eta_0$, the three components of $\hat{A}$ in terms of $q = \frac{k_z}{2\sqrt{R_1^2 - R_2^2}}$ are:
\[ A_n = 2\alpha \pi^2 \sum_{r=\text{odd}} \int_{-\infty}^{+\infty} dk_2 J_0(k_2) e^{ik_2z} \]

\[
\begin{align*}
\left[ \frac{\eta \sqrt{\eta^2 - 1}}{\sqrt{\eta^2 - \cos^2 \theta}} \sin \theta B_1^{(r)}(s'_r)^{-2} c_{e,r}(\theta, -q) C e_r(\eta, -q) F e k_r(\eta_0, q) \\
- \frac{\eta_0 \sqrt{\eta^2 - 1}}{\sqrt{\eta^2 - \cos^2 \theta}} s_{e,r}(\theta, -q)(p'_r)^{-2} \cos \theta A_1^{(r)} \right] \\
\right] ^{2r+1} \left[ \eta s_r(\eta, -q) G e k_r(\eta_0, -q) \right] ^{r+1} \right]
\end{align*}
\]

(6.8)

\[ A_\theta = 2\alpha \pi^2 \sum_{r=\text{odd}} \int_{-\infty}^{+\infty} dk_2 J_0(k_2) e^{ik_2z} \]

\[
\begin{align*}
\left[ \frac{\sqrt{\eta^2 - 1}}{\sqrt{\eta^2 - \cos^2 \theta}} \cos \theta c_{e,r}(\theta, -q) B_1^{(r)}(s'_r)^{-2} C e_r(\eta, -q) F e k_r(\eta_0, q) \\
+ \frac{\eta_0 \eta}{\sqrt{\eta^2 - \cos^2 \theta}} \sin \theta s_{e,r}(\theta, -q) A_1^{(r)} \\
(p'_r)^{-2} s_{e,r}(\eta, -q) G e k_r(\eta_0, -q) \right] \\
\right] ^{2r+1} \left[ \eta s_r(\eta, -q) G e k_r(\eta_0, -q) \right] ^{r+1} \right]
\end{align*}
\]

(6.9)

\[ A_z = 0 \]

(6.10)

where we have used the orthonormality conditions for the Mathieu functions

\[ \sum_{r=0}^{\infty} \int_0^{2\pi} d\theta' \sin \theta' s_{e,r}(\theta', -q) = \pi \sum_{r=\text{odd}}^{\infty} A_1^{(r)} \]

(6.11)
\[
\sum_{r=0}^{\infty} \int_0^{2\pi} d\theta' \cos \theta' e_{r,-}(\theta', -\eta) = \pi \sum_{r=\text{odd}}^{\infty} B_1^{(r)} \tag{6.12}
\]

The next step is the calculation of the \( z \) component of the magnetic field and the stored magnetic energy. Combining the expressions of the three components of \( \vec{A} \) (6.8)-(6.10), with the \( \vec{\nabla} \times \vec{A} \) and the equation (4.5), we obtain for the magnetic field and the dissipated energy the following expressions:

\[
B_z = (\vec{\nabla} \cdot \vec{A}) = \pi \sum_{r=\text{odd}}^{\infty} \int_{-\infty}^{\infty} dk_2 \frac{J(k_2) e^{ik_2z}}{(\eta^2 - \cos^2 \theta)}
\]

\[
\left[ B_1^{(r)} \epsilon_t k_r(\eta_0, -\eta)(s'_r)^{-2} \sqrt{\eta_0^2 - 1} + \right.
\]

\[
\left. [c_{e_r}(\theta, -\eta) \cos \theta (\eta^2 - 1) C e'_r(\eta, -\eta) - C e_r(\eta, -\eta) \sin \theta e'_r(\theta, -\eta)] + A_1^{(r)} \epsilon_t k_r(\eta_0, -\eta)(p'_r)^{-2} \eta_0 \left[ s_{e_r}(\theta, -\eta) \sin \theta \sqrt{\eta^2 - 1} S e'_r(\eta, -\eta) + \sqrt{\eta^2 - 1} \cos \theta s_{e'_r}(\theta, -\eta) S e_r(\eta, -\eta) \right] \right] \tag{6.13}
\]

\[
W_m = \alpha^2 \pi^2 \sum_{r=\text{odd}}^{\infty} \int_{-\infty}^{\infty} dk_2 J(k_2) J^*(k_2)
\]

\[
\left[ C e_r(\eta_0, -\eta) F e k_r(\eta_0, -\eta)(\eta^2 - 1)(s'_r)^{-2}(B_1^{(r)})^2 + S e_r(\eta_0, -\eta) G e k_r(\eta_0, -\eta) \eta_0^2 (A_1^{(r)})(p'_r)^{-2} \right] \tag{6.14}
\]
In succession with the methodology described in chapter 4, we create the functional $\mathcal{E}$ as

$$
\mathcal{E} = W_m - \sum_{j=1}^{J} \lambda_j [B_z(\vec{r}_j) - B_{z*}(\vec{r}_j)]
$$

(6.15)

where $\lambda_j$ are Lagrange multipliers, $B_z(\vec{r}_j)$ is the calculated value of the magnetic field at the point $\vec{r}_j$, and $B_{z*}(\vec{r}_j)$ represents the constraint value of the field at the same points. Minimizing $\mathcal{E}$ with respect to $J(k_2)$, we get an expression for the $J^*(k_2)$. Since we are interested in the $z$ gradient coil, the current density must be asymmetric around the origin of the ellipse. Therefore, the implementation in the Fourier domain is $J^*(k_2) = -J(k_2)$. With this in mind, the expression of $J(k_2)$ is

$$
J(k_2) = \frac{-2i}{\pi^2} \sum_{r=odd} \sum_{j=1}^{N} \lambda_j \frac{\sin k_2 z_j}{\eta_j^2 - \cos^2 \theta_j} \left[ \left( B^{(r)}_j \right)_y (Gk_r(\eta_0))_y (p'_r)_j \eta_0 H_j \right] \left/ \left( \sum_{r=odd} D_r \right) \right.
$$

(6.16)

with

$$
J_\theta(z) = \frac{1}{\pi^2} \int_0^\infty dk_2 \sin k_2 z \ J(k_2)
$$

(6.17)
and

\[ E_l = \epsilon \varepsilon_r(\theta_l) \cos \theta_l (\eta_l^2 - 1) \epsilon' \varepsilon_r(\eta_l) - \epsilon \varepsilon_r(\theta_l) \sin \theta_l \eta_l \epsilon_c(\eta_l) \]

\[ H_l = \epsilon \varepsilon_r(\theta_l) \sin \theta_l \eta_l \sqrt{\eta_l^2 - 1} \sec(\theta_l) + \epsilon \varepsilon_r(\theta_l) \cos \theta_l \sqrt{\eta_l^2 - 1} \sec(\eta_l) \]

\[ D_r = (\epsilon \varepsilon_r(\eta_0) F_{r} k_{r}(\eta_0)(\eta_0^2 - 1)(s_r')^{-2}(B_{l}^{(r)}E_l^2 + \sec(\eta_0) G_{r} k_{r}(\eta_0)^2 p_r' )^{-2}(A_{l}^{(r)}E_l^2) \]

Then the expressions of the magnetic field (6.13) and the energy (6.14) are

modified as

\[ B_{m} = \frac{\mu_0}{4 \pi a^2} \int_{-\infty}^{\infty} dk_2 \sum_{r=0}^{\infty} \frac{\sin k_2 z_j}{\eta_j^2 - \cos^2 \theta_j} \left[ (B_{l}^{(r)})(F_{r} k_{r}(\eta_0)), (s_r')^{-2} \sqrt{\eta_0^2 - 1} E_l + \right. \]

\[ \left. (A_{l}^{(r)}),(G_{r} k_{r}(\eta_0)),(p_r')^{-2} \eta_0 H_l \right] \cdot \]

\[ \left( k_2 \right)^{N} \frac{\sin k_2 z_j}{\eta_j^2 - \cos^2 \theta_j} \left( (B_{l}^{(r)}),(F_{r} k_{r}(\eta_0)),(s_r')^{-2} \sqrt{\eta_0^2 - 1} E_l + \right. \]

\[ \left. (A_{l}^{(r)}),(G_{r} k_{r}(\eta_0)),(p_r')^{-2} \eta_0 H_l \right) / \left( \sum_{r=0}^{\infty} D_r \right) \] (6.18)

\[ W_{m} = \frac{\mu_0}{1 \pi a^2} \int_{-\infty}^{\infty} dk_2 \left[ \sum_{r=0}^{\infty} \sum_{j=1}^{N} \frac{\sin k_2 z_j}{\eta_j^2 - \cos^2 \theta_j} \left( (B_{l}^{(r)}),(F_{r} k_{r}(\eta_0)),(s_r')^{-2} \sqrt{\eta_0^2 - 1} E_l + \right. \]

\[ \left. (A_{l}^{(r)}),(G_{r} k_{r}(\eta_0)),(p_r')^{-2} \eta_0 H_l \right) \right]^2 / \left( \sum_{r=0}^{\infty} D_r \right) \] (6.19)
Again, the determination of the Lagrange multipliers \( \lambda_j \) can be made using the constraint equation (4.2) and the expression of the magnetic (6.18). Then the matrix coefficient \( C'_{ij} \) can be written as

\[
C'_{ij} = \frac{\mu_0 16}{\pi a^2} \int \frac{dk_1}{2\pi i} \sum_{r=\text{odd}}^{\infty} \frac{\sin k_2 z_i}{\eta_i^2 - \cos^2 \theta_i} \left[ (B_1^{(r)})_i(Fek_r(\eta_0))_i(s'_r)^{-2} \sqrt{\eta_i^2 - 1} E_i + \right.
\left. \eta_0 H_i \right] \cdot
\left[ \sum_{j=1}^{\infty} \frac{\sin k_2 z_j}{\eta_j^2 - \cos^2 \theta_j} \left( (B_1^{(r)})_j(Fek_r(\eta_0))_j(s'_r)^{-2} \sqrt{\eta_j^2 - 1} E_j + \eta_0 H_j \right) \right] \left/ \left( \sum_{r=\text{odd}}^{\infty} D_r \right) \right. \quad (6.20)
\]

Thus, we have established the theoretical background for the development of the axial gradient coil.

### 6.2.2 Mathematical Development of Mathieu Functions

As it has been indicated, Mathieu functions are the representative functions for the elliptic coordinates. There also exists an entire bibliography for their definitions and their properties [75]. But, we feel that because of the uniqueness of their properties and the lack of their extensive use, a brief presentation
about Mathieu functions will help to better understand the material of this chapter.

As a beginning, let us assume Laplace’s equation

$$\nabla^2 \Psi = 0$$  \hspace{1cm} (6.21)

The above equation can be separated as

$$\nabla^2_{x,y} \Psi(x, y, z) + \frac{\partial^2}{\partial z^2} \Psi(x, y, z) = 0$$  \hspace{1cm} (6.22)

Assuming that $\Psi(x, y, z) = R(x, y)T(z)$ and $\frac{\partial^2 T}{\partial z^2} = -k_1^2 T(z)$. Then equation (6.22) becomes

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) R(x, y) + k_1^2 R(x, y) = 0$$  \hspace{1cm} (6.23)

Using the transformation from the cartesian to the elliptic coordinates (6.3) the above equation becomes

$$\left[ \left( \frac{\partial^2}{\partial \mu^2} + \frac{\partial^2}{\partial \theta^2} \right) + 2k^2 (\cosh 2\mu - \cos \theta) \right] R(\mu, \theta) = 0$$  \hspace{1cm} (6.24)

where $2k = k_1 h$, with $h$ the interfocal distance of the ellipse. Applying the method of separation of variables for $R(\mu, \theta)$, with $R(\mu, \theta) = M(\mu)\Lambda(\theta)$. 

equation (6.21) splits into two differential equations

\[ \frac{\partial^2 M(\mu)}{\partial \mu^2} - \left( a - 2q \cosh 2\mu \right) M(\mu) = 0 \]  \hspace{1cm} (6.25)

\[ \frac{\partial^2 \Lambda(\theta)}{\partial \theta^2} - \left( a - 2q \cos \theta \right) \Lambda(\theta) = 0 \]  \hspace{1cm} (6.26)

where \( q = k^2 \), and \( a \) is an arbitrary constant.

The solutions of the equation (6.26) are Mathieu functions. For a positive sign of \( q \), the entire set of the solutions are

\[ ce_{2n}(\theta, q) = \sum_{r=0}^{\infty} A^{(2n)}_{2r} \cos(2r\theta) \]

\[ ce_{2n+1}(\theta, q) = \sum_{r=0}^{\infty} A^{(2n+1)}_{2r+1} \cos(2r + 1)\theta \]

\[ se_{2n+1}(\theta, q) = \sum_{r=0}^{\infty} B^{(2n+1)}_{2r+1} \sin(2r + 1)\theta \]

\[ se_{2n+2}(\theta, -q) = (-1)^n se_{2n+2}(\frac{\pi}{2} - \theta, q) \]  \hspace{1cm} (6.27)

where \( A, B \) are constants and are dependent only on \( q \). If the sign of \( q \) is negative, then the set of solutions (6.27) is modified as follows

\[ ce_{2n}(\theta, -q) = (-1)^n ce_{2n}(\frac{\pi}{2} - \theta, q) \]

\[ ce_{2n+1}(\theta, -q) = (-1)^n se_{2n+1}(\frac{\pi}{2} - \theta, q) \]

\[ se_{2n+1}(\theta, -q) = (-1)^n ce_{2n+1}(\frac{\pi}{2} - \theta, q) \]
\[ se_{2n+2}(\theta, -q) = (-1)^n se_{2n+2}(\frac{1}{2} \pi - \theta, q) \quad (6.28) \]

Furthermore, the family of solutions for the differential equation (6.25), when the sign of \( q \) is negative, can be represented by a set of Modified Mathieu functions as follows:

\[
C'_{2n}(\mu, -q) = (-1)^n \frac{c_{2n}(0, q)}{A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} I_{2r}(2\sqrt{q} \cosh \mu) \\
F'_{2n}(\mu, -q) = (-1)^n \frac{c_{2n}(0, q)}{A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} K_{2r}(2\sqrt{q} \cosh \mu) \\
C'_{2n+1}(\mu, -q) = (-1)^n \frac{s_{2n+1}(0, q)}{\sqrt{q} B_1^{(2n+1)}} \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2n+1)} I_{2r+1}(2\sqrt{q} \cosh \mu) \\
F'_{2n+1}(\mu, -q) = (-1)^n \frac{s_{2n+1}(0, q)}{\sqrt{q} B_1^{(2n+1)}} \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2n+1)} K_{2r+1}(2\sqrt{q} \cosh \mu) \\
S_{2n+1}(\mu, -q) = (-1)^n \frac{s_{2n+1}(0, q)}{\sqrt{q} A_1^{(2n+1)}} \tanh \mu \sum_{r=0}^{\infty} (-1)^r (2r + 1) A_{2r+1}^{(2n+1)} I_{2r+1}(2\sqrt{q} \cosh \mu) \\
G_{2n+1}(\mu, -q) = \frac{c_{2n+1}(0, q)}{\sqrt{q} A_1^{(2n+1)}} \tanh \mu \sum_{r=0}^{\infty} (-1)^r (2r + 1) A_{2r+1}^{(2n+1)} K_{2r+1}(2\sqrt{q} \cosh \mu) \\
S_{2n+2}(\mu, -q) = (-1)^n \frac{s_{2n+2}(0, q)}{q B_2^{2n+2}} \tanh \mu \sum_{r=0}^{\infty} (-1)^r (2r + 2) B_{2r+2}^{2n+2} I_{2r+2}(2\sqrt{q} \cosh \mu) \\
G_{2n+2}(\mu, -q) = (-1)^n \frac{s_{2n+2}(0, q)}{q B_2^{2n+2}} \tanh \mu \sum_{r=0}^{\infty} (-1)^r (2r + 2) B_{2r+2}^{2n+2} I_{2r+2}(2\sqrt{q} \cosh \mu) \quad (6.29) \]

where \( A, B \) are the same coefficients which appeared in equation (6.27). We now proceed to evaluate these coefficients. Since it is easier to deal with the
trigonometric functions instead of the Bessel functions, we will describe the entire process for the determination of $A'$s, $B'$s based on the regular Mathieu functions. The normality conditions for the Mathieu functions are

$$\frac{1}{\pi} \int_0^{2\pi} c_{e_n}^2(\theta, q) d\theta = \int_0^{2\pi} s_{e_n}^2(\theta, q) d\theta = 1,$$

(6.30)

yielding the condition for the coefficients

$$1 = \sum_{r=0}^{\infty} \left[ A_{2r+1}^{(2n+1)} \right]^2 = 2 \left[ A_0^{2n} \right]^2 + \sum_{r=1}^{\infty} \left[ A_{2r}^{(2n)} \right]^2$$

$$= \sum_{r=0}^{\infty} \left[ B_{2r+1}^{(2n+1)} \right]^2 = \sum_{r=0}^{\infty} \left[ B_{2r+2}^{(2n+2)} \right]^2$$

(6.31)

Using also the recurrence relations for the Mathieu functions (McLachlan, Chapter III, Section 3.10), we get

- for $c_{e_{2n}}(x, q)$ and $r \geq 2$

\[ aA_0 - qA_2 = 0 \]
\[ (a - 4)A_2 - q(A_4 + 2A_0) = 0 \]
\[ (a - 4r^2)A_{2r} - q(A_{2r+2} + A_{2r-2}) = 0 \]

- for $c_{e_{2n+1}}(x, q)$ and $r \geq 1$

\[ (a - 1 - q)A_1 - qA_3 = 0 \]
\[ a - (2r + 1)^2 \] \[ A_{2r+1} - q(A_{2r+3} + A_{2r-1}) = 0 \]

(6.32)

• for se\(_{2n+1}(x, q)\) and \( r \geq 1 \)

\[ (a - 1 + q)B_1 - qB_3 = 0 \]

\[ a - (2r + 1)^2 \] \[ B_{2r+1} - q(B_{2r+3} + B_{2r-3}) = 0 \]

(6.33)

• for se\(_{2n}(x, q)\) and \( r \geq 2 \)

\[ (a - 4)B_2 - qB_4 = 0 \]

\[ (a - 4r^2)B_{2r} - q(B_{2r+2} + B_{2r-2}) = 0 \]

(6.34)

the expressions of the coefficients \( A, B \) for small \( q \) are

\[
A_0 = -\frac{1}{2} q^2 + \frac{7}{128} q^4 - \frac{29}{2304} q^6 + \frac{68687}{18874368} q^8 + O(q^{10})
\]

\[
B_1 = 1 - q - \frac{1}{8} q^2 + \frac{1}{64} q^3 - \frac{1}{1536} q^4 - \frac{11}{36864} q^5 + \frac{49}{58924} q^6 - \frac{55}{9437184} q^7
\]

\[- \frac{265}{13246208} q^8 + O(q^9)\]

\[
A_1 = b_1(-q)
\]
\[ B_2 = 4 - \frac{1}{12} q^2 + \frac{5}{13824} q^4 - \frac{289}{79626240} q^6 + \frac{21391}{458647142400} q^8 + \mathcal{O}(q^{10}) \]
\[ A_2 = 1 + \frac{5}{12} q^2 - \frac{763}{13824} q^4 + \frac{1002401}{79626240} q^6 - \frac{1669068401}{458647142400} q^8 + \mathcal{O}(q^{10}) \]
\[ B_3 = 9 + \frac{1}{16} q^2 - \frac{1}{64} q^3 + \frac{13}{20480} q^4 + \frac{13}{20480} q^5 + \frac{5}{16384} q^6 - \frac{1961}{23592960} q^7 + \frac{609}{1041857600} q^7 + \mathcal{O}(q^8) \]

\[ A_3 = b_1(-q) \]
\[ B_4 = 16 + \frac{1}{30} q^2 - \frac{317}{864000} q^3 + \frac{10049}{2721600000} q^4 + \mathcal{O}(q^5) \]
\[ A_4 = 16 + \frac{1}{30} q^2 + \frac{133}{864000} q^3 - \frac{5701}{2721600000} q^5 + \mathcal{O}(q^6) \]
\[ B_5 = \frac{1}{18} q^2 + \frac{11}{774144} q^4 - \frac{1}{147456} q^5 + \frac{37}{891813888} q^6 + \mathcal{O}(q^7) \]
\[ A_5 = b_5(-q) \]
\[ B_6 = 36 + \frac{1}{70} q^2 + \frac{187}{43904000} q^4 - \frac{5861633}{92935987200000} q^6 + \mathcal{O}(q^8) \]
\[ A_6 = 36 + \frac{1}{70} q^2 + \frac{187}{43904000} q^4 + \frac{6743617}{92935987200000} q^6 + \mathcal{O}(q^8) \quad (6.35) \]

and for \( r \geq 7 \).

\[ A_r, \ B_r = r^2 + \frac{1}{2(r^2 - 1)} q^2 + \frac{5r^2 + 7}{32(r^2 - 1)^2(r^2 - 4)} q^4 + \frac{9r^4 + 58r^2 + 29}{64(r^2 - 1)^5(r^2 - 4)(r^2 - 9)} q^6 + \ldots \quad (6.36) \]

The previous expressions (6.35), (6.36) are valid only for small values of
When \( q \) becomes larger, the series representation starts to diverge. In order to obtain an accurate estimation for the coefficients \( A_r, B_r \), we use linear interpolation methods. For more details for the use of this method, see McLachlan (Chapter 3, Sections 3.14, 3.15).

Since there is nowhere in the known mathematical libraries (SLATEC, MATHEMATICA) a library for evaluation of the coefficients of Mathieu functions, we have created our own. The numerical evaluation was implemented in FORTRAN code at a DEC3100 workstation. The calculation of the coefficients has been pursued to a point where the divergence between our estimation and the values from the tables of the National Bureau of Standards [81] was on the computers double precision accuracy. This comparison is illustrated in Table 6.1.

### 6.2.3 Design of an Axial Elliptic Coil

We now proceed with the design of the elliptical \( z \)-gradient coil. The dimensions of the ellipse have been chosen to be is compatible with the human torso. Following the minimization approach, the outcome is the continuous
Table 6.1: Comparison between the leading coefficients obtained in our numerical evaluation and those from the National Bureau of Standards (NBS)

<table>
<thead>
<tr>
<th>Mathematic function</th>
<th>s</th>
<th>Numer. Eval.</th>
<th>Table of NBS</th>
</tr>
</thead>
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<td>$ct_0$</td>
<td>0.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>1.080051647</td>
<td>1.080051649</td>
</tr>
<tr>
<td></td>
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<td>5.0</td>
<td>2.021867523</td>
<td>2.021867488</td>
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<tr>
<td></td>
<td>10.0</td>
<td>3.985870106</td>
<td>3.985870094</td>
</tr>
<tr>
<td>$ct_1$</td>
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<td>1.0</td>
<td>1.0</td>
</tr>
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<td>1.016040711</td>
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<td></td>
<td>5.0</td>
<td>1.206778130</td>
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<tr>
<td></td>
<td>10.0</td>
<td>1.560266057</td>
<td>1.560266051</td>
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<td>$ct_2$</td>
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<td>1.0</td>
<td>1.0</td>
</tr>
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<td>0.5</td>
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<td>1.047942943</td>
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<td>$ct_2$</td>
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<td>0.621263217</td>
<td>0.621261118</td>
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<tr>
<td></td>
<td>10.0</td>
<td>0.783138562</td>
<td>0.783137441</td>
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</tbody>
</table>

current distribution for the elliptical coil. In order to get an actual design of the $z$-gradient coil, we must generate the discrete current distribution which contains information about the number of the individual current loops, their positions relative to the center of the ellipse and the magnitude of the current which each one has to carry. Thus, we must extract the discrete current
distribution from the continuous one. This is feasible, using the "stream function" technique, $\Phi(\mu, \theta, z)$ [51]. Since the current density satisfies the continuity equation, in exact analogy with the vector potential and the magnetic field, there exists a function $\tilde{S}(\mu, \theta, z)$ such that

$$\vec{J} = \vec{\nabla} \cdot \tilde{S}(\mu, \theta, z)$$  \hspace{1cm} (6.37)

In the axial elliptic case, the current density is confined to lie on the surface of the ellipse and is directed along the $\hat{c}_\theta$ direction. Expressing the curl in terms of the elliptic coordinates, we obtain the relation between the current density and the stream function for the case of an axial elliptic coil

$$J_\theta(z) = \frac{\partial}{\partial z} S_i$$  \hspace{1cm} (6.38)

The next step is to break the continuous current distribution into regions of positive and negative current. Then, by integrating expression (6.38) over the line segments, we determine the positions $S_i$ of the individual loops as

$$S_i = \int_{z_i}^{z_{i+1}} dz J_\theta(z)$$  \hspace{1cm} (6.39)

The number of wires in each region is determined using the relation

$$NW = S_{\text{max}}/I$$, where $S_{\text{max}}$ is the maximum value of the current in each re-
gion and \( I \) is the desired magnitude of the current for each loop. For today's gradient amplifiers, we use as a reference point, with current of magnitude 100 Amps in order to generate a gradient field of 15\( mT/m \).

We now continue with the design of the \( z \)-elliptical gradient coil. Two coil configurations will be discussed in this section. First, we consider an ellipse with major axis length of 0.55\( m \) and minor axis length of 0.45\( m \). The strength of the gradient field is specified to be 15\( mT/m \); its linearity is chosen to vary less than 10\% from its actual value at a distance of 0.125\( m \) away from the center of the gradient at both directions. Furthermore, a third constraint is introduced to ensure that the magnetic field is homogeneous within 10\% of its actual value to a plane perpendicular to the gradient axis and up to the distance of 0.125\( m \) away from the origin. The set of these three constraints are shown in Table 6.2 and appear in the same order as we have described them above.

The second geometry is an ellipse with a major axis length of 0.55\( m \) and minor axis length of 0.10\( m \). The set of constraints are identical with the set presented in Table 6.2. Finally, both these geometries have been compared
Table 6.2: Constraint points of the magnetic field for the 55 cm × 45 cm elliptical z gradient coil. The values of the magnetic field are in T.

<table>
<thead>
<tr>
<th>$\mu_0$</th>
<th>$\theta_i$</th>
<th>$z_i$</th>
<th>$B_{z,i}$</th>
</tr>
</thead>
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<td>1.570796</td>
<td>0.001</td>
<td>0.000015</td>
</tr>
<tr>
<td>0.55838</td>
<td>1.570796</td>
<td>0.001</td>
<td>0.000015</td>
</tr>
<tr>
<td>1.0e-05</td>
<td>1.012416</td>
<td>0.001</td>
<td>0.000015</td>
</tr>
</tbody>
</table>

with results obtained from a cylinder with 0.55 m diameter and with the same number of constraints for the field specification.

6.2.4 Results

In this section, we will present the results from the design of the axial elliptic gradient coils. First, we will consider the ellipse with minor axis of 0.45 cm. For the evaluation of the current density, the magnetic field and the stored energy, the first 9 terms for the expansion of the Mathieu functions were considered. The reason is that the higher order coefficients in the summation for the Mathieu expression are very small and their contribution to the final result is insignificant, depending on the level of accuracy with which the experiment is performed. We also compare the outcome from the design of the (55 cm × 45 cm) elliptical coil with a 55 cm diameter cylindrical coil.
The current density behavior along the z axis for both coils are shown in Figure 6.2. The resulting magnetic fields from both these coils are shown in Figure 6.3. The magnetic field generated from the elliptical coil (solid line) is more linear, pushing the rollover point higher compared to the magnetic field corresponding to the cylindrical gradient coil (dashed line). Furthermore, Figure 6.4 displays the uniformities of the magnetic field of the elliptical gradient coil along the x direction (solid line) and y direction (dotted line), and for the cylindrical coil (dashed line) along the transverse direction. We notice that inside the imaging volume of 20 cm DSV the uniformity of the magnetic field generated both by the elliptical and the cylindrical coils is identical. As we move further away from the origin, the uniformity of the field along the horizontal (x) direction is better in the elliptical coil design than in the cylindrical one. Along the vertical (y) direction the magnetic field is more uniform in the cylindrical coil compared to the elliptical one, since in the elliptical design the current patterns are closer to the imaging volume in the vertical direction. Although the position of the current effects the uniformity of the field in the vertical direction, at the same time, it helps
in the reduction of eddy current effects, because it is placed further away from the main magnet shield.

We have also evaluated the dissipated magnetic energy for both systems, where the same number of field constraints have been enforced over the same imaging volume. For the elliptical coil, the stored magnetic energy necessary to generate a 15 mT/m over a 25 cm DSV is found to be 0.971 Joules. The corresponding energy for the cylindrical coil and for similar field specifications over the same DSV is 1.200 Joules [56]. Thus, we notice a reduction of the energy in favor of the elliptical coil by an amount of 19%. This decrease in the energy is slightly larger than the 18% reduction of the volume of the ellipse compared with the cylindrical coil. A reasonable explanation of this behavior is due to the fact that the current in the elliptical coil is closer to the imaging volume, since it requires less effort to generate the magnetic field with the desired specifications.

We now continue our discussion, considering another elliptical coil, where the length of its minor axis is reduced from 0.45 m to 0.40 m while the length of the major axis remains the same. Figure 6.5 displays both current
densities of the elliptical coil (solid line) and the cylindrical coil (dashed line).

We notice that for the elliptical coil necessary to generate the required field specifications over a 25 cm DSV, the negative current lobe has been reduced significantly compared with the current of the cylindrical coil. Furthermore, the magnetic field profiles which are generated from both the elliptical (solid line) and the cylindrical coil (dashed line) are shown in Figure 6.6. Examining both profiles, we find a significant increase of the linearity of the field in favor of the elliptic design, with its rollover point to be pushed further up. Figure 6.7 illustrates the comparison of the uniformity of the magnetic field along the horizontal and vertical directions for both coils. The uniformity of the field for both coils is identical inside a 20 cm DSV. Moving further away from the center point, the field generated from the elliptic design is more uniform in the horizontal direction than that of the cylindrical coil. In the vertical direction the field from the elliptic coil starts diverging faster since the current patterns are closer to the imaging region than before. Again, we also calculated the dissipated energy for the second elliptic coil as being 0.755 Joules. Thus, we have achieved a reduction in the energy over the
cylindrical coil by 37%, while the reduction of the volume is 27%.

Using the stream function methodology, we have discretized the continuous current distribution of the 55 cm x 40 cm elliptic coil. The discretized coil contains 48 current loops with 125 Amps per loop as shown in Figure 6.8. Finally, Table 6.3 displays the energy comparison between the two elliptical coils and the cylindrical coil for the same number of field constraints.

In conclusion to this section, we have presented a prototype design of an axial elliptic gradient coil. We compared this coil with an ordinary cylindrical design, and presented the advantages and disadvantages between these two geometries. Apart from the reduced uniformity of the field generated from the elliptic design in the vertical direction and in regions significantly further away from the center of the imaging volume, the overall design of the elliptic gradient coil is rather superior to the cylindrical one.

6.3 Transverse Elliptic Gradient Coils

Following the development of the axial gradient coil, we will continue our discussion with the mathematical formulation of the transverse elliptical gra-
Figure 6.2: Continuous current distributions for the \((55 cm \times 45 cm)\) elliptical coil (solid line) and the \((55 cm)\) diameter cylindrical coil (dashed line).

dient coils. The creation of the transverse coils combines the knowledge for the development of the axial gradients, the analogy between these coils and the cylindrical coils, and the properties of Mathieu functions.

In this section, we will analyze the formalism which is necessary for the development of elliptic transverse gradient coils. Particularly, we will concentrate our efforts to the development of the \(x\)-gradient coil. The theo-
Figure 6.3: $B_z$ vs $z$ for the (55 cm x 45 cm) elliptical coil (solid line) and the (55 cm) diameter cylindrical coil (dashed line).

Theoretical formalism for the $y$-gradient coil can be derived following the same methodology for the $x$-gradient coil, assuming a 90° rotation of the coordinate system around the $z$ direction of the ellipse. The antithesis with the cylindrical coil where the theoretical development is identical for both transverse directions is that in the elliptical geometry the difference in the length between the major and minor axes leads to separate theoretical formalisms.
Figure 6.4: Plot of the uniformity of the magnetic field in the two geometries. $B_z$ versus $x$ for the 55cm $\times$ 45cm ellipse (solid line), $B_z$ versus $y$ for the 55cm $\times$ 45cm ellipse (dotted line) and $B_z$ versus the radial distance $\rho$ for the (55 cm) diameter cylindrical coil (dashed line).

for the the design of elliptical coils in both transverse axes.

6.3.1 Theory

We now proceed with the mathematics of the transverse $x$ -- elliptic gradient coil. Unlike the axial gradient design (section 6.2), the current density distribution for the transverse coil is composed of two components, one directed
Figure 6.5: Continuous current distributions for the (55 cm × 40 cm) elliptical coil (solid line) and the (55 cm) diameter cylindrical coil (dashed line).

on the axial \( \hat{a}_z \) direction while the other one on the elliptic azimuthal \( \hat{a}_\theta \) direction. These two components of the current density are assumed to have both angular and axial dependence. Restricting the current density to lie on the surface of the ellipse (\( \mu \approx \mu_0 \)), the expression of the current distribution is

\[
\tilde{J}(\mu, \xi, z) = [j_0(\xi, z) \hat{a}_\theta + j_z(\xi, z) \hat{a}_z] \delta(\mu - \mu_0)
\]

(6.40)
Figure 6.6: $B_z$ vs $z$ for the (55 cm x 40 cm) elliptical coil (solid line) and the (55 cm) diameter cylindrical coil (dashed line).

where $\xi = \cos \theta$. Then the expression for the vector potential is

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \int_{0}^{\infty} d\mu' \, d\theta' \, dz' \, \frac{u}{2} (\cosh^2 \mu' - \cosh^2 \theta') \left[ j_{\theta}(\xi', z') \hat{\imath}_\theta + j_z(\xi', z') \hat{\jmath}_z \right] \delta(\mu' - \mu_0) \sqrt{\frac{1}{\cosh^2 \mu_0 - \cos^2 \theta'}} \quad (6.41)$$

with

$$\hat{\alpha}_\theta = -\frac{\cosh \mu \sin \theta \hat{i} + \sinh \mu \cos \theta \hat{j}}{\sqrt{\cosh^2 \mu - \cos^2 \theta}}$$
Figure 6.7: Plot of the uniformity of the magnetic field in the two geometries. $B_z$ versus $r$ for the 55 cm x 40 cm ellipse (solid line), $B_z$ versus $y$ for the 55 cm x 40 cm ellipse (dotted line) and $B_z$ versus the radial distance $\rho$ for the (55 cm) diameter cylindrical coil (dashed line).

$$\hat{a}_z = \hat{z}$$

Integrating equation (6.41) over the delta function, the expression of the vector potential becomes

$$\vec{A} = \frac{\mu_o}{4\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{d\theta' d\zeta'}{2} \frac{1}{|\vec{r} - \vec{r}'|} \mu_{\mu_{\alpha}} \left[ j_o(\zeta', z') \left( - \cosh \mu_o \sin \theta' \hat{z} \right) 
+ \sqrt{\cosh^2 \mu_o - 1 \cos \theta'} \hat{z} \right] + j_z(\zeta', z') \sqrt{\cosh^2 \mu_o - \cos^2 \theta'} \hat{z} \right] \quad (6.42)$$
Figure 6.8: Discrete current distribution for the 55 cm × 40 cm elliptical coil. There are 48 wire loops with current of 125 Amps per wire.

We now replace the expression of the elliptical Green function into equation (6.42), and obtain the three components of the vector potential by multiplying (6.42) with the unit vectors $\hat{a}_{\mu}, \hat{a}_{\theta}, \hat{a}_{z}$. These expressions are

$$A_{\mu} = \frac{\mu_0}{4\pi} \int_{0}^{2\pi} \int_{-\infty}^{\infty} d\theta' d\zeta' \frac{a - \frac{1}{2} \cosh^2 \mu - 1 \cosh \mu \sin \theta' \cos \theta}{\sqrt{\cosh^2 \mu - \cos^2 \theta}}$$

$$+ \frac{\sqrt{\cosh^2 \mu - 1 \cos \theta' \sin \theta}}{\sqrt{\cosh^2 \mu - \cos^2 \theta}} \left\{ 2 \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dk_2 e^{ik_2(z - \zeta')} j_0(\theta', z') \right\}$$
\[
\left\{ \begin{align*}
\mathcal{A}_\theta &= \mu_0 \frac{a}{4\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} d\theta' d\zeta' \frac{a}{2} \cosh \mu_0 \cosh \mu \sin \theta' \sin \theta \left\{ 2 \sum_{r=0}^{\infty} \int_{-\infty}^{\infty} dk_2 e^{ik_2(z-z')} j_0(\theta', \zeta') \right\} \\
\mathcal{A}_z &= \mu_0 \frac{a}{4\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} d\theta' d\zeta' \frac{a}{2} \sqrt{\cosh^2 \mu_0 - \cos^2 \theta'} \left\{ 2 \sum_{r=0}^{\infty} \int_{-\infty}^{\infty} dk_2 e^{ik_2(z-z')} j_1(\theta', \zeta') \right\}
\end{align*}\]
\]

Our main goal is to express these components of the potential in terms of the Fourier transform of the current density. If we make use the generalized expression of the current distribution, $\tilde{J}$, attempting to obtain an expression of its Fourier transform in both axial and azimuthal directions is very difficult. To make the calculations more feasible and to obtain a closed form expression for the vector potential, we must consider a known angular dependence for
either the axial or azimuthal directions. The angular dependence for the other component can be derived using the continuity equation. In agreement with the behavior of the azimuthal current density in cylindrical coordinates, and for the design of the $x$ gradient coil, the expression of the azimuthal component of the current density is chosen to be

\[ j_\theta(\theta, z) = r \cos \theta j_{\theta}(z) \]  

(6.46)

Considering the continuity equation $\vec{\nabla} \cdot \vec{\mathcal{J}}$, in elliptic coordinates, the relation between the azimuthal and the axial component of the current density is

\[ j_z(k_2, \theta) = j_\theta(k_2) \frac{\sin \theta (\cosh^2 \mu - 2 \cos^2 \theta)}{k_2 (\cosh^2 \mu - \cos^2 \theta)^{1/2}} \]  

(6.47)

where, we have already performed the Fourier transform along the axial $z$ direction. Thus, with the above equation (6.47), the expressions of $A_\mu, A_\theta$ are

\[
A_\mu = \frac{2 \pi^2 h_2}{l \pi} \sum_{r=s}^\infty \int_{-\infty}^\infty dk_2 e^{ik_2s}(-1)^{\frac{s}{2}+1}j_{\theta}(k_2) \\
\left\{ c e_r(\theta, -q)(s'_1)^2 \sin \theta \sqrt{\eta^2 - \cos^2 \theta} - \eta \sqrt{\eta^2 - 1} C e_r(\eta, -q) F e_k(\eta, -q) A_2^{(r)} \right. \\
- \frac{\sqrt{\eta^2 - 1} \eta}{\sqrt{\eta^2 - \cos^2 \theta}} s e_r(\theta, -q)(p'_1)^2 \cos \theta S e_r(\eta, -q) G e_k(\eta, -q) B_2^{(r)} \right\}
\]
\[ A_\theta = 2a\pi^2 \frac{\mu_o}{4\pi} \sum_{r=\text{even}}^{\infty} \int_{-\infty}^{\infty} dk_2 e^{ik_2 \eta} (-1)^{\tilde{z}+1} j_0(k_2) \]

\[
\left\{ \frac{\eta \eta_o \sin \theta}{\sqrt{\eta^2 - \cos^2 \theta}} B_2^{(r)} \right. \left. \right. \right. \left. Se_c(\theta, -q)(s')^{-2} Se_r(\eta, -q)G_e(\eta_o, -q) \right.
\]

\[
+ \frac{\sqrt{(\eta_o^2 - 1)(\eta^2 - 1)\cos \theta}}{\sqrt{\eta^2 - \cos^2 \theta}} A_2^{(r)} ce_c(\theta, -q)(s')^{-2} C_e(\eta, -q)F_e(\eta_o, -q) \right\} \] (6.48)

where the following orthonormality conditions have been enforced

\[
\int_0^{2\pi} \cos \theta' \sin \theta' ce_c(\theta' - q)d\theta' = 0
\]

\[
\int_0^{2\pi} \cos \theta' \sin \theta' se_c(\theta' - q)d\theta' = \pi \sum_{r=\text{even}}^{\infty} (-1)^{\tilde{z}+1} B_2^{(r)}
\]

\[
\int_0^{2\pi} \cos \theta' \cos \theta' se_c(\theta' - q)d\theta' = 0
\]

\[
\int_0^{2\pi} \cos \theta' \cos \theta' ce_c(\theta' - q)d\theta' = \pi \sum_{r=\text{even}}^{\infty} (-1)^{\tilde{z}+1} A_2^{(r)}
\]

and we have replaced \( \eta = \cosh \mu \).

Although, the expressions of these two components of the vector potential can be derived in a closed mathematical form, the derivation of the third component is more difficult than we anticipated. We will present the steps for getting the final expression for \( A_z \) in a closed mathematical form. Let us
start with the expression

\[
A_z = \frac{2a_\pi \mu_\omega}{4\pi} \sum_{r=0}^{\infty} \int_{-\infty}^{\infty} dk_2 e^{ik_2 z} \frac{i}{k_2} j_\delta(k_2) \int_0^{2\pi} d\theta' \frac{\sin \theta' (\eta_0^2 - 2 \cos^2 \theta')}{\eta_0^2 - \cos^2 \theta'} \left[ c_{e_r}(\theta', q) c_{e_r}(\theta, -q)(s_r')^{-2} C_{e_r}(\eta, -q) F_{e_r}(\eta_0, -q) + s_{e_r}(\theta', q) s_{e_r}(\theta, -q)(p_r')^{-2} S_{e_r}(\eta, -q) G_{e_r}(\eta_0, -q) \right] \quad (6.49)
\]

Examining equation (6.49), we must evaluate the following integrals

\[
\sum_{r=0}^{\infty} \int_0^{2\pi} d\theta' \frac{\sin \theta' (\eta_0^2 - 2 \cos^2 \theta')}{\eta_0^2 - \cos^2 \theta'} c_{e_r}(\theta', -q) \quad (6.50)
\]

and

\[
\sum_{r=0}^{\infty} \int_0^{2\pi} d\theta' \frac{\sin \theta'}{1 - \frac{1}{\eta_0^2} \cos^2 \theta'} s_{e_r}(\theta', -q) \quad (6.51)
\]

Starting with the expression (6.50), and with the help of the integral 3.631 of Gradshteyn & Ryzhnik [52], the result is

\[
\sum_{r=0}^{\infty} \int_0^{2\pi} d\theta' \frac{\sin \theta' (\eta_0^2 - 2 \cos^2 \theta')}{\eta_0^2 - \cos^2 \theta'} c_{e_r}(\theta', -q) = 0 \quad (6.52)
\]

for any order of \( r \) in the Mathieu function representation.

After an extensive manipulation, the most nearly closed mathematical form which we are able to obtain for the the integral (6.51) is

\[
\sum_{r=0}^{\infty} \int_0^{2\pi} d\theta' \frac{\sin \theta'}{1 - \frac{1}{\eta_0^2} \cos^2 \theta'} s_{e_r}(\theta', -q) = I_1 + I_2 \quad (6.53)
\]
where

\[
I_1 + I_2 = -\frac{\pi \eta_0^2}{(\eta_0^2 - 1)} \sum_{r=0}^{\infty} \left\{ A_1^{(2r+1)} \left[ 1 - \frac{1}{4(\eta_0^2 - 1)} - \frac{3}{8(\eta_0^2 - 1)^2} + \frac{5}{8(\eta_0^2 - 1)^3} + \ldots \right] \\
- A_1^{(2r+1)} \left[ \frac{1}{4(\eta_0^2 - 1)} - \frac{3}{8(\eta_0^2 - 1)^2} + \ldots \right] \\
- A_5^{(2r+1)} \left[ \frac{3}{8(\eta_0^2 - 1)^2} - \frac{5}{8(\eta_0^2 - 1)^3} + \ldots \right] \\
- A_7^{(2r+1)} \left[ \frac{5}{16(\eta_0^2 - 1)^3} + \frac{35}{32(\eta_0^2 - 1)^4} + \ldots \right] \right\}
\]

Therefore, the expression (6.49) for the vector potential has been altered as follows

\[
A_z = 2a \pi^2 \frac{\mu_0}{\rho} \sum_{r=0}^{\infty} \int_{-\infty}^{\infty} dk_2 e^{ik_2 x} j_z(k_2) \left[ s e_r(\theta, -q)(p')^{-2} s c_r(\eta, -q) G e_r(\eta_0, -q)(-1)^r A_1 \right] \\
\sum_{r=0}^{\infty} \int_{-\infty}^{\infty} dk_2 e^{ik_2 x} j_z(k_2) \left[ c e_r(\theta, -q)(s')^{-2} c e_r(\eta, -q) F e_r(\eta_0, -q)(-1)^r \\
(I_1 + I_2) \right]
\]

Knowing the behavior of the vector potential, we proceed to evaluate the expression of the \(z\)-component of the magnetic field, as

\[
B_z = (\vec{\nabla} \times \vec{A})_z = \frac{\alpha}{h_\mu h_\theta} \left\{ \frac{\partial}{\partial \mu}(h_\theta A_\theta) - \frac{\partial}{\partial \theta}(h_\mu A_\mu) \right\}
\]

with

\[
h_\mu = \frac{\alpha}{\frac{1}{2} \sqrt{\cosh^2 \mu - \cos^2 \theta}}
\]
\[ h_\theta = \frac{\alpha}{2} \sqrt{\cosh^2 \mu - \cos^2 \theta} \]

Thus, the expression of the magnetic field is

\[ B_z = \mu \pi \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} (-1)^{n+1} \frac{d k_2 e^{i k_2 z} j_0(k_2)}{n^2 - \cos^2 \theta} \left[ A_{2}^{(r)}(\eta, -q)(s')^{-2} - \sqrt{\eta^2 - 1} \right] \]

\[ \{ r c_{\nu}(\theta, -q) \cos \theta \eta^2 - 1) C e_r(\eta, -q) - C e_r(\eta, -q) \sin \theta \eta e_r(\theta, -q) \} \]

\[ + B_2^{(r)} c_k(\eta, -q)(\eta')^{-2} \eta \left\{ s e_{\nu}(\theta, -q) \sin \theta \eta \sqrt{\eta^2 - 1} - 1 S e_{\nu}(\eta, -q) \right\} \]

\[ + \sqrt{\eta^2 - 1} S e_{\nu}(\eta, -q) \cos \theta s e_{\nu}(\theta, -q) \} \]

(6.55)

where ' indicates the derivative with respect to the argument inside the parenthesis.

We continue our effort with the evaluation of the dissipated energy \( W \).

