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The lattice approaches for pricing path-dependent mortgage-related products

Liou, Ching-Pin, Ph.D.

Case Western Reserve University, 1994
THE LATTICE APPROACHES FOR PRICING PATH-DEPENDENT
MORTGAGE-RELATED PRODUCTS

by

CHING-PIN LIOU

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy

Thesis Adviser: Dr. Peter Ritchken

Department of Operations Research
CASE WESTERN RESERVE UNIVERSITY
January, 1994
CASE WESTERN RESERVE UNIVERSITY

GRADUATE STUDIES

We hereby approve the thesis of

CHING - P.N. LIOU

candidate for the PH. D

degree.*

(signed)  

(Kitchen)

(chair)

(\[Signature\])

[Signature]

Hamilton Commerce

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Lattice Approaches for Pricing
Path-Dependent Mortgage-Related Contracts

Abstract
by
Ching-Pin Liou

Mortgages can be viewed as risk-free assets plus various contingent claims, which are frequently modeled as options. Based on arbitrage arguments, and the characteristics of the particular mortgage analyzed, one can derive a partial differential equation for the mortgage. The exact form of the valuation equation and the methods required to solve it depend on the type of stochastic process used to model the underlying uncertainties. In general, no closed-form solutions can be obtained to the partial differential equations for these mortgage-related products. Numerical methods must be employed. Numerical methods for pricing contingent claims may be classified as forward based or backward based. For example, the most common technique, Monte Carlo simulation, is forward based. However, forward based methods cannot be applied to all types of mortgage-related products, because some option features embedded in these contracts, such as termination of the contract before it matures, depend critically on assessing future cash flows. Backward based methods, such as the lattice approaches, overcome such difficulty. In particular, the lattice approaches can be readily modified to incorporate additional option like features. However, backward
based methods have difficulty in dealing with path dependence since cash flows depending on state variables occurring earlier in time cannot be determined.

The purpose of this thesis is to develop lattice-based models for pricing the following path-dependent mortgage-related products: GNMA pass-throughs, index amortization swaps, lookback mortgages, and adjustable-rate mortgages. For these products, we show the history of the process, relevant for pricing, can be captured by a single additional state variable. Specifically, this additional statistic, together with the current interest rate is sufficient for capturing all information along the path. The usual single variable lattice based models are then adapted to handle two state variables and dynamic programming is used to obtain values.
In memory of my Yeh-Yeh (grandfather) and Neh-Neh (grandmother) for their love and integrity.
Dedicated to my mother, Song, Ere-May, Father, Liou, Yung-Kang, my mother-in-law, Lu, Yu-Chung, father-in-law, Miao, Yung-Shu, and my smart, persevering, and deferential wife, Miao, Fang-Lei.
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CHAPTER 1

INTRODUCTION

Mortgages represent the largest single sector of the US debt market. Total mortgage debt outstanding exceeds $4 trillion dollars. Daily trading alone runs in the billions of dollars. Primarily, due to the importance of housing in the US economy, numerous mechanisms have been established to facilitate credit to this sector. The most important method by which this is accomplished is securitization, where individual mortgages are pooled together. These pools are then used as collateral to issue mortgage-backed securities (MBS). These securities, mostly chartered by or issued from three major government (or government-like) agencies, namely Government National Mortgage Association (GNMA), Federal National Mortgage Association (FNMA), and Federal Home Loan Mortgage Corporation (FHLMC), comprise the major sector of the secondary mortgage market. Often times, these MBSs are themselves used as collateral again to issue collateralized derivative securities (CDSs) spanning a wide maturity and risk propensity spectrum. As a result of this repackaging, the complexity of mortgages, MBSs, and CDSs can appear quite daunting. Nonetheless, most of the products can be analyzed as variations on the basic fixed rate mortgages (FRM) and adjustable rate mortgages (ARM).
Homeowners of typical mortgages have the right to prepay their loans at any time. This American-call option is referred to as a prepayment option. In addition, homeowners have a default option as well. Specifically, by discontinuing monthly payments, homeowners can "put" their home back to the mortgagee. The exercise price of this option corresponds to the actual value of the property. Of course, additional option features may be present. In analyzing ARMs for example, periodic caps and floors, together with lifetime caps, make the analysis quite difficult. Specifically, the value of such mortgages may well depend on the entire path of interest rates.

Recently, substantial innovations have been developed to analyze the values of the mortgage-related products. Specifically, option pricing methodologies have come to be widely applied. This approach was developed originally to evaluate stock options, hence its name; but in fact it can be applied to any security whose value is affected by fluctuating economic variables. The starting point is a stochastic process model of underlying variables. In mortgage analysis, the key variable is the short-term interest rate. Then the mortgages can be viewed as risk-free assets plus various contingent claims, which are frequently modeled as options. For instance, the right to buy back or call the mortgage at par can be modeled by a call option. The right to sell or put the house to the lender at a price equal to the market value of the mortgage can be modeled by a put option. Then, based on arbitrage arguments, and the characteristics of the particular mortgage, a partial differential equation for the
mortgage can be derived. The exact form of the valuation equation and the methods required to solve it depend on the type of stochastic process used to model the underlying uncertainties.


In general, no closed-form solutions can be obtained to the partial differential equations for these mortgage-related products. Numerical methods must be employed. Unfortunately, since values of most mortgage-related contracts are dependent on the evolution of underlying uncertainties over time, or in other words they are path-dependent, they have been very difficult to price. As a result, Monte Carlo simulation is the principal solution technique.

The purpose of this thesis is to develop new lattice-based models for pricing a variety of mortgage related products which have the property that their values depend on the entire history of the underlying variables. To date, lattice-based methods have been lacking from the literature, because no efficient algorithms for computing prices
have been designed. The algorithms developed in this dissertation can be easily modified to price a variety of mortgage-related contracts.

The thesis is organized as follows. Chapter 2 introduces the mortgage products and reviews the literature on pricing mortgages and mortgage-related products. Chapter 3 begins by pricing straight amortization and callable amortization bonds. Such securities are similar to FRMs and GNMA pass-throughs, yet easier to analyze since cash flows are based on financial decisions alone. We use these models as starting points to develop efficient pricing models for FRMs and GNMA pass-throughs, the cash flows of which are based not only on financial decisions but also private information. Chapter 4 prices index amortization swaps. These contracts belong to a class of "synthetic" mortgages, and are one of the most rapidly growing contracts with notional principal exceeding one hundred billion dollars. Chapter 5 prices lookback mortgages. Such mortgages provide the homeowner with cash obligations linked to the minimum interest rate that occurs over the lifetime of the contract. Chapter 6 prices ARMs with a variety of option features, such as life and yearly caps/floors. Chapter 7 gives conclusions and considers extensions that can be done from the algorithms developed in this thesis.
CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

This chapter reviews the literature in pricing mortgages and their related products. We begin by reviewing these products. We then turn attention to the pricing literature. In particular, the procedure for establishing the fundamental partial differential equation for fixed-rate mortgages, when the stochastic processes of the underlying uncertainties are specified, is reviewed. Usually, numerical procedures for solving partial differential equations are required. In Section (2.4), we focus on some of these procedures. We also review an efficient lattice-based approximations on continuous time interest rate process. Lattice-based approaches are well-known in the pricing literature in finance, and will be the key workhorse in the following chapters. Section (2.5) gives conclusions.

2.2 An Overview of Basic Mortgage Products

By definition, a mortgage is a pledge of property to secure payment of a debt. The mortgage gives the lender (mortgagee) the right to foreclose the loan and seize the property in order to ensure that the loan is paid off if the borrower (mortgagor) fails to make the contracted payments.
In this section, we introduce various mortgages and mortgage-related products that are available in the market. We classify these products into three categories, namely, conventional mortgages, mortgage-backed securities, and synthetic mortgages as shown in Figure (2.1). In the following discussion, we give the details associated with each of these categories.
Mortgages and Mortgage-Backed Securities and Related Products

Figure 2.1

- Principal Only
- Interest Only
- Collateralized Mortgage Obligations
- Strip
- CMO
- Synthetic
- Swaps
- CMO Swaps
- Collateralized Derivative
- Pass-Throughs
- Others
- Adjustable-Rate
- Fixed-Rate
- Government, Fannie, Freddie, FHA, VA

Mortgage Products
2.2.1 Conventional Mortgages

There are two main conventional mortgage types, namely fixed-rate mortgage (FRM), and adjustable-rate mortgage (ARM) in the mortgage market. Many other mortgages are simply variations of them.

FRMs allow the homeowners to make equal payments over an agreed-upon period of time, called the maturity or term of the mortgage. The frequency of payment is typically monthly, and is usually referred as fixed-level monthly payment. The prevailing term of the mortgage is 20 to 30 years. Payment in excess of the fixed-level monthly payment is called prepayment. Prepayment occurs for some non-financial reasons such as relocation, change of job, death, or divorce. The effect of prepayment is that the cash flow from a mortgage is uncertain. This is true not only for FRMs, but for all mortgages. Because of the critical importance of prepayment in the valuation of mortgages and mortgage-related products, we discuss prepayment dynamics in more detail in Section (2.3).

FRMs suffer from a serious shortcoming, the so-called "mismatch" problem. This problem exists because mortgages, which are long-term assets, have largely been financed by institutions that obtained their funds through deposits, primarily of a short-term nature. If interest rates rise very rapidly, then the institutions may experience losses. In order to shift the interest rate risk from the institutions to the homeowners, a solution is to redesign the structure of FRMs to produce a new mortgage contract to have returns that match the short-term market rates. One
instrument that satisfies this requirement, and has won considerable popularity is ARM. The details of the contractual features of an ARM will be discussed in Chapter 6.

Other conventional mortgages, such as graduated payment mortgages, reverse mortgages, share appreciation mortgages, and balloon mortgages etc., are simply variations of FRMs and ARMs in contractual features.

2.2.2 Mortgage-Backed Securities

As just discussed, the conventional mortgages are exposed to prepayment risk that the homeowner will prepay the mortgage at any time.

To reduce the prepayment risk, a lot of so-called mortgage-backed securities (MBSs) have been created. The most common one is pass-through. The following scenario explains a pass-through. Suppose some entity purchases and pools 100 identical mortgage loans together at their origination. These loans are used as collateral to issue a security, with the cash flow from that security reflecting the cash flow from the 100 loans, as shown in Figure (2.2).
Figure 2.2
Structure of pass-through security and its cash flows.

Suppose that 100 units of this security are issued. Each unit is entitled to 1% of the cash flow. The security created is called a pass-through. If we assume that the systematic risk associated with prepayment is insignificant, it is obvious from this financial innovation that each security, with same amount of investment in the individual mortgage loan, is incurred only 1% of the prepayment risk. In addition, the liquidity of a pass-through is considerably better than that of an individual mortgage loan.

Mortgage loans which are included in a pool to issue pass-throughs is said to be securitized, and the whole process is referred as securitization.
An investor in a pass-through is still exposed to the total prepayment risk associated with the underlying mortgage loans. Securities can be further created, in which investors do not share prepayment risk equally. This can be achieved by distributing the cash flows on a prioritized basis, not a pro rata one. As an example, suppose four classes (A, B, C, and D) of bonds are created from the collateral of 100 loans as shown in Figure (2.3). For each of the four classes, there will be units representing a proportionate interest in a class. The distribution rule for cash flows is that class A is entitled to all of the principal payments until its par value is paid off.

![Diagram of mortgage classes and cash flows]

**Figure 2.3**

Structure of collateralized mortgage obligations security and its cash flows.
Then, class B receives all principal payments until it is retired, etc. Interest payments for these classes are based on the amount of par value outstanding. Such MBS is one of the collateralized derivative securities (CDS) and is referred as collateralized mortgage obligations (CMO). The collateral may be either pass-throughs or mortgage loans. The process of using pass-throughs to create CMOs is called resecuritization.

By redirecting the cash flows, classes of bonds have been created that satisfy the asset/liability objectives of certain institutional investors. For example, class A absorbs prepayments first, then Class B, etc.. The result of this is that Class A will effectively be a shorter-term security than the other classes. Certain institutional investors may be attracted to the different classes given the nature of their financial positions.

Another type of CDS is the so-called stripped MBS, where principal and interest are divided among two classes unequally. An extreme case is that one class is to receive all of the principal (principal-only security) and the other class all of the interest (interest-only security). Since the collateral for stripped MBS is pass-throughs, this is another example of resecuritization. For further discussion on these products, see Fabozzi and Modigliani (1992).

2.2.3 Mortgage Index Swaps

Many institutional investors have been trying to reduce prepayment risk. Recently a new mortgage-related product, named mortgage-indexed swap (MIS), has emerged to further decrease the prepayment risk. This contract has similar cash flows
to ordinary pass-throughs, but having interest payments based on a hypothetical balance (or notional principal) that is agreed upon at the origination.

In general, there are two kinds of MIS, namely collateralized mortgage obligation (CMO) swap, and index amortization swap (IAS). In a CMO swap, the amortization of notional principal is linked to prepayment speeds on mortgage securities, whereas, in an IAS security, the amortization of notional principal is based on some interest rate index designed to mimic the prepayment characteristics of mortgage securities. We will be discussing pricing issues of IAS in Chapter 4. Specific contractual features will be introduced there.

In this dissertation, we shall be concerned with pricing GNMA pass-throughs, one of the MIS products, ARM-related contracts, and IASs. However, the models developed can easily be adapted to handle all of the above products.

2.3 Pricing Mortgage Contracts

Pricing mortgage contracts, in fact, involves three phases. First, the underlying uncertainties need to be modeled; second, specific pricing models need to be formulated; third, appropriate solution techniques to get the mortgage prices have to be identified.

In this section, we discuss each of the above three phases. Then we illustrate these three phases by considering the basic fixed-rate mortgage. Other mortgage contracts can be analyzed along similar lines.
2.3.1 Modeling the Underlying Uncertainties

It is well-known that, when pricing mortgage-related contracts, the most important aspects that need careful attention are the interest rate process, the prepayment process, and for some mortgages, the market value of underlying property. (Hendershott (1987), Kau, Keenan, Muller and Epperson (1991), and Schwartz and Torous (1989)).

(a) Interest Rate Processes

The value of a mortgage clearly fluctuates with interest rates. As interest rates increase, the present value of all future obligations decrease, pushing the value of the mortgage down. In addition, the refinancing decision depends critically on how interest rates have changed.

Modeling interest rate behavior is non-trivial. Specifically, it is necessary to model the behavior of the entire term structure of interest rates in such a way that bonds of differing maturities cannot be meshed together to create riskless arbitrage opportunities. The traditional approach used by researchers to model the term structure involves starting with a plausible stochastic process for the short-term rate, $r$, and exploring what the process implies for bond prices and option prices. In a number of the models that have been developed, the process for $r$ is of the form

$$dr = m(r)dt + \sigma(r)dZ$$

(2.1)
where $m(r)$ and $\sigma(r)$ are instantaneous drift, and instantaneous standard deviation. $dZ$ is the Wiener increment. One of the most celebrated models of the term structure is given by Cox, Ingersoll and Ross (CIR (1985)). They established a general equilibrium model of the economy under which all forward rate evolutions are linked to the instantaneous spot interest rate. The dynamics of the spot rate were shown to be

$$dr = \kappa(\mu - r)dt + \sigma \sqrt{r} dZ$$

(2.2)

Here $\kappa$ is the speed of adjustment factor, called the mean reversion factor. $\mu$ is called the average long-run value of the short rate. We can see that when $r=\mu$, the expected incremental change is zero. When interest rates are high ($r>\mu$), the drift term is negative, and conversely, when rates are low ($r<\mu$), the drift term is positive. $\sigma \sqrt{r}$ is instantaneous standard deviation. $dZ$ is the Wiener increment. If $2\kappa \mu \geq \sigma^2$, the drift is sufficiently large to make the origin inaccessible. Under their model, analytical solutions for the prices of default-free bonds of all maturities are obtained.

**Bond Price Processes**

Let $P(r,t,T)$ be the price at time $t$ of a pure discount bond that matures at time $T$. It is assumed that at date $t$ the price of this bond depends on the state variables, $r(t)$ and $t$, and the face value of the bond is $1$. Using Ito's rule, we have
\[ dP = P_r dr + P_t dt + \frac{1}{2}(\sigma \sqrt{r})^2 P_r dt \]
\[ = P_r \left[ \kappa (\mu - r) dt + \sigma \sqrt{r} dZ \right] + P_t dt + \frac{1}{2}(\sigma \sqrt{r})^2 P_r dt \]
\[ = \left[ P_t + P_r (\kappa (\mu - r)) + \frac{1}{2}(\sigma \sqrt{r})^2 P_r \right] dt + [\sigma \sqrt{r} P_r] dZ \]

Hence,
\[ \frac{dP}{P} = \mu_p dt + \sigma_p dZ \] (2.3)

where
\[ \mu_p = \frac{\left[ P_t + P_r (\kappa (\mu - r)) + \frac{1}{2}(\sigma \sqrt{r})^2 P_r \right]}{P} \] (2.3.a)
\[ \sigma_p = \frac{\sigma \sqrt{r} P_r}{P} \] (2.3.b)

If the instantaneous expected return, \( \mu_p \), was known, then from Equation (2.3.b) the bond price could be obtained by solving the following partial differential equation subject to the appropriate boundary conditions

\[ \frac{1}{2}(\sigma \sqrt{r})^2 P_r + \kappa (\mu - r) P_r - \mu_p P + P_t = 0 \] (2.4)
\[ P(r, T, T) = 1 \]
\[ 0 \leq t \leq T \]

The problem inherent in Equation (2.4) is that the expected return on the bond, \( \mu_p \), must be exogenously determined. Under one form of the expectations hypothesis,
the so-called *local expectations hypothesis*, it could be postulated that $\mu_r = r$. In this case, Equation (2.4) reduces to

$$\frac{1}{2}(\sigma \sqrt{T})^2 p_r + \kappa(\mu - r)p_r - rP + \lambda T = 0$$

$$P(r, T, T) = 1$$

$$0 \leq t \leq T$$

The solution of this equation can be expressed as

$$P(r, t, T) = E_{r,t} \left[ e^{-\int_{t}^{T} r(s)ds} \right]$$ (2.5)

We have seen that under the local expectations hypothesis the bond price is the solution of the Equation (2.4). The fact that the expectation, given in Equation (2.5), can be obtained by solving the partial differential equation (Equation (2.4)) and vice-versa, is called the *Feynman-Kac result*.

**Equilibrium Pricing of Bonds**

In the above analysis, to obtain bond prices we assumed the local expectations hypothesis held true. In a more general setting the yield curve will depend not only on expectations about future levels of the interest rate, but also upon investor attitudes toward risk. In order to solve for the bond pricing equation without invoking the local expectations hypothesis, an equilibrium model is required.

In our one factor setting, the entire yield curve, and hence all prices are fully determined by the short rate. In any meaningful equilibrium there should be no
possibility of forming a portfolio of bonds whose return is free of risk unless its rate of return is equal to the current riskless rate. This necessary condition imposes a particular structure for the expected return $\mu_p$ on a bond.

$$\frac{\mu_p(t, T) - r(t)}{\sigma_p(t, T)} = \lambda(t)$$  \hfill (2.6)

where $\lambda(t)$ is the market price of interest rate risk and is the same for all bonds regardless of maturity. CIR (1985) showed that claims can be valued as if the local expectations hypothesis is true provided expectations were taken under the equivalent martingale measure given by

$$dr = \tilde{\mu}(r)dt + \sigma \sqrt{r} dZ_r$$  \hfill (2.7)

where $\tilde{\mu}(r) = m(r) - \lambda(r) \sigma \sqrt{r}$

and $\lambda(r) = \frac{\lambda \sqrt{r}}{\sigma}$.  \hfill (2.8)

Equation (2.7) is referred to as the risk neutralized process. Once this process is identified, valuing bonds can be accomplished by "pretending" the local expectation hypothesis holds.

In this dissertation, we shall assume that the assumptions of the CIR model hold. In this case interest rates follow Equation (2.7), and the market price of risk follows Equation (2.8).\footnote{There are numerous alternative representations for the evolution of the term structure of interest rates. An example of another approach is provided by Brennan and Schwartz (1979). They suggest that all yields can be generated from two factors, the short rate, $r$, and the long rate, $l$, where...}
(b) Prepayment Process

When interest rates decline sufficiently, mortgagors have the propensity to fully or partially prepay the remaining balance. Effectively, the lender has granted the homeowner the right to extinguish the mortgage balance at any time. The situation is similar to call options bondholders grant corporations, that provide them the right to call in the debt. Because of the complications prepayment causes in pricing mortgages, it is important to understand why homeowners prepay. Rational investors will maximize their wealth. Thus, if markets are frictionless, a mortgagor will exercise the call option whenever the existing loan can be refinanced with an identical loan which has a lower contract rate than the existing rate. The valuation of this rational prepayment option can in theory be accomplished once the dynamics of the underlying interest rate process is provided. However, prepayment constraints and mortgagors' noneconomic behavior lead to otherwise suboptimal prepayment strategies. Often times, homeowners will prepay their mortgages even when interest rates are high. This occurs for personal reasons such as relocation, divorce, death, etc. Most mortgages have due-on-sale clauses which require full payment of principal upon sale.

\[
\begin{align*}
    dr &= \beta_1 (r, l, t) dt + \eta_1 (r, l, t) dZ_1 \\
    dl &= \beta_2 (r, l, t) dt + \eta_2 (r, l, t) dZ_2
\end{align*}
\]

\(\beta_1\) and \(\beta_2\) are the instantaneous means, \(\eta_1\) and \(\eta_2\) are the instantaneous standard deviations. and \(dZ_1\) and \(dZ_2\) are the Wiener increments of the short and long rates respectively. The correlation coefficient of \(dZ_1\) and \(dZ_2\) is \(\rho\). Unfortunately, since their model is not a general equilibrium model, the market price of risk associated with each factor is not endogenous to the model and has to be externally specified. As CIR show, this may lead to internal inconsistencies. For alternative interest rate models, see Hull (1993) and the references therein.
of the house. That is, the loan is not assumable by the new homeowner. Also, individuals lacking sufficient assets may not be able to refinance even when interest rates have fallen sufficiently low. These reasons cause the homeowners' exercise policies to deviate from their optimal policy based on financial reasons alone.

The prepayment models that have been established can be categorized into deterministic, statistical, and stochastic models (see Figure (2.4)). Deterministic models simply assume a constant prepayment
Figure 2.4

Prepayment Models
rate in advance. Although simple to use, such models do not capture the complex prepayment behavior. Green and Shoven (1986) pioneered the Proportional Hazard Model, the first statistical model to describe the mortgage prepayment rate. In particular, the prepayment rate is given by

$$\theta(t, X) = \psi(t) \pi(X) = \theta(t) \exp\left[ \sum_{i=1}^{n} (\beta_i x_i) \right]$$

(2.9)

Here $X=(x_1, x_2, \ldots, x_n)$ are exogenous factors that influence prepayment rate, and $\beta$'s are associated coefficients. $\psi(t)$ is the baseline hazard or the mortgage prepayment rate under completely stationary, homogeneous conditions. We can observe $\pi(X)$ could be greater than or less than one, depending on which direction the factors influence the prepayment rate. Schwartz and Torous (1989) refined (2.9) by using the following log-logistic hazard function as the baseline hazard function

$$\psi(t) = \frac{\gamma p(\gamma t)^{p-1}}{1 + (\gamma t)^p}$$

(2.10)

with $\gamma$ and $p$ constant parameters, to improve its predictive ability. An example of a stochastic prepayment model is given by Dunn and McConnell (1981). In their model, prepayments follow by a Poisson process.

