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Dynamics and control of open- and closed-chained multibody systems

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Case Western Reserve University, 1992
DYNAMICS AND CONTROL OF
OPEN- AND CLOSED-CHAIN MULTIBODY SYSTEMS

by

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Submitted in partial fulfillment of the requirements
for the Degree of Doctor of Philosophy

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DYNAMICS AND CONTROL OF
OPEN- AND CLOSED-CHAINED MULTIBODY SYSTEMS

Abstract

by

NANJOU LIN

Locally optimal trajectory management (LOTM) performance for
kinematically-redundant open-chained manipulators is improved, by
using a low order polynomial Rayleigh-Ritz expansion to reduce the
base reaction fluctuations and by adding a kinetic energy term into
the cost function to cause joint motions to come to a full stop
without extra effort at the end of the manipulation period. The
LOTM approach may be used for operating robots in space station
laboratories without disturbing the micro-gravity environment or for
managing the joint motions of the legs of walking vehicles. The
dynamics of a walking hexapod with each leg having two segments and
two revolute joints for a total of three degrees of freedom is
investigated. Both a complete model and a simplified model are
formulated using Lagrange's equations for quasi-coordinates. The
simplified model is based on the assumption that the inertia of each
leg is much less than the inertia of the central body. The
simplified model is shown to have superior computational efficiency
in performing dynamic simulation of a hexapod. Using the simplified
model, a dynamic simulation with closed-loop feedback control is
introduced. In the dynamic model of a walking machine, the
closed-loop kinematic constraints yield infinite solutions of the feedforward joint moment calculation. Hence, the feedforward joint moments are neglected and a closed-loop control is used. The output of the dynamic simulation provides the ground reaction forces and active joint moments. Numerical results show that this method is useful for the study of the mechanics of legged locomotion. It provides an alternative to analytical techniques such as the optimal force distribution method and can be used in conjunction with experimental techniques using force-plates and high-speed cameras.
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CHAPTER ONE
INTRODUCTION

1.1 PRELIMINARY REMARKS

This dissertation is concerned with the dynamics and control of open- and closed-chained multibody systems. It is divided into three major parts: (1) The locally optimal trajectory management method is enhanced so that it has better performance in reducing the base reactions of kinematically-redundant open-chained manipulators. This method can be used for planning joint trajectories for space robots and individual legs of walking systems. (2) A simplified dynamic model of a hexapod walking system is formulated and is shown to be much more computationally efficient for dynamic simulations than the complete dynamic model. And, (3) Dynamic simulation with closed-loop feedback control is shown to be a viable approach for determining the joint torques and studying the mechanics of legged locomotion. This problem has been previously studied by many researchers using techniques such as optimal force distribution analysis and experimental methods using force-plates and high-speed cameras.

1.2 LITERATURE SURVEY

In recent years, research into legged locomotion has become a topic of particular interest in robotics. There are two motivations for building walking machines [Ref. 30]. The first one is the need
for vehicles that can travel over rough terrain, where existing vehicles can not go. Only about 50% of the earth's landmass is accessible to wheeled and tracked vehicles, while a much larger fraction can be reached by animals on foot. One aim then is to build walking vehicles that are capable of travelling to the places that have been already reached by legged animals. Legs also provide improved vehicular mobility where the ground is soft, steep, slippery or where existing paths are narrow or discontinuous.

Another motivation for building walking machines is to provide an alternative way of understanding how animals use their legs for locomotion. Biologist have learned a great deal about the behavior of living systems by measuring and manipulating their actions. Walking machines can provide new insights into problems that all legged locomotion systems must solve and the possible mechanisms for their solution. In addition, walking machines provide the opportunity to formulate and test theories about animal locomotion and behavior. Moreover, the demands of building a walking system often call attention to aspects of the biological problem that were overlooked when one was using purely analytical or empirical methods.

There are several fields related to the research of legged systems. One of those is the mechanics of legged locomotion. McMahon [Ref. 17] reviewed the energetic and mechanical principles of walking and running, using information available from experimental studies with human subjects walking on a force-plate.
He formulated a ballistic walking model that conserves the sum of the kinetic and gravitational potential energies of the body. In running, energy is stored transiently in the elastic deformations of stretched muscles and tendons. Using these principles and others, McMahon combined results from theory and experiments to find the range of stiffness values for a running track so that both the potential for human injuries is lowered and running speed is increased.

Sutherland and Ullner [Ref. 37] described their experiences with a man-carrying, 1600-lb, gasoline-powered, hydraulically actuated, six-legged walking machine without using any sophisticated servos. Leg position and loading information were input to an on-board computer to select one of a few possible hydraulic connections of the leg actuators by switching directional control valves. Passive parallel connections of hydraulic actuators distribute loads equally among several legs, creating the effect of a smaller number of virtual legs. The load-sharing property of the parallel connection created a foot stamp problem. This problem was avoided by using leg-loading data to determine which legs are being raised and must not share parallel circuits with loaded legs. Series connections of the hydraulic actuators coordinate the leg motions during walking.

Waldron, Vohnout, Pery and McGhee [Ref. 38] described their plans to build a large self-contained walking truck for use in terrain that is not passable for conventional vehicles. That paper
concentrated on the mechanical design of the legs, on the hydrostatic power transmission system and on the overall configuration of the vehicle for the adaptive suspension vehicle (ASV). That machine is not an autonomous robot in the sense that it carries an operator, however, it has a very high level of machine intelligence and environmental sensing capability.

Raibert [Ref. 31] studied symmetries of motion of trotting cats and running humans in terms of even and odd functions of time. He claimed that techniques based on symmetry are helpful in understanding legged locomotion in animals and, in addition, can simplify the control of legged machines.

Full, Blickhan and Ting [Ref. 10] used a miniature force-plate to measure the ground reaction forces of individual legs of cockroaches. It was found that each leg pair of the cockroach running with an alternating tripod gait was characterized by a unique ground reaction force pattern, but the combined whole-body dynamics is similar to running bipeds and trotting quadrupedal mammals. Also, ground reaction force vectors are oriented towards coxal joints due to nonzero horizontal and lateral components, and hence have the function of minimizing joint moments and muscle forces.

Another major field is the control of legged systems. Alexander [Ref. 1] discussed an optimality criteria at work in bipeds and quadrupeds. The gaits of reptiles, birds, mammals, turtles and people were reviewed. It is shown that mammals of
different sizes tend to move in dynamically similar fashion whenever they have equal Froude numbers $u^2/gh$. Here $u$ and $h$ are characteristic speed and length, and $g$ is the gravitational acceleration. This observation could have implications for designing effective walking machines. He also argued that slowly moving turtles minimize unwanted movement of their bodies (pitch, roll, etc.), while humans minimize energy cost. The patterns of force exerted in human walking and running minimize the work required of the muscles at each speed. Much of the energy, that is needed for running by people and other large mammals, is conserved by tendon elasticity.

Miura and Shimoyama [Ref. 20] had developed a series of small electrically powered biped locomotive robots: BIPER-3 is a stilt-type robot whose foot contacts occur at a point and who can walk sideways, backward and forward; BIPED-4 has legs which have the same degrees of freedom as human legs. All of them are statically unstable but can perform a dynamically stable walk. They approximated the motion of either robot during the single-leg support phase by the motion of an inverted pendulum and, accordingly, dynamic walk is considered to be a series of inverted-pendulum motions with appropriate conditions of connection. They also argued that free-swinging leg control (stride-length control), which can be realized by simple linear feedback, is effective for the dynamic walk of stilt-type bipeds with feet that touch the ground at a point.
Raibert, Brown and Cheponis [Ref. 32] explored the balance in legged locomotion by studying systems that hop and run on one springy leg. They extended an algorithm that originally provided balance in two dimensions to a three-dimensional one-legged machine that runs and balances on an open floor without physical support. Control of the machine is decomposed into three separate parts: control of forward running, control of the body attitude and control of the hopping height. It is claimed to be simply implemented and powerful enough to permit hopping in place, running at a desired rate and travelling along a simple path.

Hemami and Chen [Ref. 12] investigated a computer-simulated two-link planar biped system under the influence of holonomic constraints. Quadratic Lyapunov functions are used to construct the feedback strategies for maintaining stability in the vicinity of an operating point.

Song and Waldrón [Ref. 36] described an analytical approach of determining the gait stability margins of legged locomotion systems. This analytical approach defines the foot positions by means of the concept of local phase, which is the fraction of a cycle period by which the current foot position follows the placement of that foot. Based on this approach, they derived a general equation for the longitudinal stability margin for the 2n-legged wave gait.

Another field of interest is the issue regarding walking on rough terrain. Pearson and Franklin [Ref. 22] conducted a cinematographic analysis of locusts walking on a variety of
terrains. In that paper, three tactics are identified to be used by single legs to find support sites on rough terrain: (1) rhythmic searching movements which are initiated when the leg fails to find any support at the end of a swing phase, (2) elevator tactile reflex which is initiated when legs strike objects during the swing phase and which usually results in the object being used as a site for support, and (3) local searching movements which are initiated once the leg had contacted a potential supporting surface. They also discovered that animals do not adopt rigid gaits when walking on rough terrain and there is a tendency for the stepping movements of opposite legs to be either 180° out of phase or exactly in phase. In-phase stepping of the middle legs is observed frequently when the animal is walking over a ditch or onto an elevated object.

Hirose [Ref. 13] presented a paper addressing problems of energy efficiency, design and adaptive gait control of quadruped walking vehicles. He argued that the gravitationally decoupled actuator (GDA) design principle for a leg is indispensable for realizing energetically efficient walking motion. Based on the GDA principle, a lightweight leg mechanism called the three-dimensional Cartesian-Coordinate pantograph (PANTOMEC) was developed. Based on energy considerations, a quadruped walking machine was constructed using tactile sensors on the feet and a posture detector on the body. It was shown to be able to climb steps and negotiate obstacles. Hirose also presented an algorithm for terrain adaptive control which was shown to produce an efficient gait while avoiding
deadlock positions on less regular terrain using computer simulation.

