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ANALYSIS AND SYNTHESIS TOOLS

FOR A CLASS OF ACTUATOR-LIMITED

MULTIVARIABLE CONTROL SYSTEMS

By

VINCENT R. MARCOPOLI

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

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DEPARTMENT OF ELECTRICAL ENGINEERING AND APPLIED PHYSICS

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GRADUATE STUDIES

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Vincent Marcopoli
ANALYSIS AND SYNTHESIS TOOLS FOR A CLASS OF ACTUATOR-LIMITED MULTIVARIABLE CONTROL SYSTEMS

Abstract

By

VINCENT R. MARCOPOLI

It is well known that actuator limits can introduce severe performance degradation and possible instability in control systems. Thus some form of limit protection is required in practical control problems. Though there exist schemes which successfully address this problem in single-input, single-output systems, there are comparatively few methods for multi-input multi-output systems. This work investigates a class of actuator-limited multivariable control systems proposed in the literature, characterized by a static gain limit protection matrix parameter, \( \Lambda \). The application of traditional high-gain antiwindup methods to such multivariable systems is first considered. It is shown that the issue of plant directionality
in saturated systems can be addressed by appropriate treatment of the nonlimited actuators during saturation. However, limitations on the applicability of the simple directionality philosophy motivate the study of a more general approach to limit protection design. To this end, the design of limit protection is cast as a nonlinear robust control design problem. Recent results from the study of linear matrix inequalities provide analysis tools which give nonlinear measures of stability and performance. These tools are shown to give rise to an iterative approach to the synthesis of $\Lambda$. This design method is seen to agree very closely with the earlier directionality approach, indicating that directionality issues are considered transparently. Furthermore, successful limit protection is achieved for a more complex F8 aircraft H-infinity control design, for which the simple directionality approach is not applicable. As a result of this work, it is seen that effective multivariable limit protection is indeed achievable via the static parameter, $\Lambda$. Finally, a surprising observation is made: High-gain antiwindup behavior is not necessary for successful multivariable limit protection.
To Mom, Dad, Angela, and Lisa
I asked God for strength
    that I might achieve...
I was made weak,
    that I might learn humbly to obey.
I asked for health,
    that I might do greater things ...
I was given infirmity,
    that I might do better things.
I asked for riches,
    that I might be happy ... 
I was given poverty,
    that I might be wise.
I asked for power,
    that I might have the praise of men ...
I was given weakness,
    that I might feel the need of God.
I asked for all things,
    that I might enjoy life ...
I was given life,
    that I might enjoy all things.
I got nothing that I asked for,
    but everything I had hoped for.
Almost despite myself,
    my unspoken prayers were answered.
I am among all men, most richly blessed!

—Anonymous
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Chapter 1

Introduction

Nearly all practical control problems involve plants whose actuators are limited by inherent physical constraints. Examples include constraints on valve openings in chemical process control; motor current in servomechanism control; flight control surface angle deflections, engine nozzle openings, and fuel flows in aerospace systems. This research has been specifically motivated by work at NASA Lewis Research Center on the integrated flight and propulsion control of a Short Take-Off, Vertical Landing (STOVL) aircraft [GOLM93, Gar93]. This application presents a particularly challenging control problem, especially with respect to the transition to hover landing mode of operation. This is due to strong multivariable coupling between the flight and propulsion systems during an approach to hover landing. It is in this operating condition that propulsion system safety limits are typically encountered. Thus the focus of this research is on the accommodation of actuator limits in highly-coupled linear, time-invariant multivariable control systems, with particular emphasis on systems having properties similar to the NASA STOVL system.

A key feature of this work is the manner in which the multivariable nature of
the actuator limits is handled. Specifically, the control vector will be considered as consisting of two (vector) components, \( u_{nl} \) and \( u_l \), which represent the non-limited and \( \ell \) limited actuators, respectively. The actuator limits are modeled by a multivariable limiting nonlinearity, \( N \), operating on \( u_l \). This limited system representation is depicted in Figure 1.1, where \( G \) and \( K \) represent the plant and controller transfer functions, respectively. This work will consider a widely studied class of limiting nonlinearities, namely, those which are decoupled, sector-bounded, and time-varying. More precisely, the multivariable nonlinearity, \( N \), is characterized as follows:

\[
N(u_l, t) = \begin{bmatrix}
    n_1(u_{l,1}, t) \\
    n_2(u_{l,2}, t) \\
    \vdots \\
    n_\ell(u_{l,\ell}, t)
\end{bmatrix}
\]

where \( n_j \) belongs to the sector \([0, 1]\) i.e.,

\[
0 \leq u n_j(u, t) \leq u^2, \quad j = 1, \ldots, \ell.
\]

The sector requirement on \( n_i \) is depicted graphically by requiring \(|u, n_j[u, t]|\) to lie in the shaded region shown in Figure 1.2. A commonly used example of a limit
nonlinearity is the saturation nonlinearity, which has the additional property of being \textit{memoryless}, i.e., it can be represented by an input-output characteristic. The transfer characteristic of the saturation nonlinearity is also shown in Figure 1.2. The sector bound requirement restricts this characteristic to lie entirely within the shaded region.

The effect of a limiting nonlinearity on system behavior can be quite significant, especially in the multi-input, multi-output (MIMO) situation. A natural starting point for this investigation is to consider systems containing a \textit{single} limited actuator, i.e., $\ell = 1$. This restriction will be imposed on all the examples studied in this work, although the design techniques proposed in Chapter 3 apply for any $\ell$. To illustrate the pathological behavior that actuator limits can introduce in control systems, as well as the methods proposed in the sequel to alleviate such problems, consider the following two example control systems, taken from [DSE87], where
$N$ is a unit saturation nonlinearity. To simplify the determination of stability for these systems, an additional parameter, $\epsilon = 0.001$, has been introduced in the plant description.

**System 1**

$$G = \frac{s + 0.1}{2(s + \epsilon)}, \quad K = \frac{2}{s + 0.1}$$

This linear single-input, single-output (SISO) control design, based on inversion of a portion of the plant dynamics, provides the closed-loop transfer function $H_{cr} = \frac{s + \epsilon}{s + 1 + \epsilon}$. The linear response to a unit step reference command is shown in Figure 1.3, where the plant output, $y_p$, and the control signal, $u$, are shown. The response of the system with $u_l = u$ limited at 1 is shown in Figure 1.4, where it is seen that the plant output now overshoots the desired steady-state value. Note that the controller command, $u$, is significantly different from the true actuator input during limiting conditions. The signal discrepancy produces the overshoot in $y$. This controller behavior is commonly known as “windup.” Methods of alleviating such behavior will be discussed in Chapter 2.

**System 2**

$$G = \frac{4(s + 0.1)}{s + \epsilon} \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}, \quad K = \frac{1}{4(s + 0.1)} \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$$

As in System 1, this is an inverse-model based control system. However, the inversion now involves multivariable coupling. The closed-loop linear system possesses the decoupled transfer function $H_{cr} = \frac{s + \epsilon}{s + 1 + \epsilon}$. The nonlimited response to a step
reference command \( r_{wc} = \begin{bmatrix} 0.6247 & 0.7809 \end{bmatrix}^T \) (the reason for the "wc" notation will be seen in Chapter 2) is shown in Figure 1.5, where \( y_{p1}, y_{p2} \) are depicted in the left plots, and \( u_1, u_2 \) are shown in the right plots. The output responses for \( u_i = u_1 \) limited at 1 are shown in Figure 1.6. The control signal, \( u_1 \) (dotted line), is again seen to be very different from the true actuator command during limited conditions. Comparing qualitatively Figure 1.6 with Figure 1.4 illustrates that saturation in MIMO systems can cause even more troublesome behavior than in the SISO case.

The above examples illustrate that actuator limits can significantly degrade system behavior with respect to the nominal design. This observation motivates the well-known fact that the implementation of practical control systems requires some means of explicitly accommodating actuator limits. A trivial way to accomplish this is to design the controller so that its actuator commands never violate the limits for any expected reference commands. While this approach avoids the problems that limits introduce into the system, the conservatism necessary to accomplish this in the control design often sacrifices performance in the nonlimited system.

A commonly used alternative approach employs a two-step design procedure. First, a linear control design is performed to yield a closed-loop system having desirable properties in the absence of actuator limits. Many effective methods
of linear control design for MIMO systems have been and continue to be developed. Any practical control design will include specifications on actuator effort. In contrast to the conservative approach of the previous paragraph, a good design will use as much of the available actuator effort as possible, assuming “typical” reference command inputs, in order to maximize performance capability. In this step, even though actuator limits are considered in the design, stability and performance are evaluated only for linear operation. In any linear control design, however, there is always the potential of violating actuator limits given a particularly large or difficult reference command.

The second step in this approach thus deals with the possibility of encounters with actuator limits. As was seen for Systems 1 and 2, limit violations occurring in the nominal system can be very troublesome. Furthermore, it is well known that limits can even destabilize the closed-loop system. Thus the potential problems introduced by limits can range from nuisances to catastrophes. In order to combat such adverse behavior in the presence of limits, the second step involves the introduction of limit protection. Limit protection refers to additional compensation “wrapped around” the nominal controller, in order to provide desired stability and performance properties considering the true nonlinear nature of the actuators. An attractive feature of this philosophy is that it does not require modification of the nominal control design. Thus limit protection can be used to augment any existing control system.
The idea that limit protection should provide desirable performance properties in addition to stability is commonly referred to in the literature as graceful performance degradation. This concept of performance acknowledges the fact that during limited conditions, some loss of performance must be accepted. The goal of limit protection is to ensure that performance degradation appears in a well-behaved manner. This notion will be made more precise in the sequel.

The focus of this work is on the second step of the two-step design procedure, i.e., the design of limit protection. The framework adopted here to achieve limit protection is now described, defining the particular class of actuator-limited multivariable control systems considered in the sequel.

1.1 Limit Protection Framework

The framework considered here for the implementation of limit protection was introduced in [CMN89]. Figure 1.7 illustrates how this approach modifies the original closed loop system of Figure 1.1, and is described as follows. The state space description of the nominal controller is specified by the strictly proper realization \((A_c, B_c, C_c)\), giving rise to the following modified controller description:

\[
\dot{x} = A_c x + B_c e + u_A,
\]
\[
u_{nl} = C_{c,nl} x,
\]
\[
u_l = C_{c,l} x.
\]
Though the restriction to a strictly proper nominal controller is not necessary in the sequel, the focus of this work is exclusively on such control structures. Controller modification takes place through the input $u_\Lambda = \Lambda y_\Lambda$, where $\Lambda$ is a static matrix gain parameter, and modifies the controller state if and only if limits are active. Note that the only if statement is due to the static $\Lambda$ restriction. Thus $\Lambda$ has the property of instantaneously modifying the closed loop from the nominal system when limits occur, and back to the nominal system when limits cease. This framework thus reduces the multivariable limit protection design problem to one of selecting $\Lambda$.

