INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

UMI

A Bell & Howell Information Company
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
313-761-4700  800-521-0600
COMPUTATIONAL ASPECTS OF PARTICLE IMAGE SIZE AND VELOCITY MEASUREMENTS

by

SEPEHR SANAYE

Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy

Thesis Advisor: Dr. Robert V. Edwards

Department of Mechanical and Aerospace Engineering

CASE WESTERN RESERVE UNIVERSITY

January 1995
Copyright © (1995) by
Sepehr Sanaye
We hereby approve the thesis of

Sepehr Senaye

candidate for the Doctor of Philosophy

degree.*

(signed) 

Robert J. Krauss

(chair)

[Signatures]

D. John Weathersby

date 11-7-94

*We also certify that written approval has been obtained for any proprietary material contained therein.
COMPUTATIONAL ASPECTS OF SIZE AND VELOCITY MEASUREMENTS
OF PARTICLE IMAGES

Abstract
by
SEPEHR SANAYE

The measurement of the shape, size, concentration and velocity of particles or bubbles moving with a fluid flow is studied by processing computer stored images. A single image of the particles illuminated by a laser sheet is used to determine the size and shape of the objects including irregularities and holes. Cross-correlation technique using a Fast Fourier Transform algorithm is applied to consecutive images of the particles to determine the displacement vectors and velocities. A computer simulated experimental flow with particles is used to demonstrate the ability of the developed techniques to handle:

a random number and random initial position for the particles in the interrogation area when the expected particle number density is known,

particle images that overlap, enter, leave or get cut by the edges of interrogation area,
particles of a non-integer number of pixels in size and displaced by a non-integer number of pixels in either direction within the light-sheet.

An approximate Maximum Likelihood Estimation (MLE) procedure is developed to predict the displacement error obtained from cross-correlation measurements. There are various parameters such as particle number density, particle image size, image displacement values and the size of interrogation area, that affect the displacement error. Variations of the displacement error introduced by parameter changes is studied and estimates of displacement errors obtained from simulation are compared with those predicted by MLE theory.
DEDICATION:

To my parents,

to my wife,

to my daughters,

for their love, patience, support and encouragement.
ACKNOWLEDGEMENTS:

I wish to express my sincere thanks to:

Prof. Robert V. Edwards for sharing his time and knowledge.

and

Prof. Joseph M. Frahl chairman of the department for his support and advice.
<table>
<thead>
<tr>
<th>TABLE OF CONTENTS</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>ii</td>
</tr>
<tr>
<td>Dedication</td>
<td>iv</td>
</tr>
<tr>
<td>Acknowledgements</td>
<td>v</td>
</tr>
<tr>
<td>Table of Contents</td>
<td>vi</td>
</tr>
<tr>
<td>List of figures</td>
<td>viii</td>
</tr>
<tr>
<td>Nomenclature</td>
<td>xiii</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>1- CHAPTER ONE - INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>2- CHAPTER TWO - SHAPE AND BOUNDARY RECOGNITION</td>
<td>13</td>
</tr>
<tr>
<td>2-1- Thresholding</td>
<td>14</td>
</tr>
<tr>
<td>2-2- Particle Identification</td>
<td>20</td>
</tr>
<tr>
<td>2-3- Boundary and Shape Recognition</td>
<td>23</td>
</tr>
<tr>
<td>3- CHAPTER THREE - VELOCITY MEASUREMENT</td>
<td>28</td>
</tr>
<tr>
<td>3-1- Cross-correlation Techniques</td>
<td>30</td>
</tr>
<tr>
<td>3-1-1- Without Using FFT</td>
<td>31</td>
</tr>
<tr>
<td>3-1-2- With Using FFT</td>
<td>33</td>
</tr>
<tr>
<td>3-2- Implementation</td>
<td>34</td>
</tr>
<tr>
<td>3-3- Some Practical Features</td>
<td>35</td>
</tr>
<tr>
<td>3-4- Interpolation Scheme for Peak Finder</td>
<td>37</td>
</tr>
</tbody>
</table>
4- CHAPTER FOUR - ERROR ANALYSIS

4-1- Maximum Likelihood Estimation Method 40
4-2- Model Function 43
4-3- Variance of Cross-Correlation Measurements 47
4-4- Simulation 50
4-5- Effects of Various Parameters 54
   4-5-1- Random Position of Particle Images 55
   4-5-2- Number and Diameter of Particles 57
   4-5-3- The Size of Interrogation Area and Displacement Values. 61

5- CHAPTER FIVE - CONCLUSION 65

References 73
Figures 77
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>An experimental setup for object size and velocity measurement using a laser sheet and camera.</td>
<td>77</td>
</tr>
<tr>
<td>2.1</td>
<td>The histogram of an image composed of bright particle images before smoothing.</td>
<td>78</td>
</tr>
<tr>
<td>2.2</td>
<td>The histogram of an image composed of bright particle images after smoothing.</td>
<td>79</td>
</tr>
<tr>
<td>2.3</td>
<td>The histogram of an image drawn based on estimated parameters.</td>
<td>80</td>
</tr>
<tr>
<td>2.4</td>
<td>The Region Of Interest (R.O.I) for an unexpected irregular shape boundary.</td>
<td>81</td>
</tr>
<tr>
<td>2.5</td>
<td>Several adjacent objects with irregular boundaries.</td>
<td>82</td>
</tr>
<tr>
<td>2.6</td>
<td>Input binary image.</td>
<td>83</td>
</tr>
</tbody>
</table>
2.7- The Particle Identification Scheme (PIS). (TOP-DOWN PASS)

2.8- The Particle Identification Scheme (PIS). (BOTTOM-UP PASS)

2.9- Organizing the objects' unique numbers.

2.10- The boundary detecting scheme.

2.11- The detected boundary pixels for three objects with irregular boundaries.

2.12- The detected boundary pixels in character format.

3.1- Randomly positioned particle images with Gaussian intensity distribution in the first interrogation area. D=2 pixels, A=32*32 pixels.

3.2- The particle images in figure 3.1 have been shifted 5.25 and 3.75 pixels in X and Y directions respectively. Some particles have left the interrogation area and some
new particles have entered to this region.

3.3- The cross-correlation of two interrogation areas shown in figures 3.1 and 3.2. Displacement values are 5.25 and 3.75 pixels in the X and Y directions respectively.

4.1- The distribution of displacement estimates obtained from simulation for 100 samples. D=1 pixel, A=32*32 pixels, \( \langle N \rangle = 10 \) DX=5.25 pixels, DY=3.75 pixels.

4.2- The distribution of displacement error obtained from Maximum Likelihood Estimation method for 100 samples. DX=5.25 pixels, DY=3.75 pixels.

4.3- The distribution of number of particle images in the first interrogation area created by a Poisson random number generator for \( \langle N \rangle = 10 \) and 100 samples.

4.4- The distribution of number of particle images in the second interrogation area for 100 samples. Some particle images have left the
interrogation area after shifting them 5.25 and 3.75 pixels in the X and Y directions respectively.

4.5- The variation of displacement error with particle image diameter and expected value of particle image number. The number of samples used for each measurement point is 100.

4.6- The variation of displacement error with particle image diameter and expected value of particle image number. The number of samples used for each measurement point is 100.

4.7- The variation of displacement error with particle image diameter and expected value of particle image number. The number of samples used for each measurement point is 100.

4.8- The variation of displacement error with particle image diameter and expected value of particle image number. The number of samples used for each measurement point is 100.

xi
4.9- The variation of displacement error with particle image diameter and expected value of particle image number. The number of samples used for each measurement point is 100.

4.10- The variation of displacement error with expected value of particle image number. The number of samples used for each measurement point is 100.

4.11- The variation of displacement error with expected value of particle image number for two window sizes. The number of samples used for each measurement point is 100 for A=32*32 pixels and 150 for A=16*16 pixels.

4.12- The variation of displacement estimation error with displacement values from simulation and computation. The number of samples used for each measurement point is 100.
**Nomenclature:**

- $\mathbf{A}$: Image area
- $C_x, C_y$: Centroid position in the X and Y directions
- $f(x, y)$: Binary image
- $f_*$: Total photon count
- $F_1, F_2$: Two consecutive recorded images from particle movements
- $F_{ij}$: Fisher matrix
- $F_{ii}^{-1}$: $i^{th}$ diagonal element of the matrix inverse $F_{ij}$
- $g(x, y)$: Gray level image
- $G$: Gray level
- $h(n, r_a)$: Model function in MLE method
- $N$: Number of particle images in interrogation area
- $P$: The log of the probability of the data set
- $P_1(G)$: The brightness probability density function for dark or background region.
- $P_2(G)$: The brightness probability density function for light or object region.
- $Q_1, Q_2$: A priori probabilities of dark and light regions respectively
- $r_p, r_q$: Random position of particle images in $F_1$ and $F_2$
$R_{12}$ The cross-correlation measurements
$R_{n12}$ The normalized cross-correlation
$R_{f12}$ The cross-correlation measurements obtained from Fourier Transform

$R_{12}(\tau), R_{12}(\tau')$ The cross-correlation measurement matrices at the two arbitrary locations of $\tau$ and $\tau'$ in the colorogram.

