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COMPUTATIONAL SCHEMES FOR EXACT LINEARIZATION OF DISCRETE-TIME SYSTEMS USING A GEOMETRIC APPROACH

by

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Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy

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COMPUTATIONAL SCHEMES FOR
EXACT LINEARIZATION OF DISCRETE-TIME SYSTEMS
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Abstract

by

GANGADHAR JAYARAMAN

In this work we present a theory for exact linearization of discrete-time systems. Using this theory, a nonlinear system can be transformed to a linear, reachable system through a state coordinate change and a nonlinear state feedback law. This permits the controller design to be carried out in the new coordinates using linear system theory. We assume that the system map $f$ possesses an equilibrium point. and that its Jacobian is full rank around this point. We develop a mathematical framework for the following problems: i) Exact linearization using state coordinate change, and ii) Exact linearization using state coordinate change and feedback, also called feedback linearization. Vector fields and one-forms are defined using the tangent map and the co-tangent map induced by the system map. The necessary and sufficient conditions for the first problem are geometric conditions on $f$ related vector fields. The second problem is solved using two different methods. In one method, we define a
nested sequence of involutive and constant dimensional distributions using the $f$-related vector fields. Sufficient conditions for feedback linearizability are expressed in terms of these distributions. Using Frobenius's Theorem we obtain a set of partial differential equations representing the integrability conditions for these distributions. The linearizing transformation and feedback law are constructed from the solution to this set of partial differential equations. In a second approach to the feedback linearization problem, we define column vectors and row-vectors for the discrete-time systems. We show that the necessary and sufficient conditions for feedback linearizability can be rewritten in an easily verifiable form using these column vectors. We define relative degree for the discrete-time system using the row-vectors, and show that the linearizing coordinate transformation and feedback law can be constructed if the nonlinear system has full relative degree. We illustrate our theory through two models of electrically stimulated muscle from biomedical control engineering.
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LIST OF NOTATIONS

$C^\infty$ differentiable to arbitrary order
$M$ $C^\infty$ manifold (state-space)
$U$ $C^\infty$ manifold (input-space)
$M \times U$ $C^\infty$ product manifold
$\hat{\varphi}_M$ local coordinate map on $M$
$\hat{V}_M$ local coordinate neighborhood on $M$
$\hat{\varphi}_{M \times U}$ local coordinate map on $M \times U$
$V_{M \times U}$ local coordinate neighborhood on $M \times U$
$(V_{M \times U}, \hat{\varphi}_{M \times U})$ coordinate chart for $M \times U$
$(\hat{V}_M, \hat{\varphi}_M)$ coordinate chart for $M$
$x$ state vector. or local coordinates for a point on $M$
$u$ input vector. or local coordinates for a point on $U$
$(x, u)$ local coordinates for a point $p$ on $M \times U$
$f : M \times U \to M$ $C^\infty$ system map
$f_u : M \to M$ map $f$ parameterized by input argument $u$
$f^{-1}(x)$ inverse image of a point $x$ on $M$ under the map $f$
$(x^0, u^0)$ equilibrium point for the map $f$
$\pi_M : M \times U \to M$ projection on to the $x$ coordinates
$\pi_U : M \times U \to U$ projection on to the $u$ coordinates
$\pi^{-1}_M(x)$ inverse image of a point $x$ on $M$ under the map $\pi_M$
$T_p(M \times U)$ tangent space at point $p$ on the manifold $M \times U$
$T_{f(p)}M$ tangent space at point $f(p)$ on $M$
$T(M \times U)$ tangent bundle on $M \times U$
$TM$ tangent bundle on $M$
\( T^*_p(M \times U) \) cotangent space at point \( p \) on \( M \times U \)

\( T^*_{f(p)}M \) cotangent space at point \( f(p) \) on \( M \)

\( T^*(M \times U) \) cotangent bundle on \( M \times U \)

\( T^*M \) cotangent bundle on \( M \)

\( X, Y \) vector fields on \( M \times U \)

\( \frac{\partial}{\partial u_j} \) unit vector field along the \( u_j \) coordinate

\( \frac{\partial}{\partial x_i} \) unit vector field along the \( x_i \) coordinate

\( \frac{\partial}{\partial z_i} \) unit vector field along the \( z_i \) coordinate

\( f_\ast p \) tangent map induced by \( f \) at point \( p \) on \( M \times U \)

\( f_{f(p)}^\ast \) adjoint map induced by \( f \) at point \( f(p) \) on \( M \)

\( (\pi_M)_\ast p \) tangent map induced by \( \pi_M \)

\( (\pi_M)^\ast_{f(p)} \) cotangent map induced by \( \pi_M \) at point \( f(p) \)

\( (\pi_U)_\ast p \) tangent map induced by \( \pi_U \)

\( G_i, \bar{G}_i \) distributions defined on \( M \times U \)

\( P_j \) codistribution defined on \( M \times U \)

\( \ker f_\ast \) distribution spanned by vector fields in kernel of \( f_\ast \)

\( \ker P_j \) kernel of the codistribution \( P_j \)

\( \text{ann } G_i \) annihilator of the distribution \( G_i \)

\( D^i f_{\frac{\partial}{\partial u_j}} \) \( f \)-related vector field

\( h, h_i, \lambda, \lambda_i \) real valued \( C^\infty \) functions defined on \( M \times U \)

\( dh, dh_i, d\lambda, d\lambda_i \) exact one-forms induced by the respective \( C^\infty \) functions

\( P^k_f d\lambda \) \( f \)-related exact one-forms induced by \( \lambda \)

\( \langle \cdot, \cdot \rangle \) lie derivative

\([\cdot, \cdot] \) lie bracket

\( \ll \cdot, \cdot \rr \) formal calculation of lie derivative

\([\cdot, \cdot] \) formal calculation of lie bracket
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1.1 Dissertation Overview

In this dissertation we study the problem of designing controllers for discrete-time nonlinear systems. In order to enhance the closed loop performance of these systems, we require that these controllers take into account and exploit the inherent nonlinearities in the model, rather than ignoring or approximating them away. The controller proposed here uses the principle of feedback linearization. By this technique, a nonlinear system can be transformed to a linear controllable system through a nonlinear state coordinate change and nonlinear state feedback. Controllers are then designed for the feedback linearized system, using linear system theory. The primary goal of this dissertation is to develop the theory and techniques for the design of feedback linearizing controllers for discrete-time systems.

We develop a mathematical framework for the feedback linearization problem for discrete-time systems by using tools from the theory of differentiable manifolds. Vector fields and one-forms are defined using the tangent map and co-tangent map induced by the system map $f$ of the discrete-time system. We
assume that the system is operated around an equilibrium point. Also, technical assumptions are made on the system map to ensure that the $f$-related vector fields are well-defined.

These $f$-related vector fields are used to formulate and solve the following problems for discrete-time systems.

i) **Exact Linearization by State Space Transformation.** Determine a change of state coordinates that will render a given nonlinear system input-state equivalent to a linear, reachable system at every sample instant.

ii) **Exact Linearization by State Space Transformation and Nonlinear State Feedback.** Determine a change of state coordinates and a nonlinear state feedback law that will render a given nonlinear system input-state equivalent to a linear, reachable system at each sample instant.

We solve the first problem by applying the *Flow Box Theorem* to a set of $n + m$ linearly independent $f$-related vector fields. This results in a set of $n^2 + nm$ linear, first-order partial differential equations. The coordinate transformation is obtained from the $n + m$ solutions to this set of partial differential equations. To solve the second problem, the $f$-related vector fields are used to construct a nested sequence of involutive and constant dimensional distributions characterizing feedback linearizability. *Frobenius’s Theorem* is used to construct the annihilators of these distributions, which turn out to be codistributions spanned by $f$-related exact one-forms. The integrability conditions on this sequence of involutive and constant dimensional distributions results in a set of linear first-order partial differential equations. The linearizing transformation and feedback are then constructed from the solution to this set of partial differential equations.
The contributions of this work are as follows.

i) The problem of feedback linearization for discrete-time systems is formulated in a geometric setting. In addition to providing mathematical precision and geometric intuition, this approach allows direct comparisons with existing results for continuous-time systems.

ii) We have derived for multi-input discrete-time systems necessary and sufficient conditions for exact linearization by state-space transformation. A procedure for constructing the linearizing transformation is also developed.

iii) We have obtained necessary and sufficient conditions for feedback linearization of single-input discrete-time systems and a construction for the linearizing coordinate transformation and feedback law.

The important practical implication of this work is that it represents a first step in digital implementation of feedback linearizing controllers for nonlinear systems. We demonstrate this by applying our theory to a nonlinear controller design problem in the area of functional electrical stimulation for orthoses. From a theoretical standpoint, our approach is elegant and insightful because it treats single-input and multi-input systems in a unified way, and the results obtained using this approach are analogous to those for continuous-time systems. It is also interesting, but not entirely surprising, that the solution to the discrete-time problem should come from the solution to a set of partial differential equations.

The rest of this chapter is organized as follows. In Section 1.2, we discuss two examples of nonlinear systems for which linear control methods do not
yield satisfactory results. The purpose is to highlight the need for discrete-time nonlinear controllers to improve the performance of these systems. A precise statement of the problems addressed in this dissertation and the techniques used to solve them are given in Section 1.3. In Section 1.4 we provide a brief survey of current research on linearization of discrete-time systems, with emphasis on the geometric approach. The objectives and scope of this dissertation are listed in Section 1.5, and the significant contributions of this work are given in Section 1.6.

1.2 Background and Motivation

In many engineering applications we encounter systems that are inherently nonlinear. That is, ignoring or approximating away these nonlinearities results in significant deterioration in the performance of the closed loop control system. In this section we consider two examples of such nonlinear systems, and discuss the difficulties in designing digital controllers for these systems. One example is a continuous-time model of an industrial robot manipulator arm. The conventional approach to implement nonlinear controllers for these systems is to use a digital computer to evaluate a discretized version of the continuous-time feedback linearizing law, and apply this control law to the nonlinear plant through a zero-order hold. It has been shown that this can degrade the performance of the closed-loop system if the sampling rate is not high enough. A second example is a discrete-time model of electrically stimulated muscle. Due to the physiological properties of the muscle, and also for practical reasons of implementation, this system is modeled in discrete-time.
We believe that the design of feedback linearizing controllers in discrete-time will solve the control problems common to both these systems.

1.2.1 Robot Manipulator Arm

Industrial robot manipulators are, in essence, interconnections of rigid members through revolute and prismatic joints resulting in multiple degree-of-freedom kinematic chains governed by nonlinear kinematic and nonlinear dynamical equations. The dynamics of the manipulator are such that the motion of any one joint is influenced strongly by the motion of every other joint and the inertial load reflected on any one joint varies significantly as the configuration of the manipulator arm changes with time. At high speeds of operation of the manipulator arm, the coriolis and centrifugal effects become significant. The effect of the nonlinear terms is somewhat minimized when the manipulator arm is driven indirectly through a system of gears. This is because the inertia terms are reduced by a factor equal to the square of the gear ratio, and some other nonlinear terms such as interaction torque and external disturbances are reduced by a factor equal to the gear ratio. However, the present trend favors a mechanical design where high torque motors are directly coupled to each joint. This has the advantages of high mechanical stiffness, no backlash and low friction. But direct-drive manipulator design implies full coupling; therefore the effects of the nonlinear dynamics are enhanced by the high speeds of operation. Currently, the popular method of controller design for robot arms is the individual joint PID, in which local, decoupled PID’s are employed at each joint. Though this method works well for simple position control, it causes overshoot in the transient performance which is unacceptable in such applications as plasma welding, laser cutting or high speed operations in
the presence of obstacles. In order to better exploit the mechanical capabilities of high performance robots, it is necessary to understand how to account for the nonlinearities effectively in the process of designing the controller.

Feedback linearization as a method for designing controllers for robot manipulator arms was pioneered by [5] and [9]. These controllers use state feedback and state coordinate transformation for an exact cancellation of the nonlinear terms. We can make this statement more precise. Suppose that the given nonlinear system is described by the following differential equation

\[ \dot{x} = \xi_0(x) + \sum_{j=1}^{m} \xi_j(x)u_j \quad (\Sigma_{c\text{NLS}}) \]

where \( \xi_j \), for \( 0 \leq j \leq m \), are smooth vector fields defined on the state-space. Consider the coordinate transformation and the nonsingular feedback given by

\[ z = S(x) \]
\[ u_j = \alpha_j(x) + \sum_{i=1}^{m} \beta_{ij}(x)v_i \quad (1.1) \]

for \( 1 \leq j \leq m \), where the matrix \( S(x) \) is nonsingular. The vector fields of the feedback transformed system are given by \( \dot{\xi}_j(x) \) for \( 0 \leq j \leq n \), where

\[ \dot{\xi}_0(x) \overset{\text{def}}{=} \xi_0(x) + \sum_{j=1}^{m} \xi_j(x)\alpha_j(x) \]
\[ \dot{\xi}_j(x) \overset{\text{def}}{=} \sum_{i=1}^{m} \left( \sum_{j=1}^{m} \xi_j(x)\beta_{ij}(x) \right)v_i \]

The nonlinear system \( \Sigma_{c\text{NLS}} \) is feedback linearizable if, in the new coordinates, the dynamics are given by the linear system

\[ \dot{z} = Az + Bv \quad (\Sigma_{c\text{LS}}) \]
where $B = [b_1, \ldots, b_m]$ and $(A, B)$ is controllable. The matrices $A$ and $B$ are given by

$$Az = (\frac{dS}{dz})_{x=S^{-1}(z)} \hat{\xi}_0 (S^{-1}(z))$$

$$b_j = (\frac{dS}{dz})_{x=S^{-1}(z)} \hat{\xi}_j (S^{-1}(z))$$

The compensated system is input-state equivalent to a linear controllable system, that is, assuming that the same input signal is applied to both systems, the states of the compensated system and the states of the reference linear system are identical. Therefore, one may now use methods from linear system theory to design controllers to meet some performance specification for the compensated system such as tracking or regulation. The design of the controller to meet these objectives is done in the new coordinates with $u(t)$ as the new input. This technique has been used to compute, analytically, the linearizing transformation and feedback law for nonlinear models of many engineering systems (see [7], [15], and [31]). The feedback law is usually quite complex and requires a digital computer implementation which is illustrated below.

![Diagram](image)

**Fig. 1.1:** Discretization of continuous-time feedback linearizing controller

This description of discretization ignores the effects of quantization and the finite computation time of the digital computer. But even from this simplified model, we observe that the control input $\tilde{u}(t)$ to the nonlinear system is held constant by the zero-order hold for the time interval $kT \leq t < kT+T$, for $k \geq 0$. 
Evidently, this is not equal to the input $u(t)$ obtained by inverting the feedback law: hence the system from input $v(t)$ to the state $z(t)$ is not linear. In fact, simulation results in [14] and [27] show that a direct digital implementation of the continuous-time feedback laws can cause undesirable effects in the closed loop system, when the hold-time/sample-time is large relative to the simulation step-size. Therefore, to implement feedback linearizing controllers digitally, one must ensure that the sampling frequency is sufficiently large.

![Block diagram of feedback linearizing controller](image)

**Fig. 1.2:** Feedback linearizing controller in discrete-time.

In this work we propose to overcome the problems of discretization by designing the feedback linearizing controller for the sampled model (see Fig. 1.2) of the continuous-time system. This technique requires that the sampled model be computable. We must first check if the sampled model is feedback linearizable. Then, the linearizing transformation and feedback law are computed. Finally, digital controllers are designed for the compensated system in the new coordinates.

### 1.2.2 Electrically Stimulated Muscle

Control theory has recently been used in the area of Functional Electrical Stimulation (FES) orthoses to restore functional muscle contractions in para-
lyzed individuals. Small electrical pulses are used to activate motor axons in limbs, and act as a substitute for control signals lost as a result of spinal cord injury or brain damage (see [25]). The controller for such a system must use information of the joint angles, joint velocities and a reference set point, and compute the pulse width signal which will drive the joint through a desired trajectory.

A twitch is produced when a single electrical pulse applied to a muscle causes an excitation in individual fibers in the muscle. When a train of pulses is applied, there is a temporal summation of the force produced by each fiber: the net force produced is proportional to the stimulation frequency, that is, the frequency of the pulse train. At a sufficiently large stimulation frequency, called the fusion frequency, the force produced is constant and relatively free of ripples. In practice, the experiment is done at a stimulation frequency which results in less than 10 percent ripple in the mean output force. At constant stimulation frequency, muscle force can be modulated by recruitment modulation. This is done by varying the pulse width of the stimulation signal, by which we “recruit” extra fibers in the force-generation activity. The recruitment characteristic is the relationship between the pulsewidth of the stimulation signal and the steady-state force at constant stimulation frequency. This characteristic is nonlinear and time varying due to uneven distribution, orientation and innervation ratios of motor axons within the muscle.

It is reasonable to model stimulated muscle in the discrete-time domain (see [3]), because the all-or-none behavior of the neuromuscular system causes muscle dynamics to be independent of the stimulus pulse shape. In addition, the dominant time constants of muscle are short in relation to the pulse duration and the interpulse interval. Another advantage of these models is that
recursive parameter estimation algorithms may be implemented using digital computers for on-line identification.

Fig. 1.3: Uncoupled model of electrically stimulated muscle

We study two types of models [34], for describing the dynamics of electrically stimulated muscle: the uncoupled model, and the coupled model. According to the uncoupled model, the force generated by the stimulated muscle is a product of three factors: muscle activation, torque-angle and torque-velocity. Each factor is a linear approximation of the actual behavior of the muscle. The dynamics of muscle activation are described by a first- or second-order discrete-time linear model. The model is called uncoupled because the activation dynamics are independent of the torque-angle and the torque-velocity factors. The torque-angle and torque-velocity factors approximate the normalized torque-angle and torque-velocity characteristics by straight lines of slopes $c$ and $d$ respectively. Since the muscle acts on a skeletal joint, the torque produced by muscle contraction causes an angular displacement of the limb, relative to some reference position of the limb. It is this angle that appears in the torque-angle and torque-velocity factors. The limb dynamics are described by a second order linear differential equation. For the purposes of digital control, it is assumed that a sampled model of the limb dynamics is available. In the coupled model, the force generated by the muscle is a product of two
factors: muscle activation and torque-angle. It has been demonstrated that the dynamics of electrically stimulated muscle display memory of the velocity history. The primary reason for developing this model is to accommodate this property, and possibly improve the accuracy of the coupled model. Moreover, it has been shown that the coupled model is less sensitive to parameter errors. The activation dynamics for the coupled model are given by a first- or second-order difference equation in which activation and torque-velocity terms are multiplied together. The torque-angle and torque-velocity factors are given by the same input-output relationship as for the uncoupled model. The load model is identical to that used for the uncoupled model.

![Fig. 1.4: Coupled model of electrically stimulated muscle](image)

The parameters of the models of electrically stimulated muscle have been obtained in [33], [34] using a nonlinear recursive least square parameter estimation algorithm. The objective of the controller is to generate the value of the pulsewidth of the stimulation signal which, when applied to the muscle, will ensure satisfactory response of the closed loop system.

The discrete-time model of the muscle, coupled with the limb dynamics, yields a hybrid system for which digital control methods must be employed. Linear controllers have been used with satisfactory results in [6], [38] only for
the special case of isometric contractions. For this case a digital controller was
designed to regulate the muscle output force. In this dissertation, we design
nonlinear controllers for these systems by developing a theory of feedback
linearization for these systems. The results on exact linearization derived in
[8] are extended to include cases when the model does not possess full relative
degree.

1.3 Problem Statement

In this work we address the following problems for discrete-time nonlinear
systems.

\textbf{P1} Exact Linearization by State Space Transformation. Suppose that the
nonlinear system is given by

\[ x(k+1) = f(x(k),u(k)) \quad (\Sigma_{DNS}) \]

where the dimension of the state space is \( n \) and that of the input space
is \( m \). Determine a change of state coordinates given by \( z(k) = T(x(k)) \),
such that

\[ z(k+1) = T \circ f(T^{-1}(z(k)), u(k)) = Az(k) + Bu(k) \]

where \((A,B)\) is a reachable pair.

\textbf{P2} Exact Linearization by State Space Transformation and Nonlinear State
Feedback. Suppose that the nonlinear system is given by

\[ x(k+1) = f(x(k),u(k)) \quad (\Sigma_{DNS}) \]
where the dimension of the state space is $n$ and that of the input space is $m$. Determine a change of state coordinates given by $z(k) = T(x(k))$, and a nonsingular feedback law $v(k) = \gamma(x(k), u(k))$, such that

$$z(k + 1) = T \circ f \left( T^{-1}(z(k)), \gamma^{-1}(T^{-1}(z(k)), v(k)) \right) = Az(k) + Bv(k)$$

where $(A, B)$ is a reachable pair.

We are interested in obtaining the following theoretical results.

i) A set of necessary and sufficient conditions to check if problem P1 is solvable.

ii) A procedure to construct the coordinate transformation required for P1.

iii) A set of necessary and sufficient conditions to check if problem P2 is solvable.

iv) A procedure to construct the coordinate transformation and feedback law required for P2.

We define $f$-related vector fields and $f$-related one-forms for the discrete-time system. These are used for a geometric description of both P1 and P2. For problem P1 we construct a Lie algebra using these vector fields. The proof of the main result follows from an application of the Flow-Box Theorem (see [16], [32]) to $n + m$ linearly independent vector fields selected from this Lie algebra. The required transformation is obtained by solving a set of $n^2 + nm$ first-order linear partial differential equations.

Problem P2 is solved in two different ways. In one approach, we define a nested sequence of distributions in terms of the $f$-related vector fields.
Sufficient conditions for feedback linearizability are obtained for single-input and multi-input systems, and these are stated in terms of this sequence of distributions. We then apply Frobenius’s Theorem (see [16], [32]) to this set of distributions, and obtain the integrability conditions expressed by a set of \( n + m \) linear, first-order partial differential equations. The required coordinate transformation and feedback law are obtained by solving this set of partial differential equations.

In a second approach to problem P2, we obtain, for single-input systems, necessary and sufficient conditions for feedback linearizability. These conditions are expressed in terms of column vectors and row-vectors, defined in local coordinates using the system map. The conditions thus derived are easier to verify than existing geometric conditions given by distributions. The linearizing transformation and feedback law are obtained from the solution to a set of linear, first-order partial differential equations. These partial differential equations are written down using the row-vectors and column vectors. We demonstrate the equivalence of our conditions to existing geometric conditions.

1.4 Literature Survey

The class of discrete-time systems studied here is given by

\[
x(k + 1) = f(x(k), u(k)) \quad (\Sigma_{\text{DNLs}})
\]

where \((x, u)\) and \(x\) are the local coordinates for \( M \times U \) and \( M \) respectively. Following standard convention, we denote the dimension of the state manifold \( M \) by \( n \) and that of the input manifold \( U \) by \( m \). The map \( f \) is called the system map.
The problems of accessibility, observability, invariant distributions and feedback linearizability have been formulated for continuous-time systems given by \( \Sigma_{CNS} \). In terms of Lie algebras of vector fields, or distributions defined by the drift and input vector fields. For discrete-time systems, operations commonly used for continuous-time systems such as the Lie bracket and the Lie derivative, and the way coordinate transformations act on the system dynamics, do not carry over directly. To study discrete-time systems in a geometric framework, one has to define vector fields and distributions artificially, using the system map \( f \).

This has been done by some authors by defining vector fields using the idea of infinitesimal perturbations of the control variable. For example, in a recent expository paper [18], these vector fields are defined by taking partial derivatives with respect to the control variable of certain compositions of the map \( f \) and its inverse. Lie algebras are then defined by the action of diffeomorphisms on the vector fields defined above. The authors use these Lie algebras to investigate accessibility, the effect of sampling on accessibility, and the geometry of non-accessible sets. A geometric treatment of a strong form of local observability of discrete-time systems is given in [31]. A sufficient condition for observability is derived using one-forms, and a canonical form for observable systems is given. An important concept in the geometric theory of control systems is that of an invariant distribution. In [10], [12] a geometric condition for local invariance is given in terms of distributions which are defined recursively by the system map \( f \). In complete analogy with continuous-time theory, necessary and sufficient conditions for local controlled invariance are derived, and an algorithm for computing the maximal controlled invariant distribution is given. This theory of invariant distributions is used to formulate.

One reason for developing necessary and sufficient conditions for feedback linearizability of discrete-time systems is the possible loss of feedback linearizability of a continuous-time system through sampling. Let the $T$-sampled system associated with the system $\Sigma_{CNLS}$ be given by

$$x(k+1) = f_T(x(k), u(k)) \quad (\Sigma_{DNLS}^T)$$

where $f_T$ is a mapping satisfying $f_T(\zeta, \mu) \overset{\text{def}}{=} \text{solution } x(t) \text{ at time } t = T \text{ of the continuous-time system } \Sigma_{CNLS}, \text{ with initial condition } x(0) = \zeta, \text{ and constant control } u_k = \mu \text{ on the interval } (kT, k+1T)$. The system $\Sigma_{CNLS}$ is said to be sampled feedback linearizable if the discrete-time system $\Sigma_{DNLS}^T$ is feedback linearizable for all $T \in (0, \delta)$, for some $\delta > 0$. [13] has given the following example of a system that is not sampled feedback linearizable.

$$\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= (1 + x_2^2)u
\end{align*} \quad (1.2)$$

Notice that the system (1.2) is linearizable by a simple feedback law. It is conjectured in [13] that the system $\Sigma_{CNLS}$ is sampled feedback linearizable if and only if it is state-equivalent to a linear system. The conjecture has been confirmed for the special case of $n = 2$, and $m = 1$ in [1] and [20] using a proof based on a series expansion of the flow of the continuous-time system.

We now review the existing results on discrete-time feedback linearization. In [13] a nested sequence of distributions is defined using the tangent maps $f_*$ and $(\pi_M)_*$ induced by the system map $f$ and the projection map $\pi_M$ respectively. The distributions are defined recursively; at stage $k$ the distribution $\hat{G}_k$
is required to satisfy a well-definedness condition so that a basis of vector fields is guaranteed for the distribution $\tilde{G}_{k+1}$ defined at the next stage. The author shows that the system is feedback linearizable if and only if the dimension of the distribution $\tilde{G}_n$ is equal to $n$, the state dimension. The proof is completely analogous to the proof of the corresponding theorem for continuous-time systems. The main drawback of this paper is that the computations required to check for involutivity and constant-dimensionality of these distributions are quite involved. We need to solve a continuous-time controlled invariance problem at each stage to compute a basis for the distribution $\tilde{G}_k$. The method of successively modifying the coordinates to put the map $f$ in linear form is also very cumbersome.

[17] has characterized and solved the problem of feedback linearization in three different ways. The first approach uses a sequence of nested distributions defined on the state manifold. The distributions are defined recursively using the matrices $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial u}$, where $f$ is the system map. The conditions for feedback linearizability are mainly dimension requirements on these distributions. In the second version, feedback linearizability of the system is shown to be equivalent to the linearizability of a pair of submersions $(f, \pi_M)$, for which necessary and sufficient conditions are given. Finally, the author characterizes feedback linearizability in terms of attainable sets of the discrete-time system. The attainable set $A_i(x)$ is defined as the set of points in the state space reachable from $x$ in $i$ steps. For feedback linearizability, these attainable sets are required to be submanifolds of constant dimension, with $\dim A_n(x) = n$. Unlike [13], the conditions for feedback linearizability do not require checking for
involutivity of distributions. However, a method for constructing the coordinate transformation and feedback law is not given. Hence, this result, too, is useful only as a set of testable conditions for feedback linearization.

Exact linearization and approximate linearization of single-input and multi-input discrete-time systems is discussed in [21]. In approximate linearization, one looks for a state-coordinate change such that in the new coordinates, the system map is a sum of two components—a linear part and a part consisting of higher order nonlinear terms. Necessary and sufficient conditions for both types of linearization are derived through a Taylor series expansion of the system map around an equilibrium point. The attractive feature is a sufficient condition that is relatively easy to check. The problem of linearization by coordinate change, and the problem of linearization by coordinate change and feedback are studied in [22] for the single-input case. The necessary and sufficient conditions given are easy to check, and the coordinate transformation is obtained in a straightforward way from the system map $f$. However, the single-input and multi-input cases are not treated in a unified way, and the connection to continuous-time results is not clear.

Consider the problem of feedback linearization for the class of systems given by

$$x_{k+1} = G_{u_k} \circ \mathcal{H}(x_k)$$

(1.3)

where $\mathcal{H}$ is a diffeomorphism, and $G_{u_k}$ is a one-parameter group of diffeomorphisms. Necessary and sufficient conditions under which the system (1.3) is feedback linearizable are given in [30]. It is shown that every linearizable system is necessarily of the form given above. Vector fields associated with the system map are then defined as follows. Define $g$ to be the vector field such that $G(x_k, u_k) = \Phi^g_{u_k}(x_k)$, where $\Phi^g_{u_k}$ denotes the flow of the vector field
Then $H$-related vector fields are defined by the action of the tangent map $H_*$ on the vector field $g$. The necessary and sufficient conditions are stated in terms of linear independence and involutivity of certain distributions spanned by these vector fields. The coordinate transformation and feedback are then obtained by solving a set of partial differential equations arising from integrability conditions for the above distributions. The drawbacks of this work are:

i) It is not applicable to general nonlinear systems $\Sigma_{DNS}$, and ii) The results hold only for single input systems.

If a given sampled model of the continuous-time system is not feedback linearizable it is useful to investigate if a multi-rate sampled model [14] is feedback linearizable. To obtain this model, the input signal $u \in \mathcal{R}^m$ is sampled $N$-times faster than the states, with $n \leq Nm$. Sufficient conditions for feedback linearizability of the multi-rate model are given; these are based on its approximate linearization. The use of multi-rate sampled model was used in [28] to study the preservation of zero dynamics under sampling.