---

2 When $X=0$, the prepayment function reduces to the baseline hazard function.

3 When $p > 1$, the probability of prepayment rate increases from zero to a maximum and then decreases to zero thereafter. This is consistent with the observation that, all other things being equal, prepayment rates are typically low in the early years of a mortgage, increase as the age of the mortgage increases, and then diminish with further seasoning. Thus, it impounds our prior knowledge of seasoning's influence on mortgage prepayments.
\[
d y = \begin{cases} 
1 & \text{w.p. } \eta dt \\
0 & \text{w.p. } (1 - \eta) dt 
\end{cases} \tag{2.11}
\]

where \( \eta \) is the prepayment intensity, and \( \eta dt \) is the probability that a Poisson event occurs in time interval \( dt \). \( y \) is defined so that \( y \) starts at 0 and jumps to 1 when a Poisson event occurs and we characterize the random amount that the mortgage price will jump whenever a Poisson event occurs by \( \xi \). In particular, given a Poisson event at date \( t \), prepayment occurs and the homeowner pays off the balance of the loan and in return receives the value of the mortgage.

(c) Property Process

The value of a particular mortgage may also be determined by the value of the underlying property. This is because when the property value is lower than the outstanding balance, the mortgagor has an incentive to default. This is particularly important when considering commercial mortgages. The value of the property, \( V \), is often modeled by the following Ito process

\[
d V = V(\mu V - Q) dt + V \sigma V dZ_v \tag{2.12}
\]

with \( \mu_v \), representing the expected total rate of return on the value of the property, \( Q \), the rate of cash distributions to the property owner, \( \sigma_v V \), the instantaneous standard deviation, and \( dZ_v \), the Wiener increment. For many mortgage related products, default risk is of secondary concern and is often ignored. This may be due to the fact
that many mortgage contracts are protected against default risk. For example, GNMA
are backed by the good faith of the U.S. government.

2.3.2 Mortgage Pricing Models

Given the processes for interest rates, prepayment, and property, we now can
establish continuous-time valuation models for pricing fixed-rate mortgage contracts.
We assume that at origination the mortgage has a principal of \(B(0)\), a continuously
annualized compound contract rate \(c\), and a term to maturity of \(T\) years. As a result,
the total payout rate per year is

\[
F = cB(0)/[1 - e^{-cT}] \tag{2.13}
\]

and the principal outstanding at time \(t\) assuming no prepayments occurred prior to date
\(t\) is

\[
B(t) = B(0)[1 - e^{-c(T-t)}]/[1 - e^{-cT}] \tag{2.14}
\]

Actually, the real balance outstanding at date \(t\) will depend on prepayment experience.
Since prepayments in general depend on the age of the loan and on the path of interest
rates, the actual outstanding balance at date \(t\) is path dependent.

The usual assumptions that are made follow:

(A1) Perfect markets with no transaction costs, no taxes, and no short selling
constraints.
(A2) Dynamics about the term structure of interest rates can be summarized by the short rate, which is described by mean-reverting square root process of (2.2), with \( r_0 \) given, and \( 4\kappa \mu > \sigma^2 \).

(A3) Prepayment follows a Poisson process of (2.11), with the jump size equal to the residual amount after extinguishing the remaining balance by the market value of the mortgage. We further assume the timing of the jump and the jump size are independent of each other. Moreover, this jump risk is fully diversifiable. That is the jump process is uncorrelated with the market portfolio. This assumption is equivalent to assuming that residual risk associated with the jump is not relevant for pricing. Only expectation matters.

(A4) The dynamics of mortgaged property is described by the lognormal diffusion process of (2.10). The unanticipated changes in the value of the property are assumed to be correlated with unanticipated changes in the instantaneous risk-free interest rate, \( dz_r dz_v = \rho dt \), where \( \rho \) denotes the instantaneous correlation coefficient.

Given the above assumptions, the value of any fixed-rate mortgage can be expressed as

\[
M = M(r, y, V, t).
\]  

(2.15)
A direct application of Ito's rule and its extension for jump process leads to the stochastic process for the mortgage price $M$ as follows

$$
dM = M_r dr + M_V dV + (M_t + F) dt \\
+ rac{1}{2} [M_r dr^2 + 2M_r V dr dV + M_{VV} dV^2] + \xi dy
$$

(2.16)

where $M_r, M_t, M_v, M_{vv}$ and $M_{vv}$ are partial derivatives.

With (A3), the random component $\xi dy$, can be replaced by its expectation. That is, for pricing purpose, we can assume

$$\xi dy = \eta [M - B(t)] dt
$$

(2.17)

With (A2) and (A4), (2.16) simplifies to

$$
dM = \mu_M dt + \sigma_r M_r dZ_r + \sigma_v M_V dZ_v
$$

(2.18)

where

$$
\mu_M = m(r) M_r + (\mu V - Q) M_V + M_t \\
+ \frac{1}{2} \sigma(r)^2 M_{rr} + \rho \sigma(r) \sigma_v M_{rv} + \frac{1}{2} \sigma_v^2 M_{vv} + F + \eta E[\xi]
$$

represents the instantaneous mean of the mortgage process and $\eta$ is the prepayment rate.
Given the mortgage process given by (2.18), property process given by (2.12), and default-free bond process given by (2.3), an instantaneously risk-free portfolio can be constructed by hedging the interest rate risk of the mortgage with a short position in the default-free bond and by hedging its risk due to changes in the value of the building with a short position in shares of the building. An absence of arbitrage opportunities requires that the instantaneous return on this risk-free portfolio be the same as the instantaneous risk-free rate of interest. This allows us to derive the following partial differential equation that all fixed rate mortgages must satisfy

\[
\begin{align*}
(m(r) - \lambda(r)\sigma(r))M_r + (r - Q)VM_r + M_t + \rho\sigma(r)\sigma_v VM_v \\
\frac{1}{2}\sigma^2_v V^2 M_{vv} + F + \frac{1}{2}\sigma(r)^2 M_{rr} + \eta[B(t) - M] = rM + \eta M
\end{align*}
\]  

(2.19)

Here \( \lambda(r) \) is the market price of interest rate risk, defined earlier.

This is the basic partial differential equation governing all of the fixed-rate mortgages and their derivative securities. Solving the partial differential equation can be accomplished once the boundary conditions have been specified. With specific interest rate and prepayment representations, a variety of mortgage pricing models have been developed. Table 2.1 provides a table of existing models.
Table 2.1
Models in Pricing Mortgage-Related Products

<table>
<thead>
<tr>
<th>Author</th>
<th>Contract</th>
<th>Interest Rate Model</th>
<th>Prepayment Model</th>
<th>Property Process</th>
<th>Solution Technique</th>
<th>Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dunn &amp; McConnell</td>
<td>GNMA Passsthrough</td>
<td>CIR</td>
<td>Poisson (FHA)</td>
<td>NO</td>
<td>Simulation</td>
<td>1981</td>
</tr>
<tr>
<td>Brennan &amp; Schwartz</td>
<td>GNMA Passsthrough</td>
<td>BS</td>
<td>Optimal</td>
<td>NO</td>
<td>Simulation</td>
<td>1985</td>
</tr>
<tr>
<td>Kau &amp; Associates</td>
<td>Commercial</td>
<td>CIR</td>
<td>NO</td>
<td>YES</td>
<td>Finite Difference</td>
<td>1986</td>
</tr>
<tr>
<td>Hendershott &amp; Van Order</td>
<td>FRM</td>
<td>CIR</td>
<td>Optimal</td>
<td>NO</td>
<td></td>
<td>1987</td>
</tr>
<tr>
<td>Singh</td>
<td>CMO, PAC &amp; Strips</td>
<td>BS</td>
<td>Proportional Hazard &amp; Threshold</td>
<td>NO</td>
<td>Simulation</td>
<td>1988</td>
</tr>
<tr>
<td>Schwartz &amp; Torous</td>
<td>GNMA</td>
<td>BS</td>
<td>Proportional Hazard</td>
<td>NO</td>
<td>Simulation</td>
<td>1989</td>
</tr>
<tr>
<td>Titman &amp; Torous</td>
<td>Commercial</td>
<td>CIR</td>
<td>NO</td>
<td>YES</td>
<td>Finite Difference</td>
<td>1989</td>
</tr>
<tr>
<td>Kau &amp; Associates</td>
<td>ARM</td>
<td>CIR</td>
<td>Optimal</td>
<td>NO</td>
<td>Finite Difference</td>
<td>1990</td>
</tr>
<tr>
<td>Zipkin</td>
<td>MBS</td>
<td>CIR</td>
<td>Proportional Hazard</td>
<td>NO</td>
<td>Simulation</td>
<td>1990</td>
</tr>
<tr>
<td>McConnell &amp; Singh</td>
<td>ARM</td>
<td>BS</td>
<td>Proportional Hazard</td>
<td>NO</td>
<td>Simulation</td>
<td>1992</td>
</tr>
<tr>
<td>Schwartz &amp; Torous</td>
<td>Passsthrough</td>
<td>CIR</td>
<td>Proportional Hazard</td>
<td>YES</td>
<td>Finite Difference + Simulation</td>
<td>1992</td>
</tr>
</tbody>
</table>

Note: BS is the abbreviation for Brennan and Schwartz's two factor interest rate model.
2.3.3 Solution Techniques

Usually no closed-form solutions exist to Equation (19) and numerical methods are needed to approximate mortgage prices. Numerical methods for pricing contingent claims may be classified as forward looking or backward looking. For example, Monte Carlo methods are forward looking while lattice approaches and finite difference methods are generally applied in a backward fashion. Both approaches can be used to value claims with complex payouts and/or boundary conditions. However, backward methods have difficulty in dealing with path dependence since cash flows depending on state variables occurring earlier in time cannot be determined. Forward methods, on the other hand evolve over a time path and thus have no difficulty in simulating cash flows which are a function of the path of the state variables. However, forward methods are generally unable to deal with the early exercise decision of options since these rational decisions are based on valuing future cash flows. In short, forward methods determine price from past and current information while backward methods determine price from current and future information.

Lattice-based approaches have been lacking from the literature as can be seen from Table 2.1 for two reasons. First, until recently, no efficient lattice-based approximations of the underlying stochastic variables had been established. Second, even efficient lattice approximations of interest rate are available, the fact that mortgage prices depend on the specific path leading to current state creates additional
problems. As a result, no lattice based approximations have been used in pricing mortgage-related products.

In this dissertation, we shall use a linearly growing lattice approximation for interest rate process that was developed by Li (1992). This lattice overcomes the first problem. The main contribution to the literature will be to address how the path dependence can be captured by single sufficient statistic, in a way that permits dynamic programming methods to be used to establish values for mortgage related products. Before proceeding further, we review Li approximation of interest rate process.

2.4 Binomial Lattice Approximation of CIR Process

Since its development by Sharpe (1978), and Cox, Ross, and Rubinstein (1979), binomial models have become one of the most frequently used numerical tools in option pricing.

In principal, the binomial lattice can be constructed to approximate many types of diffusion processes. However, when the volatility term is not constant, then the nodes of the binomial lattice suggested by Sharpe (1978), and Cox, Ross, and Rubinstein (1979), do not reconnect. This means with just 20 time increments, for example, $2^{20}$ or over 1 million paths are generated.
Recently, Nelson and Ramaswamy (1990) have shown that an efficient binomial lattice can be achieved by transforming the original process such that the volatility of the transformed process is constant. Their transformation requires the approximating binomial process to have jump sizes and probabilities such that the local drift and variance converge to the drift and variance of the desired diffusion, with the jump sizes diminishing to zero as the jumps become more frequent. Singularity problems in their approach are fixed by modifying the transformation to allow multiple jumps. The Simplified Binomial Model developed by Tian (1991) differs from Nelson and Ramaswamy binomial model in that it does not permit multiple jumps in interest rates in a single time period, and forces the up-jump probability to 1 if its computed value exceeds 1, and to 0 if it falls below 0. Li (1992) extended their work by a so-called Binomial Approximation of Singular Diffusion (BASD) which considers a binomial approximation of diffusions with various boundary characteristics: natural, entrance, regular, and exit. His approach not only overcomes the instability encountered in singular diffusion approximation, but also trims the lattice in a systematic manner. This results in a smaller state space for discrete processes, which makes the approximation more efficient and accurate than existing model. Numerical results also show that Li's approach maintains both accuracy and speed. In this section, we shall illustrate how BASD approach can be implemented by a three step procedure. The first step involves transforming the square root mean reverting process to one
where the volatility is constant, and setting up an approximating lattice for the transformed variable. Towards this goal let

\[ x = 2 \sqrt{r/ \sigma} . \]  

(2.20)

Using Ito's Lemma, the transformed process \( x \) follows

\[ dx = \left( \frac{\phi - x^2}{2x} \right) dt + dz, \ x \geq 0. \]  

(2.21)

where

\[ \phi = 4 \kappa \mu / \sigma^2 - 1 \]

The idea now is to approximate the \( x \)-lattice by splitting the time interval \([0, T]\) into \( n \) equal lengths, each of \( \Delta t \) and then over each time increment allowing \( x \) to change by \( \pm \sqrt{\Delta t} \). Given \( r_0 \), define \( x_0 = 2 \sqrt{r_0 / \sigma} \). We now can construct a lattice for the \( x \) and \( r \) values. Over the first two periods, for example, we have

Given the values on the \( x \)-lattice, the interest rate \( r \), can be recovered by

\[ r = f(x) = \begin{cases} \sigma^2 x^2 / 4 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \]  

(2.22)

and of course, the \( r \)-lattice can also be easily recovered too.
The second step involves establishing the probability of up and down jumps at each node of the lattice. This must be accomplished so that as the partition $\Delta t$ goes to 0, the first two moments of the binomial increments should match those of the diffusion increments locally. In other words, the resulting drift and volatility terms move to $\kappa(\mu - r)$ and $\sigma \sqrt{r}$ respectively. To achieve this, the probabilities are computed off the $r$-process using the formula

$$q(r) = \frac{\kappa(\mu - r)\Delta t + (r - r^-)}{(r^+ - r^-)}$$  \hspace{1cm} (2.23)

Finally, the third step involves modifying the probabilities on the lattice when their computed values exceed 1 (or fall below 0). In particular as $r$ moves to 0, the above probability may exceed 1. When this occurs, the BASD algorithm forces the probabilities to 1 (or 0), with the up jump and down jump on the transformed process modified as follows

$$\begin{align*}
x^+ &= x_0 + \sqrt{\Delta t} \\
x^- &= x_0 - \sqrt{\Delta t} \\
x^{++} &= x_0 + 2\sqrt{\Delta t} \\
x^{--} &= x_0 - 2\sqrt{\Delta t}
\end{align*}$$

**Figure 2.5**

A two-period lattice for the transformed process

$$\begin{align*}
x^+ &= x + i_u \sqrt{\Delta t} & \text{if an up jump occurs} \\
x^- &= x + i_d \sqrt{\Delta t} & \text{if an down jump occurs}
\end{align*}$$  \hspace{1cm} (2.24)
where \( i_u \) and \( i_d \) are the smallest odd integers such that the up-jump probability

\[
q(r) = \frac{\kappa(\mu - r)\Delta t + (r - r^-)}{(r^+ - r^-)}
\]

does not fall below 0 (or exceed 1).\(^6\)

**Convergence of the BASD**

The computational efficiency of BASD in computing bond prices on the lattice can be shown in Table 2.2. The prices of discount bonds are compared to true values for different partition sizes when the interest rate follows CIR process,

<table>
<thead>
<tr>
<th>Number of partition</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>100</th>
<th>500</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>0.2855</td>
<td>0.2876</td>
<td>0.3000</td>
<td>0.304</td>
<td>0.305</td>
<td>0.306</td>
<td>0.310</td>
<td>0.3103</td>
</tr>
</tbody>
</table>

*Note: T=20 years. CIR parameters are \( \mu=0.08 \), \( \kappa=0.50 \), \( \sigma=0.50 \), and \( r_v=0.05 \). Face value of the bond is $1. Analytic CIR value=0.31087.*

We can see that the prices of discount bond converges to the analytical CIR prices very quickly. For a more complete tests of convergence see Li. As discussed earlier, we shall use this approximation in later chapters.

\(^6\) For a complete discussion of singular problems in binomial approximation, and their solutions, see Appendix A.
2.5 Conclusions

In this chapter, we reviewed mortgages and their related contracts, and relevant pricing literature. The binomial approximation of mean-reverting square root process was also discussed by using the BASD algorithm. Mortgage-related contracts have been difficult to price using "backward" based algorithms, because their values are path-dependent. As a result, most existing approaches are forward based. Indeed, the majority of models solve for the mortgage prices by using Monte Carlo simulation. Such simulations are valid if the mortgage contains no option features, and prepayment structures are taken to be exogenous. However, if there are any option features, then the method fails. Simulation models may also be quite time consuming. As a result, other numerical approximating procedures such as finite difference methods and lattice approaches would be desirable. As discussed, lattice approaches have been difficult to implement, primarily because of the path dependence.
CHAPTER 3

FIXED-RATE MORTGAGE

3.1 Introduction

In recent years, techniques of continuous time arbitrage theory have been used to value fixed-rate mortgages and their backed-securities. Among the most noted are the works of Dunn and McConnell (1981) who established equilibrium GNMA prices. Such a model turns out to be a special case of the partial differential equation in the underlying state variables in equation (2.18) of Chapter 2. In their approach, uncertainties in interest rates are captured by CIR process and suboptimal prepayments, which apply equally well to all individuals, are modeled by jump process. Brennan and Schwartz (1985) used a two-factor instantaneous short-rate process as the underlying uncertainty to model and price the same contract under the assumption that prepayments are based on financial decision alone. Other models which consider the property process as another source of uncertainty to describe the default provision or different specification of prepayment experiences can also be found in Hendershott and Van Order (1987), Kau Keenan, Muller, and Epperson (1987), Kau, ct. al. (1990), Schwartz and Torous (1989), and Schwartz and Torous (1992). For many mortgage-related products, default risk is of a secondary concern and is often ignored. Furthermore, a great portion of fixed-rate related mortgage contracts, such as GNMA
pass-throughs are essentially insured against default, since they are backed by U.S. government. In this chapter, we shall concentrate on such contracts.

In general, no closed form solutions can be obtained to the partial differential equations which GNMA pass-throughs satisfy, even if strict restrictions are imposed upon the stochastic processes describing the economic environment and the boundary conditions embodying the contractual features of the claims. Inevitably, numerical methods must be used.

As discussed in Chapter 2, numerical procedures are either "forward" based or "backward" based. The standard approach, Monte Carlo simulation, falls in the former category. While this approach is flexible and can be used to value mortgages with a variety of contract specifications, extensive time is required to obtain a solution. Further, simulation approaches have difficulty in dealing with the mortgage which provides the holder with intermediate options. In the "backward" approach, two popular procedures commonly used, namely, the lattice approach and finite difference methods, exist. Lattice approaches approximate the underlying processes, while finite difference methods approximate the partial differential equations.

Lattice approaches have been lacking from the mortgage pricing literature for two main reasons. First, an efficient lattice approximation for the underlying interest rate process has, until recently, not been available. Second, even if an efficient approximation were available for interest rates and bond prices, lattice approach for mortgages still has the difficulty of dealing with path dependence. Specifically, the
price of a mortgage contract at any point in time depends on the history of interest rates taken to that date. Hence backward procedures have to condition on the path, and this causes computational problems. If the approximation of the interest rate process on the lattice grows linearly in the number of periods, then the number of distinct interest rate paths grows exponentially.

Of course, the path-dependence can be overcome in a backward procedure setting if a few additional auxiliary state variables can be identified, that are sufficient statistics for the history of the process. Then the problem again becomes Markovian and we can proceed as usual. If computations are to be feasible in practice, however, it is necessary that the number of these auxiliary variables be kept quite small. In this chapter, we shall develop an efficient lattice-based model for fixed-rate mortgages and GNMA pass-throughs (MBS for short, from now on), where the history of the process, relevant for pricing, is captured by a single additional state variable. Specifically, this additional statistic, together with the current interest rate is sufficient for capturing all information along the path.

Given this model, we then conduct tests to measure the sensitivity of mortgage contracts to specific assumptions. In particular, we capture the sensitivity of mortgages to key parameters driving interest rate uncertainty. In addition we identify the equilibrium contract rate for MBS under specific prepayment assumptions.

The remainder of this chapter is organized as follows. Section 3.2 introduces how GNMA mortgage-backed pass-through securities are structured. Section 3.3 lists
the assumptions and notation for model specification. Section 3.4 develops a lattice-based approach to valuing straight amortization bonds (SABs), and callable amortization bonds (CABs). Such securities are similar to MBSs, yet easier to analyze. Indeed, CABs are identical to MBSs, if the assumption is made that prepayments only occur for rational financial reasons. Since mortgages are often prepaid for non-financial reasons, the equilibrium rate of a MBS will be lower. Equivalently, the equilibrium rate of a CAB will be higher. CABs will be used as a starting point to develop pricing models for MBSs. Section (3.5) discusses the path-dependent problem associated with pricing MBSs on a lattice. A 3-period example is provided to illustrate the difficulty resulting from pass-dependence. Section (3.6) establishes a binomial lattice valuation model for MBS, that efficiently handles the path-dependent problem. Section (3.7) provides numerical produced by the model. We conduct sensitivity analysis and show how equilibrium contract rates can be computed. Section (3.8) gives conclusions.

3.2 Overview of GNMA Mortgage-Backed Pass-through Securities

Pass-through securities are created when mortgages are pooled together, and undivided interests or participation in the pool are sold. The originator continues to service the mortgages, collecting payments and "pass through" the principal and interest, less the servicing, and other fees, to the security holders. The security holders receive pro rata shares of the resultant cash flows. A portion of the outstanding
principal is paid each month according to the amortization schedule established for each individual mortgage. In addition, and this is a critical feature of mortgage pass-through securities, the principal on individual mortgages in the pool can be prepaid without penalty in whole or in part at any time before the stated maturity of the security.

Mortgage originators (savings and loans, commercial banks, and mortgage bankers) actively pool mortgages and issue pass-throughs. The largest and best-known group of pass-through securities is guaranteed by GNMA. The mortgage pools underlying GNMA pass-through securities are made up of FHA (Federal Housing Administration)-insured or VA (Veteran Administration)-guaranteed mortgage loans. GNMA pass-throughs are backed by the full faith and credit of the U.S. government.

3.3 Assumptions and Notation

We have the following assumptions:

--- *Interest rate follows the square root process, and all bonds are priced as in the general equilibrium model of CIR.*

--- *CIR process is approximated by BASD (binomial approximation for singular diffusions) (Li, 1992).*

--- *The market is perfect. This means no transaction and refinancing costs.*

--- *The mortgages underlying the pools are identical.*
Mortgage loans can be prepaid at any time without a prepayment penalty. Suboptimal prepayment is only dependent on the instantaneous short rate and on time.

The discrete-time equation for the monthly fixed-level payment, $F$ with contract rate $c$, original principal $B(0)$, and maturity $T$ in years, can be represented by

$$F = B(0)(c/12) \frac{(1 + c/12)^T - 1}{(1 + c/12)^{12} - 1}$$  \hspace{1cm} (3.1)$$

$F$ is referred to as fixed-level payment.