Ozguner, Tsai and McGhee [Ref. 21] presented both hardware and software experiments with a hexapod walking machine using a triangulation ranging system making use of two charge-injection-device (CID) television cameras and a hand-held laser. Coupled with a follow-the-leader gait, the laser is used by the operator to designate candidate footholds which are accepted or rejected automatically by the machine. Accepted footholds guide the machine to traverse rough terrain with little disturbance to body attitude.

1.3 OUTLINE

The object of Chapter Two is to improve the performance of the locally optimal trajectory management (LOTM) approach which is used to minimize the base reactions of kinematically-redundant open-chained manipulators. The LOTM approach may be useful for operating robots in space station laboratories in a manner that minimizes disturbance of the micro-gravity environment.

The LOTM scheme may also be useful for managing the joint motions of the legs of walking vehicles. These legs are often kinematically redundant and their motions may be best designed by optimizing a performance function.

The redundant variables are expressed in terms of Rayleigh-Ritz expansion over discrete time intervals throughout the manipulation
time period. Numerical optimization scheme determines those Rayleigh-Ritz coefficients which minimize the cost function over each discrete time step. In the numerical examples the cost function is taken as the base reactions of the robot. Results of both two- and three-dimensional numerical examples show that (1) a low order polynomial Rayleigh-Ritz expansion can reduce the fluctuations on the base reaction pattern, and (2) the cost function must include a kinetic energy term to cause joint motions to come to a full stop without extra effort at the end of the manipulation period.

Chapter Three deals with the mechanics of a hexapod walking machine with each leg having two segments and two revolute joints for a maximum total of six degrees of freedom. The equations of motion are formulated using Lagrange's equations in terms of quasi-coordinates. Transformation matrices are then used to transform the equations of motion into joint configuration space. The resulting equations of motion are in terms of true coordinates and have the minimum number of degrees of freedom. In general, the equations of motion for a hexapod are indeterminate because the constraint forces (ground reaction forces) are included making the number of unknowns (joint accelerations or joint torques and ground reaction forces) greater than the number of equations. In dynamic simulations, constraint equations are used in addition to the equations of motion to make the system determinate. However, constraint equations are trivially satisfied in the control problem.
of legged locomotion. In such a case, experimental data from a force-plate can be used to determine the constraint forces so the equations of motion can be solved given the prescribed motion. Otherwise, an optimization scheme is necessary to solve for the joint torques.

A hexapod can be treated as having six individual manipulators, whose bases are attached to a common body and some of which are subjected to kinematic constraints on the end-effector. Therefore, the number of degrees of freedom associated with a hexapod is generally much larger than that associated with a single robotic manipulator. This implies that the computational load in simulating a hexapod is much heavier than in simulating a single robotic manipulator. In Chapter Four, a hexapod dynamic formulation which has three degrees of freedom for each leg is decomposed into parts associated with the central body and six individual legs, hence greatly reducing the computational burden. This decomposition is based on the assumption that the inertia of each leg is much less than the inertia of the central body. Based on this assumption, (1) the central body is decoupled from the legs which are in the recovery phase, (2) recovering legs are treated as manipulators with a moving base which is coupled with the central body, and (3) stance legs are treated as massless rigid beams. This approach should also prove useful in studying the mechanics of the locomotion of insects which have very light legs as compared with their body.
Chapter Five introduces a new approach of studying the mechanics of hexapod locomotion in which the walking gait pattern is given in terms of joint angles and their time derivatives. This approach makes use of the dynamic simulation of the decomposed hexapod model of Chapter Four. The walking gait pattern is referenced by the dynamic simulation module as the desired joint trajectories. A closed-loop feedback controller based on a perturbation technique is used to suppress deviations from the desired trajectories. One result of this method is the set of joint torques needed to control the walking system. Numerical results are included using rear-to-front walking gaits. The results show that each pair of legs displays a unique ground reaction force pattern while the central body is moving at a constant speed.

In Chapter Six, the solution techniques, numerical results and the conclusions are summarized and future work is discussed.
CHAPTER TWO
OPTIMAL CONTROL OF
KINEMATICALLY REDUNDANT OPEN-CHAINED MANIPULATORS

2.1 INTRODUCTION

For legged walking machines, the central body's dynamics is fully dominated by the reactions between the central body and the individual legs. These reactions should be controlled so that the central body does not undergo large accelerations. Similarly, for robots used in the micro-gravity environment of space station laboratories, the base reactions must be monitored and controlled or the reactions must be eliminated in order not to disturb the micro-gravity environment. The motion of a leg of a walking machine can be viewed as a manipulator with a moving base, in which the central body is the base and the foot is the end-effector of the manipulator. Therefore, both cases can be treated as open-chained multi-body dynamic systems.

A manipulator is said to be kinematically redundant when the number of the total degrees of freedom of the manipulator is larger than the maximum number of independent degrees of freedom of a rigid body in Cartesian space. In such a situation, the solution of the inverse kinematics problem is not unique. This permits an optimization scheme to be conducted which minimizes a cost function, which is defined to be a weighted sum of the base reactions.
Sections 2.2 and 2.3 describe the dynamics and the kinematics of a kinematically redundant manipulator. Section 2.4 reviews a locally optimal trajectory management scheme. It is demonstrated that the cost function must be modified in order to smooth motions between discrete optimization points and also to bring the robot's joints to a halt smoothly as the end-effector is brought to a stop. Modified cost functions are developed and their relative merits are demonstrated with numerical examples for robots operating in two and three dimensional space in Sections 2.5 and 2.6.

2.2 DYNAMICS

The dynamics of a generalized open-chained manipulator with rigid links and with multidegree-of-freedom revolute joints, as shown schematically in Figure 2.1, had been studied by Chang and Chen [Refs. 4,5]. From their works, the total kinetic energy, the total potential energy and the equations of motion of the manipulator are

\[
T = \sum_{j=1}^{\ell} \left( \sum_{k=1}^{j-1} \frac{1}{2} m_j \dot{R}_j^T \dot{R}_j + \frac{1}{2} m_j \sum_{k=1}^{l} L_k^T \dot{\omega}_k L_j + \frac{1}{2} \omega_j^T \omega_j \right)
\]

\[
+ m_j \sum_{k=1}^{l} \hat{R}_k^T \dot{\omega}_k L_j \dot{L}_k + \hat{R}_k^T \dot{\omega}_k S_j + m_j \sum_{t=k+1}^{j} L_k^T \dot{\omega}_k L_t \dot{L}_t
\]

\[
+ \sum_{k=1}^{l} L_k^T \dot{\omega}_k S_j
\]

\[(2.1)\]
\[
V = \sum_{j=1}^{\ell} \left( - m_j \dot{\mathbf{R}}^T \mathbf{R} - m_j \dot{\mathbf{g}}^T \sum_{k=1}^{j-1} C_{Nk} L_k - g^T C_{Nj} \mathbf{S}_j \right)
\]

(2.2)

\[
\sum_{j=1}^{\ell} \left( m_j \ddot{\mathbf{R}} + m_j \sum_{k=1}^{j-1} C_{Nk} \left( \ddot{\mathbf{R}}_k + \ddot{\mathbf{S}}_k \right) \mathbf{L}_k + C_{Nj} \left( \ddot{\mathbf{R}}_j + \ddot{\mathbf{S}}_j \right) \mathbf{S}_j - m_j \dot{\mathbf{g}} \right) = C_{N1} \mathbf{F}_1
\]

(2.3)

\[
I_{j-1} \ddot{\mathbf{R}}_j + \ddot{\mathbf{S}}_j + \ddot{\mathbf{S}}_{j-1} \mathbf{C}_{Nj} \mathbf{R} + \ddot{\mathbf{S}}_j \sum_{k=1}^{j-1} C_{Nj} \left( \ddot{\mathbf{R}}_k + \ddot{\mathbf{S}}_k \right) \mathbf{L}_k - \ddot{S}_j \mathbf{C}_{Nj} \dot{\mathbf{g}}
\]

\[
+ L_j \sum_{k=j+1}^{k-1} \left( m_k \mathbf{C}_{Nj} \mathbf{R} + m_k \sum_{t=1}^{k-1} C_{Nj} \left( \ddot{\mathbf{R}}_t + \ddot{\mathbf{S}}_t \right) \mathbf{L}_t + C_{Nj} \left( \ddot{\mathbf{R}}_j + \ddot{\mathbf{S}}_j \right) \mathbf{S}_k \right)
\]

\[
- m_k \mathbf{C}_{Nj} \dot{\mathbf{g}} \right) = M_{j-1} - C_{j,j-1} M_j, \quad j = 1, \ldots, \ell
\]

(2.4)

where \( \ell \) is the total number of links of the manipulator, as shown in Figure 2.1; \( \mathbf{R} \) is the position vector of the base with respect to the (inertial) N-frame; \( \mathbf{L}_j, m_j, S_j, I_j \) and \( \omega_j \) are the length vector, the mass, the first moment of inertia, the second moment of inertia and the absolute angular velocity of link \( j \) expressed with respect to the (body-fixed) j-frame; \( C_{Nj} \) is the configuration coordinate transformation matrix from the j-frame to the N-frame; \( \mathbf{g} \) is the gravitational acceleration vector expressed with respect to the N-frame; \( \mathbf{F}_j \) and \( \mathbf{M}_j \), expressed with respect to the j-frame, are the joint force vector and the joint moment vector of joint \( j \).
Eqs. (2.1) and (2.2) are the total kinetic energy and potential energy of the manipulator. Eq. (2.3) is the equation of motion associated with the translational degrees of freedom of the base. Eq. (2.4) is the equation of motion associated with the rotational degrees of freedom of joint $j$. The base reaction force can be obtained directly by solving Eq. (2.3) algebraically. The recursive implementation of Eq. (2.4) from $j=l$ to $j=1$ yields the base reaction moments. Consequently, in view of Eqs. (2.3) and (2.4), the base reactions are functions of the configuration coordinate transformation matrices, the absolute angular velocities and the absolute angular accelerations or

\[
\begin{align*}
F_B &= F_B(C, \omega, \dot{\omega}) \\
\mathbf{M}_B &= \mathbf{M}_B(C, \omega, \dot{\omega})
\end{align*}
\]  

(2.5)

(2.6)

where $F_B = F_1$ and $\mathbf{M}_B = \mathbf{M}_1$.