To facilitate the analysis, the limiting nonlinearity will be equivalently represented via the perturbation $\Delta = I - N$ [BB91]. More explicitly, $\Delta$ is expressed as

$$
\Delta(q, t) = \begin{bmatrix}
    \delta_1(q_1, t) \\
    \delta_2(q_2, t) \\
    \vdots \\
    \delta_\ell(q_\ell, t)
\end{bmatrix} = \begin{bmatrix}
    1 - n_1(q_1, t) \\
    1 - n_2(q_2, t) \\
    \vdots \\
    1 - n_\ell(q_\ell, t)
\end{bmatrix}.
$$

Note that $\Delta$ is also a decoupled nonlinear operator, with $\delta_j$, $j = 1, \ldots, \ell$, belonging to sector $[0, 1]$. Figure 1.7 is thus equivalently represented as shown in Figure 1.8, where the signals $q$ and $p$ are defined as the input and output, respectively, of $\Delta$. Finally, it is noted here that methods of selecting $\Lambda$ are proposed in the sequel based on closed-loop input-output properties. In order to accommodate such formulations in the current framework, the standard notation is adopted.
which defines an input signal, $w$, containing all exogenous inputs, and an output signal, $z$, containing all desired regulated outputs. By “pulling out” the signals $p, q, w, z, u_A,$ and $y_A$, the limited system is put into the standard form shown in Figure 1.9(a). The system $H^{\text{nom}}$ is thus determined by the nominal closed loop design. It will be helpful in the sequel to consider also the system shown in Figure 1.9(b), where $H^\lambda$ is defined as the $(H^{\text{nom}}, \Lambda)$ feedback combination, with inputs $p$ and $w$, and outputs $q$ and $z$.

Casting the limit protection implementation of Figure 1.7 in the standard form of Figure 1.9(b) puts the limit protection problem into a framework that has received much attention in the literature. Specifically, linear, time-invariant systems containing a nonlinearity in a feedback configuration have been widely studied and many results have been and continue to be developed for such systems. Some fundamental concepts are briefly summarized in the next section.

1.2 Technical Preliminaries

In order to discuss the important concepts of stability and performance of the limited system in a precise manner, some relevant technical formalisms are now presented. See [Vid93] for a thorough treatment of these and many other concepts in nonlinear system analysis.
Definition 1  
(i) The $L_2$-norm of a signal $x$ is defined as
\[
\|x\|_2 = \left( \int_0^\infty x^T(\tau)x(\tau)d\tau \right)^{1/2}.
\]

(ii) The $L_2$-gain of a (possibly nonlinear) system $H$ having input $w$ and output $z$ is defined as
\[
\gamma(H) = \sup_{w \neq 0, \|w\|_2 < \infty} \frac{\|z\|_2}{\|w\|_2}.
\]

(iii) The system $H$ is said to be $L_2$-stable if
\[
\gamma(H) < \infty.
\]

Physically, the $L_2$-norm of a signal is equivalent to the notion of signal energy. This signal measure is used to define a system measure via the $L_2$-gain, which specifies the maximum energy amplification of a system, over all possible input signals. Finally, an $L_2$-stable system has the interpretation that inputs of bounded energy give rise to outputs of bounded energy. Two special cases of the $L_2$-gain are of particular interest in this work, and are shown in the following simple Lemma.

Lemma 1

(i) If $H$ is a stable LTI system, then $\gamma(H) = \|H\|_\infty \overset{\Delta}{=} \sup_{\omega} \sigma [H(j\omega)]$.

(ii) If $H$ belongs to sector $[0, 1]$, then $\gamma(H) = 1$.

A critical issue in the analysis and design of any control system is the stability of the system interconnections. This assessment typically takes the form of determining appropriate sufficient conditions which must be satisfied for each component in the system. Two fundamental results, pioneered by Zames and others
(see [Zam66a, Zam66b] and the references therein) which establish such conditions are now presented which directly apply to the limited system formulation of Figure 1.9(b).

**Theorem 1** (Small Gain Theorem) *The system of Figure 1.9(b) is $L_2$-stable if*

$$\gamma(H_{qp}^\Delta)\gamma(\Delta) < 1,$$

*or, equivalently,*

$$\|H_{qp}^\Delta\|_\infty < 1.$$  

Unfortunately, Theorem 1 will prove to be excessively conservative for much of the current work. A less conservative approach is obtained by incorporating the sector properties of the limiting nonlinearity, $\Delta$. The following result is a special case of the circle criterion, applicable for the case of a single actuator limit, i.e. $\ell = 1$.

**Theorem 2** *Given that $\ell = 1$, the system of Figure 1.9(b) is $L_2$-stable if*

$$\text{Re}\{-H_{qp}^\Delta(j\omega)\} > -1.$$  

Theorem 2 therefore provides an easily checked stability condition in terms of the Nyquist diagram of the transfer function, $-H_{qp}^\Delta$. The application of this result can be shown to prove stability of both of the limited example systems introduced in this chapter, using no limit protection. All relevant Nyquist diagrams for the examples considered in this work are collected in Chapter 5, in order to graphically compare the various limit protection schemes studied in the sequel.
1.3 Summary

This chapter has introduced the problem of limit protection design for a class of actuator-limited control systems containing limit protection parameterized by a static gain parameter, $\Lambda$. A unique characteristic of the formulation presented here is the explicit distinction between the limited and nonlimited actuators. This feature will be exploited in Chapters 2 and 3 to provide new strategies for multivariable limit protection. Two classic analysis techniques, the small gain theorem and (a special case of) the circle theorem, were also presented to address stability of the complete nonlinear feedback system. The examples illustrate further that stability of limited systems does not in general imply desirable performance properties. Thus the current investigation must utilize additional considerations for alleviating such adverse system behavior.

1.4 Overview and Historical Perspective

To preview and further motivate this work, an overview of the sequel is now given in the context of a brief historical summary of some important developments in this area of research.

The adverse behavior introduced by actuator limits in control systems was first addressed in the context of PID control of SISO systems, and was observed to be due to the integration operation. Terms such as "integrator windup" and
“anti-reset windup” were used to describe the pathological controller behavior due to actuator limiting. Many ad-hoc procedures were developed to alleviate such behavior. More details about the early approaches to this problem can be found in the survey article [ÅR89], and the references therein. One such approach, known as “back-calculation,” provided a conceptual basis for a more general treatment of the problem. The fundamental property of this method is the feedback of the difference between the calculated control and the actual limited control applied to the plant [Wit89].

The back calculation concept was expanded upon by Åström to provide a framework applying to controllers having an observer structure, and later to the general state-space control structure [ÅW84]. This generalization is significant, since it applies equally well to both SISO and MIMO systems. This framework is in fact equivalent to that described in Section 1.1. However, a general design method did not emerge—pole placement of the controller dynamics was all that was suggested. While such a design approach can provide reasonable limit protection schemes for SISO systems, it does not generalize to the MIMO situation. Chapter 2 develops a design procedure for this multivariable case, based on the notion of maintaining a “critical” control direction during limited conditions. Directional considerations in saturated systems were introduced by Doyle et al. in [DSE87], where nonlinear methods were used to address this issue. Though these approaches lend much insight into the problem of multivariable limit protection,
their applicability is limited to systems having simple properties, and motivates the investigation into more general design methods.

Perhaps the two most significant design developments in the literature are due to Hanus et al. and Kapasouris et al. [HKH87, KAS88]. The technique of Hanus, called "controller conditioning," is also based on the concept of back-calculation, and applies to nominal control structures having a direct-feedthrough "$D$" term in the state-space realization. This technique has a very natural MIMO generalization, particularly for 2-DOF control structures [HK89]. Though this provides an excellent method of accommodating actuator limits, it is not applicable for the systems of interest in the current investigation, due to the necessity of the direct-feedthrough term in the controller realization.

Kapasouris et al. developed a general limit protection design method based on maintaining the directional properties of the control. Though this method does indeed provide graceful performance degradation for systems with strictly proper control structures, it requires rather involved nonlinear computations for its design and implementation. Since it has not been demonstrated that such nonlinear approaches are necessary for effective limit protection, it is of interest to examine the viability of the simpler linear approach of Section 1.1.

To this end, a major contribution was made by Campo et al. in [CMN89], where precise design goals were formulated for a general linear multivariable antiwindup problem, of which that given in Section 1.1 is a special case. The design goals
included closed-loop stability and performance considerations. This represented a significant departure from previous work, which focused solely on the open-loop controller dynamics during limits. Though this work introduced general design goals for the problem, no method existed to optimize such criteria. This is due to the multi-objective nature of the design goals which were proposed, as well as the formulation as a reduced-order output feedback design problem. Chapter 3 proposes the use of recently developed numerical optimization tools to address a reduced-order output feedback design problem for limit protection. These techniques are illustrated on two examples in Chapter 4. Finally, Chapter 5 concludes and proposes some possibilities for future research.
Figure 1.3: System 1 nominal responses

Figure 1.4: System 1 limited responses
Figure 1.5: System 2 nominal responses

Figure 1.6: System 2 limited responses
Figure 1.7: Linear multivariable limit protection implementation

Figure 1.8: Alternative representation of limit protection system

Figure 1.9: Standard form representations of limit protection system
Chapter 2

Plant Directionality and Antiwindup Methods

For LTI SISO systems, gain is a function of the input frequency. However, it is well known from the singular-value decomposition that the gain of an LTI MIMO system is a function of both the input frequency and direction. The notion of $L_2$-gain defined in Chapter 1 accounts for such directional variation via the supremum operation over all possible input signals. In comparison to the SISO case, this directional property of MIMO systems is an additional mechanism by which limit nonlinearities can degrade system performance. This observation was first made by Doyle et al. in [DSE87], and can be described geometrically as follows. Consider a two-input plant, whose “actuator space,” $u_2$ versus $u_1$, is shown in Figure 2.1. Actuator saturation limits are shown via the box. Consider a controller command, $u_c$, which violates the $u_1$ limit. The saturation operation will result in the application of $\hat{u}$ to the plant, which has a different direction than $u_c$. In order to avoid such a control direction modification, it was proposed in [DSE87] that during limits, the controller be nonlinearly modified to provide the control $\hat{u}(t) = \alpha u_c(t)$, where the scalar $0 < \alpha < 1$ is calculated such that $\hat{u}(t)$ is just inside the actuator constraints. This method was further investigated in [CM90], and
was shown to provide increased stability robustness to actuator limits. A slightly different approach, which scales the controller input was presented in [KAS88]. This nonlinear scaling philosophy was shown to provide desirable performance properties as well as stability in the limited MIMO systems which were studied.

The goal of this chapter is to address the directionality issue via the LTI limit protection parameter, Λ. To this end, Section 2.1 introduces the traditional notion of actuator windup, along with a common way of alleviating this problem. The multivariable nature of the LTI antiwindup problem and its relation to the directionality issue is subsequently addressed in Section 2.2.
2.1 The High-Gain Antiwindup Concept

A common means of accommodating actuator limits in control systems is known as the “antiwindup” philosophy. In the literature, the term *windup* is given various interpretations. In the current context, however, a straightforward use of this term is sufficient.

**Definition 2** The term *windup* refers to the discrepancy between the controller output and plant input in a limited control system. Antiwindup refers to the reduction of windup in such systems.

Excessive windup transients in a control system can be undesirable, typically causing overshoot in the controlled variables and possible system instability. Examples of windup occurring in Systems 1 and 2 and the resulting adverse effects on system behavior is illustrated in Figures 1.4 and 1.6. The goal of antiwindup methods is to prevent such windup during limited conditions, in hopes of recovering performance. This chapter will describe a widely-used method of alleviating windup in limited systems. Although many of the concepts outlined here are well-known, in practice they are usually applied in a heuristic manner.