We have shown the general expression of the stored energy in terms of the current density and the vector potential in chapter 4. The transformation of this expression in elliptic coordinates is

\[ W = \frac{\alpha}{4} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \int_{0}^{\infty} d \mu' \ d \theta' \ dz' \ \frac{\cosh^2 \mu' - \cos^2 \theta'}{\sqrt{\cosh^2 \mu' - \cos^2 \theta'}} \]

\[ \delta(\mu' - \mu_0) \left[ A_{\theta} j_0(\theta', z') + A_{\theta} j_1(\theta', z') \right] \]
and by replacing $A_0, A_1, \ldots$ the form of the dissipated energy in terms of Mathieu functions is

$$W = \frac{\mu_0 a^2 \pi^3}{1} \sum_{r \text{ even}}^{\infty} \int_{-\infty}^{\infty} dk_2 |j_0(k_2)|^2 \left[ \eta_0^2 \left(B_2^{(r)}\right)^2 S\epsilon_r(\eta, -q)G e_k(\eta, -q)\left(p_r\right)^{-2} \right.
\left. + \left(\eta_2^2 - 1\right) \left(A_2^{(r)}\right)^4 C\epsilon_r(\eta, -q) F e_k(\eta, -q) \right]$$

$$\frac{\mu_0 a^2 \pi^3}{1} \sum_{r \text{ odd}}^{\infty} \int_{-\infty}^{\infty} dk_2 |j_0(k_2)|^2 \frac{1}{k_2^2} \left[ \left(p_r\right)^{-2} S\epsilon_r(\eta, -q)G e_k(\eta, -q) \left(A_1^{(r)}\right)^2 \right.
\left. + (l_1 + l_2)^2 \left(s_r\right)^{-2} C\epsilon_r(\eta, -q) F e_k(\eta, -q) \right]$$

(6.56)

Following Turner’s method, we construct the functional $\mathcal{E}$

$$\mathcal{E}(j_0(k_2)) = W - \sum_{j=1}^{J} \lambda_j \left[ B_2(r_j) - B_0(r_j) \right]$$

where the role of all components in the right hand side of this expression have been explained in detail in chapter 4. Minimizing $\mathcal{E}$ with respect to $j_0(k_2)$, the expression of the conjugate Fourier component of the current is

$$j_0^*(k_2) = \sum_{j=1}^{J} \frac{2\lambda_j e^{ik_2 s_j}}{a^2 \pi^2} \sum_{r \text{ even}}^{\infty} \frac{(-1)^{\frac{k_2}{2} + 1}}{\eta_j^2 - \cos^2 \theta_j} \left[ A_2^{(r)} F e_k(\eta, -q)(s_r)^{-2} \sqrt{\eta_2^2 - 1} \right.$$

$$\left. \left\{ ce_r(\theta, -q) \cos \theta_j (\eta_j^2 - 1) C e_r'(\eta, -q) - C e_r(\eta, -q) \sin \theta_j \eta_j e_r'(\theta, -q) \right\} \right.$$  

$$+ B_2^{(r)} G e_k(\eta, -q)(p_r)^{-2} \eta_0 \left\{ se_r(\theta, -q) \sin \theta_j \eta_j \sqrt{\eta_2^2 - 1} S e_r(\eta, -q) \right.$$  

$$+ \sqrt{\eta_j^2 - 1} S e_r(\eta, -q) \cos \theta_j \eta_j e_r'(\theta, -q) \right\} \right] / (D + Q)$$

(6.57)
where

\[
D = \sum_{r=even}^{\infty} \left[ \eta_o^2 \left( B_2^{(r)} \right)^2 \right] S_{e_r}(\eta_o, -q) G_{ek_r}(\eta_o, -q) (p'_r)^{-2} \\
+ \left( \eta_o^2 - 1 \right) \left( A_2^{(r)} \right)^2 C_{e_r}(\eta_o, -q) F_{ek_r}(\eta_o, -q) \right] \tag{6.58}
\]

\[
Q = \left[ (p'_r)^{-2} S_{e_r}(\eta_o, -q) G_{ek_r}(\eta_o, -q) \left( A_1^{(r)} \right)^2 \\
+ (l_1 + l_2) \left( s'_r \right)^{-1} C_{e_r}(\eta_o, -q) F_{ek_r}(\eta_o, -q) \right] \tag{6.59}
\]

In order to simplify the previous expressions, we take advantage of the symmetry conditions which are present for the design of an \( x \) gradient coil.

Specifically, such a coil must be asymmetric with respect to the \( x \) direction and symmetric with respect to the \( y, z \) directions. With these conditions, the behavior of the current density in the Fourier domain is

\[
j_{\theta}(k_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz \cos k_2 z \ j_{\theta}(z) \tag{6.60}
\]

and

\[
j^{*}_{\theta}(k_2) = j_{\theta}(-k_2) = j_{\theta}(k_2) \tag{6.61}
\]

Thus, the expression of the current density is modified as

\[
j_{\theta}(k_2) = \sum_{j=1}^{J} \frac{2 \lambda_j \cos k_2 z_j}{\alpha^2 \pi^2} \sum_{r=even}^{\infty} \frac{(-1)^{j+1}}{\eta_j^2 - \cos^2 \theta_j} \left[ A_2^{(r)} F_{ek_r}(\eta_o, -q) (s'_r)^{-2} \sqrt{\eta_o^2 - 1} \right]
\]
\[ \{ c e_r(\theta, -q) \cos \theta_j (\eta_j^2 - 1) C e'_r(\eta_j, -q) - C e_r(\eta_j, -q) \sin \theta_j \eta_j c e'_r(\theta_j, -q) \} + B_2^{(r)} C e k_r(\eta_0, -q) (p'_r)^{-2} \eta_0 \left\{ s e_r(\theta, -q) \sin \theta_j \eta_j \sqrt{\eta_j^2 - \sqrt{\eta_j^2 - 1} S e'_r(\eta_j, -q)} + \sqrt{\eta_j^2 - 1} S e_r(\eta_j, -q) \cos \theta_j s e'_r(\theta_j - q) \right\} \right\} / (D + Q) \] (6.62)

The expression of the \( z \) component of the magnetic field for the \( x \) gradient

elliptical coil is in terms of the Lagrange multipliers

\[ B_z = \frac{1}{\mu_0} \sum_{r \geq e \text{ven}} \int_{-\infty}^{\infty} (-1)^{\frac{r}{2} + 1} \frac{dk_2 \cos k_2 z}{\eta^2 - \cos^2 \theta} \left[ A_2^{(r)} F c k_r(\eta_0, -q)(s'_r)^{-2} \sqrt{\eta_0^2 - 1} \right] E + B_2^{(r)} C e k_r(\eta_0, -q)(p'_r)^{-2} \eta_0 H \]

\[ \left\{ \sum_{j=1}^{J} \mathcal{L}_j \cos k_2 z_j \sum_{r \geq e \text{ven}} \frac{(-1)^{\frac{r}{2} + 1}}{\eta_j^2 - \cos^2 \theta_j} \left[ A_2^{(r)} F c k_r(\eta_0, -q)(s'_r)^{-2} \sqrt{\eta_0^2 - 1} E_j \right] + B_2^{(r)} C e k_r(\eta_0, -q)(p'_r)^{-2} \eta_0 H_j \right\} / (D + Q) \] (6.63)

where

\[ E = \{ c e_r(\theta, -q) \cos \theta (\eta^2 - 1) C e'_r(\eta, -q) - C e_r(\eta, -q) \sin \theta \eta c e'_r(\theta, -q) \} \]

\[ H = \{ s e_r(\theta, -q) \sin \theta \eta \sqrt{\eta_j^2 - \sqrt{\eta_j^2 - 1} S e'_r(\eta_j, -q)} + \sqrt{\eta_j^2 - 1} S e_r(\eta_j, -q) \cos \theta_j s e'_r(\theta_j - q) \} \]

\[ E_j = \{ c e_r(\theta_j, -q) \cos \theta_j (\eta_j^2 - 1) C e'_r(\eta_j, -q) - C e_r(\eta_j, -q) \sin \theta_j \eta_j c e'_r(\theta_j, -q) \} \]

\[ H_j = \{ s e_r(\theta_j, -q) \sin \theta_j \eta_j \sqrt{\eta_j^2 - \sqrt{\eta_j^2 - 1} S e'_r(\eta_j, -q)} + \sqrt{\eta_j^2 - 1} S e_r(\eta_j, -q) \cos \theta_j s e'_r(\theta_j - q) \} \]
Again, the Lagrange multipliers $\lambda_j$ can be determined using the constraint equation

$$\sum_{j=1}^{J} C_{ij} \lambda_j = B_i$$

where the expression of the coefficient matrix $C_{ij}$ is

$$C_{ij} = \frac{1}{\pi u^2} \sum_{z=\text{even}}^{\infty} \int_{-\infty}^{\infty} (-1)^{i+1} \frac{dk_2 \cos k_2 z_i}{\eta^2 - \cos^2 \theta_i} \left[ A_{2}^{(r)} F e k_r(\eta_o, -q) (s')^{-2} \sqrt{\eta^2_o - 1} - B_i^{(r)} e k_r(\eta_o, -q) (p')^{-2} \eta_o H_i \right]$$

$$\left\{ \sum_{j=1}^{I} \cos k_2 z_j \sum_{r=\text{even}}^{\infty} \frac{(-1)^{j+1}}{\eta_j^2 - \cos^2 \theta_j} \left[ A_{2}^{(r)} F e k_r(\eta_o, -q) (s')^{-2} \sqrt{\eta^2_o - 1} E_j + B_i^{(r)} e k_r(\eta_o, -q) (p')^{-2} \eta_o H_j \right] \right\} / (D + Q) \quad (6.64)$$

Furthermore, substituting the values of $\lambda_j$'s into equation (6.62), the expression of $\theta(k_2)$ is obtained in terms of quantities describing the geometry of the system. Substituting $\theta(k_2)$ into equation (6.47), we get the expression of $j_\ell(k_2, \theta)$. Performing the inverse Fourier transform for both these components, we acquire an expression for the continuous current distribution.

Although this procedure for the design of an $x$ gradient coil is simple theoretically, computationally it is a formidable task. There is an extra summation present in the formulas due to the presence of the current density,
magnetic field and stored energy, and it takes a significantly large computer.
time and design effort to evaluate these quantities. At the present time, the
computer power available at our facilities are not capable of handling such a
load in a reasonable time frame.

The relative expressions for the $y$-gradient elliptical coil can be derived is
a similar way. Besides the change of the angular dependence of the azimuthal
component of the current density $j_0(\theta, z)$ from $\cos \theta$ to $\sin \theta$, the expressions
for the three components of the vector potential will change drastically and
will be represented by different combinations of Mathieu functions. We resist
presenting to the reader another avalanche of the same type of equations. We
feel that the additional presentation of these formulas will not provide any
extra information of the procedure or the design of transverse gradient coils.

In conclusion, we have presented in this chapter the complete theoretical
methodology, design, and results for the axial elliptic gradient coil. We have
shown that we can achieve a reduction in the total energy by almost 40%
compared with the axial cylindrical gradient coil with similar dimensions.
We have also described the discretized version of the axial elliptical coil. We
have then proceeded with the presentation of the mathematical methodology for the design of the transverse gradient elliptical coil. Particularly, we have presented the entire theoretical development for the design of the $x$ gradient coil. We have also argued why the present computer power makes the design of this coil very difficult.
Table 6.3: Comparison of the dissipated energy between the elliptic design and the cylindrical one, for the same number of constraints (one and three). The values of the energy correspond to 15 mT/m gradient field with 10% on-axis linearity and 10% off-axis uniformity for a 20 cm DSV volume.

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Energy in Joules</th>
<th>Percentage Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>From Constraints</td>
<td>Volume Reduction</td>
</tr>
<tr>
<td>Circle</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ellipse(55×45)</td>
<td>0.97</td>
<td>19%</td>
</tr>
<tr>
<td>Ellipse(55×40)</td>
<td>0.76</td>
<td>37%</td>
</tr>
</tbody>
</table>
Chapter 7

Finite Symmetric Cylindrical Coils

7.1 Introduction

Up to this point, we have presented the methodology for the design of magnetic coils with various shapes. The theoretical development which was presented extensively in chapter 4 assumes that the length of the coils is infinite. The determination of the final length of the coil can be done after the discretization procedure, where by controlling the magnitude of the current we can reduce the total length of the coil. This procedure is a crude technique for reduction of the length of the coil. Another idea which was introduced by Turner in 1988 [1], was the presence of apodization factors. According
to this methodology the Fourier component was multiplied by a Gaussian function at the point where the current was not desired and we wanted to reduce its value drastically to zero. The center point of the Gaussian function was chosen far away from the region of high uniformity of the magnetic field in order for its effect to be as small as possible. Furthermore, the Gaussian function is constructed with a variance unity in order to eliminate any scaling problems in the real space domain. Even though this approach is able to significantly reduce the length of the coil, it increases the inhomogeneity of the coil at regions near the boundary of the DSV. Furthermore, for the current distribution for the transverse gradient coil any multiplication of one component by a Gaussian function will affect the behavior of the second component as well. Thus, the discretization procedure will be affected by this Gaussian and the results can be devastating. For example, in the biplanar coil configuration and especially for the $x$ biplanar coil, the current patterns are extended from the center for more than 1 m in each direction, which makes the application of this coil for imaging purposes not feasible. The discretization of this current puts the current patterns in their original
position and the resulting magnetic field strength diverges for the ideal value by an amount of 1% [55]. When the length of the coil is reduced, by multiplying the current density by a Gaussian function, we were able to reduce the farthest distance of the last current pattern to 0.3 m from the center of the coil in each direction, but the reduction of the strength of the coil was on the order of 30%, while its homogeneity on the prescribed volume was decreased by a comparable amount of 20% [55].

In order to avoid this problem, we present a new family of gradient coils, where their length is restricted by a prescribed value. In this case, the choice of the length of the coil can be preconditioned and it no longer depends on the magnetic field strength and the homogeneity of the coil. The advantages of this approach are the following. At first, it confines the current density to lie inside the boundaries of the coil. Second, we avoid the use of the Gaussian factorization which results in the reduction of the strength, the linearity and the uniformity of the coil inside the DSV. Third, we can generate a set of coils with reduced dimensions in order to avoid claustrophobic effects. Fourth, by reducing the length of the coil, we are able to decrease the total volume of
the coil, and thus, to lower the total dissipated energy of the system.

On the contrary, the disadvantages of this finite geometry design are the following. First, since we deal with finite structure, we no longer can express the current density as an integral representation on the Fourier domain and along the axial direction. In this case, we must consider a discrete Fourier series expansion. This limits the picture of the ideal representation for the current density. Although the contribution of the higher order Fourier components is negligible, the exclusion of these components of the series expansion will generate oscillations at the points where the current behavior approaches zero. This problem can be easily avoided with the discretization mechanism, by choosing magnitudes for the currents to be above the threshold of these oscillations.

The approach to the self-shielded design is introduced in order to reduce the effects which gradient coils generate on the main magnet shield. During the application of an imaging sequence, the gradient coils are switched on and off periodically. This causes a change in the flux and thus creates an EMF on the magnet shields and the heating shields. The eddy currents which are
generated produce a magnetic field which destroys the quality of the magnetic field inside the imaging volume. The major concern for eddy current effects is during the initial application of the gradient coils. When the gradients are switched on, eddy currents appear and tend to increase the rise time towards the maximum value of the gradient strength. To account for this effect an amount of “pre-emphasis” on the amplifier is used. According to this technique, the gradients are not driven by a square pulse but by a pulse which has more current at the beginning and systematically reduces, until it reaches a constant value. Although this technique generally reduces the effects from the eddy currents, we must confront a more difficult problem. The shape of the eddy currents and their time variation are strongly dependent on the shape of the surfaces that are generated. Thus, it is possible to have opposite directions at two different points. This means that the “pre-emphasis” will reduce the behavior of the eddy currents at the first point, but it will enhance their effects at the other point. Furthermore, in order for the “pre-emphasis” technique to be effective, we must know the exact behavior of the spatial and time variation of the eddy current fields. There exists another approach
which tries to compensate for the eddy current on the shields. According
to this technique, the design of the shields is not composed of a continuous
sheet of copper, but from smaller individual parts which are interconnected
with a small gap of insulation between them. Although the total area of the
surface remains the same, the area on which eddy currents can be created has
been reduced significantly. Thus, the magnitude and shape of eddy currents
is also reduced and their effects becomes insignificant. Although this design
mechanism reduces the effects from eddy currents, it does not eliminate them.

The idea for elimination of eddy current effects for the gradient coils is
based on the introduction of a second coil. The contribution of the second
coil is twofold. First, it helps to eliminate the magnetic field in the region
outside the system of coils. Second, at the same time it helps to maintain the
quality of the magnetic field inside the imaging volume. The importance of
the second gradient coil was realized by Turner and Bowley [47], where, the
effects of the secondary coil were evaluated inside the imaging volume and
the positions of the current were readjusted in order to compensate for this
change. Furthermore, Martens in his dissertation [56] used Turner's target
field method to generate a compact coil design where the field outside the secondary coil is zero. In this chapter, we will follow the same methodology for the design of self-shielded finite cylindrical coils.

Starting with this chapter and for the rest of part II of this dissertation, we will present gradient coil geometries associated with a restriction to the length of the coils. In addition, for the self-shielded designs the only length which is restricted is that of the inner coil, while the length of the outer coil is left unrestricted. Usually, the resulting length of the outer coil is no more than 25% larger than the length of the inner coil. Certainly, we can create a mathematical formalism in order to contain the restriction of the length of the outer coil as a constraint. Although this can be mathematical feasible, computationally it is tough to generate, since an additional summation is present, which is originated from the restriction of the length of the inner coil.

Particularly, in this chapter, we will present the mathematical methodology, the design strategy and the generated result for self-shielded cylindrical gradient coils of a restricted length, where the magnetic field satisfies certain
symmetric conditions around the geometrical origin of the coil.

### 7.2 Self-Shielded Axial Finite Symmetric Cylindrical Coils

In this section, we will focus on the design and development of an axial self-shielded cylindrical gradient coil with restricted length. We will present the variation of the continuous current distribution of the components of the current density for the inner and outer coils along the main axis of the cylinder, the resulting discrete positions of the wire loops, and the resulting shape of the magnetic field evaluated from the discrete current distribution, using the Biot-Savart law.

#### 7.2.1 Theory

We now start with the development of the analytical background necessary for the design of an axial cylindrical gradient coil. The geometry of a self-shielded gradient coil is shown in Figure 7.1.

For the axial gradient coil, the expression of the magnetic field in the three regions can be simplified, due to the behavior of the current density.
Figure 7.1: Cylindrical geometry for a self-shielded axial gradient coil. The current patterns lie on the surface of the two coils, and are azimuthally directed, while varying along the z direction.

Specifically, no angular dependence is considered for the current and it is only restricted to vary along the z direction. Considering these symmetry conditions for the current and for a set of two coils with radii $a$ and $b$ respectively, with $a < b$, the expression of the $z$ component of the magnetic field in the three regions separated by the two coils is
\[ B_z = \frac{\mu_0}{2\pi} \int_{-\infty}^{\infty} dk \ e^{ikz}k \left[ a j_0^I(k) K_1(ka) I_0(k \rho) + b j_0^A(k) K_1(kb) I_0(k \rho) \right] \quad \text{for } \rho < a \] (7.1)

\[ B_z = \frac{\mu_0}{2\pi} \int_{-\infty}^{\infty} dk \ e^{ikz}k \left[ -a j_1^A(k) K_0(k \rho) I_1(ka) + b j_1^I(k) K_1(kb) I_0(k \rho) \right] \quad \text{for } a < \rho < b \] (7.2)

\[ B_z = \frac{\mu_0}{2\pi} \int_{-\infty}^{\infty} dk \ e^{ikz}k \left[ -a j_0^I(k) K_0(k \rho) I_1(ka) - b j_1^A(k) K_0(k \rho) I_1(kb) \right] \quad \text{for } b < \rho \] (7.3)

where \( j_0^I(k), j_1^A(k) \) are the Fourier transforms of the current densities for the inner coil radius \( a \) and the outer coil (radius \( b \)), respectively.

Our main objective is to generate a magnetic field which is linear in the internal region of both coils and zero in the region outside the two coils.

In order to satisfy the second of these two conditions, the expression of the magnetic field in equation (7.3) must be zero. This can be achieved by relating the current densities of the inner and outer coils. The expression which relates the Fourier components of the current is

\[ a j_0^I(k) K_0(k \rho) I_1(ka) + b j_0^A(k) K_0(k \rho) I_1(kb) = 0 \] (7.4)

or

\[ j_0^A(k) = -\frac{a I_1(ka)}{b I_1(kb)} j_0^I(k) \] (7.5)
Substituting equation (7.5) into equation (7.1), we get the expression of
the magnetic field for the region inside the two coils \( \rho < a \) as

\[
B_r = \frac{\mu_0 a}{2\pi} \int_{-\infty}^{\infty} dk \, k e^{ikz} j_3^a(k)
\]

\[
l_0(kp)K_1(ka) \left( 1 - \frac{l_1(ka)K_1(kb)}{l_1(kb)K_1(ka)} \right)
\]

(7.6)

Also, using equation (7.5), the expression of the dissipated energy from
equation (5.27) is modified as

\[
W = \frac{n^2 \mu_0}{2} \int_{-\infty}^{\infty} dk \, |j_3^a(k)|^2
\]

\[
l_1(ka)K_1(ka) \left( 1 - \frac{l_1(ka)K_1(kb)}{l_1(kb)K_1(ka)} \right)
\]

(7.7)

We now proceed to evaluate the behavior of the Fourier component of the
current density, where the length of the coil is finite. As we have mentioned
previously the current density can be expressed in terms of the Fourier se-
ries. Before we write down the representative expression of the current, we
must consider the symmetric properties of the magnetic field. Since we are
interested in an axial gradient coil, the shape of the magnetic field must vary
linearly along the z direction of the cylinder and has to be antisymmetric
around the geometric origin of the cylinder. Also, the z component of the
field must be homogeneous in the plane which is perpendicular to the $z$ axis, i.e., in the transverse direction. Since there exists a linear relation between the current and magnetic field, the current density of the coil must satisfy similar symmetric conditions with the magnetic field. Furthermore, the finite size of the cylinder prohibits the current to have any value other than zero outside the cylinder.

Assuming that the length of the cylinder is $L$, the expression of the Fourier expansion around the geometric center for the current for the inner coil in a self-shielded design is

$$j_0^x(z) = \sum_{n=1}^{\infty} j_n \sin k_n z \text{ for } |z| \leq \frac{L}{2}$$  \hspace{1cm} (7.8)$$

$$j_0^x(z) = 0 \text{ for } z > \frac{L}{2}$$ \hspace{1cm} (7.9)

where $j_n^x$ are the Fourier coefficients in the expansion and $\sin k_n z$ represents the antisymmetry condition of the current around the origin. The demand that the current is zero outside the coil restricts the set of values that $k_n$ can take. Thus, the set of allowable values for $k_n$ are

$$j_0^x(z) = 0 \text{ for } |z| = \frac{L}{2} \implies k_n = \frac{2n\pi}{L}$$ \hspace{1cm} (7.10)
Knowing the behavior of the current \( j_\phi^a(z) \), we are interested to find its expression in the Fourier domain. Thus we define the Fourier transform of \( j_\phi^a(z) \) as

\[
j_\phi^a(m,k) = j^a_\phi(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\phi \int_{-\infty}^{\infty} dze^{-ikz} j_\phi^a(z) \tag{7.11}
\]

Substituting the expression (7.9) for the current and following the finite length restrictions, the expression of \( j_\phi^a(k) \) becomes

\[
j_\phi^a(m,k) = \sum_{n=-L}^{L} j_n^a \int_{-\frac{L}{Z}}^{\frac{L}{Z}} dze^{-ikz} sin k_n z \tag{7.12}
\]

Performing the integration on the above equation, the expression of \( j_\phi^a(k) \) is

\[
j_\phi^a(m,k) = \sum_{n=1}^{\infty} \frac{F}{2} j_n^a \psi_n(k) \tag{7.13}
\]

with

\[
\psi_n(k) = \left[ \frac{-sin(k-k_n) \frac{L}{2}}{(k-k_n) \frac{L}{2}} + \frac{sin(k+k_n) \frac{L}{2}}{(k+k_n) \frac{L}{2}} \right] \tag{7.14}
\]

We also notice that the dependence of \( \psi_n(k) \) on the sign of \( k \) is

\[
\psi_n(-k) = -\psi_n(k) \tag{7.15}
\]
Having derived the expression for the Fourier component of the current, the expression of the magnetic field is in terms of the Fourier coefficients

\[
B_z = \frac{\mu_0 a}{2\pi} \int_{-\infty}^{\infty} dk \, e^{i k z} \left( k \sum_{n=1}^{\infty} \frac{i}{2} j_n^2 \psi_n(k) \right) I_0(kp)K_1(ka) \left(1 - \frac{I_1(ka)K_1(kb)}{I_1(kb)K_1(ka)} \right)
\]

(7.16)

Since \( \psi_n(k) \) is an odd function in \( k \), the above integral is zero unless the complex exponential is only an odd function of \( k \). Thus, replacing \( e^{ikz} \) by \( i \sin(kz) \), the previous expression of the magnetic field becomes

\[
B_z = -\frac{\mu_0 a}{2\pi} \int_{-\infty}^{\infty} dk \sin(kz) \left( k \sum_{n=1}^{\infty} \frac{i}{2} j_n^2 \psi_n(k) \right) I_0(kp)K_1(ka) \left(1 - \frac{I_1(ka)K_1(kb)}{I_1(kb)K_1(ka)} \right)
\]

(7.17)

The dissipated magnetic energy, using the relation

\[
\left| j_n^a(k) \right|^2 = \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} j_n^a j_n^{a'} \frac{L^2}{4} \psi_n(k) \psi_n'(k)
\]

(7.18)

is found to be

\[
W_m = \frac{a^2 \mu_0}{2} \int_{-\infty}^{\infty} dk I_1(ka)K_1(ka) \left(1 - \frac{I_1(ka)K_1(kb)}{I_1(kb)K_1(ka)} \right) \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} j_n^a j_n^{a'} \frac{L^2}{4} \psi_n(k) \psi_n'(k)
\]

(7.19)
Knowing the expressions of the magnetic field and the stored energy, we construct the functional $\mathcal{E}$ as previously described

$$\mathcal{E}(j_n) = W - \sum_{j=1}^{\infty} \lambda_j (B_z(\vec{r}_j) - B_{ZSC}(\vec{r}_j))$$

Minimizing $\mathcal{E}$ with respect to $j_n^a$, we obtain a matrix equation for the $j_n^a$ as

$$\sum_{n'=1}^{\infty} j_n^{a*} \left\{ aL \pi \int_{-\infty}^{\infty} dk I_1(ka) K_1(ka) \left( 1 - \frac{I_1(ka)K_1(kb)}{I_1(kb)K_1(ka)} \right) \psi_n(k) \psi_{n'}(k) \right\}$$

$$= -\sum_{j=1}^{\infty} \lambda_j \int_{-\infty}^{\infty} dk \sin k z \int_{-\infty}^{\infty} dk \sin k \phi_j(k) \left( 1 - \frac{I_1(ka)K_1(kb)}{I_1(kb)K_1(ka)} \right) \psi_n(k) \psi_{n'}(k) \quad (7.20)$$

or

$$\sum_{n'=1}^{\infty} j_n^{a*} \psi_{n'} = \sum_{j=1}^{\infty} \lambda_j D_{jn} \quad (7.21)$$

Truncating the infinite summations at $M$ terms, the matrix representation of the previous equation (7.21) becomes

$$J^a \mathcal{C} = \Lambda D \text{ or } J^a = \Lambda \mathcal{D} \mathcal{C}^{-1} \quad (7.22)$$

where $J^a$ is a $1 \times M$ matrix, $\mathcal{C}$ is a $M \times M$ matrix, $\lambda$ is a $1 \times N$ matrix and $D$ is a $N \times M$ matrix. The Lagrange multipliers can be found by substituting (7.20)
into the expression of the magnetic field. Then the matrix representation of
the magnetic field is

\[ B_z(r_j) = \sum_{n=1}^{N} J_n^t D_{nm} \text{ or } B_z = J^t D^t \]  \hspace{1cm} (7.23)

where \( B_z \) is a \( 1 \times N \) matrix and the superscript \( t \) is the symbol for the
transpose matrix. Replacing \( J^t \) from equation (7.22) to equation (7.23), the
expression of the magnetic field becomes

\[ B_z = AD(C^{-1}D) \]  \hspace{1cm} (7.24)

which leads to the determination of Lagrange multipliers as

\[ \lambda = B_z \left[ DC^{-1}D^t \right]^{-1} \]  \hspace{1cm} (7.25)

providing that the inverse matrix for the expression \( [DC^{-1}D^t] \) exists. Upon
determination of Lagrange multipliers, the expression for the Fourier com-
ponents of the current density for the primary (inner) coil is in matrix form

\[ J^v = B_z \left[ DC^{-1}D^t \right]^{-1} DC^{-1} \]  \hspace{1cm} (7.26)

Thus, we were able to find the expression for the Fourier components of the
current for the inner coil. Using the equation (7.10), we obtain the continuous
distribution of the current density for the inner coils. To determine the
current for the secondary (outer) coil, we use the relation between the Fourier transforms for both current densities. Since $j_a^k$ are independent of $k$, substituting the result of $j_t^k$ into equation (7.5), we get the expression of the Fourier transform for the secondary coil. Performing an inverse Fourier transform for that expression, we obtain the continuous current distribution for the outer coil. Since we know the current densities for both coils, we use them in order to calculate the final expression of the energy from equation (7.7).

7.2.2 Design

With the theoretical development complete, we now proceed with the design of an axial self-shielded coil with a finite length. The design consists of two cylindrical coils. The inner coil has radius of 0.1 m and is 0.25 m long. The radius of the outside coil is equal to 0.13 m and its length is unrestricted. For the Fourier series expansion, 40 terms were initially chosen, to ensure the reduction of the effects which are generated by the absence of higher-order terms. The dimensions of the gradient coils which are presented in this section are chosen in order to create a coil set necessary for imaging the human
ankle. We now continue our presentation, establishing the properties of the gradient field. Four constraint points were chosen to define the characteristics of the field inside a 15 cm DSV. The first constraint point establishes a gradient field with strength of 10 mT/m. The second constraint defines the linearity of the field along its gradient axis. Specifically, at a distance of 0.075 m from the center of the coil, the magnetic field is confined to vary no more than 2% from its actual value. The remaining two constraints define the uniformity of the field across the plane perpendicular to the gradient axis. According to these two constraints, the magnetic field is limited to a value within 10% from its actual value at a radial distance of 0.075 m away from the center of the coils. This set of constraints is displayed in Table 7.1.

7.2.3 Results

Applying the set of constraints which is described in the previous section, we obtain the continuous current distributions for the inner and the outer coil. The final number of Fourier coefficients which are used at the evalu-
Table 7.1: Constraint set used for the design of a self-shielded axial finite cylindrical gradient coil. Values for $\rho$ and $z$ are in m, values for $B_{z_n}(2n)$ are in T.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\rho_1$</th>
<th>$z_1$</th>
<th>$B_{z_n}(2n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0000</td>
<td>0.001</td>
<td>0.00004000</td>
</tr>
<tr>
<td>2</td>
<td>0.0000</td>
<td>0.075</td>
<td>0.00294000</td>
</tr>
<tr>
<td>3</td>
<td>0.0375</td>
<td>0.001</td>
<td>0.00003800</td>
</tr>
<tr>
<td>4</td>
<td>0.0750</td>
<td>0.001</td>
<td>0.00003600</td>
</tr>
</tbody>
</table>

The axial variation of the magnetic field, the current density and the stored energy was reduced from 10 to 10 terms. We have compared the results which are generated for more than 10 terms in the Fourier expansion and no significant change in the values of these quantities has been observed. The axial variation of the current distribution for the inner coil is displayed in Figure 7.2.

Furthermore, the axial variation of the current distribution for the outer coils is shown in Figure 7.3 The dissipated energy of this coil system is found to be $0.1721$ Joules.

As a next step, we must discretize the continuous current distribution for both coils. We follow a discretization technique which was first introduced by Martens [56]. According to this procedure, the continuous current distri-
Figure 7.2: One-half of the continuous current distribution $j_\phi(z)$ for the inner coil of a self-shielded axial gradient coil with a restricted length.

bution is divided into positive and negative current regions. Integrating the area underneath each region, we find the amount of the total current which is contained in each region. When we have calculated the current for all regions of the cylinder, we proceed with placing discrete current loops in order to mimic the behavior of the continuous current pattern. Then, we fill each region with discrete wires carrying a prescribed amount of current which is
Figure 7.3: One-half of the continuous current distribution \( j^b_0(z) \) for the outer coil of a self-shielded axial gradient coil with unrestricted length. The total length of the coil is also 0.25 m.

the same for all the discretized regions. The decision of the common amount of current for each wire loop, is determined by considering the module of the current from all regions. In this case, the most probable value of the current can be found and we can relate the discrete behavior closer to the continuous current density. Specifically, each region at the continuous current density is
divided into smaller segments which correspond to equal amount of current. Then each wire is placed in the middle point of this segment, in order to obtain an equal contribution from both sides of the segment, when we calculate the magnetic field.

Returning to our design for the finite self-shielded $z$-gradient coil, for the inner coil, we notice that there were 8 negative and positive regions in the continuous current density. The amount of total current in each region is found to be $-967.40, 10.83, -69.10, 14.72, -14.72, 69.10, -10.83, 967.40$, respectively. The closest choice of the common current per wire was $70$ Amps. The total number of wires which were used to simulate the continuous current, were 32. Similarly, for the current density of the outer coil, there are 2 negative and positive regions with a total current of $404.35, -404.35$, respectively. Using $70$ Amps per wire loop, the total number of wires necessary to simulate the continuous current distribution for the outer coil was 12. Thus the combined system consists of 44 wire loops with 70 Amps per wire, as shown in Figure 7.1.

Using the Biot-Savart law, the $z$ component of the magnetic field is eval-
Figure 7.4: Illustration of the discrete current distribution for both coils. The positive currents are shown with a solid line while the negative loops are displayed with a dotted line.

uated and is varying along the gradient axis as displayed in Figure 7.5. The strength of the field is estimated to $42 \, mT/m$ and corresponds to a 5% difference for the actual value of $40 \, mT/m$. At a axial distance of $0.075 \, m$ away from the center of the coil and along the $z$ axis, the divergence of the field is 5% from its constraint value. Furthermore, at a radial distance $\rho = 0.075 \, m$, the divergence of the field is 12.85% from the ideal value and 3.5% from the
Figure 7.5: Plot of the magnetic field $B_z$ vs $z$, resulting from the discrete current distribution. The strength of the gradient field is $42 \text{ mT/m}$. 
constraint value.

We next proceed to evaluate the total inductance of the coil system.

A crude estimation of the total inductance of the coil \( L_{\text{total}} \), can be done using the relationship between the total energy, the total inductance and the current of the wires. Specifically,

\[
L_{\text{total}} = \frac{2W}{I^2}
\]  

(7.27)

With \( W = 0.1721 \) joules and \( I = 70 \) Amps, the first estimation of the total inductance of the system is \( L_{\text{total}} = 70.24 \) \( \mu \text{H} \). A more sophisticated approach involves the evaluation of the self inductance for each of the two coils and the mutual inductance between these two coils. First, it is well known (Smythe [31], Stratton [29]), that the self inductance of a loop with radius \( a \) is

\[
L_{\text{self}} = \mu_0 a \left[ \ln \left( \frac{8a}{\rho} \right) - 1.75 \right]
\]  

(7.28)

where \( \rho \) is the radius of the thickness of the wire. Furthermore, the mutual inductance between two coils with radii \( a, b \) which are separated by a distance
\( d \) is given by (Smythe [31])

\[
M = \frac{2\mu_0 \sqrt{a^2 + b^2}}{k} \left[ \left(1 - \frac{1}{2}k^2\right) K(k) - E(k) \right]
\]  
(7.29)

where

\[
k^2 = \frac{1 + h}{(a^2 + b^2) - d^2}
\]

and \( K(k) \), \( E(k) \) are complete elliptic integrals of the first and the second kind. Therefore, for the inner coil considering 32 coils of radius 0.1 m with a circular cross section of \( \rho = 0.001 \) m, the self inductance is

\[
L_{\text{a, self}} = 19.84 \, \mu H
\]  
(7.30)

The outer coil is assumed to have the same wire cross section, and consists of 12 loops with radius of 0.13 m. Its self inductance is

\[
L_{\text{b, self}} = 10.19 \, \mu H
\]  
(7.31)

Thus, the total self inductance of the system is

\[
L_{\text{a+b, self}} = 30.03 \, \mu H
\]  
(7.32)
The mutual inductance of the combined system is calculated from equation (7.29) and is found to be

\[ M_{ab} = 15.09 \, \mu H \]  

(7.33)

Thus, the total inductance of the system is

\[ L_{\text{total}} = 75.12 \, \mu H \]  

(7.34)

This is 6.75% bigger than the estimated inductance using the expression of the energy.

In conclusion, we have presented a theoretical model, the design and the results for a design of a self-shielded axial cylindrical coil with a restricted length. The discrete version of the current density for both coils has also been presented, and the total energy and inductance of the system have also been evaluated.

### 7.3 Self-shielded Transverse \((X-\text{ and } Y-\text{)}\) Gradient Coil

We now proceed to the design of the transverse gradient coils. Since we deal with cylindrical coordinates, we need to develop only one of the two coils
configurations, say the $x$-gradient coil. The $y$-gradient coil is exactly the $x$-gradient coil rotated by $90^\circ$ with respect to the main axis of the cylinder.

Unlike, the axial self shielded gradient coil, the transverse gradient coils possess different symmetric properties with respect to the geometric center of the coil. They must behave like a Golay pair configuration. In this case the gradient magnetic field for the $x$-gradient coil must vary linearly with $x$. If we consider the transformation from the cartesian to the cylindrical coordinates, the $x$ is proportional to the $r \cos \phi$. Therefore, there is a dependence on the expression for the magnetic field.

In this section, we will present the theoretical development, the design, and the results for a finite $x$-gradient coil configuration. The discretization procedure using the continuous current distribution will be presented, and the magnetic field is evaluated using the Biot-Savart law.

### 7.3.1 Theory

We have argued previously that the magnetic field of the transverse gradient coils must have azimuthal dependence. Since there is one-to-one corre-
spondence between the distance and the magnetic field, and the distance is proportional to the \(r \cos \phi\) the magnetic field must also have similar dependence. Therefore, the current that is necessary to generate such a magnetic field must behave in the same way. In order to be able to produce such a magnetic field, we must consider two components for the current density; one along the azimuthal and the other along the axial direction. The expression for the current density for the inner coil, restricted to lie on the surface of the cylinder \(\rho = \rho_i\), is

\[
\vec{J}^q(\vec{r}) = \left[ j^i_\phi(\phi, z) \hat{\phi} + j^i_z(\phi, z) \hat{z} \right] \delta(\rho - \rho_i)
\]  

(7.35)

We also consider a similar expression for the current of the outside coil \(J_b\).

For the design of the \(x\)-gradient coil, we are interested in a magnetic field which varies linearly along the \(x\) direction and is uniform along the \(y\) and \(z\) directions. These properties define the expression for the current density distribution for both coils. Again, we will consider that only the primary coil is restricted in length, while the length of the secondary is unrestricted. In this case, the expansion of the current density around the geometric center
of the inner coil is given

\[
\begin{align*}
   j^{a}_{\phi}(\phi, z) &= \cos \phi j^{a}_{\phi}(z) = \cos \phi \sum_{n=1}^{\infty} j^{a}_{\phi n} \cos k_{n} z \text{ for } |z| \leq \frac{L}{2} \tag{7.36} \\
   j^{a}_{z}(\phi, z) &= \sin \phi j^{a}_{z}(z) = \sin \phi \sum_{n=1}^{\infty} j^{a}_{z n} \sin k_{n} z \text{ for } |z| \leq \frac{L}{2} \tag{7.37} \\
   j^{a}_{\phi 0}(\phi, z) &= 0 \text{ for } |z| > \frac{L}{2} \\
   j^{a}_{z 0}(\phi, z) &= 0 \text{ for } |z| > \frac{L}{2}
\end{align*}
\]

where \( j^{a}_{\phi n}, j^{a}_{z n} \) are the Fourier coefficients for the \( j^{a}_{\phi}, j^{a}_{z} \) respectively. The expression of the azimuthal component of the current is chosen to follow the behavior of the magnetic field. Furthermore, the current of the inner coil is restricted to obey the continuity equation \( \nabla \cdot \vec{J}^{a} = 0 \). Transforming this relation into cylindrical coordinates leads to the equation

\[
\frac{1}{a} \frac{\partial j^{a}_{\phi}(\phi, z)}{\partial \phi} + \frac{\partial j^{a}_{z}(\phi, z)}{\partial z} = 0 \tag{7.38}
\]

where, we have performed the integration over the delta function. Substituting the expressions for \( j^{a}_{\phi} \) and \( j^{a}_{z} \) from equation (7.38) into equation (7.38), we obtain a relation between the Fourier coefficients of the two current com-
ponents as

\[
\sin \phi \sum_{n=1}^{\infty} \left[ \left\{ \frac{-j_{1n}}{k_n a} + j_{2n} \right\} \cos k_n z \right] = 0 \tag{7.39}
\]

Since \( \sin \phi \) is a variable and can get any value, \( \cos k_n z \) is also an arbitrary variable and cannot be zero, the only choice for which the above expression is zero, is when the quantity inside the brackets is zero. This leads to

\[
j_{2n} = \frac{j_{2n}}{k_n a} \tag{7.40}
\]

Substituting equation (7.40) into equation (7.38), the expressions for the two components are modified as follows

\[
j_{1n}(\phi, z) = \cos \phi \sum_{n=1}^{\infty} j_{1n} \cos k_n z \text{ for } |z| \leq \frac{L}{2} \tag{7.41}
\]

\[
j_{2n}(\phi, z) = \sin \phi \sum_{n=1}^{\infty} \frac{j_{2n}}{k_n a} \sin k_n z \text{ for } |z| \leq \frac{L}{2} \tag{7.42}
\]

and both components are zero for \( |z| > \frac{L}{2} \). As a next step, we must find the spectrum of values which \( k_n \) satisfies. Since the current can not flow outside from the cylinder, it implies that

\[
j_{1n}(\phi, \pm \frac{L}{2}) = 0 \implies k_n(\pm \frac{L}{2}) = \pm n \pi \implies k_n = \frac{2n \pi}{L} \tag{7.43}
\]
Since, the expression of the magnetic field and the energy contain the Fourier components of the current density, we need to find the expression for them in the Fourier space. The Fourier transform of the azimuthal component of the inner coil is defined as

\[ j_\phi^2(m, k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi e^{-im\phi} \int_{-\infty}^{\infty} dz e^{-ikz} j_\phi^2(\phi, z) \]  

or

\[ j_\phi^2(m, k) = \frac{1}{2\pi} \sum_{n=\pm1} j_{\phi n}^2 \int_{-\pi}^{\pi} d\phi \left( \cos m\phi \cos \phi - i \sin m\phi \cos \phi \right) \]

\[ \int_{-\infty}^{\infty} dz e^{-ikz} \cos k_n z \]  

\[(7.45)\]

Since trigonometric functions are orthonormal functions, in the \( \phi \) integration in equation (7.45), the second term is zero, while the first term inside the parentheses obey the relationship

\[ \int_{-\pi}^{\pi} d\phi \cos m\phi \cos \phi = \begin{cases} \pi & \text{for } m = \pm1 \\ 0 & \text{otherwise} \end{cases} \]  

\[(7.46)\]

With this condition in mind, and integrating equation (7.45) over \( z \), the formula for the current density (7.44) is

\[ j_\phi^2(\pm m, k) = \frac{1}{2} \sum_{n=\pm1} \frac{L}{2} j_{\phi n}^2 \psi_n(k) \]  

\[(7.47)\]
with

\[ \psi_n(k) = \left[ \frac{\sin(k - k_n) \frac{1}{2}}{(k - k_n) \frac{1}{2}} + \frac{\sin(k + k_n) \frac{1}{2}}{(k + k_n) \frac{1}{2}} \right] \quad (7.48) \]

where \( \psi_n(k) \) is an even function of \( k \).

Examining closer the relation (7.47), it is obvious that

\[ j_\phi^n(-m, k) = j_\phi^m(m, k) \quad (7.49) \]

since \( \cos m\sigma \) is an even function of \( \sigma \). Furthermore, applying the continuity equation to both Fourier representations of the components of the current for the inner coil, the relation between them is

\[ j_z^m(m, k) = -\frac{m}{ka} j_\phi^m(m, k) \quad (7.50) \]

Besides the expression that is demonstrated above (7.50), the relationship (7.5) between the current densities of the primary and the secondary coil still holds. Having this in mind, the general expression of the magnetic field for a transverse gradient coil in the region where \( \rho < a \) and is given by

\[ B_z = -\frac{\mu_0 a}{2\pi} \sum_{m=-\infty}^{\infty} e^{im\phi} \int_{-\infty}^{\infty} dk \ k \ e^{ik\rho} j_\phi^m(m, k) \]

\[ l_m(k\rho) K_m(k\rho) \left( 1 - \frac{L_m'(ka) K_m'(kb)}{L_m'(kb) K_m'(ka)} \right) \quad (7.51) \]
Since only \( m = \pm 1 \) values of \( m \) are allowed, and the modified Bessel functions satisfy the condition

\[
I_1(kp) = I_{-1}(kp)
\]
\[
K'_1(ka) = K'_{-1}(ka)
\]
\[
l'_1(ka) = l'_{-1}(ka)
\]

the expression of the magnetic field is

\[
B_z = -\frac{\mu_o d}{2\pi} \cos \Phi \int_{-\infty}^{\infty} dk e^{ikz} k j_0^*(1,k) I_1(kp)K'_1(ka) \left(1 - \frac{l'_1(ka)K'_1(kb)}{l'_1(kb)K'_1(ka)}\right)
\]

(7.52)

Since the combination of all the quantities inside the integral except the exponential term is an even function of \( k \), the only time this integral will be non-zero is when \( e^{ikz} = \cos kz \). Thus, the final expression for magnetic field using equation (7.47) is

\[
B_z = -\frac{\mu_o d L}{1\pi} \cos \Phi \sum_{n=1}^{\infty} j_{on}^* \int_{-\infty}^{\infty} dk \cos kz \psi_n(k) k
\]
\[
l_1(kp)K'_1(ka) \left(1 - \frac{l'_1(ka)K'_1(kb)}{l'_1(kb)K'_1(ka)}\right)
\]

(7.53)
In a similar fashion, the dissipated energy of the system is

\[
W = -\frac{n^2 \mu \mu L^2}{16} \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} j_{\alpha n} j_{\alpha n'}^* \int_{-\infty}^{\infty} dk \psi_n(k) \psi_{n'}(k) \left( 1 - \frac{l_{i}(ka)K_{i}^{'}(ka)}{l_{i}(kb)K_{i}^{'}(ka)} \right)
\]

(7.54)

We now proceed to construct the functional \( \mathcal{E} \) which has the identical form with equation (7.20). The only difference in this case is that the dependence of \( \mathcal{E} \) is over \( j_{\alpha n} \). Thus, minimizing \( \mathcal{E} \) with respect to \( j_{\alpha n} \), we obtain a matrix equation for \( j_{\alpha n'} \), which is

\[
\sum_{n'=1}^{\infty} j_{\alpha n'} \left\{ \frac{\alpha L \pi}{2} \int_{-\infty}^{\infty} dk l_{i}^{'}(ka)K_{i}^{'}(ka) \left( 1 - \frac{l_{i}(ka)K_{i}^{'}(ka)}{l_{i}(kb)K_{i}^{'}(ka)} \right) \psi_{n}(k) \psi_{n'}(k) \right\}
\]

\[
= \sum_{j=1}^{N} \lambda_j \psi_{\alpha}(\mathbf{r}_j) \int_{-\infty}^{\infty} dk k \cos k \int_{-\infty}^{\infty} \psi_{\alpha}(\mathbf{r}_j) \psi_{\alpha}^{*}(\mathbf{r}_j) \left( 1 - \frac{l_{i}(ka)K_{i}^{'}(ka)}{l_{i}(kb)K_{i}^{'}(ka)} \right)
\]

(7.55)

Truncate the previous infinite summations at \( M \) terms, and using compact notation, the previous expression is modified

\[
\sum_{n'=1}^{M} j_{\alpha n'} C_{n'n} = \sum_{j=1}^{N} \lambda_j D_{jn}
\]

(7.56)

or

\[
J^a \mathbf{C} = \Delta \mathbf{D} \text{ or } J^a = \Delta \mathbf{D} \mathbf{C}^{-1}
\]

(7.57)
where $J'$ is a $1 \times M$ matrix, $C$ is a $M \times M$ matrix, $A$ is a $1 \times N$ matrix and $D$ is a $N \times M$ matrix. Since equation (7.57) is identical to the expression (7.22), we can determine the expressions of the current for both coils following exactly the steps which are described at the last part of the section 7.2.1.