1.5 Objectives

i) To formulate the exact linearization problem for discrete-time nonlinear systems in a geometric framework, using the theory of differentiable manifolds.

ii) We wish to use the above framework to obtain necessary and sufficient conditions for exact linearization by state coordinate transformation. We would also like to derive a method to construct the coordinate transformation.
iii) To use the above framework to obtain necessary and sufficient conditions for exact linearization by state coordinate transformation and nonlinear state feedback. We would also like to derive a method to construct the coordinate transformation and feedback law.

1.6 Contributions

i) We developed a mathematical framework to study the problem of exact linearization of single-input and multi-input discrete-time systems in a unified way. The key elements in this framework are one-forms and vector fields for the discrete-time nonlinear system; these were defined using the system map $f$.

ii) For multi-input systems, we obtained necessary and sufficient conditions for exact linearization using state-space transformation. A method for constructing the linearizing transformation was also derived.

iii) A sufficient condition for feedback linearizability of single-input and multi-input discrete-time systems was obtained. This was done by defining a sequence of distributions spanned by $f$-related vector fields. By imposing integrability conditions on this sequence of distributions, we obtained a system of linear, first-order partial differential equations. The solution to this set of partial differential equations was used to construct the required coordinate transformation and feedback law.

iv) For single-input systems, we developed an algorithm for the coordinate calculations of the distributions that characterize feedback linearizability. A set of necessary and sufficient conditions for feedback linearizability was
derived: these conditions are more easily verified than those existing in the literature.

u) The theory developed here was used to feedback linearize two nonlinear models of electrically stimulated muscle: the uncoupled three-factor model with first-order activation dynamics and the coupled two-factor model with first-order activation dynamics.

1.7 Dissertation Outline

In Chapter 2, we develop a geometric framework to investigate the exact linearization problem for discrete-time systems. Vector fields and one-forms are defined using the tangent map, and the local coordinate calculations to compute these objects. A set of necessary and sufficient conditions for problem $P_1$ is stated in terms of these vector fields, and a procedure to construct the desired coordinate transformation is derived. For problem $P_2$ the vector fields are used to construct a sequence of nested distributions. We derive sufficient conditions for feedback linearizability: these conditions are stated for single-input and multi-input systems in terms of these distributions. The one-forms are used to obtain a set of linear, first-order partial differential equations representing integrability conditions on this sequence of distributions.

We solve the single input case of Problem $P_2$ in Chapter 3. Column vectors and row-vectors are defined in local coordinates for the discrete-time system. Necessary and sufficient conditions are derived by showing that the geometric conditions for feedback linearizability can be re-written in terms of these simple-to-compute column vectors. The row-vectors are used to write
a set of partial differential equations; we show that the solution to this set of partial differential equations yields the required coordinate transformation and feedback law.

In Chapter 4, we apply the techniques developed in Chapter 3 to nonlinear models representing the dynamics of electrically stimulated muscle. We verify that these discrete-time models are indeed feedback linearizable, and compute the linearizing coordinate transformation and feedback law by solving the appropriate set of partial differential equations. We summarize our work in Chapter 5, and identify some related research problems for future work.
GEOMETRIC RESULTS ON
EXACT LINEARIZATION

In this chapter we present a geometric framework which we use to investigate the exact linearization problem for discrete-time nonlinear systems. This framework allows us to study, in a unified way, the related problems of exact linearization using state coordinate change only, and exact linearization using state coordinate change and feedback, for both single-input and multi-input systems. By defining vector fields and one-forms for discrete-time systems, we are able to compute distributions and codistributions that provide a geometric characterization of feedback linearizability. The proofs of the main results provide a method to construct the coordinate transformation and feedback.

In this paragraph, we briefly describe the advantage of the geometric approach for the analysis of nonlinear control systems. A property characteristic of a nonlinear control system is that its dynamics evolve in a state space that is not necessarily a Euclidean space $\mathbb{R}^n$, but a "curved" space of dimension $n$. Hence, the dynamical equations describe the system behavior only in a part of the state space. The approach we take to studying nonlinear systems, called the geometric approach, models the state space as a differentiable manifold which is "locally like" $\mathbb{R}^n$. In the language of differentiable manifold theory (see [4], [36]), we say that the given dynamical equation is a local coordinate representation of the system, and in order to cover the entire state-space more than one coordinate representation may be required. The different coordinate
representations are then pieced together using the notion of coordinate transformation. Using the geometric approach, it is possible to understand certain properties of nonlinear control systems without resorting to cumbersome local coordinate calculations. These properties are said to be coordinate-free. Feedback linearizability is one such property. We investigate this property for discrete-time systems, using tools from differentiable manifolds.

The rest of this chapter is organized as follows. In Section 2.1 we introduce some key definitions and concepts for the geometric study of discrete-time systems. Coordinate-free definitions of f-related vector fields and f-related exact one-forms for discrete-time systems are given. We also outline the steps for local coordinate computation of these objects and establish some useful properties. In Section 2.2 we use these vector fields to formulate and solve the problem of exact linearization of discrete-time systems by state coordinate transformation. The techniques and the results developed are illustrated using an example of a single-input second order nonlinear system, given at the end of the section. In Section 2.3, we use these vector fields to construct recursively a nested sequence of distributions $G_0, G_1, \ldots, G_n$, where $n$ is the dimension of the system. This sequence of distributions, used to characterize feedback linearizability of discrete-time systems, is more easily computable than the distributions $G_0, \ldots, G_n$ given in [13]. This point is discussed in detail. The main result in this section is a sufficient condition for feedback linearizability of single input systems. We show that if $G_0, G_1, \ldots, G_n$ are all involutive and constant dimensional with $\dim G_n = n + m$, where $m$ is the number of inputs, then the system is feedback linearizable. The solution to the set of partial differential equations arising from the integrability conditions is used to construct the linearizing transformation and feedback law. The technique
is illustrated using a simple example. These results are extended to square multi-input systems by making certain additional assumptions.

2.1 Geometric Framework for Discrete-Time Systems

In this section we develop a geometric framework for discrete-time systems, largely along the lines of [10]. However, we have introduced several new tools which are useful for computing the distributions and codistributions that characterize feedback linearizability. For our purposes, the state space and input space for the discrete-time system are modeled as smooth manifolds. The system map is a smooth map \( f : \mathcal{M} \times U \to \mathcal{M} \) where \( \mathcal{M} \times U \) is a smooth manifold. The canonical projection maps \( \pi_M : \mathcal{M} \times U \to \mathcal{M} \) and \( \pi_U : \mathcal{M} \times U \to U \) have the property that in local coordinates \((x, u)\), \( \pi_M(x, u) = x \) and \( \pi_U(x, u) = u \).

There is a subtle difference between discrete-time systems and continuous-time systems in the way we attach a coordinate chart to the state space. The technique used for continuous-time systems does not work, because for discrete-time systems, the point obtained by applying \( f(\cdot, u) \) to a point \( x \) in a local coordinate chart may leave the chart even for small inputs. Hence, to do local coordinate calculations for discrete-time systems, we introduce a pair of coordinate charts: one in the domain of \( f \) and one in its image. Suppose that \((x^0, u^0) \in \mathcal{M} \times U\), and consider the point \( f(x^0, u^0) \in \mathcal{M} \). We select a coordinate chart \((\tilde{V}_M, \tilde{\varphi}_M)\) around \( f(x^0, u^0) \) and consider the open set \( f^{-1}(\tilde{V}_M) \) around \((x^0, u^0)\). Now choose a coordinate chart \((V_{M \times U}, \varphi_{M \times U})\) around \((x^0, u^0)\) such that \( V_{M \times U} \subset f^{-1}(\tilde{V}_M) \). We call \((V_{M \times U}, \varphi_{M \times U}), (\tilde{V}_M, \tilde{\varphi}_M)\) a coordinate chart pair. The coordinates for \((V_{M \times U}, \varphi_{M \times U})\) are denoted as \((x, u)\) and those
for \((\tilde{V}_M, \hat{x}_M)\) are denoted by \(x\). With this slight abuse of notation one may perform local coordinate calculations as though one were working with a single coordinate chart. A point in this chart is therefore simply denoted by \((x,u)\). This is illustrated in Figure 2.1.

**Fig. 2.1:** Coordinate chart pair for discrete-time systems

In general, the sets \(\pi_M(V_{M \times U})\) and \(\tilde{V}_M\) may or may not have an intersection. and they may or may not be equal. However, in the neighborhood of an equilibrium point \((x^0, u^0)\) it is possible to choose \(V_{M \times U}\) such that \(\pi_M(V_{M \times U}) \subset \tilde{V}_M\).

**Fig. 2.2:** Coordinate charts around an equilibrium point
This is because, in the neighborhood of an equilibrium point, $\tilde{V}_M$ and $f^{-1}(\tilde{V}_M)$ have a nonempty intersection, and the set $V_{M \times U}$ may be chosen small enough such that $\pi_M(V_{M \times U}) \subset \tilde{V}_M$. This means that $\pi_M \circ \varphi_{M \times U} = \varphi_M|_{\pi_M(V_{M \times U})}$, so that we are essentially using one coordinate chart. The construction is illustrated in Figure 2.2. Unless otherwise stated, it is to be assumed in this work that all coordinate calculations are done using a single coordinate chart. That is, it is assumed that the discrete-time system is in a neighborhood of an equilibrium point. We define the discrete-time systems of interest as follows.

**Definition 2.1  Discrete-Time Nonlinear System Without Outputs.** A nonlinear discrete-time system is a 3-tuple $\Sigma(M \times U, M, f)$ where $M$ and $U$ are $C^\infty$ manifolds of dimensions $n$ and $m$ respectively, and $f : M \times U \rightarrow M$ is a $C^\infty$ map. The points of $M$ are the state-space and those of $U$ are the inputs. The system dynamics are described by

\[ x(k+1) = f(x(k), u(k)) \quad (\Sigma_{DNS}) \]

where $x(\cdot)$ represents coordinates for $M$, and $u(\cdot)$ represents coordinates for $U$.

With some abuse of notation we let $f(x,u)$ be the coordinate representation of the map $f : M \times U \rightarrow M$. The point $(x^0, u^0)$ will be called an equilibrium point for the system $\Sigma_{DNS}$ if $f(x^0, u^0) = x^0$. A standard assumption, as in [13] is that the Jacobian $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial u})$ is full-rank at the equilibrium point.

**Definition 2.2 Discrete-Time System With Outputs.** A nonlinear discrete-time system with outputs is a 4-tuple

\[ \Sigma(M \times U, M, f, \{h_i\}_{i=1}^m) \]
where $M, U$ and $f : M \times U \to M$ are as previously defined, and $h_i : M \times U \to \mathbb{R}$ for $0 \leq i \leq m$ are $C^\infty$ functions. The system dynamics are described by

$$x(k+1) = f(x(k), u(k))$$

$$y(k) = h(x(k), u(k))$$

where $h \overset{\text{def}}{=} (h_1(x,u), \ldots, h_m(x,u))^T$ is the coordinate representation of the function $h : M \times U \to \mathbb{R}$. The point $(x^0, u^0)$ will be called an equilibrium point for the system $\Sigma^h_{D\text{NLS}}$ if $f(x^0, u^0) = x^0$ and $h(x^0, u^0) = 0$. It will be assumed that the Jacobian $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial u})$ is full-rank at the equilibrium point. □

Since vector fields do not occur naturally in discrete-time systems, we must construct them artificially using the system map, $f$, if we are to develop a geometric framework for these systems. What we use are $f$-related vector fields; and it is necessary to know when these vector fields are well-defined. A necessary and sufficient condition is given by the following lemma of [22].

**Lemma 2.3** ([22]) *Geometric Condition for Well-Definedness of $f$-Related Vector Fields.*

Suppose $f : M \times U \to M$ is a $C^\infty$ map such that $f_* : T(M \times U) \to TM$ is onto. For a vector field $X \in T(M \times U)$, the vector field $Y$ defined pointwise by $Y_{f(p)} = f_* p X_p$ is a well-defined vector field on some open neighborhood of $f(p)$ if and only if

$$[X, \ker f_*] \subset \ker f_*$$

(2.1)

on some open neighborhood of $p$. □

We use this lemma to construct $f$-related vector fields for discrete-time systems. These vector fields are given by the following recursive algorithm.
Definition 2.4 Coordinate-Free Definition of Vector Fields.

Step 0. Set $D^0_f \frac{\partial}{\partial u_i} = \frac{\partial}{\partial u_i}$ for $1 \leq i \leq m$.

Step $k + 1$. Given vector fields $D^k_f \frac{\partial}{\partial u_i}$ for $1 \leq i \leq m$, check if

$$\left[ D^k_f \frac{\partial}{\partial u_i}, \ker f_* \right] \subset \ker f_*$$

for $1 \leq i \leq m$. If (2.2) is true, then by Lemma 2.3 the vector fields

$$\overline{D}^{k+1}_f \frac{\partial}{\partial u_i} \overset{\text{def}}{=} f_\ast D^k_f \frac{\partial}{\partial u_i}$$

are well-defined. Since $\ker (\pi_M)_\ast \cap \ker (\pi_U)_\ast = \emptyset$, we can define a unique vector field $D^{k+1}_f \frac{\partial}{\partial u_i} \in T(M \times L')$ such that $D^{k+1}_f \frac{\partial}{\partial u_i} \in \ker (\pi_U)_\ast$ and satisfies

$$(\pi_M)_\ast D^{k+1}_f \frac{\partial}{\partial u_i} = \overline{D}^{k+1}_f \frac{\partial}{\partial u_i}$$

If (2.2) is not true at stage $k + 1$, then the procedure stops. \(\square\)

This coordinate-free approach to defining the $f$-related vector fields of interest does not require explicit computation of Jacobians, and clearly reveals the underlying structure of the tangent map operation. In the following remark we repeat the definition, but explicitly show all the local coordinate calculations.

Remark 2.5 Local Coordinate Computation of Vector Fields.

Step 0. Set $D^0_f \frac{\partial}{\partial u_i}(x, u) = \frac{\partial}{\partial u_i}$ for $1 \leq i \leq m$. Let the distribution $\ker f_\ast$ be given in local coordinates by

$$\ker f_\ast = \text{span} \{ \alpha_1(x, u), \ldots, \alpha_m(x, u) \}$$
where the vector fields $\alpha_1(x,u), \ldots, \alpha_m(x,u)$ are obtained by solving the equation

$$f_* \alpha \overset{\text{def}}{=} \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial u} \right) \alpha(x,u) = 0$$

**Step k + 1.** Given vector fields $D_f^k \frac{\partial}{\partial u_i}(x,u)$ for $1 \leq i \leq m$, check if

$$\left[ D_f^k \frac{\partial}{\partial u_i}(x,u), \alpha_j(x,u) \right] \in \text{span} \left\{ \alpha_1(x,u), \ldots, \alpha_m(x,u) \right\} \tag{2.3.1}$$

for $1 \leq i \leq m$ and $1 \leq j \leq m$. If (2.3) is true, then we define the vector field

$$D_f^{k+1} \frac{\partial}{\partial u_i}(x,u) \overset{\text{def}}{=} \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial u} \right)_{f^{-1}(x)} \left( D_f^k \frac{\partial}{\partial u_i} \right)_{f^{-1}(x)}$$

Since $\ker(\pi_{M,*}) \cap \ker(\pi_{U,*}) = \emptyset$, we then define a unique vector field $D_f^{k+1} \frac{\partial}{\partial u_i}(x,u) \in T(M \times U)$ such that $D_f^{k+1} \frac{\partial}{\partial u_i} \in \ker(\pi_{U,*})$ and satisfies

$$\left( \frac{\partial \pi_M}{\partial x}, \frac{\partial \pi_M}{\partial u} \right)_{\pi_M^{-1}(x)} \left( D_f^{k+1} \frac{\partial}{\partial u_i} \right)_{\pi_M^{-1}(x)} = D_f^{k+1} \frac{\partial}{\partial u_i}(x,u)$$

□

In the following lemma we show that the vector fields given by Definition 2.4 are invariant under a state coordinate transformation. This result is needed in the proof of Theorem 2.10 where necessary and sufficient conditions are derived for exact linearization using state-coordinate transformation.

**Lemma 2.6** Invariance of the $f$-Related Vector Fields Under Coordinate Transformation. Consider the nonlinear system

$$x(k + 1) = f(x(k), u(k)) \quad (\Sigma_{DNLS})$$

and define a change of state coordinates given by

$$\begin{pmatrix} z \\ u \end{pmatrix} = S(x,u) \overset{\text{def}}{=} \begin{pmatrix} S(x) \\ u \end{pmatrix} \tag{2.4}$$
such that the map \( f \) transforms to \( \tilde{f}(z,u) \overset{\text{def}}{=} S \circ f \circ S^{-1}(z,u) \). Then, the following vector field equivalence relation holds,

\[
D^k_{f} \frac{\partial}{\partial u_j} = \tilde{S} \circ D^k_{f} \frac{\partial}{\partial u_j} \circ S^{-1}
\]

for \( 1 \leq j \leq m \) and \( k \geq 0 \).

**Proof.** Since the map \( S \) is a coordinate transformation, its inverse exists, and hence the map \( \tilde{S} \) is invertible. Under the coordinate transformation (2.4), the Jacobian \( \tilde{f} \) transforms as

\[
\left( \frac{\partial \tilde{f}}{\partial x} \quad \frac{\partial \tilde{f}}{\partial u} \right)_{f^{-1}(z)} = \left( \frac{\partial S}{\partial x} \quad \frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial u} \right)_{f^{-1}(x)} \left( \frac{\partial S^{-1}}{\partial (z,u)} \right)_{S^{-1}(f^{-1}(x))}
\]

To prove that (2.6) is true, we first show that

\[
\tilde{S}(f^{-1}(x)) = \tilde{f}^{-1}(z)
\]

The set

\[
f^{-1}(x') \overset{\text{def}}{=} \{(x,u) \text{ such that } f(x,u) = x'\}
\]

Therefore,

\[
\tilde{S} \circ f^{-1}(x') = \{(S(x),u) \text{ such that } f(x,u) = x'\}
\]

\[
= \{(S(x),u) \text{ such that } S \circ f(x,u) = S(x')\}
\]

\[
= \{(z,u) \text{ such that } S \circ f(\tilde{S}^{-1}(z,u)) = z'\}
\]

\[
= \{(z,u) \text{ such that } \tilde{f}(z,u) = z'\}
\]

\[
= \tilde{f}^{-1}(z')
\]

Then (2.6) follows by evaluating \( \frac{\partial}{\partial (z,u)} \circ S \circ f \circ S^{-1}(z,u) \) at \( \tilde{S}(f^{-1}(x)) \). Now we prove the lemma. We note that (2.5) is trivially true for \( k = 0 \). Suppose that it is true for \( k = n - 1 \). Now,

\[
f \ast D^{n-1}_{f} \frac{\partial}{\partial u_j} (x,u) = \left( \frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial u} \right)_{f^{-1}(x)} \left( D^{n-1}_{f} \frac{\partial}{\partial u_j} \right)_{f^{-1}(x)}
\]
where $f^{-1}(x)$ denotes the inverse image of $x$ under the map $f$. Then, we may define $D^n f \frac{\partial}{\partial u_j}$ by
\[
\left( \frac{\partial \pi_M}{\partial x} \frac{\partial \pi_M}{\partial u} \right)_{f^{-1}(x)} D^n f \frac{\partial}{\partial u_j}(x, u) = f \ast D^n f -1 \frac{\partial}{\partial u_j}(x, u)
\]

Now consider
\[
S \left( f \ast D^n f -1 \frac{\partial}{\partial u_j} \right) (S(x))
\]
\[
= \left( \frac{\partial S}{\partial x} \right)_{f^{-1}(z)} \left( \frac{\partial f}{\partial x} \frac{\partial f}{\partial u} \right)_{f^{-1}(x)} \left( D^n f -1 \frac{\partial}{\partial u_j} \right)_{f^{-1}(x)}
\]
\[
= \left( \frac{\partial S}{\partial x} \right)_{f^{-1}(z)} \left( \frac{\partial f}{\partial x} \frac{\partial f}{\partial u} \right)_{f^{-1}(x)} \cdot \left( \frac{\partial S}{\partial (x, u)} \right)_{f^{-1}(x)} \left( D^n f -1 \frac{\partial}{\partial u_j} \right)_{f^{-1}(x)}
\]

Now we use (2.6) and the induction hypothesis to reduce the right hand side of the above equation.
\[
\text{r.h.s.} = \left( \frac{\partial f}{\partial z} \frac{\partial f}{\partial u} \right)_{f^{-1}(z)} \left( D^n f -1 \frac{\partial}{\partial u_j} \right)_{f^{-1}(z)}
\]
\[
= f \ast D^n f -1 \frac{\partial}{\partial u_j}(z, u)
\]

(2.8)

We must finally show that
\[
\left( \frac{\partial \pi_M}{\partial z} \frac{\partial \pi_M}{\partial u} \right)_{f^{-1}(z)} \left( S \ast D^n f -1 \frac{\partial}{\partial u_j} \right)_{f^{-1}(z)} = f \ast D^n f -1 \frac{\partial}{\partial u_j}(z, u)
\]

We evaluate the vector field on the left hand side of the above equation
\[
\text{l.h.s.} = \left( \frac{\partial \pi_M}{\partial z} \frac{\partial \pi_M}{\partial u} \right)_{f^{-1}(z)} S \ast D^n f -1 \frac{\partial}{\partial u_j}(z, u)
\]
\[
= \left( \frac{\partial S}{\partial z} \right)_{f^{-1}(z)} \left( \frac{\partial \pi_M}{\partial z} \frac{\partial \pi_M}{\partial u} \right)_{f^{-1}(z, u)} \left( \frac{\partial S}{\partial (z, u)} \right)_{f^{-1}(z, u)}
\]
\[
\cdot \left( \frac{\partial S}{\partial (z, u)} \right)_{f^{-1}(z, u)} \left( D^n f -1 \frac{\partial}{\partial u_j}(x, u) \right)_{f^{-1}(z, u)}
\]
\[
= \left( \frac{\partial S}{\partial z} \right)_{f^{-1}(z)} \left( f \ast D^n f -1 \frac{\partial}{\partial u_j} \right)_{f^{-1}(z)}
\]
\[
= f \ast D^n f -1 \frac{\partial}{\partial u_j}(z, u)
\]

where the last step is from (2.8). This completes the proof. □
To obtain our main results on feedback linearization, we apply Frobenius's Theorem to the nested sequence of involutive and constant dimensional distributions $G_0, \ldots, G_n$, which are defined later. This theorem can be used to prove that this sequence of distributions is integrable. But to express the integrability conditions we must first define one-forms on the cotangent bundle $T^*(M \times U)$.

This is done as follows. Let $d\lambda \in T^*(M \times U)$ be an exact one-form such that $d\lambda(x^0, u^0) \neq 0$. We let $\pi_M : T^*M \to T^*(M \times U)$ and $f^* : T^*M \to T^*(M \times U)$ denote the adjoint maps induced by $\pi_M : M \times U \to M$ and $f : M \times U \to M$ respectively. We use the standard notation for the operation of a one-form on a vector field $X \in T(M \times U)$

$$d\lambda(X) = \langle d\lambda, X \rangle = L_X \lambda$$  \hspace{1cm} (2.9)

We then define a sequence of one-forms recursively by the following algorithm.

**Definition 2.7 Coordinate-Free Definition of One-Forms.**

*Step 0.* Set $P_f^0 d\lambda = d\lambda$.

*Step k + 1.* Suppose that the exact one-form $P_f^k d\lambda$ satisfies $\langle P_f^k d\lambda, \frac{\partial}{\partial u_j} \rangle = 0$, for $1 \leq j \leq m$. We let $P_f^{k+1} d\lambda \in T^*M$ be the unique exact one-form such that

$$\pi_M^* P_f^{k+1} d\lambda = P_f^k d\lambda$$  \hspace{1cm} (2.10a)

Then we define

$$P_f^{k+1} d\lambda = f^* P_f^{k+1} d\lambda$$  \hspace{1cm} (2.10b)

The algorithm may be carried out until $k = n$, in order to obtain the exact one-forms $\{d\lambda, P_f d\lambda, \ldots, P_f^n d\lambda\}$. \qed
In local coordinates the adjoint operators take the form of Jacobian matrices. This is clear from Remark 2.8. Note the alternate representation of the exact one-forms $P_f^k d\lambda$ in terms of the $C^\infty$ functions $\varphi_0, \ldots, \varphi_n$. These functions will be used in Corollary 2.22 and Corollary 2.29 to construct the linearizing coordinate transformation and feedback law.

**Remark 2.8**  Local Coordinate Computation of One-Forms.

*Step 0.* We let $\varphi_0(x, u) \overset{\text{def}}{=} \lambda(x, u)$ and set $P_f^0 d\lambda(x, u) = d\varphi_0(x, u) \overset{\text{def}}{=} d\lambda(x, u)$.

*Step $k + 1.*$ Suppose that the exact one-form $P_f^k d\lambda(x, u)$ satisfies

$$\langle P_f^k d\lambda(x, u), \frac{\partial}{\partial u_j} \rangle = \langle d\varphi_k(x, u), \frac{\partial}{\partial u_j} \rangle = \frac{\partial}{\partial u_j} \varphi_k = 0$$

for $1 \leq j \leq m$. We let $P_f^{k+1} d\lambda(x) = d\tilde{\varphi}_{k+1} \in T^* M$ be the unique exact one-form such that

$$P_f^{k+1} d\lambda(x) \left( \frac{\partial \pi_M}{\partial x} \frac{\partial \pi_M}{\partial u} \right)^{-1}(x) = P_f^k d\lambda(x, u)$$

Or, in terms of the functions $\varphi_k(x)$.

$$d\tilde{\varphi}_{k+1}(x) \left( \frac{\partial \pi_M}{\partial x} \frac{\partial \pi_M}{\partial u} \right)^{-1}(x) = d\varphi_k(x, u)$$

Then the one-forms at step $k + 1$ are given by

$$P_f^{k+1} d\lambda(x, u) = P_f^{k+1} d\lambda(x) \left( \frac{\partial f}{\partial x} \frac{\partial f}{\partial u} \right)^{-1}(x)$$

Or alternately,

$$d\varphi_{k+1}(x, u) = d\tilde{\varphi}_{k+1}(x) \left( \frac{\partial f}{\partial x} \frac{\partial f}{\partial u} \right)^{-1}(x)$$

□
In the proofs for our results on feedback linearization, we encounter Lie derivatives of the form \( \langle P_f^{k-l} d\lambda, D_f^l \frac{\partial}{\partial u_i} \rangle \). Since the exact one-forms \( P_f^k d\lambda \) and the vector fields \( D_f^l \frac{\partial}{\partial u_i} \) are defined using the tangent map \( f_* \), this Lie derivative can be simplified. This is demonstrated in Lemma 2.9. First, we need the following equivalence in representing Lie derivatives. Let \( f : M \times U \to M \) be a smooth map, \( X \in T(M \times U) \) a smooth vector field and \( d\lambda \in T^\ast M \) a smooth one-form. We have the following equivalence.

\[
\langle f^* d\lambda, X \rangle = \langle d\lambda, f_* X \rangle
\]  

(2.11)

The proof for the above result is straightforward and may be found in [32].

**Lemma 2.9** Consider the nonlinear system \( \Sigma_{DNS} \). For this system, let the vector fields \( D_f^i \frac{\partial}{\partial u_i} \) for \( 0 \leq i \leq n \) and \( 1 \leq l \leq m \) be well-defined. Let \( \lambda : M \times U \to \mathbb{R} \) be a \( C^\infty \) function. Then, the statement

a) \( \langle d\lambda, D_f^i \frac{\partial}{\partial u_i} \rangle = 0 \) for \( 0 \leq i \leq n - 1 \) and \( 1 \leq l \leq m \)

implies the following statements

b) The one-forms \( P_f^j d\lambda \) are well-defined for \( 0 \leq j \leq n \).

c) \( \langle P_f^j d\lambda, D_f^i \frac{\partial}{\partial u_i} \rangle = 0 \), for \( 0 \leq i + j \leq n - 1 \) and \( 1 \leq l \leq m \).

d) \( \langle d\lambda, D_f^n \frac{\partial}{\partial u_i} \rangle = \langle P_f^k d\lambda, D_f^{n-k} \frac{\partial}{\partial u_i} \rangle \)

**Proof.** (a)\( \Rightarrow \) (b). We must show that \( \langle P_f^j d\lambda, \frac{\partial}{\partial u_i} \rangle = 0 \), for \( 0 \leq j \leq n - 1 \) and \( 1 \leq l \leq m \). Clearly, this is true for \( j = 0 \). Assume that it is true for all
\[ j < n - 1. \text{ Then} \]
\[
\langle P_f^{n-1} d\lambda, \frac{\partial}{\partial u_i} \rangle = \langle f^* P_f^{n-1} d\lambda, \frac{\partial}{\partial u_i} \rangle
\]
\[
= \langle P_f^{n-1} d\lambda, f^* \frac{\partial}{\partial u_i} \rangle
\]
\[
= \langle P_f^{n-1} d\lambda, (\pi_M)_* D_f \frac{\partial}{\partial u_i} \rangle
\]
\[
= \langle (\pi_M^* P_f^{n-1}) d\lambda, D_f \frac{\partial}{\partial u_i} \rangle
\]
\[
= \langle P_f^{n-2} d\lambda, D_f \frac{\partial}{\partial u_i} \rangle
\]

where the last equality follows because \( \langle P_f^{n-2} d\lambda, \frac{\partial}{\partial u_i} \rangle = 0 \) for \( 1 \leq l \leq m \).

Repeating the procedure we obtain finally

\[
\langle P_f^{n-1} d\lambda, \frac{\partial}{\partial u_i} \rangle = \langle d\lambda, D_f^{n-1} \frac{\partial}{\partial u_i} \rangle
\]

The r.h.s. of the above equation is zero by (a), and this completes the proof.