The remaining balance at time $t$ assuming no prepayments occurred prior to time $t$, is

$$B(t) = B(0)\frac{(1 + c/12)^{T - t} - (1 + c/12)^{t - 12}}{(1 + c/12)^{T - 12} - 1}.$$  \hspace{1cm} (3.2)$$

In addition, the following notation is used

$n$ = Number of periods (in months).

$\Delta t$ = Length of the interval (one month).

$(i, j)$ = Node on the binomial lattice, constructed by the 3-step procedure proposed in Section (2.4).

$r_{ij}$ = Spot rate node $(i, j)$. The initial value $r_{00}$ simplifies to $r_0$.

$q_{ij}$ = Risk-adjusted probability of an up (down) jump in interest rates from node $(i, j)$. Under this measure, all bonds can be priced as if the local expectations hypothesis holds.

$M_{ij}(c)$ = The price of mortgage-backed security at node $(i, j)$ with contract rate $c$, calculated by backward recursion. The
initial price $M_{\alpha}(c)$ is $M_{\alpha}(c)$. When no confusion is cause, arguments will be dropped.

$P_{ij}(k) = \text{Price of a discount bond viewed from node } (i,j) \text{ maturing in period } k, k > i, \forall k$. The initial bond prices at node $(0,0)$ are represented by $P_{0}(k)$.

$y(r,t) = \text{Fraction of the mortgagors that will prepay when the short rate is } r \text{ and the age of the mortgage (months) is } t.$

In particular, $0 \leq y(r,t) \leq 1$, and $y(r,n) = 1$.

$F_{ij}(c) = \text{Fixed-level payment from period } i+1 \text{ to } n \text{ with } \$1 \text{ as the remaining balance at node } (i,j), \text{ and contract } c$. When no confusion is caused, argument will be dropped.

$B_{ij}(1,c) = \text{Remaining balance in next period } (i+1) \text{ with } \$1 \text{ as the remaining balance at node } (i,j), \text{ and contract rate } c$. When no confusion is caused, arguments will be dropped.

Additional notation will be introduced when necessary.

### 3.4 Pricing Straight and Callable Amortization Bonds on a Binomial Lattice

Given the binomial lattice of interest rates, the entire term structure can be calculated, using the CIR model (1985). Moreover, using standard backward recursion techniques, we can price many types of default free interest rate claims. The price of a straight amortization bond, $SAB_0$, in period 0 is given by
\[ SAB_0 = F \sum_{i=1}^{N} P_0(t) \] (3.3)

Now consider a **callable amortization bond** (CAB), where the call feature can be exercised at the beginning of each period. Since the underlying security makes periodic principal and interest payments, early exercise of the option may be optimal. At node \((i,j)\), with the spot short rate \(r_{ij}\), the price of a CAB, \(CAB_{ij}\), is given by

\[ CAB_{ij} = \min[CAB_{ij}^{\text{GO}}, CAB_{ij}^{\text{STOP}}] \] (3.4)

where \(CAB_{ij}^{\text{GO}} = [q_{ij}CAB_{i+1,j+1} + \overline{q}_{ij}CAB_{i+1,j} + F]e^{-r_{ij} \Delta t}\) (3.4.a)

and \(CAB_{ij}^{\text{STOP}} = B(t)\) , (3.4.b)

with boundary mortgage prices, \(CAB_{n,j} = F\). Here \(CAB_{ij}^{\text{GO}}\) is the market price if the call feature is not exercised, while \(CAB_{ij}^{\text{STOP}}\) is the remaining balance paid if the option is exercised. When \(CAB_{ij}^{\text{GO}} < CAB_{ij}^{\text{STOP}}\) the bond will be called.

A callable amortization bond is like an MBS except in the latter contract the call feature may not be exercised on the basis of financial information alone.

### 3.5 Path-Dependence Associated with MBS

In order to price a financial contract, first we need to identify future cash flows and then discount them back to present. The future cash flows associated with MBS consist of two parts, namely, fixed-level payments, and principal prepayments.
From Equation (3.1), we can get fixed-level payment, $F_o$, for the remaining terms of the contract. The remaining balance before prepayment at the next period, calculated by Equation (3.2) is denoted by $B_o(1)$. If we subtract directly from $B_o$ the principal payment component, $B_o(1)$ can also be directly calculated by the following equation\(^1\)

$$B_o(1) = B_o - [F_0 - B_o(c/12)]$$  \hspace{1cm} (3.5)

For illustrative purpose, we begin by pricing a trivial one-period MBS. For a one-period problem as shown in Figure (3.1), we assume that in the next period interest rate will go up to $r_{i,1} = r^*$ with probability $q_o$ and go down to $r_{i,0} = r^-$, with probability $1 - q_o$. With $B_o(1)$ principal remaining, and the specification of prepayment

\[\begin{bmatrix} B_o \
- \
- \end{bmatrix} \xrightarrow{r} \begin{bmatrix} B_o(1) \
F_0 \
B_o(1)\nu(r^*, 1) \end{bmatrix}
\]

\[\begin{bmatrix} B_o(1) \
F_0 \
B_o(1)\nu(r^-, 1) \end{bmatrix} \xrightarrow{r^-} \]

**Figure 3.1**
A one-period lattice showing the cash flows of MBS.

---

\(^1\) Equation (3.5) tells that fixed-level payment consists of two parts: One is interest payment, which is equal to contract rate times previous remaining balance. The amount left belongs to principal payment.
function, \( B_0(1)\gamma(r, 1) \) will be prepaid at the up node, and \( B_0(1)\gamma(r, 1) \) will be prepaid at the down node. Since, all the balance is paid off at maturity, \( \gamma(r, 1) = \gamma(r, 1) = 1 \), and

\[
M_0 = q_0[F_0 + B_0(1)\gamma(r^+, 1) + B_0(1)(1 - \gamma(r^+, 1))]e^{-r\Delta t} \\
+ (1 - q_0)[F_0 + B_0(1)\gamma(r^-, 1) + B_0(1)(1 - \gamma(r^-, 1))]e^{-r\Delta t} \tag{3.6}
\]

Now we want to extend the above one-period problem to a two-period one and show how path-dependence comes into play in pricing MBS. At period 2 we assume the possible interest rates are as shown in Figure (3.2) with associated probabilities coming from period 1. We use abbreviated notation to denote the principals remaining at up node and down node at period 1, \( B_0(1)(1 - \gamma(r^+, 1)) \) and

\[
\begin{bmatrix}
B_0(1) \\
F_0 \\
B_0(1)\gamma(r^+, 1)
\end{bmatrix}
\begin{bmatrix}
B_1(1) \\
F_1 \\
B_1(1)\gamma(r^+, 1)
\end{bmatrix}
\]

\[
\begin{bmatrix}
B_0(1) \\
F_0 \\
B_0(1)\gamma(r^-, 1)
\end{bmatrix}
\begin{bmatrix}
B_2(1) \\
F_2 \\
B_2(1)\gamma(r^-, 2)
\end{bmatrix}
\]

\[
\begin{bmatrix}
B_0(1) \\
F_0 \\
B_0(1)\gamma(r^-, 1)
\end{bmatrix}
\begin{bmatrix}
B_3(1) \\
F_3 \\
B_3(1)\gamma(r^-, 2)
\end{bmatrix}
\]

\[
\begin{bmatrix}
B_0(1) \\
F_0 \\
B_0(1)\gamma(r^-, 1)
\end{bmatrix}
\begin{bmatrix}
B_4(1) \\
F_4 \\
B_4(1)\gamma(r^-, 2)
\end{bmatrix}
\]

**Figure 3.2**
A two-period lattice showing the cash flows of MBS.
$B_0(1)(1-y(r^,1))$, by $B_1$ and $B_2$ respectively. With $B_1$ and $B_2$ as new principals, the relevant fixed-level payment and principal prepayments can be calculated as shown in Figure (3.2) and the price of the MBS can be calculated by discounting the cash flows using the local expectations hypothesis under the equivalent martingale measure. Specifically, it can be calculated by

$$M_0 = q_0\{F_0 + B_0(1)y(r^, 1) + (q_1[F_1 + B_1(1)] + (1 - q_1)[F_1 + B_1(1)])e^{-r\Delta t}\}e^{-r\Delta t}$$

$$+ (1 - q_0)\{F_0 + B_0(1)y(r^, 1) + (q_2[F_2 + B_2(1)] + (1 - q_2)[F_2 + B_2(1)])e^{-r\Delta t}\}e^{-r\Delta t}$$

$$= q_0\{F_0 + B_0(1)y(r^, 1) + (F_1 + B_1(1))e^{-r\Delta t}\} + (1 - q_0)\{F_0 + B_0(1)y(r^, 1) + (F_1 + B_1(1))e^{-r\Delta t}\}e^{-r\Delta t}$$

(3.7)

Notice, however, that Equation (3.7) requires searching all four paths along the interest rate lattice. For a three period problem $2^3 = 8$ paths need to be calculated. In general, for an n period problem, $2^n$ paths need to be computed. For a small problem, this is not difficult. But pricing a mortgage usually involves contracts which could extend to 30 years with monthly payments. This then requires searching for $2^{360}$ paths. Computationally, this is impossible. An efficient algorithm has to be developed to solve this problem.

### 3.6 The Valuation Models

We can see that the path-dependence associated with MBS results from periodic prepayments of principal. We now show how the complexity induced by path-dependent remaining balances, thus the cash flows, can be eliminated. This
simplification is based on the following fact: The functions for calculating fixed-level payment and remaining balances in the future are homogeneous of degree one in the original principal. Thus the value of the MBS at any point in time is also homogeneous of degree one in the original principal. Using this fact, a backward procedure can be developed to price an MBS by assuming a hypothetical remaining balance of $1 at any point in time. The value calculated will be rescaled as required when the pricing mechanism moves backward. This is best illustrated by an example.

Figure (3.3) displays a binomial lattice of interest rate together with
Figure 3.3
A 3-period numerical example on lattice showing the cash flows when the remaining balances are scaled by respective mortgage prices in the following periods as pointed out in the algorithm for pricing MBS.

*Note:* The CIR parameters are $\kappa=0.2$, $\mu=8\%$, $\sigma=0.15$ and $y_0=0.11$. Contract rate =0.10, $\Delta t=1/3$ year, and prepayment function $=\exp(-20r_{ij})$, where $r_{ij}$ is the short rate at node $(i,j)$.

probabilities of going up shown at each node. The relevant CIR parameters are: $\kappa=0.2$, $\sigma=0.15$, $\mu=8\%$, and initial rate is 11%. Contract rate of the MBS=10%, $\Delta t=1/3$ year, and prepayment function, $\exp(-20r_{ij})$. Associated with each node is a 5-dimensional vector, representing the fixed-level payment, remaining balance, prepayment at the up
node, and prepayment at the down node in the next period respectively, assuming $1 of remaining balance in the present node. The fifth row represents the mortgage price.

As an example, assume the backward recursion had established the vectors at nodes (1,1) and (1,0). We show how the vector of information at node 0 is obtained. First, assume a $1 balance remains. The first bit of data, fixed-level payment, which is scheduled to be made in periods 1, 2, and 3, is calculated by Equation (3.1) and is equal to 0.354. The second bit of data, remaining balance in the next period, is calculated by Equation (3.2), and is equal to 0.676. The fraction of the remaining balance that will be prepaid, if an up jump occurs, is equal to $e^{-20(0.11)}=0.041$, and the actual amount of prepayment is equal to 0.041(0.676)=0.028. Similarly, the fraction of the remaining balance that will be prepaid, if a down jump occurs, is equal to $e^{-20(0.06)}=0.128$, and the actual amount of prepayment is equal to 0.128(0.676)=0.087. The remaining balances after the prepayment occurs are 0.676-0.028=0.648, and 0.676-0.087=0.589, at the up node and down node, respectively. We know from the fifth bit of data at node (1,1) that the price of MBS is worth 0.976, if it is originated from node (1,1) with a $1 remaining balance, same contract rate (10%), and 2 periods till maturity. Now, we have $0.648 as remaining balance. With the homogeneous property of degree in the principal, we scale $0.976 by 64.8% to get the price of MBS, 0.632, if the principal of the mortgage loan here is $0.648 at origination. Similarly, the scaled remaining balance at node (1,0) is 1.003(58.9%)= 0.590. In summary, three pieces of cash flows, if an up jump occurs, are
<table>
<thead>
<tr>
<th>Component</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>fixed-level payment</td>
<td>0.354</td>
</tr>
<tr>
<td>prepayment</td>
<td>0.028</td>
</tr>
<tr>
<td>pass-through price</td>
<td>0.632</td>
</tr>
</tbody>
</table>

**Expected value at node (1,1):** 1.014

And the cash flows, if a down jump occurs, are

<table>
<thead>
<tr>
<th>Component</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>fixed-level payment</td>
<td>0.354</td>
</tr>
<tr>
<td>prepayment</td>
<td>0.087</td>
</tr>
<tr>
<td>pass-through price</td>
<td>0.590</td>
</tr>
</tbody>
</table>

**Expected value at node (1,0):** 1.031

Weighted by the associated probabilities, and discounted by the risk-adjusted interest rate, \( r_o = 0.11 \), the price of MBS at date 0 is equal to

\[
[0.433(1.014)] + (1 - 0.433)(1.031)] e^{-0.11(1/3)} = 0.987
\]

### 3.6.1 Algorithm for Pricing MBS

Now we develop a general algorithm for pricing MBS. A 2-step procedure is employed. In the first step, the forward procedure is used to establish the binomial lattice of interest rate process as discussed in Section 2.4. In the second step, the backward recursion is constructed to establish the price of MBS at date 0. We shall focus on the second step.

Figure (3.4) illustrate the procedure for a 3-period problem to find the price of an MBS. We have explained the information contained in the
5-dimensional vector at each node in the above example. The general backward recursion for the price of MBS at node \((i,j)\), with \$1 as remaining balance, and \((n-i)\) as the life of the contract, can be calculated by discounting the fixed-level payment, principal prepayment, and rescaled remaining balance in the next period using risk-adjusted interest rate. Specifically,

\[
M_{i,j} = (F_{i,j} + B_{i,j}(1)q_{i,j}[(y(r_{i+1,j+1} + 1) + (1 - y(r_{i+1,j+1} + 1))M_{i+1,j+1})]e^{-r_{i,j} \Delta t} +

|B_{i,j}(1)\overline{q}_{i,j}[y(r_{i+1,j} + 1) + (1 - y(r_{i+1,j} + 1))M_{i+1,j}]e^{-r_{i,j} \Delta t}
\]

(3.8)
Recall that the BASD has two types of jumps, a regular up jump and a reflecting up jump. By regular up jumps, we meant those up jumps with probabilities between 0 and 1. If this is the case, Equation (3.8) can be employed. However, if a reflecting up jump occurs, the jump size needs to be fixed according to BASD algorithm in Appendix A. As an example, often times, the up-jump will go to \((i+1,j+3)\) and the associated "down" jump will go to \((i+1,j+1)\). Thus, the backward recursion must be fixed by

\[
M_{ij} = \{F_{ij} + B_{ij}(1)q_{ij}[y(r_{i+1,j+3}, i+1) + (1 - y(r_{i+1,j+3}, i+1))M_{i+1,j+3}]e^{-\tau_{ij}U}\}
\]

\[
|B_{ij}(1)F_{ij}[y(r_{i+1,j+1}, i+1) + (1 - y(r_{i+1,j+1}, i+1))M_{i+1,j+1}]e^{-\tau_{ij}U}| \tag{3.9}
\]

### 3.6.2 Equilibrium Contract Rate

In the discussion so far, we assumed that the contract rate, \(c\) is given, and it is used to amortize the loan to get the fixed level payment using Equation (3.1). In this section, we identify an equilibrium value for \(c\). In an equilibrium, this rate must be chosen in such a way that no arbitrage opportunities exist. In other words, the initial mortgage price has to be priced at par. Specifically, we want the contract rate, \(c^*\), to justify

\[
M_0(c^*) = B(0) \tag{3.10}
\]
Since the mortgage price $M_p(c)$ is monotonically increasing function of the contract rate $c$, the equilibrium contract rate can be found by iterative search as outlined in Figure (3.5).

![Image of a diagram](image)

**Figure 3.5**
Procedure for Finding Equilibrium Contract Rate Using Iterative Search

The striking point of our algorithm is the reduction it brings about in the amount of computation required. Our algorithm is written in Pascal language. We use IBM PC 486 DX to compute the contract rate of MBS and the price can be reached within one minute for a 20-year contract amortized monthly. Equilibrium contract rate can be typically found within about 7 iterations. This means we establish equilibrium contract rates less than 10 minutes, which is quite attractive.
3.7  Sensitivity Analysis

In this section, equilibrium contract rates are provided when the values of key CIR parameters are given by

$$\kappa = 0.1, 0.2, 0.3$$
$$\sigma = 0.05, 0.075, 0.1, 0.125, 0.15, 0.175, 0.20$$
$$\mu = 8\%$$
$$r_0 = 0.05, 0.11$$

These parameters cover the range of key CIR parameters that are recently used in the literature for pricing mortgages. For example, in valuing pass-throughs, Schwartz and Torous use $\kappa=0.1$, $\sigma=0.075$, and $\mu=0.065$, which are suggested by Bunce, MacRae, and Szymanoski (1988). Kau et.al. (1990) use $\kappa=0.15$, $\sigma=0.05, 0.15$, $r_0=0.08, 0.12$, and $\mu=0.1$ in pricing adjustable-rate mortgages.

The parameters in our sensitivity analysis yield term structures, which revert to the long-run average short rate of 8%. We also assume that prepayment dynamics is governed by

$$y(r,t) = \begin{cases} 
e^{-20r} & t = 0, 1, 2, ..., n - 1 \\ 1 & t = n \end{cases}$$

(3.11)

The prepayment function is plausible, for when interest rate decreases, the prepayment fraction is increasing. When interest rate reaches 0, all of the remaining balance will be
paid down. Figure (3.5) provides the procedure used to establish the equilibrium contract rates.

Table 3.1 shows results produced by the above parameters and prepayment function. We can observe that the equilibrium contract rates for 10 year maturity are lower than those for 20 year maturity. However, their direction of changes in changes of the parameters are quite similar. In the following discussion, we shall concentrate on the analyses of 10-year contracts.
### TABLE 3.1
Comparison of Contract Rates between Straight Amortization Bond, Callable Amortization Bond and MortgageBacked Security

<table>
<thead>
<tr>
<th>Year</th>
<th>SX</th>
<th>CY</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5%</td>
<td>11%</td>
</tr>
<tr>
<td>10</td>
<td>0.1</td>
<td>0.050</td>
</tr>
<tr>
<td>10</td>
<td>0.1</td>
<td>0.050</td>
</tr>
<tr>
<td>10</td>
<td>0.1</td>
<td>0.075</td>
</tr>
<tr>
<td>10</td>
<td>0.1</td>
<td>0.100</td>
</tr>
<tr>
<td>10</td>
<td>0.1</td>
<td>0.125</td>
</tr>
<tr>
<td>10</td>
<td>0.1</td>
<td>0.150</td>
</tr>
<tr>
<td>10</td>
<td>0.1</td>
<td>0.175</td>
</tr>
<tr>
<td>10</td>
<td>0.1</td>
<td>0.200</td>
</tr>
<tr>
<td>10</td>
<td>0.2</td>
<td>0.025</td>
</tr>
<tr>
<td>10</td>
<td>0.2</td>
<td>0.050</td>
</tr>
<tr>
<td>10</td>
<td>0.2</td>
<td>0.075</td>
</tr>
<tr>
<td>10</td>
<td>0.2</td>
<td>0.100</td>
</tr>
<tr>
<td>10</td>
<td>0.2</td>
<td>0.125</td>
</tr>
<tr>
<td>10</td>
<td>0.2</td>
<td>0.150</td>
</tr>
<tr>
<td>10</td>
<td>0.2</td>
<td>0.175</td>
</tr>
<tr>
<td>10</td>
<td>0.2</td>
<td>0.200</td>
</tr>
<tr>
<td>10</td>
<td>0.3</td>
<td>0.025</td>
</tr>
<tr>
<td>10</td>
<td>0.3</td>
<td>0.050</td>
</tr>
<tr>
<td>10</td>
<td>0.3</td>
<td>0.075</td>
</tr>
<tr>
<td>10</td>
<td>0.3</td>
<td>0.100</td>
</tr>
<tr>
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<td>0.3</td>
<td>0.125</td>
</tr>
<tr>
<td>10</td>
<td>0.3</td>
<td>0.150</td>
</tr>
<tr>
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<td>0.3</td>
<td>0.175</td>
</tr>
<tr>
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<td>0.200</td>
</tr>
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</tr>
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</tr>
<tr>
<td>10</td>
<td>0.3</td>
<td>0.275</td>
</tr>
<tr>
<td>10</td>
<td>0.3</td>
<td>0.300</td>
</tr>
<tr>
<td>10</td>
<td>0.3</td>
<td>0.325</td>
</tr>
<tr>
<td>10</td>
<td>0.3</td>
<td>0.350</td>
</tr>
<tr>
<td>10</td>
<td>0.3</td>
<td>0.375</td>
</tr>
<tr>
<td>10</td>
<td>0.3</td>
<td>0.400</td>
</tr>
</tbody>
</table>

The instantaneous short rate follows CIR process. The long-term mean, and time interval are 8% and one month respectively. The initial rates are 5% and 11% as shown.
Now we compare the equilibrium contract rates of MBSs to those of SABs and CABs as the volatility changes, given $\kappa$ is held constant at 0.1. Figure (3.6) shows their relationship. As expected, equilibrium rates associated with CABs are higher than those of MBSs and SABs. This makes sense because in CABs, the investors have the full flexibility of exercising the prepayment option in an opportune time. Investors who purchase CABs, will therefore demand a higher rate than contracts without this call feature. The difference of equilibrium rates between CAB and SAB, where no prepayment is allowed, is the "value of optimal prepayment" as shown in the figure. The prepayment option in mortgages is less valuable than the prepayment option in CABs. The difference between the values of these two prepayments is the value of "irrational prepayment" as also shown. If all of the investors in mortgages act rationally, the value of irrational prepayment would be 0, and the middle curve would disappear.
Figure 3.6
Cross-Security Analysis

Note: Interest rate follows CIR process with long-term mean = 8%. Binomial approximation uses BASD. Prepayment function $y(r,t) = \exp(-20t)$. Initial rate = 0.11. Kappa = 0.1.
Figure (3.7) shows a sensitivity analysis when $r_o=0.05$. Equilibrium contract rates of MBSs are not higher than SABs in small volatility. This is plausible, because when initial rate is small, the fraction of mortgagors, who will prepay, is larger than that, when initial rate is high, during the initial period of the contract. If the price used to compensate for the irrational prepayment is larger than the "value of suboptimal prepayment", then the contract rates of MBSs will be smaller than those of SABs.
Figure 3.7
Cross-Security Analysis

Note: Interest rate follows CIR process with long-term mean = 8%. Binomial approximation uses BASD. Prepayment function $y(r,t) = \exp(-20r)$. Initial rate = 0.05. Kappa = 0.1.
Effective maturity date decreases as interest rates decrease. As a result, the value of a mortgage will depend quite heavily on the levels of interest rates on the short term. This behavior is confirmed in Figure (3.8). We can also see that the contract rates of MBS's don't change too much with variations in volatility and kappa, when initial rate is small. This tells that in a small initial rate environment, a great portion of mortgagors will prepay earlier in time when the interest rate is expected to revert to the long-term mean that is higher than the initial rate. Since, as discussed, the interest rates in the near future count more than those in the long-run, the equilibrium rates don't change too much in changes of kappa and volatility.