Moreover, if we let $\alpha_j$ denote the Euler angles specifying the orientation of the $(j+1)$-frame relative to the $j$-frame. Then, the absolute angular velocities and the absolute angular accelerations can then be expressed in terms of the time derivatives of the Euler angles in recursive form, or

\[
\begin{align*}
\omega_j &= C_{j, j-1} \omega_{j-1} + D_{j} \alpha_j, \quad j = 1, \ldots, l \\
\dot{\omega}_j &= \omega_j^T C_{j, j-1} \omega_{j-1} + C_{j, j-1} \dot{\omega}_{j-1} + D_{j} \alpha_j + D_{j} \ddot{\alpha}_j, \quad j = 1, \ldots, l
\end{align*}
\]  

(2.7)

(2.8)

where $D_j$ is a transformation matrix between the relative angular
velocity of link \( j \) and the time derivative of the Euler angles of joint \( j \). Substituting Eqs. (2.7) and (2.8) into the equations of motion, the base reactions can be expressed as functions of \( C, D, \dot{\alpha} \) and \( \ddot{\alpha} \). Moreover, the transformation matrices \( C \) and \( D \) are all trigonometric functions of Euler angles. Therefore, the base reactions can be expressed in terms of the Euler angles and their time derivatives or

\[
F_B = F_B(\alpha, \dot{\alpha}, \ddot{\alpha})
\]  
(2.9)

\[
M_B = M_B(\alpha, \dot{\alpha}, \ddot{\alpha})
\]  
(2.10)

2.3 KINEMATICS

Let \( \mathbf{P} \) denote the configuration of the end-effector in Cartesian space, and \( \mathbf{\alpha} \) denote the joint-space configuration of the manipulator. In three dimensional space, \( \mathbf{P} \) is a set of six variables describing the position and the orientation of the end-effector. Also, \( \mathbf{\alpha} \) is a set of \( n \) variables if the manipulator has \( n \) degrees of freedom totally. The relationship between \( \mathbf{P} \) and \( \mathbf{\alpha} \) depends on the kinematics of the manipulator. In general, the path variable \( \mathbf{P} \) is a nonlinear function of the joint variables \( \mathbf{\alpha} \) or

\[
\mathbf{P} = \mathbf{P}(\mathbf{\alpha})
\]  
(2.11)

Differentiating Eq. (2.11) with respect to time yields the velocity form and the acceleration form of the forward kinematics, which can be expressed as
\[
\dot{p} = J_\dot{q} \\
\dot{q} = J_\dot{\theta} + J_\dot{\tilde{\theta}}
\]

where \( J \) is called the Jacobian matrix. For a generalized open-chained manipulator, the form of the Jacobian matrix had been derived in [Ref. 5].

For a manipulator of \( n \) degrees of freedom working in three dimensional space with both the position and the orientation of the end-effector taken into consideration, the Jacobian matrix is of dimension 6 by \( n \). If \( n=6 \), the Jacobian matrix is square and, hence, the inverse kinematics problem is straightforward, assuming the Jacobian matrix to be nonsingular. However, if \( n>6 \), then the manipulator has redundant degrees of freedom and the inverse kinematics problem has an infinite number of solutions. In such a case, a particular solution may be chosen on the basis of some optimization strategy.

In the Generalized Inverse Method [Ref. 9], the Jacobian matrix is partitioned into two matrices; one is a nonredundant square matrix and the other is referred to as a redundant matrix. The associated joint variables are also partitioned into nonredundant and redundant components. That is,

\[
J = \begin{bmatrix} J_n & J_r \end{bmatrix}
\]

\[
\alpha^T = \begin{bmatrix} \alpha_n^T & \alpha_r^T \end{bmatrix}
\]

(2.12)

(2.13)

(2.14)

(2.15)
where \( J_n, J_r, \alpha_n \) and \( \alpha_r \) are of dimension 6 by 6, 6 by \((n-6)\), 6 by 1 and \((n-6)\) by 1, respectively. Substituting Eqs. (2.14) and (2.15) into Eqs. (2.12) and (2.13) and assuming \( J_n \) is nonsingular, the so-called nonredundant joint variables can be expressed in terms of the redundant joint variables and the specified end-effector trajectory or

\[
\dot{\alpha}_n = J_n^{-1} \left( \dot{\beta} - J \dot{\alpha}_r \right) 
\]

\[
\ddot{\alpha}_n = J_n^{-1} \left( \ddot{\beta} - J \ddot{\alpha}_r - J \dot{\alpha}_n - J \dot{\alpha}_r \right) 
\]

(2.16)

(2.17)

The nonredundant part of \( \alpha \) can be obtained by solving Eq. (2.11) directly or, it can be approximated by using the variational form of Eq. (2.16),

\[
\dot{\alpha}_n = \dot{\alpha}_{n0} + J_n^{-1} \left( \delta \dot{\beta} - J \delta \dot{\alpha}_r \right) 
\]

\[
\ddot{\alpha}_n = \ddot{\alpha}_{n0} + J_n^{-1} \left( \delta \ddot{\beta} - J \delta \ddot{\alpha}_r \right) 
\]

(2.18)

where the subscript 0 denotes the initial values.

2.4 LOCALLY OPTIMAL TRAJECTORY MANAGEMENT

The joint trajectory of a kinematically redundant manipulator may be managed so that base reactions are minimized while the desired end-effector trajectory is accomplished. For this purpose, the base reactions are included in a quadratic cost function,

\[
G = F_W^T W_F F_B + M_W^T M_M M_B 
\]

(2.19)

where \( F_B \) and \( M_B \) are the base reaction force and the base reaction moment, respectively. \( W_F \) and \( W_M \) are 3 by 3 weighting matrices which are positive definite. The weighting matrices are used to weight
the relative importance of the components of the base reactions. According to the base reaction force and the base reaction moment given by Eqs. (2.9) and (2.10), the cost is a function of the joint variables and their time derivatives or

$$ G = G(\alpha, \dot{\alpha}, \ddot{\alpha}) $$ (2.20)

In the previous section, the nonredundant joint variables had been expressed in terms of the redundant joint variables and the end-effector trajectory. Hence, with substitution of Eqs. (2.16) through (2.18) into Eq. (2.20) and with the end-effector trajectory prescribed, the cost may be expressed as a function of the redundant joint variables and their time derivatives or

$$ G = G(\alpha_r, \dot{\alpha}_r, \ddot{\alpha}_r) $$ (2.21)

However, the nonredundant joint variables and their time derivatives, $\alpha_r$, $\dot{\alpha}_r$ and $\ddot{\alpha}_r$, are not independent variables. This makes the optimization scheme a difficult task. In order to conduct the optimization scheme over a set of independent variables, the Rayleigh-Ritz method [Ref. 33] is used to discretize the joint trajectories in time. The redundant joint variables and their time derivatives can be expressed as

$$ \alpha_r(t_{i-1} + \tau) = \alpha_r(t_{i-1}) + A \int_0^\tau \Phi(\mu) d\mu $$ (2.22)

$$ \dot{\alpha}_r(t_{i-1} + \tau) = A \dot{\phi}(\tau) $$ (2.23)

$$ \ddot{\alpha}_r(t_{i-1} + \tau) = A \ddot{\phi}(\tau) $$ (2.24)
where $0 \leq \Delta t = (t_i - t_{i-1})$. $\dot{A}$ is a $(6-n) \times m$ matrix of unknowns, except those determined by initial conditions. $\dot{\phi}(\tau)$ is a known set of $m$ shape functions that is used to represent the variation of $\dot{q}$ within a trajectory time step. The shape functions are chosen as the set of polynomials,

$$
\dot{\phi}(\tau) = \begin{bmatrix} 1 & \tau & \frac{\tau^2}{2!} & \cdots & \frac{\tau^{m-1}}{(m-1)!} \end{bmatrix}^T
$$

Substituting Eqs. (2.22) through (2.24) into Eq. (2.21), the cost may be expressed as a function of the coefficients of the matrix $A$, which is a set of independent unknowns. That is,

$$
G = G(A)
$$

At the beginning of each time step, a numerical optimization scheme is used to solve for the matrix $A$ which minimizes the cost function at the end of that time step. It should be noted that the first column of $A$ is defined as the vector of redundant joint velocities at time $t_{i-1}$, so that joint velocities remain continuous.

The redundant joint variables and their time derivatives at the next time step $t_i$ are computed by using Eqs. (2.22), (2.23) and (2.24). The nonredundant joint variables can be computed by solving Eq. (2.11) or can be approximated incrementally by using Eq. (2.18), and their time derivatives are computed by using Eqs. (2.16) and (2.17). This procedure is repeated in an open-loop fashion until the end of the manipulation time period. The end result is the history of joint motions which minimize base reactions at discrete
points while causing the end-effector to follow approximately the desired motion.