(a) \(\Rightarrow\) (c). It is easy to verify that (c) is true for \( j = 0 \), with \( i \) such that \( i + j < n \). Assume that (c) holds for some \( j \) with \( i \) such that \( i + j < n - 1 \). Now, by using the definition of \( P_f^{j+1} d\lambda \) and (2.11) we have that

\[
\langle P_f^{j+1} d\lambda, D^i_f \frac{\partial}{\partial u_i} \rangle = \langle P_f^j d\lambda, f^* D^i_f \frac{\partial}{\partial u_i} \rangle
\]

Using the definition of \( D^i_{f+1} \frac{\partial}{\partial u_i} \), this is equivalent to

\[
\langle P_f^j d\lambda, (\pi_M)_* D^i_{f+1} \frac{\partial}{\partial u_i} \rangle
\]

Using (2.11), and the definition of \( P_f^j d\lambda \), we can write this as

\[
\langle P_f^j d\lambda, D^i_{f+1} \frac{\partial}{\partial u_i} \rangle
\]

which is zero by the induction assumption, since \( i + j + 1 < n \).
(a) ⇒ (d). It can be easily verified that the statement is true for \( k = 0 \). Assume that it is true for some \( k < n \). Then, using the definitions of \( D_f^k \frac{\partial}{\partial u_i} \), \( P_f^k d\lambda \) and equation (2.11) successively, we can show that

\[
\langle P_f^{k+1} d\lambda, D_f^{n-k-1} \frac{\partial}{\partial u_i} \rangle = \langle P_f^k d\lambda, f_\ast D_f^{n-k-1} \frac{\partial}{\partial u_i} \rangle \\
= \langle P_f^k d\lambda, (\pi_M)_\ast D_f^{n-k} \frac{\partial}{\partial u_i} \rangle \\
= \langle P_f^k d\lambda, D_f^{n-k} \frac{\partial}{\partial u_i} \rangle \\
= \langle d\lambda, D_f^n \frac{\partial}{\partial u_i} \rangle
\]

Note that the last equality follows from the induction assumption.

In this section we introduced a geometric framework for discrete-time nonlinear systems. Vector fields and one-forms were artificially constructed for the discrete-time system of interest using the tangent map \( f_\ast \). A technical lemma from [22] was used to determine the conditions under which this construction is legitimate. We also established some key results that will be used in Section 2.2 and Section 2.3 to analyze the problem of exact linearization.

### 2.2 Exact Linearization by State Coordinate Change

In this section we investigate the conditions under which a discrete-time nonlinear system is equivalent, under state-space transformation, to a linear, reachable system. The problem is formulated for multi-input discrete-time systems using the vector fields given by Definition 2.4. It is interesting that the statement of the problem, and the solution technique, are directly analogous to corresponding continuous-time results. It is also remarkable that the same vector fields may be used to formulate the feedback linearization problem.
This problem has been studied in [21], [22] and [30]. Necessary and sufficient conditions for linearizability of single-input discrete-time systems are given in [22]. Provided the matrix $\frac{\partial f}{\partial x}(x^0, u^0)$ is nonsingular, the conditions may be expressed in a form that are more easily verifiable. The multi-input case is solved in [21], where the treatment is more algebraic than geometric. The single-input case is also addressed in [30]. The main assumption is that the matrix $\frac{\partial f}{\partial x}(x^0, u^0)$ is nonsingular. Necessary and sufficient conditions are stated in terms of $F$-related vector fields, where $F$ is a diffeomorphism constructed from the system map $f$.

Our main result is the following theorem.

**Theorem 2.10** Necessary and Sufficient Conditions for Exact Linearization of Discrete-Time Systems Using State Coordinate Transformation. Consider the discrete-time nonlinear system given by

$$x(k + 1) = f(x(k), u(k)) \quad (\Sigma_{DNS})$$

with equilibrium point $(x^0, u^0)$. Let the Jacobian $\frac{\partial f}{\partial x}(x^0, u^0)$ be nonsingular. There exists a coordinate transformation around $(x^0, u^0)$, given by $z = S(x)$, under which $\Sigma_{DNS}$ is transformed to the linear system given by

$$z(k + 1) = Az(k) + Bv(k) \quad (\Sigma_{DLS})$$

where $(A, B)$ is reachable, if and only if

i) The vector fields $D^k_f \frac{\partial}{\partial u_j}$ are well-defined for $1 \leq j \leq m$ and $k \geq 0$

ii) $[D^k_f \frac{\partial}{\partial u_i}, D^l_f \frac{\partial}{\partial u_i}] = 0$ for $1 \leq i, j \leq m$ and $k, l \geq 0$

iii) $\dim \text{span} \{D^k_f \frac{\partial}{\partial u_j} \text{ for } 1 \leq j \leq m \text{ and } 0 \leq k \leq n\} = n + m$. 

---
Proof (Necessity).

Assume that the system $\Sigma_{DNL5}$ is linearizable. To prove (i), we will show that the vector fields of interest are well-defined in the coordinates $(z, u)$. The result will follow from invariance of these vector fields under coordinate transformation. In the coordinates $(z, u)$, because $A$ is nonsingular, we have the following representation for the distribution ker $\tilde{f}_*$.

$$\ker \tilde{f}_* = \text{span} \left\{ \left( \frac{\partial \tilde{\alpha}}{\partial \tilde{u}_j} \right) \text{ such that } \frac{\partial \tilde{\alpha}}{\partial \tilde{u}_j} = -A^{-1}B \tilde{\beta}(z, u) \right\}$$

In the same coordinates,

$$\left[ \frac{\partial}{\partial u_j}, \ker \tilde{f}_* \right] = \left( \frac{\partial \tilde{\alpha}}{\partial u_j} \right) = \left( -A^{-1}B \frac{\partial \tilde{\beta}}{\partial u_j} \right)$$

It can be verified that $\tilde{f}_* \left[ \frac{\partial}{\partial u_j}, \ker \tilde{f}_* \right] = 0$. This implies that $\tilde{f}_* \frac{\partial}{\partial u_j}$ is a well-defined vector field. In coordinates, $\tilde{f}_* \frac{\partial}{\partial u_j} = b_j$, where $b_j$ is the $j^{th}$ column of $B$. We can choose

$$D_{\tilde{f}} \frac{\partial}{\partial u_j} = \begin{pmatrix} b_j \\ 0 \end{pmatrix}$$

Next, for some $k$,

$$\left[ D_{\tilde{f}}^k \frac{\partial}{\partial u_j}, \ker \tilde{f}_* \right] = \begin{pmatrix} \frac{\partial \tilde{\alpha}}{\partial \tilde{z}} & \frac{\partial \tilde{\alpha}}{\partial \tilde{u}_j} \\ \frac{\partial \tilde{\beta}}{\partial \tilde{z}} & \frac{\partial \tilde{\beta}}{\partial \tilde{u}_j} \end{pmatrix} \begin{pmatrix} A^{k-1}b_j \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial \tilde{\alpha}}{\partial \tilde{z}}A^{k-1}b_j \\ \frac{\partial \tilde{\beta}}{\partial \tilde{z}}A^{k-1}b_j \end{pmatrix} = \left( -A^{-1}B \frac{\partial \tilde{\beta}}{\partial \tilde{z}}A^{k-1}b_j \right)$$

One can easily verify that $\tilde{f}_* \left[ D_{\tilde{f}}^k \frac{\partial}{\partial u_j}, \ker \tilde{f}_* \right] = 0$, and therefore $\tilde{f}_* D_{\tilde{f}}^k \frac{\partial}{\partial u_j}$ is a well-defined vector field. In local coordinates,

$$\tilde{f}_* D_{\tilde{f}}^k \frac{\partial}{\partial u_j} = \begin{pmatrix} A \\ B \end{pmatrix} \begin{pmatrix} A^{k-1}b_j \\ 0 \end{pmatrix} = \begin{pmatrix} A^kB_j \end{pmatrix}$$
We compute
\[ D^{k+1}_f \frac{\partial}{\partial u_j} = \left( A^k b_j \right) \]
Hence, the vector fields
\[ \left\{ \frac{\partial}{\partial u_j}, D_f \frac{\partial}{\partial u_j}, \ldots, D^k_f \frac{\partial}{\partial u_j}, \ldots \right\} \]
are well-defined for \( 1 \leq j \leq m \). Then, by a change of coordinates given by
\[ z = S(x), \nu = id(u) \]
the vector fields in the coordinates \((x, u)\)
\[ \left\{ \frac{\partial}{\partial u_j}, D_f \frac{\partial}{\partial u_j}, \ldots, D^k_f \frac{\partial}{\partial u_j}, \ldots \right\} \]
are also well-defined for \( 1 \leq j \leq m \).

To prove that (ii) is true, we note that the Lie bracket
\[ [D_f^k \frac{\partial}{\partial u_j}, D_f^l \frac{\partial}{\partial u_i}] = \tilde{S}^{-1} [D_f^k \frac{\partial}{\partial u_j}, D_f^l \frac{\partial}{\partial u_i}] \circ \tilde{S} \]
and
\[ [D_f^k \frac{\partial}{\partial u_j}, D_f^l \frac{\partial}{\partial u_i}] = \left[ \begin{pmatrix} A^{k-1} b_j \\ 0 \\ A^{l-1} b_i \end{pmatrix}, \begin{pmatrix} 0 \\ A^{k-1} b_j \\ 0 \end{pmatrix} \right] = 0 \]
Hence condition (ii) is true. Finally, condition (iii) is true by the reachability assumption.

(Sufficiency). Assume that conditions (i) through (iii) hold. From (i) the vector fields \( D^k_f \frac{\partial}{\partial u_j} \) are well-defined for \( k \geq 0 \) and \( 1 \leq j \leq m \). Since the conditions (ii) and (iii) hold, we may choose from the set
\[ \{ D^k_f \frac{\partial}{\partial u_j} \text{ for } 1 \leq j \leq m \text{ and } 0 \leq k \leq n \} \]
\( n + m \) vector fields denoted by

\[
\left\{ X_1, \ldots, X_n, \frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_m} \right\}
\]

such that

a) \( \dim \, \text{span} \{X_1, \ldots, X_n, \frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_m}\} = n + m \)

b) \([X_i, X_k] = [\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}] = [X_i, \frac{\partial}{\partial u_j}] = 0 \) for \( 1 \leq k, i \leq n \) and \( 1 \leq j, l \leq m \).

From the Flow-Box Theorem (see [32]), there must exist a diffeomorphism

\[
\tilde{S}(x, u) = \begin{pmatrix} S(x) \\ z \end{pmatrix} = \begin{pmatrix} z \\ u \end{pmatrix}
\]

such that

\[
\tilde{S} \cdot X_i = \frac{\partial}{\partial z_i}
\]

\[
\tilde{S} \cdot \frac{\partial}{\partial u_j} = \frac{\partial}{\partial u_j}
\]

for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \). We show that \( \frac{\partial f}{\partial u_j} \) and \( \frac{\partial f}{\partial z_i} \) are constant in the new coordinates: hence \( \tilde{f}(z, u) \) is a linear function. Note that

\[
\left[ \frac{\partial}{\partial u_i}, D_f \frac{\partial}{\partial u_j} \right] = \left[ \tilde{S} \cdot \frac{\partial}{\partial u_i} \circ \tilde{S}^{-1}, \tilde{S} \cdot D_f \frac{\partial}{\partial u_j} \circ \tilde{S}^{-1} \right]
\]

\[
= \tilde{S} \cdot \left[ \frac{\partial}{\partial u_i}, D_f \frac{\partial}{\partial u_j} \right] \circ \tilde{S}^{-1}
\]

\[
= 0
\]

since \( \left[ \frac{\partial}{\partial u_i}, D_f \frac{\partial}{\partial u_j} \right] = 0 \) by (ii). Also.

\[
\left[ \frac{\partial}{\partial z_i}, D_f \frac{\partial}{\partial u_j} \right] = \left[ \tilde{S} \cdot X_i \circ \tilde{S}^{-1}, \tilde{S} \cdot D_f \frac{\partial}{\partial u_j} \circ \tilde{S}^{-1} \right]
\]

\[
= \tilde{S} \cdot [X_i, D_f \frac{\partial}{\partial u_j}] \circ \tilde{S}^{-1}
\]

\[
= 0
\]

since \( [X_i, D_f \frac{\partial}{\partial u_j}] = 0 \) by (ii). In local coordinates, the vector fields \( D_f \frac{\partial}{\partial u_j} \) for \( 1 \leq j \leq m \) are given by

\[
D_f \frac{\partial}{\partial u_j} \overset{\text{def}}{=} \begin{pmatrix} \frac{\partial f}{\partial u_j} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial u_j}(f^{-1}(z), u) \\ 0 \end{pmatrix}
\]
where the map $\hat{f}_u(z)$ is defined as the map $\hat{f}(z,u)$ for fixed $u$. From our assumptions, the map $\hat{f}_u$ is invertible. Since,

$$
[\frac{\partial}{\partial z_k}, D_{\hat{f}_u} \frac{\partial}{\partial u_j}] = 0
$$

for $1 \leq j \leq m$ and $1 \leq k \leq n$, this implies that

$$
\frac{\partial}{\partial z_k} \left( \frac{\partial \hat{f}_u}{\partial u_j}(\hat{f}_u^{-1}(z), u) \right) = 0
$$

for $1 \leq j \leq m$ and $1 \leq k \leq n$. To evaluate the partial derivatives, we let $z' = f_u^{-1}(z)$ for convenience. Then,

$$
\frac{\partial}{\partial z_k} \left( \frac{\partial \hat{f}_u}{\partial u_j}(\hat{f}_u^{-1}(z), u) \right) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial z'_i \partial u_j} \frac{\partial z'_i}{\partial z_k} = 0
$$

for $1 \leq k \leq n$, where $(f_u^{-1})_i$ denotes the $i$th component of $f_u^{-1}$. But the column vectors $\frac{\partial f_u^{-1}}{\partial z_k}$ are linearly independent for $1 \leq k \leq n$ since $f_u$ is nonsingular. Hence,

$$
\frac{\partial^2 \hat{f}_u}{\partial z'_i \partial u_j} = 0
$$

(2.12)

for $1 \leq i \leq n$. This means that $\frac{\partial \hat{f}_u}{\partial u_j}$ is independent of $z$ for $1 \leq j \leq m$.

The statement

$$
[\frac{\partial}{\partial u_l}, D_{\hat{f}_u} \frac{\partial}{\partial u_j}] = 0
$$

for $1 \leq j \leq m$ and $1 \leq l \leq m$, implies that

$$
\frac{\partial}{\partial u_l} \left( \frac{\partial \hat{f}_u}{\partial u_j}(\hat{f}_u^{-1}(z), u) \right) = 0
$$

for $1 \leq j \leq m$ and $1 \leq l \leq m$. Evaluating the partial derivatives we get

$$
\sum_{i=1}^{n} \frac{\partial^2 \hat{f}_u}{\partial z'_i \partial u_j} \frac{\partial z'_i}{\partial u_l} + \frac{\partial \hat{f}_u}{\partial u_l} \frac{\partial \hat{f}_u}{\partial u_j} = 0
$$
From (2.12) this implies that

$$\frac{\partial^2 \tilde{f}}{\partial u_l \partial u_j} = 0$$  \hspace{1cm} (2.13)$$

for $1 \leq l \leq m$. Hence, $\frac{\partial \tilde{f}}{\partial u_j}$ is independent of $u_j$, for $1 \leq j \leq m$.

Next, we consider Lie brackets of the form $[\frac{\partial}{\partial z_k}, D_{\tilde{f}} \frac{\partial}{\partial z_i}]$, for $1 \leq k \leq n$ and $1 \leq i \leq n$ and $[\frac{\partial}{\partial u_i}, D_{\tilde{f}} \frac{\partial}{\partial z_i}]$, for $1 \leq l \leq m$ and $1 \leq i \leq n$. We observe that

$$\frac{\partial}{\partial z_i} = \tilde{S}_* D_{\tilde{f}}^{i_1} \frac{\partial}{\partial u_{j_1}} \circ \tilde{S}^{-1}$$

for some $i_1, j_1$ such that $1 \leq j_1 \leq m$ and

$$\frac{\partial}{\partial z_k} = \tilde{S}_* D_{\tilde{f}}^{i_2} \frac{\partial}{\partial u_{j_2}} \circ \tilde{S}^{-1}$$

for some $i_2, j_2$ such that $1 \leq j_2 \leq m$. Hence

$$[\frac{\partial}{\partial z_k}, D_{\tilde{f}} \frac{\partial}{\partial z_i}] = \tilde{S}_* [D_{\tilde{f}}^{i_2} \frac{\partial}{\partial u_{j_2}}, D_{\tilde{f}}^{i_1} \frac{\partial}{\partial u_{j_1}}] \circ \tilde{S}^{-1}$$

$$= 0$$

by condition (ii). In local coordinates,

$$D_{\tilde{f}} \frac{\partial}{\partial z_i} = \begin{pmatrix} \frac{\partial \tilde{f}}{\partial z_i}(\tilde{f}_u^{-1}(z), u) \\ 0 \end{pmatrix}$$

Therefore, $[\frac{\partial}{\partial z_k}, D_{\tilde{f}} \frac{\partial}{\partial z_i}] = 0$, for $1 \leq i \leq n$ and $1 \leq k \leq n$. This implies that

$$\frac{\partial}{\partial z_k} \left( \frac{\partial \tilde{f}}{\partial z_i}(\tilde{f}_u^{-1}(z), u) \right) = 0$$

for $1 \leq i \leq n$ and $1 \leq k \leq n$. Evaluating the partial derivatives, we get

$$\frac{\partial}{\partial z_k} \left( \frac{\partial \tilde{f}}{\partial z_i}(\tilde{f}_u^{-1}(z), u) \right) = \sum_{l=1}^{n} \frac{\partial^2 \tilde{f}}{\partial z_l \partial u_j} \frac{\partial z_l}{\partial z_k}$$

$$= \sum_{l=1}^{n} \frac{\partial^2 \tilde{f}}{\partial z_l \partial u_j} \frac{\partial (f_u^{-1})_l}{\partial z_k} = 0$$
for $1 \leq k \leq n$, where $(f_u^{-1})_l$ denotes the $l$th component of $f_u^{-1}$. But the column vectors $\frac{\partial f_u^{-1}}{\partial z_k}$ are linearly independent for $1 \leq k \leq n$ since $f_u$ is nonsingular. Hence,

$$\frac{\partial^2 \tilde{f}}{\partial z_l \partial z_i} = 0$$

for $1 \leq l \leq n$. This means that $\frac{\partial \tilde{f}}{\partial z_k}$ is independent of $z$ for $1 \leq k \leq n$. Finally, we consider the brackets $[\frac{\partial}{\partial u_l}, D_f \frac{\partial}{\partial z_i}]$ for $1 \leq l \leq m$ and $1 \leq i \leq n$. Following a similar technique, it is possible to show that $\frac{\partial \tilde{f}}{\partial z_k}$ is independent of $u$ for $1 \leq k \leq n$. Hence, we have shown that $\tilde{f}(z, u)$ is a linear function of $z$ and $u$.

Next,

$$A = \frac{\partial}{\partial z} S \circ f(S^{-1}(z), u)$$

$$= (\frac{\partial}{\partial x}) f(x, u) \left( \frac{\partial f}{\partial x} \right)_{(x, u)} \left( \frac{\partial S^{-1}}{\partial z} \right)$$

Since $S$ is a diffeomorphism and $\frac{\partial f}{\partial x}$ is nonsingular, $A$ must be nonsingular. In the coordinates $(z, u),

$$D_f^k \frac{\partial}{\partial u_j} = \begin{pmatrix} A^{k-1} b_j \\ 0 \end{pmatrix}$$

for $1 \leq k \leq n$ and

$$\frac{\partial}{\partial u_j} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where the 1 is at the $(n + j)^{th}$ position in the column vector. By (iii),

$$\dim \text{ span } \left\{ \frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_m}, \begin{pmatrix} b_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} b_m \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} A^{n-1} b_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} A^{n-1} b_m \\ 0 \end{pmatrix} \right\} = n + m$$
This implies that the pair \((A, B)\) is reachable. We have thus shown that in the coordinates \((z, u)\), the discrete-time system is linear, reachable and has a nonsingular system matrix. This completes the proof. \(\square\)

There are two remarkable features of the results derived above which are really a consequence of the geometric framework developed in Section 2.1. The necessary and sufficient conditions are expressed in terms of the vector fields \(D^i_j \frac{\partial}{\partial u_i}\), for \(0 \leq j \leq n\) and \(i \geq 0\); these are the same vector fields that are used to express the conditions for feedback linearizability. Next, the conditions (i) and (ii) in the statement of Theorem 2.10 are directly analogous to continuous-time results [32]. The extra statement (i) is needed to ensure that the vector fields \(D^i_j \frac{\partial}{\partial u_i}\) are well-defined.

We now apply the results of Theorem 2.10 to an example of a second-order discrete-time nonlinear system. The vector fields \(D^i_j \frac{\partial}{\partial u}(x, u)\) for \(i \geq 0\) are constructed for this system. We use Theorem 2.10 to show that the system is indeed linearizable. The coordinate transformation is constructed by solving a set of nine linear, first-order partial differential equations.

**Example 2.11** Consider the nonlinear system

\[
x(k + 1) = f(x(k), u(k))
\]

where, dropping the time index \(k\),

\[
f(x, u) = \begin{pmatrix} (1 + x_1)x_2 + u \\ x_1 \\ \frac{x_1 - u}{1 + x_2(1 + x_1)x_2} \end{pmatrix}
\]

Clearly, \((0, 0, 0)^T\) is an equilibrium point. The set

\[
f^{-1}(x) = (x_2(1 + x_1) \quad \frac{x_1 - u}{1 + x_2(1 + x_1)} \quad u)^T
\]
The Jacobian of the map \( f \) is given by

\[
\left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial u} \right) = \left( \frac{x_2}{\Delta - x_1 x_2}, \frac{1 + x_1}{\Delta^2}, \frac{1}{\Delta^2} \right)
\]

where \( \Delta = (1 + u + (1 + x_1) x_2) \). We note that \( \text{rank} \ f_* (0,0,0) = 2 \). The Jacobian of \( f \) evaluated at \( f^{-1}(x) \) is given by

\[
\left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial u} \right)_{f^{-1}(x)} = \left( \frac{x_1 - u}{D}, \frac{D}{1 + x_2}, \frac{1}{1 + x_1} \right)
\]

where \( D \overset{\text{def}}{=} 1 + x_2(1 + x_1) \). Next, we compute the vector fields that form the basis for the distribution \( \ker f_* \). Since \( \dim \ker f_* = 1 \), we look for a vector field \( \alpha(x_1, x_2, u) = (\alpha_1(x_1, x_2, u) \quad \alpha_2(x_1, x_2, u) \quad \alpha_3(x_1, x_2, u))^T \) such that

\[
\frac{\partial f}{\partial x_1} \alpha_1(x_1, x_2, u) + \frac{\partial f}{\partial x_2} \alpha_2(x_1, x_2, u) + \frac{\partial f}{\partial u} \alpha_3(x_1, x_2, u) = 0
\]

Calculations show that \( \ker f_* = \text{span} \ \{ \alpha \} \) where

\[
\alpha = \begin{pmatrix}
0 \\
\alpha_2 \\
-(1 + x_1) \alpha_2
\end{pmatrix}
\]

and \( \alpha_2 \) is a smooth function of its arguments \( (x_1, x_2, u) \). For the convenience of subsequent calculations, we compute the following Jacobian.

\[
\frac{\partial \alpha}{\partial (x, u)} = \begin{pmatrix}
0 \\
\frac{\partial \alpha_2}{\partial x_1} \\
-(1 + x_1) \frac{\partial \alpha_2}{\partial x_1} - (1 + x_1) \frac{\partial \alpha_2}{\partial x_2}
\end{pmatrix}
\]

We check if \( [\frac{\partial}{\partial u}, \alpha] \in \ker f_* \)

\[
[\frac{\partial}{\partial u}, \alpha] = \begin{pmatrix}
0 \\
\frac{\partial \alpha_2}{\partial u} \\
-\frac{\partial \alpha_2}{\partial u} (1 + x_1)
\end{pmatrix}
\]
It can be verified that \( f_\ast [\frac{\partial}{\partial u}, \alpha] = 0 \); hence, \( f_\ast \frac{\partial}{\partial u} \) is a well-defined vector field. In local coordinates,

\[
f_\ast \frac{\partial}{\partial u} (x) = \left( \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_1} \right)_{f^{-1}(x)} \frac{\partial}{\partial u} \frac{1}{1 + x_1}
\]

From above,

\[
D_f \frac{\partial}{\partial u}(x_1, x_2, u) = \left( f_\ast \frac{\partial}{\partial u}(x) \right) = \left( \frac{1}{-x_2}, \frac{1}{1 + x_1} \right)
\]

Now we check if \([D_f \frac{\partial}{\partial u}, \alpha] \in \ker f_\ast \). The Jacobian of \( D_f \frac{\partial}{\partial u}(x, u) \) is

\[
\frac{\partial}{\partial (x, u)} D_f \frac{\partial}{\partial u}(x, u) = \begin{pmatrix}
0 & 0 & 0 \\
\frac{x_2}{(1 + x_1)^2} & \frac{1}{1 + x_1} & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Upon evaluation,

\[
[D_f \frac{\partial}{\partial u}, \alpha] = \begin{pmatrix}
\frac{\partial \alpha_2}{\partial x_1} - \frac{x_2}{1 + x_1} \frac{\partial \alpha_2}{\partial x_2} + \frac{\alpha_2}{1 + x_1} \\
-\alpha_2 - (1 + x_1) \frac{\partial \alpha_2}{\partial x_1} + x_2 \frac{\partial \alpha_2}{\partial x_2}
\end{pmatrix}
\]

It can be checked that \( f_\ast [D_f \frac{\partial}{\partial u}, \alpha] = 0 \). Hence, \( f_\ast D_f \frac{\partial}{\partial u} \) is a well-defined vector field. In local coordinates.

\[
f_\ast D_f \frac{\partial}{\partial u}(x, u) = \left( \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_1} \right)_{f^{-1}(x)} (D_f \frac{\partial}{\partial u})_{f^{-1}(x)}
\]

Since

\[
(D_f \frac{\partial}{\partial u})_{f^{-1}(x)} = \begin{pmatrix}
\frac{(-(x_1 - u))}{D^2} \\
0
\end{pmatrix}
\]

we get

\[
f_\ast D_f \frac{\partial}{\partial u}(x, u) = \left( \frac{0}{1 + x_1} \right)
\]

From above we evaluate

\[
D^2_f \frac{\partial}{\partial u}(x_1, x_2, u) = \begin{pmatrix}
0 \\
\frac{1}{1 + x_1}
\end{pmatrix}
\]
It can be verified that the vector field \( D_f^2 \frac{\partial}{\partial u} \) satisfies the condition

\[
[D_f^2 \frac{\partial}{\partial u}, a] \in \ker f.
\]

and therefore the vector field \( f_* D_f^2 \frac{\partial}{\partial u} \) is well-defined. In fact,

\[
f_* D_f^2 \frac{\partial}{\partial u} = \left( \begin{array}{c} 1 \\ -\frac{x_2}{1+x_1} \end{array} \right)
\]

and we obtain

\[
D_f^3 \frac{\partial}{\partial u} = \left( \begin{array}{c} 1 \\ -\frac{x_2}{1+x_1} \\ 0 \end{array} \right)
\]

Notice that \( D_f^3 \frac{\partial}{\partial u} = D_f \frac{\partial}{\partial u} \) and that the vector fields \( \frac{\partial}{\partial x}, D_f \frac{\partial}{\partial u}, D_f^2 \frac{\partial}{\partial u} \) are linearly independent. This means that we need only check the following Lie bracket conditions.

\[
[\frac{\partial}{\partial u}, D_f \frac{\partial}{\partial u}] = [\frac{\partial}{\partial u}, D_f^2 \frac{\partial}{\partial u}] = [D_f \frac{\partial}{\partial u}, D_f^2 \frac{\partial}{\partial u}] = 0
\]

Hence, these vector fields locally span \( T_p M \), and can be used to construct the linearizing transformation denoted as \( (z, u) = S(x, u) \). We let

\[
\tilde{S} \cdot \frac{\partial}{\partial u} = \frac{\partial}{\partial u}
\]

\[
\tilde{S} \cdot D_f \frac{\partial}{\partial u} = \frac{\partial}{\partial z_2}
\]

\[
\tilde{S} \cdot D_f^2 \frac{\partial}{\partial u} = \frac{\partial}{\partial z_1}
\]

In local coordinates, this reduces to the following set of partial differential equations:

\[
\begin{align*}
\frac{\partial \tilde{S}_1}{\partial x_1} &+ \frac{x_2}{1+x_1} = 1: & \frac{\partial \tilde{S}_1}{\partial x_2} &- \frac{x_2}{1+x_1} \frac{\partial \tilde{S}_1}{\partial x_2} = 0: & \frac{\partial \tilde{S}_1}{\partial u} = 0: \\
\frac{\partial \tilde{S}_2}{\partial x_1} &+ \frac{x_2}{1+x_1} = 0: & \frac{\partial \tilde{S}_2}{\partial x_2} &- \frac{x_2}{1+x_1} \frac{\partial \tilde{S}_2}{\partial x_2} = 1: & \frac{\partial \tilde{S}_2}{\partial u} = 0: \\
\frac{\partial \tilde{S}_3}{\partial x_1} &+ \frac{x_2}{1+x_1} = 0: & \frac{\partial \tilde{S}_3}{\partial x_2} &- \frac{x_2}{1+x_1} \frac{\partial \tilde{S}_3}{\partial x_2} = 0: & \frac{\partial \tilde{S}_3}{\partial u} = 1
\end{align*}
\]
A solution to the above equations is
\[ \mathbf{\hat{S}}_3(x, u) = u \]
\[ \mathbf{\hat{S}}_2(x, u) = x_1 \]
\[ \mathbf{\hat{S}}_1(x, u) = x_2(1 + x_1) \]

The coordinate transformation is then chosen as
\[
\begin{pmatrix} z \\ u \end{pmatrix} = \mathbf{S}(x, u) = \begin{pmatrix} x_2(1 + x_1) \\ x_1 \\ u \end{pmatrix}
\tag{2.15}
\]

The inverse transformation is given by
\[
\begin{pmatrix} x \\ u \end{pmatrix} = \mathbf{S}^{-1}(z, u) = \begin{pmatrix} \frac{z_2}{1 + z_2} \\ \frac{z_1}{1 + z_2} \\ u \end{pmatrix}
\tag{2.16}
\]

The coordinate transformation (2.15) linearizes the nonlinear system (2.14) to the following linear reachable system.
\[
z(k + 1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} z(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k)
\]

\[ \square \]

**Remark 2.12** In the coordinates \((z, u)\), the system map \(\mathbf{\hat{f}}(z, u) = \begin{pmatrix} z_2 \\ z_1 + u \end{pmatrix}\) and the Jacobian
\[
\mathbf{\hat{f}}_* = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}
\]

In these coordinates, it is easy to calculate and obtain
\[
D_{\mathbf{\hat{f}}} \frac{\partial}{\partial u} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad D_{\mathbf{\hat{f}}}^2 \frac{\partial}{\partial u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\]

Using the coordinate transformation (2.17) defined above, we obtain
\[
\mathbf{\hat{S}}_{\ast}^{-1} \circ D_{\mathbf{\hat{f}}} \frac{\partial}{\partial u} \circ \mathbf{\hat{S}} = \begin{pmatrix} \frac{1}{(1 + z_2)^2} \\ \frac{-z_1}{1 + z_2} \\ 0 \end{pmatrix} \quad \mathbf{S}(z, u) = \begin{pmatrix} 1 \\ \frac{-z_2}{1 + z_2} \\ 0 \end{pmatrix} = D_{\mathbf{\hat{f}}} \frac{\partial}{\partial u}
\]
Similarly, we get

\[
\hat{S}_*^{-1} \circ D_f^2 \frac{\partial}{\partial u} \circ \hat{S} = \begin{pmatrix} \frac{1}{1+\xi_2} \\ 0 \end{pmatrix} \bigg|_{(z,u)=\hat{S}(x,u)} = \begin{pmatrix} \frac{1}{1+\xi_2} \\ 0 \end{pmatrix} = D_f^2 \frac{\partial}{\partial u}
\]

What we did above was a double-check on the local coordinate calculations of the vector fields. We computed the vector fields \(D_f^{i} \frac{\partial}{\partial u}(z,u)\), for \(i \geq 0\), in the coordinates \((z,u)\). Using the inverse coordinate transformation given by (2.16), we computed the \(\hat{S}^{-1}\)-related vector fields \(\hat{S}^{-1} D_f \frac{\partial}{\partial u} \circ \hat{S}\) and \(\hat{S}^{-1} D_f^2 \frac{\partial}{\partial u} \circ \hat{S}\). These vector fields were shown to be equal to \(D_f \frac{\partial}{\partial u}\) and \(D_f^2 \frac{\partial}{\partial u}\) respectively.