We also observe in these figures that when volatility is low, the equilibrium contract rates are influenced more by kappa than by volatility. Then the trend reverses with the increases of volatility. A typical example can be shown in Figure (3.9), with initial rate=0.05. We can see that when volatility is small, the larger kappa, the larger equilibrium rates. This trend is gradually diminishing as volatility increases, and is completely reversed after some threshold volatility value. This explanation can also be applied to other cases. For example, in the same figure with initial rate equal to 0.11, and SAB in Figure (3.10), although we observe that contract rates are higher when kappa is small, from the trend of the lines, we would expect that when the volatility is smaller than some value, kappa is a dominant effect.
Figure 3.8
GNMA Pass-Through

Note: Interest rate follows CIR process with long-term mean = 8%. Binomial approximation uses BASD. Prepayment function y(r,t) = exp (-20r).
Figure 3.9
Callable Amortization Bond

Note: Interest rate follows CIR process with long-term mean = 8%. Binomial approximation uses BASD.
Figure 3.10
Straight Amortization Bond

Note: Interest rate follows CIR process with long-term mean = 8%. Binomial approximation uses BASD.
3.8 Conclusion

In this chapter, we discussed the mechanisms for pricing mortgage-related products whose contract rates are fixed during the life of the contract. The methods for pricing SABs and CABs were first developed on a lattice of interest rates. Then path-dependence associated with MBS was presented by using one and two-period examples. An efficient algorithm was then developed for pricing MBS. In our sensitivity analysis, we assumed away transaction costs, points, taxes, servicing costs, and other market imperfections. Servicing fees can easily be incorporated into the analysis. Specifically, Equation (3.1) needs to be modified to reflect that a fraction of this fixed level cash payment flows to the institution that services the mortgages. The cash flows from GNMA’s is not directly influenced by the transaction costs associated with prepayments. Of course, the decision of any homeowner to prepay may well depend on these transaction costs. In all existing models, this behavior is captured in the prepayment function. Finally, if the contract being valued is an individual fixed-rate mortgage, where prepayment is done for financial reasons alone (i.e., the mortgage is really a CAB), then the transaction costs (points, etc.) can be modeled in the stopping rule. In particular, Equation (3.4.b) needs to be modified to include the transaction costs.

The last part of this chapter provided a sensitivity analysis of equilibrium mortgage rates to key parameters driving interest rate uncertainties. The analysis revealed that the prepayment feature increased in values as the volatility of rates
expanded. The values of prepayments due to suboptimal non-financial reasons was also captured. In the next few chapters, models will be developed for more complex mortgage-related products.
CHAPTER 4
INDEX AMORTIZATION SWAP

4.1 Introduction

In a conventional mortgage product, the cash flows come from three sources, namely, scheduled interest payments, scheduled principal paydowns, and unscheduled principal prepayments. As discussed in Section 2.3, modeling unscheduled prepayments involve understanding "optimal" and "suboptimal" prepayments. "Optimal" prepayments are based on financial information alone, whereas "suboptimal" prepayments stem from noneconomic factors, such as relocation, death, change of job, or divorce. The latter are quite difficult to predict. In fact, as discussed in Chapter 2, a lot of effort in pricing conventional mortgages is spent on developing models describing the suboptimal prepayments.

In order to eliminate the uncertainties associated with suboptimal prepayments, recently a new mortgage-related product, named mortgage-indexed swap (MIS), has emerged. This contract has similar cash flows to ordinary pass-throughs. In particular, interest payments are based on a hypothetical balance (or notional principal) that is agreed upon at the origination. "Prepayments" on this notional principal have the effect of reducing the total notional principal. The prepayment experience, however, is linked analytically to financial variables that are observable to all market participants. For example, the notional principal of such contract can be amortized based upon the
prepayment experience of a specific pool of pass-throughs. As a second example, the amortization could be based upon the future levels of an interest rate index, such as three-month London inter-bank offer rate (LIBOR). In particular, the notional principal could amortize rapidly if LIBOR falls, slowly if LIBOR rises. The payoffs of these contracts are thus highly correlated to payouts of pass-throughs. As such, we call such contracts synthetic mortgages. The above synthetic security is known as index amortization swap (IAS). With approximately one hundred billion notional principal outstanding, this is perhaps the fastest growing contracts in the swap market.\footnote{As an example, Society National Bank currently in Cleveland has 1.5 billion notional principal invested in index amortization swaps.}

Pricing an IAS can be accomplished using Monte Carlo simulation. This methodology is useful, especially when the security has complex features, but does not take into account any American style options. For example, an IAS contract with an option to cancel at any point in time could not be valued using simulation.

To date, there is no literature discussing the pricing issues of IAS in the presence of option features. Our contribution in this research is to develop an efficient algorithm to price the IAS on a binomial lattice of interest rates. The algorithm developed can easily be adapted to include a wide variety of option features.

The remainder of this chapter is organized as follows. Section (4.2) gives a discussion of IASs. Section (4.3) introduces notation and assumptions. Section (4.4) identifies the path-dependent issues that arise when pricing an IAS. Section (4.5)
develops efficient valuation models for IAS. A four-period numerical example is established to illustrate the valuation process. Section (4.6) discusses the sensitivity of IAS rates to changes in key model parameters, when a typical amortization schedule is employed. Section (4.7) gives conclusions.

4.2 Overview of Index Amortization Swaps

An IAS can be treated as a sequence of cash flows where the notional principal amortizes over the life of the contract. On each payment date a fixed-rate payor makes an interest payment at a fixed coupon rate in exchange for the receipt of a floating rate payment from the fixed-rate receiver. The floating coupon rate is equal to the periodic floating rate index plus a floating rate spread. Specific interest rate payments are based on a swap notional amount, which is amortized along the life of the swap according to the amortization schedule linked to changes in a specific interest rate index, such as LIBOR, five- or ten-year constant maturity Treasury during the term, ranging from 2 to 10 years. No principal is exchanged. Usually, there is a lockout period associated with IAS, during which no amortization takes place, regardless of movements of interest rate index and a clean-up call provision that allows the issuer to call the contract when the notional amount outstanding falls below a certain threshold percentage of the original notional principal (typically about 5%).
4.3 Notation and Assumptions

The amortization of the notional principal is a key feature of an IAS. We consider IASs with geometric amortization, where a specific proportion of the notional amount is reduced, at each amortization application date.

Let \( b_0 < b_1 < ... < b_n \) be the amortization application dates. The notional amount at time \( b_i \) is the product of the previous notional amount at time \( b_{i-1} \) and a term, \( \lambda(b_{i-1}, L_{i-1}) \), which, at date \( b_{i-1} \), depends on the level of an interest index, \( L_{i-1} \). An example of the functional form of \( \lambda \) is given in Table (4.1). This function is referred to as the notional percentage remaining (NPR).
Table 4.1

2-Year Lockout/5-Year Final Maturity
Index Amortization Swap

<table>
<thead>
<tr>
<th>First 7 Quarters:</th>
<th>No Amortization (Lockout Period)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quarters 8 to 19:</td>
<td>Amortization According to the</td>
</tr>
<tr>
<td></td>
<td>Schedule Shown Below.</td>
</tr>
<tr>
<td></td>
<td>Reference Rate = 3.5% (3 month LIBOR</td>
</tr>
<tr>
<td></td>
<td>Interest Rate Index).</td>
</tr>
<tr>
<td>Last Quarter (20):</td>
<td>100% of Remaining Balance Amortized</td>
</tr>
<tr>
<td></td>
<td>(Maturity).</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>3-Month LIBOR Less</th>
<th>Remaining Balance</th>
<th>Average Life</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reference Rate</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Unchanged or Below</td>
<td>100.00%</td>
<td>2.0 years</td>
</tr>
<tr>
<td>100 bp Increase</td>
<td>11.26 %</td>
<td>3.5 years</td>
</tr>
<tr>
<td>200 bp Increase</td>
<td>2.84 %</td>
<td>4.5 years</td>
</tr>
<tr>
<td>300 bp Increase or</td>
<td>0.00 %</td>
<td>5.0 years</td>
</tr>
<tr>
<td>More Amortization of</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table (4.1) shows a typical amortization schedule used in the market. The amortization is based upon changes in interest rates. A reference interest rate index is employed to determine whether interest rates are low or high. The reference rate is

---

determined when the IAS becomes effective. Like the maturity date, the reference rate does not change thereafter. In this example, the reference rate is a 3.5% three-month LIBOR. During the two-year through five-year amortization period and on any of the quarterly amortization dates, if three-month LIBOR is unchanged or has fallen, or with a 3.50% reference rate if three-month LIBOR is 3.50% or less, then 100% of the IAS's outstanding balance will amortize in that quarter. At the other extreme, if three-month LIBOR has risen by 300 basis point, or with a 3.50% reference rate if three-month LIBOR is 6.50% or above, then none of the swap's outstanding balance will amortize. The amortization schedule also specifies the amortization for interest rate changes that are intermediate between these two extremes. For interest rate changes that are not explicitly shown in this table, we use a linear interpolation. For example, if during the two-year through five-year amortization period interest rates had increased by 50 basis points on any of the quarterly amortization dates, then 55.63% of the swap's outstanding balance would amortize.

From the amortization of notional amount, indeed, the notional amount outstanding at date \( b_i \), depends on the entire evolution of interest rates, \( (r_0, r_1, ..., r_{i-1}) \) from date 0 to date \( i-1 \) of previous amortization application date. Specifically, the notional amount at time \( b_i \), is represented by

\[
N(b_i, r_0, r_1, ..., r_{i-1}) = N(b_{i-1}, r_0, r_1, ..., r_{i-2}) \lambda(b_{i-1}, L_{i-1}), \quad i \geq 1
\]
For ease of using notation, denote $N(b) = N(b, r_0, r_1, ..., r_n)$. We treat $N(b)$ as a jump process in continuous time. As a result, the notional amount is constant over each time interval $[b_i, b_{i+1})$. Let $t_0 < t_1 < ... < t_n$, be all the dates at which coupon payments are made. The coupon payment at time $t_i$ is based on the coupon rates at time $b_i$ ($b_i < t_i$). In particular, the size of the coupon payment is

$$CF(t_i) = C(b_i, L_i)N(b_i)$$  \hspace{1cm} (4.2)

where $C(b, L_i)$ is the coupon rate based on some interest rate index, $L$, at time $b$, $N(b)$ is the notional amount at date $b$. Normally, $t_i = b_{i-1}$.

Toward the goal of developing pricing models, we make the following assumptions.

- **Interest rate follows the square-root process, and all bonds are priced as in the general equilibrium model of CIR.**
- **CIR process is approximated by BASD (binomial approximation for singular diffusions)**;
- **The market is perfect. This means no transaction and refinancing costs exist.**

In addition, the following notation is used

- $\Delta t$ = Length of the interval.
- $(i, j)$ = Node on the binomial lattice, constructed by the 3-step procedure proposed in Section (2.4).
- $r_{ij}$ = Spot rate at node $(i, j)$. The initial value $r_{0,0}$ is $r_0$.

\[\text{In fact, } CF(t_i) \text{ also depends on the entire path of interest rates.}\]
\( C_{ij} \)  = Interest rate index (coupon rate) at node \((i,j)\). The initial value \( C_{a0} \) is represented by \( C_a \). Under the CIR model, once the instantaneous rate is known, the entire term structure of interest rates can be recovered. The interest rate index may be a single point taken off this resulting yield curve. If the index is based on LIBOR, then an appropriate known risk premium is added.

\( \lambda(r_{ij}) \)  = At node \((i,j)\) fraction of notional amount remains in the next period, with prevailing short rate equal to \( r_{ij} \).

\( q_{ij}(\tilde{q}_{ij}) \)  = Risk-adjusted probability of an up (down) jump in interest rates from node \((i,j)\). Under this measure, all bonds can be priced as if the local expectations hypothesis holds.

\( P_{ij}(k) \)  = Price of a discount bond viewed from node \((i,j)\) maturing in period \( k, k>i \). The initial bond prices at node \((0,0)\) are \( P_{a}(k) \).

Additional notation will be introduced when necessary.

### 4.4 Path-Dependence Associated with IAS

The notional amount at each point in time depends on the interest rates path taken from date 0 to present. This means that the cash flows of an IAS are path-dependent. In this section, we show how path-dependence manifests itself.
Specific algorithms for resolving this difficulty will be discussed in the following sections.

Path-dependence is best illustrated by an example. Figure (4.1) shows a 3-period lattice of interest rates. We start off at period 0, with $N$ as the original notional principal. The cash flow in period 1 is equal to the original principal times the current coupon rate, $C_0$. The cash flow, $NC_0$ is indicated at nodes (1,1) and (1,0). The next payment can be computed as an example. At node (1,1), the notional amount now is $N\lambda(r_{1,1})$, and the coupon rate is $C_{1,1}$, where $C_{1,1}$ is fully determined by the yield curve at that time. The cash flow for the next period is then given by $N\lambda(r_{1,1})C_{1,1}$. Similarly, at node (1,0), payment, which equals $N\lambda(r_{1,0})C_{1,0}$, is to be made in period 2.
For the two period problem, the possible cash flow sequences are $NC_0$ followed by $N\lambda(r_{1,1})C_{1,1}$ or $NC_0$ followed by $N\lambda(r_{1,0})C_{1,\sigma}$. However, when extended to three periods as shown in Figure (4.1), four different paths need to be searched, namely

$$\{NC_0, N\lambda(r_{1,1})C_{1,1}, N\lambda(r_{1,1})\lambda(r_{2,2})C_{2,2}\},$$
$$\{NC_0, N\lambda(r_{1,1})C_{1,1}, N\lambda(r_{1,1})\lambda(r_{2,1})C_{2,1}\},$$
$$\{NC_0, N\lambda(r_{1,0})C_{1,0}, N\lambda(r_{1,0})\lambda(r_{2,1})C_{2,1}\},$$
$$\{NC_0, N\lambda(r_{1,0})C_{1,0}, N\lambda(r_{1,0})\lambda(r_{2,0})C_{2,0}\}.$$

When this problem extends to $n$ periods, the number of different cash flows is $2^{n-1}$. The fair price of an IAS is obtained by computing the expected cash flows for each path and discounting them to the initial date. If $n$ is large, the computational effort is very heavy and even impossible.

### 4.5 The Valuation Models

In this section, the algorithms for pricing IAS will be developed. We first analyze IAS with no clean-up provision, and then extend it to a algorithm with a clean-up provision. The algorithm incorporating lockout period will be shown as a simple extension of these algorithms. Here we consider valuing only the sequence of cash flows coming from the floating-rate side. Once this is established, equilibrium fixed rate can be searched by equating the present value of cash flows from fixed-rate payments to that from floating-rate payments.
4.5.1 Pricing IAS with no Clean-up Call

It is obvious that the IAS has the same path-dependent structure as MBS does, for the notional amount at each point in time depends on the cumulative amortization of the notional amount, thus on the stochastic evolution of interest rates.

First consider an IAS with notional principal of $1 at the beginning. The notional amount at date \( b_i \) is then given by

\[
N(b_i) = \prod_{k=0}^{i-1} \lambda(b_k, L_k)
\]  

The present value of the sequence of the cash flows is

\[
V(0) = \tilde{E}_0 \left\{ \sum_{i=0}^{n-1} N(b_i)C(b_i, L_i)P_0(b_{i+1}) \right\}
\]

where \( C(b, L) \) is the coupon rate at time \( b \), when the interest rate index level is \( L \), and expectation is taken under the equivalent martingale measure. Our goal is to value Equation (4.4) using backward recursion.

We solve this problem by the noting that the price of IAS is homogeneous of degree 1 in notional amount. As a result, if we can solve this problem for a notional amount of $1, then we have solved the problem. In addition, we do not need to keep track of the notional amount.

Now consider pricing an IAS with initial notional principal of $1 right at date \( b_i \). It has no coupon payments and amortization before time \( b_{i+1} \). Other than these, it
has the same features as those of the IAS originated from date 0. Denote $V(b, r)$ the time-$b$, value with notional amount of $1$ outstanding, then

$$V(b, r) = E_i \left[ \sum_{j=i}^{n-1} \left( \prod_{k=i}^{j-1} \lambda(b_k, L_k) \right) C(b_j, L_j) P_i(b_{j+1}) \right]$$

$$= \left\{ C(b_i, L_i) + \lambda(b_i, L_i) E_i \left[ V(b_{i+1}, \cdot) | r_i \right] \right\} P_i(b_{i+1}) \quad (4.5)$$

Equation (4.5) says the price of IAS at time $b$, when the interest rate is $r$, is to discount the payment and expected scaled notional amount in the next period to present. The direct implementation of Equation (4.5) on a binomial lattice of interest rates is shown in Figure (4.2). The price of IAS at node $(i, j)$ can be evaluated by

![Figure 4.2](image)

The cash flows of IAS on a binomial lattice at node $(i, j)$.

$$V_{ij} = [C_{ij} + \lambda_{ij} (r_{ij} (q_{ij} V_{i+1,j+1} + \overline{q}_{ij} V_{i+1,j}))] e^{-r_i \Delta t} \quad (4.6)$$

with $V_{n, j}=0$. When node $(i, j)$ is a singular node, then Equation (4.6) must be fixed. As an example, if node $(i, j)$ is on the entrance boundary, then
\[ V_{i,j} = [C_{i,j} + \lambda_{i,j}(r_{i,j})(q_{i,j}V_{i+1,j+3} + q_{i,j}V_{i+1,j+1}')]e^{-r_i \Delta t} \]

### 4.5.2 Pricing IAS with Clean-up Call

Unfortunately, if a clean-up call is involved, then it is important to know the notional amount at each node. The reason for this is that the clean-up call is exercised whenever the notional amount falls below the prespecified number. In order to cope with the difficulty, the state space is augmented by adding an indicator, \( q \), which describes the proportion that the notional amount, \( S_1 \), consists of the original notional principal. Obviously, \( q \) is ranging from the clean-up threshold to 1. A discretization procedure is then used to split the interval \([0,1]\) into \( h \) equal sub-intervals, each of \( \Delta q = 1/h \). Denote \( q_f = f \Delta q \), for \( f = 0,1,2,\ldots,h \). Let \( V(b_i,r_i,q_f) \) be the time-\( b_i \) value when the interest rate is \( r_i \) and indicator is \( q_f \). The value of the indicator in the next amortization application date \( b_{i+1} \) is

\[
\bar{q}_f \equiv q_f \lambda(b_i). \tag{4.7}
\]

We then have

\[
V(b_i,r_i,q_f) = E \left[ \sum_{j=1}^{n} \left( q_f \prod_{k=1}^{j-1} \lambda(b_k, L_k) \right) C(b_j, L_j) P_i(b_{j+1}) \right]
\]

\[
= \{ q_f C(b_i, L_i) + E[V(b_{i+1}, \cdot, \bar{q}_f)] | r_i \} P_i(b_{i+1}) \tag{4.8}
\]

Similarly, Equation (4.8) can be implemented on a binomial lattice of interest rate. However, linear interpolation has to be employed, for \( \bar{q}_f \) may not exactly fall on the values of the integral multiple of \( \Delta q \). Define \( s_f \) be the distance from \( \bar{q}_f \) to the lower
end, \( \overline{q_f}' \) of this particular grid, and \( t_f \) be the distance from \( \overline{q_f} \) to the high end, \( \overline{q_f}'' \) of the same grid as shown in Figure (4.3). We then have the following recursive equation

\[
V_{ij}(q_f) = \left[ q_{ij}C_{ij} + q_{ij} \left[ \frac{s_f}{\Delta q} V_{i+1,j+1}(\overline{q_f}') + \frac{t_f}{\Delta q} V_{i+1,j+1}(\overline{q_f}'') \right] + q_{ij} \left[ \frac{s_f}{\Delta q} V_{i+1,j}(\overline{q_f}') + \frac{t_f}{\Delta q} V_{i+1,j}(\overline{q_f}'') \right] \right] e^{-\kappa \Delta t}
\]

(4.9)

Observe that the above mechanism for pricing IAS when clean-up provision is considered or not considered can easily be modified to incorporate a lockout period is included. This would be just to discount the values of IAS on the lattice from the period at the end of the lockout to date 0 as of valuing a zero-coupon bond.

Figure (4.4) presents a numerical example. First, a binomial lattice of interest
A 4-period example on lattice showing the cash flows when the remaining balances are scaled by respective IAS prices in the following period as pointed out in the algorithm for pricing IAS.

Note: The CIR parameters are $\kappa=0.2$, $\sigma=0.1$, and $\gamma_0=0.05$. Base rate=$0.035$; $\Delta t=1/4$ year; amortization schedule is as shown in Table 4.1.

rate together with probabilities of going up shown at each node is constructed. The relevant CIR parameters are: $\kappa=0.2$, $\sigma=0.10$, $\mu=8\%$, and initial rate is 5%. $\Delta t=1/4$ year. The first bit of data provides the coupon payment that will be made in the next period. The second bit of data gives the notional amount in the next period after the notional amount in the current period is amortized. The last bit of data then provides the price of IAS calculated by backward procedure. As an example, let's focus on period 1 and 2. Coupon payment needs to be made at node (1,0) and (1,1) is 0.051,
with probability 0.539 going up.\(^4\) The proportion that will be amortized in the next period can be calculated by

\[
11.26\% - \frac{(0.051 - 0.035) - 0.01}{(0.02 - 0.01)}(11.26\% - 2.84\%) = 0.06208
\]

Then NPR then is equal to 1 - 0.06208 = 0.938. Scale the notional amount by the corresponding prices of IAS in period 1, and discount them back with coupon payment, we have price of IAS at date 0 as follows

\[
[0.051 + 0.539(0.938 \cdot 0.184) + 0.461(0.938 \cdot 0.109)]e^{-0.050(1/4)} = 0.195
\]

4.6 Numerical Examples

Using the algorithms developed in Section (4.5), we provide numerical results in this section. First, specific practices of implementing LIBOR rate, and amortization schedule, and procedure for finding fixed-rate are described. Second, the results produced are then discussed.

4.6.1 LIBOR Rate

*LIBOR* rate is "the rate at which major money center banks are willing to place Eurodollar time deposits at other major banks." Eurodollar time deposits are non-negotiable, fixed-rate U.S. dollar deposits in banks that are not subject to U.S. banking regulations. These banks are not located only in Europe.

The LIBOR is an important benchmark rate, for U.S. banks commonly charge the LIBOR plus a certain number of basis points on their floating-rate loans. The LIBOR is quoted on an add-on yield basis, meaning that it is a percentage of the

\(^4\) The formula for calculating the LIBOR rate, thus coupon payment is shown in Section (4.6.1). In particular, LIBOR rate is equal to 5.1%. 
Eurodollar time deposit purchase amount. It is an annualized rate based on a 360-day year. For example, if the 3-month (90 day) LIBOR is 8%, the interest on $1 million is

\[ \text{Interest} = (0.08)(90/360)(\$1 \text{ million}) = \$20,000. \]

4.6.2 Swap Rate

A no-arbitrage condition requires that an equilibrium swap rate be selected such that the present values of the payments of two sides are equal. Since the value of the fixed-side payments is monotonically increasing function of the fixed rate, the equilibrium fixed rate can be found by iterative search. Figure (4.5) outlines the procedure for establishing the equilibrium fixed rate.