2.5 EXAMPLE ONE

Consider a planar manipulator performing a two-dimensional position task in a micro-gravity work space, as shown in Figure 2.2. The manipulator consists of three rigid links and three joints. Each joint has one degree of freedom and, hence, the total number of degrees of freedom for the planar manipulator is three. As two independent variables are enough to describe the position of the end-effector in two-dimensional space, there is one redundant degree of freedom allowing joint trajectory management to be performed for minimization of the base reactions. The properties and dimensions of each link are

\[ m_1 = m_2 = m_3 = 1 \text{ Kg}, \]

\[ L_1 = L_2 = L_3 = \begin{bmatrix} .5 & 0 \\ 0 & 0 \end{bmatrix} \text{ m}, \]

\[ S_1 = S_2 = S_3 = \begin{bmatrix} .25 & 0 \\ 0 & 0 \end{bmatrix} \text{ Kg} \cdot \text{m}, \]

\[ I_1 = I_2 = I_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & .08333 & 0 \\ 0 & 0 & .08333 \end{bmatrix} \text{ Kg} \cdot \text{m}^2. \]

The task of the manipulator is to move the end-effector from the initial position to the final position along a straight line and, also, all links must begin and end motionless. An important
step in limiting base reactions is permitting only smooth, finite jerk end-effector motion whenever possible. Curtate cycloidal motion has the desired properties of limited acceleration and jerk [Ref. 34] and, hence, is used to design the end-effector motion. In [Ref. 14], four combinations of cost function and Rayleigh-Ritz expansions were used and compared in the locally optimal trajectory management method. For the i-th time step with \( t \in [t_{i-1}, t_i] \), they are

case 1:

\[
\alpha_r(t_{i-1} + \tau) = \alpha_r(t_{i-1}) + \ddot{\alpha}_r(t_{i-1}) \tau + \frac{1}{2} A_1 \tau^2 + \frac{1}{3} A_2 \tau^3 \tag{2.27}
\]

\[
\dot{\alpha}_r(t_{i-1} + \tau) = \dot{\alpha}_r(t_{i-1}) + A_1 \tau + A_2 \tau^2 \tag{2.28}
\]

\[
\ddot{\alpha}_r(t_{i-1} + \tau) = A_1 + 2A_2 \tau \tag{2.29}
\]

\[
G = \begin{bmatrix} F^T W F + M^T W M \end{bmatrix}_{t=t_i} \tag{2.30}
\]

case 2:

\[
\alpha_r(t_{i-1} + \tau) = \alpha_r(t_{i-1}) + \ddot{\alpha}_r(t_{i-1}) \tau + \frac{1}{2} \dddot{\alpha}_r(t_{i-1}) \tau^2 + \frac{1}{3} A_1 \tau^3 \tag{2.31}
\]

\[
\dot{\alpha}_r(t_{i-1} + \tau) = \dot{\alpha}_r(t_{i-1}) + \ddot{\alpha}_r(t_{i-1}) \tau + A_1 \tau^2 + A_2 \tau^3 \tag{2.32}
\]

\[
\ddot{\alpha}_r(t_{i-1} + \tau) = \dddot{\alpha}_r(t_{i-1}) + 2A_1 \tau + 3A_2 \tau^2 \tag{2.33}
\]

\[
G = \begin{bmatrix} F^T W F + M^T W M \end{bmatrix}_{t=t_i} \tag{2.34}
\]
case 3:

\[
\alpha_r(t_{1-1} + \tau) = \alpha_r(t_{1-1}) + \alpha_r(t_{1-1})\tau + \frac{1}{2} A_1 \tau^2 \tag{2.35}
\]

\[
\dot{\alpha}_r(t_{1-1} + \tau) = \dot{\alpha}_r(t_{1-1}) + A_1 \tau \tag{2.36}
\]

\[
\ddot{\alpha}_r(t_{1-1} + \tau) = A_1 \tag{2.37}
\]

\[
G = \left[ F^T_{W_B \dot{F}_B} + M^T_{W_B M_B} \right]_{t=t_1} \tag{2.38}
\]

case 4:

\[
\alpha_r(t_{1-1} + \tau) = \alpha_r(t_{1-1}) + \alpha_r(t_{1-1})\tau + \frac{1}{2} A_1 \tau^2 + \frac{1}{3} A_2 \tau^3 \tag{2.39}
\]

\[
\dot{\alpha}_r(t_{1-1} + \tau) = \dot{\alpha}_r(t_{1-1}) + A_1 \tau + A_2 \tau^2 \tag{2.40}
\]

\[
\ddot{\alpha}_r(t_{1-1} + \tau) = A_1 + 2A_2 \tau \tag{2.41}
\]

\[
G = \left[ F^T_{W_B \dot{F}_B} + M^T_{W_B M_B} \right]_{t=t_1} + \left[ F^T_{W_B \dot{F}_B} + M^T_{W_B M_B} \right]_{t=t_1} \tag{2.42}
\]

Figure 2.3 shows the prescribed motion of the end-effector and Figures 2.4 through 2.15 show numerical results for the above four cases with the following conditions:

- Initial joint angles, \( \alpha_0 = \begin{bmatrix} 45^\circ & 45^\circ & 45^\circ \end{bmatrix}^T \).

- Final end-effector position, \( p_r = \begin{bmatrix} .3536 & .8536 & 0 \end{bmatrix}^T \) m.

- Weighting coefficients, \( W_F = W_M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \).

- Maximum linear acceleration of the end-effector = 1 m/sec^2.
· Manipulation time = 3 sec.

· Number of time segments, \( N = 30 \).

· Choose \( \alpha_1 \) as the redundant degree of freedom.

In case 1 and case 2, the base reactions were found to fluctuate significantly. Case 3 and case 4 were shown to provide better performance than case 1 or case 2 from the viewpoint of base reaction profile. The local optimization method using the strategies of cases 1, 2 and 3 ensure that the base reactions are minimized at the end of each discrete time interval. During the time interval, the base reactions are permitted to attain any value.

In case 1, the coefficients \( A_1 \) and \( A_2 \) are the initial joint acceleration and jerk, respectively, which are chosen by the optimization strategy. The joint displacements and velocities are continuous, but the accelerations and jerks are discontinuous at the beginning of each time step. Clearly, discontinuities in acceleration can cause large base reactions.

In case 2, joint accelerations are ensured to be continuous but jerk remains discontinuous. Also, the joint accelerations can attain large values during the interval, causing large base reactions.

In case 3, the joint accelerations are constant throughout the discrete time interval due to the simple one coefficient Rayleigh-Ritz expansion. This prevents the base reactions from fluctuating significantly.
In case 4, the cost function is redefined as a summation of the base reactions at the start and at the end of the discrete time interval. This smooths the base reactions throughout the interval.

However, in cases 3 and 4, the joint velocities do not go to zero at the end of the manipulation as desired. In case 3, this is due to the fact that the Rayleigh-Ritz expansion cannot cause \( \ddot{\alpha}_r = 0 \) and \( \dddot{\alpha}_r = 0 \) simultaneously. In case 4, because of the definition of the cost function which includes the base reactions both at the start and at the end of discrete time step, the optimization scheme does not require \( \ddot{\alpha}_r = \dddot{\alpha}_r = 0 \) at the end of the manipulation.

The problem encountered in case 3 is a phenomenon associated with kinematically redundant manipulators: Joint velocities can be nonzero while the end-effector velocity is zero. To correct this problem, at least two unknowns are required in the Rayleigh-Ritz expansion for the last time segment because two conditions must be satisfied at the last time step: \( \dot{\alpha}_r(t_f) = 0 \) and \( \ddot{\alpha}_r(t_f) = 0 \). Hence, a new version modified from case 3 is proposed as follows:

**case 5:**

For \( i = 1, \ldots, N-1 \)

\[
\alpha_r(t_{i-1} + \tau) = \alpha_r(t_{i-1}) + \dot{\alpha}_r(t_{i-1}) \tau + \frac{1}{2} A_1 \tau^2
\]  

(2.43)

\[
\dot{\alpha}_r(t_{i-1} + \tau) = \dot{\alpha}_r(t_{i-1}) + A_1 \tau
\]  

(2.44)

\[
\ddot{\alpha}_r(t_{i-1} + \tau) = A_1
\]  

(2.45)

\[
G = \begin{pmatrix}
F^T W F + M^T W M \\
\mathcal{B}_F F - \mathcal{B}_M M - \mathcal{B}_M B
\end{pmatrix}_{t=t_1}
\]  

(2.46)
For \( i = N \)

\[
\alpha_r(t_{i-1} + \tau) = \alpha_r(t_{i-1}) + \dot{\alpha}_r(t_{i-1}) \tau + \frac{1}{2} A_1 \tau^2 + \frac{1}{3} A_2 \tau^3
\]  
(2.47)

\[
\dot{\alpha}_r(t_{i-1} + \tau) = \dot{\alpha}_r(t_{i-1}) + A_1 \tau + A_2 \tau^2
\]  
(2.48)

\[
\ddot{\alpha}_r(t_{i-1} + \tau) = A_1 + 2A_2 \tau
\]  
(2.49)

which is same as case 3 except at the last time step in which the optimization is not performed. Instead, the unknown coefficients \( A_1 \) and \( A_2 \) are solved directly from the conditions that the joint velocities and joint accelerations must be zero at the end of the manipulation. Figures 2.16 through 2.18 show the numerical results for case 5. Unconditionally stopping all motion of the joints at the last time step may require large control moments and lead to large disturbances of the base reactions. Relatively large peaks in base reactions occur at the last time step.