### 7.3.2 Design

Having completed the mathematical formulation for the $x$-gradient coil, we will present in this section the design strategy for this transverse cylindrical finite coil. In absolute agreement with the section 7.2.2, the radius of the primary coil is chosen to be equal to $0.1 \, m$, and its length is restricted to $0.25 \, m$. For the secondary coil, its radius is $0.13 \, m$, while its length is unrestricted. Again, 10 terms in the Fourier representation for both components of the current distributions of the inner coil were considered. In addition, three constraint points were chosen to specify the quality of the magnetic field inside a $15 \, cm$ DSV. The first constraint sets the strength of the magnetic field to $40 \, mT/m$. The second constraint specifies a $10\%$ variation from the actual value of the magnetic field along the gradient axis, $x$, and at a
<table>
<thead>
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<th>$n$</th>
<th>$\rho$</th>
<th>$z$</th>
<th>$B_{z,e}(2n)$</th>
</tr>
</thead>
<tbody>
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<td>1</td>
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<td>0.000</td>
<td>0.00004000</td>
</tr>
<tr>
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<td>0.0750</td>
<td>0.000</td>
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<tr>
<td>3</td>
<td>0.0010</td>
<td>0.075</td>
<td>0.00003800</td>
</tr>
</tbody>
</table>

Table 7.2: Constraint set used for the design of a self-shielded transverse finite cylindrical gradient coil. Values for $\rho$ and $z$ are in m, values for $B_{z,e}(2n)$ are in T.

distance of 0.075 m from the sweet spot of the gradient. The third constraint ensures that the uniformity of the magnetic field is not worse than 10% from its actual value at a distance of 0.075 m from the center of the coil. This set of constraints is shown in Table 7.2. The presence of this set of constraints will generate continuous current distributions for both components of the inner and outer coils of the design. Our goal is to use these current distributions in order to generate discrete current patterns which are necessary for the construction of the coils.

The discretization mechanism for a transverse gradient coil design is different from the discretization procedure of the axial gradient coil. The same methodology has been also introduced before by Martens [56], but we feel that the presentation of this procedure is necessary to understand the mech-
anism of creating a discrete current pattern from a continuous current distribution.

At first, we consider the continuity equation for the current density

$$\vec{\nabla} \cdot \vec{J} = 0$$  \hspace{1cm} (7.58)

In analogy with the magnetic field, where a vector potential is introduced, the current density can be expressed as a curl of a function $\vec{S}$, called a "stream function". Specifically

$$\vec{J} = \vec{\nabla} \times \vec{S}$$  \hspace{1cm} (7.59)

Since the current is restricted to flow on the surface of a cylinder with radius $a$ and has only angular and axial dependence, the relation between the current density and the stream function in cylindrical coordinates is

$$j^2_\phi(\phi, z)\hat{\alpha}_\phi + j^2_z(\phi, z)\hat{\alpha}_z = \frac{\partial S_\rho}{\partial z}\hat{\alpha}_\phi - \frac{1}{a}\frac{\partial S_\rho}{\partial \phi}\hat{\alpha}_z$$  \hspace{1cm} (7.60)

and $S_\rho$ is found from

$$S_\rho(\phi, z) = -a \int_{-\pi}^{\phi} d\phi' j^2_z(\phi', z)$$  \hspace{1cm} (7.61)
The contour plots of the current density are determined by

\[ S_{n}(x, z) = \left( n - \frac{1}{2} \right) S_{\text{inc}} + S_{\text{min}} \text{ for } n = 1, ..., N \]  

(7.62)

where \( N \) is the number of the current contours, \( S_{\text{min}} \) is the minimum value of the current density and \( S_{\text{inc}} \) represents the amount of the current between two contours lines. The determination of \( S_{\text{inc}} \) is

\[ S_{\text{inc}} = \frac{S_{\text{max}} - S_{\text{min}}}{N} \]  

(7.63)

with \( S_{\text{max}} \) representing the maximum value of the current density. The contours which are generated by this method follow the flow of the current and the distance between them corresponds to a current equal to an amount of \( S_{\text{inc}} \) in Amps. Finally, the discrete wires are positioned in such way as to coincide with these contour lines.

### 7.3.3 Results

Applying the design requirements, we obtain the expression for the continuous displays the axial variation of \( j_{r}^{a} \) and \( j_{r}^{b} \), since only the \( z \) component of the current density is necessary for the creation of the discrete current patterns.
Figure 7.6: One-half of the plots of the continuous current distributions of $j^i_z$ and $j^s_z$ versus $z$. The behavior of the current for the inner coil is illustrated in solid line, while the current for the outer coil is in dashed lines.
Figure 7.7: One quadrant of the discrete current pattern for the primary coil. There are 16 current loops with 122.28 Amps per loop.

For the evaluation of these current densities, only 20 Fourier coefficients were used for the same reasons which are explained in section 7.2.3.

With the help of the continuous current, we obtain the discrete current patterns, employing the stream function technique. Figure 7.7 displays the discrete current distribution for the primary gradient coil. There are 16 current loops, each one carrying an amount of current equal to 122.28 Amps.
Figure 7.8: One quadrant of the discrete current pattern for the secondary coil. There are 5 current loops with 122.28 Amps per loop.

Performing a similar procedure for the outer coil, we obtain a discretized version of the continuous current containing 5 current loops with an amount of current equal to the current of the loops for the inner coil, as shown in Figure 7.8. In the next step, we use the discrete current patterns to calculate the magnetic field using the Biot-Savart law. The magnetic field is re-evaluated is to ensure that the discretization mechanism is the correct
one and the choice of the number of loops is adequate to generate the desired quality of the magnetic field. Of course, by increasing the number of coils, we approach the continuous distribution and the quality of the magnetic field is improved. The only parameter which limits the number of coils lies on the engineering aspect. Specifically, there are limitation on the number of coils, imposed from the thickness of the wires and the spacing which is necessary between each wire. Using both discrete current patterns. Figure 7.9 displays the variation of the $z$ component of the magnetic fields along the gradient axis ($x$) for both coils. The net magnetic field strength $39 \, mT/m$, differs by $2.5\%$ from the actual field strength, while varying by $12\%$ from its ideal on-axis value and $10.5\%$ from its ideal off-axis uniformity value.

Furthermore, the total energy of the system has been calculated and is equal to $1.01 \, Joules$. In analogy with section 7.2.3, the knowledge of the total energy will give us an estimation of the total inductance of the system. Although this estimation is not accurate, it indicates the magnitude of the total inductance, which can be very close to the actual value. Therefore, for energy of $1.01 \, Joules$ and current of $122.28 \, Amps$ per wire, the total
Figure 7.9: One-half plot of the $B_z$ vs $x$ for the transverse gradient coil.
inductance of the system is \( L_{\text{total}} = 135.09 \, \mu H \).

Finally, we have presented in this chapter two gradient coil designs, one axial and the other transverse. We have discussed the mathematical methods and the design techniques. We have also presented the results associated with these designs, using discretization mechanisms which are useful for the engineering development of such coils.
Chapter 8

Non-Shielded Asymmetric Gradient Coils

8.1 Introduction

In previous chapters, we have argued for the need of high efficiency gradient coils. Specifically, the design of new and improved sequences for faster acquisition of data and improve resolution in MRI can lead to a major improvement to real time cardiac imaging, Magnetic Resonance Angiography (MRA), Echo-Planar Imaging (EPI) and fast imaging. Among the requirements for the design of the state of the art sequences are high quality gradient coils with excellent homogeneity, reduced inductance, better slew rates and low $\frac{\partial B}{\partial t}$ effects.
Up to this point, the geometries which have been presented in this dissertation were restricted to the cases where the center of the gradient field coincides with the geometric center of the coil. Thus, imaging the region between the lower part of the head and the upper portion of the neck using conventional gradient coils, requires that the dimensions of the gradient are large enough in order to account for the patient’s shoulders. This will result in a large increase of the dissipated energy of the coil and hence to an increase of its total inductance which will result in slower rise times for the gradient coil. Furthermore, the requirement for larger coils will place them closer to the magnet. Thus, for the non-shielded coils, the eddy current effects to the main magnet and the heat dissipated shield will be large and will result in an additional decrease of the rise time of the gradient due to the “pre-emphasis” pulse. In addition, whole-body gradient coils are energy inefficient when we perform imaging to a human head. The first reason is that we deploy a large gradient coil to image a region which is less than 30 cm DSV. The second reason is that the current patterns are farther away from the homogeneous region and it takes an additional effort on their part to generate the high
quality magnetic field. The third reason is that in order to generate high quality gradient fields from the whole-body gradient coils, we generate magnetic fields at regions which are not of interest to us for imaging purposes, and thus, we waste dissipated energy.

In this chapter we will present the design of a new generation of gradient coils where the center of the gradient field has been shifted axially towards the edge of the coil. This will provide us the freedom to design smaller gradient coils, which are energy efficient, have higher field strengths and generate less eddy current effects. This new generation of gradient coils will be referred to in this dissertation as “asymmetric” gradient coils. For example, let us consider a comparison between a gradient coil which contains the patient’s shoulders with the asymmetric gradient coil having the same field specifications over the same imaging volume. For the whole-body gradient coil the dimensions should be no less than 0.60 m in diameter, while the asymmetric gradient coil is required to be a bit larger than the diameter of the human head which is 0.35 m. Knowing that the energy of the gradient coil is scaled like the radius of the coil risen to the fifth power and assuming that the diam-
eter of the asymmetric coil is 0.40 m, the energy needed from the asymmetric design is 7.6 times less than the corresponding energy for the whole body coil. In addition, and especially for the axial gradient coils, the relation for the Maxwell pair coils indicates that the total length whole-body gradient coil must be at least 1.2 m in order to satisfy the Maxwell pair relation. On the contrary, for the asymmetric gradient coil, the total length can be reduced to 0.6 m, which will provide an additional decrease for the energy of the system since the volume of the coil has been reduced. Thus, instead of 15 mT/m gradient strength employed to the whole-body gradient coil, we can design the asymmetric gradient coil with identical field specifications and gradient strength of 10 mT/m. This significant reduction of the dissipated energy provides the necessary features that are required for short echo sequences, EPI, functional imaging. Besides the requirement for high quality gradient field inside the imaging volume, the reduction of the total length of the coil will help the reduction of the $\frac{\partial B}{\partial t}$ effects. In addition, the decrease of the diameter of the coil will reduce the eddy current effects at the main magnet and at the heating shield.
In the proceeding sections of this chapter, we will introduce the complete set of asymmetric gradient coils. We will present the theoretical methodology, the design, and the results for the axial and transverse gradient coils.

The procedure which we will follow in this design is based on a variation of Turner's [13, 1] method for minimization of the dissipated energy with magnetic field constraints and the calculation of the continuous current distribution. Knowing the current density and employing the stream function technique, the discretized version of the current density is created. Finally, we make use of the Biot-Savart law to ensure that this current generates the appropriate magnetic field with the desired characteristics. Because the theoretical development of the axial gradient coil differs from the transverse one, we will present the analysis and the results from both these coils separately.

We will also present a comparison between the theoretical predictions and the experimental results, and we will provide pictures which are obtained by performing imaging with this set of coils.

Even though the size of the non-shielded gradient coils help in the reduction of the eddy current effects, it does not eliminate them. Thus, in order
to avoid any pre-emphasis gradient pulse, we must design a self-shielded asymmetric gradient coil set. Although the energy of this coil set is a bit higher than the non-shielded case, it provides the additional feature of the elimination of eddy current effects. Furthermore, the asymmetric design from nature appears to generate torque effects, because the current is placed non-symmetrically around the geometric center of the coil. Although the self-shielded design does not eliminate the problem, the appearance of the second gradient coil helps in the reduction of the overall torque of the system by a factor which is roughly proportional to the ratio of the radii between the inner and the outer coils. In the last section of this chapter, we will present the theoretical analysis, the design, and the results for the self-shielded set of gradient coils. Again, the approach which we will follow is based on Turner's method for minimizing the stored energy, providing a set of field constraints. The discretized version of the continuous current distribution is also created using the stream function technique and the resulting field is calculated using the Biot-Savart law.
8.2 Non-shielded Axial Asymmetric Gradient Coil

As we have mentioned in the introduction of this chapter, asymmetric gradient coils provide an alternative of generating a high quality gradient field while maintaining their prescribed specifications. We have also mentioned the energy and eddy current advantage. One thing which was not specified was that these coils are designed with a restricted length. Thus, they provide us with an extra degree of freedom: in essence, we can control the overall length of the coil and design in order to maintain magnetic field integrity while avoiding claustrophobic problems with the patients. In this section we will present in every detail the methodology of developing non-shielded gradient coils along the axial and transverse direction.

The problem under consideration is illustrated in Figure 8.1. The coordinate system has been chosen such that its origin coincides with the geometric center of the coil. The center of the gradient field, which will be referred as a "sweet spot", is shifted along the axial direction and towards one edge of
Figure 8.1: Illustration of the asymmetric non-shielded cylindrical gradient coil design. It shows the geometric center as well as the “sweet spot”. The cylinder has length $L$. The distance of the sweet spot from the end of the cylinder is defined with $l_{ss}$, while the radius of the cylinder is denoted as $a$.

8.2.1 Theory

Following the discussion which was presented in the introduction, for the advantages of the non-shielded asymmetric coils in this section, we are interested for the design of the $z$ (axial) gradient coil. The current density must
be azimuthally oriented and can be only a function of \( z \). This is in exact analogy of what we have presented for the symmetric case in chapter 7, and is consistent with the continuity equation. Furthermore, the asymmetricity of the field and hence the current along \( z \), combined with the restriction of the length of the cylinder, demands that it must be an additional term in the expansion of the current density around the origin of the coordinate system.

Thus the current density has the form

\[
j_\phi(z) = \sum_{n=1}^{\infty} \left[ j_{1n} \sin k_{1n}z + j_{2n} \cos k_{2n}z \right] \quad |z| \leq \frac{L}{2}
\]

(8.1)

\[
j_\phi(z) = 0 \quad |z| > \frac{L}{2}
\]

(8.2)

where \( L \) represents the length of the coil, \( j_{1n}, j_{2n} \) are the Fourier coefficients associated with the two terms at the current expansion, and \( \cos k_{2n}z \) is the extra term which is introduced to the Fourier expansion in order to account for the asymmetric condition of the current. The demand that the current density must vanish at both ends of the coils, puts a restriction to the spectrum of values which \( k_{1n} \) and \( k_{2n} \) can get. Thus, the condition (8.2),
defines the spectrum of values of \( k_{1n} \) and \( k_{2n} \) as

\[
\sin k_{1n}L = 0 \quad \Rightarrow \quad k_{1n} = \frac{2\pi n}{L} \quad (8.3)
\]

\[
\cos k_{2n}L = 0 \quad \Rightarrow \quad k_{2n} = \frac{(2n - 1)\pi}{L} \quad (8.4)
\]

Since the expressions of the magnetic field and the dissipated energy involve the Fourier transform of the current density, we now proceed with its evaluation as

\[
J_\omega^1(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dc \int_{-\infty}^{\infty} dz e^{-ikz} j_\omega^1(z)
\]

or

\[
J_\omega^2(k) = \int_{-\frac{L}{2}}^{\frac{L}{2}} dz e^{-ikz} \sum_{n=1}^{\infty} \left[ j_{1n}^1 \sin k_{1n}z + j_{2n}^1 \cos k_{2n}z \right] \quad (8.5)
\]

Replacing the trigonometric functions with their relative Euler formula in the complex representation, and performing the integration over \( z \), the expression of \( J_\omega^1(k) \) is

\[
J_\omega^1(k) = \sum_{n=1}^{\infty} \left[ \frac{1}{i} \left[ j_{1n}^1 \psi_{1n}(k) - i j_{2n}^1 \psi_{2n}(k) \right] \right] \quad (8.6)
\]

with

\[
\psi_{1n}(k) = \left[ \frac{-\sin(k - k_{1n})\frac{k}{2}}{(k - k_{1n})\frac{k}{2}} + \frac{\sin(k + k_{1n})\frac{k}{2}}{(k + k_{1n})\frac{k}{2}} \right] \quad (8.7)
\]
\[ v_{2n}(k) = \left[ \frac{\sin(k - k_{2n}) \frac{L}{2}}{i(k - k_{2n}) \frac{L}{2}} + \frac{\sin(k + k_{2n}) \frac{L}{2}}{(k + k_{2n}) \frac{L}{2}} \right] \] (8.8)

We have also found that the dependences of \( v_{1n}(k) \) and \( v_{2n}(k) \) on the sign of \( k \) is

\[ v_{1n}(-k) = -v_{1n}(k) \]

\[ v_{2n}(-k) = v_{2n}(k) \] (8.9)

We now proceed with the determination of the mathematical expressions for the magnetic field and the dissipated energy. Since we have only one coil and are interested in the region inside the coil, the expression of the magnetic field is

\[ B_z = \frac{\mu_0 a}{2\pi} \int_{-\infty}^{\infty} dk \, k e^{ikz} j_0^2(k) I_0(k \rho) K_1(ka) \text{ for } \rho < a \] (8.10)

Substituting equation (8.6) into the above equation, the expression of the magnetic field changes to

\[ B_z = \frac{\mu_0 a}{2\pi} \int_{-\infty}^{\infty} dk \, e^{ikz} k I_0(k \rho) K_1(ka) \]

\[ - \sum_{n=1}^{\infty} \frac{L}{2} \left[ j_{1n}^2 v_{1n}(k) - i j_{2n}^2 \psi_{2n}(k) \right] \] (8.11)
Since \( \psi_{1n}(k) \), \( K_1(ka) \) and \( \psi_{2n}(k) \), \( I_0(k\rho) \) are odd and even functions of \( k \) respectively, the above integration over \( k \) is zero unless the complex exponential is modified such that the integrand to be an even function of \( k \) overall.

Replacing \( e^{ikz} \) by \( \cos kz - i \sin kz \), the previous expression becomes

\[
B_z = -\frac{\mu_t \mu_0}{2\pi} \int_{-\infty}^{\infty} d k \sin kz \ k \ I_0(k\rho) K_1(ka) \sum_{n=1}^{\infty} \frac{L_n^2}{2} j_{2n} \psi_{1n}(k) \\
-\frac{\mu_t \mu_0}{2\pi} \int_{-\infty}^{\infty} d k \cos kz \ k \ I_0(k\rho) K_1(ka) \sum_{n=1}^{\infty} \frac{L_n^2}{2} j_{2n} \psi_{2n}(k) \quad (8.12)
\]

For a single coil, we know that the dissipated energy is

\[
W = \frac{a_t^2 \mu_0}{2} \int_{-\infty}^{\infty} d k I_1(ka) K_1(ka) \left| j_\phi(k) \right|^2 \quad (8.13)
\]

Using the relation

\[
\left| j_\phi(k) \right|^2 = \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} \frac{L_n^2}{1} \left[ j_{1n}^2 j_{1n'}^2 \psi_{1n}(k) \psi_{1n'}(k) + j_{2n}^2 j_{2n'}^2 \psi_{2n}(k) \psi_{2n'}(k) \right] \quad (8.14)
\]

equation (8.13) becomes

\[
W_m = \frac{a_t^2 \mu_0}{2} \int_{-\infty}^{\infty} d k I_1(ka) K_1(ka) \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} \frac{L_n^2}{1} \left[ j_{1n}^2 j_{1n'}^2 \psi_{1n}(k) \psi_{1n'}(k) + j_{2n}^2 j_{2n'}^2 \psi_{2n}(k) \psi_{2n'}(k) \right] \quad (8.15)
\]
In order to obtain the expression of the current density which minimizes the total energy of the system, we follow Turner's method and construct the functional $\mathcal{E}$

$$
\mathcal{E}(j_{in}^a, j_{in}^b) = W - \sum_{j=1}^{N} \lambda_j \left( B_2(\bar{r}_j) - B_{2,SC}(\bar{r}_j) \right)
$$

To obtain the expression for the current density, we must derive similar expressions for the two Fourier coefficients. Thus, we have to minimize the $\mathcal{E}$ with respect to $j_{in}^a$ and $j_{in}^b$. Minimizing $\mathcal{E}$ with respect to $j_{in}^a$, we obtain a matrix equation for $j_{in}^a$, which is

$$
\sum_{n'=1}^{\infty} j_{1n}^a \left\{ a I_n \pi \int_{-\infty}^{\infty} dk I_1(ka) K_1(ka) \psi_{1n}(k) \psi_{1n'}(k) \right\} = -\sum_{j=1}^{N} \lambda_j \int_{-\infty}^{\infty} dk \sin k z_j I_0(k \rho_j) K_1(ka) \psi_{1n}(k)
$$

or

$$
\sum_{n'=1}^{\infty} j_{1n'}^a C_{1n'1n} = \sum_{j=1}^{N} \lambda_j D_{j1n}
$$

(8.16)

Truncating the infinite summations at $M$ terms, the matrix representation of the previous equation (8.17) is

$$
J_1^a C = \Delta D \text{ or } J_1^a = \Delta D C^{-1}
$$

(8.18)
where $J_1^n$ is a $1 \times M$ matrix, $C$ is a $M \times M$ matrix, $A$ is a $1 \times N$ matrix and $D$ is a $N \times M$ matrix.

We now follow the same procedure for the $j_{2n}^2$ coefficient. Thus, minimizing $\mathcal{E}$ with respect to $j_{2n}^2$, we obtain a matrix equation for $j_{2n}^2$ which is

$$
\sum_{n' \neq 1} \sum_{\nu \neq 1} \int \int_{-\infty}^{\infty} dk l_0(k) \int_{-\infty}^{\infty} dk l_0(k) K_1(k \alpha) \omega_{2n}(k) \omega_{2n'}(k) \right) \\
= \sum_{n \neq 1} \int_{-\infty}^{\infty} dk \int \int_{0,1} \sin \xi \xi l_0(k) K_1(k \alpha) \omega_{2n}(k)
$$

or

$$
\sum_{n \neq 1} j_{2n}^2 P_{2n}^2 = \sum_{n \neq 1} \lambda_n Q_{2n}^2
$$

Restricting the summation to $M$ terms, the matrix equation for the second Fourier component is

$$
J_2^2 P = \lambda Q \text{ or } J_2^2 = \lambda Q P^{-1}
$$

where $J_2^2$ is a $1 \times M$ matrix, $P$ is a $M \times M$ matrix, $\lambda$ is a $1 \times N$ matrix and $Q$ is a $N \times M$ matrix.

The determination of the Lagrange multipliers can be done with the help of the constraint equation. The matrix representation of the magnetic field
at the constraint points, involving both Fourier components is

\[ B_{2SC}(r) = \sum_{n=1}^{M} \left[ j_{1n}^2 D_{1n} + j_{2n}^2 Q_{2n} \right] \]

or

\[ B_{2SC} = j_1^t D' - j_2^t Q' \]  \hspace{1cm} (8.22)

where \( B_{2SC} \) is a \( 1 \times N \) matrix and the superscript \( t \) is the symbol for the transpose matrix. Substituting the expressions of \( J_1^* \), \( J_2^* \) into equation (8.22), the matrix solution for the Lagrange multipliers is

\[ \lambda = B_{2SC}^{-1} \left[ D C^{-1} D' + Q P^{-1} Q' \right]^{-1} \]  \hspace{1cm} (8.23)

providing that the inverse matrix for the expression \( [D C^{-1} D' + Q P^{-1} Q'] \) exists. When we determine the values of the Lagrange multipliers, the solutions for both components of the current density in a matrix representation is

\[ J_1^* = B_{2SC} \left[ D C^{-1} D' + Q P^{-1} Q' \right]^{-1} D C^{-1} \]  \hspace{1cm} (8.24)

\[ J_2^* = B_{2SC} \left[ D C^{-1} D' + Q P^{-1} Q' \right]^{-1} Q P^{-1} \]  \hspace{1cm} (8.25)

Again, since the Fourier coefficients have been determined, the expression of the Fourier transform of the current density can be derived easily. We
then perform an inverse Fourier transform to this expression and get the continuous current distribution for the asymmetric coil. Knowing the values of the Lagrange multipliers and the Fourier coefficients of the current density, we proceed with the final step to calculate the total dissipated energy of the system.

8.2.2 Design

We now continue with the design of the axial asymmetric gradient coil with a finite length. This design consists of a cylindrical coil with a radius of 0.192 m. The total length of the coil is restricted to 0.60 m and the position of the sweet spot of the gradient field is placed 0.145 m away from the cylinder's end. Furthermore, 10 Fourier terms were considered in order to eliminate the effects which are generated by the absence of the higher order terms in the Fourier series expansion. Because the center of the imaging region has been moved away from the center of the coil, seven constraint points have been chosen to define the linearity of the $z$ component of the magnetic field, $B_z$ along the $z$ axis, the asymmetricity of $B_z$ about the sweet spot.
<table>
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<th>$r_i$</th>
<th>$z_i$</th>
<th>$B_{z,i}$</th>
</tr>
</thead>
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<td>0.1560</td>
<td>0.000004000</td>
</tr>
<tr>
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<td>-0.000004000</td>
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<td>0.2000</td>
<td>0.00183000</td>
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<td>4</td>
<td>0.0000</td>
<td>0.1000</td>
<td>-0.00224000</td>
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</tr>
<tr>
<td>7</td>
<td>0.1336</td>
<td>0.1560</td>
<td>0.000004200</td>
</tr>
</tbody>
</table>

Table 8.1: Constraint set used to design an asymmetric axial $z$-gradient coil. Values for $r$ and $z$ are in m, values for $B_{z,i}$ are in T.

and the homogeneity of the magnetic field inside the prescribed imaging region. Specifically, the first constraint sets the strength of the gradient field to 40 $mT/m$. The following two constraint points define the asymmetricity of the magnetic field about the sweet spot. The next two constraints limit the variation of the magnetic field to within 5% from its ideal value and at a distance of 0.125 $m$ away from the sweet spot for both directions along the $z$ axis. The last constraint defines the homogeneity of the gradient field over the 25 $cm$ DSV. Especially, the magnetic field is restricted to a value of within 5% of its ideal value at the borders of the imaging volume. This set of constraints is illustrated Table 8.1. In addition, the set of constraints are chosen such
that the distance between of the two rollover points (the point where the magnetic field gets its largest value) to be greater than 0.32 m in order to avoid aliasing effects from the patient's neck to his head. Furthermore, extra precautions have been taken in order for the divergence of the local gradient (the gradient strength between two points which are positioned on the same direction inside the DSV) between two points inside the imaging volume to be less than 10% from the prescribed value of 40 mT/m. This is chosen such that we can avoid significant distortions of the image inside the imaging volume. The reason is that in the Fourier domain, the points which are placed near to the edge of the DSV will be assigned to a gradient field which is larger or smaller than its ideal value. Therefore the spins will precess faster or slower than expected. Thus, by performing the inverse Fourier transform, these points will be placed at higher or lower distances from their ideal location. Therefore, the image will appear distorted.
8.2.3 Results

Solving the inverse problem with the set of constraints described in Table 8.1, we obtain the continuous current density. The axial variation of the continuous current density is shown in Figure 8.2. The final number of Fourier terms which are used for the calculation of the current density, magnetic field and dissipated energy was reduced to 7. The choice of this number of Fourier terms was made after comparing the results which are generated from Fourier series which are generated with fewer and larger terms. No significant change to the values of these quantities has been observed, except to the significant reduction of computer time.

As a next step, we must obtain the discrete current patterns from the continuous current distribution. We follow the discretization procedure which was presented in the section 7.2. There are 12 positive and negative current regions. The amount of current corresponding to each of these regions is found to be $-209.94, 287.87, -104.90, 19.11, -101.45, -249.89, 19.79, -2591.41, 1045.43, -1090.06, 1593.28, -3544.21, 4193.23$ respectively, in $Amps$. The com-
Figure 8.2: Plot of the variation of the azimuthal component of the continuous current density along the $z$ axis. This current distribution is necessary to generate a gradient field with $40 \text{ mT/m}$ at a distance of $0.145 \text{ m}$ from the edge of the cylinder.

The common value of the current for each wire loop was chosen to be equal to $328.50 \text{ Amps}$. Furthermore, the total number of wires needed to simulate closer the behavior of the continuous current density is found to be 46. The discretization scheme for the axial asymmetric gradient coil containing the positions of the wires along the length of the cylinder is displayed in Fig-
Using the Biot-Savart law, the $z$ component of the magnetic field is evaluated and its variation along the $z$ axis is displayed in Figure 8.4. The strength of the field is estimated to $43.25 \, mT/m$, which corresponds to an $8\%$ difference from the ideal strength of $40 \, mT/m$. At an axial distance of $0.125 \, m$ away from the sweet spot, the value of the magnetic field is measured to be within $2.7\%$ from the value of the magnetic field which was set by the constraint point. Also, at a radial distance of $0.125 \, m$ from the sweet spot, no significant discrepancy was measured between the constraint value of the magnetic field and the calculated value using the Biot-Savart law.

In addition, the distance between the two rollover points was measured to be $0.35 \, m$. This is greater than the demanded distance and thus, aliasing is avoided. Also, the variation of the local gradient reaches to $10.5\%$ from the ideal strength of the gradient field at the outermost points of the imaging volume.

Since we have achieved a very good agreement between the results which are obtained from the discretized version of the current density and those
that were originally assigned as constraint values to the magnetic field, we
now proceed with the evaluation of the coil inductance. The value of the
dissipated energy of the system was found to be 7.93 Joules. Using the
relationship between the energy and the total inductance of the system, the
estimated value for the total inductance of the system for an amount of
current equal to 328.50 Amps is

\[ L_{\text{total, est}} = \frac{2W}{I^2} = 114.56 \mu H \quad (8.26) \]

Following the procedure which presented in chapter 7, a more accurate value
for the total inductance of the coil can be obtained. Assuming a set of 46
circular wire loops with radius of 0.192 m and circular cross section of 0.001 m,
the self inductance of the coil is

\[ L_{\text{s, self}} = 62 \mu H \quad (8.27) \]

The mutual inductance of the coil using the formula (7.29) is calculated to

\[ M_s = 93.11 \mu H \quad (8.28) \]

Thus, the calculated total inductance of the system is

\[ L_{\text{tot, calc}} = L_{\text{s, self}} + M_s = 155.11 \mu H \quad (8.29) \]
which is 6.8% higher than the value of the total inductance estimated from the dissipated energy $L_{d,estim}$.

8.3 Non-shielded Asymmetric Transverse ($X-$, $Y-$) Finite Cylindrical Gradient Coils

We now move to the presentation of the transverse asymmetric gradient coil geometry. We will discuss the theoretical methodology, the design and the results for the development of the $x$ gradient coil. The $y$ gradient coil configuration can be easily created by rotating counterclockwise the current by 90° with respect to the $z$ axis of the cylinder. The sweet spot of the transverse gradient coil has also been displayed towards the end of the cylinder in order to coincide with the sweet spot of the axial gradient. Although the design of the transverse gradient coil is asymmetric in the sense that the sweet spot of the gradient coil is shifted axially, its design strategy is totally different from the design of the axial gradient coil. The reason is that the gradient field in the transverse case varies linearly along the $x$ or $y$ direction. Thus, the axial shift of the sweet spot does not affect its behavior. This means that the
symmetric conditions of the magnetic field which were employed in chapter 7 still hold for this case. Specifically, the transverse gradient coil must also behave like a Golay pair configuration, which means that the variation of the magnetic field is proportional to the \( \rho \cos \phi \) in the cylindrical coordinate system.

In this section, we will analyze the transverse asymmetric gradient coil geometry. Specifically, we will discuss the mathematical development of the problem, the design strategy and the generated results. The discretization procedure using the continuous current distribution will be presented, and the magnetic field is evaluated using the Biot-Savart law.

### 8.3.1 Theory

In a parallel fashion with the design of a finite symmetric gradient coil in chapter 7, and since the magnetic field has an azimuthal \( \phi \) dependence in the cylindrical geometry, the current density must behave the same way. Because there exists one-to-one correspondence between the magnetic field and the current density, the current density must also be proportional to \( \cos \phi \) along
the azimuthal direction of the cylinder. Since the angular dependence is present in this design, the construction of the total current density for this coil must contain two components: One component along the azimuthal direction and the other along the axial direction of the cylinder. The expression of the current density which is also restricted to lie on the surface of the cylinder with a radius \( a \) is

\[
\vec{J}(\vec{r}) = \left[ j_1(\rho/a) \vec{\omega}_\varphi + j_2(\phi, z) \vec{\omega}_z \right] \delta(\rho - a)
\]  

(8.30)

For the design of the symmetric transverse gradient coil, we demand that the sweet spot of the gradient field must be axially displaced from the center of the cylinder, the magnetic field must be antisymmetric along the \( r \) direction and around the sweet spot of the gradient, and the field must be uniform around the \( y \) and \( z \) directions. These properties of the magnetic field reflect the behavior of the current density. Since we deal with a finite size geometry, we must expand the two components of the current density in Fourier series around the geometry center of the cylinder and their expression
is

\[ J_2^2(\phi, z) = \cos(\phi) \sum_{n \geq 1} \left[ j_{1n}^2 \cos k_{1n} z + j_{2n}^2 \sin k_{2n} z \right] \quad |z| \leq \frac{L}{2} \tag{8.31} \]

\[ J_2^1(\phi, z) = \sin(\phi) \sum_{n \geq 1} \left[ j_{1n}^1 \sin k_{1n} z + j_{2n}^1 \cos k_{2n} z \right] \quad |z| \leq \frac{L}{2} \tag{8.32} \]

\[ J_2^0(\phi, z) = 0 \quad \text{for} \quad |z| > \frac{L}{2} \tag{8.33} \]

\[ J_2^{-1}(\phi, z) = 0 \quad \text{for} \quad |z| > \frac{L}{2} \tag{8.34} \]

where \( L \) represents the length of the coil, \( j_{1n}^a, j_{2n}^a \) are the Fourier coefficients associated with the two terms of \( J_2^a(\phi, z) \) and \( j_{1n}^a, j_{2n}^a \) are the Fourier coefficients associated with the two terms of \( J_2^a(\phi, z) \). Again, \( \cos k_{2n} z \) is the extra term which is introduced to the Fourier expansion in order to account for the axial shift of the gradient's sweet spot. Furthermore, the current density is restricted between the two edges of the cylinder and must vanish at both ends of the coil. This restriction defines the spectrum of the allowable values for the \( k_{1n} \) and \( k_{2n} \). From equations (8.33),(8.34), the two quantities must satisfy

\[ \sin k_{1n} L = 0 \quad \Rightarrow k_{1n} = \frac{2\pi n}{L} \tag{8.35} \]

\[ \cos k_{2n} L = 0 \quad \Rightarrow k_{2n} = \frac{(2n - 1)\pi}{L} \tag{8.36} \]
Again we note there are no free-charges present in the gradient coil, and the current density must obey the continuity equation $\nabla \cdot \vec{J} = 0$, which in cylindrical coordinates is

$$\frac{1}{\mu} \left( \frac{\partial j_\phi^a(\phi, z)}{\partial \phi} - \frac{\partial j_z^a(\phi, z)}{\partial z} \right) = 0 \quad (8.37)$$

Substituting the expressions for $j_\phi^a$ and $j_z^a$ from equations (8.31), (8.32), the relations between the Fourier coefficients for the two components are

$$\sin \phi \sum_{n=1}^\infty \left\{ \frac{-j_\phi^{1n}}{k_{1n} a} + j_z^{1n} \right\} \cos k_{1n} z + \left\{ \frac{-j_\phi^{2n}}{k_{2n} a} - j_z^{2n} \right\} \sin k_{2n} z = 0 \quad (8.38)$$

In order for the previous summation to be zero, the quantities inside the brackets which define the $\cos k_{1n} z$ and $\sin k_{2n} z$ must vanish. This leads to

$$j_z^{1n} = \frac{j_\phi^{1n}}{k_{1n} a} \quad (8.39)$$

$$j_z^{2n} = \frac{-j_\phi^{2n}}{k_{2n} a} \quad (8.40)$$

$$j_z^{2n} = \frac{-j_\phi^{2n}}{k_{2n} a} \quad (8.41)$$

Thus, the expressions (8.31), (8.32) for the two components of the current density are modified as

$$j_\phi^a(\phi, z) = \cos(\phi) \sum_{n=1}^\infty \left[ j_k^{1n} \cos k_{1n} z + j_k^{2n} \sin k_{2n} z \right] \quad (8.42)$$
\[ J_z^a(\varphi, z) = \sin(\varphi) \sum_{n=1}^{\infty} \left[ \frac{j_{1n}^a}{k_{1n}a} \sin k_{1n}z - \frac{j_{2n}^a}{k_{2n}a} \cos k_{2n}z \right] \] (8.43)

Furthermore, the magnetic field and the dissipated energy are expressed in terms of the Fourier transform of the current density. Thus, we must find a way to transform the \( j_z^a \) and \( j_\varphi^a \) into Fourier space. In general the Fourier transform and the Inverse Fourier transform of \( j_z^a \) are defined as

\[ j_z^a(m, k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\varphi e^{-im\varphi} \int_{-\infty}^{\infty} dz e^{-ikz} j_z^a(\varphi, z) \] (8.44)

or

\[ j_\varphi^a(\varphi, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\varphi e^{im\varphi} \int_{-\infty}^{\infty} dz e^{ikz} j_\varphi^a(m, k) \] (8.45)

and similar expressions are used for the \( j_z^a \). Applying once again, the continuity equation to the Fourier transforms of the two current density components, we obtain the relation between them in the Fourier space as

\[ j_z^a(m, k) = -\frac{m}{ka} j_\varphi^a(m, k) \] (8.46)

We now proceed to calculate the expression for the azimuthal component of the current density. Since the expression for the axial one can be determined by equation (8.44). Using equations (8.42), (8.44) and keeping in mind that
the current is bounded on a surface of a cylinder with length \( L \). The Fourier transform of the \( j_\phi(\phi, z) \) becomes

\[
j_\phi^m(m, k) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} d\phi (\cos m\phi - i \sin m\phi) \cos \phi \\
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dz e^{-ikz} [j_{1n}^q \cos k_{1n}z + j_{2n}^q \sin k_{2n}z]
\]

(8.47)

Since trigonometric functions are orthonormal functions, in the \( \phi \) integration in equation (8.47) the second term is zero, while the first term inside the parentheses obey the relationship

\[
\int_{-\pi}^{\pi} d\phi \cos m \phi \cos \phi = \begin{cases} 
\pi & \text{for } m = \pm 1 \\
0 & \text{otherwise}
\end{cases}
\]

(8.48)

Using this condition, and performing the integration of equation (8.47) over the variable \( z \) by replacing the trigonometric functions with their relative Euler formula in the complex representation, the expression of \( j_\phi^m(\pm 1, k) \) is

\[
j_\phi^m(\pm 1, k) = \sum_{n=1}^{\infty} \frac{L}{2} [j_{1n}^q \psi_{1n}(k) + i j_{2n}^q \psi_{2n}(k)]
\]

(8.49)

with

\[
\psi_{1n}(k) = \frac{\sin(k - k_{1n})\frac{L}{2}}{(k - k_{1n})\frac{L}{2}} + \frac{\sin(k + k_{1n})\frac{L}{2}}{(k + k_{1n})\frac{L}{2}}
\]

(8.50)

\[
\psi_{2n}(k) = \frac{-\sin(k - k_{2n})\frac{L}{2}}{(k - k_{2n})\frac{L}{2}} + \frac{\sin(k + k_{2n})\frac{L}{2}}{(k + k_{2n})\frac{L}{2}}
\]

(8.51)
Examining closer the relations (8.49-8.51), it is obvious that

\[ j_0(-1, k) = j_0(1, k) \]  \hspace{1cm} (8.52)

\[ \psi_{1a}(-k) = \psi_{1a}(k) \]

\[ \psi_{2a}(-k) = -\psi_{2a}(k) \]  \hspace{1cm} (8.53)

As the next step, we will calculate the expressions of the magnetic field and the dissipated energy. In a similar fashion, the general expression of the \( z \) component of the magnetic field for a single coil and in the region inside the coil \( \rho < a \) is

\[ B_z = -\frac{\mu_0 d}{2\pi} \sum_{m<0} \int_{-\infty}^{\infty} dk \, k e^{ikz} j_0^m(m, k) I_m(k\rho) K'_m(ka) \]  \hspace{1cm} (8.54)

Since the only allowed values for \( m \) are \( m = \pm 1 \), and using the relation between modified Bessel functions of the positive and negative index

\[ I_1(k\rho) = I_{-1}(k\rho) \]

\[ K'_1(ka) = K'_{-1}(ka) \]

the expression of the magnetic field is

\[ B_z = -\frac{\mu_0 d}{2\pi} \cos \phi \int_{-\infty}^{\infty} dk \, e^{ikz} k j_0^1(1, k) I_1(k\rho) K'_1(ka) \]  \hspace{1cm} (8.55)
Furthermore, replacing \( j_\phi^s (1, k) \) from equation (8.49) to equation (8.55) and demanding that the integrand is an even function of \( k \), the expression of the magnetic field for the \( x \) gradient asymmetric coil is

\[
B_z = \frac{-\mu_0 L}{4\pi a_s} \sum_{n=1}^{\infty} j_{\phi 1n}^+ \int_{-\infty}^{\infty} dk \cos k z e_{1n}^s(k) k I_1(kp) K'_1(ka) \]

\[
-\frac{-\mu_0 L}{4\pi a_s} \sum_{n=1}^{\infty} j_{\phi 2n}^+ \int_{-\infty}^{\infty} dk \sin k z e_{2n}^s(k) k I_1(kp) K'_1(ka) \quad (8.56)
\]

The dissipated energy for this coil is

\[
W = -\alpha^2 \mu_0 \int_{-\infty}^{\infty} dk I_1'(ka) K'_1(ka) |j_\phi^s(1, k)|^2 \quad (8.57)
\]

but

\[
|j_\phi^s(1, k)|^2 = \frac{L^2}{16} \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} \left[ j_{\phi 1n}^s j_{\phi 1n}^s \psi_{1n}^s(k) \psi_{1n'}^s(k) \right. \\
\left. + j_{\phi 2n}^s j_{\phi 2n'}^s \psi_{2n}^s(k) \psi_{2n'}^s(k) \right] \quad (8.58)
\]

Therefore, the expression of the dissipative energy is

\[
W = -\frac{\alpha^2 \mu_0 L^2}{16} \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} \int_{-\infty}^{\infty} dk I_1'(ka) K'_1(ka) \\
\left[ j_{\phi 1n}^s j_{\phi 1n}^s \psi_{1n}^s(k) + j_{\phi 2n}^s j_{\phi 2n'}^s \psi_{2n}^s(k) \psi_{2n'}^s(k) \right] \quad (8.59)
\]

In a similar fashion with the section 8.2, we construct the functional \( E \),

where its expression is given in equation (8.16). Minimizing \( E \) with respect
to \( j_{\phi 1n} \), we obtain a matrix equation for \( j_{\phi 1n'} \), which is

\[
\sum_{n'=1}^{\infty} J_{n'=1}^* \left\{ -\frac{aL}{2} \int_{-\infty}^{\infty} dk I_1'(ka) K_1'(ka) \psi_{1n}(k) \psi_{1n'}(k) \right\} = -\sum_{j=1}^{N} \lambda_j \cos \phi_j \int_{-\infty}^{\infty} dk \cos k z_j I_1(k \rho_j) K_1'(ka) \psi_{1n}(k) \tag{8.60}
\]

or

\[
\sum_{n'=1}^{\infty} J_{n'=1}^* C_{1n'} = \sum_{j=1}^{N} \lambda_j D_{j1n} \tag{8.61}
\]

Following the procedure of section 8.2, we truncate the infinite summations over an arbitrary limit \( M \), and the matrix representation of the previous equation is

\[
J_{\phi 1n'}^* \mathbf{C} = \mathbf{A} \mathbf{D} \quad \text{or} \quad J_{\phi 1n'}^* = \mathbf{A} \mathbf{D} \mathbf{C}^{-1} \tag{8.62}
\]

where the sizes of the matrices are identical to those presented in section 8.2.

Similarly, minimizing \( \mathcal{E} \) with respect to the second Fourier coefficient \( j_{\phi 2n}^* \), we obtain a matrix equation for \( j_{\phi 2n'} \), which is

\[
\sum_{n'=1}^{\infty} J_{n'=1}^* \left\{ -\frac{aL}{2} \int_{-\infty}^{\infty} dk I_1'(ka) K_1'(ka) \psi_{2n}(k) \psi_{2n'}(k) \right\} = \sum_{j=1}^{N} \lambda_j \cos \phi_j \int_{-\infty}^{\infty} dk \sin k z_j I_1(k \rho_j) K_1'(ka) \psi_{2n}(k) \tag{8.63}
\]
or

\[ \sum_{n'=1}^{\infty} J_{2n'}^2 I_{2n'}^{2n} = \sum_{n=1}^{\infty} \lambda_n Q_{2n} \tag{8.64} \]

Truncating the summation to the \( M \) terms, the matrix equation for the second Fourier component is

\[ J_{2n}^2 P = \lambda Q \text{ or } J_{2n}^2 = \lambda Q P^{-1} \tag{8.65} \]

Again, the sizes of the matrices in this equation are the same with those presented in section 8.2. Furthermore, equations (8.62), (8.65) are similar with the equations (8.18), (8.21) and the procedure of determining the two Fourier coefficients is identical with the procedure which is displayed in equations (8.22) through (8.25). The expression of the current distribution is then obtained and the magnetic field and dissipated energy are evaluated.

### 8.3.2 Design

The design strategy of the \( x \) gradient asymmetric cylindrical coil is similar to the design procedure which is presented in section 7.3.3. The dimensions of this gradient coil are slightly different from the axial asymmetric coil.
Specifically, the radius of the coil is chosen to be equal to $0.182 \, m$ which is $1 \, cm$ smaller from the radius of the axial gradient coil. This choice was driven by engineering reasons, since, the axial and the transverse gradient coils cannot co-exist on the same cylindrical surface. The reason for choosing the transverse design to be inside the axial gradient coil is the following. The construction of the transverse gradient coil involves etching of the current patterns on a copper sheet instead of wire loops which are used for the axial gradient coil design. The copper sheet, a more compact surface of dielectric, will create more eddy current and heating effects to the magnet shield than the corresponding wire loops.

In addition, the total length of the coil is chosen to be equal to $0.6 \, m$ while the distance of the sweet spot of the gradient field is chosen to be $0.145 \, m$ away from the end of the cylinder in order to coincide with the sweet spot of the axial gradient coil. Furthermore, 40 Fourier terms were considered to simulate closer the behavior of the two components of the current density.