An approach commonly used to alleviate windup is to focus on the nominal controller-A inner loop of Figure 1.7, considering the limited controller commands as external inputs. This situation is illustrated in Figure 2.2, where *w* and *z* represent the limited actuator value and windup discrepancy *w* − *u_1*, respectively.
In this context, the antiwindup problem can be thought of as essentially a reference tracking problem of \( u_t \) to \( w \) [Mat93a]. For this chapter, it will be assumed that only one actuator is subject to saturation limits, i.e., \( \ell = 1 \). The saturation restriction allows \( w \), to be modeled as a step input, thus addressing a steady-state or persisting limit. The state space description of this feedback subsystem is

\[
\dot{x} = (A_c - \Lambda C_{c, l}) x + \Lambda w \tag{2.1}
\]

\[
z = w - u_t = w - C_{c, l} x.
\]

The appropriate reference tracking problem is formulated by defining the “antiwindup” transfer function from \( w \) to \( z \), \( H_{zw}^{aw} \) as

\[
H_{zw}^{aw} = -C_{c, l} (sI - A_c + \Lambda C_{c, l})^{-1} \Lambda + I \tag{2.2}
\]

Antiwindup tracking behavior clearly requires that \( |H_{zw}^{aw}(j\omega)| \) be made as small as possible. This can be accomplished by requiring \( \Lambda C_{c, l} \) to be “large” in the sense that it dominates the \( A_c \) term in (2.2). This antiwindup requirement is stated alternatively via the component-wise matrix inequality,

\[
\Lambda C_{c, l} \gg A_c, \tag{2.3}
\]
where $\Rightarrow$ denotes a comparison between the *magnitudes* of the individual matrix elements. A method which has been observed to achieve such an inequality is to choose $\Lambda$ to provide the least squares solution to the overconstrained matrix equation:

$$\Lambda C_{e,l} = kI, \quad (2.4)$$

where $k$ is a constant scalar, chosen large enough so that (2.3) is satisfied. This philosophy of antiwindup design is typically referred to in the literature as a “high-gain” approach.

The above procedure is shown to provide the desired antiwindup tracking behavior as follows. Given that the high-gain requirement (2.3) is satisfied, the following approximation can be made in (2.2):

$$H_{zw}^s \approx -C_{c,l} (sI + \Lambda C_{c,l})^{-1} \Lambda + I = -C_{c,l} \Lambda (sI + C_{c,l} \Lambda)^{-1} + I. \quad (2.5)$$

The equality above is due to a standard matrix manipulation often used in control. For completeness, its justification is given in Appendix A at the end of this chapter. Since $\Lambda C_{c,l}$ is large, the dynamics indicated in (2.5) will be “fast,” and the final value of the actuator windup error, $z$, will be reached rapidly when limited conditions persist. This value can be obtained via application of the final-value theorem, where the Laplace transform of the saturated input, $W(s)$, is represented.
by the step input command, \( L/s \).

\[
\lim_{t \to \infty} z(t) = \lim_{s \to 0} s H_{zu}^{aw} W(s) \\
= \lim_{s \to 0} s \left[ -C_{e,l}\Lambda (sI + C_{c,l}\Lambda)^{-1} + I \right] \frac{L}{s} \\
= 0,
\]

provided that \((C_{e,l}\Lambda)^{-1}\) exists (recall \( \ell = 1 \), so \( C_{e,l}\Lambda \) is actually a scalar); this is established by (2.3).

A common means of implementing the high-gain antiwindup philosophy in SISO systems is known as the “conventional antiwindup (CAW)” method, and is shown in Figure 2.3. As before, consider the state-space realization of the inner feedback loop:

\[
\dot{x} = (A_{c} - B_{c}XC_{c}) x + B_{c}Xw \\
u = C_{c}x
\]

Comparison of this expression with (2.1) reveals that the CAW approach is equivalent to the general approach for \( \Lambda = B_{c}X \). The relationship of the general approach to this and other antiwindup methods is explored in more detail in [CMN89]. The high-gain antiwindup condition (2.3) on \( \Lambda C_{c} \) now translates into a
corresponding condition on $B_c X C_c$. In the SISO case, where $X$ is a scalar, (2.3) is thus achieved if the product of $X$ and $B_c C_c$ is large.

Applying this approach to System 1 using $X = 10$ yields the response shown in Figure 2.4. The overshoot in the output response is now eliminated through the tracking property of the control signal to its limit value. Furthermore, stability can be proven via Theorem 2.

2.2 Multivariable Antiwindup

An issue that is not present in SISO systems which must be addressed in the MIMO situation is how to use the nonlimited actuators, $u_{nl}$. Insight into this
problem is obtained by considering the transfer function $H_{uw}^{aw}$, from the limit input, $w$, to the nonlimited actuators, $u_{nl}$:

$$H_{uw}^{aw} = C_{c,nl} (sI - A_c + \Lambda C_{cl})^{-1} \Lambda.$$ 

Recall that a high-gain antiwindup implementation allows $A_c$ to be neglected in the above transfer function. This permits the approximation

$$H_{uw}^{aw} \approx C_{c,nl} (sI + \Lambda C_{cl})^{-1} \Lambda = C_{c,nl} \Lambda (sI + C_{cl} \Lambda)^{-1}.$$  \hfill (2.6)

As in (2.5), if antiwindup conditions are met, (2.6) will provide a response in $u_{nl}$ having “fast” dynamics and thus the final value will be reached rapidly. This is determined via the final value theorem:

$$u_{\infty} \triangleq \lim_{t \to \infty} u_{nl}(t) = \lim_{s \to 0} sC_{c,nl} \Lambda (sI + C_{cl} \Lambda)^{-1} \frac{L}{s}$$ \hfill (2.7)

$$= C_{c,nl} \Lambda (C_{cl} \Lambda)^{-1} L.$$

Assuming $C_{cl} \Lambda \neq 0$ due to the antiwindup requirement (2.3), (2.7) can be simplified to provide an additional relationship for MIMO antiwindup design:

$$(C_{c,nl} - NC_{cl}) \Lambda = 0,$$ \hfill (2.8)

where $N = u_{\infty}/L$ represents a desired limit-normalized relationship between the limited and nonlimited actuators during limits. The expression (2.8) thus constrains $\Lambda$ based on the desired behavior of the nonlimited actuators during limits. Conditions (2.3) and (2.8) must be satisfied simultaneously to obtain a particular antiwindup scheme. The approach for MIMO antiwindup design is thus to eliminate variables by first solving the underconstrained system of equations (2.8), and
then solving for the remaining parameters via the *overconstrained* system (2.4).

Different MIMO antiwindup schemes are generated by different choices of \( N \). Two specific choices for \( N \) are now described and illustrated for System 2.

### 2.2.1 Decoupled MIMO Antiwindup

One antiwindup philosophy which may be implemented is to leave the unlimited actuators unaffected by the antiwindup compensation. This leads to the condition \( u_\infty = 0 \), or \( N = 0 \). For application to System 2, the relevant controller state-space matrices are

\[
A_c = \begin{bmatrix}
-0.1 & 0 \\
0 & -0.1 \\
\end{bmatrix}, \quad
C_c = \begin{bmatrix}
C_{c,l} \\
C_{c,nl}
\end{bmatrix} = \begin{bmatrix}
1 & 1.25 \\
0.75 & 1
\end{bmatrix}.
\]

Substitution of \( N = 0 \) into (2.8) yields

\[
C_{c,nl}\Lambda = \begin{bmatrix}
0.75 & 1 \\
\lambda_1 & \lambda_2
\end{bmatrix} = 0,
\]

which provides the relationship \( \lambda_2 = -0.75\lambda_1 \). Since this leaves only one component of \( \Lambda \) unsolved, it is straightforward to choose this component based directly on the high-gain antiwindup condition (2.3). Calculation of \( \Lambda C_{c,l} \) yields

\[
\Lambda C_{c,l} = \lambda_1 \begin{bmatrix}
1 & 1.25 \\
-0.75 & -0.9375
\end{bmatrix} = \lambda_1 \begin{bmatrix}
1 & 1.25 \\
-0.75 & -0.9375
\end{bmatrix}
\]
Condition (2.3) is satisfied, for example, by the choice $\lambda_1 = 100$, yielding the antiwindup parameter

$$\Lambda_{\text{decoup}} = \begin{bmatrix} 100 \\ -75 \end{bmatrix}.$$ 

The resulting system response is shown in Figure 2.5. Note the decoupled effect of the limit protection with respect to the limited and nonlimited actuators. Though this approach improves performance to some degree compared to doing nothing (Figure 1.6), there remains significant degradation in the output behavior. Geometrically, the decoupled philosophy can be interpreted with respect to Figure 2.1 as using the high-gain antiwindup technique to fix the $u_1$-coordinate at the limited value, while leaving $u_2$ free to vary during limited conditions. This
variation results in a corresponding undesirable change of the control direction. It is thus desirable to consider alternative antiwindup approaches which utilize $u_{nl}$ in such a way as to further improve performance. One such philosophy is now presented.

### 2.2.2 Worst-Case MIMO Antiwindup

The previous example shows that antiwindup behavior alone is not sufficient to provide desirable limited performance in multivariable systems. In particular, the manner in which $u_{nl}$ is manipulated during limited conditions plays a crucial role in obtaining desirable or undesirable performance. The decoupled approach of “doing nothing” is clearly a poor strategy for System 2. As indicated at the beginning of this chapter, performance improvements will be sought via the notion of maintaining the control direction during limited conditions. Recall that direct implementation of this philosophy requires a nonlinear scaling operation applied to the control signal. The goal of this section is to address directionality issues via the LTI limit protection parameter, $\Lambda$. In the current formulation, directionality will be addressed via selection of $N$ in (2.8). A method of determining $\bar{N}$ based on maintaining a certain “critical” control direction is now proposed. Such a control direction will be determined from nominal closed loop properties.

Consider the system obtained by modeling the saturation nonlinearity as a multiplicative plant input perturbation, $\Delta$, similar to the development in Section 1.1.
Figure 2.6: Nominal system with plant input uncertainty

The resulting closed-loop system is shown in Figure 2.6. For this discussion, the fact that $\Delta$ is actually a nonlinear perturbation operator will be ignored; it will simply be considered as a matrix of small diagonal plant input perturbations applied to $u$, i.e., $\Delta = \text{diag}(\Delta_n, \Delta_t)$, where $\Delta_n = 0$ and $\Delta_t = \delta$. From the technique of $\mu$-analysis [Doy85], it is well known that the performance of MIMO systems can suffer greatly in the presence of such diagonal, bounded input uncertainty. This issue is now investigated alternatively via the classical notion of differential sensitivity. Differential sensitivity addresses the change in a transfer function with respect to a small (infinitesimal) system perturbation. This is in contrast to robustness measures such as $\mu$, which consider the maximum possible change in a system due to (possibly large) bounded uncertainties.

The sensitivity concept, originally developed for SISO systems, is generalized to the MIMO case by considering the first order change of a transfer function with respect to a system perturbation [BB91]. More specifically, consider a transfer function of the perturbed system of Figure 2.6, $H^\Delta$, as being represented as follows:

$$H^\Delta = H^{\text{nom}} + \delta H + (\text{higher order terms in } \Delta),$$
where $H_{\text{nom}}$ is a system transfer function for $\Delta = 0$, and $\delta H$, the sensitivity function of $H$ with respect to $\Delta$, contains only first order terms in $\Delta$. For performance considerations, the square $m \times m$ transfer function from the reference input to tracking error in Figure 2.6, $H_{er}^\Delta$, will be examined. In terms of the plant and controller transfer functions, $G$ and $K$, the quantities $H_{er}^{\text{nom}}$ and $\delta H_{er}$ are given as:

$$H_{er}^{\text{nom}} = (I + GK)^{-1},$$

$$\delta H_{er} = (I + GK)^{-1}G\Delta K(I + GK)^{-1}.$$

The derivation of $\delta H_{er}$ is given in Appendix B at the end of this chapter.