To overcome the above stated problems, another modification is proposed. If the joint velocities are small at the next to last step, the control effort to have all joint motion stop at the last step is expected to be small. Hence, the cost function of case 5 is modified to include kinetic energy. The kinetic energy of the manipulator is a measurement of joint speed. A new strategy is then proposed as follows:
case 6:

For $i = 1, \ldots, N-1$

\[
\alpha_r(t_{i-1} + \tau) = \alpha_r(t_{i-1}) + \dot{\alpha}_r(t_{i-1}) \tau + \frac{1}{2} A_1 \tau^2
\]  
(2.50)

\[
\ddot{\alpha}_r(t_{i-1} + \tau) = \ddot{\alpha}_r(t_{i-1}) + A_1 \tau
\]  
(2.51)

\[
\dddot{\alpha}_r(t_{i-1} + \tau) = A_1
\]  
(2.52)

\[
G = \left(1 - Z\right) \left(F_B^T W F_B + M_B^T W M_B\right)_{t=t_1} + ZW_T
\]  
(2.53)

For $i = N$

\[
\alpha_r(t_{i-1} + \tau) = \alpha_r(t_{i-1}) + \dot{\alpha}_r(t_{i-1}) \tau + \frac{1}{2} A_1 \tau^2 + \frac{1}{3} A_2 \tau^3
\]  
(2.54)

\[
\ddot{\alpha}_r(t_{i-1} + \tau) = \ddot{\alpha}_r(t_{i-1}) + A_1 \tau + A_2 \tau^2
\]  
(2.55)

\[
\dddot{\alpha}_r(t_{i-1} + \tau) = A_1 + 2A_2 \tau
\]  
(2.56)

where $T$ is the kinetic energy of the manipulator, given by Eq. (2.1). $W_T$ is a weighting coefficient associated with the kinetic energy. $Z$ is a real number between 0 and 1, which is defined as

\[
Z = \frac{1}{2} \cos \left(\frac{2\pi i}{N}\right) + 1
\]  
(2.57)

The value of $Z$ becomes larger when the end-effector nears the desired final position. This increases the weight of the kinetic energy term in the cost function. Therefore, the speed of the joints is expected to slow due to this change in the cost function. At the last time step, as in case 5, all joints are unconditionally
forced to stop. Because the joint speed is already decreasing before the last time step due to the new cost function, the control effort to stop the joint motion at the last time step is expected to be small. Figures 2.19 through 2.21 show the results of case 6 with $W_T$ set to 10,000. As is expected, the relatively large peaks in the base reactions in the last time step in case 5 are reduced.

The modified locally optimal trajectory management of case 6 satisfies all of the requirements of the problem. In addition, it is noted that the base reaction minimization performance of case 6 is superior to the other cases. Following is a list of the CPU times required for each of the cases.

Case 1: 22 sec.
Case 2: 31 sec.
Case 3: 7 sec.
Case 4: 18 sec.
Case 5: 7 sec.
Case 6: 7 sec.

All computations were performed on an IBM VM/CMS mainframe system at NASA Lewis Research Center. The optimization was done by calling IMSL version 10 subroutine DUMINF, which minimizes a multivariate function using a Quasi-Newton method and a finite-difference gradient.
2.6 EXAMPLE TWO

The manipulator used in the three dimensional example consists of two rigid links. The base and links are connected by joints with two degrees of freedom so the manipulator has a total of four degrees of freedom, as shown in Figure 2.22. The properties and dimensions of each link are

\[ m_1 = m_2 = 1 \text{ Kg}, \]

\[ l_1 = l_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \text{ m}, \]

\[ s_1 = s_2 = \begin{bmatrix} 0 & 0 & .5 \end{bmatrix}^T \text{ Kg} \cdot \text{m}, \]

\[ I_1 = I_2 = \begin{bmatrix} .75 & 0 & 0 \\ 0 & .75 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ Kg} \cdot \text{m}^2. \]

The end-effector is to follow a straight-line path between prescribed initial and final positions. Its orientation is not to be controlled. Therefore, the manipulator has one redundant degree of freedom. The end-effector displacement, velocity and acceleration are given by curvate cycloidal motion, as in the previous two-dimensional example. Figure 2.23 shows the prescribed end-effector motion and Figures 2.24 through 2.35 show the numerical results of all six cases with the following conditions:

- Initial joint angles, \( \alpha_0 = \begin{bmatrix} 0^\circ & 45^\circ & 0^\circ & 90^\circ \end{bmatrix}^T. \)

- Final end-effector position, \( p_f = \begin{bmatrix} 0 & 1.5 & .5 \end{bmatrix} \text{ m}. \)
Weighting coefficients, $W_f = W_m = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $W_T = 1,000$.

- Maximum linear acceleration of the end-effector = $3\text{m/sec}^2$.
- Manipulation time = 3 sec.
- Number of time segments, $N = 20$.
- Choose $u_3$ as the redundant degree of freedom.

The CPU times used by each case are as follows:

- Case 1: 18 sec.
- Case 2: 22 sec.
- Case 3: 8 sec.
- Case 4: 17 sec.
- Case 5: 7 sec.
- Case 6: 7 sec.

The qualitative results for the three dimensional example are nearly identical to those for the planar example. Cases 1 and 2 display fluctuations in base reactions. In cases 3 and 4, the joint velocities do not go to zero at the end of the manipulation. In case 5, relatively large peaks in base reactions occur at the last time step due to the abrupt stopping of the joint motions. Case 6 is concluded to be superior to the other cases in both the base reaction minimization performance and the CPU time.
CHAPTER THREE
OPEN- AND CLOSED-CHAINED MULTIBODY SYSTEM
HEXAPOD MODEL I

3.1 INTRODUCTION

The mechanical model derived in this chapter for a hexapod walking machine is a generalized multi-rigid-body system consisting of a central body and six legs each with two segments and two multidegree-of-freedom revolute joints. The central body has six degrees of freedom, three for translation and three for rotation. Furthermore, each revolute joint can have one, two or three degrees of freedom, bringing a maximum total of forty two degrees of freedom for the investigated system.

When a leg is in contact with the ground, the foot translation is constrained to zero, giving three constraints. Hence, the maximum number of constraints is eighteen, which corresponds to the situation in which all six feet are in contact with the ground. Figure 3.1 illustrates one leg in contact with the ground. This leg in stance together with the central body can be viewed as a three-link manipulator where the ground is the base. The second leg segment (tibia) is link 1, the first leg segment (femur) is link 2, and the central body is link 3. The central body can be viewed as the end-effector. Joint 1 is treated as a ball joint, simulating the translational constraints associated with the foot in contact with the ground. Joint 2 and joint 3 represent the knee joint and
the hip joint, respectively. A minimum of six degrees of freedom is required for the manipulator in order to move the end-effector freely in three-dimensional working space, including both translational and rotational motion. Consequently, it turns out that joint 2 and joint 3 must have at least a total of three degrees of freedom. Hence, for the purpose of the central body having unconstrained rigid body motion in three-dimensional space, each leg must have at least three degrees of freedom.

In this chapter, the equations of motion of the hexapod model are first formulated using Lagrange's Equations in terms of absolute quasi-coordinates in Section 3.2 through Section 3.5. Then, the equations of motion are expressed in terms of relative quasi-coordinates and true coordinates by using coordinate transformations in Section 3.6 and Section 3.7, respectively.

3.2 KINETIC ENERGY

Consider one of the six legs, which is illustrated in Figure 3.1. Define the N-frame as an inertial coordinate system, the B-frame as a coordinate system fixed to the body with its origin located at the center of mass of the central body, the \( \ell_1 \)-frame as a coordinate system fixed to the femur with origin located at the hip joint, the \( \ell_2 \)-frame as a coordinate system fixed to the tibia with origin located at the knee joint. Let \( \mathbf{r}_B \) denote the position vector of the center of mass of the body with respect to the N-frame, let \( \mathbf{h}_E \) denote the length vector from the center of mass of the body to
the hip joint of leg \( l \) with respect to the B-frame, let \( \mathbf{l}_{l_1} \) denote
the length vector from the hip joint to the knee joint of leg \( l \) with
respect to the \( l_1 \)-frame, and let \( \mathbf{l}_{l_2} \) denote the length vector from
the knee joint to the foot of leg \( l \) with respect to the \( l_2 \)-frame.
\( \mathbf{p}_b \) denotes the length vector from the center of mass of the body to
a differential mass \( dm_b \) on the body with respect to the B-frame.
\( \mathbf{p}_{l_1} \) denotes the length vector from the hip joint to a differential
mass \( dm_{l_1} \) on the femur with respect to the \( l_1 \)-frame. \( \mathbf{p}_{l_2} \) denotes
the length vector from the knee joint to a differential mass \( dm_{l_2} \) on
the tibia with respect to the \( l_2 \)-frame. The position vectors,
measured with respect to the N-frame, of the differential masses on
each link can then be expressed as

\[
\mathbf{r}_b = \mathbf{r}_B + C_{NB} \mathbf{p}_B
\]

(3.1)

\[
\mathbf{r}_{l_1} = \mathbf{r}_B + C_{NB} \mathbf{p}_l + C_{Nl_1} \mathbf{p}_{l_1}, \quad \forall l \in \mathcal{L}
\]

(3.2)

\[
\mathbf{r}_{l_2} = \mathbf{r}_B + C_{NB} \mathbf{p}_l + C_{Nl_1} \mathbf{p}_{l_1} + C_{Nl_2} \mathbf{p}_{l_2}, \quad \forall l \in \mathcal{L}
\]

(3.3)

\( \mathcal{L} \) denotes the set of all six legs, \( \mathcal{L} = \{1, 2, 3, 4, 5, 6\} \). \( C_{NK} \)
is the
orientation coordinate transformation matrix from the K-frame to the
N-frame, where subscript \( K \) represents any of the body-fixed
coordinate systems, that is, \( K \in \{B, l_1, l_2\} \). The time derivative of
the orientation coordinate transformation matrix is given by

\[
\dot{C}_{NK} = C_{NK} \tilde{\omega}_K
\]

(3.4)

\( \tilde{\omega}_K \) is the skew-symmetric matrix associated with \( \omega_K \), which is the
absolute angular velocity of the K-frame expressed with respect to
the K-frame. If \( \omega_K^T = [\omega_x \omega_y \omega_z] \), the skew-symmetric matrix associated
with $\tilde{\omega}_x$ is defined as

$$
\tilde{\omega}_x = \begin{bmatrix}
0 & -\omega_z & \omega_y \\
\omega_z & 0 & -\omega_x \\
-\omega_y & \omega_x & 0
\end{bmatrix}
$$

(3.5)