![Figure 4.5](image_url)

Figure 4.5
Procedure for Finding Equilibrium Swap Rate Using Iterative Search
4.6.3 **Results and Discussion**

We use the same range for the parameters of interest rates as in Chapter 3, namely

\[
\kappa = 0.1, 0.2, 0.3 \\
\sigma = 0.05, 0.075, 0.1, 0.125, 0.15, 0.175, 0.2 \\
\mu = 0.08 \\
r_0 = 0.05, 0.11
\]

Our results produced by the programs (IAS01.PAS [for IAS without clean-up call, and lockout period], IAS02.PAS [for IAS without clean-up call, but with lockout period], IAS03.PAS [for IAS with clean-up call, but without lockout period], in the Appendix B) showed that the clean-up call plays quite an insignificant role in determining the fixed rates. In the following discussion, we shall focus only on the fixed rates between IASs without lockout provision and with lockout provision.

From Table (4.2), we see that with \( \kappa \), and \( r_0 \) fixed, swap rate increases as the volatility increases. In a volatile economy, the floating-side payor is incurred higher risk. In order to compensate for their position, the fixed-side payor is charged with higher interests. Although the amortization is automatically triggered, rather than exercised at the choice of the issuer (as in normal option), there is optionality in this structure. The premium is the investor's compensation for being exposed to potential adverse LIBOR moves, that is, to the volatility of LIBOR rates.
# TABLE 4.2
Comparison of Fixed Rates between Lockout Considered and Not Considered

<table>
<thead>
<tr>
<th>$K$</th>
<th>$\sigma$</th>
<th>5%</th>
<th>11%</th>
<th>5%</th>
<th>11%</th>
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<tbody>
<tr>
<td>0.1</td>
<td>0.050</td>
<td>0.060314</td>
<td>0.105013</td>
<td>0.065208</td>
<td>0.065208</td>
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<tr>
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<td>0.075</td>
<td>0.064298</td>
<td>0.105999</td>
<td>0.071382</td>
<td>0.071382</td>
</tr>
<tr>
<td>0.1</td>
<td>0.100</td>
<td>0.068303</td>
<td>0.108853</td>
<td>0.077842</td>
<td>0.077842</td>
</tr>
<tr>
<td>0.1</td>
<td>0.125</td>
<td>0.072329</td>
<td>0.112878</td>
<td>0.084829</td>
<td>0.084829</td>
</tr>
<tr>
<td>0.1</td>
<td>0.150</td>
<td>0.076085</td>
<td>0.117496</td>
<td>0.091377</td>
<td>0.091377</td>
</tr>
<tr>
<td>0.1</td>
<td>0.175</td>
<td>0.079675</td>
<td>0.122155</td>
<td>0.097529</td>
<td>0.097529</td>
</tr>
<tr>
<td>0.1</td>
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<td>0.083182</td>
<td>0.127239</td>
<td>0.103550</td>
<td>0.103550</td>
</tr>
<tr>
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<td>0.100604</td>
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</tr>
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</tr>
<tr>
<td>0.2</td>
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<tr>
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<td>0.094937</td>
</tr>
<tr>
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<td>0.100210</td>
</tr>
<tr>
<td>0.3</td>
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<td>0.065107</td>
<td>0.097273</td>
<td>0.070854</td>
<td>0.070854</td>
</tr>
<tr>
<td>0.3</td>
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<td>0.067224</td>
<td>0.097854</td>
<td>0.074019</td>
<td>0.074019</td>
</tr>
<tr>
<td>0.3</td>
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<td>0.069777</td>
<td>0.099867</td>
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<td>0.078127</td>
</tr>
<tr>
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<td>0.102907</td>
<td>0.082939</td>
<td>0.082939</td>
</tr>
<tr>
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<td>0.075276</td>
<td>0.106715</td>
<td>0.088081</td>
<td>0.088081</td>
</tr>
<tr>
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<td>0.077849</td>
<td>0.110471</td>
<td>0.092695</td>
<td>0.092695</td>
</tr>
<tr>
<td>0.3</td>
<td>0.200</td>
<td>0.080422</td>
<td>0.114684</td>
<td>0.097310</td>
<td>0.097310</td>
</tr>
</tbody>
</table>

The instantaneous short rate follows CIR process. The long-term mean, and time interval are 8% and one month respectively. The initial rates are 5% and 11% as shown.
From Figure (4.6) and (4.7), we can see that the equilibrium fixed rate associated with high initial rate is larger than that associated with low initial rate. This makes sense because the interest rates of near future exerts more power than those in the long-run. Note also, when initial rate equal is 0.05, the fixed rates of large kappa are higher when volatility is small. Then the trend reverses with the increase of volatility. We argue when volatility is small, kappa exerts more effect on the value of contracts. When volatility increases, its effect becomes obvious. In the same figure, with initial rate equal to 0.11, although we observe that contract rates are higher when kappa is small, from the trend of the lines, we would expect that when the volatility is smaller than some value, kappa is a dominant effect. We also show that the existence of clean-up call is of no significance. This is plausible, for as the contract matures, the contract has become less valuable. Furthermore, it happens in the future.
Figure 4.6
Fixed-Rate vs. Volatility

Note: Lockout and clean-up call is not considered. CIR interest rate process is used. Long-term mean is 8%. Binomial approximation used is BASD.
Figure 4.7
Fixed-Rate vs. Volatility

Note: Lockout is considered. Clean-up call is not considered. CIR interest rate process is used. Long-term mean is 8%. Binomial approximation used is BASD.
Figure (4.8) shows the differences of fixed rates, under the situations when lockout provision is not considered and when lockout provision is considered. In a rising yield curve, fixed rates of the former case are lower than those of the latter case, while in a downward yield curve, the trend is reversed when volatility is low. As discussed, the fixed rate applies to the whole life of the contract, and floating rate changes periodically according to the prevailing interest rate index. When initial rate is low and there is no lockout provision, the floating-side payors have already started to make lower payments, forcing the "average" fixed-rate lower than the fixed-rate when lockout is considered. However, when initial rate is high and there is no lockout provision, the floating-side payors have already started to make higher payments, forcing the "average" fixed-rate larger than the fixed-rate when lockout is considered.

We tested the convergence of the fixed-rates of IAS without clean-up call by using the model developed for pricing IAS with clean-up call by setting the clean-up threshold to 0. In so doing, this model actually tries to search all of the paths on the lattice of interest rates when the partition size is getting smaller. Our results showed that the convergence to the fixed rates calculated by our model in Table (4.2) were achieved when partition size is at around 30.
Figure 4.8
Fixed-Rate Difference vs. Volatility

Note: Lockout is considered. Clean-up call is not considered. CIR interest rate process is used. Long-term mean is 8%. Binomial approximation used is BASD.
4.7 Conclusion

In this chapter, pricing mechanisms for IAS with geometric amortization of remaining balance is discussed. While the clean-up call provision that industry generally provides is proved insignificant here, the versatility of discretization approach developed for pricing it can be extended to value IAS with various facets. For example, when the amortization schedule is based on the original notional principal, the homogeneous property that is proved efficient as discussed, is no longer applicable, for the remaining balance may go negative as time evolves in this case. Thus, carrying a vector of the full spectrum of remaining balance at each node is required. It is obvious that our algorithm, which employ the discretization mechanism for solving IAS with clean-up call, can be easily adapted to take into account such contracts.
CHAPTER 5

LOOKBACK MORTGAGE

5.1 Introduction

Recently, there has been a trend in mortgage market to create synthetic mortgage products to eliminate the undesirable prepayment risk associated with the conventional mortgages and the mortgage-backed securities. The index amortization swaps discussed in Chapter 4 is one of these products, where the prepayments on the notional principal are linked explicitly to some interest rate index which is observable by all market participants. In particular, the notional principal could amortize rapidly if this index falls, slowly if this index rises. This amortization schedule intends to mimic the actual prepayments.

In addition to capturing the dynamics of unexpected prepayment financial institutions are also attempting to have influence on the refinancing behavior, where mortgagors can contract themselves into a new rate that is lower than the previous one. An example of this is a new contract, the so-called lookback mortgage (lookback for short hereafter), where the contract rates can always be reset at the lowest short rate to date. For example, suppose that we are holding a lookback with initial payment set according to short rate at 7%. If the short rate drops to 6.75% at next reset date, the lookback will be reset according to the new rate, thus reducing

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1 For a discussion of synthetic mortgages, see Chapter 4.
payments. If the rate increases to 7.5% at next reset date, the payment remains unchanged. That is, the investor always has a rate that is the most favorable rate experienced over the life time of the loan.

It is obvious that the value of the lookback depends on the history that interest rates evolve. This feature of changing rate is difficult to value because first, the boundary that triggers the resetting of the contract rate is not constant, and second, the payoffs depend on the history of interest rates. Recently, Ritchken and Vital (1989) priced an American option whose value depends on the minimum or maximum of a stochastic process of stock price over a fixed period of time. They use binomial approximation approach and develop an efficient backward recursive procedure to price such contracts. In their method, all of the possible maximums are first identified at each node given the stock price moves from date 0 to current stock price on the lattice.\(^2\) Associated with each maximum, a pricing mechanism is developed to get the price of path-dependent American option. In so doing, their approach depends critically on the number of maximums at each node. It turns out that the number of permissible maximums given the stock price moves from date 0 to current stock price on the lattice grows linearly in respect to the number of periods.

Lookback mortgage is embedded with similar path-dependence to that of lookback option, in that the minimums of interest rates of all of the paths leading to a node on the lattice must be identified. Besides, specific mortgage-related features, such

\(^2\) The following discussion about Ritchken and Vital's work applies to "minimums" too.
as amortization schedule, and prepayments must be considered to establish the pricing models. In this chapter, the Ritchken and Vital model is modified to produce efficient pricing schemes for lookback mortgages.

The following section introduces the assumptions and notation. In Section (5.3), path-dependent issues associated with lookbacks are explored. In Section (5.4), the valuation models for pricing lookback mortgages are developed. A four-period example is used to illustrate the algorithm. In Section (5.5), sensitivity of the lookback prices to changes in key parameters are analyzed. Section (5.6) gives conclusions.

5.2 Notation and Assumptions

Toward the goal of developing the valuation model, we adopt the following notation

\[ n = \text{Number of periods (in months).} \]
\[ \Delta t = \text{Length of the interval (one month).} \]
\[ (i,j) = \text{Node on the binomial lattice, constructed by the 3-step procedure in Section (2.4).} \]
\[ r_{ij} = \text{Spot rate at node } (i,j). \text{ The initial value } r_{0,0} \text{ simplifies to } r_0. \]
\[ q_{ij}(q_{ij}) = \text{Risk adjusted probability of an up (down) jump in interest rates from node } (i,j). \text{ Under this measure, all bonds can be priced as if the local expectations hypothesis holds.} \]

We have the following assumptions

- *Interest rate follows the square root process, and all bonds are priced as in the general equilibrium model of CIR.*
· CIR process is approximated by BASD (binomial approximation for singular diffusions).

· The market is perfect. This means no transaction and refinancing costs exist.

5.3 Path-Dependence Associated with Lookback Mortgages

In this section, we show how path-dependence arises in pricing lookback when it is valued on a binomial lattice of interest rates. Since the investor always contracts into the lowest interest rate experienced, it never is favorable for them to prepay based on financial decisions alone. We start our discussion of lookback with such a simple provision. However, if floor rates were embedded into the contract, then prepayment could be optimal, and this feature can easily be incorporated.

When no prepayment occurs, the cash flows associated with lookback only come from the fixed-level payment. At any point in time, in order to determine the fixed-level payment, three factors are critical, namely the contract rate, time to maturity, and remaining balance. Denote \( F_{ij}(B,c) \), the fixed-level payment, that originates from node \((i,j)\) with remaining balance \(B\), and applies to the rest of the life of the mortgage, assuming that no interest rates that are higher than the contract rate, \(c\), have ever occurred. Similarly, denote \( B_{ij}(t,B,c) \), the remaining balance at time \(t\), \(t>i\), that originates from node \((i,j)\) with remaining balance \(B\), assuming that no interest rates that are higher than the contract rate, \(c\), have ever occurred. Now we want to establish cash flows for the lookback. This is best illustrated by an example. Figure (5.1) shows a binomial lattice
Figure 5.1
A 3-Period Lattice Showing the Path-Dependence of Lookback Mortgage.

Note: Associated with each node are vectors. The first bit of data in the vector represents the fixed-level payment, and the second bit of data represents the remaining balance in the next period.
of interest rates. Associated with each interest rate are vectors. In each vector, the first bit of data represents the fixed-level payment, and the second bit of data represents the remaining balance.

Suppose the investors start off a lookback contract with original principal equal to $B_0$ at date 0. This will yield fixed-level payment of $F_0(B_0 r_0)$, and remaining balance of $B_0(1, B_0 r_0)$ at both of the up and down nodes in period 1. Now let's focus on node $(1,1)$. Since interest rate goes up, the contract rate remains unchanged. Fixed-level payment is the same and remaining balance is decreased by regular principal payment at the associated nodes in the next period. ($(2,2)$ and $(2,1)$).

Specifically, fixed-level payment is still $F_0(B_0 r_0)$, and remaining balance is $B_0(2, B_0 r_0)$. However, when we move our focus to node $(1,0)$, we can see that the contract rate must be changed to the prevailing lower rate, $r_{1,0}$ say. With the new contract rate, $r_{1,0}$ and remaining balance, $B_0(1, B_0 r_0)$, the fixed-level payment and remaining balance on the associated interest rates in the next period ($(2,1)$ and $(2,0)$) are

$$F_{1,0}(B_0(1, B_0 r_0), r_{1,0})$$

and $B_{0,0}(1, B_0(1, B_0 r_0), r_{1,0})$.

In the discussion so far, 2 different paths of cash flows arise: for a two-period problem, namely, $F_0(B_0 r_0)$ followed by $F_0(B_0 r_0)$, and $F_0(B_0 r_0)$ followed by $F_{1,0}(B_0(1, B_0 r_0), r_{1,0})$. However, when we extend the analysis further to three periods as shown, then four different paths need to be searched. This wouldn't be a problem with lookbacks of small size number of period. However, for lookbacks with maturities of 15 or 30 years amortized monthly as the other mortgages commonly do in the market,
pricing will be very difficult, for the numbers of path need to be searched will grow exponentially to $2^{180-1}$ and $2^{350-1}$ respectively.

5.4 The Valuation Model

In this section, we develop efficient algorithms for pricing lookback on a binomial lattice of interest rates. First, the valuation models will be proposed. Second, a four-period example will be used to illustrate the valuation procedure.

From the ideas of pricing GNMA pass-throughs, we knew already the property of homogeneity of degree one in remaining balance associated with the amortization schedules. If all of the permissible contract rates at each node were known, then we can establish lookback price for each of them, assuming a $1 remaining balance, as we did for pricing MBS. The remaining balance in the next period is scaled by the associated lookback price. Then our effort in pricing lookback reduces to find all of the permissible contract rates at each node. If an efficient algorithm can be constructed to establish these permissible contract rates and the number of them is linearly growing in respect to the number of periods, then an efficient algorithm for pricing lookbacks can be developed.

5.4.1 The Algorithm

We illustrate the development of algorithm by three phases. In the first phase, the contract rates that are permissible at each node will be identified. On a lattice, the

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3 See Chapter 3 for a complete discussion.
number of the permissible rates is finite and an efficient algorithm will be developed to identify them in combination with the properties of the binomial lattice of interest rates. In the second phase, with the associated permissible rate, remaining life of the mortgage, and a $1 remaining balance, the amortization schedule is established. In the third phase, a recursive backward recursion is developed to yield the value of lookback. In the following, the details of three phases are described.

(a) Identifying the permissible contract rates

In Figure (5.2), consider an arbitrary node \((i,j)\) where the short rate is \(r_{ij}\). The smallest minimum value over all paths on the lattice that end at \(r_{ij}\) occurs if the initial consecutive jumps are all down jumps, followed by all up jumps. We can see that the

![Figure 5.2](image)

Figure 5.2

A four-period lattice of interest rates.
smallest minimum occurs at node \((i-j,0)\). Denote the total number of different possible minimums at node \((i,j)\) by \(k_{ij}\). We have

\[
k_{ij} = i+j+1
\]  
(5.1)

As an example, the number of the permissible minimums at node \((3,2)\) is \(3-2+1=2\). Specifically, they are \(r_{o\alpha}\) and \(r_{i,0}\). This is true, if the interest rates have not gone below \(r_{\min}\) as defined in Equation (A.7) in the Appendix A. Otherwise, singularity needs to be considered. As discussed in Li's method, if the transformed process goes below \(x_{\min}\) it jumps up to two different states. Then as shown in Figure (5.3), the maximum number that the interest rates can jump is

![Figure 5.3](image)

Figure 5.3
Binomial lattice for the CIR process with an reflecting boundary.
\[ L_{bound} = \left( x_0 - \frac{1 + \frac{\gamma}{1 + \gamma} - 1}{\frac{\gamma}{1 + \gamma}} \right)^{\gamma} \frac{1}{\sqrt{h}} + 1 \quad (5.2) \]

We then redefine \( k_{ij} \) by
\[ k_{ij} = \min[L_{bound} \cdot i - j + 1] \quad (5.3) \]

Let \( c_{il} \) be the \( l \)th permissible minimum at node \((i,j)\). Then
\[ c_{il} = r_{l-1,0} \quad l=1,2,...,k_{ij} \quad (5.4) \]

(b) Establishing the amortization schedule:

At node \((i,j)\), with contract rate \( c_{il} \), where
\[ c_{il} \in [c_{i1,j}, c_{i2,j}, ..., c_{ik_{ij},j}] \quad (5.5) \]
and \$1\ remaining balance, the fixed level payment, \( F_{ij}(c_{il}) \), and the remaining balance in the next period, \( B_{ij}(1, c_{il}) \), can be computed by
\[ F_{ij}(c_{il}) = (c_{il}/12) \frac{(1 + c_{il}/12)^{7.12} - 1}{(1 + c_{il}/12)^{7.12} - 1} \quad (5.6) \]
\[ B_{ij}(1, c_{il}) = 1 - [F_{ij}(c_{il}) - 1 \cdot (c_{il}/12)] \quad (5.7) \]

(c) Developing the recursive discounting pricing equation:

To value the lookback by backward procedure, we start at the expiration date. In the last period, the value of lookback equal to the remaining balance, \$1. Let \( M_{ij}(c_{il}) \) be the value of lookback at node \((i,j)\) given the minimum short rate up to node \( x_{\min} \) is defined in Appendix A. (Equation (A.5)).
(i,j) is \( c_{i,j} \), and the remaining balance is $1. We now provide the backward recursion
equations that link lookback prices in period \( i \) to those in period \( i+1 \). Figure (5.5)
shows the matrix of permissible contract rates and their associated lookback prices
at nodes (i,j), (i+1,j) and (i+1,j+1).

**Figure 5.4**
Lookback Mortgage Prices in period \( i \) and \( i+1 \)

Associated with each node is a matrix. The number of rows of the matrix indicate the number
of distinct minimum contract rates that could be achieved from the initial node to the current
node. The first column of the matrix gives the actual possible minimums. For each minimum
there exists a unique lookback price. This price is specified in the corresponding row of the
second column. Given the information in Figure (5.4), the entries in the second column of the
matrix at node (i,j) can be computed using the algorithm.
Associated with \( r_{i,j+1} \) is a \((k_{i,j+1} \times 2)\) matrix of permissible minimums and corresponding lookback values. Similarly, associated with \( r_{i,j} \) is a \((k_{i,j} \times 2)\) matrix. All values in the two matrices are known, as is the first column in the matrix at node \((i,j)\). Now we want to calculate the values in the second column of the matrix at node \((i,j)\).

Assume a minimum at node \((i,j)\) was \(c_{i,j}^{l}\) for some given \(l\), and we want to calculate its associated lookback price. When the minimum \(c_{i,j}^{l}\) is smaller than or equal to the current interest rate \(r_{i,j}\), the mortgagor will stay with the minimum, \(c_{i,j}^{l}\). However, when the minimum is higher than the interest rate, the mortgagor will contract into a new rate, \(r_{i,j}^{5}\). Thus, the payments are reduced. Specifically, at node \((i,j)\), the price of lookback, \(M_{i,j}(c_{i,j}^{l})\), is given by

\[
M_{i,j}(c_{i,j}^{l}) = \min\left[ M_{i,j}^{GO}(c_{i,j}^{l}), M_{i,j}^{SWITCH}(c_{i,j}^{l}) \right]
\]  
\(5.8\)

where

\[
M_{i,j}^{GO}(c_{i,j}^{l}) = \left[ F_{i,j}(c_{i,j}^{l}) + B_{i,j}(1, c_{i,j}^{l}) \left( q_{i,j} M_{i+1,j+1}(c_{i,j}^{l}) + \bar{q}_{i,j} M_{i+1,j+1}(c_{i,j}^{l}) \right) \right] e^{-r_{i,j} \Delta t}
\]  
\(5.9a\)

and

\[
M_{i,j}^{SWITCH}(c_{i,j}^{l}) = \left[ F_{i,j}(r_{i,j}) + B_{i,j}(1, r_{i,j}) \left( q_{i,j} M_{i+1,j+1}(r_{i,j}) + \bar{q}_{i,j} M_{i+1,j+1}(r_{i,j}) \right) \right] e^{-r_{i,j} \Delta t}
\]  
\(5.9b\)

where

\[
M_{i,j}(c_{i,j}^{l}) = \min\left[ M_{i,j}^{GO}(c_{i,j}^{l}), M_{i,j}^{SWITCH}(c_{i,j}^{l}) \right]
\]  
\(5.8\)

Equation (5.10) shows that the remaining balance in the next period is rescaled by the associated mortgage prices.

\(5\) When there is market imperfections, the mortgagor may not contract into \(r_{i,j}^{5}\).
Usually a floor rate $r_f$ will be employed to limit the contract rate from going too low. If this constraint is added, we only need to modify Equation (5.9b) to

$$M^{\text{SWITCH}}_{i,j}(c_{i,j}) = \left[ F_{i,j}(r_f) + B_{i,j}(1, r_f) \left( q_{i,j} M_{i+1,j+1}(r_f) + \overline{q}_{i,j} M_{i+1,j+1}(r_f) \right) \right] e^{-r_f \cdot t}$$

if $r_{i,j} > r_f \tag{5.10}$

With most favorable rate reset every period, rarely will mortgagor have the incentive to exercise early prepayment. However, another layer of rational prepayment consideration can be easily imposed on top of floored contract rate. Denote mortgage price at node $(i,j)$ with contract rate $c$, $M_{i,j}(c)$, we have the following equation to accommodate this variation

$$M_{i,j,t} = \min[1, M_{i,j,t}] \tag{5.11}$$

Equation 5 implies that whenever the mortgage price exceeds the remaining balance $1$, the rational prepayment call option will be exercised.

### 5.4.2 A Four-period Example

Figure (5.6) displays a binomial lattice of interest rate together with probabilities of going up shown at each node. We illustrate in this example
Figure S.5

Lookback Monte Carlo Price on a Binomial Lattice
how a lookback is valued when rational prepayment is considered. The relevant CIR parameters are: $\kappa = 0.20$, $\sigma = 0.10$, $\mu = 8\%$, initial rate is 5%, and $\Delta t = 1/4$ year. Associated with each node are 4-dimensional vectors, representing the permissible minimum, the fixed-level payment, the remaining balance in the next period, and the corresponding mortgage price.