A notion of worst-case performance is obtained by considering the maximum possible “size” of $\delta H_{er}$. This is accomplished via the singular-value decomposition (SVD), i.e.,

$$\delta H_{er}(j\omega) = U(j\omega)\Sigma(j\omega)V^H(j\omega) \tag{2.9}$$

where

$$\Sigma(j\omega) = \text{diag}[\sigma_1(j\omega), \sigma_2(j\omega), \ldots, \sigma_m(j\omega)]$$

$$U(j\omega) = \begin{bmatrix} u_1(j\omega) & u_2(j\omega) & \ldots & u_m(j\omega) \end{bmatrix},$$

$$V(j\omega) = \begin{bmatrix} v_1(j\omega) & v_2(j\omega) & \ldots & v_m(j\omega) \end{bmatrix},$$

$$U^H U = I, V^H V = I,$$ and $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m \geq 0$. The SVD is given an alternative interpretation by multiplying both sizes of 2.9 by $V(j\omega)$, yielding

$$\delta H_{er}(j\omega)V(j\omega) = U(j\omega)\Sigma(j\omega),$$
or, equivalently,

\[ \delta H_{er}(j\omega)v_i(j\omega) = \sigma_i(j\omega)u_i(j\omega), \quad i = 1, \ldots, m. \]

The singular vectors \( v_i \) and \( u_i \) are commonly referred to as the input and output principal directions, respectively [Mac89]. The largest deviation in \( H_{er}^\Delta \) will therefore occur at the frequency \( \omega_{wc} = \arg \max_\omega [\sigma_1(j\omega)] \), i.e. the frequency where \( \delta H_{er} \) has the largest gain. Define the corresponding worst-case reference input direction as

\[ r_{wc} = v_1(j\omega_{wc}). \]

Given the above notion of worst-case performance, a strategy is now proposed to determine the antiwindup parameter, \( \Lambda \), via appropriate selection of the parameter, \( N \), in (2.8). The idea is to obtain an antiwindup scheme which, during limits, will maintain the directional relationship between \( u_l \) and \( u_{nl} \) consistent with the operation of the nominal control design subject to the worst-case input, \( r_{wc} \). Such a relationship is obtained via the nominal transfer function from the reference input to the control signal, i.e.,

\[ u_{wc} = H_{ur}^{nom}(j\omega_{wc})r_{wc}. \]

where \( H_{ur}^{nom} = K(I + GK)^{-1} \). Normalization of the limited actuator to unity allows \( N \) to be determined. Note that in general, \( r_{wc} \) and \( u_{wc} \) are complex-valued. However, in order to apply (2.8), \( u_{wc} \) must be real-valued. Though this represents a fundamental limitation on the type of systems for which this approach can be
applied, insight into the problem of multivariable limit protection is obtained from its application.

To illustrate the worst-case antiwindup design philosophy presented here, consider again System 2. Upon substitution of the specific transfer functions for $G$ and $K$, along with $\Delta = \text{diag}(\delta, 0)$ (recall for System 2 that $u_l = u_1$), the closed-loop transfer functions of interest are:

$$H_{er}^{\text{nom}} = \frac{s + \epsilon}{s + \frac{1}{\epsilon}} f,$$

$$H_{ur}^{\text{nom}} = \frac{s + \epsilon}{4(s + 0.1)(s + 1 + \epsilon)} \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix},$$

$$\delta H_{er} = \frac{s + \epsilon}{(s + 1 + \epsilon)^2} \begin{bmatrix} 16 & 20 \\ -12 & -15 \end{bmatrix} \delta.$$

Singular value decomposition of $\delta H_{er}$ provides the worst-case reference direction (constant for all frequencies), of

$$r_{wc} = \begin{bmatrix} 0.6247 \\ 0.7809 \end{bmatrix}.$$

This reference direction in fact coincides with the plant output principal direction corresponding to the minimum singular value. Thus the corresponding control will be in the direction $y$ in Figure 2.1, which is the direction where the control effort will be largest. This control direction is calculated alternatively via $u_{wc} = H_{ur}^{\text{nom}} r_{wc}$. Performing this calculation and normalizing with respect to the limited
Figure 2.7: System 2 worst-case antiwindup design

actuator, \( u_1 \), yields:

\[
\mathbf{u}_{wc} = \begin{bmatrix} 1 \\ 0.7805 \end{bmatrix}.
\]

Substituting \( N = 0.7805 \) in (2.8), leads to the relationship \( \lambda_2 = 1.25\lambda_1 \). The choice \( \lambda_1 = 350 \) yields the following high-gain antiwindup parameter:

\[
\Lambda_{wc} = \begin{bmatrix} 350 \\ 437.5 \end{bmatrix}.
\]

The simulation responses are shown in Figure 2.7. Note the significantly improved behavior of the output responses. This improved performance is due to both the antiwindup tracking behavior of \( u_1 \), as well as a corresponding leveling of \( u_{nl} \). This is in contrast to the decoupled approach shown in Figure 2.5, which
only addresses the windup characteristic of \( u_1 \). Viewing this situation geometrically, it is seen that the worst-case antiwindup scheme attempts to preserve the control vector direction during limits, whereas the decoupled approach results in a change of the control direction. With respect to Figure 2.1, this is accomplished by 1) fixing \( u_1 \) at its limit as was done in the decoupled approach, which prevents horizontal movement of the control vector, and 2) "freezing" \( u_2 \), which prevents vertical movement of the control vector.

### 2.3 Conclusions/Open Issues

The multivariable antiwindup design strategies developed in this chapter illustrate the importance of appropriately manipulating \( u_{nl} \) during limited conditions to achieve desirable performance. This issue is addressed here via the notion of maintaining the control direction during limited conditions, first proposed in [DSE87]. Implementation of such a philosophy to System 2 requires a nonlinear scaling operation applied to each actuator. However, it has not been demonstrated that nonlinear approaches are necessary to provide desirable properties in limited systems. Indeed, this is a primary motivation for the current investigation into the potential of the simpler LTI approach to limit protection.

To this end, a design procedure is given in Section 2.2.2, which maintains the control direction assuming a "worst-case" direction reference input, \( r_{wc} \). The notion of multivariable sensitivity is used to define such a reference direction. This
Figure 2.8: System 2 worst-case antiwindup design response to $r_{nwc}$

directional consideration is combined with the traditional high-gain antiwindup philosophy and is shown to provide successful LTI limit protection. However, it must be pointed out that due to the LTI nature of this limit protection scheme, the control direction will not be maintained if the reference command direction is different from that which $\Lambda$ was designed. To illustrate this point, consider the response of the antiwindup system for a reference input command which is not in the worst-case direction, e.g.,

$$r_{nwc} = \begin{bmatrix} 0.2356 \\ 1.3822 \end{bmatrix}.$$  

The response is shown in Figure 2.8. Note that there is indeed degradation present, due to the inability of the limit protection to maintain the appropriate control
direction for this reference input. Therefore, the worst-case philosophy of Section 2.2.2 should thus be viewed as providing a compromise among all possible reference inputs. This interpretation is similar to that of the modern robust control strategy of minimizing the $L_2$-gain of a system, which provides a guaranteed system gain bound for all possible inputs—a more precise notion of "worst-case" performance. In fact, the concept of $L_2$-gain will be used in Chapter 3 to provide a generalization of the worst-case antiwindup method.

Though the worst-case design procedure does indeed provide a method of obtaining the static output feedback antiwindup design parameter $\Lambda$, a fixed worst-case control direction, $u_{wc}$, is required to apply design equation (2.8). This requirement is in fact true for System 2, due to the nondynamic multivariable coupling in the plant and controller. Unfortunately, most realistic systems do not possess this property—$u_{wc} = u_{wc}(j\omega)$ is typically a complex-valued function of frequency. It remains an open issue as to whether the antiwindup techniques described in Section 2.2.2 can be extended to this more general case.

Another drawback of the high-gain antiwindup design approaches discussed here (and indeed, many which have appeared in the literature) is that closed loop stability considerations are neglected. Stability must be verified after the design process. For the examples considered, all are proven stable via Theorem 2. However, in the general MIMO case, it may be necessary to manipulate $u_{nl}$ in order to maintain stability as well as performance. Therefore, $u_{nl}$ behavior must
be based on nonlinear stability as well as performance considerations. At this point, it is not clear whether the worst-case design method guarantees stability, or can incorporate such a consideration. However, it has been suggested in [HG93] that stability issues might be addressed in the context of high-gain antiwindup methods by the use of dynamic limit protection.

Though the antiwindup concepts described in this chapter leave important questions unanswered, they provide insight into issues which must be addressed in any successful limit protection scheme. In fact, the issues of stability and worst-case performance are the central concepts to be expanded upon in a more formalized approach to limit protection design. It is the goal of Chapter 3 to provide such an approach.

2.4 Appendices

The appendices of this section will verify two formulas used in this chapter. Both rely on the matrix inversion lemma:

\[(A + BEF)^{-1} = A^{-1} - A^{-1}B(FA^{-1}B + E^{-1})^{-1}FA^{-1}.\]

2.4.1 Appendix A

The goal of this appendix is to show that

\[(sI + \Lambda C)^{-1}\Lambda = \Lambda(sI + CA)^{-1},\]
in order to justify the expressions (2.2) and (2.6). To show this, first note that
the following expression easily follows from the matrix inversion lemma:

\[(I + X)^{-1} = I - X(I + X)^{-1}.\]

Application of the above expression to \((sI + \Lambda C)^{-1}\) yields

\[(sI + \Lambda C)^{-1} = \frac{1}{s} I - \frac{1}{s} \Lambda C(sI + \Lambda C)^{-1}. \tag{2.10}\]

A direct application of the matrix inversion lemma to \((sI + \Lambda C)^{-1}\), using \(A = sI, \ B = \Lambda, \ E = I, \ F = C\) yields

\[(sI + \Lambda C)^{-1} = \frac{1}{s} I - \frac{1}{s} \Lambda (sI + CA)^{-1} C. \tag{2.11}\]

Comparing the second terms on the right sides of (2.10) and (2.11) leads to the
desired result.

### 2.4.2 Appendix B

The expression is now derived for the multivariable sensitivity function, \(\delta H_{\text{er}}\), used
in Section 2.2.2. With reference to Figure 2.6, the system perturbation, \(\Delta\), yields
the perturbed plant \(P(I - \Delta)\). The perturbed transfer function, \(H_{\text{er}}^{\Delta}\), is therefore

\[H_{\text{er}}^{\Delta} = [I + P(I - \Delta)K]^{-1} = [I + PK - P\Delta K]^{-1}.\]

Applying the matrix inversion lemma yields:

\[H_{\text{er}}^{\Delta} = (I + PK)^{-1} + (I + PK)^{-1} P\Delta (I - K(I + PK)^{-1} P\Delta)^{-1} K(I + PK)^{-1} \tag{2.12}\]
Note that the first term above is simply the nominal transfer function $H_{er}$. The MIMO sensitivity function, $\delta H_{er}$, is determined by computing the first order representation of the second term with respect to $\Delta$. This can be obtained via one further application of the matrix inversion lemma to the bracketed term above:

$$(I - K(I + PK)^{-1}P\Delta)^{-1} = I + K(I + PK - P\Delta K)P\Delta.$$  

Substituting the above identity into (2.12), and retaining only first order terms in $\Delta$ yields the desired expression for $\delta H_{er}$:

$$\delta H_{er} = (I + PK)^{-1}P\Delta K(I + PK)^{-1}.$$
Chapter 3

Analysis and Design Tools for Limit Protection

Chapter 2 illustrated the benefits of limit protection design based on closed-loop input-output properties. However, it is not clear how such a design approach can be applied to systems containing dynamic multivariable coupling. The goal of this chapter is to extend the basic principles of Chapter 2 to more general systems. This will be accomplished by casting the problem of limit protection design as a nonlinear robust control design problem.