$T$ Threshold value

$\mu_1, \sigma_1$ Mean and standard deviation of dark or background region.

$\mu_2, \sigma_2$ Mean and standard deviation of light or object region

$\sigma_{en}^2$ Variance of cross-correlation measurements

$\sigma_f$ The rms of the width of the particle image

$\tau$ Arbitrary two-dimensional displacement vector

$\tau_{en}, \tau_{em}$ The displacement pixels in the X and Y directions

$\epsilon(T)$ The probability of error in specifying object or background pixels

$p$ Particle number density in the image, $<N>/A$

* Conjugate sign

|| Norm sign

<> Expected value sign

xiv
CHAPTER ONE

INTRODUCTION

Experiments have always been the keystone in all branches of science including fluid mechanics. There are few problems in fluid dynamics which can be solved analytically, therefore without doing experiments our knowledge about fluid flow behavior, even in simple geometries, is incomplete.

Information obtained from direct observations in fluid mechanics can be applied in different forms:

1- Mean and fluctuations of velocity can be used to obtain the stream function, potential function, vorticity, and Reynolds Stress.

2- Empirical relations and models can be generated which, in combination with the conservation equations, can be used to solve complex engineering flow problems.

3- Numerical results can be verified by comparison with direct observations of fluid behavior.
4- When momentum, energy and species conservation equations are coupled as in combustion and reacting flows, information obtained from experiments can be used to reduce the number of unknowns in the problem. This can help convergence of the numerical solutions toward correct results.

More complicated problems such as, fluid dynamic behavior of unsteady, turbulent, three dimensional flows, flow in complex geometries, multiphase flows, heat transfer or chemical reaction involved processes, wide range of velocities from very low Reynolds number to hypersonic flows, or some combination of them, require more accurate and efficient tools to record the observations quantitatively. Better methods and instruments are being invented and developed to meet these challenges.

Instruments which measure the flow field velocity can be roughly categorized into two groups: those which measure the velocity at a specified point, and those which measure a section of flow field at once.

The Pitot Tube is one of the first instruments used to measure local velocity at a point. This instrument is
a suitable one when flow is steady and laminar. When the fluid flow is turbulent, a pitot tube can be used to measure the mean velocity, however, it disturbs the flow field and affects the velocity fluctuations. Moreover it can not measure velocity of turbulent eddies whose length scale is less than the tube size. The device time response to variations in the flow field is also very slow.

Hot wire and film anemometers measure local velocities and disturb the flow field, but since they have been designed with small elements, have shorter time response and can measure turbulent fluctuations.

Laser Anemometry (LA) uses non-invasive methods for measurement of flow velocities. In LA, two laser beams are split from the same original beam. The two beams can be caused to interfere and this interference produces fringes. The distance between the fringes is a function of the laser wave length and the angle formed by the intersecting beams (Durst F. et al. 1981). When particles immersed in the fluid cross the fringe pattern, the light scattered from the particles is modulated at the doppler frequency which is proportional to the speed of scattering particles.
The above techniques measure local flow velocities at one point and can only be used to measure the velocity distribution in the whole flow field in steady flows. However, with multiple measurements at different points and applying statistical analysis, approximate estimates can be made for the whole flow field when the flow is unsteady.

Researchers have been eager to see the whole flow field in its entirety for both research and educational reasons. Watching a flow field as a whole is interesting to observers with different levels of knowledge about fluid mechanics. This might be the case since it conveys qualitative information about the flow field. Watching water flow in a river with vortices behind stones, waves in a sea and tracing clouds in the sky are some simple examples.

In order to better understand the physics of the whole flow field, flow visualization techniques have progressed. Smoke-wire, neutrally buoyant bubbles, schlieren photography are some examples, however, these methods provide mostly qualitative and/or semi-quantitative information.

One method for increasing the accuracy of obtained quantitative information is Particle Streak Velocimetry (FSV) in which fluorescent and phosphorescent particles
are added to the flow (Dimotakis P. E. et al. 1981, Gharib M. et al. 1985, Khalighi B. 1987, Kobayashi T. et al. 1985). A laser sheet illuminates a slice of the flow field and markers become light scattering sources when they enter the laser sheet (figure 1.1). The markers' path of movement are streaks which are recorded and measured. This is the only method which can show the velocity gradient and real particle paths. Particle Streak Velocimetry provides good accuracy in measurement of velocity if the number density of markers is low and particle image diameters are small compare to their displacements. These characteristics make the method useful when sparse velocity measurement in the flow field is desired.

When the number density of light scattering sources in the flow field is large, random interference of many light scattering sources forms speckle patterns. In this case Laser Speckle Velocimetry (LSV) can be applied to measure the displacement of particle images (Meynart R. 1983, Pickering C. J. D. et al. 1984, Adrian R. 1984, Lourenco L. M. M. et al. 1984). When a laser beam illuminates a small part of the transparency of two exposures of speckle patterns which are recorded on one photographic film, Young's fringes result. This fringe pattern can be analyzed in order to find the velocity
magnitude and direction by measuring the orientation of fringes and their spacing.

Generally, for natural aerosols and hydrosols, particle images do not overlap and a speckle pattern can only be produced by adding more seeds to the flow field. In fluid dynamic measurements this is not desirable because it restricts penetration of laser light and image recording (Adrian R. J. 1986a).

When particle images are completely separated from each other, measurement of the particle displacement is done by Particle Image Velocimetry (PIV) methods. Fringe analysis can still be applied for PIV images, however, an alternative method is auto-correlation analysis which can be implemented by two dimensional Fourier Transform of fringe plane (Adrian R. J. 1986a). There are three peaks in the transformed plane: The highest peak at the center shows the correlation of the images with themselves; the displacement of the two other peaks, which are located symmetrically about the central peak, shows the magnitude of the mean velocity vector in the interrogation region. The directional ambiguity, can be removed by image shifting (Adrian R. J. 1986b). LSV and PIV negative photographic films can be saved in the archive for future reference.
Instead of recording the images on film, a CCD (Charge Coupled Device) camera and digitizer can be used to save images on the hard disk of a computer. Recorded particle images can be compared and analyzed digitally.

Particle Tracking is one of the methods which employs the digitized images. Velocity vectors obtained in the Particle Tracking method are not based on average velocity in the interrogation area, but each particle is tracked individually. When the particle number density is low, the method is fast and the chances of false detected displacement vectors, small. This is when particle displacements are smaller than spacing between particles. As the particle number density increases, so does the number of false detected displacement vectors. Moreover, for a given particle number density, a large number of false velocity vectors can be detected because of the random position of particle images (Wernet M. P. 1989). The application of this method using gray scale images has not yet described in the literature. The method works more accurately if the number of exposures increases. For this reason the Particle Tracking method usually is applied to image fields.

Particle displacement can also be determined from correlation analysis. Cenedese et al. (1990) reported
the application of correlation techniques and digitized images for multiexposure photographs.

The Cross-correlation technique is another method which uses two consecutive frames of digitized images. In this method two interrogation areas in two consecutive frames are considered. The correlation of these two interrogation areas shows a peak which is the dominant feature of the cross-correlation measurements. The position of this peak relative to the origin of the interrogation area, shows the direction and magnitude of mean velocity vector in that region.

In chapter three, the mathematical basis of this method and the employment of digitized images and cross-correlation techniques, with and without using Fast Fourier Transform (FFT), for measurement of the two dimensional velocity distribution in a section of a flow field is presented. This method can be applied to gray level images directly. The displacement errors due to the random position of particles is very low for cross-correlation methods.

Chapter four is dedicated to an analysis of the displacement measurement error. The ability of the cross-correlation method to measure the correct displacement vector when initial particle image positions
are random is determined. A computer code simulates the most important features of a real experiment in which fluid flow carries particles or bubbles through a laser sheet.

Maximum Likelihood Estimation (MLE) method is applied to estimate, analytically, the error of measured displacement obtained from cross-correlation techniques. The parameters that affect the displacement error estimation are particle image density (ratio of the expected value of the particle number in an image to the image area), size of particles, image shift values, size of interrogation area. By proposing a physical model in which all above mentioned parameters are included, a comprehensive study of the variations of the error when parameters change, is accomplished. The estimated displacement errors obtained from simulation and from approximate MLE theory, are compared.

There are situations in which knowing shape, size, and concentration of objects (particles and/or bubbles) are as important as their velocities. This situation occurs in two phase systems such as liquid-gas (nuclear power generation and bubbly systems,...), liquid-solid (slurry flows,...), gas-solid (rocket propulsion, combustion, seed transportation, filtration,...) and in
biological fluid systems and metallurgical processes.