### 2.3 Exact Linearization by State Coordinate Transformation and Feedback

In this section we derive a set of conditions under which a given nonlinear discrete-time system can be made input-state equivalent to a linear reachable system through a state coordinate change and feedback. Under these conditions, the nonlinear system is said to be feedback linearizable. In a geometric treatment of this problem (as in [13], [17]), a set of necessary and sufficient conditions for feedback linearizability is stated in terms of a sequence of nested distributions. These distributions are defined using the system map \(f\). An attractive feature of the geometric approach is that it leads to results for discrete-time systems that are directly analogous to the continuous-time results.

In our treatment of this problem, we define an alternate sequence of nested distributions to characterize feedback linearizability. These distributions are
defined in terms of $f$-related vector fields. Under certain assumptions, they are easier to compute than the distributions in [13]. We then use Frobenius's Theorem to integrate this set of distributions; that is we determine functions which induce one-forms that span the annihilators of these distributions. The integrability conditions give rise to a set of linear, first-order partial differential equations, analogous to the continuous-time result. The coordinate transformation and feedback are then constructed from the solution to the set of partial differential equations.

A standard technique for solving a set of linear, first-order partial differential equations is by Cauchy's method of characteristics. By this method, the solution to the partial differential equations is obtained from the solution to an associated set of nonlinear differential equations. The fact that the solution to a discrete-time problem comes from a set of differential equations should not be very surprising, however, since the formulation is in terms of vector fields.

We begin with some definitions.

**Definition 2.13** A feedback function $\gamma$ is a diffeomorphism such that the following diagram commutes:

\[
\begin{array}{ccc}
M \times U & \xrightarrow{\gamma} & M \times U \\
\downarrow{\pi_M} & & \downarrow{\pi_M} \\
M & & M
\end{array}
\]

**Fig. 2.3:** Coordinate free definition of feedback function
In local coordinates \((x,u)\) for \(M \times U\), one can write \(v = \gamma(x,u)\) (with some abuse of notation), where \(v\) is the new input. Since the function \(\gamma\) is nonsingular, feedback can be viewed as a state-dependent change of input coordinates. 

**Definition 2.14 Feedback Linearizability of Discrete-Time Systems.** The nonlinear system
\[
x(k+1) = f(x(k),u(k)) \tag{\Sigma_{ONLS}}
\]
is said to be feedback linearizable in a neighborhood of \((x^0,u^0)\) if there exists a coordinate transformation \(z = T(x)\) on \(M\) and a feedback \(v = \gamma(x,u)\) in a neighborhood of \((x^0,u^0)\) on \(M \times U\), such that
\[
T \circ f\left(T^{-1}(z(k)), \gamma^{-1}(T^{-1}(z(k)), u(k))\right) = Az(k) + Bu(k) \tag{2.17}
\]
where \((A,B)\) is a reachable pair. 

We define the following sequence of distributions defined using the vector fields from Definition 2.4. These distributions are used to characterize feedback linearizability of discrete-time systems.

**Definition 2.15 Nested Sequence of Distributions for Feedback Linearizability.**

*Step 0.*
\[
G_0 \overset{\text{def}}{=} \text{span} \left\{ \frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_m} \right\}
\]

*Step \(k+1\).* Assume that the vector fields \(D_f^i \frac{\partial}{\partial u_j}\) for \(0 \leq i \leq k+1\) and \(1 \leq j \leq m\) are well-defined. Then
\[
G_{k+1} \overset{\text{def}}{=} \text{span} \left\{ \frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_m}, D_f \frac{\partial}{\partial u_1}, \ldots, D_f \frac{\partial}{\partial u_m}, \ldots, D_f^k \frac{\partial}{\partial u_1}, \ldots, D_f^k \frac{\partial}{\partial u_m} \right\}
\]
It is important to see how the distributions in (2.4) compare with the distributions defined in [13]. A statement of the main theorem from that paper is given below.

**Theorem 2.16 ([13])** For the nonlinear system $\Sigma_{DNLs}$, the following statements are equivalent.

a) The nonlinear system $\Sigma_{DNLs}$ is locally feedback linearizable in a neighborhood of $(x^0, u^0)$ to a reachable system.

b) $f_*(x^0, u^0)$ has full rank and there exists an open neighborhood $V$ about $(x^0, u^0)$ such that on this neighborhood, $\dim \tilde{G}_n = n + m$ where

$$
\tilde{G}_0 = (\pi_M)_*^{-1}(0)
$$

$$
\tilde{G}_{k+1} = \begin{cases} 
(\pi_M)_*^{-1} f_*(\tilde{G}_k) & \text{if } \tilde{G}_k + \ker f_* \text{ is involutive and } \tilde{G}_k \cap \ker f_* \text{ has constant dimension} \\
\tilde{G}_k & \text{otherwise}
\end{cases}
$$

c) There exists an open set $V$ about $(x^0, u^0)$ and regular distributions $\tilde{G}_0, \tilde{G}_1, \ldots, \tilde{G}_n$ such that $\tilde{G}_0 = (\pi_M)_*^{-1}(0)$, $\tilde{G}_{k+1} = (\pi_M)_*^{-1} f_*(\tilde{G}_k)$, and $\dim \tilde{G}_n = n + m$. 

**Remark 2.17** Statement (b) in the above theorem tells us that for $\tilde{G}_{k+1}$ to be a well-defined regular distribution, the regular distribution $\ker f_*$ must be controlled invariant with respect to the vector fields that span the distribution $\tilde{G}_k$. This means that by multiplying the vector fields in $\tilde{G}_k$ by a nonsingular matrix (which must be computed by a separate algorithm), we can render the distribution $\ker f_*$ invariant with respect to the transformed vector fields of $\tilde{G}_k$. (This is discussed in greater detail in Chapter 3.) In Definition 2.4 however, the assumption at the recursive step is that $\ker f_*$ is invariant with respect to
the vector fields of the distribution $G_k$. This is a more restrictive condition. Therefore, there may be feedback linearizable systems for which the method in Definition 2.4 does not yield the desired set of distributions.

\[ \square \]

2.3.1 Feedback Linearization of Single-Input Systems

In this section we derive a set of sufficient conditions for feedback linearizability of single-input discrete-time systems. When these conditions are satisfied, the coordinate transformation and feedback are obtained by solving a set of $n$ first-order partial differential equations. We need the concept of relative degree which is defined below.

**Definition 2.18 Relative Degree for Single-Input Discrete-Time Systems.**

The single-input nonlinear system given by

\[ x(k + 1) = f(x(k), u(k)) \]  \hspace{1cm} (\Sigma_{DNS})

\[ y(k) = h(x(k)) \]

is said to have relative degree $n$ if the vector fields $D^i_f \frac{\partial}{\partial u}$, for $0 \leq i \leq n$, are well-defined and the following conditions are satisfied:

\[ \langle dh, D^i_f \frac{\partial}{\partial u} \rangle = 0 \]  \hspace{1cm} (2.18a)

for $0 \leq i \leq n - 1$. in a neighborhood of $(x^0, u^0)$ and

\[ \langle dh, D^n_f \frac{\partial}{\partial u} \rangle(x^0, u^0) \neq 0 \]  \hspace{1cm} (2.18b)

\[ \square \]

The relative degree is the minimum number of time steps by which the output $y(k)$ must be advanced until it has an explicit dependence on the input $u(k)$. We make this statement more precise.
Lemma 2.19 Suppose that the single-input nonlinear system $\Sigma_{\delta_{N,L}}^h$ has relative degree $n$ in a neighborhood of the point $(x^0, u^0)$ and define the function $\tilde{f} \overset{\text{def}}{=} f(x,0)$. Then, the functions
\[
\begin{align*}
&h \\
&h \circ \tilde{f} \\
&h \circ \tilde{f} \circ f \\
&\quad \vdots \\
&h \circ \tilde{f}^{n-2} \circ f
\end{align*}
\] (2.19)
are all independent of $u$ in a neighborhood of the point $(x^0, u^0)$ and
\[
\frac{\partial}{\partial u} h \circ \tilde{f}^{n-1} \circ f(x^0, u^0) \neq 0
\] (2.20)

Proof. We show by induction that (2.19) is true. By assumption $h(x)$ is independent of $u$. Assume that the function $h \circ \tilde{f}^{k-2} \circ f(x,u)$ is independent of $u$. Note that from (2.18a) and Lemma 2.9
\[
\langle dh, D^n_f \frac{\partial}{\partial u} \rangle(x,u) = \langle D^n_f dh, \frac{\partial}{\partial u} \rangle(x,u) = \langle d\varphi_n, \frac{\partial}{\partial u} \rangle(x,u) = \frac{\partial \varphi_n}{\partial u}(x,u) = 0
\]
for $0 \leq k \leq n - 1$. Using the definition of $\varphi_k(x,u)$, we get
\[
\frac{\partial \varphi_k(x,u)}{\partial u} = \frac{\partial}{\partial u} \tilde{\varphi}_{k-1} \circ f(x,u) = \frac{\partial}{\partial u} h \circ \tilde{f}^{k-1} \circ f(x,u) = 0
\]
Hence the function $h \circ \tilde{f}^{k-1} \circ f(x,u)$ is independent of $u$. This proves (2.19).

We now show that (2.20) is true. From (2.18b) we obtain
\[
\langle dh, D^n_f \frac{\partial}{\partial u} \rangle(x^0,u^0) = \langle D^n_f dh, \frac{\partial}{\partial u} \rangle(x^0,u^0) = \langle d\varphi_n, \frac{\partial}{\partial u} \rangle(x^0,u^0) = \frac{\partial \varphi_n}{\partial u}(x^0,u^0) \neq 0
\] (2.21)
But $\varphi_n(x,u) \overset{\text{def}}{=} \tilde{\varphi}_{n-1} \circ f(x,u)$. From the definition of $\tilde{\varphi}_{n-1}$ and (2.21) the result follows. \hfill $\square$

The concept of relative degree was used in [8] to derive necessary and sufficient conditions for feedback linearizability. If the relative degree of the
given nonlinear system with respect to the output function is \(n\). then the system may be feedback linearized using the output functions and its successive compositions with the system map. In this work we extend the results to the case where the output function is not given. Put differently, we look for artificial output functions with respect to which the nonlinear system has full relative degree. This gives rise naturally to partial differential equations.

**Proposition 2.20** If the nonlinear system \(\Sigma_{DNLS}\) has relative degree \(n\) with respect to an output function \(\lambda\) then

1. The one-forms \(d\lambda, P_f d\lambda, \ldots, P_f^n d\lambda\) are independent in a neighborhood of \((x^0, u^0)\)

2. The vector fields \(\frac{\partial}{\partial u}, D_f \frac{\partial}{\partial u}, \ldots, D_f^n \frac{\partial}{\partial u}\) are independent in a neighborhood of \((x^0, u^0)\).

**Proof.** From our assumptions the exact one-forms \(P_f^j d\lambda\) are well-defined for \(0 \leq j \leq n\). Consider the following matrix product

\[
\begin{pmatrix}
  d\lambda \\
P_f d\lambda \\
\vdots \\
P_f^n d\lambda
\end{pmatrix}
\begin{pmatrix}
  \frac{\partial}{\partial u} \\
  D_f \frac{\partial}{\partial u} \\
  \vdots \\
  D_f^n \frac{\partial}{\partial u}
\end{pmatrix}
(x^0, u^0) \tag{2.22}
\]

Multiplying out the matrices we get

\[
\begin{pmatrix}
  \langle d\lambda, \frac{\partial}{\partial u} \rangle & \langle d\lambda, D_f \frac{\partial}{\partial u} \rangle & \cdots & \langle d\lambda, D_f^n \frac{\partial}{\partial u} \rangle \\
  \langle P_f d\lambda, \frac{\partial}{\partial u} \rangle & \langle P_f d\lambda, D_f \frac{\partial}{\partial u} \rangle & \cdots & \langle P_f d\lambda, D_f^n \frac{\partial}{\partial u} \rangle \\
  \vdots & \vdots & \ddots & \vdots \\
  \langle P_f^{n-1} d\lambda, \frac{\partial}{\partial u} \rangle & \langle P_f^{n-1} d\lambda, D_f \frac{\partial}{\partial u} \rangle & \cdots & \langle P_f^{n-1} d\lambda, D_f^n \frac{\partial}{\partial u} \rangle \\
  \langle P_f^n d\lambda, \frac{\partial}{\partial u} \rangle & \langle P_f^n d\lambda, D_f \frac{\partial}{\partial u} \rangle & \cdots & \langle P_f^n d\lambda, D_f^n \frac{\partial}{\partial u} \rangle
\end{pmatrix}
(x^0, u^0)
\]
which by (2.1) and Lemma 2.6 has the following lower triangular structure

\[
\begin{pmatrix}
0 & 0 & \cdots & \langle d\lambda, D_f^n \frac{\partial}{\partial u} \rangle \\
0 & 0 & \cdots & * \\
\vdots & \vdots & \cdots & \vdots \\
0 & \langle P_f^{n-1} d\lambda, D_f \frac{\partial}{\partial u} \rangle & \cdots & * \\
\langle P_f^n d\lambda, \frac{\partial}{\partial u} \rangle & * & \cdots & *
\end{pmatrix}
\begin{pmatrix}
x^0 \\
u^0
\end{pmatrix}
\]

By (2.18b) and Lemma 2.9, the diagonal elements are nonzero and the matrix has rank \( n + 1 \). This implies that the two matrices of (2.22) are each nonsingular; (i) and (ii) follow.

Our main result for single-input systems is the following.

**Theorem 2.21** Sufficient Conditions for Feedback Linearization of Single-Input Systems. The nonlinear system \( \Sigma_{DNS} \) with output function \( \lambda \) has relative degree \( n \) if and only if the distributions \( G_1, \ldots, G_n \) given by Definition 2.15 are all involutive and constant dimensional in a neighborhood of the operating point \( (x^0, u^0) \) with \( \dim G_n = n + 1 \). Furthermore, for such a \( \lambda \) the distributions \( G_i \) are given by

\[
G_i = \ker \text{span } \{ d\lambda, P_f d\lambda, \ldots, P_f^{n-1-i} d\lambda \}
\]

(2.23)

for \( 0 \leq i \leq n - 1 \).

**Proof.** Note that the conditions are only sufficient. This is because the distributions \( G_k \) are defined under more stringent conditions than the distributions \( \tilde{G}_k \).

Suppose that the nonlinear system \( \Sigma_{DNS} \) has relative degree \( n \) with respect to an output function \( \lambda \). Then, by Proposition 2.20 the one-forms
\(d\lambda \ldots P_f^n d\lambda\) are independent for all \((x, u)\) in a neighborhood of \((x^0, u^0)\). By Lemma 2.6b and \((2.18a)\), we then have

\[ G_i \subseteq \ker \text{span \{d\lambda \ldots P_f^{n-i}d\lambda\}} \]

for \(0 \leq i \leq n - 1\). By Proposition 2.20 and the definition of \(G_i\), it follows that \(\dim G_i = i\). And by independence of \(\{d\lambda \ldots P_f^n d\lambda\}\) we have

\[ \dim (\text{span \{d\lambda \ldots P_f^{n-i}d\lambda\}}) = n - i \]

This means that \((2.23)\) is true. Hence, the distributions \(G_0 \ldots G_n\) are all involutive and constant dimensional, and \(\dim G_n(x^0, u^0) = n + 1\).

Now suppose that for the system \(\Sigma_{\mathcal{N}_L}\) the distributions \(G_i\) defined in Definition 2.2 are all involutive and constant dimensional in a neighborhood of \((x^0, u^0)\) with \(\dim G_n = n + 1\). This assumption implies that the vector fields \(D_f^i \frac{\partial}{\partial u^i}\), for \(0 \leq i \leq n\), are well-defined. In particular, the distribution \(G_{n-1}\) is involutive and constant dimensional, and \(\dim G_{n-1} = n\). By Frobenius’ Theorem, there exists a function \(\lambda\) such that \(\text{span \{d\lambda\}} = \text{ann \}(G_{n-1})\). In other words

\[ \langle d\lambda, D_f^i \frac{\partial}{\partial u^i}\rangle(x, u) = 0 \]

for \(0 \leq i \leq n - i\) and for all \((x, u)\) in a neighborhood of \((x^0, u^0)\). Since \(\dim G_n = n + 1\), we also have

\[ \langle d\lambda, D_f^n \frac{\partial}{\partial u}\rangle(x^0, u^0) \neq 0 \]

which completes the proof. \(\square\)

This leads to the following construction.
Corollary 2.22 Construction of the Linearizing Transformation and Feedback.

Suppose that there exists a function $\lambda \in C^\infty(M \times U')$ such that $\Sigma_{DLS}$ has relative degree $n$ with respect to the function $\lambda$. Then, the required coordinate transformation and nonsingular feedback are given by the pair

$$z = T(x) \overset{\text{def}}{=} \begin{pmatrix} \hat{\varphi}_0 \\ \hat{\varphi}_1 \\ \vdots \\ \hat{\varphi}_{n-1} \end{pmatrix}(x)$$

$$v = \gamma(x, u) \overset{\text{def}}{=} \varphi_n(x, u)$$

Proof. The one-forms $P_i^*d\lambda(x, u)$, for $0 \leq i \leq n - 1$, are linearly independent and, therefore, so are the one forms $d\hat{\varphi}_i(x)$, for $0 \leq i \leq n - 1$. Hence, we conclude that $T$ is a diffeomorphism. Also,

$$\langle P_i d\lambda, \frac{\partial}{\partial u} \rangle(x^0, u^0) = \langle d\varphi_n, \frac{\partial}{\partial u} \rangle(x^0, u^0) = \frac{\partial \varphi_n}{\partial u}(x^0, u^0) \neq 0$$

Using the Implicit Function Theorem, we infer that $\gamma$ is nonsingular. To see that the transformed system is indeed $\Sigma_{DLS}$, note that

$$z_i(k + 1) = \hat{\varphi}_{i-1} \circ f(x(k), u(k)) = \varphi_i(x(k), u(k))$$

Since $\frac{\partial \varphi_i}{\partial u} = 0$, in local coordinates we have $\varphi_i(x, u) = \hat{\varphi}_i(x)$. This means that $z_i(k + 1) = z_{i+1}(k)$, for $1 \leq i \leq n - 1$, and

$$z_n(k + 1) = \hat{\varphi}_{n-1} \circ f(x(k), u(k)) = \varphi_n(x(k), u(k)) = v(k)$$

The last step is possible because the feedback function $\gamma$ is nonsingular. □

Corollary 2.23 If the nonlinear system $\Sigma_{DLS}$ has an output function $h(x)$ that satisfies the conditions of Theorem 2.21, then the function $h(x)$ may be used to construct the local coordinates transformation as in Corollary 2.22. In
this case the output of the system will be a linear function of the new state variables.

The following example serves to illustrate the results of Theorem 2.21 and Corollary 2.22.

**Example 2.24** Consider the nonlinear system

\[
x(k + 1) = \begin{pmatrix} 1 + x_2(k) + u(k) \\ x_1(k) \end{pmatrix}
= f(x(k), u(k))
\tag{2.24}
\]

We first construct the vector fields \( \frac{\partial}{\partial u}, D_f \frac{\partial}{\partial u}, D^2_f \frac{\partial}{\partial u} \) (provided that they are well-defined), and check if the corresponding distributions \( G_0, G_1, G_2 \) are involutive and constant-dimensional with \( \dim G_2 = n + m = 2 + 1 = 3 \). For clarity of expression, we shall drop the time index \( k \) from our computations.

For this system,

\[
f_* = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}
\]

The point \((x^0, u^0) = (0, 0, -1)^T\) is a fixed point of the system and notice that rank \( f_* = 2 \) in a neighborhood of the point \((x^0, u^0)\). For a point \((x, u) \in \mathcal{M} \times \mathcal{U}\), let \( x' = f(x, u) \). For the system (2.24).

\[
f^{-1}(x) = \left\{ \begin{pmatrix} x_2 \\ x_1 - u - 1 \\ u \end{pmatrix} \right\} \text{ for } (x, u) \in \mathcal{M} \times \mathcal{U}\]

By solving the equation

\[
f_* \alpha = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial u}) \alpha(x, u) = 0
\]

for \( \alpha(x, u) \), we obtain the following expression for the distribution

\[
\ker f_* = \text{span} \left\{ \alpha \frac{\partial}{\partial x_2} - \alpha \frac{\partial}{\partial u} \right\}
\]
where $\alpha(x, u)$ is an arbitrary function. We set $G_0 = \text{span} \{ \frac{\partial}{\partial u} \}$. Then

$$\left[ \frac{\partial}{\partial u}, X \right] = \frac{\partial x}{\partial u} \frac{\partial}{\partial x_2} - \frac{\partial x}{\partial u} \frac{\partial}{\partial u}$$

and $f_*\left[ \frac{\partial}{\partial u} \right] = 0$. Hence $f_*\frac{\partial}{\partial u}$ is well-defined. In coordinates,

$$f_*\frac{\partial}{\partial u}(x, u) = \left( \frac{\partial f}{\partial x} \right) \left( \frac{\partial f}{\partial u} \right)_{f^{-1}(x)} f^{-1}(x) \frac{\partial}{\partial u} = \frac{\partial}{\partial x_1}$$

and the vector field $D_f \frac{\partial}{\partial u} = \frac{\partial}{\partial x_1}$. We set $G_1 = \text{span} \{ \frac{\partial}{\partial u}, D_f \frac{\partial}{\partial u} \}$ and observe that $G_1$ is involutive with $\text{dim} G_1 = 2$ in a neighborhood of $(x^0, u^0)$. Next,

$$\left[ D_f \frac{\partial}{\partial u}, X \right] = \frac{\partial x}{\partial x_1} \frac{\partial}{\partial x_2} - \frac{\partial x}{\partial u} \frac{\partial}{\partial u}$$

Since $f_*[D_f \frac{\partial}{\partial u}, X] = 0$ this implies that the vector field $f_*D_f \frac{\partial}{\partial u}$ is well-defined. This vector field is computed in local coordinates as follows.

$$f_*D_f \frac{\partial}{\partial u}(x, u) = \left( \frac{\partial f}{\partial x} \right) \left( \frac{\partial f}{\partial u} \right)_{f^{-1}(x)} \left( D_f \frac{\partial}{\partial u} \right)_{f^{-1}(x)} = \frac{\partial}{\partial x_2}$$

and the vector field $D_f^2 \frac{\partial}{\partial u} = \frac{\partial}{\partial x_2}$. Define $G_2 = \text{span} \{ \frac{\partial}{\partial u}, D_f \frac{\partial}{\partial u}, D_f^2 \frac{\partial}{\partial u} \}$. It is seen that $\text{dim} G_2 = 3$ in a neighborhood of $(x^0, u^0)$ and $G_2$ is involutive. Therefore, we conclude that the nonlinear system is feedback linearizable.

To determine the coordinate transformation and feedback law we use Corollary 2.22. We know that there must exist a function $\lambda : M \times U \to \mathbb{R}$ such that

$$(d\lambda, \frac{\partial}{\partial u}) = 0, \quad (d\lambda, D_f \frac{\partial}{\partial u}) = 0$$

and $(d\lambda, D^2_f \frac{\partial}{\partial x})(x^0, u^0) \neq 0$. In local coordinates this means that we must solve the following set of partial differential equations:

$$\frac{\partial \lambda}{\partial u} = 0, \quad \frac{\partial \lambda}{\partial x_1} = 0$$
It is easy to see that $\lambda = x_2$ satisfies these conditions. Further, $\frac{\partial \lambda}{\partial x_2}(x^0, u^0) \neq 0$ which means that we can define locally a nonsingular feedback law.

The coordinate transformation is constructed from the function $\lambda$ as follows.

$z_1 \overset{\text{def}}{=} \hat{\varphi}_0(x) = \varphi_0(x, u) \overset{\text{def}}{=} \lambda(x, u) = x_2$

$z_2 \overset{\text{def}}{=} \hat{\varphi}_1(x) = \varphi_1(x, u) \overset{\text{def}}{=} \varphi_0 \circ f(x, u) = x_1$

The feedback is given by $v = \hat{\varphi}_1 \circ f(x, u) = (1 + x_2 + u)$. This feedback linearizes the nonlinear system to

$z(k + 1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v(k)$

\[ \square \]

### 2.3.2 Feedback Linearization of Multi-Input Systems

In this section we generalize the results obtained in the previous section to multi-input nonlinear systems. We begin by introducing the concept of vector relative degree for multi-input multi-output (MIMO) discrete-time nonlinear systems.

**Definition 2.25** Vector Relative Degree for Multi-Input Systems. Consider the nonlinear system

$x(k + 1) = f(x(k), u(k))$ \hspace{1cm} (\Sigma_{\text{DNLs}}^h)

$y(k) = h(x(k), u(k))$

The system $\Sigma_{\text{DNLs}}^h$ has vector relative degree $\{r_1, \ldots, r_m\}$ if the vector fields $D_f^i \frac{\partial}{\partial u_j}$, for $0 \leq i \leq r_j$ and $1 \leq j \leq m$ are well-defined and the following conditions are satisfied.

$\langle dh_i, D_f^j \frac{\partial}{\partial u_j} \rangle = 0$
for $0 \leq k \leq r_1 - 1$, $1 \leq j \leq m$ and $1 \leq i \leq m$ in the neighborhood of $(x^0, u^0)$, and the matrix

$$A(x^0, u^0) = \begin{pmatrix}
\langle dh_1, D_f^r \frac{\partial}{\partial u_1}\rangle & \ldots & \langle dh_1, D_f^r \frac{\partial}{\partial u_m}\rangle \\
\vdots & \ddots & \vdots \\
\langle dh_m, D_f^r \frac{\partial}{\partial u_1}\rangle & \ldots & \langle dh_m, D_f^r \frac{\partial}{\partial u_m}\rangle
\end{pmatrix} (x^0, u^0)$$

is nonsingular. □

**Lemma 2.26** Suppose that the nonlinear system $\Sigma^h_{O,NLS}$ has vector relative degree $\{r_1, \ldots, r_m\}$ at $(x^0, u^0)$. Then, the one-forms

$$dh_1, \quad P_f^1 dh_1, \quad \ldots \quad P_f^{r_1} dh_1,$$

$$dh_2, \quad P_f^1 dh_2, \quad \ldots \quad P_f^{r_2} dh_2,$$

$$\vdots \quad \vdots \quad \ldots \quad \vdots$$

$$dh_m, \quad P_f^1 dh_m, \quad \ldots \quad P_f^{r_m} dh_m$$

are independent at $(x^0, u^0)$.

**Fig. 2.4**: Structure of the matrix $QP$

**Proof.** It is clear, from our assumptions, that the one-forms $P_f^k dh_i$, for $0 \leq k \leq r_i$, and $1 \leq i \leq m$, are well-defined. Without loss of generality assume that $r_1 \geq r_i$, for $2 \leq i \leq m$. Let

$$Q = \text{col} \{dh_1, \ldots, P_f^{r_1} dh_1, dh_2, \ldots, P_f^{r_2} dh_2, \ldots, dh_m, \ldots, P_f^{r_m} dh_m\}$$

$$P = \{\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_m}, D_f \frac{\partial}{\partial u_1}, \ldots, D_f \frac{\partial}{\partial u_m}, \ldots, D_f^{r_1} \frac{\partial}{\partial u_1}, \ldots, D_f^{r_1} \frac{\partial}{\partial u_m}\}$$
and consider the matrix product $QP$. After some rearrangement, this product matrix may be written in the form given in Fig. 2.4. Using Lemma 2.6 we observe that this matrix has block, right-lower triangular structure with the diagonal blocks consisting of the rows of the matrix $A(x^0, u^0)$. From the nonsingularity assumption of $A(x^0, u^0)$ we conclude that the matrix product $QP$ is nonsingular, which implies the independence of the rows of $Q$. 