Having established the geometric specifications for the coil, we now proceed by defining the properties of the gradient field through a set of constraint
points. As we have mentioned earlier in this section, the design of the \( x \) asymmetric gradient coil contains additional symmetric features which are absent in the axial asymmetric coil. Although the sweet spot of the gradient field is shifted axially, during the theoretical development for the \( x \) gradient coil, we still consider the fact that the magnetic field is antisymmetric in the \( x \) direction and about the sweet spot. Thus, a significantly smaller number of field constraints can be used to enforce the desired quality of the magnetic field. Four constraint points were considered in order to define the quality of the magnetic field inside the 25 cm DSV. The first constraint sets the strength of the gradient field to 40 \( mT/m \) along the \( x \) direction. The second constraint limits the variation of the magnetic field to within 10% from its ideal value along the \( x \) gradient axis and at a distance of 0.135 m away from the sweet spot of the coil. The next two constraints restrict the variation of the magnetic field to 10% at a plane perpendicular to the gradient axis and at a distance of 0.145 m from the sweet spot of the gradient coil. This set of constraints is illustrated in Table 8.2.
\[
\begin{array}{|c|c|c|c|}
\hline
n & \rho_i & z_i & B_{z,m}(2n) \\
\hline
1 & 0.0010 & 0.155 & 0.00004000 \\
2 & 0.1350 & 0.155 & 0.00486000 \\
3 & 0.0010 & 0.300 & 0.00003800 \\
4 & 0.0010 & 0.010 & 0.00003600 \\
\hline
\end{array}
\]

Table 8.2: Constraint set used for the design of a non-shielded transverse finite asymmetric cylindrical gradient coil. Values for \(\rho\) and \(z\) are in m, values for \(B_{z,m}\) are in T.

### 8.3.3 Results

Employing Turner’s approach with the previous set of constraints, the azimuthal and axial components of the continuous current distribution are evaluated. Figure 8.5 illustrates the variation of the \(z\) component of the current density, \(j_z\), along the \(z\) axis, since only the \(z\) component of the current density is adequate to generate the discrete current pattern, as we have described in chapter 7. Furthermore, 10 Fourier terms were used for the evaluation of the continuous current density behavior, since the results are similar to those which are obtained with higher order terms and computer time is reduced.

Knowing the continuous current distribution, we obtain the discrete current patterns employing the stream function technique. Figure 8.6 displays
the current pattern for the $x$ gradient coil. There are 16 current loops, and each loop carries 333 $Amps$ of current. With the help of the Biot-Savart law and using the discrete current distribution, the $z$ component of the magnetic field is re-evaluated, in order to ensure the accuracy of the discretization procedure. The variation of $B_z$ along the gradient axis $x$ is illustrated in Figure 8.7. The magnetic field strength is calculated to 40.3 $mT/m$. a 0.75% difference from the ideal value. The variation of the magnetic field was estimated to 10.02% from the ideal on-axis value and at distance equal to 0.135$m$ from the sweet spot. In addition the off-axis uniformity of the magnetic field reaches to 10.78% at a distance of 0.125 $m$ away from the sweet spot.

Furthermore, the total stored magnetic energy of the system has been evaluated and is equal to 7.93 $Joules$. Thus, taking advantage of the relation between the total energy and the total inductance of the coil, a rough estimation of the total inductance of the coil is feasible. When the dissipated energy is equal to 7.93 $Joules$ and the amount of current for each current loop is 333 $Amps$, the total inductance of the $x$ asymmetric gradient coil is

$L_{\text{tot, estim}} = 143 \mu H$. 
8.4 Experimental Verification

Following the theoretical development of the design for the complete set of the asymmetric gradient model, an identical complete set of coils with the prescribed dimensions have been constructed at Picker International. John Patrick, David Lampman and Haying Liu were responsible for the construction of these coils. Specifically, the axial asymmetric gradient coil was built from 46 wires with a current of 330 Amperes passing through them. For the construction of the transverse gradient coil, a copper sheet is used with 17 line boundaries, where the current flowing through two consecutive etched lines is equal to 330 Amperes.

Following the construction of these coils, the magnetic field and the local gradient of the coil have been evaluated. The measurement of the magnetic field was performed using a search coil, which is described in more detail in Martens [56]. The basic design of the search coil consists of large number of turns which are wrapped around a cylindrical surface. A sinusoidal current form is applied to the gradient coil with a frequency of 100 Hz when the
<table>
<thead>
<tr>
<th>Position of Search Coil Along z axis</th>
<th>$dB/dt$ Induced Signal</th>
<th>$dB/dt$ Induced Signal</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>New Coil Design</td>
<td>Old Coil Design</td>
</tr>
<tr>
<td>$z = -8$ cm</td>
<td>37.2 mV (2.20%)</td>
<td>189.0 mV (14.10%)</td>
</tr>
<tr>
<td>$z = -6$ cm</td>
<td>27.3 mV (1.46%)</td>
<td>150.0 mV (9.10%)</td>
</tr>
<tr>
<td>$z = -4$ cm</td>
<td>18.3 mV (0.55%)</td>
<td>105.0 mV (4.55%)</td>
</tr>
<tr>
<td>$z = -2$ cm</td>
<td>9.1 mV (0.00%)</td>
<td>55.0 mV (0.00%)</td>
</tr>
<tr>
<td>$z = 0$ cm</td>
<td>1.7 mV</td>
<td>1.0 mV</td>
</tr>
<tr>
<td>$z = +2$ cm</td>
<td>9.8 mV (0.00%)</td>
<td>53.3 mV (0.00%)</td>
</tr>
<tr>
<td>$z = +4$ cm</td>
<td>19.8 mV (0.51%)</td>
<td>104.8 mV (1.69%)</td>
</tr>
<tr>
<td>$z = +6$ cm</td>
<td>28.7 mV (2.38%)</td>
<td>145.6 mV (8.94%)</td>
</tr>
<tr>
<td>$z = +8$ cm</td>
<td>38.2 mV (2.55%)</td>
<td>180.0 mV (15.57%)</td>
</tr>
</tbody>
</table>

Table 8.3: Induced signal in mV for the new and the old gradient coil design. The normalized value of the signal for the negative values of $z$ were assumed to be 9.1 mV for the new coil and 55.0 mV for the old coil. The normalized value of the signal for the positive values of $z$ were assumed to be 9.8 mV for the new coil and 53.3 mV for the old coil. The percentage difference from the ideal values of the induced signal are shown inside the parentheses.

A search coil is placed inside the gradient. The voltage which is induced in the search coil gives an indication for the value of the magnetic field. Table 8.3 illustrates the values induced to the search coil for two coil designs. Initially an archetypal gradient coil design was constructed where the linearity and the uniformity of the gradient coil exceeded the desired field qualifications although the strength of the gradient field was high enough. Following this design, a new asymmetric gradient coil design was built, which was more
faithful to the desired specifications of the magnetic field while its gradient strength was smaller than the previous coil. We notice that in the old coil design and at an axial distance of +0.08 m the difference between the measured value and the actual value of induced voltage in the search coil is 15.57% while for the new coil design at the same distance the difference is 2.55%. The reference induced voltage for the negative values of z was considered to be to 55 mV for the old gradient coil design and 9.1 mV for the new coil design. For the positive z values the values were 53.3 mV and 9.8 mV respectively.

Furthermore, the complete new coil set was placed inside the magnet and a spherical phantom object with diameter of 191 mm was positioned inside the gradient coil set. Performing imaging the diameter of the phantom object was re-evaluated on the xy plane for different positions along the z axis. Table 8.1 illustrates the measured values of the diameter of the phantom object along the angular positions of 0°, 15°, 90°, 135° of the coils, starting at the position of z = -47.5 mm and ending at the position of z = 32.5 mm about the sweet spot of the gradient field and at successive intervals of 10.0 mm. We notice
<table>
<thead>
<tr>
<th>Position Along ( z ) axis</th>
<th>0°</th>
<th>45°</th>
<th>90°</th>
<th>135°</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z = -47.5 \text{ mm} )</td>
<td>195 mm</td>
<td>192 mm</td>
<td>191 mm</td>
<td>191 mm</td>
</tr>
<tr>
<td>( z = -37.5 \text{ mm} )</td>
<td>193</td>
<td>191</td>
<td>189</td>
<td>189</td>
</tr>
<tr>
<td>( z = -27.5 \text{ mm} )</td>
<td>192</td>
<td>189</td>
<td>187</td>
<td>187</td>
</tr>
<tr>
<td>( z = -17.5 \text{ mm} )</td>
<td>191</td>
<td>189</td>
<td>187</td>
<td>188</td>
</tr>
<tr>
<td>( z = -7.5 \text{ mm} )</td>
<td>191</td>
<td>189</td>
<td>187</td>
<td>187</td>
</tr>
<tr>
<td>( z = +2.5 \text{ mm} )</td>
<td>190</td>
<td>189</td>
<td>186</td>
<td>187</td>
</tr>
<tr>
<td>( z = +12.5 \text{ mm} )</td>
<td>191</td>
<td>189</td>
<td>187</td>
<td>187</td>
</tr>
<tr>
<td>( z = +22.5 \text{ mm} )</td>
<td>193</td>
<td>191</td>
<td>188</td>
<td>189</td>
</tr>
<tr>
<td>( z = +32.5 \text{ mm} )</td>
<td>194</td>
<td>192</td>
<td>189</td>
<td>191</td>
</tr>
</tbody>
</table>

Table 8.1: Measurement of the diameter of a phantom object along the azimuthal directions and at different positions along the \( z \) axis around the sweet spot of the gradient field. The actual diameter of the object is 191 mm and the measurements were taken along the azimuthal direction start at \( z = -47.5 \text{ mm} \) and end at \( z = 32.5 \text{ mm} \). The largest differences were observed at \( z = -17.5 \text{ mm}. \) 0° and \( z = +2.5 \text{ mm}. \) 90°. The percentage difference was on the order of 2.6%.

that the variation of the diameter of the object is very small and only at two points reaches to 2.6% difference from its ideal value.

When the verification of the quality of the magnetic field was completed, the gradient set was inserted to a 1.0 \( T \) magnet in order to perform imaging.

As a preliminary test, a fast 3D gradient echo sequence was used. Coronal and transverse images of the human wrist and figures were obtained in order to ensure that both axial and transverse coils performed to the
level of accuracy for which they were designed. Figure 8.8 shows a coronal image of a human wrist using 3D Fast Gradient Echo sequence, with $TR = 33\, msec, TE = 1\, msec, FOV = 8\, mm$. Slice Thickness = 1.1 mm.

Figure 8.9 displays a transverse image of the human wrist employing the same parameters as in Figure 8.8. Furthermore, Figure 8.10 illustrates the coronal image of the human figures, where a 3D Fast Gradient echo is used with $TR = 36\, msec, TE = 7\, msec, FOV = 7\, mm$. Slice Thickness = 1 mm.

In Figure 8.11, the transverse image of the human figures is shown, using the same sequence and identical values for the imaging parameters as in Figure 8.10.

In conclusion, we have presented the theoretical development, the design strategy, the theoretical results and the experimental measurements for the complete set of the asymmetric gradient coil. We have also presented images which were acquired with the help of these gradient coils. Although, the testing and upgrading of these gradient coils continue, the preliminary results are very encouraging and indicate the possible use of the complete set of asymmetric gradient coils for human imaging.
Figure 8.3: Illustration of the discrete current distribution for the axial asymmetric cylindrical gradient coil laid out flat. There are 46 wire loops carrying an amount of current equal to $328.50\text{ Amps}$. The abscissa corresponds to the positions of the wires across the cylinder, while the ordinate is normalized to the ratio of the current and the radius of the wire loop. The wires with the positive current are noted by a solid line, while the negative wires are marked with dashed lines.
Figure 8.4: Plot of the variation of the $z$ component of the magnetic field along the $z$ axis which is generated using the discrete current distribution. The plot shows the sweet spot of the gradient field at 0.465 m for the beginning of the coil. The gradient strength is 43.25 mT/m and the distance between the two rollover points is 0.35 m.
Figure 8.5: Plot of the $z$ component of the continuous current distribution versus $z$ for the $x$ transverse non-shielded asymmetric gradient coil. This current density is adequate to generate the field specifications which are described in the design section.
Figure 8.6: Illustration of the discrete current distribution for the $x$ asymmetric cylindrical gradient coil. There are 16 current turns where each one carries an amount of current equal to 333 Amps.
Figure 8.7: Plot of the magnetic field $B_z$ vs $x$ resulting from the discrete current distribution. The strength of the gradient field is $40.3 \, mT/m$. 
Figure 8.8: Coronal image of the human wrist, using a 3D Fast Gradient echo sequence with TE=1 msec. TR=33 msec. FOV = 8 cm. and Slice Thickness=1.1 mm.
Figure 8.9: Transverse image of the human wrist using a 3D Fast Gradient echo sequence with $TE=1$ msec, $TR=33$ msec, $FOV=8$ cm and Slice Thickness=1.1 mm.
Figure 8.10: Coronal image of the human figures using a 3D Fast Gradient echo sequence with $TE=7$ msec, $TR=36$ msec, FOV = 7 cm and Slice Thickness = 1 mm.
Figure 8.11: Transverse image of the human figures using a 3D Fast Gradient echo sequence with TE=7 msec, TR=36 msec, FOV = 7 cm and Slice Thickness=1 mm.
Chapter 9

Self-Shielded Asymmetric Gradient Coils

9.1 Introduction

Although the dimensions of the asymmetric gradient coils are small enough compared with the dimensions of the main magnet, some eddy current and heating effects still appear on the main magnet shield. Furthermore, the position of the current patterns on the surface of the cylindrical geometry create torque due to the interaction between the magnetic field from the main magnet and the current of the gradient coil. Since the gradients are turned on and off many times during an imaging experiment, the direction of the torque will follow the sign of the gradient pulse. Thus, the transverse
gradient set will start to oscillate with a period equal to the period of turning on and off the gradient coils in the phase encoding and read direction.

One way to reduce such a problem is the introduction of the second cylindrical coil outside from the primary gradient coil. The purpose of this second coil is to shield the magnetic field which is generated from the inner coil in the region that is outside of both coils. Thus the sign of the current for the outer coil will be opposite to that of the current for the inner coil. Thus, the eddy current effects produced on the magnetic shield are eliminated. Using this design, we can also reduce the total overall torque of the coil system, by an amount which is roughly proportional to the ratio of the radii between the inner and the outer coil. The exact expression of the torque for both the axial and transverse gradient coils will be presented in a separate section of this chapter.

Specifically, in this chapter, we will present the mathematical development, the design strategy and the results which are derived using a complete set of self-shielded gradient coils. The design of this set of gradient coils assumes that the sweet spot of the gradient field is shifted axially towards
the end of the cylinder. Furthermore, the length of the inner coil is considered to have a restriction on its length, while the length of the outer coil is left unrestricted. We will introduce a section in which we will provide the mathematical methodology for evaluating the torque due to the presence of the main magnetic field in both axial and transverse gradient coils. The discretized versions of the continuous current densities will be presented and the $z$ component of the magnetic field will be evaluated using the Biot-Savart law.

9.2 Self-shielded Axial Asymmetric Gradient Coil

The design of a self shielded axial gradient coil is illustrated in Figure 9.1. The origin of the coordinate system was chosen to coincide with the geometric center of the coil. The total length of the coil is assumed to be equal to $L$, with the distance between the sweet spot and the end of the cylinder denoted by $l_{as}$. Furthermore, the radii of the inner and outer cylinders are $a$ and $b$ respectively.
Theory

In this section, we will present the mathematical methodology for the design of an axial and finite self-shielded asymmetric gradient coil. Once again, due to the properties of the axial design, the current densities for both coil are restricted to be oriented azimuthally, and can be only functions of $z$. Furthermore, since the asymmetricity occurs only along the $z$ direction, the Fourier expansion about the origin of the gradient coil for both current densities will contain sine and cosine terms. Therefore, the mathematical form of the current density for the inner coil is

$$J_\phi^i(z) = \sum_{n=1}^{\infty} [j_{1n}^i \sin k_{1n} z + j_{2n}^i \cos k_{2n} z] \quad |z| \leq \frac{L}{2} \quad (9.1)$$

$$J_\phi^o = 0 \quad |z| > \frac{L}{2} \quad (9.2)$$

where $j_{1n}^i, j_{2n}^i$ are the coefficients associated with the two terms in the Fourier expansion, for the current of the inner coil. The determination of the current density for the outer coil can be done using the continuity equation. The requirement that the current density of the inner coil must vanish at both
ends of the cylinder, defines the spectrum of values for \( k_{1n} \) and \( k_{2n} \) as

\[
\begin{align*}
\sin k_{1n}L & = 0 \implies k_{1n} = \frac{2\pi n}{L} \\
\cos k_{2n}L & = 0 \implies k_{2n} = \frac{(2n - 1)\pi}{L}
\end{align*}
\] (9.3) (9.4)

As we have previously mentioned, the expressions of the magnetic field and the dissipated energy involve the Fourier transform of the current density. In exact analogy with chapter 8, the Fourier transform for the current density of the inner and outer coil is

\[
\begin{align*}
J^a_\phi(k) & = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \int_{-\infty}^{\infty} dz e^{-ikz} j^a_\phi(z) \\
J^b_\phi(k) & = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \int_{-\infty}^{\infty} dz e^{-ikz} j^b_\phi(z)
\end{align*}
\] (9.5) (9.6)

The next step is to evaluate the Fourier transform of the current density for the inner coil, using the expression (9.5). Following the same procedure as in section 8.2.1, the expression of the current density in the Fourier space is

\[
J^a_\phi(k) = \sum_{n=1}^{\infty} \frac{L}{2} [j^a_{1n} \psi_{1n}(k) - i j^a_{2n} \psi_{2n}(k)]
\] (9.7)
with

\begin{align}
\psi_{1n}(k) &= \left[ -\sin(k - k_{1n}) \frac{k}{2} + \sin(k + k_{1n}) \frac{k}{2} \right] \quad (9.8) \\
\psi_{2n}(k) &= \left[ \sin(k - k_{2n}) \frac{k}{2} + \sin(k + k_{2n}) \frac{k}{2} \right] \quad (9.9)
\end{align}

where the dependence of \( \psi_{1n}(k) \) and \( \psi_{2n}(k) \) on the sign of \( k \) is

\begin{align}
\psi_{1n}(-k) &= -\psi_{1n}(k) \\
\psi_{2n}(-k) &= \psi_{2n}(k) \quad (9.10)
\end{align}

The presence of the second coil alters the expression of the magnetic field for each of the three areas which are divided from the two coils. The expression of the magnetic field for each one of these areas has been presented in equations (5.24)- (5.26). In order to eliminate the magnetic field for the region outside from both coils, equation (5.26) indicates that the current density of the outer coil must be related to the current density of the outer coil as follows

\[ J_{\phi}^o(k) = -J_{\phi}^e(k) \frac{a I_1(ka)}{b I_1(kb)} \quad (9.11) \]

where \( I_1(ka), I_1(kb) \) are modified Bessel functions evaluated at the surface of each cylinder respectively.
Using this relation, the expression of the $z$ component of the magnetic field in the region inside both coils is

$$B_z = \frac{\mu_0 a}{2\pi} \int_{-\infty}^{\infty} dk k e^{ikz} j^z_0(k) I_0(k\rho) K_1(ka) \left(1 - \frac{I_1(ka)K_1(kb)}{I_1(kb)K_1(ka)}\right) \text{ for } \rho < a \quad (9.12)$$

Replacing $j^z_\phi(k)$ with its Fourier space expression, the expression of the magnetic field becomes

$$B_z = \frac{\mu_0 a}{2\pi} \int_{-\infty}^{\infty} dk e^{ikz} k I_0(k\rho) K_1(ka) \left(1 - \frac{I_1(ka)K_1(kb)}{I_1(kb)K_1(ka)}\right) \sum_{n=1}^{\infty} \frac{L}{2} j^z_{1n} \psi_{1n}(k) - i j^z_{2n} \psi_{2n}(k) \quad (9.13)$$

Our next goal is to eliminate the complex exponential from the expression of the magnetic field. Knowing that $\psi_{1n}(k), K_1(ka)$ and $\psi_{2n}(k), I_0(k\rho)$ are odd and even functions of $k$ respectively, the above expression of the magnetic field becomes

$$B_z = -\frac{\mu_0 a}{2\pi} \int_{-\infty}^{\infty} dk \sin kz k I_0(k\rho) K_1(ka) \left(1 - \frac{I_1(ka)K_1(kb)}{I_1(kb)K_1(ka)}\right) \sum_{n=1}^{\infty} \frac{L}{2} j^z_{1n} \psi_{1n}(k)$$

$$+ \frac{\mu_0 a}{2\pi} \int_{-\infty}^{\infty} dk \cos kz k I_0(k\rho) K_1(ka) \left(1 - \frac{I_1(ka)K_1(kb)}{I_1(kb)K_1(ka)}\right) \sum_{n=1}^{\infty} \frac{L}{2} j^z_{2n} \psi_{2n}(k) \quad (9.14)$$
The expression for the dissipated energy is

\[ W = \frac{a^2 \mu_0}{2} \int_{-\infty}^{\infty} dk I_1(ka) K_1(ka) \left( 1 - \frac{I_1(ka)K_1(kb)}{I_1(kb)K_1(ka)} \right) |j_0^2(k)|^2 \]  

(9.15)

and using the relation (8.14), the expression (9.15) becomes

\[ W_m = \frac{a^2 \mu_0}{2} \int_{-\infty}^{\infty} dk I_1(ka) K_1(ka) \left( 1 - \frac{I_1(ka)K_1(kb)}{I_1(kb)K_1(ka)} \right) \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} \frac{L^2}{4} [j_{1n}^a j_{1n'}^a \psi_{1n}(k) \psi_{1n'}(k) + j_{2n}^a j_{2n'}^a \psi_{2n}(k) \psi_{2n'}(k)] \]  

(9.16)

Once again, we are looking for the expression of the current density which minimizes the total energy of a system, while the magnetic field must satisfy a prescribed set of constraints inside the desired imaging volume. Following Turner's method, we construct the functional \( \mathcal{E} \)

\[ \mathcal{E}(j_{1n}^a, j_{2n}^a) = W - \sum_{j=1}^{N} \lambda_j (B_z(\vec{r}_j) - B_{zSC}(\vec{r}_j)) \]

Since, the current density is expressed in terms of both Fourier components, we must minimize \( \mathcal{E} \) with respect to \( j_{1n}^a \) and \( j_{2n}^a \). The minimization of \( \mathcal{E} \) with respect to \( j_{1n}^a \) gives

\[ \sum_{n'=1}^{\infty} j_{1n'}^a \left\{ a L \pi \int_{-\infty}^{\infty} dk I_1(ka) K_1(ka) \left( 1 - \frac{I_1(ka)K_1(kb)}{I_1(kb)K_1(ka)} \right) \right\} \]
\[
\psi_{1n}(k)\psi_{1m}(k) = -\sum_{j=1}^{N} \lambda_j \int_{-\infty}^{\infty} dk \ k \sin kx \ I_0(k \rho_j) K_1(ka) \psi_{1n}(k) \\
\left(1 - \frac{I_1(ka)K_1(ka)}{I_1(kb)K_1(kb)}\right)
\]

Putting a limit on the infinite summations to \(M\) terms, the matrix representation of the previous equation is

\[
J_1^* \mathcal{C} = \Delta \mathcal{D} \text{ or } J_1^* = \Delta \mathcal{D} \mathcal{C}^{-1}
\]  

where \(J_1^*\) is a \(1 \times M\) matrix, \(\mathcal{C}\) is a \(M \times M\) matrix, \(\Delta\) is a \(1 \times N\) matrix and \(\mathcal{D}\) is a \(N \times M\) matrix.

Following the same methodology for the second Fourier component \(j_{2n}^z\), its corresponding relation is

\[
\sum_{n'=1}^{\infty} j_{2n'}^z \left\{ aL \pi \int_{-\infty}^{\infty} dk I_1(ka) K_1(ka) \left(1 - \frac{I_1(ka)K_1(ka)}{I_1(kb)K_1(kb)}\right) \right\} \\
\psi_{2n}(k)\psi_{2m}(k) = -\sum_{j=1}^{N} \lambda_j \int_{-\infty}^{\infty} dk \ k \sin kx \ I_0(k \rho_j) K_1(ka) \psi_{2n}(k) \\
\left(1 - \frac{I_1(ka)K_1(ka)}{I_1(kb)K_1(ka)}\right)
\]

Putting a threshold on the infinite summation in equation (9.19), the matrix equation for \(j_{2n}^z\) is

\[
J_2^z \mathcal{P} = \Delta \mathcal{Q} \text{ or } J_2^z = \Delta \mathcal{Q} \mathcal{P}^{-1}
\]
where $J_2^i$ is a $1 \times M$ matrix, $P$ is a $M \times M$ matrix, $\lambda$ is a $1 \times N$ matrix and $Q$ is a $N \times M$ matrix.

Thus we have derived the formulas for both components of the current density for the inner coil in terms of Lagrange multipliers. The determination of the Lagrange multipliers can be done using the constraint equation for the magnetic field, which is similar to the expression (8.22). For the final determination of the current density we follow exactly the same procedure presented in chapter 8.

9.2.1 Design

Two cylinders with circular cross section are used for the design of the self shielded asymmetric axial gradient coil. The radius of the inner cylinder is chosen to be equal to $a = 0.192 \, m$, while the radius of the outer cylinder is $b = 0.25 \, m$. The total length of the inner cylinder is restricted to $L = 0.6 \, m$, but the length of the outer cylinder is left unrestricted. Seven constraint points were considered in order to define the properties of the gradient field at a distance of $0.145 \, m$ away from the geometric center of the coil. Specifically,
the first constraint defines the strength of the field at 40 mT/m. Since no antisymmetric behavior along the z axis has been included in the theoretical development of the coil, the purpose of the next two constraints is to establish the antisymmetricity of the field about the sweet spot of the gradient field and along the z axis. Furthermore, the following two constraints limit the variation of the field to within 5% at an axial distance of 0.135 m about the sweet spot. This set of the four constraints define the “on-axis” linearity of the magnetic field. The reason for choosing the last two of the “on-axis” constraints is because we want the distance between the two rollover points to exceed 0.32 m, in order to avoid aliasing of the lower parts of the neck into the head. Finally, the last two constraints are related with the uniformity of the magnetic field inside the imaging volume. Specifically, they limit the fluctuation of the magnetic field to within 10% from its ideal value inside the DSV. In agreement with the previous five constraints, the last two constraints define the “off-axis” uniformity of the magnetic field. The presence of these two constraints is due to the requirement that the variation of the gradient strength between two points along the same axis inside the DSV cannot
<table>
<thead>
<tr>
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<th>$B_{z,i}$</th>
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<td>-0.00517333</td>
</tr>
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<td>0.1560</td>
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<tr>
<td>7</td>
<td>0.1336</td>
<td>0.1560</td>
<td>0.00004200</td>
</tr>
</tbody>
</table>

Table 9.1: Constraint set used to design a self-shielded asymmetric axial $z$-gradient coil. Values for $\rho$ and $z$ are in m, values for $B_{z,i}$ are in T.

be larger than 10% from the ideal gradient strength. the entire set of the constraint points is displayed at the Table 9.1. As we have mentioned in previous chapters, our purpose is to convert the continuous current density into a set of discrete wire loops for both coils. Although the procedure of discretization the current density of the primary coil has been discussed in chapter 8, we must generate the continuous current density for the outer coil.

From the section 9.2, the continuity equation relates the Fourier coefficients between the inner and the outer coils as

$$j_0^a(k) = -\frac{a}{b} \frac{I_1(ka)}{I_1(kb)} j_0^b$$  \hspace{2cm} (9.21)

Using the expression from equation (9.7) the expression of the Fourier trans-
form for the secondary coil is

\[ j_\phi^s(k) = \frac{iL}{4} \sum_{n=1}^{\infty} \left[ j_{1n}^s \psi_{1n}(k) - i j_{2n}^s \psi_{2n}(k) \right] \left( -\frac{a}{b} \frac{I_1(ka)}{I_1(kb)} \right) \]  

(9.22)

where the expression of \( \psi_{1n}(k) \) and \( \psi_{2n}(k) \) are given from equations (9.8),(9.9).

Our goal is to derive the behavior of the current density into the real space.

Thus,

\[ j_\phi^s(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikz} \, dk \, j_\phi^s(k) \]  

(9.23)

Using equation (9.21), the integral representation of equation (9.23) becomes

\[ j_\phi^s(z) = \frac{aL}{4b} \sum_{n=1}^{M} \left[ j_{1n}^s \int_{-\infty}^{\infty} \psi_{1n}(k) \sin k z \frac{I_1(ka)}{I_1(kb)} \, dk ight. \\
- j_{2n}^s \int_{-\infty}^{\infty} \psi_{2n}(k) \cos k z \frac{I_1(ka)}{I_1(kb)} \, dk \]  

(9.24)

The final evaluation of the above integral can be done using the Gauss-Legendre integration formalism. Thus the continuous current density for the outer coil is obtained. Finally, the discretization of the current density of the outer coil is similar to the procedure which is used at the inner gradient coil.
9.2.2 Results

Applying Turner's approach to the set of constraints from the Table 9.1, the continuous current density for both the inner and outer coil are obtained. For the creation of these quantities 10 Fourier terms were used for the same reasons which are explained previously. Figure 9.2 illustrates the variation of the current density of the inner coil along the $z$ axis of the cylinder. Furthermore, using the methodology which is described in section 9.2.1, the variation of the current density for the outer coil is displayed in Figure 9.3.

We now proceed by converting both continuous current densities into a set of discrete wire loops. Starting with the inner coil, there are 8 positive and negative areas, and the total current density which corresponds to each area is $-3401.60$ Amps, $59.71$ Amps, $-2175.59$ Amps, $1633.58$ Amps, $-1213.04$ Amps, $949.54$ Amps, $-2550.83$ Amps, $4375.19$ Amps. The best choice for the common amount of current flowing in each wire is found to be $323$ Amps. Thus 53 loops are needed to simulate the behavior of the continuous current density for the inner coil. For the outer coil there exist 2 positive
and negative regions, where the total current corresponding to each region is 2323.60 \textit{Amps}, \(-715.22\). To simulate the continuous current distribution of the secondary coil, 8 wire loops with the same amount of current per loop were considered. Figure 9.4 displays the distribution of the discretized wires along the length of the cylinder for both coils of the self-shielded axial asymmetric gradient coil.

With the help of the discrete current distribution, the \( z \) component of the magnetic field is calculated using the Biot-Savart law. The variation of the magnetic field along the gradient axis is shown in Figure 9.5. The strength of the magnetic field is estimated to 41.3 \( mT/m \) which corresponds to 3.25\% difference from the ideal strength of 40 \( mT/m \). At an axial distance of 0.125 \( m \) away from the sweet spot, the value of the magnetic field differs by 4\% from its constraint value. As Figure 9.5 illustrates, the distance between the two rollover points is equal to 0.35 \( m \), which is 0.03 \( m \) longer than the required one. An additional check for the local gradient along the gradient axis indicates that the biggest divergence from the ideal value appears at a distance of 0.12\( m \) away from the sweet spot. The local gradient at this point
is $37 \text{ mT/m}$ which is 10.41% smaller than the value of the gradient strength at the sweet spot of the gradient coil.

We next evaluate the amount of the dissipated energy, which is needed to generate the desired magnetic field. From equation (9.16), the total stored energy is 8.66 Joules which corresponds to a 9.2% increase in the stored energy of the non-shielded axial asymmetric coil. Using the relationship between the total energy and the inductance of the coils, the projected total inductance of the coil set is

$$L_{\text{tot,estim}} = 166.01 \mu H$$ (9.25)

A more accurate estimation of the total inductance of the coils can be derived from the formulas which are presented in chapter 7. Specifically, for the inner coil which consists of 53 wires with radius of $a = 0.192 \text{ m}$ and current equal to 323 Amps, its self inductance is

$$L_{a,\text{self}} = 71 \mu H$$ (9.26)

For the outer coil which consists of 9 wires with a radius of $b = 0.25 \text{ m}$ and carries the same amount of current with the wires of the inner coil, its self
inductance is

\[ L_{b,\text{self}} = 16.54 \, \mu H \]  \hspace{1cm} (9.27)

Furthermore, the mutual inductance for the combined set of the discrete wire loops is equal to

\[ M_{ab} = 103.82 \, \mu H \]  \hspace{1cm} (9.28)

Therefore the total inductance of both coils is

\[ L_{\text{tot, calc}} = L_{a,\text{self}} + L_{b,\text{self}} + M_{ab} = 191.36 \, \mu H \]  \hspace{1cm} (9.29)

which corresponds to a 15.2% increase from the estimated value of the total inductance. The inductance of the coils, and especially their self inductance, can be reduced by using wires with rectangular cross section instead of wires with circular cross section.

### 9.3 Self-shielded Asymmetric Transverse (X−, Y−) Finite Cylindrical Gradient Coils

In this section, we will present the mathematical methodology, the design and the theoretical results for the design of a self-shielded transverse asymmetric coil. Although the sweet spot of the gradient has been shifted along
the $z$ axial of the coil, it does not affect the symmetry conditions of the transverse gradient coil. In this section we will deal with the $x$ asymmetric gradient coil. The design strategy of the $y$ gradient coil is similar to the design of the $x$ gradient coil, where the current is rotated by $90^\circ$ with respect to the $z$ axis of the cylinder. The design of the $x$ gradient coil demands a magnetic field which is antisymmetric along the $z$ axis and about the sweet spot, while it is symmetric along the other two directions. Since at cylindrical coordinates the $x$ is proportional to $\rho \cos \phi$, the azimuthal component of the current density must behave in a similar way. The $\phi$ dependence on the expression for the magnetic field changes the behavior of the total current density. To account for this dependence two components of the current density must be considered. Furthermore, the discrete current distribution for the current density will be presented, and the magnitude of the magnetic field will calculated using the Biot-Savart law.
9.3.1 Theory

The geometric illustration of a $z$ transverse gradient coil is shown in Figure 9.1. The coil consists of two cylindrical gradient coils. The radius of the inner coil is denoted by $a$ and its total length is restricted to $L$. For the outer coil, its radius is denoted by $b$, while its length is left unrestricted. As we have mentioned previously, the angular dependence on the magnetic field and hence the current density requires the presence of two components for the current distribution one along the axial direction of the cylinder and the other along the azimuthal direction. Furthermore, the axial shift of the sweet spot of the gradient along the $z$ axis of the coil alters the Fourier expansion for the current density. In this situation, the expression for both azimuthal and axial components is

$$J_{\phi}^z(\phi, z) = \cos(\phi) \sum_{n=1}^{\infty} \left[ j_{\phi 1n}^z \cos k_{1n}z + j_{\phi 2n}^z \sin k_{2n}z \right] \quad |z| \leq \frac{L}{2} \quad (9.30)$$

$$J_{z}^z(\phi, z) = \sin(\phi) \sum_{n=1}^{\infty} \left[ j_{z 1n}^z \sin k_{1n}z + j_{z 2n}^z \cos k_{2n}z \right] \quad |z| \leq \frac{L}{2} \quad (9.31)$$

$$J_{\phi}^z = 0 \quad |z| > \frac{L}{2} \quad (9.32)$$
\[ J_x^a(\phi, z) = 0 \quad |z| > \frac{L}{2} \quad (9.33) \]

where \( j_{\phi 1n}^a, j_{\phi 2n}^a \) are the Fourier coefficients associated with the two terms of \( J_\phi^a(\phi, z) \) and \( j_{z1n}^a, j_{z2n}^a \) are the Fourier coefficients associated with the two terms of \( J_z^a(\phi, z) \). The restriction of the current density to the total length of the inner cylinder defines the group of values which \( k_{1n} \) and \( k_{2n} \) can take. Specifically, these values are

\[
\sin k_{1n}L = 0 \quad \Rightarrow \quad k_{1n} = \frac{2\pi n}{L} \quad (9.34)
\]

\[
\cos k_{2n}L = 0 \quad \Rightarrow \quad k_{2n} = \frac{(2n - 1)\pi}{L} \quad (9.35)
\]

Since there is no free charge present in the inner coil, the requirement that the current density of the inner coil satisfies the continuity equation relates the Fourier coefficients for both components of the current density as

\[
j_{z1n}^a = \frac{j_{\phi 1n}^a}{k_{1n} a} \quad (9.36)
\]

\[
j_{z2n}^a = -\frac{j_{\phi 2n}^a}{k_{2n} a} \quad (9.37)
\]

Thus the expression of the axial component of the current distribution be-
comes

\[ J_z^\circ(\phi, z) = \sin(\phi) \sum_{n=1}^{\infty} \left[ \frac{j_{1n}}{k_{1n} a} \sin k_{1n} z - \frac{j_{2n}}{k_{2n} a} \cos k_{2n} z \right] \]  
\hspace{2cm} (9.38)

We now proceed with the evaluation of the Fourier transform of the azimuthal component of the current density. Starting with the definition of the Fourier transform in two coordinates

\[ j_\phi^o(m, k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi e^{-im\phi} \int_{-\infty}^{\infty} dz e^{-ikz} j_\phi^o(\phi, z) \]  
\hspace{2cm} (9.39)

or

\[ j_\phi^o(\phi, z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi e^{im\phi} \int_{-\infty}^{\infty} dz e^{ikz} j_\phi^o(m, k) \]  
\hspace{2cm} (9.40)

where a similar expression can be obtained for the axial component \( j_z^o \). Since the current for the inner coil is restricted to flow on a surface of the cylinder with a total length \( L \), the expression of the Fourier transform is modified as

\[ j_\phi^o(m, k) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} d\phi (\cos m\phi - i \sin m\phi) \cos \phi \int_{-\frac{L}{2}}^{\frac{L}{2}} dz e^{-ikz} \left[ j_{1n}^o \cos k_{1n} z + j_{2n}^o \sin k_{2n} z \right] \]  
\hspace{2cm} (9.41)

Since there is a specific angular dependence on the expression of the angular current density, and considering the orthonormality conditions of
the trigonometric functions, the integration over the angular components at
the equation (9.41) gives

\[
\int_{-\pi}^{\pi} d\phi \cos \phi \cos m \phi = \begin{cases} 
\pi & \text{for } m = \pm 1 \\
0 & \text{otherwise}
\end{cases} \quad (9.42)
\]

With this condition present, and performing the z integration of equation (9.41),
the Fourier expression of the azimuthal component of the current density is

\[
j_\phi^z(\pm 1, k) = \sum_{n=1}^{\infty} \frac{L}{4} [j_{1n}^z \psi_{1n}(k) + i j_{2n}^z \psi_{2n}(k)] \quad (9.43)
\]

with

\[
\psi_{1n}(k) = \frac{\sin(k - k_{1n})L}{(k - k_{1n})L} + \frac{\sin(k + k_{1n})L}{(k + k_{1n})L} \quad (9.44)
\]

\[
\psi_{2n}(k) = \frac{-\sin(k - k_{2n})L}{(k - k_{2n})L} + \frac{\sin(k + k_{2n})L}{(k + k_{2n})L} \quad (9.45)
\]

with

\[
j_\phi^z(-1, k) = j_\phi^z(1, k) \quad (9.46)
\]

\[
\psi_{1n}(-k) = \psi_{1n}(k) \text{ and } \psi_{2n}(-k) = -\psi_{2n}(k) \quad (9.47)
\]

Having completed the evaluation of the Fourier transform of the azimuthal
component of the current, the corresponding Fourier transform for the axial
component can be related to the azimuthal one with the help of the continuity equation. Thus, this relation is

\[ j_z^a(m, k) = -\frac{m}{ka} j_\phi^a(m, k) \] (9.48)

Our next step is the evaluation of the \( z \) component of the magnetic field. Again the introduction of the second coil changes the expression of the magnetic field for each of the three regions which are divided by the two coils. Also, the angular dependence of the current density combined with the requirement of the equation (9.42) indicates that in order for the \( z \) component of the magnetic field to be zero outside of both coils, the relation between the Fourier transforms of the azimuthal component of the current density for the outer coil and the corresponding one for the inner coil must be

\[ J_\phi^a(k) = -J_\phi^b(k) \frac{a I'_1(ka)}{b I'_1(kb)} \] (9.49)

where \( I'_1(ka), I'_1(kb) \) are derivatives of the modified Bessel functions with respect to the argument and evaluated at the surface of each cylinder respectively.
Thus the $z$ component of the magnetic field becomes

\[
B_z = -\frac{\mu_0 a L}{4\pi} \cos \phi \sum_{n=1}^{\infty} j_{1n}^n \int_{-\infty}^{\infty} dk \cos k \psi_{1n}(k) k I_1(kp) K'_1(ka)
\left(1 - \frac{I_1'(ka) K'_1(kb)}{I_1'(kb) K'_1(ka)}\right) \tag{9.50}
\]

and replacing $j_n^0(1, k)$ from equation (9.43), the expression (9.50) becomes

\[
B_z = -\frac{\mu_0 a L}{4\pi} \cos \phi \sum_{n=1}^{\infty} j_{1n}^n \int_{-\infty}^{\infty} dk \cos k \psi_{1n}(k) k I_1(kp) K'_1(ka)
\left(1 - \frac{I_1'(ka) K'_1(kb)}{I_1'(kb) K'_1(ka)}\right)
+ \frac{\mu_0 a L}{4\pi} \cos \phi \sum_{n=1}^{\infty} j_{2n}^n \int_{-\infty}^{\infty} dk \sin k \psi_{2n}(k) k I_1(kp) K'_1(ka)
\left(1 - \frac{I_1'(ka) K'_1(kb)}{I_1'(kb) K'_1(ka)}\right) \tag{9.51}
\]

Then, the expression of the dissipated energy is

\[
W = -a^2 \mu_0 \int_{-\infty}^{\infty} dk I_1'(ka) K'_1(ka) \left(1 - \frac{I_1'(ka) K'_1(kb)}{I_1'(kb) K'_1(ka)}\right) |j_n^0(1, k)|^2 \tag{9.52}
\]

but

\[
|j_n^0(1, k)|^2 = \frac{L^2}{16} \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} \left[j_{1n}^n j_{1n}^{n'} \psi_{1n}(k) \psi_{1n'}(k) + j_{2n}^n j_{2n}^{n'} \psi_{2n}(k) \psi_{2n'}(k)\right] \tag{9.53}
\]

which modifies the equation (9.52) as

\[
W = -\frac{a^2 \mu_0 L^2}{16} \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} \int_{-\infty}^{\infty} dk I_1'(ka) K'_1(ka) \left(1 - \frac{I_1'(ka) K'_1(kb)}{I_1'(kb) K'_1(ka)}\right)
\]
\[ \left[ j_{\phi,1n}^a j_{\phi,1n'}^a \psi_{1n}(k) \psi_{1n'}(k) + j_{\phi,2n}^a j_{\phi,2n'}^a \psi_{2n}(k) \psi_{2n'}(k) \right] \] (9.54)

As a next step, we construct the functional \( \mathcal{E} \) with its expression to be identical to the expression presented in the section 9.2. The variation of \( \mathcal{E} \) with respect to \( j_{\phi,1n}^a \) gives us an expression for the \( j_{\phi,1n'}^a \) as

\[
\sum_{n'=1}^{\infty} j_{\phi,1n'}^a \left\{ -\frac{aL\pi}{2} \int_{-\infty}^{\infty} dk I_1'(ka) K_1'(ka) \left( 1 - \frac{I_1'(ka) K_1'(kb)}{I_1'(kb) K_1'(ka)} \right) \psi_{1n}(k) \psi_{1n'}(k) \right\} \\
= -\sum_{j=1}^{N} \lambda_j \cos \phi_j \int_{-\infty}^{\infty} dk k \cos k z_j I_1(k \rho_j) K_1'(ka) \left( 1 - \frac{I_1'(ka) K_1'(kb)}{I_1'(kb) K_1'(ka)} \right) \psi_{1n}(k) \\
(9.55)
\]

Furthermore, the variation of \( \mathcal{E} \) with respect to the second Fourier coefficient \( j_{\phi,2n}^a \) provides a matrix equation for \( j_{\phi,2n'}^a \), which is

\[
\sum_{n'=1}^{\infty} j_{\phi,2n'}^a \left\{ -\frac{aL\pi}{2} \int_{-\infty}^{\infty} dk I_1'(ka) K_1'(ka) \left( 1 - \frac{I_1'(ka) K_1'(kb)}{I_1'(kb) K_1'(ka)} \right) \psi_{2n}(k) \psi_{2n'}(k) \right\} \\
= \sum_{j=1}^{N} \lambda_j \cos \phi_j \int_{-\infty}^{\infty} dk k \sin k z_j I_1(k \rho_j) K_1'(ka) \psi_{2n}(k) \left( 1 - \frac{I_1'(ka) K_1'(kb)}{I_1'(kb) K_1'(ka)} \right) \\
(9.56)
\]

Truncating the infinite summations to an upper limit \( M \), the matrix representation of equations (9.55),(9.56) respectively become

\[
J_{\phi,1}^a C = \Delta D \text{ or } J_{\phi,1}^a = \Delta DC^{-1} \\
(9.57)
\]
\[
J_{\phi,2n}^a P = \Delta Q \text{ or } J_{\phi,2n}^a = \Delta QP^{-1} \\
(9.58)
\]
where the sizes of the matrices are identical to those presented in section 8.2.

Since we have obtained the expression of the two coefficients for the azimuthal component of the current density, the procedure of determining the Lagrange multipliers as well as the final expression for the current density, the magnetic field, and the dissipated energy is identical to the methodology which is presented in the previous chapters.

9.3.2 Design

We now proceed with the design of the $x$ transverse self-shielded asymmetric gradient coil. Two cylindrical coils are considered. The radius of the inner coil is $a = 0.182 \, m$, while the radius of the outer coil is $b = 0.24 \, m$. The total length of the inner coil is restricted to $L = 0.6 \, m$, whereas the length of the outer coil is left unrestricted. Unlike the self-shielded asymmetric coil, the design specifications for the transverse asymmetric coil are different. The reason is that although the sweet spot has been shifted axially, the magnetic field is designed to be asymmetric along the transverse direction which in our case is the $x$ direction. Therefore, we do not need to introduce a larger
number of constraint points in order to specify the behavior of the magnetic field inside the imaging volume. Specifically, four constraint points are chosen to define the quality of the $z$ component of the magnetic field inside a 25 cm DSV and are shown in Table 9.2. The first constraint specifies the strength of the magnetic field to $40 \, mT/m$ at a distance of $0.145 \, m$ from the edge of the cylinder. The second constraint limits the variation of the magnetic field to within 10% from its ideal value at a radial distance of $0.135 \, m$ away from the sweet spot. This constraint characterizes the "on-axis" linearity of the gradient coil. The remaining two constraints define the "off-axis" uniformity of the gradient field. Particularly, they restrict the variation of the magnetic field to within 10% from its ideal value at the outermost area which is perpendicular to the gradient axis. Finally, 40 Fourier terms were used in order to resemble the behavior of the two components of the current density for the inner coil of the transverse design.

Our next concern is the evaluation of the current density for the outer coil. From equations (9.43), (9.48), the expression of the Fourier component
Table 9.2: Constraint set used for the design of a self-shielded transverse finite asymmetric cylindrical gradient coil. Values for $\rho$ and $z$ are in m. Values for $B_{z_{sc}}$ are in T.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\rho_i$</th>
<th>$z_i$</th>
<th>$B_{z_{sc}}(2n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0010</td>
<td>0.155</td>
<td>0.00004000</td>
</tr>
<tr>
<td>2</td>
<td>0.1350</td>
<td>0.155</td>
<td>0.00486000</td>
</tr>
<tr>
<td>3</td>
<td>0.0010</td>
<td>0.300</td>
<td>0.00003800</td>
</tr>
<tr>
<td>4</td>
<td>0.0010</td>
<td>0.010</td>
<td>0.00003600</td>
</tr>
</tbody>
</table>

For the $z$ component of the current density is

$$j_{zb}(\pm 1, k) = \sum_{n=1}^{\infty} \frac{L}{4} \left[ j_{z,1n}^b \psi_{1n}(k) - i j_{z,2n}^b \psi_{2n}(k) \right]$$  \hspace{1cm} (9.59)

with

$$\psi_{1n}(k) = \left[ \frac{-\sin(k - k_{2n}) \frac{k}{2}}{(k - k_{2n}) \frac{k}{2} + \sin(k + k_{2n}) \frac{k}{2}} + \frac{\sin(k + k_{2n}) \frac{k}{2}}{\sin(k - k_{2n}) \frac{k}{2} + \sin(k + k_{2n}) \frac{k}{2}} \right]$$  \hspace{1cm} (9.60)

$$\psi_{2n}(k) = \left[ \frac{\sin(k - k_{1n}) \frac{k}{2}}{(k - k_{1n}) \frac{k}{2} + \sin(k + k_{1n}) \frac{k}{2}} + \frac{\sin(k + k_{1n}) \frac{k}{2}}{(k - k_{1n}) \frac{k}{2} + \sin(k + k_{1n}) \frac{k}{2}} \right]$$  \hspace{1cm} (9.61)

But

$$j_{\phi}^b(\phi, z) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im\phi} \int_{-\infty}^{+\infty} dk \ e^{ikz} j_{zb}(+1, k)$$  \hspace{1cm} (9.62)

and since

$$j_{\phi}^b(-1, k) = -j_{\phi}^b(+1, k)$$
Then, the expression (9.62) becomes

\[ j^b_\phi(\phi, z) = \frac{-L}{4\pi} \sin \phi \int^{+\infty}_{-\infty} dk \left[ \sin k z j^b_{z,1n} \psi_{1n}(k) - j^b_{z,2n} \psi_{2n}(k) \right] \quad (9.63) \]

Thus, we have derived the expression of the continuous current distribution for the outer coil.