Phase Doppler Analyzers (PDA), which is a point measurement instrument, have been successful in measuring the size of spherical particles in a flow field. When a spherical transparent droplet is moving, the fringe pattern of the probe volume is projected into space by that particle. The first photodetector sees the projected fringe pattern, a burst with a frequency proportional to the velocity of the particle. The two other photodetectors placed at different places detect the same projected fringe pattern with the same frequency but a different phase (phase shift). This phase shift is proportional to the ratio of the distance between two projected fringes over the distance between the detectors. This ratio is in turn proportional to the diameter of the droplet.

Some restrictions apply to this method. It can not measure the size of bubbles or particles larger than PDA probe. It is also restricted to measurement of the size of spherical particles.

Software for measurement of shape, size, and concentration of irregular shape objects (particles and/or bubbles) in multiphase flows is developed. When objects enter the laser sheet, they become visible. One
frame of particle or bubble images is used to recognize the shape and size. This frame is a gray level image composed of two main regions of bright objects on dark background. The histogram of such an image has two main peaks and a valley in between them. The gray level corresponding to this valley provides the threshold value used to convert the gray level image into a binary image. The pixels of an image can be grouped into object and background pixels: the gray levels higher than the threshold value, are object pixels; and those less than the threshold value, are background pixels. A unique number is assigned to each object in the image, providing the total number of objects for determining the particle concentration. This is also an important step before using the boundary detecting method, when object images are very close to each other. Once all objects are identified, this information provides input to the boundary detecting process. There are two objectives of this step: to detect and put in order clockwise, the boundary elements, so that an analytical description and closed form expression for the boundary, using a suitable interpolation scheme can be formed; and to measure the area, circumference and centroid position of the objects. The software can be applied to irregular, unexpected shape objects, not smaller than one pixel and not larger
than the whole frame even when there are holes in the objects. There is no restriction on irregularity of the shape or number of objects as well as the number of holes in them.

In chapter two the process of measuring the shape, size and concentration of objects is explained.
CHAPTER TWO

SHAPE AND BOUNDARY RECOGNITION

There are situations in which knowing shape, size, and concentration of objects (particles and bubbles) are critical: the design of two phase systems such as liquid-gas (nuclear power generation and bubbly systems,...), liquid-solid (slurry flows,...), gas-solid (rocket propulsion, combustion, seed transportation, filtration,...), three phase systems like fluidized beds and others like biological fluid systems, metallurgical processes.

The measurement of shape, size, and concentration of irregular shape objects (particles or bubbles) is broken down into three steps.

1-Thresholding

2-Particle Identification

3-Boundary Detecting
2-1-Thresholding:

Thresholding changes the gray level image to a binary image. The pixel amplitudes of gray scale images are between (0) and (255), whereas binary images contain only (0) and (1) pixel amplitudes. Binary images are black and white and can employ a fast and accurate boundary detecting scheme to obtain the object boundaries.

Thresholding is one of the most important steps in size and shape recognition of objects in an image. In particle image velocimetry, when particles enter the laser sheet, they scatter light and their bright images are recorded. Such an image \( f(x,y) \), is composed of light objects on a dark background and therefore there are two dominant groups of gray levels in the image.

These two groups of gray levels are separated by choosing a threshold value \( (T) \). The pixels with gray levels more than \( T \) are taken as object pixels and the pixels with gray levels less than \( T \), as background pixels. A thresholded image \( g(x,y) \) is defined as:

\[
g(x,y) = \begin{cases} 
0 & \text{If } f(x,y) < T \\
1 & \text{If } f(x,y) \geq T 
\end{cases}
\]  

(2.1)
Those pixels with the new substituted intensity values equal to one, are object pixels and those specified with new intensity values equal to zero, are background pixels. However, one can also choose other convenient intensity values instead of (1) and (0) in the above definition.

Thresholding and separating objects from background can be accomplished by scanning the image pixel by pixel and substituting one or zero based on definition (2.1) and whether \( f(x,y) \) is greater or less than \( T \) in that pixel. The thresholded image is also named the binary image.

A good threshold value is a gray level that minimizes confusing object pixels as background pixels and background pixels as object pixels.

A PIV image contains two major brightness regions, one for light regions (object) and one for black regions (background), therefore its histogram is bimodal (Change T. P. et al. 1985). In the following discussion the histogram of an image is used to determine the threshold value which minimizes errors of specifying the object and background pixels.

Suppose the image contains two mean values \( \mu_1 \) and \( \mu_2 \) corresponding to dark and light regions respectively and the probability distribution function for these two
regions are Gaussian and combined with additive noise.

Then:

$$P_1(G) = \frac{1}{\sqrt{2\pi} \sigma_1} \cdot \exp\left[-\frac{(G-\mu_1)^2}{2 \sigma_1^2}\right] \quad (2.2)$$

$$P_2(G) = \frac{1}{\sqrt{2\pi} \sigma_2} \cdot \exp\left[-\frac{(G-\mu_2)^2}{2 \sigma_2^2}\right] \quad (2.3)$$

Where;

$P_1(G)$ = Brightness probability density function for dark or background region.

$P_2(G)$ = Brightness probability density function for light or object region.

$\mu_1, \sigma_1$ = Mean and standard deviation of dark or background region.

$\mu_2, \sigma_2$ = Mean and standard deviation of light or object region.

and $\mu_1 < \mu_2$. 
The probability of specifying an object pixel erroneously as a background pixel is:

\[ e_1(T) = \int_{-\infty}^{\infty} P_2(G) \, dG \]  \hspace{1cm} (2.4)

The probability of specifying a background pixel erroneously as an object pixel is:

\[ e_2(T) = \int_{-\infty}^{\infty} P_1(G) \, dG \]  \hspace{1cm} (2.5)

The total probability of error is:

\[ e(T) = Q_2 \cdot e_1(T) + Q_1 \cdot e_2(T) \]  \hspace{1cm} (2.6)

Where \( Q_1 \) and \( Q_2 \) are the a priori probabilities of dark and light regions respectively and:
\[ Q_1 + Q_2 = 1 \quad (2.7) \]

The threshold value for which the total error \( \epsilon(T) \) is minimum results from setting the derivative of \( \epsilon(T) \) with respect to \( T \) to zero. Using Leibnitz's rule to differentiate the integral then;

\[ Q_1 \cdot P_1(T) = Q_2 \cdot P_2(T) \quad (2.8) \]

When \( Q_1 = Q_2 = 1/2 \), a pixel has the same likelihood to be in the light or dark region,

\[ P_1(T) = P_2(T) \quad (2.9) \]

Corresponds to the minimum error for threshold value,

\[
\frac{1}{\sqrt{2\pi}\sigma_1} \cdot \text{EXP}\left[-\frac{(T-\mu_1)^2}{2\sigma_1^2}\right] = \frac{1}{\sqrt{2\pi}\sigma_2} \cdot \text{EXP}\left[-\frac{(T-\mu_2)^2}{2\sigma_2^2}\right]
\]

\[ (2.10) \]

The best threshold value \( T \), is a function of \( \mu_1, \sigma_1, \mu_2, \sigma_2 \) which must be found from the histogram of the image.
Figure 2.1 illustrates the histogram of an image containing light particle images. The histogram is bimodal and the biggest peak corresponds to background pixels. This is because the dark background usually occupies more area than light particle images. The following steps are followed to achieve desirable parameters from the image histogram.

1- The histogram is smoothed using a low pass filter, to remove oscillations and noise (figure 2.2).

2- The gray level of the largest peak is chosen for the first estimation of the mean value of the dark region.

3- A non-linear regression scheme is applied to fit a Gaussian curve to the data points. The scheme starts with given values for mean and variance, and iteratively changes those values until it converges toward the final estimated parameters. In each iteration the difference between data points from measurement and estimated values is checked and the decreasing of this parameter guarantees the convergence of the scheme.
4- With final estimated parameters, the first peak is subtracted from the image histogram and the process is repeated for the second peak of histogram which belongs to the object pixels.

The final estimated mean and variance of the two Gaussian peaks in the histogram are substituted in (2.10), which gives a threshold value of (116) for the histogram shown in figure 2.1. Figure 2.3 illustrates the histogram drawn based on the estimated parameters.

2-2-Particle Identification(PI):

After thresholding and separating object and background pixels, particle identification is the next step before applying the boundary detection scheme.

Particle identification is applied to recognize the different particles in an image. At the end of this process, each object (particle or bubble) has a unique number which under that number, information obtained about that specific object is filed. As figure 2.4 shows each object in the image has a Region Of Interest (R.O.I), which specifies the lowest and highest values of X and Y coordinates of those pixels which belong to one
specific image. After applying the boundary detection scheme (which will be discussed in the next section), one can find the R.O.I for each object and simply find the particle image area by scanning non zero pixels in that region. When all information needed from one particle image is obtained, all pixels in the R.O.I of that image are substituted with zeros. This is equivalent to omitting the object from the image. The process continues with scanning the image and finding the next particle image boundaries.