**Remark 2.27** Since the set $Q$ has $\sum_{i=1}^m r_i + m$ elements and the dimension of $T^\ast(M \times U)$ is $m + n$, we have the inequality $r_1 + r_2 + \cdots + r_m \leq n$. 

**Theorem 2.28** *Sufficient Conditions for Feedback Linearization of Multi-Input Systems.* For the nonlinear system

$$x(k + 1) = f(x(k), u(k)) \quad (\Sigma_{ONLS})$$

there exist $m$ real-valued functions $\lambda_i : M \times U \rightarrow \mathbb{R}$, for $0 \leq i \leq m$, defined on a neighborhood of the fixed point $(x^0, u^0)$ such that $(\Sigma_{ONLS})$ with output functions $y_i \overset{def}{=} \lambda_i$ has vector relative degree $\{r_1, \ldots, r_m\}$ at $(x^0, u^0)$ with

$$r_1 + r_2 + \cdots + r_m = n$$

if the distributions $G_0, \ldots, G_n$ are involutive and constant dimensional with $\dim G_1 = 2m$ and $\dim G_n = n + m$.

**Proof.** The proof can be found in Appendix-A. 

**Remark 2.29** *Construction of the Linearizing Transformation and Feedback.* The nonlinear system

$$x(k + 1) = f(x(k), u(k)) \quad (\Sigma_{ONLS}^h)$$

$$y(k) = h(x(k), u(k))$$

...
is feedback linearizable if it has vector relative degree \( \{r_1, \ldots, r_m\} \) such that \( r_1 + r_2 + \cdots + r_m = n \). The required coordinate transformation and nonsingular feedback are given by the pair

\[
\begin{align*}
  z &= T(x) = \begin{pmatrix}
    \hat{\varphi}^1(x) \\
    \vdots \\
    \hat{\varphi}^m(x)
  \end{pmatrix} \\
  v &= \Gamma(x,u) = \begin{pmatrix}
    \gamma^1(x,u) \\
    \vdots \\
    \gamma^m(x,u)
  \end{pmatrix}
\end{align*}
\]

where

\[
\begin{align*}
  \hat{\varphi}^i(x) &\overset{\text{def}}{=} \begin{pmatrix}
    \hat{\varphi}^i_0(x) \\
    \hat{\varphi}^i_1(x) \\
    \vdots \\
    \hat{\varphi}^i_{r_i-1}(x)
  \end{pmatrix} \\
  \gamma^i(x,u) &\overset{\text{def}}{=} \varphi^i_{r_i}(x,u)
\end{align*}
\]

for \( 1 \leq i \leq m \).

**Proof.** From linear independence of the one-forms \( P^j_i dh_i(x,u) \). and hence that of \( d\hat{\varphi}^j_i(x) \) for \( 0 \leq i \leq r_i - 1 \) and \( 1 \leq j \leq m \). we conclude that \( T \) is a diffeomorphism. From nonsingularity of the matrix \( A(x^0,u^0) \) in the relative degree condition, we infer that the matrix

\[
\begin{pmatrix}
  \langle P^r_i dh_1, \frac{\partial}{\partial u_1} \rangle & \cdots & \langle P^r_i dh_1, \frac{\partial}{\partial u_m} \rangle \\
  \vdots & \ddots & \vdots \\
  \langle P^r_m dh_m, \frac{\partial}{\partial u_1} \rangle & \cdots & \langle P^r_m dh_m, \frac{\partial}{\partial u_m} \rangle
\end{pmatrix}
(x^0,u^0)
\]

is nonsingular. But \( \langle P^r_i dh_i, \frac{\partial}{\partial u_j} \rangle(x^0,u^0) = \langle d\varphi^j_i, \frac{\partial}{\partial u_j} \rangle(x^0,u^0) = \frac{\partial \varphi^j_i}{\partial u_j}(x^0,u^0) \). Hence, the Jacobian matrix given by

\[
\begin{pmatrix}
  \frac{\partial \varphi^1_i}{\partial u_1} & \cdots & \frac{\partial \varphi^1_i}{\partial u_m} \\
  \vdots & \ddots & \vdots \\
  \frac{\partial \varphi^m_i}{\partial u_1} & \cdots & \frac{\partial \varphi^m_i}{\partial u_m}
\end{pmatrix}
(x^0,u^0)
\]

is also nonsingular. Applying the implicit function theorem, we conclude that \( \Gamma \) is nonsingular. To see that the transformed system is indeed \( \Sigma_{\alpha LS} \), note that for the \( i \)th subsystem

\[
\hat{z}^i_j(k + 1) = \hat{\varphi}^i_{j-1} \circ f(x(k), u(k)) = \varphi^i_j(x(k), u(k))
\]
and because \( \frac{\partial \varphi_j^l}{\partial u_i} = 0 \) for \( 1 \leq l \leq m \) and \( 0 \leq j \leq r_i - 1 \). in local coordinates \( \varphi_j^l(x, u) = \hat{\varphi}_j^l(x) \) for \( 0 \leq j \leq r_i - 1 \). This means that

\[
z_j^l(k+1) = z_{j+1}^l(k)
\]

for \( 1 \leq j \leq r_i - 1 \) and

\[
z_{r_i}^l(k+1) = \hat{\varphi}_{r_i-1}^l \circ f(x(k), u(k)) = \varphi_{r_i}^l(x(k), u(k)) = v_i(k)
\]

where the last step is possible because of nonsingularity of the map \( \Gamma \). \( \square \)

2.4 Summary

In this chapter we developed a geometric framework for discrete-time nonlinear systems. The essential features of this framework are vector fields and one-forms for discrete-time systems. This framework was used to study the problem of exact linearization using state coordinate change, and the problem of feedback linearization, for both single-input and multi-input systems. Necessary and sufficient conditions for the first problem were obtained. The required transformation was obtained by solving a set of \( n^2 + nm \) linear, first-order partial differential equations. Sufficient conditions were obtained for feedback linearization of single-input and multi-input systems. The required transformation and feedback are obtained in this case by solving a set of \( n \) linear, first-order partial differential equations.
3.1 Introduction

In this chapter we present a set of necessary and sufficient conditions for feedback linearizability of single-input discrete-time systems. The results are expressed in terms of certain column vectors and row-vectors which are closely related to the vector fields and one-forms defined in the previous chapter. Our contributions are i) A set of conditions for feedback linearizability that are simpler to verify than the conditions in [13], and ii) A simple procedure to construct the linearizing transformation and the feedback law. In addition, we resolve the problem of computing the distributions that characterize feedback linearizability in [13].

The conditions in [13] are expressed in terms of a nested sequence of distributions which are defined recursively using the tangent map $f_*$. We first show that the well-definedness condition for the distribution $\tilde{G}_k$ may be interpreted as a controlled invariance condition imposed on an associated continuous-time nonlinear system. To compute this sequence of distributions one must, in principle, solve a continuous-time controlled invariance problem at each step $k$, for $k = 0, 1, \ldots, n-1$. We demonstrate, however, that it suffices
to check at each step $k$ that the controlled invariance problem is solvable. It is not necessary to compute the feedback matrix.

To recognize this, a recursive, closed form expression for the vector fields that span the distribution $\tilde{G}_k$ is first derived. Then, the column vectors $D^i_f \frac{\partial}{\partial u}(x,u)$, for $0 \leq i \leq n$, are defined on the local coordinate chart. The point to note is that although the conditions for the existence of these column vectors are the same as the conditions for the existence of the vector fields $D^i_f \frac{\partial}{\partial u}$, for $0 \leq i \leq n$ in Chapter 2, the column vectors are easier to compute. These vectors may be used to obtain an equivalent representation of the distributions $\tilde{G}_k$, for $0 \leq k \leq n$. Also, the well-definedness condition for the distribution $\tilde{G}_k$ is expressed entirely in terms of these column vectors.

If the distributions $\tilde{G}_k$, for $0 \leq k \leq n$, are all involutive and constant dimensional, we may apply Frobenius's Theorem to integrate them simultaneously. These integrability conditions are given by a set of partial differential equations, using the vector fields that span the bases of these distributions. We show that the integrability conditions may be given by an alternate set of partial differential equations using the column vectors $D^i_f \frac{\partial}{\partial u}(x,u)$, for $0 \leq i \leq n$. The solution to this set of partial differential equations is used to construct the required coordinate transformation and feedback law.

This chapter is organized as follows. In Section 3.2 we show that the condition for well-definedness of the distributions $\tilde{G}_k$ is equivalent to a controlled invariance condition on a continuous-time nonlinear system associated with these distributions. Using this interpretation, we obtain a simple procedure for computing these distributions. In Section 3.3, we prove some technical lemmas
to simplify the well-definedness condition for the distribution $\tilde{G}_k$. These intermediate results are combined to obtain a new set of necessary and sufficient conditions for feedback linearizability of single-input discrete-time systems. In Section 3.4, we express the integrability conditions on the sequence of involutive distributions $\tilde{G}_k$, for $0 \leq k \leq n - 1$, in terms of an alternate set of linear first-order partial differential equations, using the column vectors $D^k \frac{\partial}{\partial u}(x, u)$. The solution to this set of partial differential equations is used to construct the coordinate transformation and feedback law according to the method outlined in Corollary 3.16. Our results are illustrated in Section 3.5 through an example of a second-order nonlinear system.

3.2 The Controlled Invariance Condition

In this section we show that the geometric condition (3.1) for well-definedness of the $f$-related distribution $\tilde{G}_k$ is really a controlled invariance condition on an associated continuous-time nonlinear system whose dynamics are given by the vector fields that span the distribution $\tilde{G}_k$. This interpretation allows us to identify possible simplifications in the local coordinate computations involved in constructing the basis vector fields for the distributions $\tilde{G}_k$, for $0 \leq k \leq n$. These distributions were used in [13] in the theorem below.

**Theorem 2.16** ([13]) **Geometric Condition for Feedback Linearizability.** For the nonlinear system $\Sigma_{DNLs}$, the following statements are equivalent.

a) The nonlinear system $\Sigma_{DNLs}$ is locally feedback linearizable about $(x^0, u^0)$ to a reachable system.
b) \( f_\ast(x^0,u^0) \) has full rank and there exists an open neighborhood \( V \) about \( (x^0,u^0) \) such that on this neighborhood. \( \dim \tilde{G}_n = n + 1 \) where

\[
\begin{align*}
\tilde{G}_0 &= (\pi_M)_\ast^{-1}(0) \\
\tilde{G}_{k+1} &= \begin{cases} \\
(\pi_M)_\ast^{-1}f_\ast(\tilde{G}_k) & \text{if } \tilde{G}_k + \ker f_\ast \text{ is involutive} \\
\tilde{G}_k & \text{and } \tilde{G}_k \cap \ker f_\ast \text{ has constant dimension} \\
\end{cases}
\end{align*}
\]

if \( \tilde{G}_k + \ker f_\ast \) is involutive and \( \tilde{G}_k \cap \ker f_\ast \) has constant dimension otherwise.

\[ \tag{3.1} \]

c) There exists an open set \( V \) about \( (x^0,u^0) \) and regular distributions \( \tilde{G}_0, \tilde{G}_1, \ldots, \tilde{G}_n \) such that \( \tilde{G}_0 = (\pi_M)_\ast^{-1}(0) \), \( \tilde{G}_{k+1} = (\pi_M)_\ast^{-1}f_\ast(\tilde{G}_k) \), and \( \dim \tilde{G}_n = n + 1 \).

From the above definition it is clear that a basis for the distribution \( \tilde{G}_0 \) is \( \frac{\partial}{\partial u} \). Hence \( \tilde{G}_0 = \text{span} \{ \frac{\partial}{\partial u} \} \). The distribution \( \tilde{G}_k \) is computed recursively. At step \( k \), we must be able to compute \( \tilde{G}_{k+1} \) by operating the tangent map \( f_\ast \) on the vector fields that form the basis for the distribution \( \tilde{G}_k \). A necessary and sufficient condition for such an operation to give well-defined vector fields is given by the following lemma.

**Lemma 2.3** ([22]) Geometric Condition for Well-Defined Vector Fields. Suppose that \( f : M \times U \to M \) is a \( C^\infty \) map such that the induced tangent map \( f_\ast : T(M \times U) \to TM \) is onto. Given a vector field \( X \in T(M \times U) \), the vector field \( Y \) defined pointwise by \( Y_{f(p)} = f_\ast p X_p \) is a well-defined vector field on some open neighborhood of \( f(p) \) if and only if

\[ [X, \ker f_\ast] \subset \ker f_\ast \tag{3.2} \]

on some open neighborhood of \( p \). \( \square \)
The geometric interpretation of (3.2) is that the distribution \( \ker f_* \) is \textit{invariant} with respect to the vector field \( X \). In the next paragraph we give a geometric interpretation of the condition in (3.1).

Assume that a basis is available for \( \mathcal{G}_k \) at stage \( k+1 \) of the recursive procedure outlined in Theorem 2.16. and denote the distribution \( \mathcal{G}_k \) by

\[
\mathcal{G}_k = \text{span} \{X_1, \ldots, X_{k+1}\}
\]

Suppose further, that at stage \( k \) the conditions given in (3.1) hold. That is,

i) The distribution \( \mathcal{G}_k + \ker f_* \) is involutive, and

ii) The distribution \( \mathcal{G}_k \cap \ker f_* \) has constant dimension.

For the distribution \( \mathcal{G}_k \), we can define an associated continuous-time nonlinear system whose dynamics are given by

\[
\frac{d\zeta}{dt} = X_1(\zeta)\nu_1 + \cdots + X_{k+1}(\zeta)\nu_{k+1}
\]

(3.3)

Here, the augmented state vector \( \zeta \overset{\text{def}}{=} (x, u)^T \). Note that the drift vector field is absent in this system. Conditions (i) and (ii) imply that the distribution \( \ker f_* \) is \textit{controlled invariant} (see [16] and [32]) with respect to the input vector fields \( X_i \), for \( 1 \leq i \leq k+1 \). Hence, we conclude that there must exist a nonsingular matrix \( \beta^k(x, u) \), of dimension \( k+1 \times k+1 \), such that the vector fields \( X_i^{\beta^k} \), for \( 1 \leq i \leq k+1 \), obtained by the equation

\[
\begin{pmatrix}
X_1^{\beta^k} \\
\vdots \\
X_{k+1}^{\beta^k}
\end{pmatrix}
= \begin{pmatrix}
X_1 \\
\vdots \\
X_{k+1}
\end{pmatrix} \beta^k(x, u)
\]

satisfy the conditions

\[
[X_i^{\beta^k}, \ker f_*] \subset \ker f_*
\]
for $1 \leq i \leq k + 1$. In other words, the distribution ker $f_*$ is invariant with respect to the vector fields $X^j_\ast$ for $1 \leq i \leq k + 1$. From Lemma 2.3, we observe that this is also the condition that the vector fields $X^j_\ast$ must satisfy such that $f_\ast X^j_\ast$ are well-defined vector fields for $1 \leq i \leq k + 1$.

Consider the definition of vector fields in Definition 2.4. The assumption at the recursive step is that the distribution ker $f_*$ is invariant with respect to the vector fields that span the distribution $G_k$. From Lemma 2.3 it follows that the tangent map $f_\ast$ operated on the vector fields that span the distribution $G_k$ will give well-defined vector fields. Hence, we did not have to solve a controlled invariance problem to construct a basis for the distribution $G_{k+1}$. On the other hand, this means that the distributions $G_k$ may not be defined even if the distributions $\tilde{G}_k$ are well-defined.

We adopt the controlled invariance interpretation of the geometric condition (3.1) and give the following procedure for computing the distributions $\tilde{G}_k$, for $0 \leq k \leq n$ (which are defined in a coordinate-free way in Theorem 2.16.) Note that the local coordinate representation of each vector field in the basis for the distribution $G_k$ is given by a recursive, closed form expression.

**Definition 3.1 Local Coordinate Calculation of the Distributions that Characterize Feedback Linearizability.** We begin the recursive definition by setting $\tilde{G}_0 = \text{span} \{ \frac{\partial}{\partial u} \}$. The distribution ker $f_*$ is given by ker $f_\ast = \text{span} \{ \alpha(x, u) \}$, where $\alpha(x, u)$ is a vector field on $M \times U$ such that $f_\ast \alpha = (\frac{\partial f}{\partial x} \frac{\partial f}{\partial u}) \alpha(x, u) = 0$.

**Step 0.** Check if $\tilde{G}_0 + \ker f_\ast$ is involutive and if $\tilde{G}_0 \cap \ker f_\ast$ is constant dimensional. In local coordinates, this means that we have to check if

$$\left[ \frac{\partial}{\partial u}, \alpha(x, u) \right] \in \text{span} \{ \alpha(x, u) \} + \text{span} \{ \frac{\partial}{\partial u} \} \quad (3.4)$$
and the second condition means that we have to check if

$$\dim \text{span} \{ \alpha(x,u) \} \cap \text{span} \{ \frac{\partial}{\partial u} \} = \text{constant} \quad (3.5)$$

If (3.4) and (3.5) are satisfied, we compute a $1 \times 1$ nonsingular matrix $J_0(x,u)$ such that $[\mathcal{G}_0 J_0(x,u), \ker f_*] \subset \ker f_*$. In local coordinates this means that

$$[J_0(x,u) \frac{\partial}{\partial u}, \alpha(x,u)] \in \text{span} \{ \alpha(x,u) \} \quad (3.6)$$

The distribution $\mathcal{G}_0$ may be written as

$$\mathcal{G}_0 J_0(x,u) \overset{\text{def}}{=} \text{span} \{ X^0_1 \} \quad (3.7)$$

where $X^0_1 \overset{\text{def}}{=} J_0(x,u) \frac{\partial}{\partial u}$. Since (3.6) holds, we may compute the distribution $\mathcal{G}_1$ as follows.

$$\mathcal{G}_1 = \text{span} \{ \frac{\partial}{\partial u}, D_f X^0_1 \} \quad (3.8)$$

where the vector field $D_f X^0_1$ is computed using the method outlined in Remark 2.5.

**Step k.** Let the distribution $\mathcal{G}_k$ be given by

$$\mathcal{G}_k = \text{span} \{ \frac{\partial}{\partial u}, D_f X^{k-1}_1, \ldots, D_f X^{k-1}_k \} \quad (3.9)$$

where,

$$X^{k-1}_i = J^{k-1}_{1,i}(x,u) \frac{\partial}{\partial u} + \sum_{j=1}^{k-1} J^{k-1}_{j+1,i}(x,u) D_f X^{k-2}_j \quad (3.10)$$

for $1 \leq i \leq k$. This expression (3.10) for the vector field $X^{k-1}_i$ was derived by observation. We note that the vector field $D_f X^{k-1}_1$ is obtained from the vector field $X^{k-1}_1$ by the $D_f$ operation defined in Remark 2.5. To construct a basis for $\mathcal{G}_{k+1}$, we must first check if $[\mathcal{G}_k, \ker f_*] \subset \ker f_* + \text{span} \{ \mathcal{G}_k \}$, and if
the distribution $\mathcal{G}_k \cap \ker f_*$ is constant dimensional. In local coordinates, this means that we must check if
\[
\left[ \frac{\partial}{\partial u}, \alpha(x, u) \right] \in \text{span} \{ \alpha(x, u) \} + \text{span} \left\{ \frac{\partial}{\partial u}, D_f X_i^{k-1}, \ldots, D_f X_k^{k-1} \right\}
\quad (3.11)
\]
and
\[
\left[ D_f X_i^{k-1}, \alpha(x, u) \right] \in \text{span} \{ \alpha(x, u) \} + \text{span} \left\{ \frac{\partial}{\partial u}, D_f X_i^{k-1}, \ldots, D_f X_k^{k-1} \right\}
\quad (3.12)
\]
for $1 \leq i \leq k$, and if
\[
\dim \left\{ \text{span} \{ \alpha(x, u) \} \cap \text{span} \left\{ \frac{\partial}{\partial u}, D_f X_i^{k-1}, \ldots, D_f X_k^{k-1} \right\} \right\} = \text{constant}
\quad (3.13)
\]
in a neighborhood of the point $(x^0, u^0)$. If the conditions given by (3.11) through (3.13) are satisfied, compute a nonsingular matrix $J^k(x, u)$ of dimension $k + 1 \times k + 1$ such that $[\mathcal{G}_k J^k(x, u), \ker f_*] \subset \ker f_*$. In local coordinates, this implies that
\[
[X_i^k, \alpha(x, u)] \in \text{span} \{ \alpha(x, u) \}
\]
where
\[
X_i^k = J^k_{i, i}(x, u) \frac{\partial}{\partial u} + \sum_{j=1}^{k} J^k_{j+1, i}(x, u) D_f X_j^{k-1}
\]
for $1 \leq i \leq k + 1$. In the new basis, the distribution $\mathcal{G}_k$ is given by
\[
\mathcal{G}_k J^k(x, u) \overset{\text{def}}{=} \text{span} \left\{ X_1^k, \ldots, X_{k+1}^k \right\}
\]
This allows us to compute the distribution $\mathcal{G}_{k+1}$ using the $D_f$ operation on the set of vector fields given above. Thus,
\[
\mathcal{G}_{k+1} = \text{span} \left\{ \frac{\partial}{\partial u}, D_f X_1^k, \ldots, D_f X_{k+1}^k \right\}
\quad (3.14)
\]
This completes the definition. \hfill \square
Observe that at each stage $k$ in the above definition, we must solve a controlled invariance problem in continuous-time in order to compute $\bar{G}_{k+1}$. Since the matrix $J^k(x,u)$ is nonsingular, we can relax the conditions given by (3.11), (3.12) and (3.13). This is discussed in detail in the next section.

### 3.3 Simplified Testable Conditions

In this section we derive a set of necessary and sufficient conditions for feedback linearizability of single-input discrete-time systems. These conditions are expressed in terms of the column vectors $D^i_f \frac{\partial}{\partial u}(x,u)$, for $0 \leq i \leq k$, which are easier to compute than the distributions $\bar{G}_k$. The main results of this section are developed as follows. We begin by defining, loosely, the column vectors $D^i_f \frac{\partial}{\partial u}(x,u)$, for $0 \leq i \leq n$. We show that these vectors can be used to construct, in local coordinates, the basis for the distribution $\bar{G}_k$. The conditions for well-definedness of the distribution $\bar{G}_k$ are expressed in terms of these column vectors. We then define the column vectors $D^i_f \frac{\partial}{\partial u}(x,u)$, for $0 \leq i \leq n$, more rigorously, by incorporating well-definedness conditions. We show that the column vectors $D^i_f \frac{\partial}{\partial u}(x,u)$, for $0 \leq i \leq k$, are computable by this definition if and only if the distribution $\bar{G}_k$ is well-defined. This result is the basis for Theorem 3.10 which gives necessary and sufficient conditions for feedback linearizability in terms of the column vectors $D^i_f \frac{\partial}{\partial u}(x,u)$, for $0 \leq i \leq n$.

Consider the following definition for column vectors. Note that the definition only provides a method to calculate the column vectors $D^i_f \frac{\partial}{\partial u}(x,u)$, for $0 \leq i \leq n$. It does not check if these calculations are really permissible.
Definition 3.2  The vectors $D^k_f \frac{\partial}{\partial u}(x,u)$ are computed recursively as follows.

i) Let $D^0_f \frac{\partial}{\partial u}(x,u) \overset{\text{def}}{=} \frac{\partial}{\partial u}.$

ii) Given $D^k_f \frac{\partial}{\partial u}(x,u),$ compute the vector

$$D^{k+1}_f \frac{\partial}{\partial u}(x,u) = \left( \frac{\partial f}{\partial x} \frac{\partial f}{\partial u} \right)_{f^{-1}(x)} \left( D^k_f \frac{\partial}{\partial u}(x,u) \right)_{f^{-1}(x)}$$

Let $D^{k+1}_f \frac{\partial}{\partial u}(x,u)$ be the column vector such that

$$(\frac{\partial \pi_{\xi}}{\partial x} \frac{\partial \pi_{\xi}}{\partial u})(x,u) D^{k+1}_f \frac{\partial}{\partial u} = D^{k+1}_f \frac{\partial}{\partial u}(x,u)$$

\[\square\]

Using Definition 3.2. and the linearity property of the operator $D_f,$ the following properties can be shown to be true.

i) $D_f(X_1 + X_2) = D_fX_1 + D_fX_2$ (3.10)

ii) $D_f \left( \beta(x,u)X(x,u) \right) = \tilde{\beta}(x,u)D_fX(x,u)$ (3.11)

where $\tilde{\beta}(x,u) = \beta(x,u) \big|_{f^{-1}(x)}$ \[\square\]

The following lemma shows that a vector field that belongs to the basis for the distribution $G_k,$ for $0 \leq k \leq n,$ can be expressed as a linear combination of the column vectors $D^i_f \frac{\partial}{\partial u}(x,u),$ for $0 \leq i \leq k.$ This observation helps in understanding the proofs of Lemma 3.6 and Lemma 3.7.

Lemma 3.4  Suppose that the vector fields $D_fY^{k-1}_i(x,u),$ for $0 \leq i \leq k,$ have been computed according to Definition 3.1. and that the column vectors $D^j_f \frac{\partial}{\partial u}(x,u),$ for $0 \leq j \leq n,$ have been computed according to Definition 3.2.
Then, the coordinate representation of the vector fields $D_j X_i^{k-1}(x, u)$, for $1 \leq i \leq k$, may be written as

$$D_j X_i^{k-1}(x, u) = \sum_{j=1}^{k} \mu_{i,j}^{k-1}(x, u) D_j \frac{\partial}{\partial u}(x, u)$$  \hspace{1cm} (3.19)$$

where $\mu_{i,j}^{k-1}(x, u)$ are smooth functions defined on a neighborhood of the point $(x^0, u^0)$.

**Proof.** The proof is based on induction on $k$. Note that

$$X_i^{k-1}(x, u) = J_i^{k-1}(x, u) \frac{\partial}{\partial u} + \sum_{j=i}^{k-1} J_i^{k-1}(x, u) D_j X_j^{k-2}(x, u)$$  \hspace{1cm} (3.20)$$

for $1 \leq i \leq k$ and $1 \leq k \leq n$. Note that $J_i^{k-1}(x, u)$ is the $(i,j)^{th}$ element of the feedback matrix $J^{k-i}(x, u)$. We first test the hypothesis for $k = 1$. Substituting $k = 1$ in (3.20) we get

$$X^0(x, u) = X_1^0 = J^0(x, u) \frac{\partial}{\partial u}$$

Using the $D_j$ operator we get,

$$D_j X^0_1(x, u) = J^0(x, u) D_j \frac{\partial}{\partial u}(x, u) = \mu^0(x, u) D_j \frac{\partial}{\partial u}(x, u)$$

where $\mu^0(x, u) \overset{\text{def}}{=} J^0(x, u)$. Next, assume that (3.19) holds for $k - 1$. That is

$$D_j X_i^{k-1}(x, u) = \sum_{j=1}^{k-1} \mu_{i,j}^{k-2}(x, u) D_j \frac{\partial}{\partial u}(x, u)$$

for $1 \leq i \leq k - 1$. For the induction step $k$, we obtain $D_j X_i^{k-1}(x, u)$, for $1 \leq i \leq k$. using (3.20) as follows.

$$D_j X_i^{k-1}(x, u) = J_i^{k-1}(x, u) D_j \frac{\partial}{\partial u} + \sum_{j=1}^{k-1} J_i^{k-1}(x, u) D_j D_j X_j^{k-2}(x, u)$$  \hspace{1cm} (3.21)$$
Note that \( D_f D_j X_j^{k-2} \) must be interpreted simply as a column vector obtained by the \( D_f \) operation (as in Definition 3.2) on \( D_f X_j^{k-2}(x,u) \). Using the induction hypothesis we can expand \( D_f X_j^{k-2}(x,u) \) and rewrite (3.21) as

\[
D_f X_i^{k-1}(x,u) = \tilde{j}_{1,i}^{k-1}(x,u) D_f \frac{\partial}{\partial u}(x,u) \\
+ \sum_{j=1}^{k-1} \tilde{j}_{j+1,i}^{k-1}(x,u) D_f \left( \sum_{l=1}^{k-1} \mu_{j,l}^{k-2}(x,u) D_f^l \frac{\partial}{\partial u}(x,u) \right) \\
= \tilde{j}_{1,i}^{k-1}(x,u) D_f \frac{\partial}{\partial u}(x,u) \\
+ \sum_{l=1}^{k-1} \sum_{j=1}^{k-1} \mu_{j,l}^{k-2}(x,u) \tilde{j}_{j+1,i}^{k-1}(x,u) D_f D_f^l \frac{\partial}{\partial u}(x,u) \\
= \sum_{l=1}^{k} \mu_{i,l}^{k-1}(x,u) D_f^l \frac{\partial}{\partial u}(x,u)
\]

where \( \mu_{j,l}^{k-2}(x,u) \overset{\text{def}}{=} \mu_{j,l}^{k-2}(x,u) \mid_{f^{-1}(x)} \) and

\[
\mu_{i,l}^{k-1}(x,u) \overset{\text{def}}{=} \sum_{j=1}^{k-1} \mu_{j,l}^{k-2}(x,u) \tilde{j}_{j+1,i}^{k-1}(x,u)
\]

Hence (3.19) holds for all \( k \) for \( 1 \leq k \leq n - 1 \). This completes the proof. \( \square \)

We will later encounter column vectors of the form \( D_f^l X_i^{k-1}(x,u) \), where the \( D_f \) operation is a formal calculation given in Definition 3.2. It is useful to know that a column vector of this form too may be represented as a linear combination of the vectors \( D_f^l \frac{\partial}{\partial u}(x,u) \), for \( 0 \leq i \leq k \). This is summarized in the following corollary to Lemma 3.4.