### 9.3.3 Results

Employing Turner's approach to the set of constraint points described at the design section, we obtain the continuous current densities for both coils.

Recalling our discussion in section 7.3.2 only the \(z\) component of the current density is necessary for the discretization process. Figure 9.6 shows the variation of the \(z\) component of the current density for the inner coil along the \(z\) axis. Furthermore, the variation of the \(z\) component of the current density for the outer coil along the \(z\) axis is shown in Figure 9.7.

Using the stream function technique, the discrete current patterns were derived from the continuous current distributions. Figure 9.8 illustrates the discrete current loops for the inner coil of the self-shielded \(x\) gradient design. There are 16 current loops and each one carries an amount of current equal
to 522 Amps. The discrete current loops for the outer coil in this transverse design are shown in Figure 9.9. Particularly, there are 8 current loops where each loop carries a current equal to 534 Amps.

As a next step, we generate the $z$ component of the magnetic field applying the Biot-Savart law to the discrete current patterns for both the inner and the outer coil. The variation of the magnetic field along the $x$ axis for the inner coil (solid line) and the outer coil (dashed line) is shown in Figure 9.10. The gradient strength for this coil set is calculated to be $37 \text{ mT/m}$ which differs from the ideal value of $40 \text{ mT/m}$ by 7.5%. The gradient strength of the on-axis variation of the magnetic field near the rollover point is $32 \text{ mT/m}$ which differs by 13.5% from the ideal value. The off-axis variation of the magnetic field at the borders of the DSV was estimated to be 13.42% from the prescribed value which was set by the constraint points.

The final step in this section corresponds to the evaluation of the total inductance of the coils. The calculated energy for both coils is 11.84 Joules. Using the relationship between the inductance and the energy, the estimated total inductance of the system is estimated to be $L_{tot,estim} = 85 \mu H$. 
9.4 Torque

As we have discussed previously, one of the problems that the asymmetric design introduces is the presence of torque due to the interaction between the magnetic field of the main magnet and the current density of the individual coil. Since the value of the $z$ component of the magnetic field for the main magnet is the dominant quantity, the effects on the torque which are generated from the presence of the other components of the magnetic field and gradient field are almost negligible.

In this section, we will discuss the effects of the torque in both axial and transverse asymmetric coils, and we will compare the reduction which we will obtain using the self-shielded design over the non-shielded design. Furthermore, the only component of the torque which is of major interest to us is the one which is directed along the $y$ direction of the cylinder. The reason is that this component of the torque will tend to rotate the gradient coil away from the imaging table and towards the main magnet which will make the operation of the coil very difficult.
9.4.1 Axial Asymmetric coil

Let us consider a self-shielded asymmetric coil design which generates a \( z \) component of the magnetic field varying along the \( z \) direction. As we have seen in the section 9.2.2, the shape of the current density for both coils will be azimuthally oriented and shifted towards the region where the sweet spot of the gradient field is. Thus, it will be asymmetric about the geometric center of the coil. We are interested in understanding if such current configuration will generate torque.

It is well known that the expression for the torque \( \vec{N} \), generated by the interaction between a magnetic field \( \vec{B} \) and a current density \( \vec{J} \) is given as

\[
\vec{N} = \int_V \vec{r} \times (\vec{J} \cdot \vec{B}) \, dV
\]  

(9.64)

where \( V \) is the volume of interest and \( \vec{r} \) is the distance where the torque is evaluated. With the requirement that the contribution for the magnetic field is only from the main magnet \( \vec{B} = B_0 \hat{z} \), and the current density must lie on the surface of the coils,

\[
\vec{J} = J_o^z \hat{z} \delta(\rho - a) + J_o^\phi \hat{\phi} \delta(\rho - b)
\]  

(9.65)
The expression of the total torque is

\[
\vec{\mathcal{N}} = \int_0^\infty \int_0^{2\pi} \int_{-\infty}^{\infty} \left( J_0^b(\rho-a) + J_0^b(\rho-b) \right) B_0 \rho \, d\rho \, d\phi \, d\zeta \, d\phi \tag{9.66}
\]

Integrating over the delta function and using the relationship between the Fourier transform of the inner and outer coil, the expression of the total torque becomes

\[
\vec{\mathcal{N}} = \int_0^\infty \int_0^{2\pi} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} ik \, J_0^a(k) - a \, J_0^b(k) \frac{l_1(ka)}{l_1(kb)} \right] d\phi \, d\zeta \, d\phi \tag{9.67}
\]

where we have expressed the azimuthal unit vector in terms of the cartesian coordinate system. Thus, the angular integration from equation (9.67) is

\[
\int_0^{2\pi} \left[ -\sin \phi \, \vec{x} + \cos \phi \, \vec{y} \right] d\phi = 0 \tag{9.68}
\]

Therefore, the total torque for the axial asymmetric gradient coil is zero. The reason is that the current density is directed along the azimuthal direction of the cylinder and its interaction with the main magnetic field does not generate any torque effect for the gradient coil. This conclusion has been reached assuming that the only component of the magnetic field which contributes to the total torque is along the \( z \) direction. As we have mentioned
previously, there are additional components of the magnetic field present which will affect the result which we have derived in this section. Since their magnitude is very small compared with the magnitude of the main magnetic field, their contribution will be insignificant.

9.4.2 Transverse Asymmetric coil

The derivation of the total torque for the transverse self shielded asymmetric coil is slightly different from that of the axial coil. The reason is the current density is not directed along the azimuthal direction and it now has angular dependence. In this case, the total current distribution for the inner and outer coil in the gradient coil design can be analyzed in terms of two components, one along the azimuthal direction and the other along the axial one. Furthermore, the angular dependence of these two components have been previously explained. Thus, the expressions for the total current density for the inner and outer coils are

\[
\tilde{J}^t = \left[J^t_{\rho}\hat{\phi}(z) + J^t_{\sigma}(z)\hat{\sigma}_z\right] \delta(\rho - a) = \left[\cos \phi J^t_{\rho}(z)\hat{\phi} + \sin \phi J^t_{\sigma}(z)\hat{\sigma}_z\right] \delta(\rho - a)
\]

and
\[ \mathbf{J}^b = \left[ J_x^b \hat{\alpha}_x + J_y^b \hat{\alpha}_y \right] \delta(\rho - b) = \left[ \cos \phi \, J_0^b \hat{\alpha}_\phi + \sin \phi \, J_z^b \hat{\alpha}_z \right] \delta(\rho - b) \]  \hspace{1cm} (9.69)

Once again the main source for the torque in the asymmetric transverse coil design is assumed to be the magnetic field of the main magnet. From the equation (9.64), the expression of the total torque for the transverse gradient coil is

\[ \mathbf{N} = B_0 \int d\varphi \left[ \cos \varphi \, \eta - \frac{1}{2} \sin 2\phi \, \hat{\alpha}_x \right] \int_{\frac{a}{2}}^{\frac{b}{2}} dz \left( a \, J_0^b(z) + b \, J_z^b(z) \right) \]  \hspace{1cm} (9.70)

Since only the \( \eta \) component of torque is the important one, we perform the angular integration for only the \( \hat{\alpha}_y \) component of the torque and replace the current densities of the inner and outer coils with their representative Fourier transforms. Thus, the expression of the \( \eta \) component of the total torque is

\[ N_\eta = \frac{B_0}{2} \int_{\frac{a}{2}}^{\frac{b}{2}} dz \int_{-\infty}^{\infty} dk e^{ikz} \left( a \, J_0^b(k) + b \, J_z^b(k) \right) \]  \hspace{1cm} (9.71)

and using the relation between the Fourier transforms of the inner and outer coils for a self-shielded gradient coil design, equation (9.71) becomes

\[ N_\eta = \frac{B_0}{2} \int_{\frac{a}{2}}^{\frac{b}{2}} dz \int_{-\infty}^{\infty} dk e^{ikz} a \, J_0^b(k) \left( 1 - \frac{I_1(ka)}{I_1(kb)} \right) \]
or
\[
N_y = \frac{B_su}{2} \int_{-\infty}^{\infty} dk \left( 1 - \frac{I_1'(ka)}{I_1'(kb)} \right) J_2(k) i
\left[ \frac{1}{k^2} \left( 2\sin \frac{kL}{2}, -(kL)c\cos \left( \frac{kL}{2} \right) \right) \right]
\]  
(9.72)

The expression (9.72) can be modified further assuming the expression of the azimuthal component of the current density for the inner coil (equation (9.43)). The expression for the \( y \) component of the total torque is
\[
N_y = \frac{B_suL}{\pi} \int_{-\infty}^{\infty} dk \left( 1 - \frac{I_1'(ka)}{I_1'(kb)} \right) \sum_{n=1}^{\infty} j_{2n}^2 \psi_{2n}(k)
\left[ \frac{1}{k^2} \left( -i(kL)\cos \frac{kL}{2} - 2\sin \left( \frac{kL}{2} \right) \right) \right]
\]  
(9.73)

Thus, there is a total torque associated with the asymmetric transverse gradient coil. In the case where the outside coil is absent, the term \( 1 - \frac{I_1'(ka)}{I_1'(kb)} \) in equation (9.73) must be replaced by 1. Therefore, with the introduction of the second coil, we can succeed in a reduction of the total torque by an amount equal to \( \left( \frac{I_1'(ka)}{I_1'(kb)} \right) \). Using the expression of the derivatives for the modified Bessel functions in terms of Bessel functions of higher and lower order, the reduction of the \( y \) component of the total torque is proportional
\[
\frac{I_a(ka) - \frac{1}{(kn)I_1(ku)}}{I_0(kb) - \frac{1}{(kn)I_1(kb)}}
\] (9.74)

For small values of \( k \), and for the usual values of the radii of the inner and outer coils, the reduction can be on the order of the ratio between the radius of the inner coil to the radius of the outer coil, which can reach up to 75%.

In this section, we have presented the calculation of the total torque from the asymmetric gradient coils. We have shown that the contribution of the total torque for the magnetic field of the main magnet is zero for the axial asymmetric gradient coil, while it can be reduced up to 75% for the self-shielded transverse asymmetric gradient coil. This will be helpful in the consideration of the design for the transverse asymmetric gradient coils.

In conclusion for this chapter, we have presented an adequate design for the complete set of the self-shielded asymmetric coil. We have shown that the axial asymmetric gradient coil does not generate torque effects from the interaction of the current with the magnetic field of the main magnet which is assumed to be along the \( z \) direction. For the transverse self-shielded
asymmetric coils, the torque will be reduced by an amount equal to the ratio of the modified Bessel functions evaluated at the surface of the two coils respectively. Furthermore, we have presented the discrete current patterns for the inner and the outer coil as well as the resulting $z$ component of the magnetic field which was evaluated using the Biot-Savart law.
Figure 9.1: Illustration of the asymmetric self-shielded cylindrical gradient coil design. It shows the geometric center as well as the "sweet spot". The cylinder has length $L$. The distance of the sweet spot from the end of the cylinder is defined with $l_{ss}$, while the radii of the inner and outer coils are denoted as $a$ and $b$ respectively.
Figure 9.2: Plot of the variation of the azimuthal component of the continuous current density of the inner coil along the z axis. This current distribution is necessary to generate a gradient field with 40 mT/m at a distance of 0.145 m from the edge of the cylinder.
Figure 9.3: Plot of the variation of the azimuthal component of the continuous current density of the outer coil along the $z$ axis. This current distribution is necessary to generate a gradient field with $40 \, mT/m$ at a distance of $0.145 \, m$ from the edge of the cylinder.
Figure 9.4: Illustration of the discrete current distribution for the self-shielded axial asymmetric gradient coil. There are 53 wire loops carrying an amount of current equal to 323 Amps. The abscissa corresponds to the positions of the wires across the cylinder, while the ordinate is normalized to the ratio of the current and the radius of the wire loop. The wires with positive current are noted by a solid line, while the negative wires are marked with dashed lines.
Figure 9.5: Plot of the variation of the $z$ component of the magnetic field along the $z$ axis which is generated using the discrete current distribution. The plot shows the sweet spot of the gradient field at $0.465\, m$ for the beginning of the coil. The gradient strength is $41.3\, m T/m$ and the distance between the two rollover points is $0.35\, m$. 
Figure 9.6: Plot of the \( z \) component of the continuous current distribution versus \( z \) for the \( x \) transverse self-shielded asymmetric gradient inner coil. This current density is adequate to generate the field specifications which are described in the design section.
Figure 9.7: Plot of the $z$ component of the continuous current distribution versus $z$ for the $x$ transverse self-shielded asymmetric gradient outer coil. This current density is adequate to generate the field specifications which are described in the design section.
Figure 9.8: Illustration of the discrete current distribution for the inner asymmetric cylindrical gradient coil. There are 16 current turns where each one carries an amount of current equal to 522 Amps.
Figure 9.9: Illustration of the discrete current distribution for the outer asymmetric cylindrical gradient coil. There are 8 current turns where each one carries an amount of current equal to 522 Amps.
Figure 9.10: Plot of the $z$ component of the magnetic field along the $z$ axis for the inner coil (solid line) and the outer coil (dashed line) using the discrete current patterns for the inner and the outer coils respectively.
Chapter 10

Wrist Gradient Coils

10.1 Introduction

During the past decade, Magnetic Resonance Imaging has expanded to new horizons of diagnostic medicine. Magnetic Resonance Angiography, cardiac and functional imaging, as well as ultrafast imaging set the pace for the new developments in MRI. In order to design new and improved sequences for faster acquisition of data, larger and faster gradient coils are required.

Up to this chapter, we have presented in this dissertation the theoretical development, the design and the results both theoretical and experimental for a vast variety of gradient coils. Each of these coils was designed and built in order to cover a specific area of application in the field of MRI. Specifically,
asymmetric gradient coils were primarily designed to image the human head and lower parts of the neck without the shoulders restrict the size of the coil or the quality of the magnetic field associated with by these coils.

Therefore, we have presented a variety of novel gradient coils ranging from elliptical geometries to self-shielded asymmetric geometries, and the design of all these gradient coils share a common geometry. The component of the magnetic field which is responsible for generating the gradient field is chosen to be along the $z$ direction, in order to coincide with the direction of the magnetic field of the main magnet.

Although this design of gradient coils dominates the applications in the field of MRI, there are, however, specific applications where the shape of the gradient field can be generated by its $x$ component. Particularly, imaging a human wrist are among the applications where MRI can have major contributions. In general, there are two different types of gradient coils which can be used to image the human wrist. The first type is the traditional cylindrical gradient coil type where the major axis of the cylinder coincides with the direction of the main magnet's magnetic field. This configuration requires
that the patient be inserted head first inside the magnet in a prone position
lying on his stomach with his wrist extended towards the center of the main
magnet. As we can imagine, this position is rather uncomfortable, since the
patient must remain inside the imaging system for quite a large amount of
time.

An alternative way involves a gradient coil design where the main axis of
the gradient cylindrical coil forms a 90° angle with respect to the main magnet
axis. With this configuration, in order to image the human wrist, the patient
is positioned again head first and in the supine position (lying on his back)
while the patient's hand and thus his wrist can be rested on his thorax area.
positioned perpendicular to the main axis of the MRI imager. Therefore,
we must design a set of gradient coils that are generating a component of
the linearly varying magnetic field which coincides with the direction of the
main magnet’s field. In this case, the \( x \) component of the gradient field must
be considered as the primary component which varies linearly along the \( x \).
\( y \) and \( z \) axis of the coordinate system which is attached to the gradient coil
set.
In this chapter, we will present three gradient coil configurations which are necessary to image the human wrist. This set of coils will be designed such that the $x$ component of the magnetic field varies linearly along the $x$, $y$ and $z$ direction of a reference frame with respect to the gradient coil set. There are, however, two additional concerns for the design of this type of gradient coils. The first concern is their total length. The gradient coil set is placed perpendicular to the direction of the main magnet's major axis and it is also lifted upwards because it must be placed inside the area between the patient's thorax and the cover of the main magnet. Therefore there must exist a restriction for the total length of the coils, and their theoretical development must consider finite size gradient coils. This means that we must expand once again the current density of the coils in terms of Fourier series. The second concern is the eddy current generation on the main magnet and heating shield, since these coils are placed very close to the shields. Therefore, in order to avoid this problem, we consider a self-shielded gradient coil configuration where the total magnetic field outside from both coils is zero, and the eddy current effects are eliminated.
Combining all the points which have been mentioned above, we will present in this chapter the theoretical development, the design procedure, and the results for the complete set of finite size cylindrical gradient coils which are responsible for creating gradient fields appropriate for imaging the human wrist. We will also calculate the total dissipated energy for each of the coils as well as its total inductance. Finally, the discussion of the quality of the gradient field which is generated from the discrete current distribution will be presented.

10.2 Self-Shielded X Wrist Cylindrical Gradient Coil with Finite Length

As we have discussed in the introduction section, imaging of the human wrist requires a specific set of gradient coils. These coils are designed in order to generate a linearly varying $x$ component of the magnetic field. For the theoretical development of such a set of coils, we will consider a coordinate system which will be attached to the gradient coil set. Therefore, all the mathematics must be transformed to this particular coordinate system. The
generation of the $x$ component of the magnetic field is not an easy task. Since the original shape of the gradient coil is cylindrical, the generation of the $x$ component of the magnetic field requires the combination of the radial and the azimuthal component of the magnetic field in the cylindrical coordinates. Therefore, the generation of the $x$ gradient coil must be done by using a current density which is the superposition of two current components, one along the $x$ direction of the gradient coil and the other along the azimuthal one. Furthermore, the restriction on the length for the coil introduces the familiar Fourier expansion scheme for both components of the current density.

In this section, we will present the theoretical development, the design, and the results for the design of a gradient coil necessary for creating an $x$ component of the magnetic field along the self-shielded $z$ direction of the reference frame associated with the gradient coil. This gradient field configuration is analogous to the picture of the $z$ gradient coil in the traditional gradient coil design. Finally, in this section we will evaluate the total inductance for both coils.
10.2.1 Theory

The geometric illustration for the the z wrist gradient coil is displayed in Figure 10.1. There are two cylindrical gradient coils with radii \( a \) and \( b \) respectively. The length of the inner coil is considered to be \( L \), while the length of the outer gradient coil is left unrestricted. The design for this type of gradient coil involves a gradient field which is antisymmetric in the \( z \) direction around the geometric origin of the gradient coil set while symmetric in the \( y \) and \( z \) directions about the origin of the reference frame. The Fourier expansion of the current density for this coil reduces to sine series.

Before we introduce the derivation of the magnetic field and the dissipated energy of this gradient coil, it is necessary to understand the behavior and the shape of the current density which is necessary to generate both these quantities. As we have mentioned before, the \( z \) wrist gradient coil is the analogous design to the traditional \( z \) gradient coil. Therefore, the shape of its current distribution on the surface of the coils must be similar to the Maxwell coil pair design. The difference though is that in this case the
current density must be constructed from two current components instead of one. Although this is true, the symmetric conditions for the total current density must remain the same. Thus the expression of the total current density is

\[ \vec{J}^2(\vec{r}) = \left[ j_\phi^z(\phi, z)\hat{\phi} + j_z^z(\phi, z)\hat{z} \right] \delta(\rho - a) \] (10.1)

and we obtain a similar expression for the total current density for the outer coil \( \vec{J}^1 \) by simply replacing \( a \) with \( b \) in the expression (10.1). Since the behavior of the \( z \) component of the current density defines the behavior of the magnetic field generated by the gradient coil, we can build an expression for the total current density by examining the symmetric conditions which the \( j_z^z(\phi, z) \) must satisfy. First, since we are interested in the design of the \( x \) gradient wrist coil, \( j_z^z(\phi, z) \) must be symmetric along the \( z \) direction and around the origin of the coil. Therefore, only \( \cos k_n z \) terms must be present in the Fourier series expansion for the axial component of the current density.

To generate an \( x \) component of the magnetic field with the demand that it must vary linearly along the \( z \) direction, the azimuthal dependence for
$j_z^a(\phi, z)$ is chosen such that $j_z^a(\phi, z)$ is asymmetric about $\phi = \frac{\pi}{2}$ and $\phi = \frac{3\pi}{2}$ and is also antisymmetric around $\phi = 0$ and $\phi = \pi$. The correct functional behavior for this azimuthal dependence of $j_z^a(\phi, z)$ is proportional to $\sin 2\phi$. Furthermore, the functional dependence for the azimuthal component of the current density and for the inner coil must be adequate in order for the total expression of the current distribution to satisfy the continuity equation. Thus the expressions for both components of the current density for the inner coil is

\begin{align}
    j_\phi^a(\phi, z) &= \cos 2\phi \sum_{n=1}^{\infty} j_{\phi n} \sin k_n z \text{ for } |z| \leq \frac{L}{2} \\
    j_z^a(\phi, z) &= \sin 2\phi \sum_{n=1}^{\infty} j_{z n} \cos k_n z \text{ for } |z| \leq \frac{L}{2} \\
    j_z^a(\phi, z) &= 0 \text{ for } |z| > \frac{L}{2}
\end{align}

(10.2) (10.3)

where $j_{\phi n}, j_{z n}$ are the Fourier coefficients for the $j_\phi^a, j_z^a$ respectively. The restriction to the spectrum of $k_n$ is related to the restriction to the length of the inner coil. Since there is no current outside from the coil according to the equation (10.3), the group of discrete values which $k_n$ can take is

\[ \cos k_n L = 0 \implies k_n = \frac{(2n - 1)\pi}{L} \]

(10.4)
As we have mentioned previously, the current density for the inner coil must satisfy the continuity equation. Therefore, applying the continuity equation to the set of equations (10.3), we obtain the relation between the Fourier coefficients of the axial and the azimuthal components of the current density as

$$j_{zn}^a = -2 \frac{j_{\phi n}^a}{k_n a}$$  

(10.5)

Then equation (10.3) is modified, using the notation $j_{\phi n}^a = j_n^a$, as

$$j_n^a(\phi, z) = \cos 2\phi \sum_{n=1}^{\infty} j_n^a \sin k_n z \text{ for } |z| \leq \frac{L}{2}$$  

(10.6)

$$j_n^a(\phi, z) = \sin 2\phi \sum_{n=1}^{\infty} \frac{-2j_n^a}{k_n a} \cos k_n z \text{ for } |z| \leq \frac{L}{2}$$  

(10.7)

Before we proceed to the evaluation of the Fourier transform of the azimuthal component of the current, we would like to derive the expression of the $x$ component of the magnetic field. Since dealing with cylindrical coordinates, we can obtain fairly easily the expression for the radial $B_r$ and azimuthal $B_\phi$ components of the magnetic field. Taking advantage of the relation between the $z$ axis at the cartesian coordinate system and the radial and azimuthal direction for the cylindrical coordinate system, the expression
of the $x$ component of the magnetic field $B_x$ in terms of $B_\rho$ and $B_\phi$ is

$$B_x = B_\rho \cos \phi - B_\phi \sin \phi$$  \hspace{1cm} (10.8)

For a set of two cylindrical coils with current densities $j_\rho^a$ and $j_\phi^a$, the expressions of the radial and azimuthal components of the magnetic field inside both coils and outside both coils are respectively

$$B_\rho = \frac{i \mu_0}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk \ e^{im\phi+ikz} \ k \left[ a \ j_\rho^a(m, k)K'_m(ka)I'_m(k\rho) \\
+ b j_\phi^a(m, k)K'_m(kb)I'_m(k\rho) \right]$$  \hspace{1cm} (10.9)

$$B_\phi = \frac{-\mu_0}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk \ e^{im\phi+ikz} \ k \left[ a \ m \ j_\rho^a(m, k)K'_m(ka)I'_m(k\rho) \\
+ b \ m \ j_\phi^a(m, k)K'_m(kb)I'_m(k\rho) \right] \text{ for } \rho < a$$  \hspace{1cm} (10.10)

$$B_\rho = \frac{i \mu_0}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk \ e^{im\phi+ikz} \ k \left[ a \ j_\rho^a(m, k)K'_m(kp)I'_m(ka) \\
+ b j_\phi^a(m, k)K'_m(kp)I'_m(kb) \right]$$

$$B_\phi = \frac{-\mu_0}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk \ e^{im\phi+ikz} \ k \left[ a \ m \ j_\rho^a(m, k)K'_m(kp)I'_m(ka) \\
+ b \ m \ j_\phi^a(m, k)K'_m(kp)I'_m(kb) \right] \text{ for } \rho > b$$

The demand that $B_x$ must be zero at the region outside of both coils, will be satisfied if both components $B_\rho, B_\phi$ are simultaneously zero. This can be
achieved by choosing the current density of the second coil such that

\[ j^b_\phi(m, k) = -\frac{a}{b} \frac{I'_m(ka)}{I'_m(kb)} \]  

(10.11)

Using equation (10.11), the expression for both components of the magnetic field at the region inside the coils is modified as

\[
B_\rho = \frac{i\mu_0}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk \ e^{im\phi+ik\rho} k \left( 1 - \frac{I'_m(ka)K'_m(kb)}{I'_m(kb)K'_m(ka)} \right)
\]

\[ a j_\rho^a(m, k) K'_m(ka) I'_m(k\rho) \]

\[
B_\phi = \frac{-\mu_0}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk \ e^{im\phi+ik\rho} \left( 1 - \frac{I'_m(ka)K'_m(kb)}{I'_m(kb)K'_m(ka)} \right)
\]

\[ \frac{am}{\rho} j_\phi^a(m, k) K'_m(ka) I_m(k\rho) \text{ for } \rho < a \]  

(10.12)

Now we are ready to calculate the Fourier transform of the azimuthal component of the current density for the inner coil of the \( x \) wrist gradient coil. The general expression for the Fourier and Inverse Fourier transform for a two-variable function is

\[
j^a_\phi(m, k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi e^{-im\phi} \int_{-\infty}^{\infty} dz e^{-ikz} j^a_\phi(\phi, z) \]

(10.13)

and

\[
j^a_\phi(\phi, z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi e^{im\phi} \int_{-\infty}^{\infty} dz e^{ikz} j^a_\phi(m, k) \]

(10.14)
and similar expressions are used for \( j^a_\phi \). Substituting (10.7) into equation (10.13), and considering the restriction to the total length for the inner cylinder, the Fourier transform for \( j^a_\phi(m, k) \) is

\[
    j^a_\phi(m, k) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} d\phi (\cos m\phi - i \sin m\phi) \cos 2\phi \\
    \left[ \int_{-\frac{k}{2}}^{\frac{k}{2}} dz e^{-iks} j^a_n \sin k_n z \right] (10.15)
\]

Using the orthonormality conditions of the trigonometric functions, the angular integration on the above expression becomes

\[
    \int_{-\pi}^{\pi} d\phi \cos 2\phi \cos m\phi = \begin{cases} 
    \pi & \text{for } m = \pm 2 \\
    0 & \text{otherwise}
\end{cases} (10.16)
\]

where the angular integration over \( \sin m\phi \) is neglected because the integrand is odd and thus the value of the integral is zero. We now continue with the evaluation of the second integral over the \( z \) variable. Thus, the expression of the second integral in the Fourier space is

\[
    \int_{-\frac{k}{2}}^{\frac{k}{2}} dz e^{-iks} j^a_n \sin k_n z = \frac{i L}{4} j^a_n \\
    \left[ \frac{\sin(k - k_n) \frac{L}{2}}{(k - k_n) \frac{L}{2}} + \frac{\sin(k + k_n) \frac{L}{2}}{(k + k_n) \frac{L}{2}} \right] (10.17)
\]
Combining equations (10.15)-(10.17), the expression for the azimuthal component of the current density in the Fourier representation is

\[ j^\phi_n(\pm 2, k) = \frac{iL}{4} \sum_{n=1}^{\infty} j^\phi_n \psi_n(k) \]  

(10.18)

with

\[ \psi_n(k) = \left[ -\frac{\sin(k - k_n) \frac{L}{2}}{(k - k_n) \frac{L}{2}} + \frac{\sin(k + k_n) \frac{L}{2}}{(k + k_n) \frac{L}{2}} \right] \]  

(10.19)

Knowing the expression for the Fourier transform of the current density, the expression of the radial and azimuthal components of the magnetic field can be modified as

\[ B_\rho = -\frac{\mu_o a L}{4\pi} \cos 2\phi \int_{-\infty}^{\infty} dk \frac{k \cos k z}{k \cos k z} \left( 1 - \frac{I'_2(k \rho) K'_2(k b)}{I'_2(k b) K'_2(k a)} \right) I'_2(k \rho) K'_2(k a) \psi_n(k) \sum_{n=1}^{\infty} j^\phi_n \psi_n(k) \]

\[ B_\phi = \frac{\mu_o a L}{2\pi} \sin 2\phi \frac{1}{\rho} \int_{-\infty}^{\infty} dk \cos k z \]

\[ \left( 1 - \frac{I'_2(k \rho) K'_2(k b)}{I'_2(k b) K'_2(k a)} \right) I_2(k \rho) K'_2(k a) \psi_n(k) \]  

(10.20)

where the symmetry conditions for \( j^\phi_n(\pm 2, k) \) and \( \psi_n(k) \) have been employed

\[ \psi_n(-k) = -\psi_n(k) \]

\[ j^\phi_n(-2, k) = j^\phi_n(2, k) \]
Substituting (10.20) into equation (10.8), the modified expression for the \( x \) component of the magnetic field is

\[
B_x = -\frac{\mu_0 a L}{4\pi} \left[ \sum_{n=1}^{\infty} J_n \int_{-\infty}^{\infty} dk \ k \cos k z \left( 1 - \frac{I_1'(ka)K_2'(kb)}{I_2'(kb)K_2'(ka)} \right) K_2'(ka) \psi_n(k) \right] \cos 2\phi \cos \phi I_1(k\rho) - \cos 3\phi \frac{2}{k\rho} I_2(k\rho) \right]
\]

where the following identity for the modified Bessel functions

\[
I_1(k\rho) = \left( I_1'(k\rho) + \frac{2}{k\rho} I_2(k\rho) \right)
\]

was used in the derivation of the final expression of \( B_x \).

The general expression for the stored magnetic energy and for the case of the self-shielded gradient coil design has been introduced before by Martens [56]

and has the form

\[
W_m = -\frac{a^2 \mu_0}{2} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} dk |j_n^2(m, k)|^2 I_n'(ka)K_n'(ka) \left( 1 - \frac{I_n'(ka)K_n'(kb)}{I_n'(kb)K_n'(ka)} \right)
\]

The above expression can be reduced to a simple integral equation for the stored energy, by eliminating all the values of \( m \) in the summation except \( m = \pm 2 \) in order for the condition (10.16) to be satisfied. Thus, equation (10.22)
is modified as

\[ W_m = -a^2 \mu_o \int_{-\infty}^{\infty} dk |j_\nu^a(2,k)|^2 I_2'(ka)K_2'(ka) \left( 1 - \frac{I_2'(ka)K_2'(kb)}{I_2'(kb)K_2'(ka)} \right) \] (10.23)

Expressing \( |j_\nu^a(2,k)|^2 \) in terms of the Fourier coefficients

\[ |j_\nu^a(2,k)|^2 = \frac{L^2}{16} \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} j_n^a j_{n'}^a \psi_n(k) \psi_{n'}(k) \] (10.24)

the final expression of the stored energy for the \( x \) wrist gradient coil is

\[ W = -\frac{a^2 \mu_o L^2}{16} \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} j_n^a j_{n'}^a \int_{-\infty}^{\infty} dk \frac{I_2'(ka)K_2'(ka)}{I_2'(kb)K_2'(ka)} \left( 1 - \frac{I_2'(ka)K_2'(kb)}{I_2'(kb)K_2'(ka)} \right) \psi_n(k) \psi_{n'}(k) \] (10.25)

Following Turner's methodology once again, we construct the functional \( \mathcal{E} \) in terms of the stored energy and the \( x \) component of the magnetic field as

\[ \mathcal{E}(j_n^a) = W - \sum_{j=1}^{N} \lambda_j (B_x(\vec{r}_j) - B_{xSC}(\vec{r}_j)) \]

where the meaning of the quantities at the right hand side of the above equation have been introduced in previous chapters. Minimizing \( \mathcal{E} \) with respect to \( j_n^a \), we obtain a matrix equation for \( j_n^a \) as

\[ \sum_{n'=1}^{\infty} j_{n'}^a \left\{ -\frac{aLx}{2} \int_{-\infty}^{\infty} dk I_2'(ka)K_2'(ka) \left( 1 - \frac{I_2'(ka)K_2'(kb)}{I_2'(kb)K_2'(ka)} \right) \psi_{n'}(k) \psi_n(k) \right\} \]
\[
\sum_{j=1}^{N} \lambda_j \int_{-\infty}^{\infty} dk \, k \cos k z_j \, K'_{2}(ka) \left(1 - \frac{I^*_2(ka)K'_2(kb)}{I^*_2(kb)K'_2(ka)}\right) \psi_n(k) \\
\left[ \cos^2 \phi_j \cos \phi_j I_1(k \rho_j) - \cos 3 \phi_j \frac{2}{k \rho_j} I_2(k \rho_j) \right] \quad (10.26)
\]

In a compact matrix notation associated with a truncation of the infinite summation to \( M \) terms, the previous equation becomes

\[
\sum_{n=1}^{M} j^*_{n} C_{n,n} = \sum_{j=1}^{N} \lambda_j D_{j,n} \implies J^* C = \Delta D \quad \text{or} \quad J^* = \Delta D C^{-1} \quad (10.27)
\]

where \( J^* \) is a \( 1 \times M \) matrix, \( C \) is a \( M \times M \) matrix, \( \lambda \) is a \( 1 \times N \) matrix and \( D \) is a \( N \times M \) matrix.

The form of the equation (10.27) is familiar to us and has been presented in the previous chapters. Therefore in order to obtain the final expression for the continuous current density of the inner coil, we follow the methodology which has been explained in detail in section 8.2.

### 10.2.2 Design

In this section, we will present the design procedure for the creation of the \( x \) wrist gradient coil. This configuration consists of two circular cylindrical coils. The radius of the inner coil is equal to \( a = 0.07 \, m \), and the radius
of the outer coil is chosen to be \(b = 0.085\ m\). The total length of the inner coil is chosen to be equal to \(L = 0.2\ m\) in order to fit in the region between the thorax and the plastic cover of the MRI imager. Although the length of the outer coil is left unrestricted, and in general is larger than the length of the inner coil, the careful choice for the linearity and the homogeneity of the magnetic field inside the imaging volume can indirectly limit the length of the outer coil. Three constraint points help in the definition of the magnetic field inside a 10 cm DSV. The first constraint point sets the strength of the \(x\) component of the magnetic field along to \(x\) axis to \(40\ mT/m\). The next constraint point limits the on axis variation of the magnetic field to 10.5\% from its ideal value at a distance of 0.05 \(m\) from the center of the gradient field. Since the volume of the DSV is large with respect to the dimensions of the coil, an additional constraint point is necessary to limit the off-axis variation of the magnetic field to within 5\% from its ideal value at a distance of 0.05 \(m\) from the center of the coil. This set of constraints is displayed in Table 10.1. Furthermore, since we must deal with higher order trigonometric functions, 40 Fourier terms are used for the design and the calculation of the


<table>
<thead>
<tr>
<th>n</th>
<th>$\rho_i$</th>
<th>$z_i$</th>
<th>$B_{zsc}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.001</td>
<td>0.000</td>
<td>0.00004000</td>
</tr>
<tr>
<td>2</td>
<td>0.050</td>
<td>0.000</td>
<td>0.00175000</td>
</tr>
<tr>
<td>3</td>
<td>0.001</td>
<td>0.050</td>
<td>0.00003800</td>
</tr>
</tbody>
</table>

Table 10.1: Constraint set used for the design of a self-shielded $x$ wrist gradient coil. Values for $\rho$ and $z$ are in m, values for $B_{zsc}$ are in T.

current density for the inner and the outer coils. The symmetry requirements for the $x$ wrist gradient coil have already been built into the mathematical formulation of the problem and no extra measure was taken when this set of constraints was chosen.

10.2.3 Results

Solving the inverse problem with the set of constraints given in Table 10.1, the expressions for both continuous current distributions were obtained. Figure 10.2 displays the variation along the $z$ axis of the $z$ component of the inner coil (solid line) as well as the variation of the $z$ component of the outer coil (dashed line), since only $j_z^i$ and $j_z^o$ are the necessary components for the creation of the appropriate discrete current patterns for the inner and the outer coil. For the creation of these continuous current densities the first 15
of the 40 Fourier terms were used.

The discretization procedure for this gradient coil was performed using the stream function technique and is similar to the discretization mechanism which was described in chapter 7. For the inner gradient coil 16 discrete coil turns carrying 39.59 Amps per turn were considered. The discretization version for the secondary coil consists of 8 coil turns where each turn carries an amount of current equal to −41.58 Amps. Figure 10.3 displays the discrete current distribution for the inner coil, while Figure 10.4 illustrates the discrete current pattern for the outer coil. We can also notice from Figure 10.4 that the distance of the farthest current loop from the center of the coil is equal to 0.12 m, which sets the total length of the outer coil to 0.24 m. This corresponds to an increase of 20% compared with the total length of the inner coil.

As a next step, we obtain the expression of the z component of the magnetic field for both coil sets by employing the Biot-Savart law to both discrete coil patterns. This has been done in order to ensure the reliability of our discretization procedure, and to obtain a real estimate for the behavior
of the magnetic field of the gradient coil. Figure 10.5 displays the variation of the \( z \) component of the magnetic field along the \( z \) direction which coincides for both the inner (solid line) and outer coil (dashed line). The strength of the magnetic field was estimated to 44 \( mT/m \) which corresponds to a 10\% increase from the ideal gradient strength. The on-axis value of the magnetic field at a distance of 0.05\( m \) away from the center of the coil was measured at 1.801 \( \times 10^{-3} \) Tesla which corresponds to a 5.25\% decrease from the constraint value of the magnetic field at this point. Furthermore, the measured value of the \( z \) component of the magnetic field at the plane perpendicular to the gradient axis at a distance 0.05\( m \) away from the center deviates from the prescribed constraint value by 5\%.

The total dissipated energy for the \( x \) wrist self-shielded gradient coil is found to be 0.0154 Joules. Using again the relation between the total energy and the inductance of the coil, a rough estimation for the total inductance of the coil can be obtained. For the amount of the energy given above and for an average current equal to 40.585 Amps the total estimated inductance for the combined set of both coils is \( L_{\text{total,estim}} = 18.70 \mu H \).
In conclusion for this section, we have presented a novel approach for the design of a $z$ gradient coil, where the $x$ component of the magnetic field is varying along the $x$ axis of a coordinated system which is attached to the gradient coils set. The mathematical methodology, the design and the results were presented. We also generated the discrete current patterns for both the inner and the outer coil. By employing the Biot-Savart law the $x$ component of the magnetic field was calculated using the discrete current pattern distribution. As the results indicate, the quality of the magnetic field is very good and deviates from the ideal values which were set by the constraint points by less than 10%.

10.3 Self-Shielded $Y$ Wrist Cylindrical Gradient Coil with Finite Length

We now continue with the presentation of the second wrist gradient coil. The shelf-shielded $y$ wrist gradient coil is also designed in order to generate a linearly varying $x$ component of the magnetic field along the $y$ direction of a coordinate system which is associated with the gradient coil. In parallel
fashion with the section 10.2, the creation of the $x$ component of the magnetic field still requires the superposition of the radial and azimuthal component of the magnetic field with respect to the cylindrical coordinate system. This corresponds to current density distribution which must also be considered as the combination of two components, one along the axial and the other along the azimuthal direction. Furthermore, the requirement for a finite size coil, leads to the expansion of the current density in terms of Fourier coefficients.

In this section, we will present the mathematical development, the design procedure, and the theoretical results for a self-shielded gradient coil which generates a linearly varying $x$ component of the magnetic field along the $y$ axis of the reference frame attached with the gradient coil. This coil is otherwise referred to as the $y$ wrist gradient coil. The discretization scheme for the inner and outer coils of this gradient set is also presented. Furthermore, the $x$ component of the magnetic field will be re-evaluated applying the Biot-Savart law to the discrete current patterns of these two coils. Finally, the total stored energy of the system will be calculated and an estimation of the total inductance of the coil set will be presented.
10.3.1 Theory

The geometric set for the y wrist gradient coil is similar to the x wrist gradient coil and is shown in Figure 10.1. For the self-shielded design there are two cylindrical coils. Again the dimensions of the two coils are presented in detail in section 10.2.1.

We start the mathematical formulation by understanding the necessary formation of the current density in order to generate the desired specifications of the magnetic field. In order to generate an x component of the magnetic field, the current density must be considered as a superposition of two current densities: one along the axial direction and the other along the azimuthal one.

The general expression of the current density for the inner coil $\vec{J}^a$ is given in equation (10.1). A similar expression can be assumed for the continuous current density of the outer coil $\vec{J}^b$. Again, in order to generate a current density which is adequate for the creation of a linearly varying x component of the magnetic field along the y direction, the symmetry conditions which the x component of the current density $j_x(\phi, z)$ satisfies is an indication of
the behavior of the magnetic field in the region of interest. First, since we are interested in the design of the $y$ wrist gradient coil, $j^y_z(\phi, z)$ must be symmetric about the origin of the coil and along the $z$ direction. To include this condition in the mathematical methodology, the Fourier series expansion for $j^y_z(\phi, z)$ must be modified and expressed in terms of cosine terms. The demand that the $x$ component of the magnetic field varies linearly along the $y$ direction of the coil, implies that the $x$ component of the current density must be symmetric around $\phi = 0$ and $\phi = \pi$. Furthermore, the antisymmetric behavior of $B_x$ about the origin of the magnetic field and along the $y$ direction requires that $j^y_z(\phi, z)$ must be symmetric around $\phi = \frac{\pi}{2}$ and $\phi = \frac{3\pi}{2}$. To simulate this angular behavior, $j^y_z(\phi, z)$ must be proportional to $\cos 2\phi$. In addition, the functional dependence of the azimuthal component of the current density for the inner coil must have the correct form in order for the combined expression for the current distribution to satisfy the continuity equation. Thus, the Fourier expansion for both components of the current
density for the inner coil is

\[
\begin{align*}
j^s_\phi(\phi, z) &= \sin 2\phi \sum_{n=1}^{\infty} j^s_{\phi n} \sin k_n z \text{ for } |z| \leq \frac{L}{2} \\
j^s_z(\phi, z) &= \cos 2\phi \sum_{n=1}^{\infty} j^s_{zn} \cos k_n z \text{ for } |z| \leq \frac{L}{2} \\
0 &= 0 \text{ for } |z| > \frac{L}{2}
\end{align*}
\]

(10.28)

where \( j^s_{\phi n}, j^s_{zn} \) are the Fourier coefficients for the \( j^s_\phi, j^s_z \) respectively. The restriction to the total length of the inner coil defines the group of discrete values which \( k_n \) can take and is given by equation (10.4). Since the total current density of the inner coil must obey the continuity equation, we can relate the Fourier coefficients for both current components by applying the continuity equation in the set of equations (10.28) as follows

\[
j^s_{zn} = 2 \frac{j^s_{\phi n}}{k_n a}
\]

(10.29)

Then equation (10.28) is modified, using the notation \( j^s_{\phi n} = j^s_n \), as

\[
\begin{align*}
j^s_\phi(\phi, z) &= \sin 2\phi \sum_{n=1}^{\infty} j^s_{n} \sin k_n z \text{ for } |z| \leq \frac{L}{2} \\
j^s_z(\phi, z) &= \cos 2\phi \sum_{n=1}^{\infty} \frac{j^s_n}{k_n a} \cos k_n z \text{ for } |z| \leq \frac{L}{2}
\end{align*}
\]

(10.30) (10.31)

The procedure of the evaluation of the Fourier transform for the azimuthal
component of the current density of the inner coil is similar to the methodology described in detail in section 10.2.1. In this section, we will sketch the procedure for obtaining the Fourier transform of \( j_\phi^2(\phi, z) \). Substituting equation (10.30) into equation (10.13), the expression of the Fourier transform for \( j_\phi^2(\phi, z) \) becomes

\[
j_\phi^2(m, k) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} d\phi (\cos m\phi - i \sin m\phi) \sin 2\phi \\
\int_{-\frac{L}{4}}^{\frac{L}{4}} dz e^{-ikz} j_n^a \sin k_n z
\]

(10.32)

The angular integration on the right hand side of the equation (10.32) can be further simplified, by using the orthonormality condition of the trigonometric functions as follows

\[
-i \int_{-\pi}^{\pi} d\phi \sin 2\phi \sin m\phi = \begin{cases} 
-i \pi & \text{for } m = -2 \\
+ i \pi & \text{for } m = +2 \\
0 & \text{otherwise}
\end{cases}
\]

(10.33)

The determination of the axial integral in equation (10.32) has been shown in equation (10.17). Therefore, the expression for the \( j_\phi^2(m, k) \) is

\[
j_\phi^2(\pm 2, k) = \frac{\pm L}{4} \sum_{n=1}^{\infty} j_n^a \psi_n(k)
\]

(10.34)
with

$$\psi_n(k) = \left[ -\frac{\sin(k - k_n)k_k}{(k - k_n)^{1/2}} + \frac{\sin(k + k_n)k_k}{(k + k_n)^{1/2}} \right]$$  \( (10.35) \)

The expression of the Fourier transform for the outer coil can be determined using the expression (10.11) which relates the Fourier transforms for both coils.

The next step is the calculation of the \( x \) component of the magnetic field. Certainly, the functional dependence of the magnetic field changes drastically compared with the \( x \) wrist gradient coil, since different symmetry conditions are imposed. In order to determine the expression of the \( B_x \) for the \( y \) wrist gradient coil, we must first derive the expressions of the radial and azimuthal components of the magnetic field. From the set of equations (10.10), \( B_\rho \) and \( B_\phi \) are modified as follows

\[
B_\rho = -\frac{\mu_0 a L}{4\pi} \sin 2\phi \int_{-\infty}^{\infty} dk k \cos k z \left( 1 - \frac{I_2'(ka)K_2'(kb)}{I_2'(kb)K_2'(ka)} \right) L_2'(k \rho)K_2'(ka) \sum_{n=1}^{\infty} j_n^\rho \psi_n(k)
\]

\( \text{and} \)

\[
B_\phi = \frac{\mu_0 a L}{2\pi} \cos 2\phi \sum_{n=1}^{\infty} j_n^\phi \frac{1}{\rho} \int_{-\infty}^{\infty} dk \cos k z \left( 1 - \frac{I_2'(ka)K_2'(kb)}{I_2'(kb)K_2'(ka)} \right) I_2(k \rho)K_2'(ka) \psi_n(k)
\]

\( \text{(10.36)} \)

\( \text{and} \)

\( \text{(10.37)} \)
where the symmetry conditions for \( j_2^e(\pm 2, k) \) and \( \psi_n(k) \)

\[
\psi_n(-k) = -\psi_n(k)
\]

\[
j_2^e(-2, k) = -j_2^e(2, k)
\]

have been employed.