As an example, assume the backward recursion had established the vectors at nodes (1,1) and (1,0). We show how the vector of information at node 0 is obtained. First, assume a $1 balance remains. The first bit of data, 0.05, is the initial contract rate. The second bit of data, 0.258, is the scheduled fixed-level payment calculated by Equation (5.6). The third bit of data, 0.755, is the remaining balance in the next period calculated by Equation (5.7). Let's focus on node (1,0). Since interest rate drops to 0.039, we use the associated lookback, 0.998, to scale the remaining balance, 0.755, and have (0.998x0.755)=0.753. Now consider node (1,1). Since interest rates have increased, the contract rate remain unchanged and there is only one vector here. Scale 0.755 by 0.994, we then have (0.994x0.755)=0.750

<table>
<thead>
<tr>
<th>fixed-level payment</th>
<th>0.258</th>
</tr>
</thead>
<tbody>
<tr>
<td>lookback price</td>
<td>0.750</td>
</tr>
</tbody>
</table>

Expected lookback price at node (1,1) : 1.008

And the cash flows, if a down jump occurs, are

<table>
<thead>
<tr>
<th>fixed-level payment</th>
<th>0.254</th>
</tr>
</thead>
<tbody>
<tr>
<td>lookback price</td>
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Expected lookback price at node (1,0) : 1.007
Discounted by the risk-adjusted interest rate, \( r_o = 0.05 \), the price of lookback at date 0 is equal to

\[
[1.008 \cdot 0.54 + 1.007 \cdot (1 - 0.54)]e^{-(0.05)(1/4)} \equiv 0.996
\]

5.4.3 **Equilibrium Spread**

In the above example, we can see that the price we get is not equal to the par value $1. Then there is an arbitrage opportunity. In order to eliminate the arbitrage opportunity, a spread is added to the interest rates. The spread must be chosen such that

\[
M_o = $1
\]

(5.12)

Since the lookback price is monotonically increasing function of the spread, the equilibrium spread can be found by iterative bisection search.

5.5 **Sensitivity Analysis**

In this section, the equilibrium spread is established for a wide variety of parameters driving the interest rate process.

- \( \kappa = 0.1, 0.2, 0.3 \)
- \( \sigma = 0.05, 0.075, 0.1, 0.125, 0.15, 0.175, 0.2, 0.225 \)
- \( \mu = 8\% \)
- \( r_o = 0.05, 0.11 \)
- \( T = 5, 10 \) (years)

These parameters cover the range of key CIR parameters recently used in mortgage pricing literature. The maturity is chosen to reflect the maturity commonly used for synthetic mortgages.
Table (5.1) compares the equilibrium spreads of lookback mortgages of 5 and 10 year maturities, under the cases when floor rate is considered and not considered.\textsuperscript{6}

In addition, rational prepayment is allowed. As expected, equilibrium spread increases, when volatility increases. This situation is similar to callable amortization bond, where higher contract rates are charged when interest rates are high. In fact, lookback is like a callable amortization bond except that the contract is reset monthly. Thus, higher spread is imposed to reflect the investors' flexibility in exercise this option.

\textsuperscript{6} Table (5.1) is produced by computer program, entitled LOOKBACK.PAS in Appendix B. For a 10-year lookback, the program takes about 3 minutes to get the prices on IBM PC 486 DX.
## TABLE 5.1
Comparison of Equilibrium Spreads between Lookback Mortgage of 5 and 10 Year Maturity, When Floor is not Considered and Considered.

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</table>

The instantaneous short rate follows CIR process. The long-term mean, and time interval are 8% and one month respectively.
The instantaneous short rate follows CIR process. The long-term mean, and time interval are 8% and one month respectively. Binomial approximation used is BASSD. Kappa = 0.1.

We would imagine that floor rate is effective only if the initial rate is low and maturity is long. If initial rate is high and maturity is short, lookback might have not reached the floor rate before it expires. Figure (5.7) confirms this. We use 2% as the floor rate. We can see that when initial rate is 11%, both 5 and 10 year contracts showed no difference in changes of spreads. However, when initial rate is 5%, difference arises. Furthermore, contract of 10 year maturity has a spread larger than that of a 5 year contract.

Figure (5.8), and (5.9) compare the equilibrium spreads in changes of κ, σ, and
The instantaneous short rate follows CIR process. The long-term mean, and time interval are 8% and one month respectively. Binomial approximation used is BASD.
\( \mu \), given maturity fixed at five years. Recall that in Figure (3.9), when we compare the contract rates of callable amortization bonds with initial rates equal to 5% and 11%, we found that contract rates depend critically on initial rates. When initial rate is high, the contract rate is high, vice versa. However, from Figure (5.8) and (5.9), we found that the equilibrium spreads are less dependent on the initial rates. This makes sense because of the mean-reverting feature associated with the underlying interest rate process. When interest rate is high, it tends to go to a lower rate, and when interest rate is low, it tends to go to a higher rate, then the option that allows the lookback contracts into the lowest rate experienced would result in quite a stable contract rate.

It is also interesting to see that the effect of kappa versus volatility exhibits same relationship as other contracts did in previous chapters. That is when volatility is small, kappa is quite important in determining the equilibrium spreads. However, as volatility increases, it become a critical force.

### 5.6 Conclusion

In this chapter, we discussed an efficient algorithm for pricing lookback mortgages, and the equilibrium spread is established by using iterative bisection search. Sensitivity analysis is also conducted in changes of parameters of the underlying interest rate process and contractual features.

Although we apply our approach in pricing lookback, where the contract rate is reset monthly in accordance with the prevailing short rate, it is quite easy to extend
our algorithm to consider contracts with different reset interval, and interest rate index. For example, if the interest rate index is one-year Treasury rate, we only need to convert the short rate to one-year Treasury rate (see CIR (1985)), and proceed as usual. A similar contract, the adjustable-rate mortgage (ARM), where contract rates are reset annually according to the prevailing one-year Treasury rate, will be explored in the next chapter. It shows another facet of path-dependent mortgage product. Efficient lattice-based algorithms will be developed based on some findings from this chapter.
CHAPTER 6

ADJUSTABLE RATE MORTGAGES

6.1. Introduction

Adjustable-rate mortgages (ARM) are based on a simple modification of the familiar fixed-rate mortgages (FRM). Contract rates are not set for the life of the loan, but periodically change with movements in accordance with some appropriately chosen benchmark index, typically one based on a short-term interest rate. Indeed, we can treat an ARM as a series of FRMs with contract rates of these FRMs reset periodically.

This chapter uses option pricing techniques to price ARMs on a binomial lattice of interest rate. However, in attempting to use backward procedure to price ARM, we encounter the path-dependent problem, since the balance and hence price depends critically on the path of rates over time and on the prepayment experiences. The valuation of ARM is further complicated by contract features, such as caps and floors, for the contract rates and prepayment behavior are again dependent on these contractual provisions.

The complexity involved in ARM makes pricing issue quite a problem. Few studies address this problem to date. Kau, Keenan, Muller, and Epperson (1990) first considered valuing ARM using contingent claim approach. They use CIR process as
the underlying interest rate process and establish a general valuation equation, which is solved by finite difference method. Recently, McConnell and Singh (1991) developed a model by using Brennan and Schwartz's (1979) two-factor process as the underlying instantaneous interest rates process and proportional hazards model as the prepayment representation. The valuation is achieved by Monte Carlo simulation to approximate the underlying uncertainties.

Lattice approaches have been lacking from literature in pricing ARMs for reasons that have been discussed in previous chapters. To sum up, lattice approach is backward based and has difficulty dealing with path-dependence since cash flows depending on state variables occurring earlier in time can not be determined.

In this chapter, we shall develop an efficient lattice-based model for ARM, where the history of the process, relevant for pricing, is summarized by a single additional state variable. Specifically, this additional statistic, together with the current interest rate is sufficient for capturing all information along the path. Given this model, we then conduct tests to measure the sensitivity of the ARM prices to the key contractual provisions and parameters driving interest rate uncertainty.

The remainder of this chapter is organized as follows. Section (6.2) gives an overview of the ARM contract. Section (6.3) provides assumptions and notation used through this chapter. Section (6.4) develops efficient lattice-based models for pricing ARMs. Section (6.5) conducts sensitivity analysis of the ARM prices in
changes of key contractual provisions and parameters driving interest rates uncertainties. Section (6.6) gives conclusions.

6.2 Overview of the Adjustable-Rate Mortgage

From the discussion in Chapter 2, we know that FRM suffers a serious shortcoming, the "mismatch" problem. A solution to this problem yields ARM. ARMs have been popular with lenders because they shift interest rate risk from the lender to the borrower. Thus, institutions prefer to hold ARMs in their portfolio rather than FRMs, because ARMs provide a better matching with their liability.

Characteristics

In an ARM, the contract rate is reset according to the benchmark index plus a spread, or margin on successive reset dates. The spread usually ranges from 100 to 200 basis points, reflecting market conditions, the features of the ARM, and the cost of servicing an ARM. To encourage borrowers to use ARMs rather than FRMs, mortgage originators generally offer an initial contract rate that is less than the prevailing market mortgage rate. This below-market initial contract rate is commonly referred to as a teaser rate.

A pure ARM is one in which the contract rate resets periodically to the prevailing interest rate index and has no other terms that affect the monthly mortgage payment. Most ARMs, however have periodic cap/floor, and life cap/floor features, which limit the amount that the contract rate may increase or decrease at the reset
date. From an option's perspective, the caps allow the homeowner the right to borrow money at a below-market interest rate. Thus, the lender or investor has sold an option on an interest rate to the homeowner. In the case of floors, the homeowner has sold an option on the same interest rate to the lender or investor. Since periodic caps/floors go into effect each year, in fact, a package of options are sold in this case.

Thus, from the lender's view of point, an ARM can be viewed as:

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<th>ARM</th>
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<tr>
<td>Pure Floater + Long Position in a Series of Floors + Long Position in the Life Floor + Short Position in a Series of Caps + Short Position in the Life Cap</td>
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The prices of ARMs with interest rate adjusted continuously and instantaneously, would in the absence of other frictions, never vary significantly from par, and the refinancing issue is not a problem. However, caps/floors, and finite time constraints between reset periods, have the effect of pulling the price of the ARM away from par.

6.3 Assumptions and Notation

We have the following assumptions:

--- Interest rate follows the square root process, and all bonds are priced as in the general equilibrium model of CIR.
--- CIR process is approximated by BASD (binomial approximation for singular diffusions).

--- The market is perfect. This means no transaction and refinancing costs.

--- Mortgage loans can be prepaid at any time without a prepayment penalty.

--- Suboptimal prepayment is only dependent on the instantaneous short rate and on time.

The discrete-time equation for the monthly fixed-level payment, $F$ with contract rate $c$, original principal $B(0)$, and maturity $T$ in years, can be represented by

$$F = B(0)(c/12)\frac{(1 + c/12)^{T/12} - 1}{(1 + c/12)^{T/12} - 1}$$

(6.1)

$F$ is referred to as fixed-level payment.

The remaining balance at time $t$ assuming no prepayments occurred prior to time $t$, is

$$B(t) = B(0)\frac{(1 + c/12)^{T/12} - (1 + c/12)^{t/12}}{(1 + c/12)^{T/12} - 1}$$

(6.2)

**Treasury Rate**

As discussed, the contract rate of ARM is determined in accordance with the Treasury rate. CIR (1985) showed that in their model, bond prices have the form

$$P(t, T) = A(t, T)e^{-B(t, T)r}$$

(6.3)

with

$$B(t, T) = \frac{2(\epsilon^{(T-t)} \gamma - 1)}{(\gamma + \kappa)(\epsilon^{(T-t)} - 1) + 2\gamma}$$
and

\[ A(t, T) = \left[ \frac{2\gamma e^{(\kappa+\gamma)(T-t)/2}}{(\gamma+\kappa)(e^{\pi(T-t)} - 1) + 2\gamma} \right]^{2\kappa/\sigma^2} \]

where \( \gamma = \sqrt{\kappa^2 + 2\sigma^2} \).

If \( b(t, T) \) is the continuously compounded interest rate at time \( t \) for a term of \( T-t \), then

\[ P(t, T) = e^{-b(t,T)(T-t)} \] (6.4)

and

\[ b(t, T) = -\frac{1}{T-t} \ln P(t, T). \] (6.5)

### 6.4 The Valuation Models

In this section, we start developing valuation models for ARMs by considering a pure ARM, where no caps and floors exist. In addition, only rational prepayment is involved. This pure-ARM model will then be used as benchmark model to develop other models in which realism such as lifetime cap/floor, periodic cap/floor, and irrational prepayment will be included.

#### 6.4.1 Valuation of Pure ARM

ARMs without caps and floors are like FRMs except that their contract rates are reset periodically according to the prevailing index.
If a backward procedure is employed to price the pure ARM, all of the potential cash flows at each node at the last date must first be identified. The major difficulty ARM poses is apparent immediately, for these cash flows depend on the remaining balances in previous reset dates, and they in turn depend on the path of interest rates. This requires searching all paths from date 0 to the last reset date to establish all possible remaining balances on the last reset date, thus the cash flows on the last date of the ARM contract. This would cause severe computational difficulty.

As in the other chapters, we exploit the fact that the value of the mortgage is homogeneous of degree one in the remaining balance, $B(0)$. Specific procedure for establishing the algorithm is as follows. When there are no caps and floors, the contract rate at each node $(i,j)$ can be fully computed from the term structure, if the contract rate is a Treasury rate. Denote this Treasury rate, $b_{t,i}$. With $\$1$ as the remaining balance, originated from node $(i,j)$, the fixed-level payment $F(b_{t,j})$ and remaining balance $B(s,b_{t,j})$ at each period, $s\geq i$, can be calculated by Equations (6.1) and (6.2). For illustrative convenience, consider the sublattice rooted at node $(i,j)$ at some reset date $i$, extending out to the next reset date in $l$ periods. In fact, within this sublattice, the mechanism for establishing the ARM price at node $(i,j)$ is exactly the same as for a callable amortization bond in Chapter 3, if rational prepayment is considered, with the following contractual provisions
contract rate : \( b_{ij} \)  
maturity : \( l \) periods

with terminal mortgage prices at node \((i+l,k)\), \( k=j,j+1,j+2,\ldots,j+l \), on next reset date, \( i+l \), scaled as

\[
M_{i+l,k} = B(i + l, b_{ij}) \cdot \min[1, M_{i+l,k}], \tag{6.6}
\]

and boundary mortgage prices equal to \( F(b_{N,k}) \) where \( h=0,1,2,\ldots,N-l \). Equation (6.6) exploits the homogeneity property by rescaling the remaining balance at next reset date by the previously computed mortgage price at that node.

The pricing mechanism for pure-ARM price at node \((i,j)\), when irrational prepayment is considered, is also similar to that for pricing GNMA pass-throughs. The backward procedure starts at date \( i+l-1 \), with the remaining balance, in the next period after prepayment, scaled by the associated pure-ARM prices. Then the pricing mechanism proceeds as pricing a GNMA pass-through.\(^1\) Numerical results of pure-ARM for both rational and irrational prepayments are presented in Section (6.6).

6.4.2 ARM with Caps/Floors

To accommodate the variations when caps/floors are imposed, changes must be made to first identify the permissible contract rates at the nodes in each reset period. Once this is established, pricing can be achieved as we did for a pure-ARM.

---

\(^1\) For discussion of pricing callable amortization bond and GNMA pass-through, see Chapter 3.
In the simplest situation, when only the life cap, \( c_{\text{life}} \), and life floor, \( c_{\text{life}} \), are considered, permissible contract rates can be easily identified by

\[
  c_{ij} = \begin{cases} 
  c_{\text{life}} & \text{if } c_{\text{life}} < b_{ij} \\
  b_{ij} & \text{if } c_{\text{life}} \leq b_{ij} \leq c_{\text{life}} \\
  c_{\text{life}} & \text{if } b_{ij} < c_{\text{life}}
  \end{cases} \tag{6.7}
\]

and it is obvious that at each node only one contract rate is permissible and pricing is as easy as pricing a pure-ARM.

Most ARMs not only specify the life cap/floor but also the maximum periodic reset in the contract rate. As an example, the current most common yearly cap is 2 percent. The contract rate at node \((i,j)\), \( c_{ij} \), then can be found by the following relationship

\[
  c_{ij} = \max\{\min[b_{ij}, c_{\text{life}}, a_b + k], c_{\text{life}}, a_b - k\}, \tag{6.8}
\]

where \( k \) is the maximum periodic reset amount, and \( a_b \) is the contract rate at previous reset date. This last term creates path dependence in the contract rate, and complicates pricing. Specifically, the current contract rate may depend on previous contract rates and it again may depend on its predecessor. Identifying the potential set of rates at any node is, in general, difficult. This feature makes this problem more complex than
the look mortgage. Our solution is to augment the state space by a variable representing the actual contract rate. The range of this contract rate is

\[ c_{life}, \overline{c_{life}} \]

and it is approximated by a grid of values. The backward recursive pricing mechanism can be derived as follows. For each grid of the contract rate at each node, fixed-level payment and remaining balances in the future are then available. A dynamic programming recursive equation can then be established.

### 6.4.3 Numerical Example

Figure (6.1) shows a binomial lattice of interest rate together with probabilities of going up shown at each node. We price an ARM with life cap=13%, life floor=3%, and optimal prepayment. The relevant CIR parameters are: \( \kappa=0.2, \sigma=0.20, \mu=8\% \), and initial rate is 8%. Contract rate is reset periodically according to one-year Treasury rate, and \( \Delta t=1/2 \) year. Associated with each node is a 5-dimensional vector, representing the one-year Treasury rate, actual contract rate, fixed-level payment, remaining balance, and the price of ARM respectively, assuming $1 of remaining balance in the present node.
The internal CIR parameters are: $\tau = 0.2, \mu = 0.02, \phi = 0.8$, and initial rate is 8%. Contact rate is

![Diagram](image-url)
As an example, assume the backward recursion had established the vectors at nodes (2,2), (2,1) and (2,0). One thing needs to be noted is that at node (2,2), the one-year Treasury has exceeded the life cap, so the actual rate is reset according to the life cap. Now we show how the vector of information at node 0 is obtained. First, assume a $1 balance remains. We have the one-year Treasury rate and the actual rate on the top two bits of data. They are all 0.08. Fixed-level payment scheduled to be made in periods, 1, 2, 3, and 4, is 0.275, which is calculated by Equation (6.1). The remaining balance in the next reset period, period 2, is calculated by Equation (6.2), and is equal to 0.519. At node (2,2), with the homogeneous property of degree one in the principal, we scale $0.958 by 51.9% to get the price of ARM, 0.497. Similarly, the scaled remaining balance at node (2,1) is 0.994(51.9%)= 0.516, and at node (2,0) is
1.003 (51.9%) = 0.521. In summary, in period 2, we have three pieces of expected cash flows

At node (2,2)

<table>
<thead>
<tr>
<th></th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>fixed-level payment</td>
<td>0.275</td>
</tr>
<tr>
<td>scaled remaining balance</td>
<td>0.497</td>
</tr>
</tbody>
</table>

Expected cash flow at node (2,2) = 0.772

At node (2,1)

<table>
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<tr>
<th></th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>fixed-level payment</td>
<td>0.275</td>
</tr>
<tr>
<td>scaled remaining balance</td>
<td>0.516</td>
</tr>
</tbody>
</table>

Expected cash flow at node (2,1) = 0.791

At node (2,0)

<table>
<thead>
<tr>
<th></th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>fixed-level payment</td>
<td>0.275</td>
</tr>
<tr>
<td>scaled remaining balance</td>
<td>0.521</td>
</tr>
</tbody>
</table>

Expected cash flow at node (2,0) = 0.796

The expected cash flows on the lattice in the first reset interval can be summarized by the following lattice

Then we treat the valuation as of pricing a coupon bond, and establish the price of ARM at date 0, 0.987.

6.5 Sensitivity Analysis

In this section, equilibrium contract rates are provided when the values of key CIR parameters are given by

\[ \kappa = 0.1, 0.2 \]
\[ \sigma = 0.05, 0.1, 0.15, 0.2 \]
\[ \mu = 8\% \]
\[ r_0 = 0.05, 0.11 \]
These parameters cover the range of key CIR parameters that Kau et al. (1990) use in pricing adjustable-rate mortgages. Specifically, they are $\kappa=0.15$, $\sigma=0.05$, $0.15$, $r_o=0.08$, 0.12, and $\mu=0.10$. These parameters also reflect those parameters used in some other research in pricing mortgage-related products. For example, in valuing pass-throughs, Schwartz and Torous use $\kappa=0.1$, $\sigma=0.075$, and $\mu=0.065$, which are suggested by Bunce, MacRae, and Szymanoski (1988). The parameters in our sensitivity analysis yield term structures, which revert to the long-run average short rate of 8%. Life caps of 9, 10, 11, 12, and 13% are used and a life floor of 0.02 is assumed. The periodic cap/floor is 0.02. In mortgage market, life cap is set at around 0.12, life floor usually doesn't exist and if it exists, usually, it is set at around 0.02. Periodic cap/floor are commonly set at 0.02. We also assume that prepayment dynamics is governed by

$$y(r, t) = \begin{cases} 
  e^{-20r} & t = 0, 1, 2, \ldots, n - 1 \\
  1 & t = n 
\end{cases} \quad (6.9)$$

This is the prepayment function as that used for GNMA pass-throughs. The prepayment function is plausible, for when interest rate decreases, the prepayment fraction is increasing. When interest rate reaches 0, all of the remaining balance will be paid down.

The first analysis concerns the effect life cap has on the equilibrium spread. Since the life cap bounds the rate the lender can receive from the homeowner, clearly,
we would expect that the lender must be compensated with a higher spread when a life cap is present. Furthermore, the additional compensation required increases with the hikes of volatility of interest rates. Table (6.1) confirms our statements. It contains results produced by the above interest rate parameters, assuming no life floor, no periodic cap/floor, and optimal prepayment.² We can observe that with life cap and kappa fixed, the equilibrium spreads increase when volatility increases. It is also natural to see from Figure (6.2) and Figure (6.3) that the tighter the life cap constrain, the more the spread must rise. This is especially obvious when the initial rate is higher than the life cap. Under this situation, the lenders start off with a "loss" already. In order to compensate such immediate loss, a much higher spread is added.

When we include another realism for life floor, it is very interesting to note that we almost "duplicate" the results in Table (6.1). In other words, as expected from the market observation, life floor is an insignificant contractual provision and is often ignored in the industry. Although the life floor has the effect of limiting the size of payments not going to low, thus, negating the influence of life cap, however, the rational prepayment option will comes into play at an advantageous moment to the borrowers. This means that the borrowers will prepay based on financial decisions when interest rates are low.

² Table (6.1) is produced by a computer program, entitled "ARM01.PAS", in Appendix B.
### TABLE 6.1
Equilibrium Spreads of Pure Adjustable-Rate Mortgage

<table>
<thead>
<tr>
<th>K</th>
<th>σ</th>
<th>Life Cap</th>
<th>Initial Rate = 5%</th>
<th>Initial Rate = 11%</th>
</tr>
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<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>10 Years</td>
<td>20 Years</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td>0.1</td>
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<td>0.09</td>
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</tr>
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<tr>
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<tr>
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<td>0.004991</td>
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<td>0.029428</td>
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</tbody>
</table>

The instantaneous short rate follows CIR process. The long-term mean, and time interval are 8% and one month respectively.
Figure 6.2
Pure ARM (Initial Rate = 5%)

Note: Interest rate follows CIR process with long-term mean = 8%. Binomial approximation used is BASD. Kappa = 0.1.