**Corollary 3.5** The column vectors \( D_f^l X_i^{k-1}(x,u) \), for \( 1 \leq i \leq k \), may be written as

\[
D_f^l X_i^{k-1}(x,u) = \sum_{j=1}^{k} \mu_{i,j}(x,u) D_f^j \frac{\partial}{\partial u}(x,u) \tag{3.22}
\]

where \( \mu_{i,j}(x,u) \) are smooth functions defined on a neighborhood of the point \((x^0,u^0)\). \( \square \)
The next lemma gives a simple test for dimension of the distribution $\mathcal{G}_k$. It is significant that the test requires only the column vectors $D_f \frac{\partial}{\partial u}(x,u)$ for $0 \leq i \leq k$. These vectors are easier to compute than the vectors $D_f X_i^{k-1}(x,u)$, for $0 \leq i \leq k$, since the latter must be computed using Definition 3.1 (which requires one to solve a controlled invariance problem). Lemma 3.6 is used later, in Theorem 3.10, to obtain a simpler test for feedback linearizability of discrete-time systems.

**Lemma 3.6** Let $D_f X_i^{k-1}(x,u)$ for $1 \leq i \leq k$ be the local coordinate representation of the vector fields obtained using Definition 3.1, and let $D_f \frac{\partial}{\partial u}(x,u)$ for $0 \leq i \leq k$ be the vectors constructed according to Definition 3.2. Then, for each point $(x,u)$ in a neighborhood of $(x^0,u^0)$ the dimension of the distribution $\mathcal{G}_k$ is $k + 1$ if and only if

$$\dim \text{span} \left\{ \frac{\partial}{\partial u}, D_f \frac{\partial}{\partial u}(x,u), \ldots, D_f^k \frac{\partial}{\partial u}(x,u) \right\} = k + 1 \quad (3.23)$$

**Proof.** Suppose that $\dim \mathcal{G}_k = k + 1$. Then, the vectors in the set

$$\left\{ \frac{\partial}{\partial u}, D_f X_1^{k-1}(x,u), \ldots, D_f X_k^{k-1}(x,u) \right\} \quad (e_0)$$

are linearly independent in a neighborhood of $(x^0,u^0)$. We will show that the vectors $\left\{ \frac{\partial}{\partial u}, D_f \frac{\partial}{\partial u}(x,u), \ldots, D_f^k \frac{\partial}{\partial u}(x,u) \right\}$ are also linearly independent. From $(e_0)$ it is obvious that that the vector $\frac{\partial}{\partial u}$ is linearly independent of the vectors in the set $\{D_f X_1^{k-1}(x,u), \ldots, D_f X_k^{k-1}(x,u)\}$. Using the definition of $D_f X_i^{k-1}(x,u)$ and the property (3.18), we can show that

$$\{D_f X_1^{k-1}(x,u), \ldots, D_f X_k^{k-1}(x,u)\} = \left\{ D_f \frac{\partial}{\partial u}(x,u), D_f D_f X_1^{k-2}(x,u), \ldots, D_f D_f X_{k-1}^{k-2}(x,u) \right\} \tilde{\beta}^{k-1}(x,u)$$
where \( \tilde{z}^{k-1}(x, u) = z^{k-1}(x, u) |_{f^{-1}(x)} \). By assumption, the system remains in a neighborhood of \((x^0, u^0)\). Hence, if we choose this neighborhood to be sufficiently small, we may use a continuity argument and claim that the matrix \( \tilde{z}^{k-1}(x, u) \) is also nonsingular in this neighborhood. This means that the vectors in the set

\[
\left\{ \frac{\partial}{\partial u}(x, u), D_f D_f X_1^{k-2}(x, u), \ldots, D_f D_f X_{k-1}^{k-2}(x, u) \right\}
\]  

(\(e_1\))

are linearly independent. From (\(e_0\)) and (\(e_1\)) we conclude that the vectors \( \frac{\partial}{\partial u} \) and \( D_f \frac{\partial}{\partial u} (x, u) \) are linearly independent of each other, and also linearly independent of the vectors in the set \( \left\{ D_f D_f X_1^{k-2}(x, u), \ldots, D_f D_f X_{k-1}^{k-2}(x, u) \right\} \).

Now we use the recursive definition of \( X_i^{k-2}(x, u) \), for \( 1 \leq i \leq k - 1 \), and use a similar argument as above to show that the vectors in the set

\[
\left\{ D_f D_f \frac{\partial}{\partial u}(x, u), D_f D_f D_f X_1^{k-3}(x, u), \ldots, D_f D_f D_f X_{k-2}^{k-3}(x, u) \right\}
\]  

(\(e_2\))

are linearly independent. At the \(k\)th step of this procedure, we have used the \(D_f\) operation \(k\) times to obtain vector

\[
D_f D_f \cdots D_f \frac{\partial}{\partial u}(x, u) = D_f^k \frac{\partial}{\partial u}(x, u)
\]  

(\(e_k\))

Combining the partial results \(e_1, \ldots, e_k\), we conclude that the vectors in the set

\[
\left\{ \frac{\partial}{\partial u}, D_f \frac{\partial}{\partial u}(x, u), \ldots, D_f^k \frac{\partial}{\partial u}(x, u) \right\}
\]

are linearly independent.

To prove the converse statement, we use Definition 3.1 to recursively construct the following basis vectors for the distribution \( \tilde{G}_k \).

\[
\left\{ \frac{\partial}{\partial u}, D_f X_1^{k-1}(x, u), \ldots, D_f X_k^{k-1}(x, u) \right\}
\]
from the vectors $D_i^j \frac{\partial}{\partial u}(x,u)$. for $0 \leq i \leq k$, and the "feedback" matrices $J^i(x,u)$. for $0 \leq i \leq k-1$. Then, linear independence of the vectors in the set

$$\left\{ \frac{\partial}{\partial u}, D_j X_1^{k-1}(x,u), \ldots, D_j X_k^{k-1}(x,u) \right\}$$

follows from linear independence of the vectors in the set

$$\left\{ \frac{\partial}{\partial u}, D_j \frac{\partial}{\partial u}(x,u), \ldots, D_j^k \frac{\partial}{\partial u}(x,u) \right\}$$

and the fact that the matrices $J^i(x,u)$, for $0 \leq i \leq k-1$ are nonsingular in a neighborhood of $(x^0, u^0)$. This completes the proof.

Next, we derive an equivalent set of conditions to (3.11), (3.12) and (3.13). The new conditions are expressed in terms of the column vectors $D_i^j \frac{\partial}{\partial u}(x,u)$, for $0 \leq i \leq k$. For our purposes, it is useful to introduce a pair of operators, which are defined as follows. Let $Y_1(\xi)$ and $Y_2(\xi)$ be column vectors such that each element of the vector is a smooth function of $\xi$. Note that $Y_1(\xi)$ or $Y_2(\xi)$ need not be the coordinate representation of some vector field on $M \times U$. The column vector $[Y_1(\xi), Y_2(\xi)]$ is defined as follows.

$$[Y_1(\xi), Y_2(\xi)] \overset{def}{=} \frac{\partial Y_2}{\partial \xi} Y_1(\xi) - \frac{\partial Y_1}{\partial \xi} Y_2(\xi) \quad (3.24)$$

Consider a smooth function $\lambda(\xi)$ which may or may not be the coordinate representation of a function on the manifold $M \times U$. The operator $\langle \cdot, \cdot \rangle$ is defined as follows.

$$\langle d\lambda(\xi), X(\xi) \rangle \overset{def}{=} \frac{\partial \lambda}{\partial \xi} X(\xi) \quad (3.25)$$

Therefore, the operators $[\cdot, \cdot]$ and $\langle \cdot, \cdot \rangle$ are simply formal calculations on column vectors defined on $TR^{n+m}$ and row vectors defined on $T^*R^{n+m}$. Using the definitions given above, it can be verified that the following properties hold.

$$[\beta(\xi)Y_1(\xi), Y_2(\xi)] = J(\xi)[Y_1(\xi), Y_2(\xi)] - \left( \frac{\partial \beta}{\partial \xi} Y_2(\xi) \right) Y_1(\xi) \quad (3.26)$$
\[ [X_1(\xi), Y_1(\xi) + Y_2(\xi)] = [X_1(\xi), Y_1(\xi)] + [X_1(\xi), Y_2(\xi)] \]  
(3.27)

\[ \langle d\lambda(\xi), J(\xi)Y_1(\xi) \rangle = J(\xi)\langle d\lambda(\xi), Y_1(\xi) \rangle \]  
(3.28)

where \( J(\xi) \) is a smooth function of its argument. \( \square \)

As discussed earlier, the distribution \( \tilde{G}_k \) must satisfy the condition given by

\[ [\tilde{G}_k, \ker f_*] \subset \ker f_* + \text{span} \{ \tilde{G}_k \} \]

This allows us to construct a basis for the distribution \( \tilde{G}_{k+1} \). In local coordinates this means that the conditions given in (3.11), (3.12) and (3.13) must be satisfied. The following lemma uses the properties of the \([\cdot, \cdot] \) operator to obtain an equivalent set of conditions in terms of the column vectors \( D_f \frac{\partial}{\partial u} (x, u) \) for \( 0 \leq i \leq k \). The new set of conditions are easier to check.

**Lemma 3.7** For each \( 0 \leq k \leq n - 1 \) the statements

\[ [\frac{\partial}{\partial u}, \alpha(x, u)] \in \text{span} \{ \alpha(x, u) \} \]

\[ + \text{span} \left\{ \frac{\partial}{\partial u}, D_f X_i^{k-1}(x, u), \ldots, D_f X_k^{k-1}(x, u) \right\} \]  
(3.29)

\[ [D_f X_i^{k-1}(x, u), \alpha(x, u)] \in \text{span} \{ \alpha(x, u) \} \]

\[ + \text{span} \left\{ \frac{\partial}{\partial u}, D_f X_i^{k-1}(x, u), \ldots, D_f X_k^{k-1}(x, u) \right\} \]  
(3.30)

for \( 1 \leq i \leq k \) are true if and only if the statements

\[ [D_f^{i} \frac{\partial}{\partial u} (x, u), \alpha(x, u)] \in \text{span} \{ \alpha(x, u) \} \]

\[ + \text{span} \left\{ \frac{\partial}{\partial u}, D_f^{i} \frac{\partial}{\partial u} (x, u), \ldots, D_f^{k} \frac{\partial}{\partial u} (x, u) \right\} \]  
(3.31)

for \( 0 \leq i \leq k \) are true.

**Proof** (Sufficiency). Assume that the conditions given by (3.31) hold. By setting \( i = 0 \) in (3.31), and using Lemma 3.6, we can prove that (3.29) is
true. To prove (3.30), we use the expression for \( X_j^{k-1}(x,u) \) for \( 1 \leq j \leq l \), successively for \( l = k, k-1, \ldots, 0 \).

For \( l = k \), consider the bracket \([D_f X_j^{k-1}(x,u), \alpha(x,u)]\). We show that

\[
[D_f X_j^{k-1}(x,u), \alpha(x,u)] \in \text{span} \left\{ \frac{\partial}{\partial u}, D_f \frac{\partial}{\partial u}(x,u), \ldots, D_f^k \frac{\partial}{\partial u}(x,u) \right\}
\]

(3.32)

for each \( 1 \leq j \leq k \). By Lemma 3.6, this will imply that

\[
[D_f X_j^{k-1}(x,u), \alpha(x,u)] \in \text{span} \left\{ \frac{\partial}{\partial u}, D_f X_1^{k-1}(x,u), \ldots, D_f X_k^{k-1}(x,u) \right\}
\]

By successively applying properties (3.17) and (3.18) we have the following equalities.

\[
[D_f X_j^{k-1}(x,u), \alpha(x,u)]
= [D_f \left( \beta_{i,j}^{k-1}(x,u) \frac{\partial}{\partial u} + \sum_{i=1}^{k-1} \beta_{i+1,j}^{k-1}(x,u) D_f X_i^{k-2}(x,u) \right), \alpha(x,u)]
= [\left( \frac{\partial}{\partial u} + \sum_{i=1}^{k-1} \beta_{i+1,j}^{k-1}(x,u) D_f X_i^{k-2}(x,u) \right), \alpha(x,u)]
\]

(3.33)

where \( \beta_{i,j}^{k-1}(x,u) \) is defined as \( \beta_{i,j}^{k-1}(x,u)|_{f^{-1}(x)} \). Using (3.26), the right hand side of (3.33) may be expanded as

\[
[D_f X_j^{k-1}(x,u), \alpha(x,u)]
= \frac{\partial}{\partial u} + \sum_{i=1}^{k-1} \beta_{i+1,j}^{k-1}(x,u) D_f X_i^{k-2}(x,u), \alpha(x,u)\]
\]

Using (3.18), the above expression may be re-written as

\[
\beta_{i,j}^{k-1}(x,u) [\frac{\partial}{\partial u}, \alpha(x,u)] - \frac{\partial}{\partial (x,u)} \beta_{i,j}^{k-1}(x,u) D_f \frac{\partial}{\partial u}(x,u)
+ \sum_{i=1}^{k-1} \beta_{i+1,j}^{k-1}(x,u) [D_f X_i^{k-2}(x,u), \alpha(x,u)]
- \sum_{i=1}^{k-1} \left( \frac{\partial}{\partial (x,u)} \beta_{i+1,j}^{k-1}(x,u) \alpha(x,u) \right) D_f X_i^{k-2}(x,u)
\]
Using Corollary 3.5 and condition (3.31), the conditions in (3.32) are equivalent to the conditions given below.

\[
\left\{ \left[ D_f \frac{\partial}{\partial u}(x, u), \alpha(x, u) \right], \left[ D_f D_f X_i^{k-2}(x, u), \alpha(x, u) \right], \ldots, \left[ D_f D_f X_{k-1}^{k-2}(x, u), \alpha(x, u) \right] \right\} \tilde{J}^{k-1}(x, u)
\]
\[\in \text{span} \left\{ \frac{\partial}{\partial u}, D_f \frac{\partial}{\partial u}(x, u), \ldots, D_f^k \frac{\partial}{\partial u}(x, u) \right\}\]

But \( \tilde{J}^{k-1}(x, u) \) is nonsingular in a neighborhood of \((x^0, u^0)\) and by (3.31) we know that

\[
\left[ D_f \frac{\partial}{\partial u}(x, u), \alpha(x, u) \right] \in \text{span} \left\{ \frac{\partial}{\partial u}, D_f \frac{\partial}{\partial u}(x, u), \ldots, D_f^k \frac{\partial}{\partial u}(x, u) \right\}
\]

Hence, it only remains to be shown that

\[
\left[ D_f D_f X_i^{k-2}(x, u), \alpha(x, u) \right] \in \text{span} \left\{ \frac{\partial}{\partial u}, D_f \frac{\partial}{\partial u}(x, u), \ldots, D_f^k \frac{\partial}{\partial u}(x, u) \right\}
\]

for \(1 \leq i \leq k-1\). This can be accomplished by repeating the above procedure, using the recursive formula for \(X_i^{k-2}(x, u)\), for \(1 \leq i \leq k-1\). At the end of the \(k^{th}\) iteration of this procedure, we must show that

\[
\left[ D_f^k \frac{\partial}{\partial u}(x, u), \alpha(x, u) \right] \in \text{span} \left\{ \frac{\partial}{\partial u}, D_f \frac{\partial}{\partial u}(x, u), \ldots, D_f^k \frac{\partial}{\partial u}(x, u) \right\}
\]

in order to complete the proof. But this is true by (3.31). Hence, the proof of sufficiency is complete.

(Necessity). We omit the proof for this part since the technique is quite similar to that for sufficiency. \(\square\)

In addition to (3.11) and (3.12), the distribution \(\tilde{G}_k\) must also satisfy the dimensionality condition given in (3.13). Note that this condition is expressed in terms of the vector fields that span \(\tilde{G}_k\). However, the following lemma
provides an equivalent condition that is easier to check because it is expressed in terms of the column vectors $D^i_j \frac{\partial}{\partial u}(x, u)$, for $0 \leq i \leq k$.

**Lemma 3.8** Consider the discrete-time nonlinear system $\Sigma_{DNS}$ and suppose that the vector fields $D^k_j X_{1}^{k-1}(x, u)$ have been computed using Definition 3.1 and the vectors $D^j_i \frac{\partial}{\partial u}(x, u)$, for $0 \leq j \leq n$, have been computed using Definition 3.2. In local coordinates, let the distribution $\ker f_\ast$ be given by $\ker f_\ast = \text{span} \{ \alpha(x, u) \}$. Then, the distribution

$$\text{span} \left\{ \frac{\partial}{\partial u}, D^k_j X_{1}^{k-1}(x, u), \ldots, D^j_i \frac{\partial}{\partial u}(x, u) \right\} \cap \ker f_\ast$$

is constant dimensional if and only if the dimension of the subspace

$$\text{span} \left\{ \frac{\partial}{\partial u}, D^k_j \frac{\partial}{\partial u}(x, u), \ldots, D^j_i \frac{\partial}{\partial u}(x, u) \right\} \cap \text{span} \{ \alpha(x, u) \}$$

is constant at each point $(x, u)$ in a neighborhood of $(x^0, u^0)$.

**Proof.** The proof is a straightforward application of Lemma 3.6.

We now combine the results of Lemma 3.7 and Lemma 3.8. The calculations for these vectors in this definition are identical to the calculations of the vectors defined in Definition 3.2. However, the vectors below are defined only if the conditions given in Lemma 3.7 and Lemma 3.8 are satisfied.

**Definition 3.9** Consider the single-input discrete-time nonlinear system given by

$$x(k + 1) = f(x(k), u(k))$$

(\(\Sigma_{DNS}\))

and assume that the rank of the Jacobian \( \frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial u} \) is equal to $n$ in a neighborhood of the equilibrium point $(x^0, u^0)$.
Step 0. Let $D_f^0 \frac{\partial}{\partial u}(x, u) \overset{\text{def}}{=} \frac{\partial}{\partial u}$.

Step $k$. For each $0 \leq l \leq k$, check if
\[
[D_f^l \frac{\partial}{\partial u}(x, u), \alpha(x, u)] \in \text{span} \left\{ \frac{\partial}{\partial u}, D_f^l \frac{\partial}{\partial u}(x, u), \ldots, D_f^k \frac{\partial}{\partial u}(x, u) \right\}
+ \text{span} \left\{ \alpha(x, u) \right\}
\]
(3.36)
in a neighborhood of $(x^0, u^0)$, and if the subspace given by
\[
\text{span} \left\{ \frac{\partial}{\partial u}, D_f^l \frac{\partial}{\partial u}(x, u), \ldots, D_f^k \frac{\partial}{\partial u}(x, u) \right\} \cap \text{span} \left\{ \alpha(x, u) \right\}
\]
(3.37)
is constant dimensional in a neighborhood of $(x^0, u^0)$. If (3.36) and (3.37) are true, then compute the vector $D_f^{k+1} \frac{\partial}{\partial u}(x, u)$ as follows.
\[
D_f^{k+1} \frac{\partial}{\partial u}(x, u) = \left( \frac{\partial f}{\partial x} \frac{\partial f}{\partial u} \right) f^{-1}(x) \left( D_f^k \frac{\partial}{\partial u}(x, u) \right) f^{-1}(x)
\]
Then let $D_f^{k+1} \frac{\partial}{\partial u}(x, u)$ be the column vector such that
\[
(\frac{\partial \pi_m}{\partial x} \frac{\partial \pi_m}{\partial u})(x, u) D_f^{k+1} \frac{\partial}{\partial u}(x, u) = D_f^{k+1} \frac{\partial}{\partial u}(x, u)
\]
If (3.36) or (3.37) is not true, set $D_f^{k+1} \frac{\partial}{\partial u}(x, u) = D_f^k \frac{\partial}{\partial u}(x, u)$.

By comparing the above definition with Lemma 3.7 and Lemma 3.8, we observe that the column vectors $D_f^k \frac{\partial}{\partial u}(x, u)$, for $0 \leq k \leq n$, are computable by Definition 3.9 if and only if the distributions $\tilde{G}_k$, for $0 \leq k \leq n$, are well-defined. This observation is the basis for the following theorem which gives necessary and sufficient conditions for feedback linearization. Since these conditions are stated in terms of the column vectors $D_f^i \frac{\partial}{\partial u}(x, u)$, for $0 \leq i \leq k$, these conditions are simpler to verify than the conditions in Theorem 2.16.

**Theorem 3.10** Necessary and Sufficient Conditions for Feedback Linearizability of Discrete-Time Systems. The single-input nonlinear system given by
\[
x(k + 1) = f(x(k), u(k))
\]
($\Sigma_{DNLS}$)
is feedback linearizable around the equilibrium point \((x^0, u^0)\) if and only if

1) The vectors \(D_f^k \frac{\partial}{\partial u}(x, u)\), for each \(0 \leq k \leq n\), can be computed using Definition 3.9

2) \(\text{dim span} \left\{ \frac{\partial}{\partial u} \cdot D_f \frac{\partial}{\partial u}(x, u), \ldots, D_f^n \frac{\partial}{\partial u}(x, u) \right\} = n + 1.\)

**Proof.** From Lemma 3.6, Lemma 3.7 and Definition 3.9 we know that

1) The distributions \(\tilde{G}_k\) are well-defined for \(0 \leq k \leq n\) if and only if the vectors \(D_f^k \frac{\partial}{\partial u}(x, u)\) can be computed by Definition 3.9.

2) The distribution \(\tilde{G}_n\) has dimension \(n + 1\) if and only if

\[\text{dim span} \left\{ \frac{\partial}{\partial u} \cdot D_f \frac{\partial}{\partial u}(x, u), \ldots, D_f^n \frac{\partial}{\partial u}(x, u) \right\} = n + 1\]

The proof then follows from Theorem 2.16.

\[\square\]

### 3.4 The Linearizing Transformation and Feedback Law

In this section, we show that the integrability conditions on the sequence of involutive distributions \(\tilde{G}_k\), for \(0 \leq k \leq n - 1\), may be expressed as a set of linear first-order partial differential equations which are written using the column vectors \(D_f^k \frac{\partial}{\partial u}(x, u)\), for \(0 \leq k \leq n\). We show that in order to construct the coordinate transformation and feedback law, it suffices to solve this set of partial differential equations.

We first define row-vectors for the nonlinear system \(\Sigma_{DNLS}\), and establish some useful properties. We are especially interested in the operation of a row-vector on a column vector. Relative degree for single-input nonlinear systems
is defined using these operations. We show that the row-vectors $P_f^k d\lambda(x,u)$ for $0 \leq k \leq n$ and the column vectors $D_f^i \frac{\partial}{\partial u}(x,u)$, for $0 \leq i \leq n$, are linearly independent if and only if the nonlinear system has relative degree $n$. This is useful for proving the main results in Theorem 3.15 and Lemma 3.16.

**Definition 3.11**

**Step 0.** Let $\lambda(x,u)$ be a smooth function. Let $\varphi_0(x,u) \overset{\text{def}}{=} \lambda(x,u)$ and set $P_f^0 d\lambda(x,u) = d\varphi_0(x,u) \overset{\text{def}}{=} d\lambda(x,u)$.

**Step $k + 1.$** Suppose that the row-vector $P_f^k d\lambda(x,u)$ satisfies

\[
\left\{ P_f^k d\lambda(x,u), \frac{\partial}{\partial u_j} \right\} = \left\{ d\varphi_k(x,u), \frac{\partial}{\partial u_j} \right\} = \frac{\partial \varphi_k}{\partial u_j} = 0
\]

for $1 \leq j \leq m$. We let $\overline{P_f^{k+1} d\lambda(x)} = d\tilde{\varphi}_{k+1}$ be the row vector such that

\[
\overline{P_f^{k+1} d\lambda(x)}(\frac{\partial \pi_M}{\partial x} \frac{\partial \pi_M}{\partial u}) \pi_{\pi_{-1}}^{-1}(x) = P_f^k d\lambda(x,u)
\]

In terms of the functions $\tilde{\varphi}_k(x)$, this can be written as

\[
d\tilde{\varphi}_{k+1}(x)(\frac{\partial \pi_M}{\partial x} \frac{\partial \pi_M}{\partial u}) \pi_{\pi_{-1}}^{-1}(x) = d\varphi_k(x,u)
\]

The row vectors at step $k + 1$ are then given by

\[
P_f^{k+1} d\lambda(x,u) = \overline{P_f^{k+1} d\lambda(x)}(\frac{\partial f}{\partial x} \frac{\partial f}{\partial u})_{f^{-1}(x)}
\]

Or alternately,

\[
d\varphi_{k+1}(x,u) = d\tilde{\varphi}_{k+1}(x)(\frac{\partial f}{\partial x} \frac{\partial f}{\partial u})_{f^{-1}(x)}
\]

Note that the calculations are identical to the local coordinate calculations of the one-forms defined in Definition 2.8.
Lemma 3.12  For the nonlinear system $\Sigma_{\text{DNLs}}$, let the column vectors $D^l_j \frac{\partial}{\partial u_i}$ be well-defined for $0 \leq i \leq n$ and $1 \leq l \leq m$. Let $\lambda : M \times \mathcal{U} \rightarrow \mathbb{R}$ be a $C^\infty$ function. Then, the statement

a) $\langle d\lambda, D^i_j \frac{\partial}{\partial u_l} \rangle = 0$ for $0 \leq i \leq n - 1$ and $1 \leq l \leq m$

implies the following statements

b) The row-vectors $P^j_i d\lambda$ are well-defined for $0 \leq j \leq n$.

c) $\langle P^j_i d\lambda, D^i_j \frac{\partial}{\partial u_l} \rangle = 0$, for $0 \leq i + j \leq n - 1$ and $1 \leq l \leq m$.

d) $\langle d\lambda, D^n_j \frac{\partial}{\partial u_l} \rangle = \langle P^k_j d\lambda, D^{n-k}_j \frac{\partial}{\partial u_l} \rangle$

Proof. To prove this lemma, we modify the proof of Lemma 2.9 so that all calculations are in local coordinates. Then, the proof is complete if we substitute the operator $\langle \cdot, \cdot \rangle$ for $\langle \cdot, \cdot \rangle$. We omit these details.

We now define the concept of relative degree for $\Sigma_{\text{DNLs}}$ in terms of the column vectors $D^k_j \frac{\partial}{\partial u}(x, u)$, for $0 \leq k \leq n$.

Definition 3.13 Relative Degree for Discrete-Time Systems. The single-input nonlinear system given by

$$x(k + 1) = f(x(k), u(k)) \quad (\Sigma_{\text{DNLs}})$$

is said to have relative degree $n$ around the equilibrium point $(x^0, u^0)$ if

a) The column vectors $D^k_j \frac{\partial}{\partial u}(x, u)$, for $0 \leq k \leq n$, can be computed using Definition 3.9.
b) There exists a function $\lambda(x,u)$ defined on a neighborhood of $(x^0,u^0)$, such that $d\lambda(x^0,u^0) \neq 0$, and which satisfies the conditions

\begin{align}
\text{i) } & \langle d\lambda(x,u), D^k_f \frac{\partial}{\partial u}(x,u) \rangle = 0 \text{ for } 0 \leq k \leq n-1 \\
\text{ii) } & \langle d\lambda, D^n_f \frac{\partial}{\partial u}(x^0,u^0) \rangle \neq 0.
\end{align}

(3.38) (3.39)

\[
\square
\]

It can be easily shown that the interpretation of relative degree given in Remark 2.19 holds for the above definition. That is, if the relative degree of the nonlinear system $\Sigma_{DNL}$ is $n$ with respect to an artificial output function $\lambda$, then $\lambda(x(k))$ must be advanced $n$ time steps before it depends explicitly on the input function $u(k)$.

We now establish the following. If the nonlinear system $\Sigma_{DNL}$ has relative degree $n$, then the row-vectors $P^k f d\lambda(x,u)$, for $0 \leq k \leq n$, and the column vectors $D^i_f \frac{\partial}{\partial u}(x,u)$, for $0 \leq i \leq n$ are linearly independent. This result is used in the construction of the linearizing coordinate transformation and feedback law. It is also used in the proof of Theorem 3.15.

**Lemma 3.14** If the nonlinear system $\Sigma_{DNL}$ has relative degree $n$, then

\begin{align}
\text{i) } & \text{The row-vectors } P^i d\lambda, P^2 d\lambda, \ldots, P^n d\lambda \text{ are independent in a neighborhood of } (x^0,u^0) \\
\text{ii) } & \text{The column vectors } \frac{\partial}{\partial u}, D^i_f \frac{\partial}{\partial u}, \ldots, D^n_f \frac{\partial}{\partial u} \text{ are independent in a neighborhood of } (x^0,u^0).
\end{align}

**Proof.** Note that the calculations of the row-vectors $P^k f d\lambda$, for $0 \leq k \leq n$, and the column vectors $D^i_f \frac{\partial}{\partial u}(x,u)$, for $0 \leq i \leq n$, are identical to the local
coordinate calculations of the one-forms $P^k d\lambda$ for $0 \leq k \leq n$ and the vector fields $D^i f \frac{\partial}{\partial u}$. for $0 \leq i \leq n$. which are defined in Chapter 2. Hence, we can use the technique of the proof of Proposition 2.20, but with the operator $\langle \cdot, \cdot \rangle$ replacing the operator $\langle \cdot, \cdot \rangle$. The details of the proof are omitted.

We now prove the main theorem of this section. This theorem gives necessary and sufficient conditions for feedback linearization of the nonlinear system $\Sigma_{DNLSS}$. The conditions are expressed in terms of the column vectors $D^i f \frac{\partial}{\partial u}(x, u)$, for $0 \leq i \leq n$.

**Theorem 3.15** The single-input discrete-time nonlinear system given by

$$x(k + 1) = f(x(k), u(k)) \quad (\Sigma_{DNLSS})$$

is feedback linearizable in a neighborhood of $(x^0, u^0)$ if and only if it has relative degree $n$ in a neighborhood of $(x^0, u^0)$.