3.1 State Space Descriptions

The techniques proposed in this chapter utilize state space system descriptions. To this end, consider a state space realization for $H^{\text{nom}}$ in Figure 1.9. Its definition via Figure 1.8 constrains the state space realization to have the following form:

$$
\begin{aligned}
    \dot{x} &= Ax + B_x p + B_w w + B_u u_A, \\
    q &= C_q x, \\
    z &= C_z x + D_{zp} p + D_{zw} w, \\
    y_A &= -p.
\end{aligned}
$$

(3.1)
A realization for $H^\Lambda$ is obtained from $H^\text{nom}$ via the substitution $u_\Lambda = \Lambda y_\Lambda$:

$$
H^\Lambda \left\{ 
\begin{array}{l}
\dot{x} = Ax + (B_p - B_u \Lambda)p + B_w w, \\
q = C_q x, \\
z = C_z x + D_zp p + D_zw w.
\end{array}
\right.
$$

(3.2)

To describe the complete nonlinear limited system of Figure 1.9, it is first noted that the limiting nonlinearity can be equivalently modeled by considering $\Delta$ to be a time-varying linear gain matrix, $K(t) = \text{diag}(k_1(t), \ldots, k_\ell(t))$, where $0 \leq k_j(t) \leq 1$, $j = 1, \ldots, \ell$ [BY89]. This equivalence is apparent by noting that (i) each $k_j(t)$ does indeed belong to sector $[0, 1]$, and thus satisfies the original assumptions of the limiting operation, and (ii) given any nonlinear mapping $\delta_j$ belonging to sector $[0, 1]$, an equivalent linear gain, $0 \leq k_j(t) \leq 1$, can be obtained at every instant which duplicates the nonlinear output. This is illustrated in Figure 3.1, where it is seen that all possible input/output combinations due to the class of $[0,1]$ sector-bounded nonlinearities correspond to those arising from a linear, time varying gain $0 \leq k_j(t) \leq 1$. The particular case of a saturation limit, where $\delta_j$ is a deadzone, is also shown in Figure 3.1. This alternative view of the limit nonlinearity leads to the following equivalent representation of the limited system via the linear, time-varying system obtained by substituting $p = K(t)q$ in (3.2):

$$
\begin{aligned}
\dot{x} &= \tilde{A}(t)x + B_w w, \\
z &= \tilde{C}_z(t)x + D_zw w,
\end{aligned}
$$

(3.3)

where $\tilde{A}(t) = A + (B_p - B_u \Lambda)K(t)C_q$, and $\tilde{C}_z(t) = C_z + D_zp K(t)C_q$. 
Figure 3.1: Equivalent representation of nonlinearity via linear gain, \( k_j(t) \)

One final family of state space descriptions is now introduced, based on (3.3). Specifically, consider the \( 2^\ell \) LTI systems obtained by substituting all combinations of the extreme values (i.e. the sector bounds) for \( k_j(t) \), \( j = 1, \ldots, \ell \). In other words, the single time-varying matrix in (3.3) is replaced by the \( 2^\ell \) constant gain matrices \( K_i = \text{diag}(k_1, k_2, \ldots, k_\ell) \), where each \( k_j \in \{0,1\} \), yielding the following family of \( 2^\ell \) LTI systems:

\[
H_i \quad \begin{cases} 
\dot{x} = A_i x + B_w w, \\
z = C_{zi} x + D_{zw} w, 
\end{cases} \quad i = 0, \ldots, 2^\ell - 1, \tag{3.4}
\]

where \( A_i = A + (B_p - B_u \Lambda) K_i C_q \), \( C_{zi} = C_z + D_{zp} K_i C_q \), and \( H_i \) is the transfer function of the \( i \)th LTI system. An interesting interpretation of this family of linear systems with respect to the limited system is obtained by observing from Figure 1.8 that \( k_j = 1 \) is equivalent to zeroing the corresponding actuator signal to the plant, effectively removing a feedback path (and introducing the limit protection loop through \( \Lambda \)), whereas \( k_j = 0 \) leaves the nominal feedback path unchanged. Thus (3.4) can be interpreted as a family of linear representations of
the limited system considering all possible combinations of active \((k_j = 1)\) and nonactive \((k_j = 0)\) limits.

The reason for introducing the LTI family of systems (3.4) is due to the property, shown in [BY89], [BEFB94], that the time-varying parameters in (3.3) satisfy

\[
(\hat{A}(t), \hat{C}_s(t)) \in \text{Co}\{(A_0, C_s^0), (A_1, C_s^1), \ldots, (A_{2\ell - 1}, C_s(2\ell - 1))\}. \tag{3.5}
\]

where \(\text{Co}\) denotes the convex hull of the given set of matrices. Therefore, if stability or performance properties can be shown to hold for the linear, time-varying system (3.3), where \(\hat{A}(t)\) and \(\hat{C}(t)\) satisfy (3.5), the properties will also hold for the nonlinear system. It is this concept which makes possible the analysis and synthesis tools of the following sections. Such methods apply to more general systems than those addressed in Chapter 2, and also for \(\ell > 1\).

### 3.2 Robustness Analysis

The concept of robustness can be described as a desired property, i.e. stability or performance, holding for every system in a prespecified set. In the current framework, this set of systems is depicted in Figure 1.9, where \(\Delta\) can be any valid sector-bounded limiting nonlinearity, as described in Chapter 1. Recall from Section 3.1 that this set of nonlinear systems is represented equivalently via the time varying system (3.3). Therefore, relevant robust stability and performance measures can be obtained via the stability and performance analysis of this system.
Specifically, Section 3.2.1 presents a robust stability criterion equivalent to that given in Theorem 2, extended to the case of multiple limits. Section 3.2.2 defines robust performance via the $L_2$-gain from $w$ to $z$.

### 3.2.1 Robust Stability via LMI Numerical Solution

In Chapter 1, a special case of the circle criterion (Theorem 2) was presented to provide a tool for establishing stability of the limited system for $\ell = 1$. This criterion guarantees $L_2$-stability by restricting the Nyquist diagram of $-H_{qp}$ to a certain region of the complex plane via the condition $\text{Re}\{-H_{qp}(j\omega)\} > -1$. By an application of the Kalman-Yakubovich Lemma, it can be shown that this condition is in fact equivalent to the existence of a Lyapunov function of the form $V = x^TPx$ proving exponential stability of the nonlinear system [Vid93]. This provides a connection between the $L_2$-stability framework of the circle theorem and the state-space Lyapunov stability framework. The stability criterion is now stated precisely in terms of such a Lyapunov function.

**Theorem 3 [BY89]** The system (3.3) is exponentially stable if there exists a quadratic Lyapunov function $V = x^TPx$ which proves stability for each linear system in (3.4). This is equivalent to finding a matrix $P = P^T > 0$ such that the following matrix constraints are satisfied:

$$A_i^T P + PA_i < 0, \quad i = 0, \ldots, 2^n - 1.$$  \hspace{2cm} (3.6)
The matrix expression shown in (3.6) is known as a Linear Matrix Inequality (LMI), and the problem of determining an appropriate matrix \( P \) which satisfies (3.6) is known as an LMI feasibility problem. In [BY89], Boyd and Yang descriptively call this problem the "simultaneous Lyapunov stability" problem, since it is required that a single Lyapunov function "simultaneously" prove stability for each individual system. The determination of such a quadratic Lyapunov function is also known as the "quadratic" stability problem. Though quadratic stability of the family of linear systems (3.4) can be conservative in determining exponential stability of the system (3.3), this concept has proven to be very useful in practice, due to its computational tractability.

It is trivial to verify that if Theorem 3 holds, then the Lyapunov inequality \( \dot{A}^T P + P \dot{A} < 0 \) is indeed satisfied for every matrix \( \dot{A}(t) \) belonging to the convex set (3.5), thus proving stability for the limited system. Note that (3.6) is not equivalent to determining the stability of the individual systems. In other words, it is not enough to simply establish that no \( A_i, \ i = 0, \ldots, 2^\ell - 1 \), has nonnegative eigenvalues. This is a stronger condition, due to the common Lyapunov function requirement.

To briefly illustrate the issues involved in the solution of the LMI feasibility problem of Theorem 3, consider the case \( n_l = 0 \) (i.e. no limited conditions). Theorem 3 reduces to the problem of determining stability for a single linear system, which is known from Lyapunov theory to be equivalent to the eigenvalues
of $A$ having negative real part. Furthermore, a matrix $P$ satisfying the above LMI condition can be obtained by choosing any $Q < 0$ and solving for the unique positive definite solution $P_Q$ of the Lyapunov equation

$$A^T P + PA = Q,$$

which is simply a system of linear scalar equations.

Unfortunately, for $n_l \geq 1$, there does not exist such analytical methods of solving the quadratic stability problem of Theorem 3. However, it has been shown that such problems can be cast as convex optimization problems, and lend themselves to reliable numerical solution. The general LMI feasibility problem takes the form $M(\xi) \leq 0$, where $\xi$ is a vector of optimization decision variables, and $M(\xi)$ is a symmetric matrix which is affine in $\xi$. In this context, affineness means that $M(\xi)$ can be put into the form

$$M(\xi) = M_0 + \sum_{i=1}^{d} \xi_i M_i,$$

where $d$ is the number of decision variables, and $M_i$, $i = 0, \ldots, d$, are constant, symmetric matrices. In (3.6), $\xi$ corresponds to the $n(n + 1)/2$ unique entries of the symmetric $n \times n$ matrix variable $P$. The optimization objective is one of determining $\xi$ such that the maximum eigenvalue of $M(\xi)$ is nonpositive. This is equivalent to solving the problem

$$\min_{\xi} \lambda_{\text{max}}(M(\xi)).$$

(3.7)
The LMI is feasible if and only if the optimal solution of (3.7) is nonpositive. Recently, efficient methods of solution have been developed for this as well as other problems involving LMIs [BE93], [GN93a], [NG94]. These developments suggest that it may be advantageous to consider solutions of such convex programs in addition to analytical or closed-form solutions as providing valid "solutions" to control problems. See [BEFB94], [BBFE93], [Boy94] for more elaboration and justification of this issue.

3.2.2 $L_2$-gain Robust Performance

Recall the goal of limit protection design is to provide for graceful performance degradation in addition to stability during limited conditions. It was shown in Chapter 2 that a design based on a notion of "worst-case" control and reference input directions provided a means to obtain such desirable performance properties for System 2. This notion of performance is addressed more formally via the concept of robust performance. The robust performance problem is commonly defined by augmenting the robust stability problem with a bound on the $L_2$-gain from $w$ to $z$. For the LTI case, such a robust performance problem is equivalent to a constraint on $\mu$. This was first proposed as a performance criterion for the general nonlinear limit protection problem in [CMN89]. Note that the Chapter 2 concept of worst-case signal directions is included in the $L_2$-gain bound specification, due to the supremum operation over all possible input signals in the definition of the
$L_2$-gain (Definition 1).

In a manner analogous to the stability analysis formulation, robust performance is specified in the current context through a bound on the $L_2$-gain from $w$ to $z$ of the time varying system (3.3). This is accomplished through a "simultaneous" bound on the $L_2$-gain of each LTI system of (3.4). This is done in the LMI framework via the following theorem.

**Theorem 4 [BEFB94]** The system (3.3) has $L_2$-gain less then $\gamma$ if there exists $P = P^T > 0$ such that

$$
\begin{bmatrix}
A_i^T P + PA_i + C_{zi}^T C_{zi} & PB_w + C_z^T D_{zw} \\
B_i^T P + D_{zi}^T C_z & -\gamma^2 I + D_{zw}^T D_{zw}
\end{bmatrix} < 0, \quad i = 0, \ldots, 2^\ell - 1
$$

(3.8)

Furthermore, a well-defined upper bound on the $L_2$-gain is obtained by solving the following optimization problem:

$$
\min \gamma
$$

subject to (3.8), $P = P^T > 0$

(3.9)

Note that the LMI condition (3.8) is stronger than the stability LMI condition of Theorem 3. Thus feasibility of (3.8) implies stability. The optimization problem (3.9) is an example of the problem of minimizing a linear objective function subject to an LMI constraint, which has the general form

$$
\min c^T \xi
$$

subject to $M(\xi) \leq 0$. 