Figure 6.3
Pure ARM (Initial Rate = 11%)

Note: Interest rate follows CIR process with long-term mean = 8%. Binomial approximation used is BASD. Kappa = 0.1.
Now we add another layer of realism to consider the suboptimal prepayment. If suboptimal prepayment is allowed, we would expect that the equilibrium spread will be reduced from that of pure ARM, which has the same contractual provisions except that the suboptimal prepayment is not permitted. This is similar to the situation, where equilibrium contract rates of GNMA's are lower than those of callable amortization bonds, in that the latter have full flexibility of exercising the call feature. Figure (6.4) justifies this argument. We can see that the equilibrium spreads of the pure ARM are consistently higher than those associated with ARMs with additional suboptimal prepayment feature.

To see how the equilibrium spreads may change when we add the periodic cap/floor provisions to the pure ARM, consider the case when life cap is equal to
13%. People might be expected that periodic cap and periodic floor would offset each other. However, Sobti and Sykes (1993) showed by using historical interest rates over the last five and ten years that periodic floor was more valuable in a downward yield curve environment than periodic cap. They argue, given the current steep yield curve, the periodic cap would be expected to have a greater weight than the periodic floor. In the former case, since it is to the advantage of the lenders, equilibrium spreads will be reduced, whereas in the latter, equilibrium spreads will increase. Figure (6.5), produced by a computer program, ARM03.PAS in Appendix B, compares the equilibrium spreads of a pure ARM with life cap equal to 13% and equilibrium spreads of the same contract with additional periodic cap/floor provisions. We can see that
Sobti and Sykes' conclusions are confirmed in this figure. When initial rate is high (downward sloping to the long-run short rate average, 8%), the equilibrium spreads are consistently lower than, when periodic caps/floors are added, and vice versa.

6.6 Conclusion

ARMs have been very difficult to price because of the contractual complexities that need considerable attention to the path-dependence arising from three sources, namely, periodic resetting of contract rates, periodic caps/floors, and suboptimal prepayments. The present effort to develop efficient lattice-based models for pricing adjustable mortgages has shown the robustness of the backward recursive procedure. Specifically, we have dealt the path-dependence associated with periodic adjustments of contract rates and unexpected suboptimal prepayments by employing the homogeneity in remaining balance, and the path-dependence associated with lagged contract rates in presence of periodic caps/floors by introducing an auxiliary state variable. Finally, we satisfy the arbitrage condition by iterating the bisection search until the equilibrium spreads reach appropriate accuracy.

Although some other contractual features such as teasers and points are not considered in present study, however, from the development of our models, we can see that these provisions can be easily incorporated. For example, teasers can be included by maintaining the same contract rate in the first periods, where they are effective. The consideration of points can be achieved in the same manner as in GNMAAs (see Section (3.8)).
CHAPTER 7

CONCLUSIONS AND FUTURE RESEARCH

7.1 Conclusions

This dissertation has proposed efficient lattice-based models for valuing path-dependent mortgage-related contracts when interest rates follow a mean-reverting square root process, which is approximated by binomial approximations for singular diffusions. Specifically, we have valued GNMA pass-throughs, index amortization swaps, lookback mortgages, and adjustable rate mortgages. These contracts exhibit a wide spectrum of path-dependence. In other words, their values depend on the history of interest rates since date 0. Specific sources that cause the path-dependence are summarized as follows.

<table>
<thead>
<tr>
<th>Contracts</th>
<th>Sources of Path-Dependence</th>
</tr>
</thead>
<tbody>
<tr>
<td>GNMA Pass-throughs</td>
<td>unexpected prepayments based on a prepayment function.</td>
</tr>
<tr>
<td>Index Amortization Swaps</td>
<td>notional principal amortization according to some index and clean-up call provision.</td>
</tr>
<tr>
<td>Lookback Mortgages</td>
<td>minimums that occur over the time</td>
</tr>
<tr>
<td>Adjustable-Rate Mortgages</td>
<td>periodic resetting of contract rates and contract provisions such as life caps and floors</td>
</tr>
</tbody>
</table>

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The difficulties arising from path-dependence cause severe computational problem in applying lattice-based solution techniques. In particular, unless auxiliary state variables representing the path can be identified, an extensive search over all the paths on a binomial lattice of interest rates is required. Specifically, if the number of periods is \( n \), then usually \( 2^n \) different paths need to be searched. For a 20-year mortgage with payments made monthly, this requires searching for \( 2^{240} \) paths. Such calculations are not possible. Our contributions have been to develop efficient mechanisms that resolved the path-dependence. Specifically, we have established efficient algorithms for pricing these path-dependent mortgage-related contracts on a binomial lattice of interest rates. Our solutions can be summarized as follows.

<table>
<thead>
<tr>
<th>Sources of Path-Dependence</th>
<th>Methods for Handling Path-Dependence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unexpected Prepayment (GNMA pass-throughs and ARM)</td>
<td>exploit the homogeneous property in the remaining balance</td>
</tr>
<tr>
<td>Amortization of Notional Principal (IAS)</td>
<td>exploit the homogeneous property in the amortization schedule.</td>
</tr>
<tr>
<td>Clean-up Call (IAS)</td>
<td>discretization of an indicator ranging from 0 to 1 showing the proportion of notional amount in the original notional principal</td>
</tr>
<tr>
<td>Minimum Contract Rates over Time (lookback)</td>
<td>establishing all of the permissible minimums at every node on the lattice and using them as state variable.</td>
</tr>
<tr>
<td>Periodic resetting of Interest Rates (lookback and ARM)</td>
<td>exploit the homogeneous property in the remaining balance</td>
</tr>
<tr>
<td>Caps and Floors (lookback and ARM)</td>
<td>life caps/floors can be taken as resetting of interest rates, hence solved by homogeneous property in remaining balance; periodic caps/floors can be solved by discretization of range from life floor to life floor</td>
</tr>
</tbody>
</table>
In fact, the procedure for pricing these four mortgage-related products on a binomial lattice of interest rates can be summarized by iterating the following flow chart until equilibria are reached.

**Figure 7.1:** Flow chart for establishing equilibrium in pricing path-dependent mortgage-related products

Our approach is quite general and can be extended to price all path-dependent financial contracts following the procedure in Figure (7.1).

The lattice approach has many advantages over the simulation approach. In particular, the lattice approach can be readily modified to incorporate additional option like features. From a pedagogical viewpoint, the lattice approaches are also intuitive and are now commonplace in finance. As a result, it is somewhat surprising that they have not been applied to mortgages.
6.2 Future Research

The research in pricing mortgage-related products is a fascinating area. A lot of work can be continued from our research. First of all, alternative models for the interest rate process could be used. Examples of recent models include Heath, Jarrow, and Morton (1990), Hull and White (1990), and Ritchken and Sankarasubramanian (1992). Extensions to two-factor interest rate models may also be desirable.

The models we used had simple prepayment functions for GNMA pass-throughs and adjustable-rate mortgages, which depended on the short rate and time alone. However, if the formal pricing models are to be successful in the competitive market, far more serious treatment of prepayments may be necessary. This could require the inclusion of new structures for prepayment models, which incorporate the history of rates as an additional state variable. Unfortunately, this extension would increase the dimensionality of the problem. Empirical tests that evaluate whether this increased sophistication is worthwhile need to be conducted.

We priced the IAS, where amortization of notional principal is based on a geometric schedule. Some of the existing IAS contracts amortize the principal according to the arithmetic schedule. In this situation, the homogeneous property can not be used. However, the algorithm developed for pricing IAS when clean-up call is considered can be extended to consider this case. In fact, in this model, we
augmented the state space so that we could track the remaining principal. If this algorithm solves IASs with clean-up call on geometric based amortization schedule, it will have no problem solving IAS on arithmetic based amortization schedule. As discussed, since Monte Carlo simulation is the principal solution technique in pricing IASs in the industry, intermediary options, such as the option to close the contract before it matures, can not be easily valued. It is obvious from the development of the backward pricing procedure that this option can be readily valued in our algorithms simply by terminating the contract at those nodes when the present value of payments investor pays exceeds what he receives.

We priced lookback mortgages, where the contract rate is reset monthly in accordance with the prevailing short rate. As discussed, it is quite easy to extend our models to consider lookbacks with different reset intervals and interest rate indices. A more interesting extension is to use an index that uses yields-to-maturity of Treasuries that match the remaining lifetime of the contract. In pricing such an innovative product, prepayment experience is not fully understood. Bayesian based pricing models that attempt to capture prepayment experiences with limited prepayment data may be helpful.
Appendix A: Techniques for Solving Singularity Problems in CIR

Recently, Nelson and Ramaswamy (1990) suggest transforming the diffusion process such that the volatility of the transformed process is constant and the approximating lattice is computationally simple. For the transformed process $X$, we can use the binomial model of Cox and Rubinstein (1985),

$$
\begin{align*}
    x_u &= x + \sqrt{h} \\
    x_d &= x - \sqrt{h}
\end{align*}
$$

(A.1)

with transition probability

$$
q(x) = 0.5 + \frac{m(x)\sqrt{h}}{2}
$$

(A.2)

where $h$ is time interval and $m(x)$ is the drift term of the transformed process.

If $X$ is transformed from the mean-reverting square root process, since $X$ cannot be negative, Equation (A.1) should be modified to

$$
\begin{align*}
    x_u &= x + \sqrt{h} \\
    x_d &= \max\{x - \sqrt{h}, 0\}
\end{align*}
$$

(A.3)

Let $\phi=4\kappa\mu/\sigma^2-1$ for the CIR process. The up-jump probability is assigned to match the mean exactly; that is

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\[ q(x) = \frac{\left(\frac{\phi - \kappa x^2}{2x}\right) h + x - x_d}{x_u - x_d} \]  

(A.4)

This transition probability is well-defined when \( x \in [x_{\min}, x_{\max}] \), where

\[
\begin{align*}
x_{\max} &= \frac{\sqrt{1 + \kappa \phi h} + 1}{\kappa \sqrt{h}} \\
x_{\min} &= \frac{\sqrt{1 + \kappa \phi h} - 1}{\kappa \sqrt{h}}
\end{align*}
\]  

(A.5)

For this case, we have

\[ p(f(x)) = q(x), \text{ and} \]  

(A.6)

\[
\begin{align*}
r_{\max} &= f(x_{\max}) \\
r_{\min} &= f(x_{\min})
\end{align*}
\]  

(A.7)

for \( x, \phi, h \geq 0 \). Here \( p(f(x)) \) is the up-jump probability for the \( r \)-lattice of the mean-reverting square root process.

If all the states on the lattice defined in Equation (A.3) fall within the range \([x_{\min}, x_{\max}]\), the binomial process is stable. Otherwise singularity problem arises. In the literature, three approaches have been proposed. We discussed each of them as follows:

**Nelson and Ramaswamy approach:**

NR use multiple jumps to fix the singularity problem. Specifically, they define

\[ J_u(x) = \text{the smallest, odd, positive, integer } j \text{ such that} \]

\[ 4h\kappa\mu\sigma^2 + x^2(1 - \kappa h) < (x + j\sqrt{h})^2 \]  

(A.8)

\[ J_d(x) = \text{the smallest, odd, positive, integer } j \text{ such that either} \]
\[ 4h\kappa\mu/\sigma^2 + x^2(1-\kappa h) \geq (x-j\sqrt{h})^2 \text{ or } x-j\sqrt{h} \leq 0 \quad (A.9) \]

Then

\[
p(f(x)) = \begin{cases} \frac{\kappa(\mu-r)+r}{\kappa-r} & \text{if } r_u > 0 \\ 0 & \text{otherwise} \end{cases} \quad (A.10)
\]

In Equation (A.8) and (A.9), \( J \) is chosen to guarantee that \( 0 \leq p(f(x)) \leq 1 \) in (A.10), in such a way that the local drift converges to the diffusion limit.

**Tian's Approach**

When \( \phi > 0, 0 \) is an entrance boundary, which cannot be reached from the interior of the state space. This means when \( r \) is near to 0, the up-jump probability will be greater than one. Tian fixed this by considering the following transition probability

\[
p(r) = \begin{cases} 0 & r > r_{\max} \\ \frac{\kappa(\mu-r)+r}{\kappa-r} & r_{\min} \leq r \leq r_{\max} \\ 1 & r < r_{\min} \end{cases} \quad (A.11)
\]

where \( r_{\max} \) and \( r_{\min} \) are defined in Equation (A.6).

**Li's Approach**

Li proposes a binomial process that matches the mean locally at any accessible node on the lattice. The up and down jumps for the transformed process \( X \) are defined by

\[
x_u = \begin{cases} \frac{x_{\min}}{x + \sqrt{h}} & x < x_{\min} \\ x + \sqrt{h} & x_{\min} \leq x \leq x_{\max} \\ x + i_u \sqrt{h} & x > x_{\max} \end{cases} \quad (A.12)
\]
\[
x_d = \begin{cases} 
\max\{x - \sqrt{h}, 0\} & x \geq x_{\min} \\
 x + i_d \sqrt{h} & x < x_{\min}
\end{cases}
\] (A.13)

where \(i_d\) (or \(i_u\)) is the smallest odd integer that the up-jump probability
\[
p(r) = \frac{\kappa(\mu - r)h + r - r_d}{r_u - r_d}
\] (A.14)
does not fall below 0 (or exceed 1).

As shown in Figure (A.1), if the binomial process goes above the range \([x_{\min}, x_{\max}]\), it jumps down to two different states in the range.

Similarly, if the process goes below the range \([x_{\min}, x_{\max}]\), it jumps to two
different states in the range. This binomial process will never reach zero if \( \phi > 1 \). It can touch zero if \( \phi = 1 \), but it will be reflected with probability 1 in the next period. In the figure, \( UB \) and \( LB \) are the smallest state above \( x_{\max} \) and the largest state below \( x_{\min} \), respectively on the lattice. Then \( UB \) and \( LB \) are reflecting barriers for the binomial process.

If -1<\( \phi < 1 \), then 0 is a regular boundary, where the diffusion can both enter and leave. Furthermore, it is a sticky barrier in the sense that when the process reaches this boundary, it will stay there for a positive time. In this case, redefine

\[
x_{\min} = \max \left\{ \frac{1 + x \phi \sqrt{h} - 1}{x \sqrt{h}}, 0 \right\}
\]

Let \( x_{last}(x) \) be the smallest state that is on the lattice greater than \( x \) in the next period. The lattice is modified to

\[
x_u = \begin{cases} 
    x_{last} & x < x_{\min} \\
    x + \sqrt{h} & x_{\min} \leq x \leq x_{\max} \\
    x - i_u \sqrt{h} & x > x_{\max}
\end{cases}
\]

\[
x_d = \max\{x - \sqrt{h}, 0\}
\]

with the transition probability given by Equation (A.14). The binomial process has a sticky boundary \( LB=0 \), where the process is absorbed with positive probability.
Appendix B: Computer Programs

(* GLOBAL VARIABLES DEFINITION *)

CONST declaration

accuracy : precision for bisection search;
partition : partition numbers;
critical : lower bound of the array;

INTEGER declaration

T : years;
n : number of payments per year;
ju : up jump size;
jd : down jump size;
jump : jump size when singularity occurs;

REAL declaration

y0 : initial rate;
x0 : initial value of the transformed diffusion process;
x : value of the transformed process;
kappa : coefficient in CIR process (mean-reversion force);
theta : coefficient in CIR process (short rates average);
sigma : coefficient in CIR process (volatility);
h : time interval (typically one month);
hh : square root of h (jump size in transformed process);
spd : equilibrium spread;
right : right-hand-side value in Bisection Search;
left : left-hand-side value in Bisection Search;
pmt : fixed-level payment for FRM and ARM;
upPrepay : unexpected prepayment when an up jump of interest rates occurs;
downPrepay : unexpected prepayment when a down jump of interest rates occurs;
pu : up-jump probability on the lattice for short rate;

ARRAY declaration

y,p,m,Q : one-dimensional vectors of real numbers for holding
temporary values in a backward recursion procedure;
Rate : one-dimensional vectors of up-jump probabilities of the whole lattice;
(* Notice that since on a binomial lattice of interest rates,
  the up-jump probabilities are the same for nodes with the
  same horizontal level (but this is not true for singular
  nodes), one-dimensional vector is sufficient for holding
  all the values. *)
Functions and Procedures

(* Recover short rate from a transformed process. *)
function R(x:real):real;
begin if (x<0) then r:=0 else r:=sigma*sigma*x*x/4;
end;

(* Calculating one-year Treasury rate on a CIR setting.*)
function Tr(x:real):real;
var gamma,A,B,P,temp: real;
begin gamma:=sqrt(kappa*kappa+2*sigma*sigma);
temp:=(gamma+kappa)*(exp(gamma)-1)+2*gamma;
B:=2*(exp(gamma)-1)/temp;
A:=exp(2*kappa*theta/sigma/sigma*ln(2*gamma*exp((kappa+gamma)/2)/temp));
P:=A*exp(-B*x);
Tr:=-ln(P);
end;

(* Establishing the fixed-level payment. *)
function fPmt(x:real;s:integer):real;
(* x: contract rate; s: remaining life in months. *)
begin
  fPmt:=(x/12)*exp(s*ln(1+x/12))/(exp(s*ln(1+x/12))-1)
end;

(* Establishing the remaining balance at time t. *)
function Bal(t:integer;x:real:s:integer):real;
(* x: contract rate; s: remaining life in months. *)
begin
  Bal:=(exp(s*ln(1+x/12)) - exp(t*ln(1+x/12)))/
       (exp(s*ln(1+x/12))-1)
end;

(* Finding minimum of two values. *)
function min(x,y:real):real; begin if x>y then min:=y else min:=x; end;
(* Bisection search *)
function flag(price:real):boolean;
    begin
        if (abs(price-par)>=accuracy) then
            begin
                if (price>par) then right:=spd else left:=spd;
                spd:=(right+left)/2;
            end
        else flag:=TRUE;
    end;

(* Calculating 3-month LIBOR rate off a CIR process *)
function LIBOR(x:real):real;
    (* x : short rate. *)
    var gamma,A,B,P,temp : real;
    begin
        gamma:=sqrt(kappa*kappa+2*sigma*sigma);
        temp:=(gamma+kappa)*(exp(gamma/4)-1)+2*gamma;
        B:=2*(exp(gamma/4)-1)/temp;
        A:=exp(2*kappa*theta/sigma/sigma*ln(2*gamma*exp((kappa+gamma)/4)/temp));
        P:=A*exp(-B*x);
        LIBOR:=(1/P-1)*4;
    end;

(* Finding the the reflecting point. *)
(* Or finding the lowest nodes the lattice can go.*)
procedure reflecting_point: label stop0;
    var i : integer;(* index *)
    high,low : integer;
    begin
        for i:=high downto low do
            if (x0+i*hh)<=2*hh then begin Lbound:=i+1;goto stop0; end
        stop0;
    end;

function fPmt(x:real:s:integer:cap:real):real;
    (* Cap : life cap *)
    begin
        if x>=cap then x:=cap;
        (* If this statement is activated, we then *)
        (* consider the floor rate *)
        if x<=floor then x:=floor;
        fPmt:=(x/12+spd/12)*exp(s*ln(1+x/12+spd/12))/
            (exp(s*ln(1+x/12+spd/12))-1) end;
function Bal(t: integer; x: real; s: integer; cap: real): real;
begin  if x < cap then x := cap;
{if x <= floor then x := floor;}  
Bal := \frac{(e^{s \ln(1+x/12+spd/12)} - e^{t \ln(1+x/12+spd/12)})}{e^{s \ln(1+x/12+spd/12)} - 1};
/* Calculating the prices of straight amortization bond. */
procedure straight_amortization_bond(cr: real);
(* cr: contract rate. *)
label stop;
var k, i, gl, dis : integer;(* index *)
begnin
pmt := fPmt(cr, 12*T);
for k:=12*T-1 downto 0 do
  begin
    for i:=k downto 0 do
      begin
        dis := 2*i-k;
        x := x0 + dis * hh;
        if (x <= 2*hh) then goto stop;
        yu := 1; jd := 1;
        if (kappa * theta * h + Rate(dis) * (1 - kappa * h) >
            Rate(dis + yu)) then yu := yu + 2;
        if (yu > 1) then jd := -1;
        pu := kappa * (theta - Rate(dis)) *
            h + Rate(dis) - Rate(dis - jd))/
            (Rate(dis + yu) - Rate(dis - jd));
        if pu < 0 then pu := 0;
        y[i] := (pmt + (1-pu) * P(i+trunc((1-jd)/2))
            + pu * P(i+trunc((1+yu)/2))) * exp(-Rate(dis)*h);
      end;
  end;
end;

/* Calculating the prices of callable amortization bond. */
procedure callable_amortization_bond(cr: real);
label stop;
var k, i, gl, dis : integer;(* index *)
tempBal : real;(* remaining balance *)
begnin
pmt := fPmt(CR, 12*T);
tempBal := 0.0;
for k:=12*T-1 downto 0 do
  begin
    for i:=k downto 0 do
      begin
        dis := 2*i-k;
        x := x0 + dis * hh;
        if (x <= 2*hh) then goto stop;
        yu := 1; jd := 1;
        if (kappa * theta * h + Rate(dis) * (1 - kappa * h) >
            Rate(dis + yu)) then yu := yu + 2;
        if (yu > 1) then jd := -1;
        pu := kappa * (theta - Rate(dis)) * h + Rate(dis)
            - Rate(dis - jd))/
            (Rate(dis + yu) - Rate(dis - jd));
        if pu < 0 then pu := 0;
        y[i] := (pmt + min(tempBal,
(1-pu)*P[i+trunc((1-ju)/2)])
-pu*min(tempBal,P[i+trunc((1+ju)/2)])
*exp(-Rate(dis)*h);
end;
stop;
for gl:=k downto 0 do p[gl]:=y[gl];
tempBal:=(tempBal+pmt)/(1+CR/12);
end;
end;  

 Procedure fixed_rate_mortgages(cr:real);
 label stop;
 var k,i,gl,dis : integer;(* index *)
tempBal : real; (* remaining balance *)
upPrepay : real; (* Prep. in up jump *)
downPrepay : real; (* Prep. in down jump *)
begin:
   for k:=12*T-1 downto 0 do
      begin
         pmt:=fPmt(CR,n-k);
tempBal:=1-(pmt-1*(CR/12));
         for i:= k downto 0 do
            begin
               dis:=2*i-k;
x:=x0+dis*hh;
               if (x<=-2*hh) then goto stop;
               ju:=1; jd:=1;
               if (kappa*theta*h+Rate(dis)*(1-kappa*h)>Rate(dis+ju)) then ju:=ju+1;
               if (ju>1) then jd:=1;
               pu:=(kappa*theta*Rate(dis)*h+Rate(dis)-Rate(dis-jd))/(Rate(dis+ju)-Rate(dis-jd));
               if pu<0 then pu:=0;
               if (rate(dis)>1) or (k=n-1) then
                  begin
                     UpPrepay:=0.0;
                     DownPrepay:=0.0;
                  end
                  else
                     begin
                        UpPrepay:=tempBal*exp(-20*rate(dis+ju));
                        DownPrepay:=tempBal*exp(-20*rate(dis-jd));
                     end;
                  y[gl]:=(pmt*(1-pu)*DownPrepay+(tempBal-DownPrepay)*P[i+trunc((1-ju)/2)])
                  +pu*(UpPrepay+(tempBal-UpPrepay)
                  *P[i+trunc((1+ju)/2)])*exp(-Rate(dis)*h);
               end;
         stop;
         for gl:=k downto 0 do p[gl]:=y[gl];
      end;
end;
IAS01.PAS