Substituting (10.36) and (10.37) into equation (10.8), the expression of the \( x \) component of the magnetic field for the \( y \) wrist gradient coil is

\[
B_x = -\frac{\mu_0 a L}{4\pi} \left[ \sum_{n=1}^{\infty} j_n^* \int_{-\infty}^{\infty} dk \, k \cos k z \left( 1 - \frac{I_2'(ka)K_2'(kb)}{I_2'(kb)K_2'(ka)} \right) K_2'(ka) \psi_n(k) \right. \\
\left. \left\{ \sin 2\phi \cos \phi \, I_1(k \rho) - \sin \phi \frac{2}{k \rho} I_2(k \rho) \right\} \right]
\]

(10.38)

The expression for the stored magnetic energy for the \( y \) wrist gradient coil is similar to the expression (10.25), since no angular dependence is presented in the final result for the dissipated energy. Thus,

\[
W = \frac{a^2 \mu_0 L^2}{16} \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} j_n^* j_{n'}^* \int_{-\infty}^{\infty} dk \, I_2'(ka)K_2'(ka) \\
\left( 1 - \frac{I_2'(ka)K_2'(kb)}{I_2'(kb)K_2'(ka)} \right) \psi_n(k) \psi_{n'}(k)
\]

(10.39)

In order to proceed with Turner's inverse approach method, we must create the functional \( \mathcal{E} \) in terms of the stored energy and the \( x \) component
of the magnetic field as

\[ E(j_n^a) = W - \sum_{j=1}^{N} \lambda_j (B_z(\bar{r}_j) - B_{zSC}(\bar{r}_j)) \]

Minimizing \( E \) with respect to \( j_n^a \), we get a matrix equation for \( j_n^a \) as

\[
\sum_{n'=1}^{\infty} j_{n'}^a \left\{ -\frac{aL_0}{2} \int_{-\infty}^{\infty} dk I_2'(ka)K_2'(ka) \left( 1 - \frac{I_2'(ka)K_2'(kb)}{I_2'(kb)K_2'(ka)} \right) \psi_{n'}(k) \psi_n(k) \right\} \\
= \sum_{j=1}^{N} \lambda_j \int_{-\infty}^{\infty} dk \ k \ \cos \kappa_j \ K_2'(ka) \left( 1 - \frac{I_2'(ka)K_2'(kb)}{I_2'(kb)K_2'(ka)} \right) \psi_n(k) \\
\left[ \sin 2\phi_j \cos \phi_j I_1(k\rho_j) - \sin \phi_j \frac{2}{k\rho_j} I_2(k\rho_j) \right]
\]

(10.40)

Again using a compact matrix notation and putting an upper threshold \( M \) on the infinite summations, the previous expression becomes

\[
\sum_{n'=1}^{M} j_{n'}^a C_{n'n} = \sum_{j=1}^{N} \lambda_j D_{j,n} \Rightarrow J^a = \Delta D \text{ or } J^a = \Delta D C^{-1}
\]

(10.41)

where \( J^a \) is a \( 1 \times M \) matrix, \( C \) is a \( M \times M \) matrix, \( \lambda \) is a \( 1 \times N \) matrix and \( D \) is a \( N \times M \) matrix.

The formalism of the equation (10.41) is identical to the equation (10.27).

Thus, in order to obtain the continuous current density for both coils in the \( y \) wrist gradient coil design, we must follow the steps which are described in the last paragraph of section 10.2.1.
10.3.2 Design

We now proceed with the design of the y wrist gradient coil. The radius of the inner coil is chosen to be \( a = 0.07 \, m \), while the radius of the outer coil is \( b = 0.085 \, m \). The total length of the cylinder for the inner coil is \( L = 0.2 \), while the length of the outer coil is left unrestricted. Although no initial limitation on the length of the outer coils is considered, the choice of the field constraints put a limitation to the total length of the outer coil. Four set of constraints were considered in order to establish the quality of the magnetic field inside the 10 cm DSV. The first constraint point sets the strength of the \( x \) component of the magnetic field to 40 \( mT/m \) along the \( y \) axis of the cylinder. The second constraint defines the on-axis variation of the magnetic field to within 2.5% from its ideal value and at a distance of 0.05 \( m \) from the center of the gradient coil. The next two constraints limit the off-axis variation of the \( x \) component of the magnetic field is no greater than 2.5% at the borders of the 10 cm DSV. This set of constraints is displayed in Table 10.2. Furthermore, in order to reduce the effects of
<table>
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<td>0.050</td>
<td>0.00003800</td>
</tr>
</tbody>
</table>

Table 10.2: Constraint set used for the design of a self-shielded y wrist gradient coil. Values for \(\rho\) and \(z\) are in m, values for \(\phi\) are radians, and values for \(B_{exc}\) are in T.

The higher order trigonometric in the Fourier series expansion, 40 terms were initially considered for the design and the evaluation for the current densities of the inner and outer coils.

10.3.3 Results

Employing the variation approach with the set of constraints from Table 10.2, the expression for the continuous current distribution of the \(z\) component of the current density is obtained. Figure 10.6 illustrates the variation of \(j_z^1\) and \(j_z^2\) along the \(z\) axis of the cylinder. For the calculation of these two current densities 15 Fourier terms are considered.

As a next step, we proceed with the discretization procedure for the \(y\) wrist gradient coil. Using the stream function technique as it was presented
in chapter 7, the discretized current patterns for the inner and the outer coil are derived. The discrete current distribution for the inner coil contains 16 loops carrying 46.81 Amps each, shown in Figure 10.7. For the outer coil, the discretized version of the current density consists of 8 coil turns carrying an amount of current equal to $-48.23$ Amps each. The discrete current distribution for the outer coil is displayed in Figure 10.8. Furthermore, the distance of the farthest current loop from the origin of the coils is equal to 0.1125 m, which indicates that the total length of the outer coil is equal to 0.225 m. This corresponds to a 12.5% increase with respect to the total length of the inner coil.

In order to ensure that the quality of the magnetic field remains unchanged, after the discretization procedure is performed, we apply the Biot-Savart law to the discrete current distribution for the inner and outer coils. Figure 10.9 illustrates the variation of the $z$ component of the magnetic field along the $y$ axis for both coils. The net strength of the magnetic field inside the region of the imaging volume is estimated to be 44.5 mT/m. This corresponds to a 11.25% increase over the ideal value for the strength of the
magnetic field. The on-axis net value of the magnetic field 0.05 m away from the center of the coil is calculated to be $1.9523 \times 10^{-3} T$ which deviates by 2.4% from the constraint value of the magnetic field at this point. Furthermore, the off-axis uniformity of the $x$ component of the magnetic field is increased to the 8% level from the ideal value of the magnetic field at the borders of the imaging volume.

The total dissipated energy of the system is found to be 0.0197 Joules. Taking advantage of the relationship between the energy and the total inductance of the system, the total estimated inductance of the $y$ wrist gradient coil is $L_{\text{tot,estim}} = 17.65 \mu H$, where the average amount of current which is equal to 47.25 Amps.

In conclusion for this section, we have presented a novel approach to the design of the $y$ wrist gradient coil, with the $x$ component of the magnetic field varying along the $y$ axis of the gradient coil. The mathematical methodology, the design and the theoretical results have also been presented, along with the discrete current distributions of the inner and outer coils. Furthermore, applying the Biot-Savart law to these discrete current distributions, the $x$
component of the magnetic field has been re-evaluated. The magnetic field inside the imaging volume still maintains its original quality with a maximum deviation from the ideal values of less than 8%.

10.4 Self-Shielded Z Wrist Cylindrical Gradient Coil with Finite Length

We wrap up the discussion of the complete wrist gradient coil set with the presentation of the self-shielded z wrist cylindrical gradient coil. The design of this coil focuses on the generation of a linearly varying z component of the magnetic field along the z direction. Since the z component of the magnetic field can be considered as the superposition of the radial and the azimuthal components of the magnetic field in cylindrical coordinates, the resulting expression of the continuous current for both coils must be analyzed into two components, one along the axial and the other along the azimuthal direction. Also, the restriction on the length of the coils enforces the expansion of the current density in terms of Fourier coefficients.

In this section, we will discuss the mathematical formalism, the design
and the theoretical results for the self-shielded $z$ gradient coil. We will also present the discretization scheme for both coils in the $z$ wrist gradient coil design. Also, employing the Biot-Savart law to these discrete current distributions, the expression of the magnetic field is re-evaluated in order to measure the accuracy of the discretization techniques. Finally, the total dissipated energy for this coil system will also be derived and an estimation of the total inductance of the system will be presented.

10.4.1 Theory

The geometric set up of the self-shielded $z$ wrist gradient coil is similar to the geometric configuration which is shown in Figure 10.1. The dimensions of the inner and outer coils in this design remain the same and are described in detail in the section 10.2.1.

As we have mentioned in the two previous chapters, the behavior of the current density defines the behavior of the magnetic field, since there is a one-to-one correspondence between them. Thus, we start the theoretical development of this $z$ wrist gradient coil, by examining the symmetry condi-
tions which the current density for both coils must satisfy. As we have also mentioned during the introduction of this section, the total current density $\vec{J}^a$ can be considered as the combination of two components. The mathematical formalism for $\vec{J}^a$ has been presented in equation (10.1). The same expression can also be considered for the total current density of the outer coil, by replacing $a$ with $b$ in the expression (10.1). In parallel fashion with the two previous sections, starting with the current density of the inner coil, the determination of the behavior of its $z$ component indicates the behavior of the total current density. Since there is a one-to-one correspondence between the magnetic field and the current density, the symmetry restrictions to the magnetic field reflect the symmetry behavior of the current density. The symmetry conditions for the $z$ component of the current density for the design of the $z$ wrist gradient coil which are necessary to generate a linearly varying $x$ component of the magnetic field along the $z$ direction of cylinder are analogous to the conditions for the conventional transverse gradient coil. Specifically, the demand that the $x$ component of the magnetic field must be antisymmetric along the $z$ direction of the cylinder and about the geometric
center of the coil indicates that the Fourier expansion must be expressed in terms of sine terms. The requirement that the magnetic field of the $z$ wrist gradient coil must be symmetric along the $x$ and $y$ directions of the cylinder and about its geometric center, defines the azimuthal behavior of the $z$ component of the current density. Specifically, $j_z(q, z)$ must be symmetric around $\phi = \frac{\pi}{2}$. In order to simulate this angular behavior for the $j_z(q, z)$, its correct functional dependence must be proportional to $\sin \phi$. The determination of the functional dependence for the azimuthal component of the current density $j_z(q, z)$, is chosen such that the final expression of the combined current density distribution for the inner coil satisfies the continuity equation. With this in mind, the Fourier expansion for $j_z(q, z)$ and $j_\phi(q, z)$

are

$$j_\phi(q, z) = \cos \phi \sum_{n=1}^{\infty} j_{\phi n} \cos k_n z \text{ for } |z| \leq \frac{L}{2}$$

$$j_z(q, z) = \sin \phi \sum_{n=1}^{\infty} j_{z n} \sin k_n z \text{ for } |z| \leq \frac{L}{2}$$

$$j_\phi(q, z) = 0 \text{ for } |z| > \frac{L}{2}$$

(10.42)

(10.43)

where $j_{\phi n}, j_{z n}$ are the Fourier coefficients for the $j_{\phi}, j_z$ respectively. Also, the
confinement for the current density of the inner coil defines the spectrum of the permeable values for \( k_n \). Since, \( j_i^a(\phi, z) \) must be zero outside of the cylinder, the allowable values of \( k_n \) are

\[
sin k_n L = 0 \implies k_n = \frac{2n\pi}{L} \tag{10.44}
\]

Employing the continuity equation to the total current density of the inner coil, we obtain the relation among the Fourier coefficients of the axial and azimuthal components of the current density as

\[
j^a_{in} = \frac{j^a_{in}}{k_n a} \tag{10.45}
\]

Using this relationship and the convention \( j^a_{in} = j^a_{n} \), the expressions (10.42),(10.43) are modified as

\[
j^a(\phi, z) = \cos \phi \sum_{n=1}^{\infty} j^a_n \cos k_n z \text{ for } |z| \leq \frac{L}{2} \tag{10.46}
\]

\[
j^a(\phi, z) = \sin \phi \sum_{n=1}^{\infty} \frac{j^a_n}{k_n a} \sin k_n z \text{ for } |z| \leq \frac{L}{2} \tag{10.47}
\]

We now proceed with the evaluation of the Fourier component of the azimuthal component of the current density for the inner coil, since the Fourier transform of the current density for the outer coil can be related to the corresponding transform of the inner coil from the equation (10.11). Considering
the general expression of the two variable Fourier transform from (10.13) along with the restriction of the total length of the inner coil, the double Fourier transform of \( j^a_\phi (\phi, z) \) is

\[
j^a_\phi (m, k) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} d\phi \left( \cos m\phi - i \sin m\phi \right) \cos \phi \int_{-\frac{L}{2}}^{\frac{L}{2}} dz e^{-ikz} j^a_n \cos k_n z
\]

(10.48)

Applying the orthonormality conditions for the trigonometric functions, the angular integration on the right hand side of (10.48) is

\[
\int_{-\pi}^{\pi} d\phi \cos \phi \cos m\phi = \begin{cases} \pi & \text{for } m = \pm 1 \\ 0 & \text{otherwise} \end{cases}
\]

(10.49)

and the angular integration over the \( \sin m\phi \) vanishes. Furthermore, the expression of the second integral over the \( z \) variable is

\[
\int_{-\frac{L}{2}}^{\frac{L}{2}} dz e^{-ikz} j^a_n \cos k_n z = \frac{L}{4} j^a_n \left[ \frac{\sin (k - k_n) \frac{L}{2}}{(k - k_n) \frac{L}{2}} + \frac{\sin (k + k_n) \frac{L}{2}}{(k + k_n) \frac{L}{2}} \right]
\]

(10.50)

Combining equations (10.49),(10.50), the expression (10.48) becomes

\[
j^a_\phi (\pm 1, k) = \frac{L}{4} \sum_{n=1}^{\infty} j^a_n \psi_n (k)
\]

(10.51)
with

$$
\psi_n(k) = \left[ \frac{\sin(k - k_n) \frac{L}{2}}{(k - k_n) \frac{L}{2}} + \frac{\sin(k + k_n) \frac{L}{2}}{(k + k_n) \frac{L}{2}} \right]
$$

(10.52)

Having derived the formalism for the double Fourier transform of \( j^a_\theta \), we now continue with the evaluation for the expression of the \( x \) component of the magnetic field. Since the symmetry conditions are different from the conditions of the two previous wrist gradient coil designs, we must determine the expression for \( B_x \), by deriving the expressions for \( B_\rho \) and \( B_\phi \). Using the condition (10.49) and the expression for \( j^a_\theta(\pm 1, k) \), the expressions (10.10) for \( B_\rho \) and \( B_\phi \) are modified as follows

$$
B_\rho = -\frac{\mu_0 a L}{4\pi} \cos\phi \int_{-\infty}^{\infty} dk \ k \ \text{sinkz} \left( 1 - \frac{I_1'(ka)K_1'(kb)}{I_1'(kb)K_1'(ka)} \right)
$$

$$
I_1'(k\rho)K_1'(ka) \sum_{n=1}^{\infty} j^a_n \psi_n(k)
$$

(10.53)

$$
B_\phi = \frac{\mu_0 a L}{4\pi} \sin\phi \sum_{n=1}^{\infty} j^a_n \frac{1}{\rho} \int_{-\infty}^{\infty} dk \ \text{sinkz}
$$

$$
\left( 1 - \frac{I_1'(ka)K_1'(kb)}{I_1'(kb)K_1'(ka)} \right) I_1(k\rho)K_1'(ka) \psi_n(k)
$$

(10.54)

where the symmetry conditions for \( j^a_\theta(\pm 1, k) \) and \( \psi_n(k) \)

$$
\psi_n(-k) = \psi_n(k)
$$

$$
j^a_\theta(-1, k) = j^a_\theta(1, k)
$$
have been employed.

Thus, the expression of $B_x$ for the z wrist gradient coil, combining equations (10.8), (10.36) and (10.37), is

$$B_x = -\frac{\mu_0 a L}{4\pi} \left[ \sum_{n=1}^{\infty} j_n^a \int_{-\infty}^{\infty} dk \sin \nu z \left( 1 - \frac{I_1'(ka)K_1'(kb)}{I_1'(kb)K_1'(ka)} \right) K_1'(ka) \psi_n(k) \right]$$

$$\left\{ \cos^2 \phi(k \rho) I_0(k \rho) - \cos 2\phi I_1(k \rho) \right\}$$

(10.55)

In section 10.2.1, we have presented a general expression of the dissipated magnetic energy for a self-shielded gradient coil (10.22). Using the restriction to the values of $m$ and the expression for $j_n^a(1, k)$ from (10.51), the expression of the stored magnetic energy is

$$W = -\frac{a^2 \mu_0 L^2}{16} \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} j_n^a j_{n'}^a \int_{-\infty}^{\infty} dk \frac{I_1'(ka)K_1'(ka)}{I_1'(kb)K_1'(ka)} \left( 1 - \frac{I_1'(ka)K_1'(kb)}{I_1'(kb)K_1'(ka)} \right) \psi_n(k) \psi_{n'}(k)$$

(10.56)

Following Turner's variation approach once again, we construct the functional $E$ in terms of the stored energy and the $z$ component of the magnetic field as

$$E(j_n^a) = W - \sum_{j=1}^{N} \lambda_j (B_x(\vec{r}_j) - B_{xSC}(\vec{r}_j))$$
Minimizing $\mathcal{E}$ with respect to $j_{n'}^a$, we get a matrix equation for $j_{n'}^a$ as

$$\sum_{n'=1}^{\infty} j_{n'}^a \left\{ -\frac{aL\pi}{2} \int_{-\infty}^{\infty} dk I_1'(ka) K_1'(ka) \left( 1 - \frac{I_1'(ka)K_1'(kb)}{I_1'(kb)K_1'(ka)} \right) \psi_{n'}(k) \psi_n(k) \right\}$$

$$= \sum_{j=1}^{N} \lambda_j \int_{-\infty}^{\infty} dk \sin k z_j K_1'(ka) \left( 1 - \frac{I_1'(ka)K_1'(kb)}{I_1'(kb)K_1'(ka)} \right) \psi_n(k)$$

$$\left[ \cos^2 \phi \left( k \rho \right) I_0(k \rho) - \cos 2\phi I_1(k \rho) \right]$$

(10.57)

Putting an upper threshold ($M$) on the infinite summations, the compact matrix notation of equation (10.57) is

$$\sum_{n'=1}^{M} j_{n'}^a C_{n'n} = \sum_{j=1}^{N} \lambda_j D_{j'n} \implies J^a C = \Delta D \text{ or } J^a = \Delta D C^{-1}$$

(10.58)

where $J^a$ is a $1 \times M$ matrix, $C$ is a $M \times M$ matrix, $\lambda$ is a $1 \times N$ matrix and $D$ is a $N \times M$ matrix.

The form of equation (10.58) is similar to the expressions (10.41),(10.27).

Thus, the methodology for determining the continuous current density for both coils is identical to the procedure which was described in detail in section 10.2.1.

10.4.2 Design

For the design of the z self-shielded wrist gradient coil, two coils with cylindrical cross section were considered. The radius of the inner coil is $a = 0.07 m$
and its total length is restricted to \( L = 0.2\, m \). The radius of the outer coil is chosen to be \( b = 0.085\, m \), while its total length is left unrestricted. Although we have not restricted the length of the outer coil, a careful choice of the constraint points about the uniformity and linearity of the magnetic field limits its total length. Four constraint points were chosen in order to define the quality of the \( x \) component of the magnetic field inside a 10 cm DSV. The first constraint point defines the strength of the magnetic field to 44 mT/m along the \( z \) axis of the cylinder. The second constraint point restricts the on-axis variation of the gradient field to within 2.5% from its ideal value at a distance of 0.05 m from the center of the gradient coil. The next two constraint points restrict the off-axis variation of the \( x \) component of the magnetic field to 2.5% at the borders of the 10 cm DSV. This set of constraints is displayed in Table 10.3. Also, for the estimation of the continuous current density of the inner and outer coil 40 Fourier terms were initially considered.
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Table 10.3: Constraint set used for the design of a self-shielded z wrist gradient coil. Values for \( \rho \) and \( z \) are in m, values for \( B_{exc} \) are in T.

### 10.4.3 Results

Solving the inverse problem for the z wrist gradient coil, the continuous current distributions for the inner and outer coils are derived. Figure 10.10 illustrates the variation of \( j_i^a \) and \( j_i^b \) along the z axis of the cylinder, since the z component of the current density is important for the determination of the discrete current patterns for both coils. For the derivation of these two current densities 15 Fourier terms are considered.

We now proceed with the discretization process for the self-shielded z wrist gradient coil. Employing the stream function technique, the discrete current patterns of the inner and outer gradient coils are obtained. The discrete current distribution of the inner coil contains 16 current loops which
carry an amount of current equal to 51.61 Amps each. Figure 10.11 shows the
discrete current pattern for the inner coil in the \( z \) wrist gradient coil design.
For the outer coil, the corresponding discrete current distribution consists
of 8 current loops which carry an amount of current equal to \(-51.61\text{ Amps}\)
each, and is shown in Figure 10.12. Furthermore, the distance of the farthest
current loop from the center of the gradient coil set is measured to be 0.145 m.
This indicates that the total length of the outer gradient coil is 0.29 m, which
 corresponds to a 45% increase with respect to a length of the inner coil.

Following the discretization procedure, the expression of the \( z \) component
of the magnetic field is evaluated for the inner and the outer coils using the
Biot-Savart law. The variation of the \( z \) component of the magnetic field along
the \( z \) direction of the cylinder is shown in Figure 10.13. The net strength
of the magnetic field was calculated to be 44.09 mT/m, which corresponds
to a 0.208% increase from the constraint value. The on-axis value of the \( z \)
component of the magnetic field at a distance of 0.05 m away from the center
of the coil is calculated to be equal to \( 2.147 \times 10^{-3} T \) which deviates by 0.09% from the constraint value of \( B_z \) at this point. Also, the off-axis uniformity
of $B_z$ shows no significant increase from the constraint value of the magnetic field at the farthest point of the DSV.

The total dissipated energy for this system is found to be 0.0970 Joules. Therefore, using the relationship between the total energy and the total inductance of the coil, for an amount of current equal to 51.61 Amps, the estimated total inductance of the $z$ wrist gradient coil is $L_{tot,estim} = 72.83 \mu H$.

In conclusion, we have introduced a novel geometry for the design of the $z$ wrist gradient coil. We have presented the mathematical formalism, the design and the theoretical results. Although the total energy of the system has been increased, this increase for this range of energies is insignificant. The discrete current patterns for both the inner and the outer gradient coils have been presented, and the $z$ component of the magnetic field has been evaluated employing the Biot-Savart law to the discrete current distributions. The quality of the magnetic field is superb and deviates from the original constraints by less than 0.03%.

With this section, we have come to the end of the second part of this dissertation, involving the design strategies of novel shapes of gradient coils.
Figure 10.1: Illustration of the symmetric self-shielded cylindrical gradient coil design necessary to generate an $x$ component of magnetic field varying along the $x$ direction of the coordinate system attached to the gradient coil set. The radius of the inner coil is denoted by $a$, and the radius of the outer coil is denoted by $b$. The total length of the inner coil is $L$, while the length of the outer coil is left unrestricted. In this figure, the relative position of the gradient coil with respect to orientation of the MRI imager is also shown.
Figure 10.2: One-half plot of the \( z \) component of the continuous current distributions for the inner (solid line) and the outer coil (dashed line) for the \( z \) wrist gradient coil. These current densities are adequate to generate the field specifications which are described in the design section.
Figure 10.3: Illustration of one quadrant of the discrete current distribution for the inner x wrist gradient coil. There are 16 current turns where each one carries an amount of current equal to 51.61 Amps.
Figure 10.4: Illustration of one quadrant of the discrete current distribution for the outer x-wrist gradient coil. There are 8 current turns where each one carries an amount of current equal to 51.61 Amps.
Figure 10.5: One-half plot of the $x$ component of the magnetic field along the $x$ axis for the inner coil (solid line) and the outer coil (dashed line) using the discrete current patterns for the inner and the outer coils, respectively.
Figure 10.6: One-half plot of the z component of the continuous current distributions for the inner (solid line) and the outer coil (dashed line) for the y wrist gradient coil. These current densities are adequate to generate the field specifications which are described in the design section.
Figure 10.7: Illustration of one quadrant of the discrete current distribution for the inner y wrist gradient coil. There are 16 current turns where each one carries an amount of current equal to 46.81 Amps.
Figure 10.8: Illustration of one quadrant of the discrete current distribution for the outer y wrist gradient coil. There are 8 current turns where each one carries an amount of current equal to \(-48.23\) Amps.
Figure 10.9: One-half plot of the $x$ component of the magnetic field along the $y$ axis for the inner coil (solid line) and the outer coil (dashed line) using the discrete current patterns for the inner and the outer coils respectively.
Figure 10.10: One-half plot of the $z$ component of the continuous current distributions for the inner (solid line) and the outer coil (dashed line) for the $z$ wrist gradient coil. These current densities are adequate to generate the field specifications which are described in the design section.
Figure 10.11: Illustration of one quadrant of the discrete current distribution for the inner z-wrist gradient coil. There are 16 current turns where each one carries an amount of current equal to 51.61 Amps.
Figure 10.12: Illustration of one quadrant of the discrete current distribution for the outer z-axis gradient coil. There are 8 current turns where each one carries an amount of current equal to $-51.61$ Amps.
Figure 10.13: One-half plot of the $x$ component of the magnetic field along the $z$ axis for the inner coil (solid line) and the outer coil (dashed line) using the discrete current patterns for the inner and the outer coils respectively.
Part III

RF Penetration
Chapter 11

Introduction to RF Penetration and Power deposition

The desire for higher quality images in MRI demands magnets with higher field strengths because of the increase of signal-to-noise. The limitation to the strength of the main magnet is based on the behavior of the rf inside the dielectric object. Due to the increase of the field strength and hence to the frequency $\omega$, two important questions arise. How does the presence of the conducting dielectric material affect the behavior of the rf field? How far can we increase the strength of the magnetic field before significant heating effects appear?
In this chapter, we will attempt to answer both these questions. Specifically, we will present an analytic approach in order to predict the behavior of the rf field inside a dielectric object for any arbitrary frequency. Particularly, we will concentrate on the derivation of the rf profile inside a single layer dielectric object where the incident magnetic field is parallel to the layer of the object. We will also discuss the behavior in the limits where either the conductivity dominates the behavior of the rf inside the dielectric object (conduction current limit), or the permittivity prevails (displacement current limit).

In order to answer the question concerning the rf power deposited, we will present an expression for the electromagnetic power deposition inside the human body containing quantities related to the imaging.

We now proceed with the determination of the shape of the magnetic field profile inside a three dimensional rectangular object. The entire mathematical methodology has been presented in [13]. The 3D layer configuration is shown in Figure 11.1 The layer is bounded along the z direction and is infinitely extended along the y and z directions and is surrounded by air.
Figure 11.1: Orientation of the RF field in a planar model of thickness $2R$. $B_1$ is the applied RF field and is a function of $x$. The static field $\tilde{B}_0$ is along $\hat{z}$. The object being imaged lies between $-R$ and $R$ while the RF is applied at $-x_0$ and $x_0$.

The thickness of the layer is chosen to be $2R$, while the conductivity $\sigma$ and permittivity $\varepsilon$ are assumed to remain constant both inside and outside the layer. Steady-state sinusoidal time dependence $e^{-i\omega t}$ is assumed for all rf fields and currents. Furthermore, the direction of the incident magnetic field $\tilde{B}_1$ is chosen to be along the $\hat{y}$ direction and depends only on $x$. The bound-
ary conditions are that at \( x = \pm x_o \) (where \( x_o > R \)), the magnitude of the magnetic field is equal to its incident value \( B_1 \). In order to obtain a solution for the spatial behavior of the magnetic field, we construct a standing wave solution along the \( x \) direction. In this case, we consider

\[
\vec{B}(x) = B_1 \ h(x) \ \hat{y}
\]  \hspace{1cm} (11.1)

and using Ampere's law with \( \vec{J} = \sigma \vec{E} \) and \( \vec{D} = \epsilon \vec{E} \), the expression for the electric field is

\[
\vec{E}(x) = -i \omega \frac{B_1 \ h'(x)}{k^2} \ \hat{z}
\]  \hspace{1cm} (11.2)

where \( h(x) \) represents the specific spatial dependence of the magnetic field, \( h' \) indicates the derivative of \( h(x) \) with respect to \( x \) and \( k^2 = \mu_0 \omega^2 \left( \epsilon - i\frac{\sigma}{\omega} \right) \).

Therefore, we obtain an expression of the electric field which is also parallel to the boundaries of the layer.

Combining Ampere's and Faraday's law with association of the \( \vec{\nabla} \cdot \vec{B} = 0 \), the magnetic field inside the dielectric satisfies the Helmholtz equation

\[
h''(x) + k^2 \ h(x) = 0
\]  \hspace{1cm} (11.3)
For the region outside the dielectric object, we replace $k \rightarrow k_o$, where $k_o^2 = \mu_o \omega^2 \epsilon_o$. Therefore the general solution of the equation (11.3) is

$$h(x) = A_i \cos k_i x + B_i \sin k_i x \text{ with } i = 1, 2, 3$$

(11.4)

where $i = 1$ corresponds to the solution for the region $-x_o < x < -R$, $i = 2$ for $-R < x < R$ and $i = 3$ for $R < x < x_o$. Therefore, we notice that $k_1 = k_3 = k_o$ and $k_2 = k$.

In order to determine the expression of the six coefficients, we consider the four continuity conditions for the tangential component of the magnetic field and for the electric field, combined with the additional two conditions that at $x = \pm x_o$, $B(\pm x_o) = B_1$. Thus, the final expression of the magnetic field inside and outside the dielectric is

$$h(x) = \frac{B_1}{\Delta} \begin{cases} \cos k_x \text{ for } |x| < R \\ P \cos k_o x + Q \sin k_o |x| \text{ for } |x| > R \end{cases}$$

(11.5)

where

$$\Delta = P \cos k_0 x_o + Q \sin k_0 x_o$$

(11.6)

$$P = \cos k R \cos k_0 R + \frac{k_0}{k} \sin k R \sin k_0 R$$

$$Q = \cos k R \sin k_0 R - \frac{k_0}{k} \sin k R \cos k_0 R$$
As we can see from equation (11.5), the behavior of the magnetic and the electric field depends on the values of the conductivity and permittivity of the dielectric object.

Let us now consider two different limiting cases. Using the expression of the wavevector $k^2$ and assuming that $\sigma$ and $\epsilon$ are real, the expression for $k$ is

$$k = \beta + i \frac{\alpha}{2}$$

(11.7)

where

$$\beta = \omega \sqrt{\mu_0 \epsilon} \left[ \sqrt{1 + \left( \frac{\omega}{\omega_c} \right)^2 + 1} \right]^{\frac{1}{2}}$$

$$\frac{\alpha}{2} = \omega \sqrt{\mu_0 \epsilon} \left[ \sqrt{1 + \left( \frac{\omega}{\omega_c} \right)^2 - 1} \right]^{\frac{1}{2}}$$

If we consider a very good conductor ($\frac{\omega}{\omega_c} >> 1$) (conduction current limit), equation (11.7) becomes

$$k \simeq (1 + i) \sqrt{\mu_0 \frac{\omega \epsilon}{2}}$$

(11.8)

In this limit the magnetic field is damped inside the dielectric with a penetration length equal to

$$\delta \simeq \frac{1}{2 \sqrt{\mu \sigma}}$$

(11.9)
Figure 11.2: A case for the conduction current limit. The frequency for the rf is 64 MHz and the values of conductivity and permittivity are $\sigma = 1 \text{ S/m}$, $\varepsilon = \varepsilon_0$, respectively. The magnitude of the rf field has been normalized to 1 at the center of the layer.

where $f$ represents the frequency expressed in MHz. An example of the damping of the magnetic field profile inside a dielectric with high value of $\sigma$ is shown in Figure 11.2. In the case where the conductivity is very small such that $\left( \frac{\sigma}{\omega_0} \right) \ll 1$ (displacement current limit), the contribution of the imaginary term in equation (11.7) is almost negligible and the solution will
oscillate inside the dielectric with wavelength

\[ \frac{\lambda}{4} \approx \frac{75}{f\sqrt{\varepsilon_r}} \text{ meters} \]  \hspace{1cm} (11.10)

where \( f \) is expressed in MHz and \( \varepsilon_r = \frac{\varepsilon}{\varepsilon_0} \). An example of the oscillatory behavior of the magnetic field inside the dielectric object with \( \sigma = 0 \) and \( \varepsilon_r = 275 \) is displayed in Figure 11.3.

Up to this point, we have shown how the conductivity and permittivity of a dielectric object affect the behavior of the rf field. The next step is the estimation of the rf power which is damped inside the dielectric object with conductivity \( \sigma \).

The time average power density distribution (or local Specific Absorption Rate (SAR)) can be estimated from the expression

\[ SAR = \frac{1}{2} \sigma |\vec{E}|^2 \]  \hspace{1cm} (11.11)

where \( |\vec{E}| \) is the magnitude of the electric field and \( \sigma \) indicates the conductivity of the dielectric object. If we consider a square linear polarized rf pulse with a duration of \( \tau \) which is designed to tip the spins by an angle \( \theta \) from their initial position, the magnitude of the incident magnetic field is
Figure 11.3: A case for the displacement current limit. The frequency for the rf is 64 MHz and the values of conductivity and permittivity are $\sigma = 0 S/m$, $\epsilon = 275\epsilon_0$, respectively. The magnitude of the rf field has been normalized to 1 at the center of the layer.

$$B_1 = \frac{2\pi}{\gamma t},$$

where $\gamma$ is the gyromagnetic ratio. Furthermore, during an imaging experiment, the rf pulse is applied for a limited amount of time $\tau$, while the application of the imaging sequence lasts time $T$. Thus in order to estimate the power deposition inside the object, we must consider the ratio $D$ which is called the duty cycle and is defined as $D = \frac{\tau}{T}$. Therefore, equation (11.11)
is modified as

\[ SAR = \frac{1}{2} \sigma |\vec{E}|^2 D \]  \hspace{1cm} (11.12)

The next step is to find the expression for the electric field. From equations (11.2),(11.5), the magnitude of the electric field is proportional to

\[ |\vec{E}| \simeq \frac{C' B_1 \omega \sin k x}{k} \]  \hspace{1cm} (11.13)

where \( C' \) represents all the constants independent of \( k \) and \( x \). For an object with dimensions smaller than the rf wavelength, \( \sin k x \rightarrow k x \). Thus, equation (11.13) becomes

\[ |\vec{E}| \simeq C' B_1 \omega x \]  \hspace{1cm} (11.14)

Combining equations (11.12),(11.14), and replacing \( B_1 \) in terms of the tip angle, we obtain the relation of the SAR as

\[ SAR \simeq C' \frac{2\sigma \theta^2 \omega^2 x^2}{\gamma^2 \tau T} \]  \hspace{1cm} (11.15)

Examining equation (11.15) closer, we notice the rf power has a quadratic dependence in terms of the rf frequency, while it is linear with respect to \( \sigma \).
Chapter 12

Higher Order Frequency Dependence and RF Penetration in Planar, Cylindrical and Spherical Models

12.1 Introduction

Imaging is being considered up to 400 MHz today for materials research and 170 MHz for human studies. Numerous articles have appeared attempting to evaluate RF penetration effects for simple shapes ([6]-[12]) with only a few attempting the more complicated heterogeneous solution ([13]-[15]). We will evaluate the fourth order correction to the field and power deposition for the planar [13], cylindrical [6, 10] and spherical models [11, 12]. These
expressions illustrate the size and the direction of the effects for various relative permittivities $\varepsilon$, conductivities $\sigma$ object size $2R$ and frequency $\omega$. Furthermore, corrections to the various equations related to the problem of the conducting sphere are presented, since previous solutions (4) appear to suggest incorrect behavior for the RF magnetic field $(B_1)$ and the power loss in the region inside the sphere.

12.2 Planar Model

The planar model has been chosen due to the existence of the heterogeneous solution [13] (a point which has not been addressed sufficiently in the literature), the ability to compare theory with experiment (albeit with certain size approximations relative to the RF wavelength), and the consideration of abdominal or phantom imaging (which may in some circumstances be considered planar in nature). To compare with the physical set up in modern whole body systems, the RF field, $B_0$, is chosen parallel to the layer and pointing in the $y$ direction while the static field points along the long axis (the $z$ direction). The solution for the local $B_1$ field in the planar model (Figure 11.1)
has the form:

\[ B_1(x) = \frac{B_0}{\Delta} \cos kx \]  

(12.1)

where the expression of \( \Delta \) is given in (11.6)

In order to evaluate the power to order \( \omega^4 \) corrections, \( B_1(x) \) must be expanded to order \( k^6x^6 \), due to the cancelation of the cross terms in the expansion. The expansion must be considered to this order because the spatial derivative of \( B_1(x) \) is required to evaluate \( E(x)\xi \) and \( k^2 \) is proportional to both \( \omega \) through the conductivity term and \( \omega^2 \) through the permittivity term. To a good approximation, the expressions for the electric and magnetic fields are reduced to more convenient forms

\[ B_1(x) = \frac{B_0}{Q}(1 - \frac{(kx)^2}{2} + \frac{(kx)^4}{24})(1 + \alpha k^2 R^2 + \gamma k^4 R^4) \]  

(12.2)

\[ E(x) = i \frac{B_0\omega x}{Q}(1 - \frac{(kx)^2}{6} + \frac{(kx)^4}{120})(1 + \alpha k^2 R^2 + \gamma k^4 R^4) \]  

(12.3)

where

\[ Q = (1 + \frac{(k_0R)^2}{2} - \frac{(k_0R)^4}{8} + \frac{(k_0R)^6}{144})(1 - \frac{(k_0x_0)^2}{2} + \frac{(k_0x_0)^4}{24} - \frac{(k_0x_0)^6}{720}) + \\
(k_0x_0 - \frac{(k_0x_0)^3}{6} + \frac{(k_0x_0)^5}{120} - \frac{(k_0x_0)^7}{5040})(\frac{(k_0R)^3}{3} - \frac{(k_0R)^5}{30} + \frac{(k_0R)^7}{840}) \]
\[ \alpha = \beta + k_0^4 \left( \frac{5x_0^4}{48} \right) + \frac{59R^4}{144} + \frac{x_0^3R}{9} - \frac{x_0R^3}{2} - \frac{x_0^3R^2}{8} \]  
(12.4)

\[ \beta = \frac{1}{2} + k_0^2 \left( \frac{-R^2}{3} + \frac{x_0R}{3} \right) \]  
(12.5)

\[ \gamma = \frac{5}{24} + k_0^2 \left( \frac{-3R^2}{10} + \frac{3x_0R}{10} \right). \]  
(12.6)

Generally, the outer boundaries of the air layers at \(-x_0\) and \(x_0\), surrounding the middle layer, do not coincide with its thickness R; \((x_0 \neq R)\). In this situation, the expression for the power deposited per unit area has been obtained by integrating the magnitude of the electric field over \(x\) for the total thickness of the middle layer, assuming that the conductivity \(\sigma\) and the permittivity \(\varepsilon\) are spatially invariant. The result of this integration gives:

\[
\frac{dP}{dA} = \frac{\sigma B_0^2 \omega^2}{2} \left( \frac{R^3}{3} + \omega^2 \left( \frac{2\alpha}{3} - \frac{1}{15} \right) + \mu_0 \sigma^2 R^7 \left( \frac{-2\gamma}{3} + \frac{\alpha^2}{3} + \frac{\beta}{15} - \frac{\alpha}{15} + \frac{1}{630} \right) \right)
\]

\[
(1 + 2k_0^2 \left( \frac{x_0^2 - R^2}{2} \right) + 2k_0^4 \left( \frac{5x_0^4}{24} + \frac{3R^4}{8} - \frac{(x_0R)^2}{4} - \frac{x_0R^3}{3} \right)) \]  
(12.7)

Another important aspect arises when the surrounding air layers are absent. According to this situation, \(x_0 = R\) and the corresponding expression for the power is:

\[
\frac{dP}{dA} = \frac{\sigma B_0^2 \omega^2}{2} \left( \frac{R^3}{3} + \omega^2 \left[ \frac{4\mu_0 \varepsilon R^5}{15} - \frac{17\mu_0^2 \sigma^2 R^7}{315} \right] \right) \]  
(12.8)
The expression for the total power indicates that there is an appropriate choice of $\sigma$ for a given $\epsilon_r$ and $R$ that can lead to a solution similar to that for the low frequency limit; specifically, when

$$\sigma^2 = \frac{84 \epsilon_r}{17 \mu_0 \rho R^4 \epsilon^2}$$  \hspace{1cm} (12.9)$$

the term in $\omega^4$ vanishes. For $R$ less than 10cm and $\frac{\omega}{2\pi}$ less than 42 MHz, the higher order terms in $\omega^6$ are also reduced by roughly another factor of 10 and have higher power dependence on $R$ as well. Evidently, for objects less than 20cm thick, these effects are very small. It should be noted that this relation is independent of frequency except through the implicit dependence of $\sigma$ and $\epsilon_r$ on $\omega$. It is remarkable that at the high frequencies of interest for human imaging, the cancellation is almost complete. It is also to be expected that, since the static solution is almost attained, the RF profile would be nearly uniform. To illustrate this return to uniformity, if equation (12.9) is substituted into equation (12.8) and we require that the field returns to its central value, the estimated value for the half thickness $R$ for this to happen is 7.2cm for $\sigma = 0.3 \text{ S/m}$ and $\epsilon_r = 58$. The maximum deviation of the
local field inside the layer from its central value occurs at the positions $x = \pm 5.1\text{cm}$ and is 7.2 % a reasonably small deviation from unity. (Recall the profile is also essentially flat for $\varepsilon_r = 1$ and $\sigma = 0.5/\text{m}$.) Clearly, oscillations are occurring inside the object, but their amplitudes are small.

As $R$ increases above $7.2\text{cm}$, deviations in the "magic" values of $\sigma$ and $\varepsilon_r$ occur. Usually, for a known $\varepsilon_r$, a higher than predicted conductivity is required, since the higher order terms $\omega^6$ are present. A plot of $\sigma$ versus $\varepsilon_r$, where the $\omega^4$ term vanishes, is shown in Figure 12.1.

### 12.3 Cylindrical Model

Let us now consider a cylinder, where the applied rf field is taken to be perpendicular to the $z$ axis of an infinite cylinder with radius $a$. The cylinder is filled with a conducting dielectric material with conductivity $\sigma$ and permittivity $\varepsilon$. Considering the relation between the magnetic field and the vector potential and assuming the Coulomb gauge condition, the vector potential must satisfy the Helmholtz equation. Expanding this differential equation into cylindrical coordinates and assuming continuity of the vector potential
Figure 12.1: A plot of $\sigma$ versus $\epsilon_r$ for the planar model (solid line) where the higher order terms in $\omega^4$ in the power vanish. Similar results are shown for the cylindrical (thin-dashed line) and spherical (thick-dashed line) models.

across the interface between the dielectric cylinder and the surrounding air, the expression of the transverse component of the magnetic field inside the cylinder is at any point $((r, \phi)$ ( [9], [10]).

$$
\vec{B}_{tr} = B_z \hat{z} + B_y \hat{y} = \left[ \frac{1}{J_0(ka)} \left( \frac{2 J_1(kr)}{kr} - J_0(kr) \right) \cos 2\phi + \frac{J_0(kr)}{J_0(ka)} \right] \hat{z} \nonumber \\
\left[ \frac{1}{J_0(ka)} \left( \frac{2 J_1(kr)}{kr} - J_0(kr) \right) \sin \phi \right] \hat{y} \tag{12.10}
$$
In this section, we present the expansions to order $\omega^4$ for the different polarizations of the RF field, along the transverse plane of the cylinder and along the long axis of the cylinder.

When the RF field is directed along the transverse plane of the cylinder, the current paths are along the infinite long axis of the cylinder (Figure 12.2), although they are more limited in one direction, and hence, lead to results similar to those in the previous section. In this transverse example $\phi = \frac{\pi}{2}$, and the approximate expression of the electric field for $\omega$ small has the form:

$$E(r) = iB_0\omega r \sin\phi (1 + k^2\left[\frac{r^2}{8} - \frac{R^2}{4}\right] + k^4\left[\frac{3R^4}{64} + \frac{r^4}{192} - \frac{R^2r^2}{32}\right])$$

(12.11)

Furthermore, the total integrated power per unit length of the cylinder for this direction of the RF field can be written as:

$$\frac{dP}{dl} = \frac{2\pi \omega B_0^2}{\mu_0||k||^2 ||J_0(kR)||^2} \text{Im}[k^* J_2(k^* R) J_1(kR)]$$

(12.12)

(This should be compared with the similar form found for the spherical solution in the next section). In the limit where the frequency is small, the approximate expression for the total power is:

$$\frac{dP}{dl} = \frac{\pi \sigma^2 B_0^2}{8} \left(R^4 + \omega^2\left[\frac{\mu_0 \sigma R^6}{3} - \frac{11\mu_0^2 \sigma^2 R^8}{384}\right]\right)$$

(12.13)
Figure 12.2: Orientation of the RF field in the cylindrical model of diameter $2R$ is either along the long axis (axial) or along the dashed line representing the $x$-axis (transverse). The calculated power is integrated over $r$ and $\phi$ along the plane perpendicular to the long axis of the cylinder.

For the situation where the RF field is along the long axis of the cylinder [10], the currents are directed azimuthally along the surface of the cylinder. In this scheme the corresponding expression for the total power deposited per unit length has the form:

$$\frac{dP}{dl} = \frac{\pi \sigma \omega^2 B_0^2}{16} (R^4 + \omega^2 \left[ \frac{\mu_0 \epsilon R^6}{3} - \frac{11 \mu_0^2 \sigma^2 R^8}{384} \right])$$  \hspace{1cm} (12.14)
which is exactly half of that for equation (12.13) (see also ref. [9]). Therefore, for the two representations, the $\omega^4$ corrections to the power are zero when,

$$
\sigma^2 = \frac{\lambda - \epsilon_r}{\mu_0 R^2 c^2}
$$

(12.15)

where $\lambda = 11.64$ for both transverse and axial RF fields. For a fixed $\epsilon_r$, this leads to a higher conductivity than in the planar model (Figure 12.1) and a decrease in the total power deposited.

### 12.4 Spherical Model

We now proceed with the determination for the behavior of the magnetic field inside a conducting dielectric sphere which is surrounded by air. Let us consider that the applied rf field is perpendicular to the $z$ axis of the sphere on a cartesian coordinate system. If we express the magnetic field in terms of the vector potential and assume the Coulomb gauge condition, the vector potential must satisfy the Helmholtz equation. We solve this differential equation in spherical coordinates by considering that the only component of the vector potential present is along the azimuthal ($\hat{\phi}$) direction for the spherical coordinates, in order to be consistent with the behavior of the
magnetic field outside the sphere. In addition, we also assume that the
contribution from the air is almost negligible, while the vector potential must
be continuous across the interfaces. Denoting R as the radius of the sphere
instead of a, the correct expressions for the three cartesian components are:

\[ B_x(r) = \frac{3}{2} B_1 \frac{1}{j_0(kR)} \frac{xz}{r^2} j_2(kr) \tag{12.16} \]
\[ B_y(r) = \frac{3}{2} B_1 \frac{1}{j_0(kR)} \frac{yz}{r^2} j_2(kr) \tag{12.17} \]
\[ B_z(r) = 3 B_1 \frac{1}{j_0(kR)} [j_1(kr) \frac{kr}{2r^2} + \frac{z^2}{z^2} j_2(kr)] \tag{12.18} \]

Furthermore, the formula for the power deposited, should have the form:

\[ P = \frac{3 \pi \omega B_0^2}{\mu_0 ||k||^2 ||j_0(kR)||^2} \text{Im}[k^* R^2 j_2(k^* R) j_1(kR)] \tag{12.19} \]

### 12.4.1 Asymptotic Solution for the Spherical Model

The sphere is chosen to simulate the finite current paths available in practice.