As in LMI feasibility problems, there are efficient numerical methods of solving such problems.

As mentioned above, Theorem 4 can be interpreted as a "simultaneous" $H_\infty$ norm bound for the $2^\ell$ linear systems of (3.4). This is due to the fact, shown in [BEFB94], that each LMI of (3.8) establishes a bound on the $H_\infty$-norm of the individual linear systems of (3.4). In fact, if only a single LTI system is considered in the problem (3.9), the minimizing value of $\gamma$ provides the exact value of the $H_\infty$ norm for each LTI system. This observation motivates a means of depicting limited system performance via the $2^\ell$ singular value plots $\sigma(H_i)$. Specifically, a useful indication of the performance degradation incurred under limits is obtained by comparing the nominal singular value plot, $\sigma(H_0)$, with that of the remaining linear systems defined in (3.4), $\sigma(H_i), i = 1, \ldots, 2^\ell - 1$. It is important to note that this comparison provides only an indication of performance degradation of the limited system. In other words, if $\sigma(H_0) \approx \sigma(H_i), i = 1, \ldots, 2^\ell - 1$, it cannot be concluded that limited performance is approximately the same as the nominal performance, since the true $L_2$-gain performance measure is based on the "simultaneous" $H_\infty$ norm bound LMI condition of Theorem 4. More concisely, desirable performance indicated by $\sigma(H_i)$ is a necessary condition for desirable performance via the $L_2$-gain performance measure, and as such can only indicate undesirable performance. This method of visualizing limited system performance is used in the examples.
3.3 Limit Protection Design

Recall that in the present framework, design of limit protection is equivalent to determining a suitable static gain, $\Lambda$, which provides acceptable system performance in the presence of actuator limits. The previous section presented notions of stability and performance for the limited system via (3.3), along with the means of numerical evaluation using LMI optimization. Unfortunately, it has been established that an optimal reduced order control design problem cannot be formulated as such a convex LMI problem [IS94]. However, it is useful to examine the $L_2$-gain analysis LMI of Theorem 4 which is obtained with $\Lambda$ appearing explicitly as another matrix variable.

In order to make $\Lambda$ explicit in the system descriptions, the $2^\ell$ LTI systems of (3.4) will be rewritten as follows:

$$
\begin{align*}
\dot{x} &= (A^{K_i} + B_u \Lambda C_y^{K_i})x + B_w w, \quad i = 0, \ldots, 2^\ell - 1, \\
z &= C_{zi} x + D_{zo} w
\end{align*}
$$

(3.10)

where $A^{K_i} = A + B_p K_i C_q$, and $C_y^{K_i} = -K_i C_q$. The optimization problem of
Theorem 4 is now restated as

\[
\begin{align*}
\min \gamma & \\
\text{subject to} & \begin{bmatrix}
[A^K_i]^T P + PA^K_i + C_{zi}^T C_{zi} \\
+ (B_u \Lambda C^K_y) P + PB_u \Lambda C^K_y \\
B_u^T P + D_{zw}^T C_{zi} \\
- \gamma^2 I + D_{zw}^T D_{zw}
\end{bmatrix} \begin{bmatrix}
P B_w + C_{zi}^T D_{zw} \\
0
\end{bmatrix} < 0,
\end{align*}
\]

\[i = 0, \ldots, 2^e - 1,
\]

\[P = P^T > 0
\]

(3.11)

It is clear from examination of (3.11) that while \(\Lambda\) appears explicitly in the optimization problem, the matrix inequality constraint for the synthesis problem is no longer a linear matrix inequality—it is now a \textit{bilinear} matrix inequality in the variables \(P\) and \(\Lambda\). This issue is present in any case of reduced order output feedback design.

Though LMI methods cannot directly address such a static output feedback design problem, a heuristic two-step iteration procedure is often employed. The two steps can be viewed as alternating between analysis and synthesis, as follows:

\textbf{Analysis}: The design parameter, \(\Lambda\), is fixed. This yields an LMI analysis problem exactly as in Section 3.2.2, whose solution indicates the quality of the design parameter.

\textbf{Synthesis}: The analysis parameter, \(P\), is fixed, yielding an LMI “synthesis” problem in the design variable, \(\Lambda\).
While this iterative strategy offers no guarantee of yielding the *optimal* design parameter, it provides a guaranteed performance level for each successive synthesis parameter. Furthermore, this approach has been observed to work well in practice—successive iterations do in fact reduce the objective function until some type of “convergence” is observed [Boy94].

### 3.3.1 Initialization

An important consideration in the above iterative design procedure is how to initialize it. A simple approach is to begin with the synthesis step, by choosing an arbitrary positive definite matrix for $P$ such as $mI, m > 0$, and solving for $\Lambda$. A minimum requirement here is that the resulting $\Lambda$ be stabilizing; otherwise, the subsequent analysis iteration will fail due to infeasibility of its LMI constraint. Experience has shown this initialization procedure to work well when Theorem 3 proves stability of the limited system for $\Lambda = 0$, i.e., when limits do not destabilize the nominal system and quadratic stability can be shown for $\{A^k\}_{i=0}^{2^{f-1}}$.

A very common occurrence, however, is for limits to destabilize the nominal closed loop, Therefore, to accommodate such systems, as well as those which are not *quadratically* stable when $\Lambda = 0$, an alternative method for obtaining an initial quadratically stabilizing value of $\Lambda$ is now proposed. Recall that Theorem 1 guarantees stability via the condition $\|H^A_{sp}\|_\infty < 1$. This condition ensures system
stability by requiring the transfer function $H^{\Lambda}_{sp}$ to be "small enough," in the appropriate sense. Thus the current initialization problem is one of choosing $\Lambda$ such that this size requirement on $H^{\Lambda}_{sp}$ is satisfied. This can be addressed via an "auxiliary" stabilizing design problem to provide the appropriate $L_2$-gain reduction on $H^{\Lambda}_{sp}$ based on the LMI formulation of Theorem 4. Since this problem involves only the single LTI system $H^{\Lambda}$, Theorem 4 reduces to a single LMI. Substitution of the appropriate state-space matrices from (3.2) yields the specific LMI form of the auxiliary stabilizing design problem:

$$\min \gamma$$

subject to

$$\left[ \begin{array}{ccc} A^T P + PA + C_q^T C_q & PB_p - P B_u \Lambda \\ B_p^T P - \Lambda^T B_u^T P & -\gamma^2 I \end{array} \right] < 0,$$

$$P = P^T > 0.$$

(3.12)

Iterative solution of this auxiliary design problem also requires initialization. Fortunately, for this problem initialization is straightforward. Since the system $H^{\Lambda}$ reduces to the stable nominal system for $\Lambda = 0$, this auxiliary design problem can be initialized by either choosing $\Lambda = 0$ as the initial value, or selecting an arbitrary $P$ matrix and solving for $\Lambda$ via an initial synthesis iteration.
3.3.2 Stabilizability

It is important to note that the above initialization procedure will fail if the limited system of Figure 1.7 is not (quadratically) stabilizable via the static output feedback design parameter \( \Lambda \). Some general issues involved in determining stabilizability of limited systems are discussed in Chapter 5. However, a simple necessary condition for quadratic stabilizability will be useful in the examples. Note from (3.10) that quadratic stabilizability is equivalent to the existence of \( \Lambda \) such that \( A^{K_i} + B_u \Lambda C^{K_i} \), \( i = 0, \ldots, 2^t - 1 \), satisfy Theorem 3. If this is true, then there obviously must also exist quadratically stabilizing state feedback. A simple consequence of this is now stated.

**Lemma 2** The linear, time-varying system (3.3) is quadratically stabilizable only if each pair \( (A^{K_i}, B_u) \), \( i = 0, \ldots, 2^t - 1 \), is stabilizable.

With respect to the limited system of Figure 1.9(a), Lemma 2 can be given the interpretation that the \( 2^t - 1 \) linear systems obtained by substituting \( k \in \{0, 1\} \) for each perturbation nonlinearity (i.e. removing all combinations of plant input connections—see Figure 1.8) must be individually stabilizable from the input \( u_A \).

3.4 Summary

This chapter has viewed limit protection design as a nonlinear robust control design of the static output feedback parameter, \( \Lambda \). To accomplish this, notions of
robust stability and performance were formulated in terms of linear matrix inequalities. The design problem was addressed via an iterative design procedure. This is of practical interest, since, to date, the problem of reduced order optimal control design remains an open issue. A systematic method of performing such an iterative design was presented, considering the cases when the system can and cannot be proven quadratically stable in the presence of limiting nonlinearities. Finally, a brief discussion was given regarding the role that the notion of stabilizability plays in this design approach.
Chapter 4

Limit Protection Design Examples

The synthesis procedure will now be demonstrated on System 2, as well as on a
more sophisticated aerospace example which has appeared in the literature. The
high degree of multivariable coupling in these systems provides a good test for
the proposed techniques. For both examples, \( \ell = 1 \), \( w = r \) and \( z = \begin{bmatrix} e^T & u^T_{nl} \end{bmatrix}^T \).
Performance analysis involves the linear systems \( H_i \), whose realizations are given
by (3.4). The case \( \ell = 1 \) implies \( i = 0, 1 \). As described in Section 3.2.2, per-
formance degradation for the various limit protection designs will be graphically
depicted and compared via the singular value plots \( \sigma(H_0) = \sigma(H_{rw}^{nom}) \), and \( \sigma(H_1) \).

The LMI optimization is accomplished here using a preliminary version of
the Matlab toolbox LMI-Lab, soon to be commercially available [GN93b]. As
mentioned in Section 3.3, application of the iteration scheme produces a sequence
of design parameters which yield decreasing \( L_2 \)-gain bounds, until some type of
"convergence" is observed. The heuristic notion used here is that the iterative
procedure is considered "converged" as soon as the objective function (the \( L_2 \)-
gain bound) does not decrease for 10 consecutive iterations. For completeness, it
should further be noted that a "feasibility radius" of \( 10^4 \) was used in the synthesis
iteration for both examples to prevent excessively large \( \Lambda \) parameters from being generated.

### 4.1 System 2

To repeat from Chapter 1, the plant and nominal controller transfer functions, \( G \) and \( K \), respectively, are

\[
G = \frac{4(s + 0.1)}{s + \epsilon} \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}, \quad K = \frac{1}{4(s + 0.1)} \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}.
\]

The case of \( \epsilon = 0 \) is treated in [DSE87]. However, this causes a difficulty in using the current approach. Specifically, application of Lemma 2 leads to the conclusion that the unstable plant mode \( s = 0 \) is uncontrollable for the pair \( (A^{K_i}, B_u) \). Thus the removal of the \( u_1 \) connection renders the system unstabilizable. This was the initial motivation to modify the plant to have a slightly negative pole, i.e. \( \epsilon = 0.001 \). With this modification, the limited system is not only stabilizable, but is actually stable without any limit protection \( (\Lambda = 0) \), which is shown by application of Theorem 3 to \( \{A^{K_i}\}_{i=0}^{1} \). This modified system behaves very similar to the original system. The nominal response to the step reference command, \( r_{wc} = \begin{bmatrix} 0.6247 \\ 0.7809 \end{bmatrix}^T \), and the severely degraded system behavior with \( u_1 \) limited at 1 were already shown in Figures 1.3 and 1.4, respectively.