**Proof (Necessity).** Suppose that the nonlinear system $\Sigma_{DNLSS}$ is feedback linearizable. Then, by Theorem 2.16, the distributions $\mathcal{G}_k$ are involutive and constant dimensional, and $\dim \mathcal{G}_n = n + 1$. This implies that the column vectors $D^k f \frac{\partial}{\partial u}(x, u)$, for $0 \leq k \leq n$, are computable using Definition 3.9, hence part (a) of Definition 3.13 holds.

We now prove that part (b) holds. Note that the theorem is stated for the single-input case. Hence, in particular, it must be true that the distribution $\mathcal{G}_{n-1}$ is involutive, and $\dim \mathcal{G}_{n-1} = n$. By Frobenius's Theorem, we infer that there exists a function $\lambda(x, u)$ defined on a neighborhood of $(x^0, u^0)$, with $d\lambda(x^0, u^0) \neq 0$, such that this function satisfies the conditions

$$\langle d\lambda(x, u), \frac{\partial}{\partial u} \rangle = 0 \quad (3.40)$$
\[
\langle d\lambda(x, u), D_f X_i^{n-2} \rangle = 0 \tag{3.41}
\]
for \(1 \leq i \leq n - 1\). Since \(\dim \mathcal{G}_n = n + 1\) and \(d\lambda(x^0, u^0) \neq 0\), the vector
\[
\left( \langle d\lambda, \frac{\partial}{\partial u} \rangle \quad \langle d\lambda, D_f X_i^{n-1} \rangle \quad \cdots \quad \langle d\lambda, D_f X_{n-1}^{n-1} \rangle \right)^T (x^0, u^0) \neq \mathbf{0} \tag{3.42}
\]
where \(\mathbf{0}\) is the zero vector. Also,
\[
X_i^{k-1}(x, u) = \mathcal{J}_1 \cdot \mathcal{J}_{k-1}(x, u) \frac{\partial}{\partial u} + \sum_{j=1}^{k-1} \mathcal{J}_{j+1} \cdot \mathcal{J}_{k-j}(x, u) D_f X_j^{k-j-1}(x, u) \tag{3.43}
\]
for \(1 \leq i \leq k\), and \(1 \leq k \leq n\). In particular, for \(k = n - 1\) we have.
\[
X_i^{n-2}(x, u) = \mathcal{J}_1 \cdot \mathcal{J}_{n-2}(x, u) \frac{\partial}{\partial u} + \sum_{j=1}^{n-2} \mathcal{J}_{j+1} \cdot \mathcal{J}_{n-j-2}(x, u) D_f X_j^{n-j-3}(x, u) \tag{3.44}
\]
for \(1 \leq i \leq n - 1\). Let us focus on (3.41). Using (3.44) in (3.41) we get
\[
\mathcal{J}_1 \cdot \mathcal{J}_{n-2}(x, u) \langle d\lambda, D_f \frac{\partial}{\partial u} (x, u) \rangle + \sum_{j=1}^{n-2} \mathcal{J}_{j+1} \cdot \mathcal{J}_{n-j-2}(x, u) \langle d\lambda, D_f D_f X_j^{n-j-3}(x, u) \rangle = 0 \tag{3.45}
\]
for \(1 \leq i \leq n - 1\). Note that the matrix \(\mathcal{J}_{n-2}(x, u)\) is nonsingular in a sufficiently small neighborhood of \((x^0, u^0)\). We may use a continuity argument to prove that the matrix \(\mathcal{J}_{n-2}(x, u)\) is also nonsingular in that neighborhood. Using (3.43), we can rewrite (3.45) as follows.
\[
\langle d\lambda, D_f \frac{\partial}{\partial u} (x, u) \rangle = 0 \tag{3.46}
\]
\[
\langle d\lambda, D_f D_f X_i^{n-3}(x, u) \rangle = 0 \tag{3.47}
\]
for \(1 \leq i \leq n - 2\). We repeat the above steps for \(k = n - 2, n - 3, \ldots, 1\).

Collecting the intermediate results, we conclude that the pair of conditions given by (3.40) and (3.41) imply the following conditions.
\[
\langle d\lambda, \frac{\partial}{\partial u} \rangle = 0
\]
\[
\langle d\lambda, D_f \frac{\partial}{\partial u} (x, u) \rangle = 0
\]
\[
\vdots
\]
\[
\langle d\lambda, D_f^{n-1} \frac{\partial}{\partial u} (x, u) \rangle = 0
\]
\[
\vdots
\]
Now consider (3.42). Using (3.40), we can rewrite (3.42) as

\[
\begin{pmatrix}
\langle d\lambda, D_f X_{n-1} \rangle & \cdots & \langle d\lambda, D_f X_{n-1} \rangle \\
\end{pmatrix}^T (x^0, u^0) \neq 0 \tag{3.49}
\]

Let \( k = n \) in the expansion formula (3.43). Using this in (3.49) we get the following nonsingularity condition.

\[
\begin{pmatrix}
\hat{J}^{n-1}_{1,1}(x, u) \langle d\lambda, D_f \frac{\partial}{\partial u} (x, u) \rangle + \sum_{j=1}^{n-1} \hat{J}^{j-1}_{j+1,1}(x, u) \langle d\lambda, D_f D_f X_{n-2}^{j-2} \rangle \\
\vdots \\
\hat{J}^{n-1}_{1,n}(x, u) \langle d\lambda, D_f \frac{\partial}{\partial u} (x, u) \rangle + \sum_{j=1}^{n-1} \hat{J}^{j-1}_{j+1,n}(x, u) \langle d\lambda, D_f D_f X_{n-2}^{j-2} \rangle 
\end{pmatrix} \neq 0
\]

at \((x^0, u^0)\). Since \( \hat{J}^{n-1}(x, u) \) is nonsingular, this implies that

\[
\begin{pmatrix}
\langle d\lambda, D_f \frac{\partial}{\partial u} \rangle & \langle d\lambda, D_f D_f X_{1}^{n-2} \rangle & \cdots & \langle d\lambda, D_f D_f X_{n-2}^{n-2} \rangle \\
\end{pmatrix}^T (x^0, u^0) \neq 0 \tag{3.50}
\]

But \( \langle d\lambda, D_f \frac{\partial}{\partial u} (x, u) \rangle = 0 \) by (3.48). Hence, we get

\[
\begin{pmatrix}
\langle d\lambda, D_f D_f X_{1}^{n-2} \rangle & \cdots & \langle d\lambda, D_f D_f X_{n-2}^{n-2} \rangle \\
\end{pmatrix}^T (x^0, u^0) \neq 0 \tag{3.51}
\]

We repeat this procedure for \( k = n - 1, n - 2, \ldots, 1 \). At the last step we obtain the following nonsingularity condition.

\[
\langle d\lambda, D_f^n \frac{\partial}{\partial u} \rangle (x^0, u^0) \neq 0 \tag{3.52}
\]

We have shown that the vectors \( D_f^k \frac{\partial}{\partial u} (x, u) \), for \( 0 \leq k \leq n \), may be computed using Definition 3.9, and that conditions (3.48) and (3.52) hold. Hence, we conclude that the nonlinear system \( \Sigma_{D, NLS} \) has relative degree \( n \). This completes the proof of necessity.

(Sufficiency). This follows from Lemma 3.16. \( \square \)
This leads to the following construction.

**Lemma 3.16** Suppose that $\Sigma_{D\times L}$ has relative degree $n$ with respect to the function $\lambda$. Then, the linearizing coordinate transformation and nonsingular feedback are given by the pair

$$z = T(x) \overset{\text{def}}{=} \begin{pmatrix} \tilde{x}_0 \\ \tilde{x}_1 \\ \vdots \\ \tilde{x}_{n-1} \end{pmatrix} (x)$$

$$v = \gamma(x, u) \overset{\text{def}}{=} \varphi_n(x, u)$$

**Proof.** We omit the proof, since it is identical to that of Corollary 2.22. □

### 3.5 An Illustrative Example

In this section we illustrate the techniques and results developed in Section 3.3 and Section 3.4 through an example. Using Definition 3.9, we construct the column vectors $D_f^i \frac{\partial}{\partial u}(x, u)$ for $0 \leq i \leq n$. The nonlinear system is tested for feedback linearizability using Theorem 3.10. The set of partial differential equations arising from the integrability conditions is written down. The linearizing transformation and feedback law are constructed from the solution to these partial differential equations. It is important to note that the algorithm for constructing the vector fields in Definition 2.5 breaks down for this system, even though this system is clearly feedback linearizable.

Consider the following example of a second-order nonlinear system.

$$x(k + 1) = \begin{pmatrix} x_2(k) \\ x_1(k) + (1 + x_2(k))u(k) \end{pmatrix}$$

$$= f(x(k), u(k))$$

(3.53)
To test for feedback linearizability, we must construct the vectors

\[ \frac{\partial}{\partial u}, D_f \frac{\partial}{\partial u}(x,u), D_f^2 \frac{\partial}{\partial u}(x,u) \]

and check if

\[ \dim \text{span} \left\{ \frac{\partial}{\partial u}, D_f \frac{\partial}{\partial u}(x,u), D_f^2 \frac{\partial}{\partial u}(x,u) \right\} = 3 \]

For clarity of expression, let us drop the time index \( k \) from our computations.

\[
\begin{pmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial u}
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & 0 \\
1 & u & 1 + x_2
\end{pmatrix}
\]

We note that the point \((x^0, u^0) = (0, 0, 0)^T\) is a fixed point of the system and that \(\text{rank} \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial u} \right) = 2\) in a neighborhood of \((x^0, u^0)\). For the system (3.53), the inverse image of the point \(x\) under the map \(f\) is given by

\[ f^{-1}(x) = \left\{ (x_2 - (1 + x_1)u, x_1, u)^T \right\} \text{ for } (x, u) \in M \times U \]

and

\[
\left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial u} \right)_{f^{-1}(x)} =
\begin{pmatrix}
0 & 1 & 0 \\
1 & u & 1 + x_1
\end{pmatrix}
\]

It can be checked that

\[ \ker \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial u} \right) = \text{span} \left\{ \alpha(x,u) \right\} \]

\[ = \text{span} \left\{ -(1 + x_2)\alpha_3(x,u) \frac{\partial}{\partial x_1} + \alpha_3(x,u) \frac{\partial}{\partial u} \right\} \]

where \(\alpha_3(x,u)\) is a smooth function defined on a neighborhood of \((x^0, u^0)\). Next, we check if

\[ \left[ \frac{\partial}{\partial u}, \alpha(x,u) \right] \in \text{span} \left\{ \alpha(x,u) \right\} + \left\{ \frac{\partial}{\partial u} \right\} \]

Using the expression for \(\alpha(x,u)\) we get

\[ \left[ \frac{\partial}{\partial u}, \alpha(x,u) \right] = -(1 + x_2) \frac{\partial \alpha_3}{\partial u} \frac{\partial}{\partial x_1} + \frac{\partial \alpha_3}{\partial u} \frac{\partial}{\partial u} \]
and it can be verified that \((\frac{\partial f}{\partial x}, \frac{\partial f}{\partial u})[\frac{\partial}{\partial u}, \alpha(x,u)] = 0\). Also,

\[
\text{span}\{\alpha(x,u)\} \cap \text{span}\{\frac{\partial}{\partial u}\} = \{0\}
\]

which has constant dimension. From Definition 3.9, we may then define the vector \((\frac{\partial f}{\partial x}, \frac{\partial f}{\partial u})(\frac{\partial}{\partial u})\). In coordinates,

\[
(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial u})_{f^{-1}(x)} \left(\frac{\partial}{\partial u}\right)_{f^{-1}(x)} = (1 + x_1) \frac{\partial}{\partial x_2}
\]

and the vector \(D_{\frac{\partial}{\partial u}}(x,u) = (1 + x_1) \frac{\partial}{\partial x_2}\). In order to compute \(D_{\frac{\partial}{\partial u}}^2(x,u)\), we must check that

\[
[\frac{\partial}{\partial u}, \alpha(x,u)] \in \text{span}\{\alpha(x,u)\} + \{\frac{\partial}{\partial u}, D_{\frac{\partial}{\partial u}}(x,u)\}
\]

Using the expressions for \(D_{\frac{\partial}{\partial u}}(x,u)\) and \(\alpha(x,u)\) we get,

\[
[D_{\frac{\partial}{\partial u}}(x,u), \alpha(x,u)] = ((1 + x_2) \frac{\partial \alpha_3}{\partial u} + \alpha_3(x,u)) \frac{\partial}{\partial x_1} - (1 + x_2) \alpha_3(x,u) \frac{\partial}{\partial x_2} - \frac{\partial \alpha_3}{\partial x_2} \frac{\partial}{\partial u} + \left(\begin{array}{c}
0 \\
\alpha_3 \\
0
\end{array}\right) + \left(\begin{array}{c}
0 \\
-(1 + x_2) \alpha_3 \\
0
\end{array}\right)
\]

Note that \(\alpha_3 \frac{\partial}{\partial x_1} \in \text{span}\{\alpha, \frac{\partial}{\partial u}\}\). Hence

\[
[D_{\frac{\partial}{\partial u}}(x,u), \alpha(x,u)] \in \text{span}\{\alpha(x,u)\} + \text{span}\{\frac{\partial}{\partial u}, D_{\frac{\partial}{\partial u}}(x,u)\}
\]

Also

\[
\text{span}\{\alpha(x,u)\} \cap \text{span}\{\frac{\partial}{\partial u}, D_{\frac{\partial}{\partial u}}(x,u)\} = \{0\}
\]

Hence we may compute, in local coordinates,

\[
(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial u})D_{\frac{\partial}{\partial u}}(x,u) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial u})_{f^{-1}(x)} \left(D_{\frac{\partial}{\partial u}}\right)_{f^{-1}(x)}
\]

\[
= (1 + x_2 - (1 + x_1)u) \frac{\partial}{\partial x_1} + u(1 + x_2 - (1 + x_1)u) \frac{\partial}{\partial x_2}
\]
and the vector field

\[ D_f^2 \frac{\partial}{\partial u} (x, u) = \begin{pmatrix} (1 + x_2 - (1 + x_1)u) \\ u(1 + x_2 - (1 + x_1)u) \\ 0 \end{pmatrix} \]

It is seen that

\[ \dim \text{span} \left\{ \frac{\partial}{\partial u}, D_f \frac{\partial}{\partial u} (x, u), D_f^2 \frac{\partial}{\partial u} (x, u) \right\} = 3 \]

in a neighborhood of \((x^0, u^0)\). To determine the coordinate transformation and feedback law we use Lemma 3.16. We know that there must exist a function \(\lambda : M \times U \to \mathbb{R}\) such that

\[ \langle d\lambda, \frac{\partial}{\partial u} \rangle = 0, \quad \langle d\lambda, D_f \frac{\partial}{\partial u} (x, u) \rangle = 0 \]

and \(\langle d\lambda, D_f^2 \frac{\partial}{\partial u} (x, u) \rangle (x^0, u^0) \neq 0\). In local coordinates this means that we must solve the following set of partial differential equations.

\[ \frac{\partial \lambda}{\partial u} = 0, \quad (1 + x_1) \frac{\partial \lambda}{\partial x_2} = 0 \]

It is easy to see that \(\lambda = x_1\) satisfies these conditions. Further, it must be true that

\[ \left( (1 + x_2 - (1 + x_1)u) \frac{\partial \lambda}{\partial x_1} + u(1 + x_2 - (1 + x_1)u) \frac{\partial \lambda}{\partial x_2} \right)_{(x^0, u^0)} \neq 0 \]

which means we can define locally a nonsingular feedback law.

The coordinate transformation is constructed from the function \(\lambda\) as follows.

\[ z_1 \overset{\text{def}}{=} \tilde{\varphi}_0 (x) = \varphi_0 (x, u) \overset{\text{def}}{=} \lambda (x, u) = x_1 \]

\[ z_2 \overset{\text{def}}{=} \tilde{\varphi}_1 (x) = \varphi_1 (x, u) \overset{\text{def}}{=} \varphi_0 \circ f (x, u) = x_2 \]
The feedback is given by $v = \tilde{\gamma}_1 \circ f(x, u) = x_1 + (1 + x_2)u$. This feedback linearizes the nonlinear system to

$$z(k + 1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v(k)$$

3.6 Summary

In this chapter we introduced column vectors $D_j^i \partial \phi(x, u)$, for $0 \leq i \leq n$, and row-vectors $P_j^i d\lambda(x, u)$, for $0 \leq j \leq n$. Necessary and sufficient conditions for feedback linearizability were expressed in terms of these column vectors. We showed that our conditions are easier to verify than similar conditions in the literature. Through the notion of relative degree for the nonlinear system, we derived a construction for the linearizing transformation and feedback law. The theory developed in this chapter was illustrated using an example of a second-order nonlinear system.
4.1 Introduction

In this chapter we use the results developed in Chapter 3 to design digital nonlinear controllers for discrete-time nonlinear models of electrically stimulated muscle. We consider two types of muscle models: the uncoupled model where the activation dynamics are independent of the torque-angle and torque-velocity terms, and the coupled model where the activation dynamics are coupled to the torque-velocity factor. These models were introduced in Chapter 1. We apply the technique of feedback linearization to render each of these models input-state equivalent to a linear reachable system. Linear digital controllers can be then designed for the compensated system.

4.2 Feedback Linearization of Muscle Models

Two types of models are investigated here: i) Discrete-time model with second-order load dynamics and first-order uncoupled activation dynamics. and ii) Discrete-time model with second-order load dynamics and first-order coupled activation dynamics. State space models are constructed from these
equations. The coordinate transformation and feedback law needed to linearize these models are calculated using the results given in Chapter 3.

### 4.2.1 Uncoupled Model

The dynamics of the uncoupled model of electrically stimulated muscle are given by the following set of difference equations (see [33], [34]).

\[
\tau(k) = A(k)T_\theta(k)T_\omega(k)
\]

\[
A(k + 1) = aA(k) + bu(k)
\]

\[
T_\omega(k) = 1 - c\omega(k)
\]

\[
T_\theta(k) = 1 + d\theta(k)
\]

\[
\omega(k) = \theta(k) - \theta(k - 1)
\]

where

\[
\begin{align*}
A(k) & = \text{activation} \\
\tau(k) & = \text{muscle output torque} \\
T_\theta(k) & = \text{torque-angle factor} \\
T_\omega(k) & = \text{torque-velocity factor} \\
\omega(k) & = \text{angular velocity of the limb acted upon} \\
\theta(k) & = \text{angular position of the limb acted upon}
\end{align*}
\]

The variables \(\theta(k)\) and \(\omega(k)\) are from the discretized model of the load given by

\[
\theta(k + 1) = l_1\theta(k) + l_2\theta(k - 1) + l_3\tau(k)
\]

The parameter values of the model are given in the next section. We choose the following state variables.

\[
x_1(k) \overset{\text{def}}{=} A(k)
\]

\[
x_2(k) \overset{\text{def}}{=} \theta(k - 1)
\]

\[
x_3(k) \overset{\text{def}}{=} \theta(k)
\]
The resulting state-space model is given by

\[ x(k + 1) = f(x(k), u(k)) \]  \hspace{1cm} \text{(4.3)}

where, dropping the time index \( k \) from the notation,

\[
f(x, u) = \begin{pmatrix} ax_1 + bu \\ x_3 \\ l_1x_3 + l_2x_2 + x_1(l_3 + dl_3x_3 - cl_3x_3 - cdl_3x_3^2 + cl_3x_2 + cdl_3x_2x_3) \end{pmatrix}
\]

Note that the point \((x^0, u^0) = (0, 0, 0, 0)^T\) is an equilibrium point for (4.3). Calculations given in Appendix-B show that the system (4.3) is feedback linearizable in a neighborhood of the point \((x^0, u^0)\). Further, the linearizing transformation \(T\) and its inverse \(T^{-1}\) are given by

\[
T(x) = \begin{pmatrix} x_2 \\ x_3 \\ l_1x_3 + l_2x_2 + x_1(l_3 + dl_3x_3 - cl_3x_3 - cdl_3x_3^2 + cl_3x_2 + cdl_3x_2x_3) \end{pmatrix}
\]

\[
T^{-1}(z) = \begin{pmatrix} z_3 - l_1z_2 - l_2z_1 \\ z_1 \\ z_2 \end{pmatrix}
\]

and the inverse feedback law is given by

\[
\gamma^{-1}(x, v) = \frac{v - l_1\Delta_1 - l_2x_3 - ax_1(l_3 + dl_3\Delta_1 - cl_3\Delta_1 - cdl_3x_3^2 + cl_3x_3 + cdl_3x_3\Delta_1)}{b(l_3 + dl_3\Delta_1 - cl_3\Delta_1 - cdl_3x_3^2 + cl_3x_3 + cdl_3x_3\Delta_1)}
\]

where

\[
\Delta_1 \overset{\text{def}}{=} l_1x_3 + l_2x_2 + x_1(l_3 + dl_3x_3 - cl_3x_3 - cdl_3x_3^2 + cl_3x_2 + cdl_3x_2x_3)
\]

The resulting linear system is given by

\[
z(k + 1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} z(k) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v(k)
\]  \hspace{1cm} \text{(\Sigma_v)}
4.2.2 Coupled Model

The dynamics of the uncoupled model of electrically stimulated muscle are given by the following set of difference equations (see [33], [34]).

\[
\tau(k) = A(k) T_\theta(k)
\]
\[
A(k+1) = aT_\omega(k)A(k) + bu(k)
\]
\[
T_\omega(k) = 1 - c\omega(k)
\]
\[
T_\theta(k) = 1 + d\theta(k)
\]
\[
\omega(k) = \theta(k) - \theta(k - 1)
\]

(4.4)

where the variables have the same physical interpretation as for the uncoupled model. As before, the variables \(\theta(k)\) and \(\omega(k)\) are from the discretized model of the load given by

\[
\theta(k+1) = l_1\theta(k) + l_2\theta(k - 1) + l_3\tau(k)
\]

(4.5)

The parameter values are given in the next section. We choose the following state variables.

\[
x_1(k) \overset{\text{def}}{=} A(k)
\]
\[
x_2(k) \overset{\text{def}}{=} \theta(k - 1)
\]
\[
x_3(k) \overset{\text{def}}{=} \theta(k)
\]

The resulting state-space model of the system (4.4) is given by

\[
x(k+1) = f(x(k), u(k))
\]

(4.6)

where, dropping the time index \(k\) from the notation,

\[
f(x, u) = \begin{pmatrix}
ax_1 - acx_3x_1 + acx_2x_1 + bu \\
x_3 \\
l_1x_3 + l_2x_2 + l_3x_1 + dl_3x_1x_3
\end{pmatrix}
\]
Note that the point \((x^0, u^0) = (0, 0, 0, 0)^T\) is an equilibrium point for (4.6). Calculations given in Appendix-B show that the nonlinear system (4.6) is feedback linearizable in a neighborhood of \((x^0, u^0)\). The coordinate transformation \(T\) and its inverse \(T^{-1}\) are given by

\[
T(x) = \begin{pmatrix} x_2 \\ x_3 \\ l_1 x_3 + l_2 x_2 + l_3 x_1 + dl_3 x_1 x_3 \end{pmatrix}
\]

\[
T^{-1}(z) = \begin{pmatrix} z_2 - l_1 z_3 - l_2 z_1 \\ l_3 + dl_3 z_2 \\ z_1 \\ z_2 \end{pmatrix}
\]

and the inverse feedback law is given by

\[
\gamma^{-1}(x, u) = \frac{1}{\Delta} \left( v - l_3 (ax_1 - acx_3 x_1 + acx_2 x_1) - l_2 x_3 - (l_1 x_3 + l_2 x_2 + l_3 x_1 +dl_3 x_1 x_3) \right)
\]

where \(\Delta \overset{\text{def}}{=} bl_3 + bdl_3 (l_1 x_3 + l_2 x_2 + l_3 x_1 + dl_3 x_1 x_3)\).

The resulting linear system is given by

\[
z(k + 1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} z(k) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v(k) \tag{\Sigma_c}
\]

### 4.3 Summary

In this chapter we applied the techniques developed in Chapter 3 to discrete-time nonlinear models of electrically stimulated muscle. Using Theorem 3.15, we verified that these models are indeed feedback linearizable. The required coordinate transformation and feedback law were then computed by applying Lemma 3.16. We have thus shown that our theory of feedback linearization for discrete-time systems can be successfully applied to systems of practical interest.
5.1 Summary of Work Accomplished

In this dissertation we developed a geometric framework for the problem of exact linearization of discrete-time nonlinear systems. The essential features of this framework are $f$-related vector fields and $f$-related one-forms. This framework was used to study the problem of exact linearization using state coordinate change, and the problem of feedback linearization, for both single-input and multi-input systems. Necessary and sufficient conditions for the first problem were obtained. The required transformation was obtained by solving a set of $n^2 + nm$ linear, first-order partial differential equations. Sufficient conditions were obtained for feedback linearization of single-input and multi-input systems. The required transformation and feedback are obtained in this case by solving a set of $n$ linear, first-order partial differential equations.

We studied the feedback linearization problem for single-input systems through the use of column vectors $D_i^j \frac{\partial}{\partial u}(x, u)$, for $0 \leq i \leq n$, and row-vectors $P_j^i d\lambda(x, u)$, for $0 \leq j \leq n$. Necessary and sufficient conditions for feedback linearizability were expressed in terms of the column vectors. We showed that our conditions are easier to verify than similar conditions in the literature.
We derived a construction for the linearizing transformation and feedback law through the notion of relative degree for the nonlinear system. The theory developed here was illustrated using an example of a second-order nonlinear system.

We applied the techniques developed in Chapter 3 to discrete-time nonlinear models of electrically stimulated muscle. It was verified that these models are indeed feedback linearizable. The required coordinate transformation and feedback law were then computed by applying Lemma 3.16. We have thus shown that our theory of feedback linearization for discrete-time systems can be successfully applied to systems of practical interest.

5.2 Future Research Topics

- Necessary and sufficient conditions for feedback linearizability of multi-input discrete-time systems must be derived by extending the results of Chapter 3.

- We believe that a geometric approach using $f$-related vector fields and $f$-related one-forms can be used to study controlled invariance, disturbance decoupling and input-output decoupling of discrete-time systems. It is then possible to derive procedures to compute, in local coordinates, the distributions that are used to characterize these problems.

- It will be useful to investigate if the framework developed here for exact linearization can be used to solve the problem of sampled feedback linearizability of continuous-time systems. For this, one must be able to relate the
vector fields of the continuous-time system to the $f$-related vector fields of the sampled system.


Theorem 2.28 Sufficient Conditions for Feedback Linearization of Multi-Input Systems. For the nonlinear system

\[ x(k + 1) = f(x(k), u(k)) \quad (\Sigma_{DNLs}) \]

there exist \( m \) real-valued functions \( \lambda_i : M \times U \to \mathbb{R} \), for \( 0 \leq i \leq m \), defined on a neighborhood of the fixed point \((x^0, u^0)\) such that (2.1) with output functions \( y_i \overset{\text{def}}{=} \lambda_i \) has vector relative degree \( \{ r_1, \ldots, r_m \} \) at \((x^0, u^0)\) with

\[ r_1 + r_2 + \cdots + r_m = n \]

if the distributions \( G_0, \ldots, G_n \) are involutive and constant dimensional with \( \dim G_1 = 2m \) and \( \dim G_n = n + m \).

Proof. From our assumptions.

i) For each \( 0 \leq k \leq n \) the distribution \( G_k \) is constant dimensional in a neighborhood of \((x^0, u^0)\).

ii) The distribution \( G_n \) has dimension \( n + m \).

iii) For each \( 0 \leq k \leq n - 1 \) the distribution \( G_k \) is involutive.

From (i) and (ii), we know that there must exist an integer \( N \leq n \) such that

\[ \dim G_{N-1} < n + m \]

\[ \dim G_N = n + m \]
Set

\[ m_1 = n + m - \dim (G_{N-1}) \]

By assumption \( G_{N-1} \) is involutive. By Frobenius' Theorem there exist \( m_1 \) functions \( \lambda_i : M \times U \to \mathbb{R} \) for \( 1 \leq i \leq m_1 \) such that

\[ \text{span} \{ d\lambda_1, \ldots, d\lambda_{m_1} \} = \text{ann} (G_{N-1}) \]

By construction these functions are such that

\[ \langle d\lambda_i, D_j^k \frac{\partial}{\partial u_j} \rangle = 0 \quad (\text{A.1}) \]

in a neighborhood of \((x^0, u^0)\) for \( 0 \leq k \leq N - 1 \), \( 1 \leq j \leq m \) and \( 1 \leq i \leq m_1 \).

Moreover, the \( m_1 \times m \) matrix

\[ A_1(x, u) = \{a^1_{ij}(x, u)\} \overset{\text{def}}{=} \{ \langle d\lambda_i, D_j^N \frac{\partial}{\partial u_j} \rangle(x, u) \} \]

has rank \( m_1 \) at \((x^0, u^0)\).

For suppose that this were not true. Then using (A.1), this implies that for some \( \{c_i : 1 \leq i \leq m_1\} \), not all \( c_i = 0 \).

\[ \sum_{i=1}^{m_1} c_i \langle d\lambda_i, D_j^k \frac{\partial}{\partial u_j} \rangle(x^0, u^0) = 0 \]

for all \( 0 \leq k \leq N \), and \( 1 \leq j \leq m \). This implies that

\[ \sum_{i=1}^{m_1} c_i d\lambda_i(x^0, u^0) \in \text{ann} (G_N(x^0, u^0)) \]

But \( \dim G_N(x^0, u^0) = n + m \). hence \( c_i = 0 \), for \( 1 \leq i \leq m_1 \) since \( d\lambda_i \) are independent by construction. We note that \( m_1 \leq m \) always, since \( A_1(x^0, u^C) \) has \( m \) columns and \( m_1 \) rows.

\[ \text{rank} \ A_1(x^0, u^0) = m \]
with
\[ r_1 = r_2 = \cdots = r_m = N \]

Thus, the system with outputs \( \lambda_i(x,u) \), for \( 1 \leq i \leq m \) has vector relative degree \( \{N,N,\ldots,N\} \). Moreover, \( r_1 + r_2 + \cdots + r_m = n \) because
\[ n + m = \dim G_N \leq mN + m = \sum r_i + m \]

and \( \sum r_i \leq n \) by Remark 2.27.