Several papers have addressed the situation where permittivity effects can
be ignored [11, 12]. In this section, we show that both terms contribute to
the power to order \(\omega^4\). The expression for the current density is

\[ J = ie_0 \sigma \omega B_0 r \sin \theta \left[ -\frac{1}{2} + k^2 \left( \frac{r^2}{20} \frac{R^2}{12} \right) + k^4 \left( \frac{-r^4}{560} \frac{7R^4}{360} + \frac{R^2 r^2}{120} \right) \right] \tag{12.20} \]
This leads to a power per unit volume of

\[
\frac{dP}{dV} = \frac{\sigma \omega^2 B_0^2 r^2 \sin^2 \theta \cdot \frac{1}{4} + \frac{\omega^2 \mu_0 \epsilon R^2}{6} - \frac{\omega^2 \mu_0^2 \sigma^2 R^4}{80}}{2} + r^4 \left( -\frac{\omega^2 \mu_0 \epsilon}{10} + \frac{\omega^2 \mu_0^2 \sigma^2 R^4}{120} \right) - r^6 \frac{3 \omega^2 \mu_0^2 \sigma^2}{2880} \tag{12.21}
\]

By integrating over the whole volume of the sphere, we obtain the expression for the total power deposited,

\[
P = \frac{\pi \sigma \omega^2 B_0^2 R^5}{15} \left[ 1 + \omega^2 \left( \frac{8 \mu_0 \epsilon R^2}{21} - \frac{\mu_0^2 \sigma^2 R^4}{35} \right) \right] \tag{12.22}
\]

The second order term vanishes when,

\[
\sigma^2 = \frac{40 \epsilon_r}{3 \mu_0 R^2 c^2} \tag{12.23}
\]

which leads to higher conductivities than the planar and cylindrical models (Figure 12.1) for a fixed value of the relative permittivity.

## 12.5 Heating Effects

As we have mentioned in chapter 11, power deposition plays a key role in the design of the imaging sequence. The presence of \( \sigma \) and \( \epsilon \) in any material will alter the behavior of the rf field inside the dielectric object. In previous
section of this chapter, we have presented the effects which the conductivity and permittivity have on the rf field profile. Besides the change in the spatial behavior which the rf profile experiences inside the dielectric object, significant heating effects can also be generated. Since the power deposited is proportional to $\omega^2$ and $\sigma$, and since the wavelength is inversely proportional to the value of the relative permittivity $\varepsilon_r$, significant resonance effects can appear inside a dielectric object with high values of $\sigma$ and $\varepsilon_r$.

Looking at the anatomy of the human body, there exist certain areas which can accumulate a significant amount of rf power. These areas include the eyes, the Cerebral Spinal Fluid (CSF) and pathologies with large conductivities such as lesions, cysts and blood clots. Another factor, which plays a key role in the estimation of the heat dissipation in an object, is the rate at which the heat diffuses to the surrounding areas inside the human body. Thus, we expect the heat to diffuse faster in tissues which are surrounded by blood vessels and slower to the tissues which are isolated from the rest of the body. The first category includes tissues such as CSF, while the latter include tissues like the eyes.
In this section, we will discuss the effects which rf power deposition generates to different parts of the human body, and we will present a mathematical method for estimating the average increase of temperature in objects with spherical symmetry.

In order to estimate the effects of the power deposition in a dielectric object, we have constructed a three dimensional phantom to mimic the human head. Different compartments inside the phantom were representing different areas of the human head, such as CSF, gray/white matter, upper jaw, lower jaw, left eye, right eye, etc.. Each of these compartments was filled by a gel like material which was doped with NaCl. The amount of NaCl which was used to dope each region of the head phantom was chosen in order to coincide with the acceptable values of CSF, gray/white matter, eyes, etc., as they were presented in the literature [7],[8]. The conversion between the amount of NaCl and the value of the conductivity was made, knowing that 3 grams/liter of NaCl corresponds to a conductivity of 0.5 S/m.

Furthermore, eight non-ferromagnetic thermocouples were placed in different areas of the head like phantom. Following this, the phantom was placed
inside the 1.0 \( T \) Magnetom Imaging system. The MRI sequence which was used to generate significant heating effects to the phantom was a multi-echo spin echo sequence. In order to generate the multiple echo scheme, four rf pulses were used. The first pulse was chosen to be a 90° one, the second a 180°, and the remaining two pulses were designed to be 135°. The imaging parameters of this sequence were the following: \( TR = 220 \text{msec} \), \( TE = 11 \text{msec} \), No. of slices=5, FOV=256 mm, slice thickness=8 mm, No. of Acquisitions=3. The total time of running this sequence was 4.5 minutes. The amplitude of the voltages in the rf amplifier which were necessary to generate these four pulses were 181.38V, 389.98V, 291.60V, 291.60V. The computed average power per unit mass was 4.32 \( W/ Kg \), while the local SAR was estimated to be 13.22 \( W/Kg \). In order to achieve a temperature equilibrium between the phantom and the magnet environment, the head phantom was left inside to the main magnet room for three days. The average initial value of the temperature for the entire phantom was 64.5° F. Running the multi-echo sequence, we measured the temperature for each gel part of the phantom immediately after the completion of the sequence. We wanted to minimize
<table>
<thead>
<tr>
<th>Time</th>
<th>CSF, ventricles</th>
<th>Left Eye</th>
<th>Right Eye</th>
</tr>
</thead>
<tbody>
<tr>
<td>20:41</td>
<td>64.5°F</td>
<td>64.9°F</td>
<td>64.6°F</td>
</tr>
<tr>
<td>20:46</td>
<td>64.7°F</td>
<td>67.7°F</td>
<td>66.8°F</td>
</tr>
<tr>
<td>20:50</td>
<td>65.2°F</td>
<td>69.2°F</td>
<td>67.3°F</td>
</tr>
<tr>
<td>20:55</td>
<td>65.7°F</td>
<td>70.7°F</td>
<td>68.1°F</td>
</tr>
<tr>
<td>20:59</td>
<td>65.9°F</td>
<td>71.0°F</td>
<td>68.5°F</td>
</tr>
<tr>
<td>21:03</td>
<td>66.4°F</td>
<td>72.4°F</td>
<td>69.0°F</td>
</tr>
</tbody>
</table>

Table 12.1: Temperature measurements for the three parts of the head phantom using non-ferromagnetic thermocouples. The sequence was a multi-echo spin echo sequence with $TR = 220\ msec$, $TE = 11\ msec$, No. of slices=5, FOV=256 mm, slice thickness=8 mm, No. of Acquisitions=3.

The heat diffusion effects from one gel part to another. Table 12.1 illustrates the change in the temperature at three different areas of the head phantom. The variation of the temperature to the remaining 5 areas of the phantom was insignificant and do not appear in this table. We also notice from Table 12.1, that the change of the temperature for the CSF at the ventricles was very small between scans, while the average rise on the temperature in the left and the right eye was on the order of 2°F per scan.

As a next step, we doubled the amplitude of the voltages in the rf amplifier using the off-line adjustment of the MRI system. In this case the average power per unit mass as well as the local SAR were quadruple, as it was
Table 12.2: Temperature measurements for the three parts of the head phantom using non-ferromagnetic thermocouples and when the voltages are doubled. The sequence was a multi-echo spin echo sequence with $TR = 220 \text{ msec}$, $TE = 11 \text{ msec}$, No. of slices=5, FOV=256 mm, slice thickness=8 mm, No. of Acquisitions=3.

Table 12.2 displays the change in the temperature at the same three areas of the head phantom, using the same MRI sequence. In this case, we observe that the increase of the temperature at the eyes was quadruple to $8^\circ F$ per scan, and is consistent with the behavior of the rf power deposited.

We now proceed to derive the mathematical formula for the estimation of the change in the temperature at the eyes. Considering a spherical shape for the eyes, where their dimensions are smaller than the wavelength of the rf, we are able to use the expansion form of the total power deposited for the spherical model, which is shown in equation (12.22). From thermodynamics, we know that the heat and the change of the temperature in an object with
mass \( m \) and specific heat \( c \) is

\[
Q = m c \Delta T
\]  \hspace{1cm} (12.24)

We also know that the heat and the power are related, assuming no additional loss in the system, as \( Q = Pt \) where \( t \) represents the time which the gradients are on. Since the rf pulse is not applied constantly during the experiment, we must multiply the expression of equation (12.22) by the Duty cycle \( D \). Also, we must include the effects for a linear polarized or a circular polarized rf coil and they will be indicated by the factor \( \alpha_{rf} \). The only values which the \( \alpha_{rf} \) can get are 2 and 4 for the circular and linear polarized coil respectively. The reason is the following. For the linear polarized coil the expression of the rf is \( 2B_1 \cos(\omega t) \). Thus, since the power deposited is proportional to the square of the magnitude of the rf magnetic field, in this situation the total power will be proportional to \( 4B_1^2 \). For the circular polarized rf coil, we have two rf pulses, one in phase and the other out of phase with respect to a counterclockwise rotating frame which contribute constructively. In this case, the rf power deposited is proportional to \( B_1^2 + B_1^2 = 2B_1^2 \). Combining
equations (12.22), (12.24), the expression for the change in the temperature is

$$\Delta T = \frac{\sigma \pi^2 (\nu B_1)^2 r^2 D a_{rf} t}{2 \times 10^3 \times 4.2} ^\circ C$$  \hspace{1cm} (12.25)

where $\sigma$ is the conductivity of the object, $r$ is the radius of the sphere, $\nu$ is the frequency in Hz of the rf, $B_1$ is the amplitude of the rf field which is evaluated, using the relationship $B_1 = \frac{1}{\gamma T}$ for a $\frac{\pi}{2}$ flip angle and $\gamma$ represents the duration of the rf pulse.

For our experimental set up, and specifically, for the eye are, the following values were used: $D = \frac{1}{2}$, $\sigma = 1.5 S/m$, $a_{rf} = 4$, $r = 1 \text{ cm}$, $B_1 = \frac{1}{\gamma}$, $2.5 \mu T$, $\nu = 64 MHz$, $t = 60 sec$, $c = 1 \frac{\text{Joule}}{Kg \circ K}$. The value of the average increase of the temperature per minute is

$$<\Delta T> = 0.27 ^\circ C \text{ per minute}$$  \hspace{1cm} (12.26)

which for a 4 minute scan corresponds to an 1 $^\circ C$ increase. This shows a very good agreement between the experiment and the theoretical prediction.
12.6 Discussion

Planar, cylindrical and spherical models play an important role in understanding the reaction of the body to an RF pulse of a specific frequency. Although most machines used today for medical purposes are limited to 2.0 T or 84 MHz, higher NMR frequencies (170 MHz) are under experimental investigation. Planar, cylindrical and spherical phantoms are used in these machines in order to obtain important information concerning the RF profile, power deposited, and the improvement of signal-to-noise ratio for high NMR frequencies.

From an imaging perspective, tissues and organs in the human body are essentially distinguished by their difference in shape and their response to an applied RF pulse. However, the power deposited also depends strongly on their conductivity \( \sigma \) and relative permittivity \( \varepsilon_r \). To estimate how significant the response of the power deposited is as a result of higher order terms, we have considered an infinitely extended plane in the \( y, z \) direction, having thickness \( 2R \); a cylinder extended to infinity along its axis of symmetry but
bounded by a radius \( R \), and a sphere limited to a radius \( R \). Asymptotic expansions of the magnetic field, the electric field and the power deposited were evaluated up to order \( \omega^4 \). The \( \omega^4 \) term in the expression of the power vanishes when the relation between the conductivity and relative permittivity is of the form

\[
\sigma^2 = \lambda \frac{\epsilon_r}{\mu_0 R^2 c^2}
\]  
(12.27)

where

\[
\lambda = 4.94 \quad \text{for the planar model}
\]
\[
\lambda = 11.64 \quad \text{for the cylindrical model}
\]
\[
\lambda = 13.33 \quad \text{for the spherical model}
\]

For physiologic values of \( \epsilon_r \) and \( \sigma \), the estimate of \( R \) at 42 MHz is roughly 10 cm which approximates the size of some portions of the human body. Furthermore, for this value of \( R \), the RF profile is found to be nearly uniform. The value of \( \lambda \) increases from the planar up to the spherical model. As mentioned before, since the plane is infinite in extension, the value of the
conductivity which is required to produce the desired current density for an applied RF field is smaller. For the cylinder, the occupied volume is smaller than that of the planar model and a higher value of conductivity is required to produce the same current density. Due to the finite size of the sphere, a higher conductivity than in the previous two models is required in order to produce an equivalent current density (Figure 12.1). In review [9], the two expressions referring to the total power deposited for the cylinder show that the total power for the RF transverse field is twofold the value of the power which is obtained when the RF is directed along the long axis of the cylinder. By comparing the expressions for the total power per unit volume between the sphere and the axial case for the cylinder, we found that the corresponding power for the sphere is 2.5 times that for the cylinder.

Although these shapes are not usually representative of the human internal structure, in that they allow the generation of large currents, they do show that there exists an intermediate solution between finite wavelength effects (dominated by large permittivities when $\sigma = 0$) and a screened system (dominated by a large conductivity) when little field distortion occurs and
power deposition is a minimum. When the system satisfies equation (12.27), the penetration can mimic the low frequency behavior, i.e., the \( B_1 \) field is nearly uniform. If this can occur for a uniform object, it is also likely to occur when non-interacting regions of the same total volume are probed. It is apparent that, at 63 MHz, the values of in \textit{vivo} permittivities and conductivities are in the region which leads to such a fortuitous cancellation of \( \omega^4 \) penetration effects. Another significant fact seen from equation (12.27), is that the conductivity required for the second order correction term to vanish is independent of the applied frequency. Therefore, the power deposited is proportional only to the square of the applied frequency for the \textit{magic} values of \( \epsilon_r \) and \( \sigma \) (which do contain some dependence on \( \omega \)).
Chapter 13

Extraction of Conductivity and Permittivity Using Magnetic Resonance Imaging

13.1 Introduction

Magnetic resonance imaging (MRI) at low fields such as 0.15 T has no difficulty obtaining uniform images with respect to radiofrequency (RF) penetration. The wavelength of the RF radiation is rather long, roughly 50 m. Even with a relative dielectric constant of 100, this reduces to only 5 m. Since the relevant dimensions of the body are the transverse distances, 5 m is much longer than either 30 cm for the head or 60 cm for the body. Finite wavelength effects are small and first order calculations of penetration and
power deposition are sufficient (Bottomley et al [6], Bottomley et al [9], Carlson [11], Glover et al [10], Petropoulos [57]). The situation changes dramatically at 1.5 T where now the internal wavelength may be as small as 50 cm, on the order of the body size. This leads to a distortion of the RF field inside the irradiated object with the consequence that the image uniformity is disrupted and high local power deposition may occur.

Electrical impedance tomography (Baber [58], Newell [61]) is one method which attempts to evaluate conductivity, $\sigma$, for different tissues in the human body. This method is based on measuring the voltage fluctuations for an applied current on the surface of the body. Using these spatial variations of the voltage, it is possible to relate voltage and conductivity (given the permittivity) through Maxwell's equations and, therefore, to obtain the values of $\sigma$ in various tissues of the body. Even though this method seems to be very reliable, albeit of low resolution, it is difficult to apply to more complicated systems, since there are mathematical limitations of solving Laplace’s equation for an arbitrary geometry.

The larger the electric properties, the more the RF profile is disrupted in
MRI. The question we address in this paper is, "How large must these effects be to allow the electric properties themselves to be measured?". The answer depends on the field strength, object size, object geometry, RF polarization and signal-to-noise. In this paper, we focus on the heterogeneous planar model (Brown 1988) to evaluate the effects of layer thickness, number of layers and signal-to-noise in extracting the conductivity and permittivity. Once these values are established, it becomes possible to determine the MR image response and local power deposition more accurately; an important issue at high field strengths. We consider 1.0 T experimental results and 1.0 T and 1.5 T simulated results and use electric properties based on those expected in human tissues.

13.2 Theory

13.2.1 Electrical Properties in Terms of the Field Behavior

The solution to the heterogeneous layer model has been presented in chapter 11. In this section, we will review the one layer model in order to understand some concepts for the behavior of non-conducting and conducting dielectrics
and the latter's affect on phase. This model has a thickness 2R and the RF fields are applied at ±x₀ (Figure 11.1). We have shown that the exact solution of the magnetic field inside the one layer model (the RF field, B₁, is parallel to the layer) can be written as:

\[ B₁(x) = \frac{B₀}{\Delta} \cos k x \]  

(13.1)

with,

\[ \Delta = F \cos k₀ x₀ + G \sin k₀ x₀ \]  

(13.2)

where,

\[ F = \cos k R \cos k₀ R + \frac{k₀}{k} \sin k R \sin k₀ R \]  

(13.3)

\[ G = \cos k R \sin k₀ R - \frac{k₀}{k} \sin k R \cos k₀ R \]  

(13.4)

and \( k₀^2 \) is determined by replacing \( \sigma \) and \( \epsilon \) with \( \sigma₀ \) and \( \epsilon₀ \) of the surrounding medium (usually air). Therefore, an alternative way is to reverse the problem and knowing the behavior of the spin density inside the dielectric object, the values of the conductivity, \( \sigma \), and the permittivity, \( \epsilon \), can be derived from:

\[ \epsilon(x) = -\frac{1}{\mu₀ \omega^2} \text{Re} \frac{B''(x)}{B₁(x)} \]  

(13.5)
\[ \sigma(x) = \frac{1}{\mu_0 \omega} \text{Im} \frac{B''_1(x)}{B_1(x)} \]  

(13.6)

where the prime indicates the derivative with respect to \( x \).

The implications of these relations are that an examination of the real and imaginary channels of the object being imaged can be used to estimate \( \sigma \) and \( \epsilon \). A straightforward method to evaluate \( \sigma \) and \( \epsilon \) would be the estimation of the second derivative of the RF field. This derivative can be directly evaluated from the image, since the image is proportional to the field in some cases (vide infra). The reason this method is not pursued further at this time is that MR images contain spurious phase effects unrelated to the RF penetration which makes a simple extraction difficult.

13.2.2 Extracting the Electrical Properties from the MR Image.

If we consider a short TR gradient-echo sequence, the expression for the magnetization is

\[ M(\theta) = \frac{\rho \sin \theta \left[ 1 - e^{-\frac{t_{2R}}{T_1}} \right]}{1 - \cos(\theta) e^{-\frac{t_{2R}}{T_1}}} \]  

(13.7)
where \( \rho \) represents the spin density of the slice, \( TR \) is the repetition time of the sequence. If the image is acquired using a low flip angle, \( \theta \), gradient echo sequence (Haacke [59], Haacke [60]). For a given TR, if \( \theta^2 \ll \frac{2TR}{T_1} \), then the transverse magnetization can be shown to depend only on \( M_0 \sin \theta \) (Haacke [60])(where \( M_0 \) is the equilibrium magnetization). The signal is proportional to \( \frac{\partial}{\partial t} (M \cdot B_1) \) where \( M = M_0 \sin \theta \). For a given field \( B_1 \), the resulting tip angle is \( \theta = \gamma B_1 \tau \) where \( \tau \) is the duration of the RF pulse \( B_1 \). Clearly, for small \( \theta \), \( M \) is proportional to \( B_1 \). Finally, the image is then proportional to \( |B_1|^2 = B_1 B_1^* \).

It is also possible to rewrite equation (13.1) to give a quadratic and quartic dependence on \( kx \). To obtain a direct correlation with the magnitude squared of the image, we take \( (B_1 B_1^*)^2 \) and retain terms to fourth order in position:

\[
(B_1 B_1^*)^2 = 1 - 2 [\mu_0 \omega^2 \epsilon] x^2 + \frac{\mu_0^2 \omega^4}{3} \left[ 5 \epsilon^2 + \left( \frac{\sigma}{\omega} \right)^2 \right] x^4
\]  

(13.8)

The term \( \left| \frac{B_1}{\Delta} \right|^2 \) has been left out so that equation (13.8) represents \( (B_1 B_1^*)^2 \) normalized to unity for \( x = 0 \). Since the images are normalized to the central value to compare with equation (13.8), \( \gamma \), \( \tau \) and other constants play
no role.

13.2.3 A General Least Squares Solution to the Multilayer Problem.

There is no such simple analytic expression for the n-layer heterogeneous case. Instead, the inverse problem is solved for the electrical properties by iterating from an initial guess assumed to be close enough to the actual solution that a linear approximation can be made and avoid local minima that can give incorrect estimations for $\sigma$ and $\epsilon$. Specifically, the boundary layers are reasonably well established (to within a pixel) as are the tissue types (or materials) from the MR image itself. Under these conditions, the vector of unknown model parameters representing $\sigma$ and $\epsilon$ as a function of position over the $n$ pixels is defined as:

$$\mathbf{u} = (\sigma, \epsilon)$$

(13.9)

where

$$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n),$$

and
Figure 13.1: The N-layer planar model used to calculate $B_1(x)$ from the heterogeneous planar layer model. The static field $\vec{B}_s$ is along $\hat{z}$.

$$\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$$

with the $i^{th}$ subscript representing the electrical properties at the $i^{th}$ pixel (Figure 13.1). For example, $\sigma_i = \sigma(i\Delta x)$, where $\Delta x$ is the pixel width. In this discussion, $n\Delta x$ represents the field of view being imaged.

The first step is to perform the forward problem i.e., for a given model to find the RF profile (i.e., $B_1(x)$) as a function of position. Second, the
predicted $B_1(x)$ profile, the complex vector of the RF field for each pixel in
the image is found and defined to be $b$. Third, when $a$ is changed to $a + \delta a$, a
second RF field is calculated and the corresponding image change to $b + \delta b$
is found. The goal is to determine $\delta a'$, the difference between the model and
the desired solution given $\delta b'$, the difference between the estimated and the
actual image intensity (which depends on $|B_1|^2$ as described in section 2.2).
These two vectors can be related by calculating the parametric matrix $M$ via

$$M = \frac{\delta b}{\delta a}.$$  \hspace{1cm} (13.10)

first and then, fourth,

$$\delta b' = M \delta a'.$$  \hspace{1cm} (13.11)

Generally, the number of layers is much less than the number of measured
image values. To solve this case, the least squares solution for $\delta a'$ is used:

$$\delta a' = (M^T M)^{-1} M^T \delta b'.$$  \hspace{1cm} (13.12)

Fifth, the updated model is:

$$\hat{a} = a + \delta a'.$$  \hspace{1cm} (13.13)
The procedure is repeated by returning to equation (13.10). Sixth, this iteration continues until there is convergence in the values of the electric properties of the material, specifically until the difference between two consecutive values is less than some threshold (chosen to be 1% in this study). At this point, the properties for that layer are fixed.

13.3 Method

13.3.1 Physical Models

As discussed in section 13.2.2, MR images acquired with a low flip angle will have a signal directly proportional to the field $B_1$, a phase term which depends on the RF penetration, and other imaging factors (Haacke [60]). When the magnitude of this complex image is taken, the result will be proportional to $|B_1|^2$ as shown in (13.8). In this simple one layer case, the two dimensional MR image normally acquired in the $xy$ plane can, in theory, be examined at any value of $y$. In practice, a finite phantom is used which is thin in the $x$-direction ($5 - 10 \text{ cm}$) and as long and wide as possible ($50 \text{ cm} \times 50 \text{ cm}$), to approximate the theoretical model. The one-dimensional
signal (referred to as a profile, shown in Figures 13.2, 13.3) as a function of $x$ is evaluated at $y = 0$ to obtain the closest comparison to the theory. After this profile is normalized to unity at $x = 0$, it can be compared to (13.8) to extract $\sigma$ and $\epsilon$.

The phantom is initially filled with distilled water having conductivity $0 \, S/m$. A second phantom, with the same dimensions, is filled with a saline solution using $3 \, gr/liter \, NaCl$. The conductivity was measured using a conducting cell with $KCl$ as a calibrator. One phantom is carefully positioned in the body coil ($1.0 \, T$ Siemens Magnetom) to be in the isocenter of the RF coil. A short $TR = 200 \, ms$, low flip angle $\theta = 20^\circ$, $TE = 10 \, ms$, $8$ acquisition gradient echo sequence with a $2 \, cm$ thick slice was acquired. The same scanning scenario was then followed for the second phantom.

### 13.3.2 Simulated Models

Two simulated heterogeneous models will also be evaluated. Specifically, the RF field is analytically generated for a leg model (Figure 13.4) and head model (Figure 13.5). The leg model, which mimics the human leg in cross
Table 13.1: Original values for the conductivity and the permittivity for the different tissues in the leg model. The table also gives the thickness of each individual layer and the corresponding number of pixels (128) for this layer.

The head model is an asymmetric model which consists of five layers, is 18.4 cm wide, and the total number of sampling points was set to 128. The
Table 13.2: Original values for the conductivity and the permittivity for the different tissues in the head model. The table also gives the thickness of each individual layer and the corresponding number of pixels (128) for this layer.

<table>
<thead>
<tr>
<th>tissue</th>
<th>$\sigma$ in S/m</th>
<th>$\epsilon$</th>
<th>thickness</th>
<th>no. of pixels</th>
</tr>
</thead>
<tbody>
<tr>
<td>fat</td>
<td>0.04</td>
<td>10.0</td>
<td>1.0 cm</td>
<td>7</td>
</tr>
<tr>
<td>White/Grey matter</td>
<td>0.30</td>
<td>100.0</td>
<td>7.2 cm</td>
<td>50</td>
</tr>
<tr>
<td>C.S.F</td>
<td>1.50</td>
<td>60.0</td>
<td>3.8 cm</td>
<td>27</td>
</tr>
<tr>
<td>White/Grey matter</td>
<td>0.30</td>
<td>100.0</td>
<td>5.4 cm</td>
<td>37</td>
</tr>
<tr>
<td>fat</td>
<td>0.04</td>
<td>10.0</td>
<td>1.0 cm</td>
<td>7</td>
</tr>
</tbody>
</table>

original values of $\sigma$ and $\epsilon$, for each layer, the thickness of each layer, and the number of pixels corresponding to each layer are shown in Table 13.2. Using again the original values for the dielectric properties of the layers (Stoy [7], Stuchly [8]), an MR image with background noise of 1% to 5% is generated. Once again, the values of $\sigma$ and $\epsilon$, for the layer of fat are assumed constant and the boundaries for each layer are fixed. The solution for $\sigma$ and $\epsilon$, was obtained for a given seed for the random number generator. The seed was changed sixteen times to generate a Monte Carlo estimate for the mean and standard deviation of $\sigma$ and $\epsilon$. 
13.4 Results

Three models will be considered to evaluate the sensitivity of $\sigma$ and $\varepsilon_r$ (the relative permittivity) on the number of pixels in the image, the number of layers, and layer thickness.

Initial comparison between theory and experiment were performed at 1.0 T. The first simulated model is simply a single layer with $\varepsilon_r = 80$ and $\sigma = 0.3 \text{ S/m}$ designed to mimic the physical model. The image parameters were set in such a way that the object was 10 cm, and 128 pixels across the object were used. Using equation (13.8), the value for $\sigma$ was determined to be $0.316 \pm 0.016$ and for $\varepsilon_r$ to be $81.5 \pm 1.2$. Clearly, the random errors are small for this one layer model. Both $\sigma$ and $\varepsilon_r$ were also extracted from experimental data acquired from two phantoms which were 10 cm thick and long enough (50 cm x 50 cm) in order to be considered infinite along the two other dimensions. The first phantom contained pure distilled water so that the conductivity of the phantom was essentially 0.0 S/m. The other phantom was doped with a conducting substance (NaCl) and its conductivity was es-
timated from the KCl conducting cell to be 1.3 S/m (vida supra). Using the magnitude squared approach of equation (13.8), the solid line in Figure 13.2 is the quartic fit to the profile. From the fit, the values for the conductivity for both of the phantoms are found to be zero for the non-conducting phantom and 1.31 for the conducting phantom, while the relative permittivities were 85.3 for the non-doped phantom and 45.3 for the doped one.

The second simulated model to be considered is a multilayer heterogeneous leg model which is symmetric with respect to the center layer, (Figure 13.4) this time at 1.5 T. Applying the multilayer least squares fit method to this data, $\sigma$ and $\epsilon_r$ for each layer were computed (Table 13.3). In Figure 13.6, the approach to equilibrium for $\sigma$ and $\epsilon_r$ for the different layers of the leg and head models are shown. The incremental changes, $\delta a$, are chosen appropriately so that the system converges after 6 to 8 iterations. Agreement with theory is excellent for the thicker layers, but poor for the narrow layer. The third simulated model to be examined is the head model (Figure 13.5) in an axial region through the ventricles again at 1.5 T. The results of performing the multilayer least squares fit for the remaining three
Table 13.3: Computed values and standard deviation for the conductivity and relative permittivity for the three layers of the leg model, assuming that the values for the fat are constant. Results are given for different levels of noise in the image.

As this table indicates, the agreement between the computed and original value of conductivity and relative permittivity is very good.

13.5 Discussion

The experimental results using the physical phantoms agree well with the expected conductivity, and for the undoped case ($\sigma = 0$), the permittivity as well (the value of $\varepsilon_r$ for water is 80.36 at 20°C ([8])). The reason that the


<table>
<thead>
<tr>
<th>tissue</th>
<th>$\sigma \pm \Delta \sigma$ in S/m</th>
<th>$\epsilon \pm \Delta \epsilon$</th>
<th>noise</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wh/Gr mat.</td>
<td>0.303±0.002</td>
<td>100.700±0.449</td>
<td>1 %</td>
</tr>
<tr>
<td></td>
<td>0.312±0.008</td>
<td>98.939±0.429</td>
<td>3 %</td>
</tr>
<tr>
<td></td>
<td>0.329±0.023</td>
<td>102.444±2.045</td>
<td>5 %</td>
</tr>
<tr>
<td>C.S.F</td>
<td>1.504±0.016</td>
<td>62.187±0.608</td>
<td>1 %</td>
</tr>
<tr>
<td></td>
<td>1.479±0.070</td>
<td>63.720±3.653</td>
<td>3 %</td>
</tr>
<tr>
<td></td>
<td>1.496±0.133</td>
<td>76.083±3.280</td>
<td>5 %</td>
</tr>
<tr>
<td>Wh/Gr mat.</td>
<td>0.291±0.002</td>
<td>98.461±0.405</td>
<td>1 %</td>
</tr>
<tr>
<td></td>
<td>0.285±0.005</td>
<td>96.321±1.098</td>
<td>3 %</td>
</tr>
<tr>
<td></td>
<td>0.278±0.012</td>
<td>94.363±1.961</td>
<td>5 %</td>
</tr>
</tbody>
</table>

Table 13.4: Computed values and standard deviation for the conductivity and relative permittivity as in Table 3 but for the three layers of the head model.

permittivity is poorly defined for $\sigma \neq 0$ is that the higher order term in equation (13.8) is more dominant and the effect of $\epsilon_r$ becomes small and can not accurately be determined with this method. Simulations show that with 5% noise $\epsilon_r$ varies dramatically for $\sigma = 1.31 \, S/m$. On the other hand, the larger changes in response to $\sigma = 0.0 \, S/m$ make it easier to extract $\epsilon_r$. As can be seen from equation (13.8), the finite wavelength effects of permittivity lead to a quadratic effect in RF penetration, causing the $B_1$ field to increase in the central region. This effect is clearly seen even at 1.0 T as shown in Figure 13.2.
Careful examination of the results for the leg model indicate that the computed values of $\sigma$ for the two muscle layers are in good agreement with the original values even with 3% noise on the data. The predicted values of $\epsilon_r$ appear to have up to a 20% discrepancy from the original values. For the central marrow layer, there is a big difference between the original and computed values for the conductivity. The reason is that the initial value for $\sigma$ for this layer is small compared with the corresponding values of the conductivity for the surrounding layers. The minimal distortion affect this has on the image, coupled with the systematic errors of $\sigma$ and $\epsilon_r$ for the two surrounding layers, increases the value of the conductivity. The permittivity is also poorly defined for thin layers.

In comparison with the previous model, the values for $\sigma$ and $\epsilon_r$ for the three layers of the head model are larger than those of the leg model. Therefore, the RF profile change is large enough that these three layers can be easily discriminated. That is, even though the CSF layer is not thick, it has a sufficiently high conductivity to enhance its affect on the RF profile. Only as the noise rises to 5% does the permittivity for the thin CSF layer begin
to significantly deviate from the correct value.

A reasonable guess close to the original values of the conductivity and permittivity has been made in order to save time in the iteration procedure ($\varepsilon_r = 50$ and $\sigma = 0.5 \, S/m$ for all layers). The same results were obtained for any initial guess of $\sigma$ and $\varepsilon_r$ (even if all the conductivities are chosen to be zero). The upper and lower limits for $\sigma$ and $\varepsilon_r$ are chosen to be between $0.0 \, S/m$ to $9.0 \, S/m$ and $1.0$ to $100$, respectively. If these limits are exceeded the electrical properties are set to the closest boundary value.

There are two reasons for choosing the incremental changes in $\delta a$ to be on the order of one-hundredth of the expected solution. First, for large matrix sizes, it will take a longer processing time to achieve a stable solution. Second, there may exist a local minima to which the solution might be driven if the incremental choice is too small. For the simulation models, the incremental changes are chosen to be: $\delta \sigma = 0.001 \, S/m$ for a layer thickness less than $1 \, cm$ and $\delta \sigma = 0.01 \, S/m$ for a layer thicker than $1 \, cm$. For $\varepsilon_r$, the corresponding changes were $0.5$ for a layer less than $1 \, cm$ thick and $1.0$ for thicker layers (thicker than $1 \, cm$).
Although the noise in the image for the head model is up to 5%, the least squares fit via equation (13.12) makes such a sensitive fit possible and gives a reasonable result for $\sigma$ and $\epsilon_r$. Clearly, either many points will be required for an accurate estimate of the electric properties or the signal-to-noise must be very high. This means layers must be thick, or $\sigma$ and $\epsilon_r$ must be in the region of parameter space where the RF profile changes rapidly. Otherwise, the matrix $M$ is likely to be ill-conditioned.

The fact that these parameters are not accurately known in some of the above circumstances (i.e., thin layers) is not a particular problem in predicting the image response. This is because the profile itself is insensitive to the choice of $\epsilon_r$ and $\sigma$ for the layer thickness. In summary, this method predicts the appropriate values of $\sigma$ and $\epsilon_r$ for a layer thickness of more than 0.5 cm and conductivity larger than 0.1 $S/m$. 
Figure 13.2: Image profile for a 10 cm thick planar phantom acquired with a 2 cm thick slice, a 20\(^\circ\) flip angle, TE = 10 ms, TR = 200 ms, field of view 12.5 cm and single slice centered on the phantom. The ordinate is an arbitrary signal amplitude while the abscissa is the pixel number in the field of view. The finite wavelength effects are clear even at 42.6 MHz, as can be seen when \( \sigma = 0 \) for this phantom.
Figure 13.3: Image profile for a 10 cm thick planar phantom acquired with a 2 cm thick slice, a 20° flip angle, TE = 10 ms, TR = 200 ms, field of view 12.5 cm and single slice centered on the phantom. The ordinate is an arbitrary signal amplitude while the abscissa is the pixel number in the field of view. When the phantom is doped with NaCl, screening is taking place, and $\sigma$ and $\epsilon$ are competing for profile changes. In both figures, the solid line represents a quartic least squares fit through the data.
Figure 13.4: Symmetric 5-layer, 15 cm thick, planar model. The values of the conductivity and relative permittivity are shown in Table 1. The layers are situated in a static magnetic field along \( \hat{z} \) and the RF field is along \( \hat{y} \).
Figure 13.5: Asymmetric 5-layer, 18.4 cm thick, planar model. The values of the conductivity and relative permittivity are shown in Table 2. The layers are situated in a static magnetic field along $\hat{z}$ and the RF field is along $\hat{y}$. 
Figure 13.6: Stability for the least squares fit method for the conductivity and relative permittivity for the leg model. The boxes represent the electrical property for the first layer of muscle and the crosses represent the second layer after the marrow.
Chapter 14

Predicting RF Field Penetration in Heterogeneous Bodies Using a Finite Element Approach

14.1 Introduction

Recent developments in 128-\textit{MHz} and 170-\textit{MHz} whole-body imaging systems have made analysis of the radiofrequency (rf) behavior inside the human body important. Access to imaging in high-field small-bore systems at 300 to 500 \textit{MHz} is also available, allowing animal studies to be performed. Images for both human and animal studies in these frequency domains will be affected by non-uniform rf penetration. As we have mentioned in chapter 13,
at high field strengths, the wavelength of the rf is reduced significantly inside the dielectric object. This is significantly larger than the dimensions of the human torso, and hence, first-order estimates of the rf power, including both specific absorption rate (SAR) for local and average conditions, are sufficient. For a 4.0 T main magnet, however, \( \lambda \sim 0.1875 \, m \) inside an object with similar dielectric properties as above. Thus, the wavelength is much less than the dimensions \( d \) of the cross-section of the human body. In practice, significant effects are expected when \( \lambda/4 \) is greater than \( d \). This results in both the distortion of the rf field inside the body, and the production of local heating effects. Using these arguments, even imaging of mice will be sensitive to rf penetration effects at frequencies of 300 to 500 MHz. To account for the nonlinearity inherent in such rf effects, an analytical theory is needed that goes beyond first-order approximations.

Besides the analytical methods which are useful to solve ordinary differential equations, there is a variety of problems which can not be handled analytically. The first numerical method, which was developed in order to deal with complicated solutions of inhomogeneous differential equations, was
the Finite Difference Method (FDM). According to this method, we replace
the differential equations by finite difference equations. Specifically, we relate
the value of the dependent variable at the point of the region of interest to
the values of some neighboring points.

Unlike the FDM, Finite Element Method (FEM) is more versatile. It
can handle complex geometrical shapes at inhomogeneous media with sharp
boundaries, as well as complex boundary conditions for the dependent vari-
able.

Numerous articles attempt to describe the behavior of the rf profile inside
the human body and to evaluate SAR for any sequence at any field strength
in MRI [6, 62, 63, 64, 10]. Brown et al. [13] introduced a heterogeneous
multilayer model, consisting of parallel planes with heterogeneous dielectric
properties, to predict the behavior of the rf profile. Foo et al. [65, 66]
analyzed a multi-concentric cylindrical model to simulate the rf profile for
any heterogeneous model with similar symmetry. Zhou [67] analyzed the rf
profile behavior for a set of heterogeneous concentric spheres.

Since the human body in general does not possess any simple symmetry,
the models described above give only an estimate of any associated rf behavior. Clearly, there exist areas inside the human body where the predictions of the above models can fail, and this can be most significant in predicting local SAR particularly at high fields [68, 69, 70].

In this chapter, we present the results for a 2D finite element solution to Laplace’s and Poisson’s equations, using the scalar potential as the independent variable at the Lagrangian expression, and a 3D finite element model which describes in detail the rf behavior inside a heterogeneous dielectric object. Divergence conditions have been enforced as constraints in order to obtain results for the rf profile consistent with Maxwell’s equations and to make the finite element problem well posed. We compare the results of the rf profile between the 3D finite element method and the three models of planes, cylinders and spheres.

14.2 Theory

The complex behavior of the heterogeneous objects requires techniques that can predict it with a very small deviation for the experimental values. The
finite element approach is up to now the best technique of solving complicated
problems that can not be tackled analytically. In the past year, we have
developed finite element programs and boundary element programs trying to
predict the behavior of rf penetration and power deposition in heterogeneous
objects, especially on the head phantom.

14.2.1 2D Finite Element Solutions for Poisson’s and
Laplace’s Equations

Our first trial involved the solution of Laplace’s equation ($\nabla^2 U = 0$, $U$ is
the scalar potential) in two and three dimensions without a source present.

Solving Laplace’s equation is equivalent to minimizing the functional

$$ W[U] = \frac{1}{2} \int |\nabla U|^2 dS $$

(14.1)

Assuming $U$ can be represented by a linear function within a triangular
element (in 2D) or tetrahedral element (in 3D), eg.,

$$ U = a_1 + a_2 x + a_3 y, \quad \text{in } 2D $$

(14.2)

$$ U = a_1 + a_2 x + a_3 y + a_4 z, \quad \text{in } 3D $$

(14.3)
Solving for the constants, using the value of the potential at the vertices 
\( (U_1, U_2, U_3 \text{ in the 2D case}) \), the above equations can be rewritten as

\[
U = \sum_{i=1}^{3} U_i \, a_i(x,y), \quad \text{in 2D} \quad (14.4)
\]

\[
U = \sum_{i=1}^{4} U_i \, a_i(x,y,z), \quad \text{in 3D} \quad (14.5)
\]

Then the functional \( W[U] \) takes the form

\[
W^e[U] = \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} U_i \int \nabla a_i \cdot \nabla a_j \, dS_j, \text{ each element (2D)} \quad (14.6)
\]

\[
W^e[U] = \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} U_i \int \nabla a_i \cdot \nabla a_j \, dV_j, \text{ each element (3D)} \quad (14.7)
\]

\[
W = \sum \text{ all elements } W^e[U] \quad (14.8)
\]

where the superscript \( e \) denotes the value of \( W \) for each element. Minimizing the above expressions of the functional with respect to \( U_k \) for the 2D and 3D cases, we get an expression for \( U \). We tried our program for solving Laplace's equation in 3-D for the cube with edges of length 1 (arbitrary units), \( U=1 \) on the top face, \( U=0 \) on the other faces, and compared our solution with that obtained analytically. Dividing the cube into a \( 7 \times 7 \times 7 \) grid of smaller cubes and each of the small cubes to 5 tetrahedrons (343 nodes in 2D, 1080 nodes
in 3D), the values of the scalar potential at the middle was

\[ U = 0.1667, \quad \text{analytically} \]  \hspace{1cm} (14.9)

\[ U = 0.1675, \quad \text{finite element approach} \]  \hspace{1cm} (14.10)

The next step was to add a source term \( \nabla^2 U = -4\pi \rho \) in 3D. We approximated the charge density by

\[ \rho(x, y, z) = \sum_{i=1}^{4} \rho_i \ a_i(x, y, z), \quad \text{in 3D} \]  \hspace{1cm} (14.11)

Following the same minimization procedure for the functional \( W[U] \), we can evaluate the scalar potential, given the boundary conditions.

We tested our program, trying to obtain the expression of the scalar potential for a square (2D) with a source, and a cube (3D) with a source, and compared this with the analytic solutions. Using a mesh of squares (2D) and a mesh of cubes (3D), we computed the the quantity

\[ \chi^2 = \sum_{\text{points}} \frac{\sum_{i=1}^{3(2D),4(3D)} \left( \frac{U_{\text{in,
\text{elem}}} - U_{\text{exact}}}{U_{\text{exact}}} \right)^2}{n} \]  \hspace{1cm} (14.12)

and we found

\[ \chi^2 = 0.05421, \quad \text{for 40x40 mesh squares(2D)} \]  \hspace{1cm} (14.13)
\[ \chi^2 = 0.07635, \quad \text{for 20x20x20 mesh cubes (3D)} \quad (14.14) \]

Thus, there is a good agreement between the finite element analysis and the analytical calculation of the potential in two and three dimensions.

### 14.2.2 Theory for the 3D FEM

The finite element method is based on optimization of a function with respect to a given variable. The choice of the optimized function and the corresponding variable strongly depend on the particular application. In practice, each problem under consideration often requires its own variation of the finite element approach. In electromagnetic theory, the optimized function is mainly chosen to be the Lagrangian and the corresponding variable can be the electric field \( \vec{E} \), the scalar potential \( \phi \), or the vector potential \( \vec{A} \).

The initial step for the creation of a finite element methodology is the decomposition of the object shape into a mesh of individual elements. The shape of each element is closely related to the choice of coordinate basis. To simplify the analysis of the finite element method and to improve the efficiency of the programming, “simplex” coordinates \( \zeta \) are considered as the
Figure 14.1: One of the decompositions of a cube into five tetrahedrons.

cordinate basis [71]. This set of coordinates always defines the optimized shape of the individual element for the finite element mesh in any dimension. For a 3D problem the most appropriate shape is the tetrahedron, in analogy with a 2D problem where the triangle is the most appropriate shape.

To analyze a 3D problem, we start building the mesh with cubes which later are decomposed into five tetrahedrons(Figure 14.1). In order to de-
Figure 14.2: Illustration of a tetrahedron with one of the simplex coordinates of an arbitrary point \( P \).

To describe a first order tetrahedron, four nodes are needed which correspond to the points of the intersections for each of the three triangular surfaces of the tetrahedron. For the second order tetrahedron, there exists 10 nodal points. Four nodal points correspond to the first order tetrahedron nodes. The remaining six nodes are positioned at the midpoints of the line boundaries of each triangular surface of the tetrahedron as shown in Figure 14.2. Thus,
the general rule for finding the number of nodes for any order tetrahedron is
given by the expression

$$N(n) = \frac{1}{3!} (n + 1)(n + 2)(n + 3)$$

where $n$ represents the order of the tetrahedron which is used in the finite
element mesh. In this chapter and for the 3D finite element model only first
order tetrahedrons are considered.

Four simplex coordinates are needed to describe each tetrahedron in the
mesh. The definition of these homogeneous coordinates for the $i^{th}$ face is:

$$\zeta_i = \frac{h_i}{H_i} \quad (14.15)$$

where $h_i$ is the height of a point from the triangular face of the tetrahedron
and $H_i$ is the height of the vertex opposite to that face (Figure 14.2). There-
fore, any point $P(x, y, z)$ inside a tetrahedron, can be described in terms of
four simplex coordinates

$$P(x, y, z) = P(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \quad (14.16)$$

We also notice from (14.15) that $\zeta_1$ is zero along the triangle $BCD$ and
1 at node $A$ of the tetrahedron of Figure 14.2. Therefore, the represent-
tion to the cartesian coordinate system of the four simplex coordinates is 

\((1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\) at the nodal points \(A, B, C, D\), respectively. In the approach presented here, the optimized quantity is the Lagrangian \(L\) and the variable is the vector potential \(\vec{A}\). The vector potential \(\vec{A}\), in general a complex quantity, can be expanded in the basis of the simplex coordinates \(\zeta_i\) as:

\[
A_j = \sum_{i=1}^{4} \zeta_i A_{ij} \tag{14.17}
\]

where \(A_{ij}\) are the values of each of the three components \((j = 1, 2, 3)\) of the vector potential at each of the 4 nodes of the individual first-order tetrahedrons.