Application of the iterative procedure to the \( L_2 \)-gain LMI problem (3.11) using
the synthesis iteration initialization $P = 10I$ produces convergence in 41 iterations. The resulting "optimized" design parameter is

$$\Lambda_{opt} = \begin{bmatrix} 752 \\ 934 \end{bmatrix}.$$ 

It is satisfying to note that this result differs by approximately a scalar multiple to that obtained via the worst-case antiwindup design procedure of Section 2.2.2, and yields virtually identical output responses. Figure 4.1 depicts $\sigma(H_0)$ (solid line), and $\sigma(H_1)$ for the system with optimized limit protection (circles), and no limit protection (dashed line). The severe degradation of the uncompensated system is evident from this perspective. Furthermore, it can be seen that for frequencies above 0.1 rad/sec, the nominal and optimized limit protection curves are very similar, whereas for low frequencies there is significant discrepancy in all the curves. An interpretation of this characteristic will be discussed in Example 2.

The windup properties are shown alternatively in Figure 4.2 via the bode magnitude plot of the transfer function $H^\text{aw}_{\text{n}}$ defined in (2.2). The large attenuation of this transfer function indicates that the optimized limit protection has strong antiwindup properties.

### 4.2 F8 Aircraft

This system, originally appearing in [KAS88], provides a more realistic $H_\infty$ setting for limit protection design. The plant is a modified model of the longitudinal
dynamics in an F8 aircraft, having the following state space realization:

\[
\begin{bmatrix}
-0.8 & -0.006 & -12 & 0 \\
0 & -0.014 & -16.64 & -32.2 \\
1 & -0.0001 & -1.5 & 0 \\
1 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x \\
u \\
\end{bmatrix}
\begin{bmatrix}
-19 & -3 \\
-0.66 & -0.5 \\
-0.16 & -0.5 \\
0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x \\
\end{bmatrix}
\]

\[
y_p = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} x
\]

A 1 DOF $H_\infty$ controller was obtained for the "design plant" shown in Figure 4.3, using the MATRIXx routine HINF_CONTR [Int89]. The $H_\infty$ design scalings and frequency weights are $W_e = \text{diag}(W_{e1}, W_{e2})$, $W_y = \text{diag}(W_{y1}, W_{y2})$, and $W_u = \text{diag}(W_{u1}, W_{u2})$, where

\[
W_{ei} = k_{ei} \frac{0.1}{s + 0.1}, \quad k_{e1} = 150, k_{e2} = 200,
\]

\[
W_{yi} = k_{yi} \frac{s}{s + a_1}, \quad k_{y1} = 10, a_1 = 10, k_{y2} = 50, a_2 = 100,
\]

\[
W_{u1} = W_{u2} = 0.25.
\]

An 8th order $H_\infty$ controller is thus generated, and is shown explicitly in the appendix at the end of this chapter. The time response of the resulting nominal closed-loop system to the reference step command $r_1 = \begin{bmatrix} 10 & 10 \end{bmatrix}^T$ is shown in Figure 4.4. Note the different time scales used in the plant output and actuator plots. The response for actuator $u_2$ limited at 25 is shown in Figure 4.5, where the nominal plant signals are shown for reference via the dotted lines in the $y_{p1}$, $y_{p2}$, and $u_1$ plots. The dotted line in the $u_2$ plot depicts the controller windup. In
contrast to System 2, analysis via Theorem 3 shows that $\{A^{K_i}\}_{i=0}^{1}$ is not quadratically stable. In this circumstance, the auxiliary small-gain LMI synthesis problem described in Section 3.3 must be employed to provide a stabilizing initialization of $\Lambda$. Using the simple initialization $\Lambda = C$ for this auxiliary problem, this procedure provides the following stabilizing value of $\Lambda$ in one iteration:

$$
\Lambda_{stab} = \begin{bmatrix}
0.3381 \\
2.167 \\
-0.9407 \\
-9.169 \\
-0.3096 \\
0.03638 \\
0.06175 \\
-0.8157
\end{bmatrix}.
$$
Iterative reduction of the $L_2$-gain via (3.11) produces the following "optimized"

limit protection in 25 iterations:

\[
\Lambda_{opt} = \begin{bmatrix}
-0.2740 \\
4.728 \\
-3.000 \\
-0.4988 \\
-0.5001 \\
-0.0001094 \\
-0.3047 \\
-2.491
\end{bmatrix}.
\]

Figure 4.6 depicts performance degradation properties via plots of $\bar{\sigma}(H_i)$ for the
nominal system (solid line), initial stabilizing limit protection (dashed line), and
optimized limit protection (circles). Significant performance degradation of the
initial stabilizing limit protection is indicated via the large peaking behavior. The
optimized limit protection, on the other hand, exhibits a much smoother charac-
teristic. Windup properties are depicted in Figure 4.7 via the bode magnitude
plot of $H_{aw}$ for the initial (dashed line) and optimized (circles) limit protection.

As opposed to the corresponding plot for System 2 shown in Figure 4.2, this plot
does not indicate strong antiwindup tracking properties. In fact, it can be seen
that both limit protection schemes amplify the windup in the frequency ranges
where the gain of this transfer function is above 0 dB. Furthermore, it can be
seen that upon iterative reduction of the $L_2$-gain, this transfer function actually increases. The simulation responses for this limit protection scheme are shown in Figure 4.8, where it appears that nominal design performance properties have been recovered to a good degree. This example indicates somewhat surprisingly that graceful performance degradation, as defined by the $L_2$-gain, does not require limit protection with high-gain antiwindup properties.

It is also insightful to examine the system behavior under the influence of a reference command which drives the actuators to their limits in the steady-state (non-transiently), which has been observed to be a problem in certain aerospace applications [Wat94]. To illustrate this situation, the reference command $r_2 = \begin{bmatrix} -10 & 10 \end{bmatrix}^T$ is applied to the system. The nominal linear response is shown in Figure 4.9, where it is seen that actuator $u_2$ is required to assume a large steady-state value. As before, $u_2$ is limited at 25. The limited system behavior without limit protection is shown in Figure 4.10, where the system now appears unstable as the $u_2$ command “winds up.” The system behavior resulting from limit protection $\Lambda_{opt}$ is shown in Figure 4.11. It is seen that the limited actuator no longer winds up in an unbounded manner. Furthermore, though performance is significantly compromised, the system appears stable and reasonably well-behaved. In fact, the degraded performance for this situation may be heuristically anticipated via the singular value plots of Figure 4.6 by an examination of the low frequency behavior of the curves, where it is seen that significant performance degradation
is indicated in both the initial and optimized schemes.

Finally, a question may be raised regarding the system behavior if the reference command is changed during the time when the windup discrepancy is large, i.e., when \( t \approx 0.1 \) seconds in Figure 4.8 or when \( t > 0.1 \) seconds in Figure 4.11. This situation is illustrated by changing the reference command from \( r_1 \) to \(-r_1\) at \( t = 0.08 \) seconds. The system responses to this command are shown for the \( \Lambda = 0 \) case and the \( \Lambda = \Lambda_{opt} \) case in Figures 4.12 and 4.13, respectively. The dashed lines again represent the nominal linear response for comparison purposes in the traces for \( y_{p1}, y_{p2}, \) and \( u_1 \). The dashed line in the \( u_2 \) plot represents the windup characteristic. It is again seen in Figure 4.13 that the limit protection results in a response which is very much improved. Furthermore, two features are immediately evident. First, when the reference command is switched, there does not exist an actuator lag time resulting from the windup discrepancy. Also note that this reference command causes the lower limit to also be encountered. Fortunately, the limit protection does indeed maintain qualitatively good behavior throughout.

### 4.3 Conclusions

This chapter has illustrated the design procedure proposed in Chapter 3 via two multivariable systems. The first example showed this design procedure to yield
limit protection nearly identical to the worst-case antiwindup method of Chapter 2. Subsequently, a more complicated aerospace system was considered, which could not be addressed using the antiwindup approach. The limit protection thus obtained was shown to achieve desirable properties with respect to stability and graceful performance degradation (via $L_2$-gain reduction). The examples also illustrate the depiction of performance degradation via the singular value plots of the LTI systems $H_i$, as described in Section 3.2.2. This is shown to provide a potentially helpful tool for evaluating limit protection schemes, since meaningful statements can be made regarding limited system behavior before running multiple simulations with various reference commands. The F8 system also provides a very interesting observation: Limit protection having the traditional "high gain antiwindup" characteristic appears not to be necessary to provide desirable stability and performance properties. Note, however, that high gain properties were in fact obtained in the limit protection design for System 2, consistent with the worst-case antiwindup method of Chapter 2. Thus the current approach does not necessarily exclude the high gain property. Rather, appropriate action is taken based on the property of most importance: Closed-loop performance. It is seen that this may or may not involve high gain antiwindup behavior.
4.4 Appendix

The state-space realization, \((A_c, B_c, C_c)\), for the nominal \(H_\infty\) controller for the F8 aircraft system, described in Section 4.2, is:

\[
A_c = \begin{bmatrix} A_1 & A_2 \end{bmatrix},
\]

where

\[
A_1 = \begin{bmatrix}
-1.00 \times 10^{-1} & 0 & 0 & 0 \\
0 & -1.00 \times 10^{-1} & 0 & 0 \\
8.86 \times 10 & -3.32 & -4.47 \times 10 & 1.74 \times 10^{-3} \\
3.18 & -2.69 & -1.34 & -1.38 \times 10^{-2} \\
8.69 \times 10^{-1} & -3.12 & 8.46 \times 10^{-1} & 7.68 \times 10^{-5} \\
0 & 0 & 1.00 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
\[ A_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-5.87 \times 10^2 & -3.46 \times 10^2 & 4.68 & -1.30 \times 10^2 \\
-8.78 \times 10 & 8.53 & 2.76 \times 10^{-1} & -1.07 \times 10 \\
-6.77 \times 10 & 6.03 \times 10 & 1.75 \times 10^{-1} & -8.59 \\
0 & 0 & 0 & 0 \\
0 & 4.00 \times 10 & -1.00 \times 10 & 0 \\
-4.00 \times 10^2 & 4.00 \times 10^2 & 0 & -1.00 \times 10^2
\end{pmatrix} ,
\]

\[ B_c = \begin{pmatrix}
3.75 \times 10 & 0 \\
0 & 5.00 \times 10 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix} ,\]
and

\[
C_e^T = \begin{bmatrix}
-4.62 & -2.58 \times 10^{-1} \\
-8.52 \times 10^{-1} & 6.50 \\
2.38 & -4.53 \times 10^{-1} \\
-7.09 \times 10^{-5} & -3.31 \times 10^{-4} \\
9.86 & 1.29 \times 10^2 \\
3.92 \times 10 & -1.33 \times 10^2 \\
-2.01 \times 10^{-1} & -2.86 \times 10^{-1} \\
4.36 & 1.58 \times 10 \\
\end{bmatrix}
\]
Figure 4.1: System 2 singular value plots $\sigma(H^I)$

Figure 4.2: System 2 Bode magnitude plot of $H_{zw}^{au}$
Figure 4.3: Example 2 $H_{\infty}$ design plant

Figure 4.4: F8 system nominal responses for $r = r_1$
Figure 4.5: F8 system limited responses for $r = r_1$

Figure 4.6: F8 system singular value plots $\sigma(H^1)$
Figure 4.7: F8 system Bode magnitude plot of $H_{zw}^{au}$

Figure 4.8: F8 system optimized limit protection responses for $r = r_1$
Figure 4.9: F8 system nominal responses for $r = r_2$

Figure 4.10: F8 system limited responses for $r = r_2$
Figure 4.11: F8 system optimized limit protection responses for $r = r_2$

Figure 4.12: F8 system limited responses for switched reference command
Figure 4.13: F8 system optimized limit protection responses for switched reference command
Chapter 5

Conclusions and Future Work

This chapter begins with a brief summary of the issues addressed in this research, and outlines some new conclusions which can be made as a result of this work. The remaining sections discuss some possibilities for future work.