The boundary detecting process can not be implemented if the R.O.I of several adjacent objects in an image overlap (figure 2.5). In this case two problems arise;

a) The area for a specified object can not be found correctly by scanning the R.O.I of that object, because all or some part of the area which belongs to the other objects is added to the one of interest.

b) Substituting zero values in the R.O.I of the object of interest deletes all or some part of other objects.
The Particle Identification Scheme (PIS) solves the above mentioned problems by recognizing and numbering the objects in an image. This gives the capability to use the boundary detecting scheme in more general and complicated situations. PIS begins by assigning a unique number to an object pixel of a specified particle image and attempts to propagate this number through all pixels which belong to that particle. This is accomplished by selecting the minimum number of all neighbors for the centroid pixel (Haralik R. M. 1981). When an object is assigned different numbers because of its irregular shape, an iterative algorithm is applied. Top-down followed by bottom-up iterative numbering propagates one unique number in an image until no change can be observed.

Figure 2.6 shows an input binary image composed of three objects in which the R.O.I of these objects overlap.

After top-down and bottom-up passes, three different objects have been completely recognized by different unique numbers which are not necessary in order and predictable (figures 2.7 and 2.8). In the last part of PIS, the object unique numbers are put in order (figure 2.9), which specifies the total number of objects in the image.
2-3 - Boundary and Shape Recognition:

The objectives of boundary and shape recognition are:

A) - To detect of boundary elements and put them in order (clockwise).

B) - To measure the area, circumference, centroid of the objects.

Putting boundary pixels in order provides useful, organized information which can be applied to describe the shape of the boundary in a closed form. If only some part of the boundary is detectable from its image, a suitable interpolation scheme is applied to find and complete the whole boundary.

Boundary following involves detecting the boundary between an object and its background (Rosenfeld A. et al. 1982). The process begins with scanning the image until it encounter a white pixel which is part of an object, it then turns left and takes a step, if the next encountered pixel is a background pixel, it turns right and takes a step. This process terminates when it returns to the starting pixel.
Figure 2.10 illustrates the operation of this algorithm on a simple binary object.

After detecting the boundary pixels for each object, area and centroid for binary image are computed from:

\[ A = \sum_{i=\text{imin}}^{\text{imax}} \sum_{j=\text{jmin}}^{\text{jmax}} G(i, j) \quad (2.11) \]

\[ C_x = \frac{\sum_{i=\text{imin}}^{\text{imax}} \sum_{j=\text{jmin}}^{\text{jmax}} X(i,j) \cdot G(i,j)}{\sum_{i=\text{imin}}^{\text{imax}} \sum_{j=\text{jmin}}^{\text{jmax}} G(i,j)} \quad (2.12) \]

\[ C_y = \frac{\sum_{i=\text{imin}}^{\text{imax}} \sum_{j=\text{jmin}}^{\text{jmax}} Y(i,j) \cdot G(i,j)}{\sum_{i=\text{imin}}^{\text{imax}} \sum_{j=\text{jmin}}^{\text{jmax}} G(i,j)} \quad (2.13) \]
All pixels which belong to an object are set to zero before starting the boundary detecting process for the next object.

The scan starts again from first up-left pixel in the image and it continuous until it hits a boundary pixel which belongs to the next object. Since after PI each particle owns its unique number, it takes less time when scanning the image and making a list of the first boundary pixels for each object. It is not necessary to scan the image the number of times equal to the number of objects present in the image. The boundary detecting method can be applied when the R.O.I of particles overlap because the PI process described in 2-2 was implemented.

As shown in figure 2.11, boundary pixels for three objects are detected and saved. The information obtained in this step is used to show the boundary pixels in character format (figure 2.12). All boundary pixels have been detected correctly. The results show that the method can detect the boundary of objects with unexpected irregular shapes.

The following techniques is implemented to handle the situations in which holes exist in the object.
1- The image is scanned until a line scan hits a boundary pixel which belongs to a hole inside the object. The coordinates saved from outer boundary detection, can be used to specify whether the border pixel belongs to a hole or to an outer boundary.

2- Apply the boundary detecting method to turn all around the boundary pixels for each hole.

3- Since transition from background to object and object to background (which can be a hole), are opposite, the instructions of white-black and white-white transitions are reversed for hole boundary identification.

To summarize the issues related to shape and size recognition of objects (particles or bubbles),

a- No information about the objects and their shapes or sizes are needed and the boundary detecting method can be applied to objects with arbitrary, unexpected, irregular shape objects.

b- The boundary of holes can be detected. The method can also be used to find several holes inside each other (hole in hole).
c- No limitations on the number of objects and holes exist.

d- If the region of interest of two objects do not overlap, the boundary detecting method performs particle identification (PI) automatically, otherwise PI facilitate the identification of objects by giving them unique numbers.

f- After organizing the boundary pixels in order, a suitable interpolation scheme is applied to different parts of the boundary to describe it in a closed form and analytically.

g- The area, circumference and centroid of each object can be found.

h- Since after particle identification all objects have a unique number, one or some of the objects can be chosen for boundary detecting.

i- The boundary detecting method is not restricted to binary images. This method also works well on output images from the PI process.
CHAPTER THREE

VELOCITY MEASUREMENT

Particle Image Velocimetry (PIV) is a non-invasive method for measuring velocity in a fluid flow. If enough small size particles are added to a flow and a slice of the flow is illuminated by a laser sheet, particles in that sheet scatter light and their movements can be recorded on consecutive frames. The position of particle images in the first frame is random and the fluid flow moves them in the direction and with the magnitude of the flow velocity during the time interval between the frames. Particle images in the new location are recorded in the second frame. If it is assumed there is no relative velocity between particles and the flow field, the magnitude of the displacement vector divided by time interval between two consecutive frames provides the velocity vector. When the displacement magnitudes are small and the time interval between two frames, short, their ratio is a good measure of the local velocity. Since there is no unique distinguishable feature for each particle in two frames, the task now is to find the corresponding particle images in the first and second frames which gives the displacement of particles and
local velocity of the flow field. The most reliable approach is to apply a statistical analysis to find the best match between particles in two consecutive images.

The statistical methods for displacement measurement can be divided into two main groups: those which detect the path of movement of each individual particle and those which determine the velocity vector in a small region of the image as the average of the velocity vectors of the particles in that region. The cross-correlation techniques are among the second group. They detect the average velocity of moving objects in an interrogation area of an image.

The cross-correlation technique is chosen because of its strong features which are presented in this and the next chapters.

This chapter continues with discussion about the mathematical basis of cross-correlation methods and its different forms. A constructed synthetic image pair is presented and its cross-correlation is illustrated. The measurement of the cross-correlation peak position to sub-pixel accuracy is discussed. Some important practical features of cross-correlation methods are listed.
3-1-Cross-Correlation techniques:

The cross-correlation is a statistical method which can be applied for estimation of particle image displacements in two consecutive frames. This method measures the average velocity of all existing particles in an interrogation area (IA). These particle images are randomly positioned in the first and second frames. Since the initial positions of particle images are random, a statistical analysis for obtaining the best match between two frames is applied. The best shift value for the best match between the first and the second images is sought.

There are two groups of cross-correlation techniques to apply on digitized images: those cross-correlation methods which do not use Fourier Transform, and those which use this transformation to give the displacement estimated values. Their mathematical basis is the same.
3-1-1-Cross-correlation Techniques Without using FFT:

In general cross-correlation of two images $F_1$ and $F_2$ is defined as (Bendat J. S. et al. 1986);

$$R_{12} = \langle F_1(x) \cdot F_2^*(x+\tau) \rangle \quad (3.1)$$

Where $\langle \rangle$ denotes expected value. For PIV images this expected value is average taken over particle initial positions.

An integral expression for cross-correlation estimation can be in the form of:

$$R_{12}(\tau) = \frac{1}{A} \int F_1(x) \cdot F_2^*(x+\tau) \, dx \quad (3.2)$$

Where $F_1$ and $F_2$ are two recorded images from particle movements and $\tau$ is an arbitrary two-dimensional displacement vector. $R_{12}$ measurements have a peak value which is the key feature of the cross-correlation measurements and shows the displacement vector at which the best match between $F_1$ and $F_2$ is obtained. In (3.2), $^*$, stands for conjugate sign, since $F$ here is a real value function, $F=F^*$.
Equation (3.2) can be applied in normalized form of;

\[
\tilde{R}_{N12}(\tau) = \frac{\int F_1(x) \cdot F_2^*(x+\tau) \, dx}{|F_1| \cdot |F_2|} \tag{3.3}
\]

which \( || \) stands for norm of \( F \). The normalized cross-correlation value is generally between -1 and +1 when the signal has positive and negative amplitude. In the case of digitized images in which pixel amplitudes are always positive, the cross-correlation estimates change between zero and one.