The expressions of the cartesian coordinates (for any \(x, y, z\)) in terms of the simplex coordinates are

\[
x = \sum_{i=1}^{4} x_i \zeta_i \tag{14.18}
\]

\[
y = \sum_{i=1}^{4} y_i \zeta_i \tag{14.19}
\]

\[
z = \sum_{i=1}^{4} z_i \zeta_i \tag{14.20}
\]

where \(x_i, y_i, z_i\) are the values of the cartesian coordinates at the node points.
The next step is to find the relationship between the simplex and the cartesian coordinates. This relationship can be determined by summing the volumes of the four tetrahedrons which are formed by connecting the vertices of the tetrahedron $ABCD$ to the point $P(x, y, z)$. Then the total volume of the tetrahedron $ABCD$ is

$$\frac{1}{3}h_1A_1 + \frac{1}{3}h_2A_2 + \frac{1}{3}h_3A_3 + \frac{1}{3}h_4A_4 = V$$

(14.21)

or from (14.15)

$$\frac{1}{3}\zeta_1 H_1A_1 + \frac{1}{3}\zeta_2 H_2A_2 + \frac{1}{3}\zeta_3 H_3A_3 + \frac{1}{3}\zeta_4 H_4A_4 = V$$

(14.22)

where $A_1 - A_4$ are the areas of the four faces of the tetrahedron $ABCD$ respectively. Since, for every value of $i = 1, 2, 3, 4$

$$V = \frac{1}{3}H_iA_i$$

(14.23)

we conclude from the relation (14.22) that

$$\sum_{i=1}^{4}\zeta_i = 1$$

(14.24)
Solving the system of equations (14.18), (14.19), (14.20), (14.24) for the $\zeta'$s, we obtain

$$\zeta_i = \frac{1}{6V} [a_i + b_i \; x + c_i \; y + d_i \; z]$$ (14.25)

with

$$a_i = (-1)^{i1} [x_{i1} (y_{i2}z_{i3} - y_{i3}z_{i2}) + x_{i2} (y_{i3}z_{i1} - y_{i1}z_{i3}) + x_{i3} (y_{i1}z_{i2} - y_{i2}z_{i1})]$$

$$b_i = (-1)^{i} [(y_{i2}z_{i3} - y_{i3}z_{i2}) + (y_{i3}z_{i1} - y_{i1}z_{i3}) + (y_{i1}z_{i2} - y_{i2}z_{i1})]$$

$$c_i = (-1)^{i} [(x_{i2}z_{i3} - x_{i3}z_{i2}) + (x_{i3}z_{i1} - x_{i1}z_{i3}) + (x_{i1}z_{i2} - x_{i2}z_{i1})]$$

$$d_i = (-1)^{i} [(x_{i2}y_{i3} - x_{i3}y_{i2}) + (x_{i3}y_{i1} - x_{i1}y_{i3}) + (x_{i1}y_{i2} - x_{i2}y_{i1})]$$ (14.26)

where $i$, $i1$, $i2$, $i3$ are the integers $1$, $2$, $3$, $4$ or any cyclic permutation of them and $6V = a_1 + a_2 + a_3 + a_4$, and $V$ is the volume of the tetrahedron.

Using the above expressions for $a_i$ and $\zeta_i$ the vector potential can be expressed in matrix form as

$$\vec{A} = \vec{a} \cdot \vec{A}$$ (14.27)

where $\vec{A}$ has matrix elements $A_{ij}$, and $\vec{a}$ is a vector with elements $a_i$. 
Ignoring effects of the magnetic permeability and assuming no external current source, the behavior of the vector potential satisfies Maxwell's equations as

\[
\frac{1}{\mu_0} \nabla \times (\nabla \times \vec{A}) = \vec{J} = (\sigma + i\omega\epsilon)\vec{E}
\]  
(14.28)

If the Coulomb gauge condition \((\nabla \cdot \vec{A} = 0)\) is enforced, and assuming sinusoidal time dependence for the magnetic field, electric field and vector potential, then from Maxwell's equation we obtain the relation

\[
\nabla \times [\vec{E} + i\omega\vec{A}] = 0
\]

or

\[
\vec{E} = -i\omega\vec{A}
\]  
(14.29)

with the assumption that there is no source present and no charge density accumulation occurs. Therefore equation (14.28) becomes

\[
\nabla \times (\nabla \times \vec{A}) = \mu_0\omega^2 \left(\epsilon - \frac{i\sigma}{\omega}\right)\vec{A}
\]  
(14.30)

We now proceed with the determination of the Lagrangian shape which describes the behavior of this quasi-static problem. Since, the FEM is based
on the minimization of the total energy of the system, the construction of the
Lagrangian involves the total magnetic stored energy of the system $\frac{1}{2}\mu_0 B^2$
and term $\vec{J} \cdot \vec{A}$, which corresponds to the work needed to bring the currents
to where they are. Thus, an initial expression of the Lagrangian is

$$\mathcal{L} = \int \int \int_V \left[ \frac{1}{2} \mu_0^{-1} \left( \nabla \times \vec{A} \right)^2 - \vec{J} \cdot \vec{A} \right] dV$$  \hspace{1cm} (14.31)

As a next step, we will investigate if (14.31) is consistent with (14.28).

Let us consider a small variation $\delta \vec{A}$ to the value of the vector potential $\vec{A}$.

Therefore, the variation on the Lagrangian $\mathcal{L}$ is

$$\delta \mathcal{L} = \int \int \int_V \left[ \frac{1}{2} \mu_0^{-1} \left( \nabla \times (\vec{A} + \delta \vec{A}) \right)^2 - \vec{J} \cdot (\vec{A} + \delta \vec{A}) \right] dV$$

$$- \int \int \int_V \left[ \frac{1}{2} \mu_0^{-1} \left( \nabla \times \vec{A} \right)^2 - \vec{J} \cdot \vec{A} \right] dV$$  \hspace{1cm} (14.32)

or

$$\delta \mathcal{L} = \int \int \int_V \left[ \frac{1}{2} \mu_0^{-1} \left( \nabla \times \vec{A} \right) \cdot \left( \nabla \times \delta \vec{A} \right) \right.$$

$$\left. - \vec{J} \cdot \vec{A} + \frac{1}{2} \mu_0^{-1} \left( \nabla \times \delta \vec{A} \right)^2 \right] dV$$  \hspace{1cm} (14.33)

Using the identity

$$\nabla \cdot \left[ \delta \vec{A} \times \left( \nabla \times \vec{A} \right) \right] = \left( \nabla \times \vec{A} \right) \cdot \left( \nabla \times \delta \vec{A} \right) - \delta \vec{A} \cdot \left[ \nabla \times \nabla \times \vec{A} \right]$$
associated with Gauss theorem, equation (14.32) becomes

$$
\delta \mathcal{L} = \int \int_S [\delta \vec{A} \times (\vec{\nabla} \times \vec{A})] \, dS \nonumber \\
+ \int \int \int_V [\delta \vec{A} \cdot (\mu_0^{-1} \vec{\nabla} \times \vec{\nabla} \times \vec{A} - \vec{j})] \, dV \nonumber \\
+ \int \int \int_V \frac{1}{2} \mu_0^{-1} (\vec{\nabla} \times \delta \vec{A})^2 \, dV \quad (14.34)
$$

If we assume that the surface integral is evaluated over an infinite surface, then the first integral in equation (14.34) vanishes. Since the last integral in equation (14.34) is always positive for any value of \( \vec{A} \), the only case where \( \delta \mathcal{L} \) is always positive is when \( \mu_0^{-1} \vec{\nabla} \times \vec{\nabla} \times \vec{A} - \vec{j} = 0 \) which is identical with the expression (14.28). Since \( \vec{\nabla} \cdot \vec{A} \) is always zero, it can be included to the form of the Lagrangian as a constraint without altering its general behavior. Thus, under the assumption that there is no free charge present, the Lagrangian that can describe the problem with the Coulomb gauge condition imposed \( (\vec{\nabla} \cdot \vec{A} = 0) \) and which is consistent with the Dirichlet boundary conditions has the form:
\[ \mathcal{L} = \mu_0^{-1} \int_V \left[ \frac{1}{2} [\vec{\nabla} \times \vec{A}]^2 - k^2 A^2 + \frac{1}{2} [\vec{\nabla} \cdot \vec{A}]^2 \right] dV \] (14.35)

with

\[ k^2 = \mu_0 \omega^2 \left[ \epsilon - i \frac{\sigma}{\omega} \right] \] (14.36)

where \( \omega, \sigma, \epsilon \) are the frequency, conductivity and permittivity, respectively, and \( A^2 = \vec{A} \cdot \vec{A} \). The term containing \( \vec{\nabla} \cdot \vec{A} \) represents the Coulomb gauge and its presence here ensures its being obeyed in a consistent fashion with the optimized solution. Expressing in matrix form and minimizing \( \mathcal{L} \) with respect to the vector potential, we find:

\[ \frac{\partial \mathcal{L}}{\partial \vec{A}} = \sum_{\text{all free nodes}} V \mu_0^{-1} \left[ (M_c^T M_c + M_d^T M_d) \vec{A} - k^2 T \right] = 0 \] (14.37)

The set of values of the vector potential at the nodes are unknowns except at the nodes where the boundary conditions are enforced.

The expressions for \( M_c \) and \( M_d \) (which are the matrix representations of the curl and divergence, respectively) and \( T \) (which contains all the informa-
tion about the source) are:

\[
M_c = \begin{bmatrix}
0 & \hat{d} & -\hat{c} \\
-\hat{d} & 0 & \hat{b} \\
\hat{c} & -\hat{b} & 0
\end{bmatrix}
\]

\[
M_d = \begin{bmatrix}
\hat{b} & \hat{c} & \hat{d}
\end{bmatrix}
\]

\[
T = \frac{1}{V} \int \hat{a}^T \hat{a} \, dV
\]

with \(\hat{a}, \hat{b}, \hat{c}, \hat{d}\) the matrix analogs of the coefficients in Eq. (14.25) and 0 representing a null vector with size 1 \(\times\) 4. The dimension of the matrices depends on the choice of the grid for the finite element mesh. For the first-order tetrahedrons, where there are 4 interpolation nodes, the dimension of the vector potential matrix \(\hat{A}\) is 12 \(\times\) 1, the size of the curl tensor \(M_c\) is 3 \(\times\) 12, the size of the divergence tensor \(M_d\) is 1 \(\times\) 12, and the size of each of the individual matrices \(\hat{a}, \hat{b}, \hat{c}, \hat{d}\) is 1 \(\times\) 4.

The normal component of the magnetic field is discontinuous across the interfaces of the materials with different dielectric properties. Since the vector potential is continuous across these interfaces, discontinuity is introduced from the curl. The final expression for the magnetic field \(\vec{B}\) from the vector potential \(\vec{A}\) depends on the choice of the order of tetrahedrons in the
mesh and on the gauge condition. Imposing no gauge condition on the vector potential leads to unacceptably large errors for the predicted value of the magnetic field [72]. The gauge-fixing is required to find a unique value for $\vec{A}$. Without it, the solution can diverge, particularly for the higher order elements. Knowing the potential at every node, we can evaluate every component of the magnetic field via the equation

$$\vec{B} = \vec{\nabla} \times \vec{A} = M_c \vec{A}.$$  \hspace{1cm} (14.38)

### 14.2.3 Methods

The 3D finite element method for a heterogeneous dielectric object was implemented in FORTRAN code and executed on a SUN SPARC2. The matrix sizes that we deal with in the finite element approach are large leading to rather lengthy calculation times. The finite element approach that we have presented is a boundary element approach. This requires the knowledge of the behavior of the vector potential at the boundaries of the model. Extending the boundaries of the whole design to surpass the boundaries of the dielectric object, we assume that a normalized rf pulse with harmonic time
dependence ($e^{-i\omega t}$) is applied at the boundaries of the finite element mesh. The interaction between the coil and the subject on the rf pulse has not been considered.

To design the finite element program for a 3D object, we have to include boundary conditions which can extend well past the boundaries of the object, and a large number of elements is needed. This results in a large matrix size. Taking advantage of the symmetry of the matrices in Eq. (14.37), we use only the upper half plus the diagonal of the matrix, and the storage space is reduced almost by half. Furthermore, most of these matrix elements are zero; they are sparse matrices. Taking advantage of this property, we can store the matrix as a vector assigning an additional 1xN matrix to account for the nonzero elements. This reduces the total storage space for each matrix by a factor of $n$, where $n$ represents the total number of nodes, and significantly reduces CPU time.

The 3D space is decomposed into a set of cubes (Figure 14.3), and each cube is divided into five tetrahedrons. The shape of the object is approximated by using an appropriate number of layers in the $x$, $y$ and $z$ direction.
Figure 14.3: Illustration of the 3D volume divided into cubes and tetrahe- 
drons.

In all cases the outside cell size is different from that in the object. Specifi-
cally, the comparison between the analytical solution from the planar model 
and the 3D finite element was made by considering a three dimensional rect-
tangle which contains any shape of the dielectric object. The directions of 
the rectangle were chosen to coincide with the directions of the MRI main 
magnet. The next step is to divide the 3D volume into smaller cubes which
are subdivided into five tetrahedrons. The number of cubes in each direction is denoted by \(n_x\), \(n_y\) and \(n_z\), respectively. Since for the planar model the variation of the rf profile was assumed along the \(\hat{z}\) direction, we introduce an extensive number of elements along this direction, but the program can be easily modified to give us the rf field in any of the other two directions. In general, there are two ways to subdivide the cube into a set of five tetrahedrons. We have run the finite element program with both decompositions and we observed no change in the values of the rf field profile. The volume outside the object is assumed to be air or any other substance with dielectric properties different from those of the interior. As a next step, we require that both the real and imaginary part of the three components of the vector potential are continuous across the interfaces of both the tetrahedrals and the cubes, and satisfy the prescribed boundary conditions. Furthermore, solving for \(\hat{A}\) from Eq.(14.37) requires finding an inverse matrix with a precision of 10ppm (parts per million). In this way, we obtain the resulting values of the vector potential at any point inside the volume. We ensure that the values of \(\hat{A}\) satisfy the Coulomb gauge condition and we evaluate the magnetic
field using the matrix expression of Eq.(14.38). The choice of the Coulomb
gauge condition significantly reduces the error of the magnetic field profile
evaluated by the finite element method [72].

The solution of the matrix equation Eq.(14.37) for the non-prescribed el-
ements of the vector potential matrix \( \hat{A} \) was performed using the Liebmann
Algorithm, with gaussian iteration, where the unknown computed matrix
elements were altered by a prescribed amount which depends on the rf fre-
quency and the values of the conductivity and permittivity as well. The
choice of the Liebmann Algorithm over the Cholesky factorization technique
was made, because the matrix elements for the boundary element method
are preconditioned and a rapid convergence towards the desired values of
the matrix elements can be achieved. We have also compared the Cholesky
decomposition technique to the Liebmann algorithm for Poisson's equation
for the scalar potential and the agreement between these two different algo-
rithms is very good (within the computer's double precision error), but the
Cholesky algorithm is much slower than the Liebmann algorithm.

The convergence for the Liebmann algorithm is accomplished by the in-
roduction of an acceleration factor. The decision of accepting a given matrix element is made by asking that the error (the difference between the old and the improved values) in the evaluation of the improved matrix element be less than 0.001% of the value of the particular element.

The discrepancy between the finite element method and the analytic solution is defined to be

\[
d_{\text{error}} = \frac{\sum_{n=1}^{N} (|B_{FE}| - |B_{AN}|)}{\sum_{n=1}^{N} |B_{AN}|} \tag{14.39}
\]

where \( N \) represents the number of points where the magnitude of the rf profile is evaluated in both the finite element \( (B_{FE}) \) and analytical \( (B_{AN}) \) approaches.

Two other approaches are used to improve the numerical evaluation. First, an outer layer is added to the object to stabilize the calculation. The reason is that with the replacement of the air with the dielectric, a phase factor that better interpolates the behavior of the vector potential across the boundaries at high frequencies has been introduced. There is still the tendency for this phase to oscillate inside the object, a factor which accounts for
the error between the rf profiles of the models and numerical results. This variation can be further reduced by the introduction of more elements (more tetrahedrons in the plane of the interfaces of the different dielectric objects).

14.3 Results

The 3D finite element method was tested by comparing the output with analytical results that were obtained using the cylindrical [66], spherical [67] and multilayer [13] models. Second, we considered a three-layer cubic model with dimensions $y = 1.0m$ (the direction of the rf field), and $z = 1.0m$ (the direction of the static field) and a thickness that is variable in the $z$ direction (the direction along which the rf penetration is evaluated) for a dielectric object with conductivities $\sigma$ ranging from $0.0\ S/m$ to $1.5\ S/m$ and permittivities $\epsilon$ ranging from $\epsilon_0$ to $78\epsilon_0$. Particularly, we compared the numerical output with the analytical results obtained from the cylindrical and spherical geometry at $42\ MHz$ through Eq.(14.39).

The first comparison of the finite element approach was with the cylindrical model. We chose the dimensions of the cylinder to be small enough
in order to be compatible with the first-order approximation of the rf field resulting from a cylindrical shape [57]. Simulations were performed at 42 MHz and 64 MHz. For the finite element case, the diameter of the cylinder was 0.2 m and its length was 1.6 m in an attempt to simulate an infinitely long cylinder in the axial direction that the analytical model assumes. The percentage error resulting from this comparison was no greater than 1.35% for an rf frequency of 42 MHz and 2.8% for 64 MHz. We have proceeded further by designing a heterogeneous multiconcentric cylindrical model, comparing the results with those presented by Foo et al. [66]. The rf frequency for this heterogeneous cylindrical model was chosen to be 170 MHz. The diameter of the inner cylinder was 0.15 m, and its length was 1.6 m. The values of the conductivity and the permittivity for the inner dielectric cylinder was chosen as $\sigma = 0.68 S/m$ and $\epsilon = 58 \epsilon_0$. The outer hollow cylinder has been simulated such that the thickness in the radial direction is 0.011 m, and its length is equal to that of the inner cylinder. The outer cylinder was also assumed to contain dielectric material with conductivity $\sigma = 0.05 S/m$ and permittivity $\epsilon = 57 \epsilon_0$. The comparison between the rf profile generated from
the finite element method and the data presented by Foo et al. [66] with the same values of the conductivity and permittivity for the phantom and the shield material at 170 MHz is illustrated in Figure 14.4. Agreement is very good; the differences are always less than 5%.

Second, we have compared the finite element solution with the analytical solution for a dielectric sphere [57] with a diameter of 0.2 m. The direction of the rf field was along the \( \hat{y} \) direction, and the change of the rf profile in the \( \hat{z} \) direction was calculated. In both analytical and finite element methods, the dielectric sphere was considered to have conductivity \( \sigma = 0.05 \text{ S/m} \) and permittivity \( \varepsilon = 58 \varepsilon_0 \) with the frequency of the oscillating rf field equal to 42 MHz. We have also compared the results of the finite element and the analytic spherical model along the \( \hat{y}, \hat{z} \) directions, using the same parameters as in the previous case (Fig. 14.5). The rf profile along the \( \hat{y} \) direction is in good agreement with this of the \( \hat{z} \) direction. The difference between the two methods for the rf field profile was less than 3% and was evaluated using the formula in Eq.(14.39).

Third, Figures(14.6-14.7) illustrate a comparison of the rf profiles across a
3-D dielectric object surrounded by air between the finite element approach, and the planar model for 42 MHz, and 64 MHz with the same number of elements, respectively. We have varied the thickness of the inside and outside layers in order to get a good estimate for the stability of the finite element program. Particularly, at 42 MHz, (Figure 14.6) the thickness of the outer layers was chosen to be 0.065 m for each one. The middle layer was 0.12 m thick in the $\hat{z}$ direction. The thickness of the entire model was chosen to be 0.25 cm. The outer layers were assumed to be filled by air and the inner layer with a dielectric material with $\sigma = 1.5 S/m$ and $\epsilon = 58\varepsilon_0$. For the 64 MHz case, the thickness of the outer layers was 0.06 m in the $\hat{z}$ direction. The inner dielectric layer was 0.08 m thick with $\sigma = 0.5 S/m$ and $\epsilon = 58\varepsilon_0$. We notice that as the frequency increases, especially at 170 MHz, the deviation from the analytical rf profiles also increases.

As the values of the dielectric properties for the heterogeneous object increase, the discontinuity in the phase of the vector potential disappears. In this case, even a small number of elements can be used to predict with reasonable accuracy the behavior of the rf profile inside the dielectric ob-
ject. Figures 14.8-14.10 represent a comparison between the finite element model and the planar analytic model, where the layers have different values of conductivity and permittivity. For the rf frequency of 64 MHz (Figure 14.8), the thickness of the outside layers was 0.05 m in the $\hat{z}$ direction with $\sigma = 0.2S/m$ and $\epsilon = 50\epsilon_0$. The inner layer was 0.1 m thick with $\sigma = 1.5S/m$ and $\epsilon = 58\epsilon_0$. The agreement of the rf profile generated with these two methods is good with the largest error less than 10%, which occurs at the intersection between the two layers.

Expanding the comparison to 170 MHz for the planar model with the conductivity and permittivity of the outer and inner layers on the same order of magnitude, there is a very good agreement between the finite element and the planar model, even with a smaller number of elements (Figure 14.9). For this model, the inner layer has thickness 0.14 m with $\sigma = 0.5S/m$ and $\epsilon = 58\epsilon_0$. The thickness of the outer layers has been reduced to 0.03 m with $\sigma = 0.2S/m$ and $\epsilon = 50\epsilon_0$. The total thickness of the model was 0.2 m. Apart from a slight shift in the scale, the results are in good agreement with the finite element calculation. Furthermore, we have compared the finite
element program with the planar model at 170 M Hz. The thickness of the inner layer is chosen to be large compared to the outer layers, and it is given a significantly higher conductivity. The total thickness of the model was 0.2 m. The inner layer was 0.175 m wide with \( \sigma = 1.5 S/m \) and \( \varepsilon = 50\varepsilon_o \). The outer layers were chosen to be 0.0125 m wide with \( \sigma = 0.2S/m \) and \( \varepsilon = 50\varepsilon_o \). The rf profiles which were generated from these two models (Figure 14.10) show an excellent agreement between the finite element and the analytic solution. The deviations between the profiles of the finite element method and the planar model are reduced compared to the previous case.

Table 14.1 also gives some additional comparison of the finite element model to the multilayer model with the surrounding medium to be air and denotes the percentage error for any point of the rf profile for three different frequencies \( \omega = 42 \text{ M Hz}, 64 \text{ M Hz} \) and 170 M Hz, respectively. Table 14.2 indicates additional comparisons of the finite element model with the multilayer model which is surrounded by a dielectric material and denotes the percentage error for any point of the rf profile for three different frequencies \( \omega = 42 \text{ M Hz}, 64 \text{ M Hz} \) and 170 M Hz, respectively. We have also observed
<table>
<thead>
<tr>
<th>model</th>
<th>$\sigma = 0.0 S/m$</th>
<th>$\sigma = 0.5 S/m$</th>
<th>$\sigma = 1.5 S/m$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\epsilon = 58\epsilon_0$</td>
<td>$\epsilon = 58\epsilon_0$</td>
<td>$\epsilon = 58\epsilon_0$</td>
</tr>
<tr>
<td></td>
<td>$n=3400$</td>
<td>$n=3400$</td>
<td>$n=3400$</td>
</tr>
<tr>
<td>$\omega=42 MHz$</td>
<td>0.03%-0.6%</td>
<td>0.03%-0.1%</td>
<td>0.01%-0.7%</td>
</tr>
<tr>
<td>$\omega=64 MHz$</td>
<td>0.1%-2.8%</td>
<td>0.01%-2.9%</td>
<td>0.001%-2.9%</td>
</tr>
<tr>
<td>$\omega=170 MHz$</td>
<td>0.9%-15%</td>
<td>0.6%-12%</td>
<td>0.6%-10%</td>
</tr>
</tbody>
</table>

Table 14.1: Relative error of finite element calculation versus analytic solution for a three-layer model, where the outer boundary layers are air ($\sigma = 0 S/m$, $\epsilon = \epsilon_0$). The percentage range illustrated in this table represents a point by point comparison of the $B_1$ profiles between the finite element program and the planar model. The mesh of the elements in the three directions are $n_x = 34$, $n_y = 10$ and $n_z = 10$, with $n = n_x n_y n_z$.

that as the number of nodes is increased, the reduction in the error is significant and proportional to $O(h/r)^2$, in accord with the discussion given by Mur [73].

14.4 Discussion

The propagation of the rf field inside a dielectric object depends strongly on the values of the conductivity and permittivity of the object as well as on those of the surrounding medium. Objects with large values of conductivity and/or permittivity produce substantial non-uniformities of the rf field.
<table>
<thead>
<tr>
<th>model</th>
<th>$\sigma = 0.0 S/m$</th>
<th>$\sigma = 0.5 S/m$</th>
<th>$\sigma = 1.5 S/m$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\epsilon = 58 \epsilon_0$</td>
<td>$\epsilon = 58 \epsilon_0$</td>
<td>$\epsilon = 58 \epsilon_0$</td>
</tr>
<tr>
<td></td>
<td>$n=3400$</td>
<td>$n=3400$</td>
<td>$n=8704$</td>
</tr>
<tr>
<td>$\omega = 42 MHz$</td>
<td>0.03%-0.45%</td>
<td>0.03%-0.15%</td>
<td>0.01%-0.96%</td>
</tr>
<tr>
<td>$\omega = 64 MHz$</td>
<td>0.1%-3.0%</td>
<td>0.01%-3.6%</td>
<td>0.001%-3.6%</td>
</tr>
<tr>
<td>$\omega = 170 MHz$</td>
<td>0.9%-10%</td>
<td>0.58%-12%</td>
<td>0.6%-11%</td>
</tr>
</tbody>
</table>

Table 14.2: Relative error of finite element calculation where the surrounding is a conducting dielectric with $\sigma = 0.2 S/m$ and $\epsilon = 50 \epsilon_0$ versus the analytic solution for a three-layer model. The percentage error range illustrated in this table represents a point by point comparison of the $B_i$ profiles between the finite element program and the planar model. The size of the mesh of cubes in the three directions is $n_x = 34$, $n_y = 10$ (except in the last case of 170 $MHz$ where $n_y = 16$), and $n_z = 10$ (except in the last case of 170 $MHz$ where $n_z = 16$).

in phase and magnitude which generates artifacts for some sequences [13].

Imaging at high fields up to 170 $MHz$ for human studies and 500 $MHz$ for animal studies is a desired goal because of the improvement in signal-to-noise. Knowing the rf profile inside the dielectric object will help the design of new rf pulses which can correct this effect. Furthermore, power deposition and the evaluation of the local SAR also play an important role in designing sequences for human imaging at high fields. The transition from 64 $MHz$ to 170 $MHz$ imaging will result in the increase of the power deposited from the
rf by a factor of 7 which will make some fast sequences inappropriate for human imaging. In the results section, we have presented rf profiles for various values of the rf frequency, the conductivity of the object, and its permittivity; we have shown the effects that each has on the rf field profile, individually or in some combination. We have considered not only the spherical and cylindrical models, but also the planar model. The latter, although less realized practically, plays a key role in that accessible analytical solutions are used to compare the finite element calculation for a heterogeneous object.

In the beginning, we considered the 3D finite element analysis free of any gauge conditions. The lack of any gauge condition resulted in a significant error of the rf field profile compared to the analytical prediction of the planar model. Furthermore, the phase behavior of the rf profile was oscillating and was not stable. Similar results, which support our observations have also been reported recently [72]. With the introduction of the Coulomb gauge condition for the vector potential as a constraint in the Lagrangian, we are able to obtain the correct phase of the rf field and to generate results for different values of $\sigma$ and $\epsilon$. The introduction of this constraint will be helpful
in the future for the design of the second-order tetrahedron elements because it ensures the uniqueness of the solution of the rf field profile.

The behavior of the standing wave, which is generated by an rf coil, inside a dielectric environment can be understood in part from the contributions of the conductivity and permittivity to the wavenumber \( k \). When there is no conductivity present in the object, then the wavenumber is a real quantity that results in the field oscillating inside the object with a quarter-wavelength \( \lambda \). Recall that \( \lambda \) is inversely proportional to the rf frequency and the square root of the relative permittivity [13]. When the conductivity is present, the wavenumber is a complex quantity. The presence of high conductivity will generate an exponential damping of the field inside the object with penetration depth \( \delta \)

\[
\delta = \frac{1}{2\sqrt{f\sigma}} \text{ meters} \tag{14.40}
\]

At 170 MHz and for a conductivity \( \sigma = 1.55 S/m \), the penetration depth \( \delta \) evaluated from Eq. (14.40) is equal to 0.0313 m. This value is in very good agreement with the calculated 0.0309 m from the rf field profile which is
described by the finite element program in Figure 14.10.

The influence on the rf profile from the combination of the effects of the conductivity and permittivity has been previously presented [37]. Thus, there is a combination of the conductivity and permittivity that depends on the frequency and the model (planar, cylindrical, spherical) which eliminates second-order effects to the power deposited and the generated rf field profile is nearly flat. For the planar model, the relationship between the two quantities σ and ε is given by the formula

$$\sigma^2 = \frac{84}{17} \frac{\varepsilon_r}{\mu_0 R^2 c^2}$$

(14.41)

where \( \varepsilon_r \) is the relative permittivity \( \varepsilon/\varepsilon_0 \), \( \mu_0 \) is the magnetic permeability, \( c \) is the velocity of light and \( R \) the thickness of the dielectric object. If we consider the values of \( \sigma \) and \( \varepsilon \) for the inner layer at 64 MHz (illustrated in Figure 14.7 with thickness of 0.08 m), the value of the conductivity that generates a flat rf profile should be 0.56 S/m for \( \varepsilon_r = 58 \). Figure 14.7 displays a nearly flat rf profile with \( \sigma = 0.5 S/m \) and \( \varepsilon_r = 58 \). Thus, there is reasonably good agreement between the predicted value of the conductivity and its input value.
in the finite element program.

The deviation that is observed between the finite element approach and the planar model in Figure 14.8 is because the conductivity of the inner layer is an order of magnitude larger than the outer layers. A more condensed mesh of elements at the interface of the layers will reduce the rapid change of the phase oscillation of the vector potential and will simulate better the continuity of the real and imaginary components of the vector potential. Because of the structure of the 3D finite element method, an additional increase of the number of elements in the other two directions will result in a significant increase of the CPU time but it will make the agreement between the finite element and the planar model better.

At 170 MHz and when the surrounding medium is filled by air the deviation between the finite element and the planar model is more drastic as Table 1 indicates. The reasons for this behavior are the following. First, in the finite element approach the object is bounded in all three directions instead of one direction which is the assumption in the planar model. Second, for the higher frequencies, the variation of the phase factor inside the dielectric
object is more drastic, while the phase of the rf field in the surrounding air
layers is zero and constant. To account for this very nonlinear behavior, we
need to introduce a large number of elements, so the fraction of the volume
which each tetrahedral occupies is reduced, and the assumption of the linear
variation of the phase of the vector potential across their surfaces and the
boundaries is more accurate.

At high frequencies (170 MHz) with the heterogeneous model (Figure
14.9), where the outside layers have conductivity values of the same order
of magnitude as the inside layer, both the finite element and the planar
model predict an oscillation of the rf profile inside the dielectric object. In
this case, even though conductivity effects are present, they are overpowered
by permittivity effects that produce the oscillations. Figure 14.9 clearly
demonstrates the combination of the two effects and is a good example of
the power behind this finite element approach. The first effect, as mentioned
previously, is the damping of the rf profile due to the conductivity with large
penetration length δ. The second effect is the oscillation in space of the rf field
with half-wavelength of 0.1 m which is in good agreement with the analytical
prediction of 0.115 m even at 170 MHz. The small difference is because of the presence of the second dielectric layer in the finite element program that has not been taken into account in the analytical evaluation. The difference in the profiles observed in Figure 14.9 away from the central layer is due to the finite element model being bounded from all three directions instead of the planar model which is restricted only in the z direction. For rf in the range we have studied, the error between the analytical solution and the finite element method is no greater than 10%, comparing well with other finite element designs [73].

Furthermore, we note a large damping of the rf field profile in Figure 14.10. Drawing upon the above explanation, it can be understood as follows. Since the conductivity for the inner layer is an order of magnitude larger than those of the outside layers, the frequency of the wave is high, and the thickness of the inner layer is large compared to the outside layers. This results in the total domination of the rf profile effects from the properties of the inner layer. The effect of the outer layers in the rf field profile is almost negligible. The rf electric field is linearly dependent on the rf frequency ω to
first approximation and on the thickness of the object. Thus, an increase in
the frequency by almost a factor of 3 with respect to the case in Figure 14.8,
combined with the simultaneous increase in the thickness of the object with
the higher conductivity by a factor of 2, results in the observed behavior of
the rf field profile.

The 3D finite element method can give reasonably accurate and definitive
predictions for the behavior of the rf profile inside heterogeneous dielectric
objects of rather general shape. It therefore will provide local SAR for the
human body in the important region of 1.5 \( T \) to 4.0\( T \). It will also prove
useful in predicting rf penetration effects on imaging at higher fields [57, 74].
We are currently evaluating the program in a general 3D model to study the
effects for both the head and abdomen.
Figure 14.4: Comparison of magnetic field profiles between the cylindrical model and the finite element model for a cylinder at frequency $\omega = 170\, MHz$. The diameter of the inner cylinder is 0.15 m with $\sigma = 0.68S/m$ and $\epsilon = 58\epsilon_0$. The outer hollow cylinder is 0.0125 m thick with $\sigma = 0.05S/m$ and $\epsilon = 57\epsilon_0$. The dashed line from Foo et al. [66] is experimental data for an object of width 0.128 m. These profiles are normalized so that $B_{rf}$ is unity at the center.
Figure 14.5: Comparison of magnetic field profiles between the finite element model (solid line) and the analytic spherical model (centered line) at frequency \( \omega = 42 \, MHz \). The diameter of the sphere is 0.2 m with \( \sigma = 0.0 \) and \( \epsilon = 58\epsilon_0 \). The surrounded medium is air. Additional comparisons are shown for the \( \hat{y} \) and \( \hat{z} \) directions for the finite element model (dashed line) and the analytic spherical model (dotted line). These profiles are normalized so that \( B_{rf} \) is unity at the outer boundary.
Figure 14.6: Comparison of the magnetic field $B_1$ profiles between the finite element and the planar model at a frequency $\omega = 42 \, MHz$. The dielectric properties of the object are $\sigma = 1.5 S/m$, and $\epsilon = 58 \, \epsilon_0$. The 3D object is surrounded by air. The thickness of the air (outer) layers are 0.065 m each. The inner (the object) layer is 0.12 m wide. The surrounding medium is air. These profiles are normalized so that $B_{r,f}$ is unity at the outer boundary.
Figure 14.7: Comparison of the magnetic field $B_1$ profiles between the finite element and the planar model at a frequency $\omega = 64\ MHz$. The dielectric properties of the object are $\sigma = 0.5 S/m$, and $\epsilon = 58 \ \epsilon_0$. The 3D object is surrounded by air. The thickness of the air (outer) layers are 0.06 m each. The inner (the object) layer is 0.08 m wide. The profile is almost flat, showing that imaging can be excellent for appropriate choices of $\sigma$ and $\epsilon$. These profiles are normalized so that $B_{ref}$ is unity at the outer boundary.
Figure 14.8: Comparison of the magnetic field $B_1$ profiles between the finite element and the planar model at a frequency $\omega = 64 \text{ MHz}$. The inner layer is $0.1 \text{ m}$ wide and its dielectric properties are $\sigma = 1.5 S/m$, and $\epsilon = 58 \epsilon_0$. The 3D object is surrounded by a dielectric material which is $0.05 \text{ m}$ thick, with $\sigma = 0.2 S/m$, and $\epsilon = 50 \epsilon_0$. These profiles are normalized so that $B_{rf}$ is unity at the outer boundary.
Figure 14.9: Comparison of the magnetic field $B_1$ profiles between the finite element and the planar model at a frequency $\omega = 170 \text{ MHz}$. The dielectric properties of the object are $\sigma = 0.55 S/m$, and $\epsilon = 58 \epsilon_0$ and is $0.14$ m wide. The 3D object is surrounded by a dielectric material which is $0.03$ m thick and with $\sigma = 0.2 S/m$, and $\epsilon = 50 \epsilon_0$. These profiles are normalized so that $B_{rf}$ is unity at the outer boundary.
Figure 14.10: Comparison of the magnetic field $B_1$ profiles between the finite element and the planar model at a frequency $\omega = 170 \text{ MHz}$. The dielectric properties of the object are $\sigma = 1.5 S/m$, and $\varepsilon = 58 \varepsilon_0$. The 3D object is surrounded by a dielectric material with values $\sigma = 0.2 S/m$, and $\varepsilon = 50 \varepsilon_0$. The thickness of the inner layer is 0.175 m and for the outer layers 0.0125 m. These profiles are normalized so that $B_{rf}$ is unity at the outer boundary.
Chapter 15
Conclusions

In this dissertation we have dealt with the three major areas of MRI. Specifically, we have presented novel approaches which can be used for main magnet design, we have introduced new geometrical shapes and design ideas for gradient coils, and finally we have studied the effects of rf penetration and power deposition inside a conducting dielectric object.

For main magnets, we have considered two different designs. For both variational techniques have been employed, using the derivatives of the magnetic field as constraints instead of its value inside the imaging volume. We have noticed that in order to eliminate up to the 10th axial derivative of the magnetic field, configurations where the current density is concentrated in
three major areas are needed. These patterns will generate a magnetic field with an inhomogeneity of 20 ppm over a 48 cm DSV. With the introduction of an extra derivative constraint, eliminating up to the 12th axial derivative of the field, we were able to reduce the level of the inhomogeneity of the magnetic field to 7 ppm inside a 50 cm DSV. Although the second design improves the behavior of the magnetic field inside the imaging volume, there is a concern for the behavior of the magnetic field at the exterior space of the coil. Even though the magnetic field outside the coil behaves like a dipole, it can maintain significant strength at distances near the main magnet coil. For this reason, we have introduced a second cylindrical coil which is placed outside the primary coil, in order to reduce the value of the resultant magnetic field in the area outside both coils. In this situation the homogeneity of the magnetic field inside the imaging volume remains the same, while the magnetic field drops to a value of 5 Gauss at a radial distance of 2.5 m away from the center of the coil system. To generate this behavior the current of the secondary coil has two major peaks, one with a negative sign and the other with a positive sign. A similar observation was made by Thompson [5].
In the second part of this thesis, we have presented different novel geometries for the design of various types of gradient coils. These designs include elliptically shaped gradient coils, self-shielded finite gradient coils, finite cylindrical gradient coils where the sweet spot is shifted axially towards the end of the cylinder, and finite self-shielded cylindrical gradient coils which generate a linearly varying component of the magnetic field, other than its $z$ component.

For the elliptical coil design, we have presented two axial gradient coils with elliptical cross section. These two coils were compared with a cylindrical gradient coil with a $55\text{ cm}$ diameter. Employing the same number of constraints inside identical imaging volumes and demanding the same quality of the magnetic field, the reduction in the energy for the $55\text{ cm} \times 40\text{ cm}$ ellipse compared with the cylinder was on the order of $37\%$ for a three constraint set, rather greater than the $27\%$ reduction from the volume alone. We then proceeded with the mathematical development of the $x$ transverse elliptical gradient coil. The mathematical formalism is different in this case since the current density must be decomposed into two different components. The
expressions of the stored magnetic energy as well as for the current density indicate the presence of a double summation over coefficients of Mathieu functions. This creates an additional problem with the design of the transverse elliptical coil, since extra computer power is needed so that the evaluation of the double summation can be done in a reasonable amount of time. At the present time this is not feasible.

Up to this point, we have presented gradient coil geometries with no restriction to the total length of the cylinder or ellipse. In chapter 7, we have presented a self-shielded design of symmetric gradient coils with a restricted length. For the design of the self-shielded axial finite gradient coil, only the total length of the inner cylinder was restricted, while the length of the outside coil was left unrestricted. We have evaluated the total energy of the coils system as well as its total inductance. A projection for the rise time \( t_r \), for these coils, assuming that the voltage for the gradient amplifier is 150 Volts, is \( t_r = 32.78 \mu \text{sec} \). Using the discretized version for the current density and applying the Biot-Savart law, the calculated strength of the gradient field was 13.5% larger than its ideal value. The on-axis linearity of the magnetic
field was increased to 5% while its off-axis uniformity was 12.85% compared
with its ideal value at the constraint points. For the transverse (x) finite
self-shielded gradient coil the results are in very good agreement with the
prescribed constraint values of the gradient field. Using the stream function
technique, we have obtained the discretized version of the current density
and we have used the Biot-Savart law to re-evaluate the magnetic field. The
difference in the strength of the magnetic field between its ideal value and the
value which was obtained by the discrete current distribution was 2.5%. The
on-axis uniformity of the field was different by 12% for its ideal value, while
its off-axis uniformity was deviated by 10.5%. Overall this design generates
a complete set of gradient coils with the desired quality characteristics for
the gradient field.

In chapter 8, we have revolutionized the design concept of the gradient
coils. We have presented a novel geometry, where the sweet spot of the gradi-
ent coil has been shifted axially towards the end of a finite length cylindrical
coil. Both axial and transverse gradient coils have been generated. For the
axial asymmetric coil, the strength of the resulting magnetic field has been
evaluated to be 43.25 mT/m which corresponds to an 8% increase over its ideal value, while no significant discrepancy was observed for the on-axis linearity and off-axis uniformity of the magnetic field. The distance between the two rollover points was measured to be 0.35 m, thus aliasing of the neck back to the head is avoided. Furthermore, the deviation of the local gradient between any two points inside the DSV was no greater than 10.5%. Actually, these values were calculated between two points which were at the borders of the imaging volume. For the transverse asymmetric coil, we have presented the discrete version of the continuous current density and we have proceeded with the evaluation of the magnetic field, using the Biot-Savart law. The gradient strength deviated by 0.75% from its ideal value, while no significant variation from the constraint values was observed for the on-axis linearity and off-axis uniformity of the magnetic field. Furthermore, a complete set of the asymmetric gradient coils has been constructed and tested at Picker International. We have shown an excellent agreement between the theoretical predictions and the experimental results. Finally, imaging was performed using this set of gradient coils inserted in the MRI imaging unit. The results
for the imaging experiment are very encouraging for the further development of the entire gradient coil set.

Although the design of the non-shielded asymmetric gradient coil opens new horizons to a generation of novel gradient coil geometries, it also creates additional problems. The first is the appearance of eddy current effects in the main magnet shield, and the second is the appearance of torque effects, due to the interaction between the asymmetric current density and the $z$ component of the magnetic field of the main magnet. In order to eliminate these two problems, we have introduced the design of the self-shielded, finite, cylindrical gradient coils. Since for the axial case, we have shown that there are no torque effects present, the introduction of the self-shielded design eliminates the eddy current effects with an additional 19.2% increase of the stored energy compared to the non-shielded design. For the transverse asymmetric gradient coil, we have shown that the self-shielded design first eliminates the eddy current effects to the main magnet shield, and second reduces the total torque of the coil system by 75% from its original value, depending on the relationship between the radii of the inner and outer coils.
The energy increase for the self-shielded transverse asymmetric gradient coil compared with the non-shielded case was on the order of 33%. Both these designs maintained the quality of the gradient field inside the imaging volume which was achieved with the non-shielded gradient coils.

Continuing the development of novel geometries for the gradient coils, we have introduced a complete set of self-shielded finite cylindrical gradient coils which are adequate for imaging the human wrist. The new characteristic in this design is that these coils are able to generate a linearly varying $x$ component of the magnetic field along the three gradient axes instead of the $z$ component. Therefore, the design of each of the three gradient coils must be treated independently, because the rotation symmetry which was applied to the conventional gradient coils is not valid any more. For the $x$ wrist gradient coil, we have presented the theoretical development and the results which are necessary for generating a linearly varying $x$ component of the gradient field along the $x$ axis. The discrete current patterns for the inner and outer coils were generated and the magnetic field was re-evaluated using the Biot-Savart law. The strength of the gradient field was raised to
44\text{mT/m} which corresponds to a 10% increase from its ideal value, while the on-axis linearity and off-axis uniformity of the magnetic field inside the DSV deviate from their ideal values by 5.25% and 5%, respectively. If we consider a gradient amplifier with a supplied voltage of 150\text{Volts}, the projected rise time \( t_r \) for this gradient coil and for a strength of 40 \text{mT/m} is \( t_r = 5.19 \text{\mu sec} \). For the \( y \) wrist gradient coil, we have presented the theoretical development, and the results for generating a linearly varying \( x \) component of the magnetic field along the \( y \) axis of the gradient coil. We have obtained the discrete version of the current density using the stream function technique for both inner and outer coils, which in succession was used to re-evaluate the magnetic field using the Biot-Savart law. The strength of the gradient field was 44 \text{mT/m} which corresponds to a 11.5% increase with respect to its ideal strength. No significant deviations from the constraint values were noticed for the on-axis linearity and off-axis uniformity of the magnetic field. Using a gradient amplifier with the same voltage as in the \( x \) wrist gradient coil, the estimated rise time for this gradient coil is \( t_r = 5.56 \text{\mu sec} \), which is compatible with the rise time of the \( x \) wrist gradient coil. The last gradient coil design which is
presented in this thesis is the z wrist gradient coil. For this coil, we have also presented the mathematical formalism, and the results necessary to generate a linearly varying $x$ component of the magnetic field along the $z$ axis of the gradient coil. Using the stream function technique, we have generated the discrete current patterns which have helped us to re-evaluate the magnetic field using the Biot-Savart law. No significant changes between the calculated and the constrained values of the magnetic field were observed. Finally, the rise time of this gradient coil for a strength of $40mT/m$ was estimated to be $t_r = 24.72\, \mu sec$.

We have also studied the effects of the rf magnetic field inside a conducting dielectric object. We have presented a brief description for the rf penetration and power deposition inside a 1D planar dielectric object. We have also discussed the behavior of the rf field inside the dielectric object at the extreme cases of either high conductivity (conduction current limit) or high permittivity (displacement current limit). Expanding the expressions of the rf field and the absorbed power to the fourth power of the frequency $\omega$, we have obtained the appropriate solutions for the power deposited for the com-
monly used models: the planar, the cylindrical, and the spherical. We have also shown that there exists an important solution between the conductivity and the permittivity for all three models. In this situation, the $B_1$ profile is nearly flat and the rf power is minimized. It turns out that at a 64 $MHz$ window, the physiologic values of the $\sigma$ and $\epsilon$ for the human body lie in the region which leads to the elimination of the $\omega^4$ term in the expression of the rf power deposited. Furthermore, this relationship between the conductivity and permittivity is also frequency independent. Using the expansion to the fourth order of $\omega$ for the magnetic field profile and for a small tip angle, we can work backwards and extract the values of $\sigma$ and $\epsilon$ using information which is obtained from the MRI data. We have applied this method in a 1D layer phantom which was doped with NaCl. The agreement between the actual value of the conductivity and the extracted one is very good (within a standard deviation). We have expanded this methodology to a multilayer symmetric planar model (leg model) and to a multilayer asymmetric planar model (head model). Employing a least square fit to a computer generated rf profile with noise levels up to 5% of its magnitude, we were able to extract
the values of the conductivity and permittivity for all layers and for both models. The agreement between the input and the extracted values for $\sigma$ and $\epsilon$ was very good, although the number of pixels which was used to design these layers was small. This method is also very stable as indicated in chapter 13 and potentially can be used to extract the values of $\sigma$ and $\epsilon$ from an rf profile in vivo.

Although the planar, cylindrical and spherical models provide valuable information for the behavior of the rf field inside a conducting dielectric object, the understanding of the behavior of the rf field inside the human torso is very difficult, since the symmetry of the human body does not coincide with any of the three previous models. Thus, we have developed a 3D Finite Element Method (FEM) which attempts to describe the rf behavior inside the human body and uses the Coulomb gauge condition of the vector potential as a constraint. The agreement between the 3D FEM and the multilayer planar heterogeneous model is very good even at 170 MHz. We have also compared the 3D FEM with experimental results obtained from a heterogeneous cylindrical model at 170 MHz (Foo et al. [66]). As it was indicated
in chapter 14, we have excellent agreement between the results generated by the FEM and the experimental results obtained by Foo et al.
Bibliography


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