Programs for Chapter 4

(* Calculating the price of IAS, when lockout is not *)
(* considered, cleanup is not considered. *)
(* If price is not equal to 0, the function for *)
(* bisection search has to be employed. *)

procedure IAS_noLockout_noCleanup(fixedrate:real);
label stop, stop1;

var
  k, i, j, dis, dis1, g, gl: integer; (* index *)
  pmt : real; (* net payment of the *)
           (* fix-side payor. *)
  tempBal : real; (* notional amount after amortization. *)

begin
  for k:=4*T-1 downto 0 do
    begin
      for i:= k*3 downto 0 do
      begin
        dis := 2*i - k*3;
        x := x0 + dis*hh;
        if (x<-2*hh) then goto stop1;
        j := 1; jd := 1;
        if (kappa*theta*h + Rate(dis)*(1-kappa*h) >
            Rate(dis+ju)) then ju := ju + 2;
        if (ju>l) then jd := -1;
        pu := (kappa*(theta - Rate(dis))*h + Rate(dis) - Rate(dis-jd)) /
              (Rate(dis+ju) - Rate(dis-jd));
        if (pu<0) then pu := 0.0;
        tempRate := LIBOR(Rate(dis));
        Pmt := (fixedrate*temprate);
        if (ju>l) then jump := 1 else jump := 0;
        tempBal := Bal(Temprate);
        for j := i + 3 downto 1 do
          P[j] := pmt + tempBal*M[j];
        for j := 3*k + 2 downto 3*k do
          begin
            for g := 1 + j - 3 * k downto 1 do
              begin
                DIS1 := 2*g - j;
                x := x0 + DIS1*hh;
                if (x<-2*hh) then goto stop1;
                j := l; jd := 1;
                if (kappa*theta*h + Rate(DIS1)*(1-kappa*h) >
                    Rate(DIS1+ju)) then
                  ju := ju + 2;
                if (ju>l) then jd := -1;
                pu := (kappa*(theta - Rate(DIS1))*h +
                       Rate(DIS1) - Rate(DIS1-jd)) /
                       (Rate(DIS1+ju) - Rate(DIS1-jd));
                if (pu<0) then pu := 0;
                y[g] :=
                  (((1-pu)*P[trunc((1-jd)/2)] +
                    pu*P[trunc((1+ju)/2)]) /
                    exp(Rate(DIS1)*h));
              end;
            stop1:
            for g1 := i + j - 3 * k downto 1 do
              P[g1] := y[g1];
            end;
          Q[i] := y[i];
        end;
      end;
    stop:
for gl:=k+3 downto 0 do m(gl):=Q(gl);
end;
end;

IAS02:PAS
(" calculating the price of IAS, when lockout is */
(" two years, cleaning is not considered. ")
(" If price is not equal to 0, the function for ")
(" bisection search has to be employed. ")
procedure IAS_withlockout_nocleanup(fixedrate:real);
lable stop,stop1,stop2;
var
  k,i,j,dis,disl,g,gl :integer; (* index *)
pmt :real; (* net payment of the *)
  pu :real; (* fix-side payor. *)
  tempBal :real; (* notional amount af-*)
  tempRate :real; (* ter amortization. *)
begin
  for k:=4*T-1 downto 8 do
    begin
      for i:= k+3 downto 0 do begin
        dis:=2*i-k-3;
        x:=x0+dis*h;
        if (x<=2*hh) then goto stop;
        yu:=i;jd:=1;
        if (kappa*theta*h+Rate(dis)*(1-kappa*h)> Rate(dis+ju)) then yu:=ju+i+2;
        if (ju>1) then jd:=1;
        pu:=(kappa*(theta+Rate(dis))*h+Rate(dis)-Rate(dis+jd))/
         (Rate(dis+ju)-Rate(dis-jd));
        if (pu<0) then pu:=0.0;
        tempRate:=LIBOR(Rate(dis));
        Pmt:=(fixedrate-tempRate);
        if (ju>1) then jump:=1 else jump:=0;
        tempBal:=Bal(TempRate);
        for j:=i+3+jump downto i do
          P[j]:=Pmt+tempBal*M[j];
        for j:=k+2 downto 3*k do
          begin
            for g:=i+j-3*k+jump downto i do begin
              disl:=2*g-1;
              x:=x0+disl*h;
              if (x<=2*hh) then goto stop1;
              yu:=i;jd:=1;
              if (kappa*theta*h+Rate(disl)*(1-kappa*h)> Rate(disl+ju)) then 
                yu:=ju+i+2;
              if (ju>1) then jd:=-1;
              pu:=(kappa*(theta+Rate(disl))*h+ Rate(disl+ju)-Rate(disl-jd))/
               (Rate(disl+ju)-Rate(disl+jd));
              if (pu<0) then pu:=0;
              Y[g]:= ((1-pu)*P[g+trunc((1-jd)/2)]-pu*P[g+trunc((1+ju)/2)])/
               exp(Rate(disl)*h);
              end;
            stop1:
            for gl:=i+j-3*k downto i do P[gl]:=Y[gl];
            end;
        Q[i]:=Y[i];
      end;
    end;
  end;
end;
for gl:=k*3 downto 0 do m(gl):=Q(gl);
end;
for k:= 23 downto 0 do
begin
for i:= k downto 0 do
begin
disl:=2*i-k;
x:=x0+disl*hh;
if (x<-2*hh) then goto stop2;
ju=1;jd=1;
if (kappa*theta*h+Rate(disl)*
(1-kappa*h)>Rate(disl+ju)) then
ju:=ju+2;
if (ju>1) then jd:=-1;
pw:=(kappa*(theta-Rate(disl))*h+
Rate(disl)-Rate(disl-jd))/
(Rate(disl+ju)-Rate(disl-jd));
if (pu<0) then pu:=0;
Q[i]:=(1-pw) *m[i+trunc((1-jd)/2)]
+ pu*m[i+trunc((1+ju)/2)]/exp(Rate(disl)*h);
end;
stop2:
for gl:=k downto 0 do m(gl):=Q(gl);
end; {end for the lockout period}
end;
IAS03.PAS

(* Calculating the price of IAS, when lockout is not *)
(* considered, cleanup is considered. *)
(* If price is not equal to 0, the function for *)
(* bisection search has to be employed. *)
procedure IAS_nolockout_with cleanup (fixedrate:real);
label stop, stop1;
var
k, i, j, dis, disl, q, gl, g2 : integer; (* index *)
pmt : real; (* net payment of the *)
fixSide : real; (* fix-side payor. *)
notional : real; (* notional amount af= *)
ter : real; (* ter amortization. *)
begin
for k:=4*T-1 downto 0 do
begin
for i:= k*3 downto 0 do
for i:= partition downto 1 do
begin
dis:=2*i-k*3;
x:=x0+dis*hh;
if (x<-2*hh) then goto stop;
ju=1;jd=1;
if (kappa*theta*h+Rate(dis)*(1-kappa*h)>Rate(dis+ju)) then ju:=ju+2;
if (ju>1) then jd:=-1;
pw:=(kappa*(theta-Rate(dis))*h+Rate(dis)-Rate(dis-jd))/
(Rate(dis+ju)-Rate(dis-jd));
if (pu<0) then goto stop;
tempRate:=LIBOR(Rate(dis));
Pmt:=1/partition*fixSide-tempRate);
if (ju>1) then jump:=1 else jump:=0;
tempBal:=1/partition*Bal(tempRate);
Bindex:=trunc(tempBal*partition);
if tempBal >0.05 then
(* This is a coarse linear interpolation. *)
(* However, it has already converges fast.*)
for j:= i*3+jump downto i do
\[ \forall (j):= \text{temp8al} \times M[j, \text{Bindex}] \]

\[ \text{else} \]
\[ \text{for } j := 4 + 3 \text{ downto } i \text{ do} \]
\[ P[j] := \text{pmt}; \]
\[ \text{for } j := 3 \times k + 2 \text{ downto } 3 \times k \text{ do} \]
\[ \text{begin} \]
\[ \text{for } g := i + j - 3 \times k + \text{jump downto } i \text{ do} \]
\[ \text{begin} \]
\[ \text{disl} := 2 \times g - j; \]
\[ x := x_0 + \text{disl} \times \text{hh}; \]
\[ \text{if } (x < -2 \times \text{hh}) \text{ then goto stop1; } \]
\[ j := 1; j_d := 1; \]
\[ \text{if } (\kappa \times \theta > \text{Rate} \times \text{disl}) \times (1 - \kappa) > \text{Rate} \times (\text{disl} + ju) \]
\[ \text{then } j := j + 2; \]
\[ \text{if } (ju > 1) \text{ then } j_d := -1; \]
\[ p := (\kappa \times (\theta - \text{Rate} \times \text{disl}) \times \text{h} + \text{Rate} \times \text{disl} - \text{Rate} \times \text{disl} - j_d) / (\text{Rate} \times \text{disl} + ju - \text{Rate} \times \text{disl} - j_d); \]
\[ \text{if } (pu < 0) \text{ then } pu := 0; \]
\[ y[g] := \]
\[ (1 - pu) \times P[g + \text{trunc}((1 - j_d) / 2)] - pu \times P[g + \text{trunc}(1 - ju) / 2)] / \exp(\text{Rate} \times \text{disl}) \times h; \]
\[ \text{end}; \]
\[ \text{stop1;} \]
\[ \text{for } g1 := i + j - 3 \times k \text{ downto } i \text{ do } P[g1] := y[g1]; \]
\[ \text{end}; \]
\[ Q[i, i] := y[i]; \]
\[ \text{end}; \]
\[ \text{stop;} \]
\[ \text{for } g1 := k \times 3 \text{ downto } 0 \text{ do} \]
\[ \text{for } g2 := \text{partition downto } 1 \text{ do } m[g1, g2] := Q[g1, g2]; \]
\[ \text{end}; \]
LBK.PAS

(* Computing t: 
procedure look: 
label stop; 
var
  k, i, j, f, dispmt 
  tempSal 
contract 
begin 
  for k:=n-1 do 
  begin 
  for i:
  begin 
  end; 
stop:
end; 
end;
LBK.PAS

Program for Chapter 5

(* Computing the lookback mortgage prices. *)

procedure lookback_mortgage(fixedrate: real);
begin
  label stop;
  var
    k, l, j, f, dis : integer; (* index *)
    pmt : real; (* fixed-level pmt *)
    (* in next period. *)
    tempBal : real; (* remaining balance *)
    (* in next period. *)
    contract : real; (* new contract rate *)
  begin
    for k := n-1 downto 0 do
      begin
        for l := k downto 0 do
          begin
            dis := 2*l-k;
            x := 0 + dis*h;
            if (x < 2*h) then goto stop;
            ju := l; jd := 1;
            if (kappa*theta*h + Rate(dis)*(1-kappa*h) >
                Rate(dis+ju)) then ju := ju+2;
            if (ju > 1) then jd := -1;
            pu := (kappa*(theta - Rate(dis))*h + Rate(dis) - Rate(dis-jd))/
                  (Rate(dis+ju) - Rate(dis-jd));
            if pu < 0 then pu := 0;
            for j := max(lbound, i-k) to 0 do
              begin (* Calculate the inner loop. *)
                if (Rate[j] < Rate[dis]) then
                  begin
                    contract := Rate[j];
                    f := j;
                  end
                else
                  begin
                    contract := Rate[dis];
                    f := dis;
                  end;
              end (* if the floor rate is imposed, then the *)
              (* following statement should be *)
              (* effective. Otherwise, no floor rate. *)
              (* if contract <= 0.05 then contract := 0.05; *)
            Pmt := fPmt(contract, T*12-k);
            tempBal := 1 - (pmt - (contract + spd))/12;
            y[j, dis] := (pmt + min(tempBal, tempBal -
                          ((1-pu)*y[f, dis + trunc((1-3*jd)/2)]
                          + pu*y[f, dis + ju])))*
                            exp(-Rate[dis]*h);
          end (* Calculate the inner loop. *)
  stop;
end;
Programs for Chapter 6

ARM01.PAS

(* Calculating the price of a pure ARM. *)
(* If price is not equal to 0, the function for *)
(* bisection search has to be employed. *)
procedure pure ARM(fixedrate:real);
label stop,stop1;
var
  k,i,j,dis,disl,gl,g1 : integer; (* index *)
  tempBal,tempBall : real; (* remaining balance *)
  pmt : real; (*fixed-level pmt *)
begin
  for k:=T-1 downto 0 do
    begin
      for i:= k*12 downto 0 do
        begin
          dis:=2*i-k*12;
          x:=x0+dis*hh;
          if (x<-2*hh) then goto stop;
          ju:=1;jd:=1;
          if (kappa*theta*h+Rate{dis}*(1-kappa*h)> Rate{dis+ju}) then ju:=ju+2;
          if (ju>1) then jd:=1;
          pu:=(kappa*(theta-Rate{dis})*h+Rate{dis}-Rate{dis-jd})/ (Rate{dis+ju}-Rate{dis-jd});
          if pu<0 then pu:=0;
          tempRate:=Tr(Rate{dis});
          Pmt:=fPmt(TempRate,12*T-12*k,lifecap);
          if (ju>1) then jump:=1 else jump:=0;
          tempBal:=Bal(12,TempRate,12*T-12*k,lifecap);
          for j:=i+1 jump downto 0 do
            begin
              P[j]:=tempBal*M[j];
              tempBall:=tempBal;
            end;
          for j:=12*k+11 downto 12*k do
            begin
              for g:=1+j-12*k+jump downto i do
                begin
                  disl:=2*g-j;
                  x:=x0+disl*hh;
                  if (x<-2*hh) then goto stop1;
                  ju:=1;jd:=1;
                  if (kappa*theta*h+Rate{disl}*(1-kappa*h)> Rate{disl+ju}) then ju:=ju+2;
                  if (ju>1) then jd:=1;
                  pu:=(kappa*(theta-Rate{disl})*h+ Rate{disl}-Rate{disl-jd})/ (Rate{disl+ju}-Rate{disl-jd});
                  if (pu<0) then pu:=0;
                  y[g]:=(pmt+(1-pu)*min(tempBall,
                                  P[g+trunc((1-jd)/2)])
                        -pu*min(tempball,
                                  P[g+trunc((1+ju)/2)])
                        )
                        *exp(-Rate{disl}*h);
                end;
          end;
        end;
      end;
    end;
  end;
end;
Q[i]:=v[i];
end;
stop:
for gl:=12*k downto 0 do m(gl):=Q[gl];
end;

(* Calculating the price of an ARM with unexpected prep.*)
(* If price is not equal to 0, the function for *)
(* bisection search has to be employed. *)
procedure ARM;
label stop, stop1;
var
k, i, j, dis, disl, g, gl : integer; (* Index *)
tempBal, tempball : real; (* remaining balance *)
pmt : real; (* fixed-level pmt *)
begin
for k:=T-1 downto 0 do
begin
for i:= k*12 downto 0 do
begin
dis:=2*i-k*12;
x:=x0+dis*hh;
if (x<-2*hh) then goto stop;
ju:=l; jd:=l1;
if (kappa*theta*h+Rate(dis)*(1-kappa*h)+
Rate(dis+ju)) then u:=ju+2;
if (ju>1) then jd:=jd-1;
pu:=(kappa*(theta+Rate(dis))*h+Rate(dis)-Rate(dis-jd))/
(Rate(dis+ju)-Rate(dis-jd));
if pu<0 then pu:=0;
tempRate:=Tr(Rate(dis));
for j:=12*k+11 downto 12*k do
begin
for g:=i+j-12*k+jump downto i do
begin
dis1:=2*g-j;
x:=x0+dis1*hh;
if (x<-2*hh) then goto stop1;
ju:=l; jd:=l1;
if (kappa*theta*h+Rate(dis1)*
(1-kappa*h)+Rate(dis+ju)) then
ju:=ju+2;
if (ju>1) then jd:=jd-1;
pu:=(kappa*(theta+Rate(dis1))*h+Rate(dis1)-Rate(dis-jd))/
(Rate(dis+ju)-Rate(dis-jd));
if (pu<0) then pu:=0;
pmt:=pmt(pmt, n-j, lifecap);
tempBal:=1-(pmt-1*(tempRate+spc)/12);
if [rate(dis1)>1] or (k=n-1) then
begin
UpPrepay:=0.0;
DownPrepay:=0.0;
end
else
begin
UpPrepay:=tempBal*
exp(-20*rate(dis1+ju));
DownPrepay:=tempBal*
exp(-20*rate(dis1-jd));
end;
\gamma[g] = \frac{pmt}{(1-pu) \cdot (DownPrepay + P[(g-trunc((1-jd)/2))] - pu \cdot (UpPrepay + (tempBal - UpPrepay)p[g-trunc((1+ju)/2)])} \cdot \exp(-\text{Rate[dis1]} \cdot h);
end;
stop1;
for gl := i+j=12 \ast k \text{ downto } 1 \text{ do } P[gl] = \gamma[gl];
end;
Q[i] := \gamma[i];
end;
stop1;
for gl := 12 \ast k \text{ downto } 0 \text{ do } m[gl] = Q[gl];
end;

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**ARM03_PAS**

(* Calculating the price of an ARM with periodic cap/ floor. If price is not equal to 0, the function for bisection search has to be employed. *)

procedure ARM_Periodic_cap_floor;
label stop, stop1;
var
k, i, j, dis, disl, g, gl : integer; (* index *)
tempBal, tempBal1 : real; (* remaining balance *)
pmt : real; (*fixed-level pmt *)
begin
for k := T-1 \text{ downto } 1 \text{ do }
for i := k \ast 12 \text{ downto } 0 \text{ do }
begin
\text{dis} := 2 \ast i \ast k; \text{ dis} := \text{dis} \ast \text{hh};
\text{if } (x<2 \ast \text{hh}) \text{ then goto stop;}
\text{jd} := 1;
\text{if } (\text{kappa} \ast \text{theta} \ast \text{Rate[dis]} \ast (1-\text{kappa} \ast \text{h}) > \text{Rate[dis]} \ast \text{ju}) \text{ then } \text{ju} := \text{ju} + 2;
\text{if } (\text{ju} > 1) \text{ then } \text{jd} := -1;
\text{pu} := \text{kappa} \ast \text{theta} \ast \text{Rate[dis]} \ast \text{h} \ast \text{Rate[dis]} \ast \text{Rate[dis]} \ast \text{jd}) /
(\text{Rate[dis]} \ast \text{ju}) \ast \text{Rate[dis]} \ast \text{jd});
\text{if } \text{pu} < 0 \text{ then } \text{pu} := 0;
\text{if } (\text{ju} > 1) \text{ then jump := 1 } \text{ else } \text{jump := 0;}
\text{Spot} := \text{Tr} \ast \text{Rate[dis]} ;
for il := \text{gt} \text{ downto } 0 \text{ do }
begin (il := \text{gt} \text{ downto } 0 )
\text{if } 2 \ast i \ast k \ast 12 > \text{bound} \text{ then CR := 0.13;}
\text{else CR := FindCR(Spot,0.01\ast il);}
\text{pmt} := \text{fPmt(CR,12\ast T-12\ast k)};
\text{tempBal} := \text{Bal}(12,CR,12\ast T-12\ast k);
for j := 1+12 \ast \text{jump} \text{ downto } i \text{ do }
begin (j := 1+12 \ast \text{jump} \text{ downto } i )
\text{tempCR} := \text{FindCR(Tr(Rate[2*]
\text{jump}-(1+12)),CR));}
\text{P[j]} := \text{tempBal} \ast b[j],
\text{trunc(CR*100)} \ast 1);
end:(j := 1+12 \ast \text{jump} \text{ downto } i)
\text{tempBal} := \text{tempBal};
for j := 12 \ast k+il \text{ downto } 12 \ast k \text{ do }
begin (j := 12 \ast k+il \text{ downto } 12 \ast k)
for g := i+j=12 \ast k \text{ jump } \text{ downto } i \text{ do }


begin { i,j = 12 k + jump downto i }
  disl := 2 * g - j;
  x := x 0 + disl * hh;
  if (x <= 2 * hh) then goto stop 1;
  ju := i; jd := 1;
  if (kappa * theta * h * Rate(disl) * (1 - kappa * h) > Rate(disl + ju)) then
    ju := ju + 2;
  if (ju > l) then jd := 1;
  pu := (kappa * (theta - Rate(disl)) * h * Rate(disl + ju) - Rate(disl + jd)) / (Rate(disl + ju) - Rate(disl - jd));
  if (pu < 0) then pu := 0;
  y[gl] := (pmt + min(tempBal, ((1 - pu) * P[g + trunc((1 - jd) / 2)] + pu * P[g + trunc((1 + ju) / 2)])))*
            exp(-Rate(disl) * h);
end; { i,j = 12 k + jump downto i }

stop 1:
  for gl := i + j - 12 k downto i
  do P[gl] := y[gl];
  tempBall := (tempBall + pmt) /
              (1 + (CR + spd) / 12);
end; { j = 12 k + 11 downto 12 k }

A[i, i] := y[i];
end; { i /= pt downto 0 }

end;

for gl := 0 to 120 do for il := 0 to pt do
  b(gl, il) := a(gl, il);

CR := tr(rate(0));
Pmt := fPmt(CR, 12 * T);
tempBal := Bal(12, CR, 12 * T);
for j := 12 downto 0 do
  begin { j := i + 12 + jump downto i }
    (tempCR := FindCR(Tr(Rate(2 * j) + jump - (i + 12)), CR));
    P[j] := tempBal * b[j, trunc(CR * 100) + 1];
  end; { j := i + 12 + jump downto i }

for := 11 downto 0 do
  begin { j := 12 * k + jump downto 12 * k }
    tempBal := tempBal;
  end; { j := 11 downto 0 }

for g := j downto 0 do
  begin { i + j - 12 k + jump downto i }
    disl := 2 * g - j;
    x := x 0 + disl * hh;
    if (x <= 2 * hh) then goto stop 3;
    ju := i; jd := 1;
    if (kappa * theta * h * Rate(disl) * (1 - kappa * h) > Rate(disl + ju)) then
      ju := ju + 2;
    if (ju > l) then jd := 1;
    pu := (kappa * (theta - Rate(disl)) * h * Rate(disl + ju) - Rate(disl + jd)) / (Rate(disl + ju) - Rate(disl - jd));
    if (pu < 0) then pu := 0;
    y[gl] := (pmt + min(tempBall, ((1 - pu) * P[g + trunc((1 - jd) / 2)] + pu * P[g + trunc((1 + ju) / 2)])))*
              exp(-Rate(disl) * h);
  end; { i + j - 12 k + jump downto i }

stop 3;
for gl:=1 downto 0
  do P[gl]:=y[gl];
      tempBall:=(tempBall+pmc)/
         (1+(CR+spd)/12);
  end; (j:=12*k+11 downto 12*k)
References


Kau, J., Keenan, D., Muller, W., and Epperson, J., A Generalized Valuation Model for Fixed Rate Residential Mortgages, *Working Paper*, Department of Insurance, Legal Studies and Real Estate, the University of Georgia.