A digital estimation expression, in normalized form, suitable for employing digital images is:

\[
\tilde{R}_{N12}(i, j) = \frac{\sum_{m=1}^{N} \sum_{n=1}^{N} F_1(m, n) \cdot F_2(m+i, n+j)}{\left( \sum_{m=1}^{N} \sum_{n=1}^{N} F_1(m, n)^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{m=1}^{N} \sum_{n=1}^{N} F_2(m+i, n+j)^2 \right)^{\frac{1}{2}}} \tag{3.4}
\]
3-1-2-Cross-correlation Techniques using FFT.

Since cross-correlation estimation can be defined in spatial domain as the following convolution between $F_1$ and $F_2$,

$$
\hat{R}_{P12}(\tau_x, \tau_y) = F_1(\tau_x, \tau_y) \otimes F_2(-\tau_x, -\tau_y)
$$

its Fourier Transform is equal to the multiplication of the Fourier Transforms of $F_1$ and $F_2$ in the Fourier domain (Brigham E. O., 1974);

$$
\hat{R}_{P12}(m, n) = FT^{-1} [ \hat{F}_1(u, v) \cdot \hat{F}_2^*(u, v) ]
$$

Where the two-dimensional digital Fourier Transform of image $F$ is defined by;

$$
\hat{F}(u, v) = \frac{1}{M \cdot N} \sum_{m=1}^{M} \sum_{n=1}^{N} F(m, n) \cdot EXP[-j \pi (\frac{m \cdot u}{M} + \frac{n \cdot v}{N})]
$$

The cross-correlation estimates have been applied using both (3.4) and (3.6), and there is no difference between these two expression. However, (3.6) is applied because of its programming advantages.
3-2-Implementation:

To test the capabilities of the cross-correlation techniques, a series of image pairs are created. Each image pair consists of two consecutive images $F_1$ and $F_2$. The first image $F_1$ is created by locating particle images randomly in a 32×32 pixels area. These particle images have Gaussian intensity distribution (figure 3.1). Particles in image $F_1$ are shifted to a specific direction and with a specific magnitude. This creates the second image $F_2$ (figure 3.2). $F_1$ and $F_2$ are now one image pair. One can realizes that, as it can happen in a real PIV experiments, some of particle images leave the interrogation area and some new particles enter this area. Cross-correlation computation now can be applied to $F_1$ and $F_2$. The dominant feature of the cross-correlation of this image pair ($F_1$ and $F_2$) is a peak (fig. 3.3). The position of the peak relative to the center of the image, shows the displacement of particles. We have now a good tool to study the capabilities and features of cross-correlation techniques for displacement estimation and can answer questions such as:

a- Does the cross-correlation method give an estimation with reasonable error when initial particle positions are random?
b- What is the least number of particles in the interrogation area to have a correct estimation of the displacements?

c- What is the minimum and maximum displacement values that the cross-correlation can measure?

Since answers to the above questions are also related to the error analysis of displacement estimation and simulation which is discussed in detail in chapter four, the answer to these questions is deferred to that chapter.

3-3-Some practical features:

There are some important practical features when cross-correlation techniques are applied:

a- If the flow velocity is fast and movement of the same particles can not be recorded in two consecutive images, cameras with higher frame rates should be used. Standard cameras can record 30 frames per second which shows that the time interval between two consecutive images is about 33 msec. Cameras with the frame rate of 200 (5 msec time interval between two frames) can be found with a
reasonable price, however the resolution of the image decreases. No matter what the camera frame rate is, the methods which are explained in this chapter can be applied for measurement of displacement and velocity.

b- Cross-correlation methods can be applied to gray scale images directly before thresholding or binary images after thresholding. When there is no need to measure the size and to recognize the shape of objects in the image the cross-correlation methods can be used directly. In this case displacement estimation of particles can be obtained directly from gray level images without thresholding.

c- The method can be applied to either two consecutive frames or fields. A field is odd or even lines of a frame. It has half of resolution of a frame in number of lines. The time interval between two fields is 16.5 msec (for standard cameras) which is half of the time interval between two frames. This provides for the ability to measure particle displacements twice fast as which cross-correlation methods applied to two consecutive frames. However, due to lower resolution of fields compare to the frames, uncertainty of displacement estimates increases.
3-4-Interpolation Scheme for the Peak Finder.

Different methods for finding the position of the peak in the cross-correlation measurements have been examined. The three-point second order interpolation technique provides the most reliable result in all experiments done using the simulation (discussed in detail in the next chapter). Finally, the position of peak in cross-correlation estimates has been obtained with a subpixel accuracy using two one-dimensional interpolation schemes in X and Y directions. In each direction, a second-order curve is fitted to three points of the data. These points are the pixel with peak value of cross-correlation estimates and two adjacent pixels in both sides of it.

The main source of error in displacement estimation, applying cross-correlation techniques is due to using two interpolation curves in both the X and Y directions. This error decreases when the displacement values are integer numbers of pixels. In this case, the exact estimation of displacement and least amount of displacement variance can be obtained. However, in a real experiment it is not expected to have only integer numbers of displacement values and the results are accompanied with errors due to using interpolation
schemes.

Error analysis due to random position of particles is discussed in detail in the next chapter.
In PIV experiments the initial position of particles in the flow field is random. A displacement measuring technique can be reliable if for some specified displacement values in X and Y directions, it can estimate the velocity vector in an interrogation area with reasonable accuracy even if the initial position of particle images were random. In other words, the method should give reasonable variance for an estimated displacement. We would like to see how the estimated displacement values and their variances change as a function of parameters such as particle size, particle number density and displacement values.

In this chapter, it will be shown how an approximate Maximum Likelihood Estimation (MLE) procedure can be applied to study the variance of displacement estimates when cross-correlation techniques are being applied. There are various important parameters that affect the displacement error estimation. These parameters are, particle number density, particle image size, image shift values, and the size of interrogation area. A physical
model is proposed in which all above mentioned parameters are included, a comprehensive study of variations of error when parameters change has been accomplished. The error values which are obtained from the Maximum Likelihood method are compared with the same error values from experiments. These experiments are implemented by applying synthetic images and a simulation discussed in detail.

4-1-Maximum Likelihood Estimation (MLE):

In this section an approximate Maximum Likelihood method is applied to obtain the variance of the displacement estimates.

If it is assumed that the data at each measurement point is statistically independent of the data in every other interval, P, the log of the probability of the data set can be obtained as (Van Trees H. L. 1968):

\[ P = \log p(\hat{R}_n | \tau_s) = \sum_{n} \log p(\hat{R}_n | \tau_s) \]  \hspace{1cm} (4.1)

\( \tau_s \) is the vector of the estimated parameters which its components are two displacement values in X and Y directions. \( \hat{R}_n \) are the estimated cross-correlation
measurements based on one of the forms (3.2) to (3.6) mentioned in the last chapter. Since the Central Limit Theorem indicates that statistics of the correlation function are Gaussian, (4.1) can be written as:

$$P = -\sum_n \left[ \frac{[\hat{R}_{12}(n) - h(n, \tau_s)]^2}{2 \sigma_{\tau_s}^2} - 0.5 \sum_n \log 2\pi \sigma_{\tau_s}^2 \right]$$

(4.2)

where $h(n, \tau_s)$ denotes the model function for cross-correlation in which all important physical parameters in the problem are included and $\sigma_{\tau_m}^2$ is the variance of cross-correlation measurements. Both $h(n, \tau_s)$ and $\sigma_{\tau_m}^2$ will be discussed in detail later.

For an estimation of the error corresponding to the displacement vector, the second derivative of $P$, log of the probability of the data set, with respect to the estimated parameters is taken. This constructs the Fisher matrix in the form of:

$$F_{ij} = \left< \frac{\partial^2 P}{\partial \tau_{s_i} \partial \tau_{s_j}} \right> = \sum_n \frac{1}{\sigma_{\tau_s}^2} \frac{\partial h}{\partial \tau_{s_i}} \frac{\partial h}{\partial \tau_{s_j}} \quad i, j = 1, 2$$

(4.3)

here $<>$ denotes expected value.
The lower limit on the error estimate of parameters can be obtained from the Cramer-Rao bound given by (Van Trees H. L. 1968):

\[ \sigma^2_{i} \geq F^{-1}_{ii} \]  \hspace{1cm} (4.4)

where \( F^{-1}_{ii} \) is the \( i \)th diagonal element of the matrix inverse of \( F_{ij} \). This shows that the expected error of the estimated parameters can never be reduced below the Cramer-Rao bound.

One can get an efficient estimate of \( \tau_s \) for measurements with Gaussian statistics if (4.4) is satisfied as the following equality:

\[ \sigma^2_{i} = F^{-1}_{ii} \]  \hspace{1cm} (4.5)

The Fisher matrix depends only on \( h \), the model function and the expected parameters. Thus it can be used to examine the influence of the parameters on the
expected errors. The variance of estimated displacement is a function of different parameters such as, number and size of the particles, image shift values, size of the interrogation area, image contrast, and particle image density. The effects of the above mentioned parameters on the variance of the displacement estimation is discussed in detail in the next sections.