5.1 Summary and Conclusions

This work has investigated a previously unsolved problem in the design of limit protection for a class of actuator-limited multivariable control systems. Chapter 1 introduced the problem by defining the class of limiting nonlinearities to be considered, and illustrating via two examples the pathological effects that such nonlinearities can have on system performance. A two-step design philosophy was presented as the means adopted in this work to address this problem. The specific implementation of this philosophy was given, defining the particular class of actuator-limited systems to be considered. A unique characteristic in the formulation is the explicit distinction between the limited and nonlimited actuators.

Chapter 2 explored the role of directionality issues with respect to saturated
control systems. The traditional high-gain antiwindup approach was investigated, and extended to the multivariable case by considering the additional effect of the limit protection on the nonlimited actuators. It was shown that nominal closed-loop considerations can be used to provide limit protection with desirable directional properties via the notion of multivariable sensitivity.

The primary contribution of this research is in Chapter 3, where limit protection design was formulated as a nonlinear robust control problem. To this end, recently developed results in the area of linear matrix inequalities were utilized. These tools provide stability and performance robustness to the class of decoupled sector-bounded limit nonlinearities described in Chapter 1. An iterative design procedure was presented based on these concepts.

This approach to limit protection design is illustrated in Chapter 4 via two examples. Based on these examples, the following conclusions may be made:

- Successful multivariable limit protection can indeed be achieved in non-trivial systems using static output feedback. It was uncertain during the course of this research as to whether this was possible using the simple parameter $\Lambda$. This work has demonstrated the viability of this approach.

- Plant directional properties appear to be exploited using the proposed design approach. This is in contrast to the philosophy of maintaining the control direction, described in Chapter 2, which can be thought of as avoiding directionality issues. Such methods can in general be conservative, since all
control commands are scaled down, independent of their effect on the output response. Moreover, the proposed approach accomplishes effective actuator usage "transparently," via closed loop input-output considerations. In other words, no constraint is placed on the controller behavior during limits—this is addressed automatically through the design procedure.

- High-gain antiwindup behavior is not necessary for successful limit protection. It is a commonly held belief that high-gain antiwindup is a fundamental property for limit protection. However, the limit protection design for the F8 example in Chapter 4 indicates that such a property is not required to provide stability and desirable performance.

Due to the relatively brief treatment of this problem in the literature, further contributions involve outlining some important issues which should be addressed in future investigations. The final sections will elaborate on some of them.

5.2 Limitations of Static $\Lambda$

Recall from Section 1.1 that the restriction of the limit protection parameter, $\Lambda$, to be static is based on the common philosophy that system performance should be modified if and only if limits are in effect. The design examples of Chapter 4 illustrate that this philosophy does indeed have the potential to significantly and positively affect both stability and closed-loop performance properties. However,
it is clear that the simplicity of this parameter may impose fundamental limitations on what can be achieved by the limit protection. This issue is now expanded on with respect to the issues of performance tradeoffs during limited operation, and stabilizability.

5.2.1 Performance Tradeoffs

Achieving design tradeoffs is of fundamental importance in any control design method. To motivate this issue with respect to the limit protection problem, consider the following scenario. Suppose that due to an additional limit on actuator $u_1$ in the F8 system of Section 4.2, it would be desirable to utilize even less actuator effort than is shown in Figure 4.8. A limit protection design that could achieve a controlled tradeoff between actuator effort and output performance would be desirable in this situation. This type of design tradeoff is typically achieved in methods based on closed-loop input-output properties via scalings and/or frequency weightings on the quantities of interest. This was in fact attempted, but to no avail. An investigation into performance limitations imposed by static limit protection may provide insight into the extent of this problem. It might be conjectured that the simplicity of static compensation imposes fairly strong limitations on the possibilities for performance tradeoffs. Alternatively, this could mean that dynamic performance specifications may require dynamic limit protection.
5.2.2 General Stabilizability Considerations

Section 3.3.2 introduced the important concept of stabilizability of the limited system. Indeed, this issue must be addressed before any design can take place. In the context of multivariable systems, this problem takes on a combinatorial nature, since many different combinations of limits can occur. Thus a complete investigation into stabilizability must include all combinations of possible limited actuators. This investigation might consist of two phases:

1. Determination of the actuator combinations that, when bounded by limits, will maintain plant stabilizability. The issue of stabilizability via bounded controls is addressed in [SS90] and the references therein.

2. Given the above notion of plant stabilizability, it must then be determined whether such stabilization can be achieved in the current limit protection framework via the parameter, \( \Lambda \). The determination of general conditions for the existence of stabilizing static output feedback is a current research issue [IS94], [GdSS94]. As in the issue of performance tradeoffs, it is plausible to conjecture that a static \( \Lambda \) constraint may not be sufficient, making necessary the consideration of dynamic \( \Lambda \).

Once these issues are addressed, it will be apparent which actuators must not encounter limits. Furthermore, these considerations can (and should) guide the design of the nominal controller.
5.3 Nyquist Diagram Comparison

Recall Theorem 2 was used to establish stability of the limited systems considered in Chapters 1 and 2, via the Nyquist diagrams of $-H_{sp}^A$. These diagrams are collected here to provide an alternative comparison of the various limit protection schemes which were implemented. Such qualitative comparisons may suggest the plausibility of a design method based on the "open-loop" transfer function, $H_{sp}^A$.

Specifically, Figure 5.1 depicts the Nyquist diagram of $H = -H_{sp}^A$ of SISO System 1 for the CAW cases $X = 0$ (i.e. no limit protection), and $X = 10$, shown via the dashed and solid lines, respectively. Figure 5.2 shows the Nyquist plots for MIMO System 2 for the cases of no limit protection (dashed line), and the decoupled antiwindup approach (solid line). The Nyquist plot for System 2 using the worst-case antiwindup scheme is shown in Figure 5.3. Finally, Figure 5.4 shows the Nyquist diagrams for the F8 example of Chapter 4 for the cases of $\Lambda = 0$ (dotted line), $\Lambda = \Lambda_{stab}$ (dashed line), and $\Lambda = \Lambda_{opt}$ (solid line).

Some interesting observations are immediately evident from the Nyquist plots. In particular, the systems exhibiting superior closed-loop limited performance are seen to give rise to Nyquist diagrams which are farther away from the critical region $\text{Re}[H(j\omega)] \leq -1$. Even more interesting behavior is shown in Figure 5.4. The cases of $\Lambda = 0$ and $\Lambda = \Lambda_{stab}$ exhibit unusual "kinking" behavior in the Nyquist diagrams, whereas the case $\Lambda = \Lambda_{opt}$, in addition to being farther away to the critical region, does not have such kinks. In fact, the frequencies at which
these kinks occur correspond to precisely those frequencies in Figure 4.6 where peaking occurs. This very interesting, because Figure 4.6 depicts limited system performance degradation via the closed-loop MIMO transfer functions $H_i$, whereas the Nyquist diagrams involve only the open-loop SISO transfer function, $H_{y_p}^A$. It would be of interest to determine if these observations are true in general, and if so, whether they can be generalized to the case of multiple limits. This may suggest the existence of a design procedure based on entirely linear considerations, eliminating the need to consider the closed-loop $L_2$-gain (via $2^e$ LTI systems), as described in Chapter 3.

5.4 Alternative Paradigm for Limited Control System Design

An alternative approach, suggested in [BB91], for dealing with plants containing actuator limiting nonlinearities is to incorporate nonlinear stability and performance considerations directly into the original control design, rather than the current philosophy of putting a layer of limit protection over an existing control design. Control design using this philosophy would thus have limit protection “built in.”

A simple way to accomplish this in the context of $H_\infty$ control design is to augment the design plant with the signals $p$ and $q$, as shown in Figure 5.5. By
Theorem 1, a successful control design would require the $H_\infty$-norm of the linear closed-loop system transfer function from $p$ to $q$ to be less than 1 to ensure stability of the limited system. However, this additional small gain specification will most likely sacrifice achievable nonlimited performance. This problem has in fact been observed in attempts to implement such an approach [Mat93b]. Indeed, the primary advantage of the limit protection concept is that nonlimited performance is optimized in a manner independent of any nonlinear considerations due to the existence of limited actuators.

The framework of Chapter 3 suggests an alternative approach for such a design philosophy which is less conservative in guaranteeing stability. Specifically, consider the $2^i$ design plants, shown in Figure 5.6. The control objective in this context is to "simultaneously" minimize the closed-loop $L_2$-gain from $w$ to $z$ for every $i = 0, \ldots, 2^f - 1$.

The potential computational benefits of such an approach are significant compared with the iterative limit protection design approach described in Section 3.3. It is shown in [IS94] that certain robust output feedback control problems can be formulated in terms of LMIs (as opposed to the BMIs of Section 3.3), when the controller order is not fixed a priori. This is a common assumption in practice, and thus allows truly optimal controllers to be obtained via solution of a convex LMI problem. Note that this issue of controller order is similar in spirit to the commonly used $H_\infty$ design methods, which generate controllers of the same order
as the "design plant," i.e. the plant along with all dynamic performance specifications. Thus it is of interest to investigate whether the simultaneous design problem described above can in fact be reduced to an LMI problem.

5.5 Application to NASA STOVL System

An important next step will be the application of the limit protection design method of Chapter 3 to the system which motivated this research. The NASA STOVL system possesses a higher degree of complexity than the F8 example studied in Chapter 4. Specifically, it is an integrated model of the longitudinal flight and propulsion dynamics during the difficult transition from the flight to hover phase of operation. This transition introduces significant multivariable coupling between the flight and propulsion subsystems. The 7th order model, containing 8 actuators and 4 controlled variables, provides a realistic context in which to study the problem of multiple limits (i.e. \( \ell > 1 \)). Furthermore, an important property of this system is the presence of actuator redundancy. It is of interest to examine how the optimization scheme would make use of this redundancy, and whether tradeoffs could be made among the redundant actuators.
5.6 Extension of Limit Protection Framework

This work has focused on the parameter $\Lambda$ to achieve limit protection by modifying the states of the controller, as shown in Figure 1.7. An extension to this framework, proposed in [CMN89], is to additionally modify the output of the controller. In the current context, this additional modification is shown in Figure 5.7, where $\Lambda_1$ modifies the controller states, and $\Lambda_2$ modifies the \textit{nonlimited} controller output, $u_{nl}$. The modification is not applied to $u_l$ in order to avoid algebraic loops.

It is of interest to determine how this extra degree of freedom might affect performance, and whether it would allow greater tradeoff possibilities. Recall from Section 5.2.1 that design tradeoffs appear not to be achievable in the F8 example using only $\Lambda_1$. 
Figure 5.1: Nyquist diagrams of $H = -H_{sp}^\Lambda$ for System 1, $X = 0$ (dashed) and $X = 10$ (solid)

Figure 5.2: Nyquist diagrams of $H = -H_{sp}^\Lambda$ for System 2, $\Lambda = 0$ (dashed) and $\Lambda = \Lambda_{decoup}$ (solid)
Figure 5.3: Nyquist diagram of $H = -H_{qp}^\Lambda$ for System 2, $\Lambda = \Lambda_{wc}$

Figure 5.4: Nyquist diagrams of $H = -H_{qp}^\Lambda$ for F8 system, $\Lambda = 0$ (dotted), $\Lambda = \Lambda_{\text{stab}}$ (dashed), $\Lambda = \Lambda_{\text{opt}}$ (solid)
Figure 5.5: $H_\infty$ design plant for a stabilizing nominal controller using small gain concepts

Figure 5.6: Alternative "simultaneous" design plant framework for stabilizing nominal control design
Figure 5.7: Extension of limit protection framework
Bibliography