Hence, by differentiating Eqs. (3.1) through (3.3) with respect to time and substituting Eq. (3.4) into the result, the absolute velocities, expressed with respect to the $N$-frame, of the differential masses on each link are given by

$$
\dot{r}_B = \dot{R}_B + C_{NB} ^{B} \tilde{\omega}_B \rho_B
$$

(3.6)

$$
\dot{r}_{\ell_1} = \dot{R}_B + C_{NB} ^{B} \tilde{\omega}_B \ell + C_{N\ell_1} ^{N} \tilde{\omega}_{\ell_1} \ell_{\ell_1}, \quad \forall \ell \in \ell
$$

(3.7)

$$
\dot{r}_{\ell_2} = \dot{R}_B + C_{NB} ^{B} \tilde{\omega}_B \ell + C_{N\ell_2} ^{N} \tilde{\omega}_{\ell_2} \ell_{\ell_2}, \quad \forall \ell \in \ell
$$

(3.8)

Here, it should be noted that $\rho_B$, $\rho_{\ell_1}$ and $\rho_{\ell_2}$ are constant vectors and, therefore, $\dot{\rho}_B = \dot{\rho}_{\ell_1} = \dot{\rho}_{\ell_2} = 0$. The kinetic energies of each link can be determined by integrating the kinetic energy of the differential mass over the mass domain. Hence, the kinetic energies of the body, the femurs and the tibias can be expressed as

$$
T_B = \frac{1}{2} \int \dot{r}_B ^{T} \dot{r}_B dm_B
$$

$$
= \frac{1}{2} m \dot{R}_B ^{T} \dot{R}_B + \dot{R}_B ^{T} C_{NB} ^{B} \tilde{\omega}_B S_B + \frac{1}{2} \dot{\omega}_{B} ^{T} \omega_{B}
$$

$$
= \frac{1}{2} m \dot{R}_B ^{T} \dot{R}_B + \frac{1}{2} \dot{\omega}_{B} ^{T} \omega_{B}
$$

(3.9)
\[ T_{l1} = \frac{1}{2} \int_{\dot{\ell}_{l1} \cdot \dot{\ell}_{l1}} \dot{\ell}_{l1} \cdot \dot{\ell}_{l1} \, dm_{l1} \]

\[ = \frac{1}{2} m_{l1} \dot{R}^T_R + \frac{1}{2} m_{l1} \dot{H}_l^T \dot{w}_B H_l + \frac{1}{2} \omega_{l1}^T I_{l1} \omega_{l1} \]

\[ + m_{l1} \dot{R}^T_{l1} \dot{w}_B H_l + \dot{R}^T_{l1} \dot{w}_l^T \, \dot{S}_{l1} + \dot{H}^T_{l1} \dot{w}_B \dot{S}_{l1}, \quad \forall \ell \in \mathcal{L} \]  

(3.10)

\[ T_{l2} = \frac{1}{2} \int_{\dot{\ell}_{l2} \cdot \dot{\ell}_{l2}} \dot{\ell}_{l2} \cdot \dot{\ell}_{l2} \, dm_{l2} \]

\[ = \frac{1}{2} m_{l2} \dot{R}^T_R + \frac{1}{2} m_{l2} \dot{H}_l^T \dot{w}_B H_l + \frac{1}{2} m_{l2} \dot{I}_{l1} \dot{w}_{l1} \dot{l}_{l1} \]

\[ + \frac{1}{2} \omega^T_{l2} I_{l2} \omega_{l2} + m_{l2} \dot{R}^T_{l2} \dot{w}_B H_l + m_{l2} \dot{R}^T_{l2} \dot{w}_l^T \, \dot{S}_{l2} \]

\[ + \dot{R}^T_{l2} \dot{w}_B \dot{S}_{l2} + m_{l2} \dot{H}^T_{l2} \dot{w}_B \dot{S}_{l2} + \dot{H}^T_{l2} \dot{w}_B \dot{S}_{l2}, \quad \forall \ell \in \mathcal{L} \]  

(3.11)

where

\[ I_{B} = \int_{\mathcal{B}} \rho_B \rho_B \, dm_B \]  

(3.12)

\[ I_{l1} = \int_{\mathcal{L}_{l1}} \rho_{l1} \rho_{l1} \, dm_{l1}, \quad \forall \ell \in \mathcal{L} \]  

(3.13)

\[ I_{l2} = \int_{\mathcal{L}_{l2}} \rho_{l2} \rho_{l2} \, dm_{l2}, \quad \forall \ell \in \mathcal{L} \]  

(3.14)

are the (second) moments of inertia of each link, which are always symmetric and positive definite. Also,

\[ S_{B} = \int \rho_B \, dm_B = 0 \]  

(3.15)
\[ S_{\ell_1} = \int \rho_{\ell_1} dm_{\ell_1} = m_{\ell_1} \bar{r}_{\ell_1}, \quad \forall \ell \in \ell \tag{3.16} \]

\[ S_{\ell_2} = \int \rho_{\ell_2} dm_{\ell_2} = m_{\ell_2} \bar{r}_{\ell_2}, \quad \forall \ell \in \ell \tag{3.17} \]

are the first moments of inertia of each link. \( \bar{r}_{\ell_1} \) is the position vector of the center of mass of the femur, measured with respect to the \( \ell_1 \)-frame. \( \bar{r}_{\ell_2} \) is the position vector of the center of mass of the tibia, measured with respect to the \( \ell_2 \)-frame.

The total kinetic energy of the hexapod dynamic system is determined by summing the kinetic energies of each link and the body or

\[ T = T_b + \sum_{\ell=1}^{6} \left( T_{\ell_1} + T_{\ell_2} \right) \tag{3.18} \]

### 3.3 POTENTIAL ENERGY

In the study of rigid body systems, the potential energy is due to gravity. Let \( g \) denote the gravitational acceleration expressed with respect to the \( \mathcal{N} \)-frame. For example, \( g^\mathcal{N} = [0 \ 0 \ -9.81] \text{m/sec}^2 \). Choose the origin of the \( \mathcal{N} \)-frame as the zero reference for gravitational potential energy. Hence, the potential energy of each link can be found by integrating the inner product of the position vector relative to the \( \mathcal{N} \)-frame and the gravitational acceleration vector. The potential energies of the body, the femurs and the tibias can be expressed as
\[
V_B = \int \left[ -g^T R_B d\mathbf{m}_B - m_B g^T R_B - g^T C_{NB} S_B \right] \\
= -m_B g^T R_B \\
V_{t_1} = \int \left[ -g^T R_{t_1} dt_1 \right] \\
= -m_{t_1} g^T R_B - m_{t_1} g^T C_{NB} t_1 - g^T C_{t_1} S_{t_1}, \quad \forall \ell \in \ell \\
V_{t_2} = \int \left[ -g^T R_{t_2} dt_2 \right] \\
= -m_{t_2} g^T R_B - m_{t_2} g^T C_{NB} t_2 - m_{t_2} g^T C_{t_2} t_2 - g^T C_{t_2} S_{t_2}, \quad \forall \ell \in \ell
\]

The minus sign inside the integrand of Eqs. (3.19) through (3.21) ensures that potential energy due to positions above the reference surface is positive while potential energy due to positions below the reference surface is negative. The reference surface is defined as the plane surface passing through the reference point, which is perpendicular to the gravitational acceleration vector.

The total potential energy of the hexapod dynamic system is the sum of the potential energies of each link and the body, or

\[
V = V_B + \sum_{\ell=1}^{6} \left( V_{t_1} + V_{t_2} \right)
\]
3.4 VIRTUAL WORK

Let $M_{\ell_1}$ and $F_{\ell_1}$, expressed with respect to the $\ell_1$-frame, denote the hip joint moment and the hip joint force, respectively. $M_{\ell_2}$ and $F_{\ell_2}$, expressed with respect to the $\ell_2$-frame, denote the knee joint moment and the knee joint force, respectively. $M_{\ell_1}$ and $F_{\ell_1}$ are applied by the central body and act on the femur. $M_{\ell_2}$ and $F_{\ell_2}$ are applied by the femur and act on the tibia. Also, $F_{\ell_3}$ denotes the constraint force which is applied by the ground and acts on the foot. $F_{\ell_3}$ is conveniently expressed with respect to the $N$-frame and is zero when leg $\ell$ is not in contact with the ground. Virtual work is defined as the summation of forces and moments multiplied by their associated virtual displacements. Therefore, the virtual work corresponding to leg $\ell$ is

$$\delta W_\ell = \left( -C_{\ell_1-\ell_1} M_{\ell_1} \right)^T \delta \beta_{\ell_1} + \left( M_{\ell_1} - C_{\ell_1\ell_2-\ell_2} \right)^T \delta \beta_{\ell_1} + M_{\ell_1}^T \delta \beta_{\ell_2} + F_{\ell_3}^T \delta R_{\ell_3}, \quad \forall \ell \in L \tag{3.23}$$

where $\delta \beta_{\ell_1}$ and $\delta \beta_{\ell_2}$ are referred to as the rotational virtual displacements expressed in terms of absolute quasi-coordinates associated with and expressed in the $B$-frame, $\ell_1$-frame and $\ell_2$-frame, respectively. Also, the absolute angular velocities of each frame are defined as

$$\omega_{\ell} = \frac{d\beta_{\ell}}{dt} \quad \tag{3.24}$$

$$\omega_{\ell_1} = \frac{d\beta_{\ell_1}}{dt}, \quad \forall \ell \in L \tag{3.25}$$
\( \omega_{t2} = \frac{d\beta_{t2}}{dt}, \forall \ell \in \mathcal{L} \) \hspace{2cm} (3.26)

\( \delta R_{t3} \) is the translational virtual displacement of the foot. In view of Figure 3.1, the position vector, expressed with respect to the N-frame, of the foot can be expressed as

\[ R_{t3} = R_B + C_{NB} R_{\ell} + C_{\ell_2} R_{\ell_2}, \forall \ell \in \mathcal{L} \] \hspace{2cm} (3.27)

The time derivative of Eq. (3.27) is the absolute velocity, expressed with respect to the N-frame, of the foot. That is,

\[ \dot{R}_{t3} = \dot{R}_B + C_{NB} \dot{R}_{\ell} + C_{\ell_2} \dot{R}_{\ell_2}, \forall \ell \in \mathcal{L} \] \hspace{2cm} (3.28)