4-2-Model function \( h(n, \tau_s) \):

From (4.5) and (4.3), we need to introduce a model \( h(n, \tau_s) \), which is a function of the estimated parameters. Since this function is the expected value of the data measurements, here \( h(n, \tau_s) \) is:

\[
    h(\tau) = \langle \hat{R}_{12}(\tau) \rangle 
\]

(4.6)

where \( \langle R_{12} \rangle \) is the expected value of cross-correlation measurements and can be obtained from:

\[
    \hat{R}_{12}(\tau) = \frac{1}{A} \int F_1(x) . F_2^*(x+\tau) \, dx 
\]

(4.7)
Two consecutive images $F_1$ and $F_2$ can be interpreted as the summation of all composed individual particle images,

$$F_1(x) = \sum_p f(x-x_p) \quad (4.8)$$

$$F_2(x) = \sum_q f(x-(x_q+r_q)) \quad (4.9)$$

Each particle image is assumed to have a Gaussian intensity distribution of the form,

$$f(x) = f_0 \exp\left[-\frac{|x|^2}{2\sigma_i^2}\right], \quad (4.10)$$

where $r_p$ and $r_q$ specify the random position of particle images in $F_1$ and $F_2$, $f_0$ is the total photon count and $\sigma_i$ is the rms of the width of the particle image. By substituting (4.8) to (4.10) in (4.7), $<R_{12}>$ is obtained as,
\[
\langle \mathcal{E}_{12}(\tau) \rangle = \frac{1}{N^2} \sum_{n=1}^{N-x} \sum_{m=1}^{N-y} \sum_{p} \sum_{q} \langle f(x_{nm} - x_{p}) \cdot f(x_{nm} - (x_{q} + \tau_{s}) + \tau) \rangle \tag{4.11}
\]

\(\tau_{xn}\) and \(\tau_{ym}\) are the displacement pixels in the X and Y directions. The expected value in (4.11) can be simplified to:

\[
\langle f(x_{nm} - x_{p}) \cdot f(x_{nm} - (x_{q} + \tau_{s}) + \tau) \rangle = \tag{4.12}
\]

\[
\frac{1}{A} \iint f(x_{nm} - x) \cdot f(x_{nm} - (x + \tau_{s}) + \tau) \, dx
\]

when \(p=q\) and:

\[
\langle f(x_{nm} - x_{p}) \cdot f(x_{nm} - (x_{q} + \tau_{s}) + \tau) \rangle = \tag{4.13}
\]

\[
\frac{1}{A^2} \iint f(x_{nm} - x) \, dx \iint f(x_{nm} - (x + \tau_{s}) + \tau) \, df
\]

When \(p \neq q\). \(A\) in (4.12) and (4.13) specifies the image area.
Finally the model function from equation (4.11) is obtained as:

\[ h(\tau) = \langle R_{12}(\tau) \rangle = \]  
(4.14)  

\[
\frac{1}{N^2} \sum_{p=1}^{N^2} \sum_{q=1}^{N^2} < p \sum_{p=1}^{N^p} f_{q_p}^2 \sigma_{q_p}^2 \exp\left[-\frac{(\tau - \tau_{q_p})^2}{4\sigma_{q_p}^2}\right] 
\]

\[
+ \frac{4\pi^2}{A^2} \sum_{p=1}^{N^p} \sum_{(q\neq p)+1}^{N^q} f_{q_p}^2 \sigma_{q_p}^2 f_{q_q} \sigma_{q_q}^2 \rangle 
\]

Where the first and second terms are results when \( p = q \) and \( p \neq q \) respectively. When the particle images in the both \( F_1 \) and \( F_2 \) are the same size, (4.14) can be replaced with the following equality:
$$h(\tau) = (\xi_{12}(\tau)) = \quad (4.15)$$

$$(1 - |\tau_{sm}| \cdot (1 - |\tau_{sn}|) \cdot (\rho f^2 \sigma^2 \pi^2 \ EXP[- \frac{(\tau - \tau_s)^2}{4\sigma^2}])$$

-4 $\rho^2 f^2 \sigma^4 \pi^2$$

Where $\rho$ is the particle number density in the image and can be defined as $<N>/A$, where $<N>$ is the expected value of particle numbers in an image and $A$ is the image area.

4-3-Variance of Cross-Correlation Measurements:

To estimate the displacement variance, one needs $\sigma^2(R_{12})$, the variance of cross-correlation measurements. To compute $\sigma^2(R_{12})$, one can start from the calculation of the covariance which has the following definition (Saleh B. 1978):
\[ \Lambda_{t_1} = (R_{12}(\tau) \cdot R_{12}(\tau')) - (R_{12}(\tau)) \cdot (R_{12}(\tau')) \] (4.16)

\( R_{12}(\tau) \) and \( R_{12}(\tau') \) are cross-correlation measurement matrices at the two arbitrary locations of \( \tau \) and \( \tau' \) in the colorogram. Substituting (4.8) and (4.9) in (4.16) provides:

\[ \Lambda_{t_1} = \frac{1}{N_d} \sum_n \sum_m \sum_k \sum_l (F_1(x_{nm}) \cdot F_2(x_{nm} + \tau)). \] (4.17)

\[ F_1(x_{kl}) \cdot F_2(x_{kl} + \xi) \]

Where the expected value in (4.17) can be written in the following form by applying (4.10):

\[ (.) = \sum_s \sum_p \sum_q (f(x_{nm} - x_p) \cdot f(x_{nm} - (x_q + \tau_q) + \tau)). \] (4.18)

\[ f(x_{kl} - x_s) \cdot f(x_{kl} - (x_u + \tau_s) + \xi) \]
The covariance can be simplified to:

$$\Lambda_{\tau} = \rho \cdot \tau^4 \cdot \sigma^4_{\tau} \cdot \pi^2 \cdot (1 - \frac{|\tau_{\text{sn}}|}{N}) \cdot (1 - \frac{|\tau_{\text{sn}}|}{N}) \cdot \text{(4.19)}$$

$$\exp[-\frac{(\tau - \tau_s)^2}{4 \sigma^2_{\tau}} + \frac{2 \cdot \rho \sigma^2_{\tau} \cdot \pi \cdot \exp[-\frac{(\tau - \tau_s)^2}{8 \sigma^2_{\tau}}]}{4}] + 2 \cdot \rho \sigma^2_{\tau} \cdot \pi \cdot \exp[-\frac{(\tau - \tau_s)^2}{8 \sigma^2_{\tau}}] +$$

$$\exp[-\frac{(\tau + \tau_s - 2 \tau)^2}{8 \sigma^2_{\tau}}] + 4 \exp[-\frac{(\tau - \tau_s)^2}{4 \sigma^2_{\tau}}] +$$

$$4 \exp[-\frac{(\tau - \tau_s)^2}{4 \sigma^2_{\tau}}] + 64 \rho^2 \cdot \sigma^4_{\tau} \cdot \pi^2$$

where the sums have been approximated by integrals. Finally the variance matrix $\sigma^2_{\text{sn}}$ of cross-correlation measurements is obtained from (4.19) by substituting $\tau = \tau'$. 
With specifying $h(\tau)$ model function from (4.15) and $\sigma_n^2$ from (4.19), $F_{ij}$ elements of the Fisher matrix can be computed from (4.3) and an approximate estimation of displacement variance can be obtained from (4.5). This theoretical error estimation is the error estimation from Maximum Likelihood method or computation.

4-4-Simulation:

In order to assess the numerical values for variance of estimated displacements and its dependency on various parameters in PIV experiments, when cross-correlation methods are being applied, a series of synthetic images are constructed. This simulates the real particle images recorded in two consecutive frames, and the effect of each parameter on the variance can be studied.

The process of displacement error estimation includes the following steps:

1- A Poisson random number generator specifies the number of particles in an image when expected values of particle number is known.

2- The position of particles in image $F_1$ can be determined by using two random number generator which
specifies two random values for X and Y, which is the center of the particle image.

3- Particle images with Gaussian intensity distribution are located at random positions specified in the last step. Each pixel amplitude has been computed by two-dimensional integrating of the Gaussian intensity distribution function over that pixel area. Because of the Gaussian intensity distribution, with increasing the distance of a pixel centroid from the center of the particle image, the pixel amplitude decreases. Particle image diameters can be chosen with a real or integer number of pixels.

4- To create image $F_2$, particle images in $F_1$ have been shifted with known values of displacements in the X and Y directions.

5- The cross-correlation estimates of a pair of constructed images $F_1$ and $F_2$ were computed applying a Fast Fourier Transform. This provides estimated displacement values in the X and Y directions from the position of cross-correlation peak as described in the last chapter.
6. Since the diameter of particles, particle number density and displacement value are specified, model function $h(r)$ and variance of the cross-correlation measurements $\sigma^2_{x_n}$ are computed using (4.15) and (4.19).

7. Information obtained from previous steps is used to compute Fisher matrix (4.3) and the variance of the displacement values from (4.5).