Suppose that \( m_1 < m \). We need additional functions to complete the required coordinate transformation. We claim that

a) the codistribution
\[ \Omega_1 = \text{span} \{ d\lambda_1, \ldots, d\lambda_{m_1}, P_j d\lambda_1, \ldots, P_j d\lambda_{m_1} \} \]
has dimension \( 2m_1 \) around \((x^0, u^0)\)

b) \( \Omega_1 \subset \text{ann} G_{N-2} \)

The one-forms \( d\lambda_i, 1 \leq i \leq m_1 \) which are in \( \text{ann} G_{N-1} \) are by construction also in \( \text{ann} G_{N-2} \), since \( G_{N-2} \subset G_{N-1} \). The one-forms \( \{ P_j d\lambda_1, \ldots, P_j d\lambda_{m_1} \} \) by Lemma 2.9 and (A.1) are such that
\[ \langle P_j d\lambda_1, D_j^k \frac{\partial}{\partial u_j} \rangle = 0 \]
for \( 0 \leq k \leq N-2, 1 \leq j \leq m, 1 \leq i \leq m_1 \) in a neighborhood of \((x^0, u^0)\). This implies that
\[ P_j d\lambda_i \in \text{ann} G_{N-2} \]
for \( 1 \leq i \leq m_1 \). This proves (b). To prove (a), assume that it is false. Then there exist real numbers \( c_i \), for \( 1 \leq i \leq m_1 \) and \( d_i \), for \( 1 \leq i \leq m_1 \) with not all \( c_i, d_i \) equal to zero simultaneously, such that
\[
\sum_{i=1}^{m_1} c_i d\lambda_i + \sum_{i=1}^{m_1} d_i P_j d\lambda_i = 0
\]
This implies that
\[
\sum_{i=1}^{m_1} c_i (d\lambda_i, D_f^{N-1} \frac{\partial}{\partial u_j}) + \sum_{i=1}^{m_1} d_i (P_j d\lambda_i, D_f^{N-1} \frac{\partial}{\partial u_j}) = 0
\]
for \( 1 \leq j \leq m_1 \). But \( d\lambda_i \in \text{ann } G_{N-1} \) for \( 1 \leq i \leq m_1 \), and using Lemma 2.9 the above equation reduces to
\[
\sum_{i=1}^{m_1} d_i (d\lambda_i, D_f^{N} \frac{\partial}{\partial u_j}) = 0
\]
But the rows of \( A^1(x^0, u^0) \) are independent and \( d\lambda_i \) for \( 1 \leq i \leq m_1 \) are independent by construction. This means that all \( c_i \) and \( d_i \) are zero, a contradiction.

From (a) and (b) we conclude that \( \dim (\text{ann } G_{N-2}) \geq 2m_1 \). Suppose that the inequality is strict. Set
\[
m_2 = \dim (\text{ann } G_{N-2}) - 2m_1
\]
and
\[
\text{ann } G_{N-2} = \Omega_1 + \text{span } \{d\lambda_i : m_1 + 1 \leq i \leq m_1 + m_2\}
\]
The new one forms are such that
\[
(d\lambda_i, D_f^{k} \frac{\partial}{\partial u_j}) = 0
\]
for \( 0 \leq k \leq N - 2 \), \( 1 \leq j \leq m \) and \( m_1 + 1 \leq i \leq m_1 + m_2 \). Also the \( m_1 + m_2 \times m \) matrix
\[
A^2(x^0, u^0) = \{a^2_{ij}(x^0, u^0)\}
\]
where
\[ a_{ij}^2(x^0, u^0) = \langle d\lambda_i, D_f^N \frac{\partial}{\partial u_j} \rangle \quad 1 \leq i \leq m_1 \]
\[ a_{ij}^2(x^0, u^0) = \langle d\lambda_i, D_f^{N-1} \frac{\partial}{\partial u_j} \rangle \quad m_1 + 1 \leq i \leq m_1 + m_2 \]
has rank \( m_1 + m_2 \).

To prove this suppose that on the contrary, there exist \( c_i \), for \( 1 \leq i \leq m_1 \) and \( d_i \) for \( m_1 + 1 \leq i \leq m_1 + m_2 \), not all zero, such that
\[ \sum_{i=1}^{m_1} c_i \langle d\lambda_i, D_f^N \frac{\partial}{\partial u_j} \rangle + \sum_{i=m_1+1}^{m_1+m_2} d_i \langle d\lambda_i, D_f^{N-1} \frac{\partial}{\partial u_j} \rangle = 0 \]
for \( 1 \leq j \leq m \). Using Lemma 2.9 we can write this as
\[ \langle \left( \sum_{i=1}^{m_1} c_i P_i d\lambda_i + \sum_{i=m_1+1}^{m_1+m_2} d_i d\lambda_i \right), D_f^{N-1} \frac{\partial}{\partial u_j} \rangle = 0 \]

Now
\[ D_f^{N-1} \frac{\partial}{\partial u_j} \in G_{N-1} \]
and
\[ \text{ann } G_{N-1} = \text{span } \{ d\lambda_i, 1 \leq i \leq m_1 \} \]
which implies that
\[ \sum_{i=1}^{m_1} c_i P_i d\lambda_i + \sum_{i=m_1+1}^{m_1+m_2} d_i d\lambda_i \in \text{span } \{ d\lambda_i, 1 \leq i \leq m_1 \} \quad (A.2) \]

But by construction the left hand side of the above relation belongs to \( \text{ann } G_{N-2} \) and
\[ \text{ann } G_{N-2} = \Omega_1 + \text{span } \{ d\lambda_i, m_1 + 1 \leq i \leq m_1 + m_2 \} \]
which contradicts \((A.2)\). This proves that \( A^2(x^0, u^0) \) has rank \( m_1 + m_2 \). It is easy to see that \( m_1 + m_2 \leq m \).
Suppose that $m_1 + m_2 = m$. Then the nonlinear system $\Sigma_{DNLs}$ with output functions defined as $y_i = \lambda_i$, for $1 \leq i \leq m$ has vector relative degree \{r_1, \ldots, r_m\} with

$$r_1 = r_2 = \cdots = r_{m_1} = N$$

$$r_{m_1+1} = r_{m_1+2} = \cdots = r_{m_1+m_2} = N - 1$$

Furthermore,

$$n + m = \dim G_{N-1} + m_1 \leq (m_1 + m_2)N + m_1$$

$$= m_1N + m_2(N - 1) + m$$

$$= \sum r_i + m$$

$$\leq n + m$$

by Remark 2.27 and thus $\sum r_i = n$. If $m_1 + m_2 < m$ we go through the procedure once again.

After $N - 1$ iterations we have the following set of one-forms.

$$\begin{align*}
\left\{ \begin{array}{ll}
d\lambda_i, P_j d\lambda_i, \ldots, P_j^{N-1} d\lambda_i & \text{for } 1 \leq i \leq m_1; \\
d\lambda_i, P_j d\lambda_i, \ldots, P_j^{N-2} d\lambda_i & \text{for } m_1 + 1 \leq i \leq m_1 + m_2; \\
\vdots & \\
d\lambda_i & \text{for } m_1 + \cdots + m_{N-2} + 1 \leq i \leq m_1 + \cdots + m_{N-1}
\end{array} \right.
\end{align*}$$

as the basis of ann $G_1$. Since by assumption $\dim G_1 = 2m$, we have $\dim (\text{ann } G_1) = n - m$. Hence

$$n - m = m_1(N) + m_2(N - 1) + \cdots + m_{N-1} \quad (A.3)$$

As before, we can show that the one-forms

$$\begin{align*}
\left\{ \begin{array}{ll}
d\lambda_i, P_j d\lambda_i, \ldots, P_j^{N} d\lambda_i & \text{for } 1 \leq i \leq m_1; \\
d\lambda_i, P_j d\lambda_i, \ldots, P_j^{N-1} d\lambda_i & \text{for } m_1 + 1 \leq i \leq m_1 + m_2; \\
\vdots & \\
d\lambda_i, P_j d\lambda_i & \text{for } m_1 + \cdots + m_{N-2} + 1 \leq i \leq m_1 + \cdots + m_{N-1};
\end{array} \right.
\end{align*} \quad (A.4)$$
are independent and contained in \( \text{ann} \ G_0 \). Since there are \( m_1(N+1) + m_2(N) + \cdots + 2m_{N-1} \) one-forms in this set, and \( \dim(\text{ann} \ G_0) = n \), this implies that

\[
m_1(N) + m_2(N-1) + \cdots + m_{N-1} + m_1 + m_2 + \cdots + m_{N-1} \leq n
\]

Using (A.3) this means that \( m_1 + m_2 + \cdots + m_{N-1} \leq m \). If the inequality is strict we choose \( m_N \) such that

\[
m_N = m - (m_1 + m_2 + \cdots + m_{N-1})
\]

Now we choose \( m_N \) independent functions \( \{\lambda_1, \ldots, \lambda_{m_N}\} \) such that the set of one-forms

\[
\{d\lambda_i; \ m_1 + m_2 + \cdots + m_{N-1} + 1 \leq i \leq m_1 + m_2 + \cdots + m_N\}
\]

when appended to the set (A.4) will annihilate \( G_0 \). The nonlinear system \( (\Sigma_{DNLSS}) \) with output functions \( y_i = \lambda_i \) for \( 1 \leq i \leq m \) has vector relative degree \( \{r_1, \ldots, r_m\} \) with

\[
r_i = \begin{cases} N & \text{for } 1 \leq i \leq m_1; \\ N - 1 & \text{for } m_1 + 1 \leq i \leq m_1 + m_2; \\ \cdots & \cdots \\ 1 & \text{for } m_1 + \cdots + m_{N-1} + 1 \leq i \leq m_1 + \cdots + m_N; \end{cases} \quad (A.5)
\]

Moreover,

\[
n + m = \dim (\text{ann} \ G_0) \leq m_1(N+1) + m_2N + m_N
\]

\[
= m_1N + m_2(N-1) + m
\]

\[
= \sum r_i + m
\]

\[
\leq n + m
\]

by Remark 2.27 and thus \( \sum r_i = n \). \qed
B.1 Feedback Linearization of Uncoupled Model

The dynamics of the uncoupled model are described by

\[ r(k + 1) = f(x(k), u(k)) \]  \hspace{1cm} (B.1)

where, dropping the time index \( k \) from the notation.

\[ f(x, u) = \begin{pmatrix} ax_1 + bu \\ x_3 \\ l_1x_3 + l_2x_2 + x_1(l_3 + dl_3x_3 - cl_3x_3 - cdll_3x_3^2 + cl_3x_2 + cdll_3x_2x_3) \end{pmatrix} \]

Note that the point \((x^0, u^0) = (0, 0, 0, 0)^T\) is an equilibrium point for \((B.1)\).

We test if \((B.1)\) is feedback linearizable in a neighborhood of \((x^0, u^0)\). For this system, the Jacobian \( \frac{\partial f}{\partial x} \) is given by

\[ \frac{\partial f}{\partial x_1} = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \]

\[ \frac{\partial f}{\partial u} = \begin{pmatrix} \frac{\partial f}{\partial u} \end{pmatrix} = \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix} \]

\hspace{1cm} (B.2)
Note that rank \( \left( \frac{\partial f}{\partial x} \frac{\partial f}{\partial u} \right) \) = 3 in a neighborhood of \((x^0, u^0)\). The inverse image of the point \(x\) under the map \(f\) is given by

\[
f^{-1}(x) = \begin{pmatrix}
g_1(x, u) \\
g_2(x, u) \\
g_3(x, u)
\end{pmatrix}
\]

(B.3)

where

\[
g_1(x, u) = \frac{x_1 - bu}{a}
\]

\[
g_2(x, u) = \frac{x_3 - l_1 x_2 - g_1(x, u)(l_3 - dl_3 x_2 + cl_3 x_2^2 + cdl_3 x_2^2)}{\Delta}
\]

\[
g_3(x, u) = x_2
\]

and where \(\Delta = l_2 + cl_3 g_1(x, u) + cdl_3 x_2 g_1(x, u)\). Using (B.3) we may write (B.2) as

\[
\left( \frac{\partial f}{\partial x} \frac{\partial f}{\partial u} \right)_{f^{-1}(x)} = \begin{pmatrix}
a & 0 & 0 & b \\
0 & 0 & 1 & 0 \\
\xi_1(x, u) & \xi_2(x, u) & \xi_3(x, u) & 0
\end{pmatrix}
\]

(B.4)

where

\[
\xi_1(x, u) = l_3 + dl_3 x_2 - cl_3 x_2 - cdl_3 x_2^2 + cl_3 g_2(x, u) + cdl_3 x_2 g_1(x, u)
\]

\[
\xi_2(x, u) = l_2 + cl_3 g_1(x, u) + cdl_3 x_2 g_1(x, u)
\]

\[
\xi_3(x, u) = l_1 + dl_3 g_1(x, u) - cl_3 g_1(x, u) + cdl_3 g_1(x, u) g_2(x, u) - 2 cdl_3 x_2 g_1(x, u)
\]

Also,

\[
\text{ker} \left( \frac{\partial f}{\partial x} \frac{\partial f}{\partial u} \right) = \text{span} \{\alpha(x, u)\} = \text{span} \left\{ \begin{pmatrix}
\alpha_1(x, u) \\
\alpha_2(x, u) \\
\alpha_3(x, u) \\
\alpha_4(x, u)
\end{pmatrix} \right\}
\]

where \(\alpha_i(x, u)\) are such that

\[
\alpha(x, u) = \begin{pmatrix}
-x_{2u}(x, u) \\
\frac{a}{\varepsilon(x) \alpha_4(x, u)} \\
\alpha_4(x, u)
\end{pmatrix}
\]
and
\[
e(x) = \frac{l_3 + dl_3x_3 - cl_3x_3 - cdl_3x_3^2 + cl_3x_2 + cdl_3x_2x_3}{l_2 + cl_3x_1 + cdl_3x_1x_3}
\]

To test for feedback linearizability we check if the column vectors
\[
\left\{ \frac{\partial}{\partial u}, D_f \frac{\partial}{\partial u}(x, u), D_f^2 \frac{\partial}{\partial u}(x, u), D_f^3 \frac{\partial}{\partial u}(x, u) \right\}
\]
are computable using Definition 3.9. and if
\[
\text{dim span } \left\{ \frac{\partial}{\partial u}, D_f \frac{\partial}{\partial u}(x, u), D_f^2 \frac{\partial}{\partial u}(x, u), D_f^3 \frac{\partial}{\partial u}(x, u) \right\} = 4 \quad (B.5)
\]
in a neighborhood of \((x^0, u^0)\). It can be shown that the procedure in Definition 3.9 yields the column vectors shown below.

\[
\frac{\partial}{\partial u} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad D_f^2 \frac{\partial}{\partial u}(x, u) = \begin{pmatrix} ab \\ 0 \\ b\xi_1(x, u) \\ 0 \end{pmatrix}
\]
\[
D_f \frac{\partial}{\partial u}(x, u) = \begin{pmatrix} b \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad D_f^3 \frac{\partial}{\partial u}(x, u) = \begin{pmatrix} ba^2 \\ b\delta_2(x, u) \\ ba\xi_1(x, u) + b\delta_2(x, u)\xi_3(x, u) \\ 0 \end{pmatrix} \quad (B.6)
\]

where
\[
\delta_2(x, u) \overset{\text{def}}{=} b\xi_1(x, u)_{f^{-1}(x)}
\]
\[
= l_3 + dl_3g_2(x, u) - cl_3g_2(x, u) - cdl_3g_2^2(x, u) + cl_3g_2(x, u)_{f^{-1}(x)} + cdl_3g_2(x, u)(g_2(x, u)_{f^{-1}(x)})
\]

For these vectors, it can be checked that the condition in \((B.5)\) is true. Therefore, by Theorem 3.10 the uncoupled muscle model \((B.1)\) is feedback linearizable.
To construct the linearizing transformation and feedback law, consider the following set of partial differential equations:

\[
\begin{align*}
\langle d\lambda, \frac{\partial}{\partial u} \rangle &= 0 \\
\langle d\lambda, D_f \frac{\partial}{\partial u} (x, u) \rangle &= 0 \\
\langle d\lambda, D_f^2 \frac{\partial}{\partial u} (x, u) \rangle &= 0
\end{align*}
\]

Substituting from (B.6) for these column vectors, we get

\[
\begin{align*}
\frac{\partial \lambda}{\partial u} &= 0 \\
\frac{\partial \lambda}{\partial x_1} &= 0 \\
\frac{\partial \lambda}{\partial x_1} + \xi(x, u) \frac{\partial \lambda}{\partial x_3} &= 0
\end{align*}
\]

(B.7)

It is easy to verify that \( \lambda = x_2 \) is a solution. Note also that

\[
\langle d\lambda, D_f^3 \frac{\partial}{\partial u} (x, u) \rangle = \frac{\partial \lambda}{\partial x_2} b \delta_2(x, u) = b \delta_2(x, u)
\]

and \( \delta_2(x^0, u^0) \neq 0 \).

The coordinate transformation and feedback law are computed as follows.

\[
\begin{align*}
z_1 &\overset{\text{def}}{=} \tilde{\varphi}_0(x) = \varphi_0(x, u) \overset{\text{def}}{=} \lambda(x, u) = x_2 \\
z_2 &\overset{\text{def}}{=} \tilde{\varphi}_1(x) = \varphi_1(x, u) \overset{\text{def}}{=} \tilde{\varphi}_0 \circ f(x, u) = x_3 \\
z_3 &\overset{\text{def}}{=} \tilde{\varphi}_2(x) = \varphi_2(x, u) \overset{\text{def}}{=} \tilde{\varphi}_0 \circ f(x, u) = \eta(x)
\end{align*}
\]

where

\[
\eta(x) \overset{\text{def}}{=} l_1 x_3 + l_2 x_2 + x_1 \left( l_3 + dl_3 x_3 - cl_3 x_3 - cdl_3 x_3^2 + cl_3 x_2 + cdl_3 x_2 x_3 \right)
\]

The feedback law is defined by

\[
v = \gamma(x, u) \overset{\text{def}}{=} \tilde{\varphi}_2 \circ f(x, u) \\
= l_1 \Delta_1 + l_2 x_3 + ax_1 \left( l_3 + \Delta_1 (dl_3 - cl_3 + cdl_3 x_3) - cdl_3 \Delta_1^2 \\
+ cl_3 x_3 \right) + ub \left( l_3 + dl_3 \Delta_1 - cl_3 \Delta_1 - cdl_3 \Delta_1^2 + cl_3 x_3 + cdl_3 x_3 \Delta_1 \right)
\]
where

\[ \Delta_1 = l_1 x_3 + l_2 x_2 + x_1 \left( l_3 + dl_3 x_3 - cl_3 x_3 - cdl_3 x_3^2 + cl_3 x_2 + cdl_3 x_2 x_3 \right) \]

Since \( \langle d\lambda, D^2 \frac{\partial}{\partial u}(x, u) \rangle(x^0, u^0) \neq 0 \) this feedback law is nonsingular.

To summarize, the linearizing transformation \( T \) and its inverse \( T^{-1} \) are

\[
T(x) = \begin{pmatrix} x_2 \\ x_3 \\ l_1 x_3 + l_2 x_2 + x_1 \left( l_3 + dl_3 x_3 - cl_3 x_3 - cdl_3 x_3^2 + cl_3 x_2 + cdl_3 x_2 x_3 \right) \end{pmatrix}
\]

\[
T^{-1}(z) = \begin{pmatrix} z_3 - l_1 z_2 - l_2 z_1 \\ \frac{l_3 + dl_3 z_2 - cl_3 z_2 - cdl_3 z_2^2 + cl_3 z_1 + cdl_3 z_1 z_2}{z_1} \\ z_2 \end{pmatrix}
\]

(B.8)

(B.9)

and the inverse feedback law is given by

\[
\gamma^{-1}(x, v) = \frac{v - l_1 \Delta_1 - l_2 x_3 - ax_1 \left( l_3 + dl_3 \Delta_1 - cl_3 \Delta_1 - cdl_3 \Delta_1^2 + cl_3 x_3 + cdl_3 x_3 \Delta_1 \right)}{b(l_3 + dl_3 \Delta_1 - cl_3 \Delta_1 - cdl_3 \Delta_1^2 + cl_3 x_3 + cdl_3 x_3 \Delta_1)}
\]

(B.10)

The resulting linear system is given by

\[
z(k + 1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} z(k) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v(k)
\]

(B.11)

\[ z(k + 1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} z(k) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v(k) \]

B.2 Feedback Linearization of Coupled Model

The dynamics of the coupled model are described by

\[
x(k + 1) = f(x(k), u(k))
\]

(B.12)
where, dropping the time index \( k \) from the notation,

\[
f(x, u) = \begin{pmatrix} ax_1 - acx_3 x_1 + acx_2 x_1 + bu \\ x_3 \\ l_1 x_3 + l_2 x_2 + l_2 x_1 + dl_3 x_1 x_3 \end{pmatrix}
\]

Note that the point \((x^0, u^0) = (0, 0, 0, 0)^T\) is an equilibrium point for \((B.12)\).

We test if \((B.12)\) is feedback linearizable in a neighborhood of \((x^0, u^0)\). For this system, the Jacobian \( \left( \frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial u} \right) \) is given by

\[
\left( \frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial u} \right) = \begin{pmatrix}
    a - acx_3 + acx_2 & acx_1 & -acx_1 & b \\
    0 & 0 & 1 & 0 \\
    l_3 + dl_3 x_3 & l_2 & l_1 + dl_3 x_1 & 0 
\end{pmatrix}
\]

\((B.13)\)

Note that rank \( \left( \frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial u} \right) \) is 3 in a neighborhood of \((x^0, u^0)\). The inverse image of the point \( x \) under the map \( f \) is given by

\[
f^{-1}(x) = \begin{pmatrix} g_1(x, u) \\
    g_2(x, u) \\
    g_3(x, u) \\
    u \end{pmatrix}
\]

\((B.14)\)

where

\[
g_1(x, u) = \frac{-B - \sqrt{B^2 - 4AC}}{2A}
\]

\[
g_2(x, u) = \frac{x_1 - ag_1(x, u) - bu - acx_2 g_1(x, u)}{acg_1(x, u)}
\]

\[
g_3(x, u) = x_2
\]

and where \( A, B \) and \( C \) are given by

\[
A = acl_3 + acdl_3 x_2
\]

\[
B = acl_1 x_2 - al_2 + acl_2 x_2 - acx_3
\]

\((B.16)\)

\[
C = l_2 x_1 - bl_2 u
\]

Using \((B.14)\), we may write \((B.13)\) as

\[
\left( \frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial u} \right) f^{-1}(x) = \begin{pmatrix}
    a - acg_3(x, u) + acg_2(x, u) & acg_1(x, u) & -acg_1(x, u) & b \\
    0 & 0 & 1 & 0 \\
    l_3 + dl_3 g_3(x, u) & l_2 & l_1 + dl_3 g_1(x, u) & 0 
\end{pmatrix}
\]

\((B.17)\)
Also,

\[
\ker \left( \frac{\partial f}{\partial x} \frac{\partial f}{\partial u} \right) = \text{span} \{ \alpha(x,u) \} = \text{span} \left\{ \begin{pmatrix} \alpha_1(x,u) \\ \alpha_2(x,u) \\ \alpha_3(x,u) \\ \alpha_4(x,u) \end{pmatrix} \right\}
\]  
(B.18)

where

\[
\alpha(x,u) = \begin{pmatrix} \alpha_1(x,u) \\ -\frac{l_3+dl_3x_3}{l_2} \alpha_1(x,u) \\ 0 \\ (l_3 + dl_3x_3) \frac{acr_1}{br_2} - \frac{a - acr_1 + acr_2}{b} \alpha_1(x,u) \end{pmatrix}
\]

To test for feedback linearizability we check if the column vectors

\[
\left\{ \frac{\partial}{\partial u}, D_f \frac{\partial}{\partial u}(x,u), D_f^2 \frac{\partial}{\partial u}(x,u), D_f^3 \frac{\partial}{\partial u}(x,u) \right\}
\]

are computable using Definition 3.9. and if

\[
\dim \text{span} \left\{ \frac{\partial}{\partial u}, D_f \frac{\partial}{\partial u}(x,u), D_f^2 \frac{\partial}{\partial u}(x,u), D_f^3 \frac{\partial}{\partial u}(x,u) \right\} = 4
\]  
(B.19)

in a neighborhood of \((x^0, u^0)\). It can be shown that the procedure in Definition 3.9 yields the column vectors shown below.

\[
\frac{\partial}{\partial u} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad D_f \frac{\partial}{\partial u}(x,u) = \begin{pmatrix} b \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]

\[
D_f^2 \frac{\partial}{\partial u}(x,u) = \begin{pmatrix} a - acg_3(x,u) + acg_2(x,u) \\ 0 \\ l_3 + dl_3g_3(x,u) \\ 0 \end{pmatrix}
\]

\[
D_f^3 \frac{\partial}{\partial u}(x,u) = \begin{pmatrix} (a - acg_3(x,u) + acg_2(x,u)) \delta_1(x,u) - acg_1(x,u) \delta_2(x,u) \\ \delta_2(x,u) \\ (l_3 + dl_3g_3(x,u)) \delta_1(x,u) + (l_1 + dl_3g_1(x,u)) \delta_2(x,u) \\ 0 \end{pmatrix}
\]  
(B.20)

where

\[
\delta_1(x,u) \overset{\text{def}}{=} a - acg_3(x,u)\|_{f^{-1}(x)} + acg_2(x,u)\|_{f^{-1}(x)}
\]

\[
\delta_2(x,u) \overset{\text{def}}{=} l_3 + dl_3g_3(x,u)\|_{f^{-1}(x)}
\]
For these vectors, it can be checked that the condition in (B.19) is true. Therefore, by Theorem 3.10 the coupled muscle model (B.12) is feedback linearizable.

To construct the linearizing transformation and feedback law, consider the following set of partial differential equations:

\[ \begin{align*}
\{ d\lambda, \frac{\partial}{\partial u} \} &= 0 \\
\{ d\lambda, D_f \frac{\partial}{\partial u}(x, u) \} &= 0 \\
\{ d\lambda, D^2_f \frac{\partial}{\partial u}(x, u) \} &= 0
\end{align*} \]

Substituting from (B.20) for these column vectors, we get

\[
\begin{align*}
\frac{\partial \lambda}{\partial u} &= 0 \\
\frac{b}{\partial x_1} &= 0 \\
\frac{\partial \lambda}{\partial x_1}(a - acg_3(x, u) + acg_2(x, u)) + \frac{\partial \lambda}{\partial x_3}(l_3 + dl_3g_3(x, u)) &= 0
\end{align*}
\] (B.21)

It is easy to verify that \( \lambda = x_2 \) is a solution. Note also that

\[ \begin{align*}
\{ d\lambda, D_f^3 \frac{\partial}{\partial u}(x, u) \} &= \frac{\partial \lambda}{\partial x_2} \delta_2(x, u) = \delta_2(x, u)
\end{align*} \]

and \( \delta_2(x^0, u^0) \neq 0 \).

The coordinate transformation and feedback law are computed as follows.

\[
\begin{align*}
z_1 &\overset{\text{def}}{=} \tilde{\varphi}_0(x) = \varphi_0(x, u) \overset{\text{def}}{=} \lambda(x, u) = x_2 \\
z_2 &\overset{\text{def}}{=} \tilde{\varphi}_1(x) = \varphi_1(x, u) \overset{\text{def}}{=} \varphi_0 \circ f(x, u) = x_3 \\
z_1 &\overset{\text{def}}{=} \tilde{\varphi}_2(x) = \varphi_2(x, u) \overset{\text{def}}{=} \varphi_0 \circ f(x, u) = \eta(x)
\end{align*}
\]

where

\[
\eta(x) \overset{\text{def}}{=} l_1x_3 + l_2x_2 + l_3x_1 + dl_3x_1x_3
\]

The feedback law is defined by

\[
u = \gamma(x, u) \overset{\text{def}}{=} \tilde{\varphi}_2 \circ f(x, u)
\]

\[
= (l_1x_1 + l_2x_2 + l_3x_1 + dl_3x_1x_3) \left( l_1 + dl_3(ax_1 - acx_3x_1 + + acx_2x_1) \right) + l_3(ax_1 - acx_3x_1 + acx_2x_1) + l_2x_3 + + u \left( bl_3 + bdl_3(l_1x_3 + l_2x_2 + l_3x_1 + dl_3x_1x_3) \right)
\]
Since $\langle d\lambda, D^3_T \frac{\partial}{\partial u}(x, u) \rangle(x^0, u^0) \neq 0$ this feedback law is nonsingular.

To summarize, the linearizing transformation $T$ and its inverse $T^{-1}$ are

\[
T(x) = \begin{pmatrix}
x_2 \\
x_3 \\
l_1 x_3 + l_2 x_2 + l_3 x_1 + dl_3 x_1 x_3
\end{pmatrix} \tag{B.22}
\]

\[
T^{-1}(z) = \begin{pmatrix}
z_3 - l_1 z_2 - l_2 z_1 \\
l_3 + dl_3 z_2 \\
z_1 \\
z_2
\end{pmatrix} \tag{B.23}
\]

and the inverse feedback law is given by

\[
\gamma^{-1}(x, v) = \frac{1}{\Delta} \left( v - l_3 (ax_1 - acx_3 x_1 + acx_2 x_1) - l_2 x_3 - (l_1 x_3 + l_2 x_2 + l_3 x_1 +
+ dl_3 x_1 x_3) (l_1 + dl_1 (ax_1 - acx_3 x_1 + acx_2 x_1)) \right) \tag{B.24}
\]

where $\Delta \overset{\text{def}}{=} bl_3 + bdl_3 (l_1 x_3 + l_2 x_2 + l_3 x_1 + dl_3 x_1 x_3)$.

The resulting linear system is given by

\[
z(k + 1) = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix} z(k) + \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix} v(k) \tag{B.25}
\]