The variational form of Eq. (3.28) is the translational virtual displacement of the foot or

\[ \delta R_{t3} = \delta R_B + C_{NB} \delta R_{\ell} + C_{\ell_2} \delta R_{\ell_2}, \forall \ell \in \mathcal{L} \] \hspace{2cm} (3.29)

Substituting Eq. (3.29) into Eq. (3.23) yields

\[ \delta W_\ell = F_{t2}^T \delta R_{t3} + \left[-C_{\ell_1} M_{t1} + \dot{R}_{t} C_{\ell H} F_{t3}\right]^T \delta \beta_{t2} \]

\[ + \left[M_{t1} - C_{\ell 2} M_{\ell_2} + \dot{R}_{t \ell_2} C_{\ell H} F_{\ell_2}\right]^T \delta \beta_{t1} \]

\[ + \left[M_{t2} - \dot{R}_{t \ell_2} C_{\ell H} F_{\ell_2}\right]^T \delta \beta_{t2}, \forall \ell \in \mathcal{L} \] \hspace{2cm} (3.30)

which is the virtual work of leg \( \ell \) in terms of the virtual displacements of absolute quasi-coordinates. The hip joint force and the knee joint force, \( F_{\ell_1} \) and \( F_{\ell_2} \), are not included in the virtual work because the revolute joints permit no relative translational motion. In other words, \( F_{\ell_1} \) and \( F_{\ell_2} \) are workless
forces. The total virtual work associated with the hexapod dynamic system is the summation of the virtual works of all six legs. That is,

\[ \delta W = \sum_{\ell=1}^{6} \delta W_{\ell} \]

\[ = \left( q_{R} + \sigma_{R} \right)^{T} \delta R_{B} + \left( q_{B} + \sigma_{B} \right)^{T} \delta B_{B} \]

\[ + \sum_{\ell=1}^{6} \left( \left( q_{t1} + \sigma_{t1} \right)^{T} \delta B_{t1} + \left( q_{t2} + \sigma_{t2} \right)^{T} \delta B_{t2} \right) \]  \hspace{1cm} (3.31)

where

\[ q_{R} = 0 \]  \hspace{1cm} (3.32)

\[ q_{B} = \sum_{\ell=1}^{6} -C_{Bt1}M_{t1} \]  \hspace{1cm} (3.33)

\[ q_{t1} = \frac{M_{t1}}{-C_{t1}t2}M_{t2}, \quad \forall \ell \in \ell \]  \hspace{1cm} (3.34)

\[ q_{t2} = \frac{M_{t2}}{-C_{t2}}, \quad \forall \ell \in \ell \]  \hspace{1cm} (3.35)

are the generalized forces due to joint moments. Also,

\[ \sigma_{R} = \sum_{\ell=1}^{6} F_{t3} \]  \hspace{1cm} (3.36)

\[ \sigma_{B} = \sum_{\ell=1}^{6} \frac{R_{t} C_{t1} t2 - F_{t3}}{C_{t1} t1} \]  \hspace{1cm} (3.37)

\[ \sigma_{t1} = \frac{L_{t1} C_{t1} t2}{-t3}, \quad \forall \ell \in \ell \]  \hspace{1cm} (3.38)
\[ \sigma_{t_2} = \frac{\mathcal{L}_{t_2}}{\mathcal{L}_{t_2}} \tau_{t_2} F_{t_3}, \quad \forall t \in \mathcal{L} \] (3.39)

are the generalized forces due to the feet in contact with the ground.

3.5 FORMULATION OF EQUATIONS OF MOTION IN TERMS OF ABSOLUTE QUASI-COORDINATES

After having derived the kinetic energy, the potential energy and the virtual work of the hexapod dynamic system, its equations of motion can then be formulated by using Lagrange's Equations in matrix notation, which can be expressed in the form

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_B} \right) - \frac{\partial T}{\partial q_B} + \frac{\partial V}{\partial q_B} = \tau_R + \tau_R \] (3.40)

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\omega}_B} \right) + \dot{\omega}_B \frac{\partial T}{\partial \omega_B} - \frac{\partial T}{\partial \omega_{NB}} - \frac{\partial V}{\partial \omega_{NB}} = \tau_B + \tau_B \] (3.41)

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{t}_1} \right) + \dot{\omega}_{t_1} \frac{\partial T}{\partial \omega_{t_1}} - \frac{\partial T}{\partial \omega_{t_1}} + \frac{\partial V}{\partial \omega_{t_1}} = \tau_{t_1} + \tau_{t_1}' \] (3.42)

\[ \forall t \in \mathcal{L} \]

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{t}_2} \right) + \dot{\omega}_{t_2} \frac{\partial T}{\partial \omega_{t_2}} - \frac{\partial T}{\partial \omega_{t_2}} + \frac{\partial V}{\partial \omega_{t_2}} = \tau_{t_2} + \tau_{t_2}' \] (3.43)

\[ \forall t \in \mathcal{L} \]

Equations (3.41) through (3.43) are known as Lagrange's equations for absolute quasi-coordinates, whereas Eq. (3.40) is for true coordinates. Substituting Eqs. (3.18) and (3.22) into Eqs. (3.40) through (3.43), in conjunction with following matrix calculus and identities,
\[ \frac{\partial}{\partial \alpha} (X^T Y) = Y \]  
(3.44)

\[ \frac{\partial}{\partial \alpha} (X^T C Y) = Y C X \]  
(3.45)

\[ \ddot{X} Y^T + \ddot{Y} X + \dddot{X} Y = 0 \]  
(3.46)

and with the definitions of the global absolute velocity vector and the global generalized force vectors

\[ \ddot{\omega}^T = \begin{bmatrix} \omega^T_B & \omega^T_\omega & \omega^T_\omega & \omega^T_\omega & \omega^T_\omega & \omega^T_\omega \end{bmatrix} \]  
(3.47)

\[ \ddot{q}^T = \begin{bmatrix} q^T_R & q^T_B & q^T_B & q^T_B & q^T_B & q^T_B \end{bmatrix} \]  
(3.48)

\[ \ddot{c}^T = \begin{bmatrix} c^T_R & c^T_B & c^T_B & c^T_B & c^T_B & c^T_B \end{bmatrix} \]  
(3.49)

Lagrange's equations can be expressed in the form

\[ \dddot{\dot{\omega}} + \dddot{\ddot{c}}^C + \dddot{\ddot{c}}^G = \dddot{q} + \dddot{c} \]  
(3.50)

where \( \dddot{\dot{\omega}} \) is referred to as the inertia matrix for absolute quasi-coordinates, which is symmetric and is of dimension 42 by 42. \( \dddot{\ddot{c}}^C \) is a vector of centrifugal and Coriolis terms. \( \dddot{\ddot{c}}^G \) is a vector of gravity terms. Both \( \dddot{\ddot{c}}^C \) and \( \dddot{\ddot{c}}^G \) are vectors of dimension 42 by 1.

The detailed descriptions of \( \dddot{\dot{\omega}}, \dddot{\ddot{c}}^C \) and \( \dddot{\ddot{c}}^G \) are listed as follows:
\[
\mathbf{\tilde{H}} = \begin{bmatrix}
\bar{H}_{00} & \bar{H}_{01} & \bar{H}_{02} & \bar{H}_{03} & \bar{H}_{04} & \bar{H}_{05} & \bar{H}_{06} \\
\bar{H}_{10} & \bar{H}_{11} & 0 & 0 & 0 & 0 & 0 \\
\bar{H}_{20} & 0 & \bar{H}_{22} & 0 & 0 & 0 & 0 \\
\bar{H}_{30} & 0 & 0 & \bar{H}_{33} & 0 & 0 & 0 \\
\bar{H}_{40} & 0 & 0 & 0 & \bar{H}_{44} & 0 & 0 \\
\bar{H}_{50} & 0 & 0 & 0 & 0 & \bar{H}_{55} & 0 \\
\bar{H}_{60} & 0 & 0 & 0 & 0 & 0 & \bar{H}_{66}
\end{bmatrix}
\]

(3.51)

\[
\bar{H}_{\ell\ell} = \frac{(\bar{m}_\ell + \Sigma m_{l_2} + \Sigma m_{l_1})I_{3\times3} + \left[\Sigma m_{l_1} + \Sigma m_{l_2}\right] C_{W_B} H^T_{\ell l}}{\left[\Sigma m_{l_1} + \Sigma m_{l_2}\right] C_{W_B} H_{\ell l} + \left[\Sigma m_{l_1} + \Sigma m_{l_2}\right] H_{\ell l}^T H_{\ell l}}
\]

(3.52)

\[
\bar{H}_{0l} = \bar{H}_{l0}^T = \frac{\bar{C}_{Nl_2, l_1} \bar{S}_{l_1} + m_{l_2} C_{Nl_1, l_1} \bar{C}_{l_2}^T}{\bar{C}_{Nl_2, l_1} \bar{S}_{l_1} + m_{l_2} \bar{C}_{Nl_1, l_1} \bar{C}_{l_2}^T}, \quad \forall l \in B
\]

(3.53)

\[
\bar{H}_{l\ell} = \frac{I_{l_2} + m_{l_2} \bar{C}_{l_2}^T \bar{C}_{l_1} \bar{C}_{l_1} \bar{C}_{l_2} \bar{C}_{l_2}}{\bar{S}_{l_2} C_{l_2 l_1} \bar{C}_{l_1}^T + I_{l_2}}, \quad \forall l \in B
\]