8. Steps 1 to 7 are repeated one hundred times. The mean and variance of the displacement values is obtained from cross-correlation measurements (simulation) for these one hundred realizations. The mean and variance of the estimated variance from the Fisher matrix is also computed (computation) for these one hundred samples.

One hundred samples are chosen for the estimation of the variance because there is no significant change in the results when the number of samples is more than 100.

We can summarize the following important characteristics of the PIV experiment which is modeled in the simulation:
a- The number of particles in an image is random, however, the ratio of the expected value of the particle numbers ($<N>$) divided by the interrogation area, which is particle number density, is known.

b- The position of particle images in an interrogation area is random.

c- The size of the particle images can be a real or integer number of pixels.

d- The displacement of the particles can be a real or integer number of pixels in both the X and Y directions.

e- Particle images can leave/disappear from $F_2$ and enter/appear into it, i.e. unpaired particle images can exist.

f- The particle images can be cut by edges of the interrogation area and only some part of them remain in the second image pair ($F_2$).

g- Particle images can overlap.
4-5-Effects of Various Parameters on displacement error:

It is apparent from (4.3), (4.5), (4.15), the shift estimated variance depends on the particle number density, the size of the particle images, the displacement values, and the size of interrogation area. The effect of each parameter on the variance is studied.

A large number of experiments applying the simulation show that the correlation coefficient of the displacement estimates in the X and Y directions change from the minimum value of (1.67E-3) to the maximum value of (1.6E-1). Therefore these two measurements can be assumed to be independent. Since the estimated displacement values in both directions show the same behavior when the particle number density \( <N> \), particle image diameter \( D \), and the shift values vary, the variation of displacement estimates in the X direction will only be discussed.

In the following sections, it will be shown how the variation of the various mentioned parameters affect the variance of the estimated displacement. Since no simple form for the Fisher matrix was possible, the expected error is computed numerically. In what follows, these
theoretical errors will be compared to errors measured using the simulation.

4-5-1-Random Position of Particle Images:

Since the position of particles in the flow field and the laser sheet is random, the question that arises is, whether the correct and same displacement vectors can be found if the position of particle images change from one image to another, while the number, size, and all other parameters are kept unchanged. In methods like Particle Tracking, the random position of particles can result in incorrect displacement values and false velocity vectors (Wernet M. P. 1989).

In this section the important parameters in displacement error analysis are kept unchanged and the results if the initial position of particle images is random are studied.

Figure 4.1 illustrates the distribution of estimated shift values obtained from the cross-correlation measurements (simulation) for 100 samples. The true shift value is 5.25 pixels in the X direction, the particle image diameter is one pixel, the window size is 32 pixels, and the particle number density is (10/1024) for all one hundred samples. The min. , max. , mean and
variance of the displacement estimated values are (5.075), (5.75), (5.274), (0.0298) respectively.

Extensive runs of the simulation show that the cross-correlation method can be applied to estimate the displacement of particle images and this method is reliable when the initial particle image positions are random. Furthermore the error of the estimated displacements is less than a pixel in all displacement measurements.

We can now analyze and compare the displacement error obtained from Maximum Likelihood Estimation (MLE) method and Cramer-Rao lower bond criteria (computation), with the above results which have been obtained from simulation.

Figure 4.2 shows the distribution of the estimated error using the Fisher matrix for 100 samples. The min., max., mean and variance of estimated error are (0.0646), (0.1234), (0.093), (0.0001) respectively.

The variance of the estimated variance from the computation is low, and the mean value of 100 samples is used to compare the displacement variance from the computation with the variance of the estimated displacement from the simulation.
4-5-2-Number and Diameter of Particles:

The number of particle images in an interrogation area of an image is not constant. During a PIV experiments this number in an interrogation area can change in each instant. However, the particle number density \( \langle N \rangle / A \) is usually known and can be controlled by the experimenter. Therefore \( \langle N \rangle \) is assumed to be known. A Poisson random number generator is used to determine the number of particles in an image when the expected value of that number is known. Fig 4.3 shows the distribution of the number of particles for one hundred samples when the expected number of particles \( \langle N \rangle \) is ten.

In PIV experiments, because the particles are carried by the flow field, they leave the interrogation area in the second exposure and new particles move into that area. From figures 3.1 and 3.2 we could observe that after shifting particle images in F1, some of them leave the interrogation area. Some new particle images also have been added to the F2 and located randomly to model those particle images which enter the interrogation area. Figure 4.4 illustrates the distribution of the remaining particle images in F2 after shifting them with
the specified magnitude in the X and Y directions before adding new particle images.

Since a Poisson random number generator is used to specify the number of particles in the first interrogation area, when the expected number of particle images in F₁ is low, it is possible to have only one or no matching particles in the second image F₂. In this case, all methods which apply the particle images including the cross-correlation technique, fail in detecting the correct displacement values. Indeed this effect is apparent in several of simulation results. With increasing number of particle images in F₁, the cross-correlation peak is more distinguishable from noise, and the uncertainty of the peak position measurement decreases. Experiments show that for at least 8 particle images in F₁, the method rarely fails to find correct displacement values.

The characteristic of the cross-correlation method that increasing the number of particles in the interrogation area increases the chance of correct estimation of the displacement values is not observed in the Particle Tracking method, where the number of detected false vectors increases as the number of particles increases (Wernet M. P. 1989).
When particle number density changes, the displacement error also varies. The variation of the displacement error with \(<N>\) in the interrogation area, shows different behaviors for different particle diameters.

When particle diameter is one pixel, with increasing particle number density the displacement error decreases (figure 4.5). This has been observed for both displacement errors obtained from simulation and from the MLE theory.

There are three curves in figure 4.5, the one specified by \((S)\), is obtained from cross-correlation measurements (simulation). Each point in this curve has been obtained by computing the variance of 100 estimated displacement values. The square root of variance (standard deviation) is the number shown as the displacement error in the figures.

The second curve with the label \((C)\) is the error estimation obtained from the MLE theory. Each point on this curve is the mean value of error, computed by the MLE method for 100 samples. The third curve named CT shows the estimated displacement error if the true value of displacement is used to compute the error from the MLE method. The two last curves are very close.
A smooth variation of error obtained from the MLE method has been observed for all particle diameters bigger than one pixel (figures 4.5 to 4.9). This is the practical range of particle diameters which is studied in digitized images.

The third curve (CT), also shows the error obtained from the MLE theory when the displacement value is correctly estimated. With increasing particle number density the difference between (C) and (CT) curves decreases.

When the particle diameter is three pixels, the slope of the both (S) and (C) curves decreases. When \( <N> \) is more than ten, these two curves become flat.

For five pixel particle diameter and when \( <N> \) is bigger than ten, the MLE method shows a mild increase in error with increasing \( <N> \).

Error obtained from both the simulation and the MLE method increases when particle images become bigger (figure 4.8). This is due to increasing the chance of particle image overlapping in the interrogation area and/or particle images being cut by the interrogation area edges.

When the particle diameter is less than one pixel, the estimated displacement values from the MLE method is higher than the simulation and the error oscillates
because of large variations in the estimated displacement values. The displacement values for the above results are 5.25 and 3.75 pixels in the X and Y directions respectively. The window size was also 32×32 pixels.

The particle image diameter can be changed to a real number of pixels. Figure 4.10 illustrates the variation of error when the particle diameter is 3.5 pixels. New displacement values (2.62 and 1.78 pixels) are used and applied to study the effect of window size on the displacement error.

4-5-3-The Size of Interrogation Area and Displacement Values:

Window size is an important parameter which directly affects the velocity measurement. This size has to be determined with regard to time and the length scale of the flow field. The maximum allowed displacement value is one half of the window size to satisfy the Nyquist frequency criteria. For a 32×32 window size, the maximum displacement estimation is limited to sixteen pixels. In practice more reliable results can be obtained if the maximum displacement value is even less than sixteen.
pixels, otherwise there will be many unpaired particle images in the two consecutive windows, resulting in erroneous particle displacement values.

Better resolution can be obtained if a smaller window size is applied because in the cross-correlation technique, the particle displacements are estimated from the average of all particle image displacements in the whole window. When there is a velocity gradient in the flow field, smaller window sizes are recommended for more accurate results.

Window size affects the displacement error. Figure 4.11 shows the variation of displacement error as $<N>$ and size of the interrogation area change. In these tests, two pixel diameter particles are moved 2.62 and 1.78 pixels in the X and Y directions, respectively. Decreasing the window size to 16×16 pixels, increases the displacement error to more than $\%70$, because of the increased chance of particle images leaving the interrogation area or being cut by the interrogation area edges.

A lower limit for the displacement error applied to the cross-correlation techniques is sought. The Cramer-Rao lower bond specifies that there is a lower limit for error corresponding to the displacement estimates. The
displacement values obtained from cross-correlation methods are not accurate for displacement values less than the error, specified for certain conditions of particle size, number density and size of the interrogation area.

There are some limitations for displacement values when other methods are applied. Particle tracking works well if the particle movements are smaller than the spacing of those particle images (Wernet M. P. 1989). Extensive tests with synthetic images show that there is no such limitation with the cross-correlation technique which can be used even if the displacement of the particles is larger than their spacings.

To study the effect of displacement values on the estimated error, a series of image particles were shifted with different displacement values from 0.7 to 7 pixels. Both simulation and MLE show that the displacement error increases when shift values increase. The MLE method predicts that the increasing error with increasing displacement is almost linear with a mild slope. However, simulation illustrates sharper variations in displacement error. This behavior has been observed for all particle diameters. Figure 4.12 shows the variation of displacement error with shift values. During these
tests, the expected number of particle images has been kept unchanged and equal to 10.
CHAPTER FIVE

CONCLUSION:

The measurement of the shape, size, concentration and velocity of particles or bubbles moving with a fluid flow is studied by processing computer stored images. A single image of the particles illuminated by sheet lighting is used to determine the shape and size of objects including irregularities and holes.

The process of shape and size detection is broken down into three steps, thresholding, particle identification and boundary detection.

To summarize the issues related to shape and size recognition of objects (particles or bubbles):

a- No information about the objects and their shapes or sizes are needed and the boundary detecting method can be applied to objects with arbitrary, unexpected, irregular shape.
b- The boundary of holes can be detected. The method can also be used to find several holes inside each other (hole in hole).

c- No limitations on the number of objects and holes exist.

d- If the region of interest of two objects do not overlap, the boundary detecting method performs particle identification (PI) automatically, otherwise PI facilitate the identification of objects by giving them unique numbers.

f- After organizing the boundary pixels in order, a suitable interpolation scheme is be applied to different parts of the boundary to describe it in a closed form and analytically.

g- The area, circumference and centroid of each object can be found.

h- Since after particle identification all objects have a unique number, one or some of the objects can be chosen for boundary detecting.
i- The boundary detecting method is not restricted to binary images. This method also works well on output images from the PI process.

Since there is no unique distinguishable feature for each particle image in two consecutive frames, a statistical analysis is applied to find the best match between particles in two images.

Cross-correlation technique using a Fast Fourier Transform algorithm is applied to consecutive images of the particles to determine the displacement vectors and velocities. This technique detects the average velocity of moving objects in an interrogation area of an image.

Different methods for finding the position of the peak in the cross-correlation measurements, to subpixel accuracy have been examined. The three point second order interpolation technique provides the most reliable result in all experiments.
To estimate the displacement measurement error two methods are applied. The first method is a computer simulated experimental flow with particles, to estimate the displacement error obtain from cross-correlation methods. A computer program is developed to handle:

- a random number and random initial position for the particles in the interrogation area when the expected particle number density is known,
- particle images that overlap, enter, leave or get cut by the edges of interrogation area,
- particles of a non-integer number of pixels in size and displaced by a non-integer number of pixels in either direction within the light-sheet.

A large number of simulation runs show that cross-correlation methods are able to estimate the correct displacement values, even when there is only a $\%30$ match between particle image pairs in the interrogation area of two consecutive frames.

The results also show that the cross-correlation method rarely fails to give the correct displacement values if the number of particle images in the
interrogation area is larger that eight.

The above mentioned characteristics of the cross-correlation methods illustrate that these methods are reliable to be applied to measure the particle image displacements when initial position of particle images is random.

An approximate Maximum Likelihood Estimation (MLE) procedure is also developed to predict theoretically the error corresponding to displacement measurements obtained from cross-correlation techniques.

To summarize issues relevant to displacement error estimated from the simulation and from the MLE;

\begin{itemize}
  \item[a] The error of displacement measurement obtained by cross-correlation techniques is less than a pixel.
\end{itemize}

For particle image diameters from one to five pixels and interrogation area, A=32x32 pixels, the range of the displacement error due to random position of particle images are:

\begin{itemize}
  \item 0.09 to 0.3 from simulation,
  \item and,
  \item 0.06 to 0.26 pixel from MLE.
\end{itemize}
b- An analytical expression for the displacement error (4.3, 4.5, 4.15, 4.19), showed that error corresponding to displacement measurement is a function of:

* Size of particle image
* Image shift values
* The size of interrogation area
* Particle image density

c- For a specific number of particle images in an interrogation area, when particle diameter increases, displacement error increases because of increased particle image overlapping and particle image truncation by interrogation area edges.

d- The window size has to be determined with regard to the time and the length scale of the flow field.

The maximum allowed displacement value in practice is about one third of the size of the interrogation area.

The displacement error increases when shift values increase. This occurs because more particle images leave the interrogation area and the displacement values are
determined based on matching fewer particle image pairs in two consecutive frames.

Better resolution in the velocity measurement can be obtained if a smaller window size is applied. However, decreasing the window size increases the displacement error because of the increased chance of particle images leaving the interrogation area or being cut by the interrogation area edges.

The displacement values obtained from cross-correlation methods are not accurate for displacement values less than the error, specified for certain conditions of particle size, number density and size of the interrogation area.

The variation of the displacement error with increasing $<N>$ shows various forms for different particle image diameters.

For $D < 3$ pixels, both the simulation and the MLE show decreasing in error.

For $5 < D < 3$ pixels, both the simulation and the MLE show that error is almost constant.

For $D > 5$ pixels, the displacement error obtained from simulation slightly increases. In this case
displacement error obtained from the simulation is almost constant.
This illustrates that with increasing <N> more than about 10, and for particle diameter larger than about five pixels, the cross-correlation measurements do not provide a better estimate for the position of the cross-correlation peak.

The above given results can be applied to assess the reliability of the displacement measurements given by the cross-correlation methods. They also provide a good guide to design an experiment in which the lowest displacement error is sought.
LIST OF REFERENCES


73


1.1- An experimental setup for object size and velocity measurement using a laser sheet and camera.
2.1 - Histogram of an image composed of bright particle images before smoothing.
2. Histogram of an image composed of bright particle images after smoothing.
2.3 - Histogram of an image drawn based on estimated parameters.
2.4- Region Of Interest (R.O.I) for an unexpected irregular shape boundary.
2.5- Several adjacent objects with irregular boundaries.
2.6- Input binary image.
2.7- Particle Identification Scheme (PIS).

(TOP-DOWN PASS)
2.8- Particle Identification Scheme (PIS).

(BOTTOM-UP PASS)
2.9- Organizing the objects' unique numbers.
2.10- Boundary detecting scheme.
2.11- Detected boundary pixels for three objects with irregular boundaries.
2.12- Detected boundary pixels in character format.
3.1- Randomly positioned particle images with Gaussian intensity distribution in the first interrogation area. 
D=2 pixels, A=32x32 pixels
3.2- Particle images in figure 3.1 have been shifted 5.25 and 3.75 pixels in X and Y directions respectively. Some particles have left the interrogation area and some new particles have entered to this region.
3.3- Cross-correlation of two interrogation areas shown in figures 3.1 and 3.2. Displacement values are DX=5.25 and DY=3.75 pixels in the X and Y directions.
4.2. Distribution of displacement error obtained from Maximum Likelihood Estimation method for 100 samples. 

- $D = 1$ pixel, $A = 128$ pixels, $< N > = 10$
- $D = 1.25$ pixels, $A = 32^2$ pixels, $< N > = 10$
- $D = 0.25$ pixels, $A = 32^2$ pixels, $< N > = 10$
- $D = 0.125$ pixels, $A = 32^2$ pixels, $< N > = 10$

Min = 0.0046, Max = 0.1234, Mean = 0.0343, Variance = 0.0001
4.3- Distribution of number of particle images in the first interrogation area created by a Poisson random number generator for $<N> = 10$ and 100 samples. $Min = 5$, $Max = 19$, $Mean = 10.35$
4.4- Distribution of number of particle images in the second interrogation area for 100 samples. Some particle images have left the interrogation area after shifting them 5.25 and 3.75 pixels in X and Y directions respectively.
4.5 Variation of displacement error with particle image diameter and expected value of particle image number. Number of samples used for each measurement point is 100.

- S = Simulation
- C = Computation
- C1 = Computation with true displacements

DX = 5.25 pixels
DY = 3.75 pixels
D = 1 pixel
A = 32x32 pixels
4.6- Variation of displacement error with particle image diameter and expected value of particle image number. Number of samples used for each measurement point is 100.
4.7 - Variation of displacement error with particle image diameter and expected value of particle image number. Number of samples used for each measurement point is 100.
4.8- Variation of displacement error with particle image diameter and expected value of particle image number. Number of samples used for each measurement point is 100.
4.8 Variation of displacement error with particle image diameter and expected value of particle image number. Number of samples used for each measurement point is 100.
4.10- Variation of displacement error with expected value of particle image number. Number of samples used for each measurement point is 100.
4.11 - Variation of displacement error with expected value of particle image number for two window sizes. Number of samples used for each measurement point is 100 for A=32x32, and 150 for A=16x16.
4.12- Variation of displacement estimation error with displacement values from simulation and computation. Number of samples used for each measurement point is 100.