(3.54)

\[
\bar{S}_{l_2} = \begin{bmatrix}
\bar{S}_{l_2}^T & \bar{S}_{l_2 l_1} & \bar{S}_{l_2 l_1}^T \\
\bar{S}_{l_2} & \bar{C}_{l_2 l_1} \bar{C}_{l_1} & \bar{C}_{l_2 l_1} \bar{C}_{l_1}^T \\
\bar{S}_{l_2 l_1} & \bar{C}_{l_2 l_1} C_{l_1} & \bar{C}_{l_2 l_1} C_{l_1}^T \\
\bar{S}_{l_2 l_1}^T & \bar{C}_{l_2 l_1} C_{l_1}^T & \bar{C}_{l_2 l_1} C_{l_1}
\end{bmatrix}
\]

(3.55)
\[ \bar{c}_{CL} = \left[ \begin{array}{c} \Sigma m_{l1} C_{\text{NB}} \tilde{\omega}_{l1} \tilde{\omega}_{l1} S_{l1} + \Sigma m_{l2} C_{\text{NB}} \tilde{\omega}_{l2} S_{l2} \\
\Sigma m_{l1} C_{\text{NB}} \tilde{\omega}_{l1} \tilde{\omega}_{l1} + \Sigma m_{l2} C_{\text{NB}} \tilde{\omega}_{l2} S_{l2} + \Sigma m_{l1} C_{\text{NB}} \tilde{\omega}_{l1} \tilde{\omega}_{l1} S_{l1} + \Sigma m_{l2} C_{\text{NB}} \tilde{\omega}_{l2} S_{l2} \\
\Sigma m_{l1} C_{\text{NB}} \tilde{\omega}_{l1} \tilde{\omega}_{l1} + \Sigma m_{l2} C_{\text{NB}} \tilde{\omega}_{l2} S_{l2} + \Sigma m_{l1} C_{\text{NB}} \tilde{\omega}_{l1} \tilde{\omega}_{l1} S_{l1} + \Sigma m_{l2} C_{\text{NB}} \tilde{\omega}_{l2} S_{l2} \\
\Sigma m_{l1} C_{\text{NB}} \tilde{\omega}_{l1} \tilde{\omega}_{l1} S_{l1} + \Sigma m_{l2} C_{\text{NB}} \tilde{\omega}_{l2} S_{l2} + \Sigma m_{l1} C_{\text{NB}} \tilde{\omega}_{l1} \tilde{\omega}_{l1} S_{l1} + \Sigma m_{l2} C_{\text{NB}} \tilde{\omega}_{l2} S_{l2} \\
\Sigma m_{l1} C_{\text{NB}} \tilde{\omega}_{l1} \tilde{\omega}_{l1} S_{l1} + \Sigma m_{l2} C_{\text{NB}} \tilde{\omega}_{l2} S_{l2} + \Sigma m_{l1} C_{\text{NB}} \tilde{\omega}_{l1} \tilde{\omega}_{l1} S_{l1} + \Sigma m_{l2} C_{\text{NB}} \tilde{\omega}_{l2} S_{l2} \\
\end{array} \right] \]  

\[ (3.56) \]

\[ \bar{c}_{CL} = \left[ \begin{array}{c} \tilde{S}_{l1} C_{\text{NB}} \tilde{\omega}_{l1} \tilde{\omega}_{l1} S_{l1} + \\
\tilde{S}_{l1} C_{\text{NB}} \tilde{\omega}_{l1} \tilde{\omega}_{l1} + \\
\tilde{S}_{l1} C_{\text{NB}} \tilde{\omega}_{l1} \tilde{\omega}_{l1} + \\
\tilde{S}_{l1} C_{\text{NB}} \tilde{\omega}_{l1} \tilde{\omega}_{l1} + \\
\tilde{S}_{l1} C_{\text{NB}} \tilde{\omega}_{l1} \tilde{\omega}_{l1} + \\
\end{array} \right] \]  

\[ (3.57) \]

\[ \bar{G}_{G} = \left[ \begin{array}{c|c|c|c|c|c} \bar{G}_{0} & \bar{G}_{1} & \bar{G}_{2} & \bar{G}_{3} & \bar{G}_{4} & \bar{G}_{5} \end{array} \right] \]  

\[ (3.58) \]

\[ \bar{G}_{0} = \left[ \begin{array}{c} -m_{l2} - \Sigma m_{l1} - \Sigma m_{l2} \\
- \Sigma m_{l2} C_{\text{NB}} - \Sigma m_{l1} C_{\text{NB}} \end{array} \right] \]  

\[ (3.59) \]

\[ \bar{G}_{CL} = \left[ \begin{array}{c} -m_{l2} - \Sigma m_{l1} - \Sigma m_{l2} \\
- \Sigma m_{l2} C_{\text{NB}} - \Sigma m_{l1} C_{\text{NB}} \end{array} \right] \]  

\[ (3.60) \]

where the symbol \( \Sigma \) denotes the summation from \( l=1 \) to \( l=6 \) and \( 3 \times 3 \) is
an identity matrix of dimension 3 by 3.

In the case of a leg in contact with the ground, the position vector of the foot cannot change. Hence, setting Eq. (3.27) equal to a constant, the position vector of leg \( \ell \) in contact with the ground can be expressed as

\[
-\mathbf{R}_B + C_{NB} \mathbf{H} + C_{NL1} \mathbf{L}_1 + C_{NL2} \mathbf{L}_2 = \text{constant}, \quad \forall \ell \in D
\]  

(3.61)

where \( D \) denotes the set of legs in contact with the ground. \( D \) is a subset of \( \mathbb{Z} \), \( D \subseteq \{1,2,3,4,5,6\} \). Differentiating Eq. (3.61) with respect to time twice yields

\[
\frac{\dot{A}_\ell \ddot{\omega}}{2} + \frac{B_\ell}{2} = 0, \quad \forall \ell \in D
\]  

(3.62)

where

\[
A_\ell = \begin{bmatrix} 3 \times 3 & C_{NB} \mathbf{H}^T & 0_{3 \times 6(\ell-1)} & C_{NL1} \mathbf{L}_1^T & C_{NL2} \mathbf{L}_2^T & 0_{3 \times 6(6-\ell)} \end{bmatrix}
\]  

(3.63)

\[
B_\ell = C_{NB} \ddot{\omega} \mathbf{H} + C_{NL1} \ddot{\omega} \mathbf{L}_1 + C_{NL2} \ddot{\omega} \mathbf{L}_2
\]  

(3.64)

in which \( 0_{3 \times 6(\ell-1)} \) and \( 0_{3 \times 6(6-\ell)} \) are null matrices of dimension 3 by \( 6(\ell-1) \) and 3 by \( 6(6-\ell) \), respectively. Therefore, Eqs. (3.50) and (3.62) represent the equations of motion and the constraint equations of the dynamic system in terms of absolute quasi-coordinates. They can also be expressed as follows:

\[
\frac{\ddot{\Pi}}{2} = \frac{\ddot{\omega}}{2} + f(\omega, C, \sigma)
\]  

(3.65)

\[
\frac{\dot{A}_\ell \ddot{\omega}}{2} = -B_\ell(\omega, C), \quad \forall \ell \in D
\]  

(3.66)

where \( f(\omega, C, \sigma) \) is a vector function of absolute angular velocities,
orientation coordinate transformation matrices and constraint forces.

3.6 FORMULATION OF EQUATIONS OF MOTION IN TERMS OF RELATIVE QUASI-COORDINATES

The equations of motion and the constraint equations can also be expressed in terms of relative quasi-coordinates. Let \( \Omega_B \) denote the angular velocity of the B-frame relative to the N-frame, expressed with respect to the B-frame. \( \omega_{t1} \) denotes the angular velocity of the \( l1 \)-frame relative to the B-frame, expressed with respect to the \( l1 \)-frame. \( \omega_{t2} \) denotes the angular velocity of the \( l2 \)-frame relative to the \( l1 \)-frame, expressed with respect to the \( l2 \)-frame. Note that relative angular velocity is defined as the time derivative of relative quasi-coordinate. The relationship between the absolute angular velocities and the relative angular velocities can be expressed as follows:

\[
\begin{align*}
\omega_B &= \Omega_B \\
\omega_{t1} &= C_{l1B} \Omega_B + \omega_{t1}, \quad \forall \ell \in \mathbb{L} \\
\omega_{t2} &= C_{l2B} \Omega_B + C_{l2l1} \omega_{t1} + \omega_{t2}, \quad \forall \ell \in \mathbb{L}
\end{align*}
\]

Defining the global relative velocity vector as

\[
\tilde{\Omega}^T = \begin{bmatrix}
\tilde{R}^T_{-B} & \Omega^T_{-11} & \Omega^T_{-12} \\
\tilde{R}^T_{-B} & \Omega^T_{-21} & \Omega^T_{-22}
\end{bmatrix}
\]

Eqs. (3.67) through (3.69) can also be expressed as

\[
\ddot{\omega} = \ddot{\Omega}
\]
where

\[
\bar{C} = \begin{bmatrix}
1_{6 \times 6} & 0 & 0 & 0 & 0 & 0 \\
\bar{C}_{10} & \bar{C}_{11} & 0 & 0 & 0 & 0 \\
\bar{C}_{20} & 0 & \bar{C}_{22} & 0 & 0 & 0 \\
\bar{C}_{30} & 0 & 0 & \bar{C}_{33} & 0 & 0 \\
\bar{C}_{40} & 0 & 0 & 0 & \bar{C}_{44} & 0 \\
\bar{C}_{50} & 0 & 0 & 0 & 0 & \bar{C}_{55} \\
\bar{C}_{60} & 0 & 0 & 0 & 0 & 0 & \bar{C}_{66}
\end{bmatrix}
\]  

(3.72)

\[
\bar{C}_L^l_0 = \begin{bmatrix}
0_{3 \times 3} & C_{L1B} \\
0_{3 \times 3} & C_{L2B}
\end{bmatrix}, \quad \forall l \in L
\]  

(3.73)

\[
\bar{C}_L^l = \begin{bmatrix}
1_{3 \times 3} & 0_{3 \times 3} \\
C_{L2L1} & 1_{3 \times 3}
\end{bmatrix}, \quad \forall l \in L
\]  

(3.74)

in which \( \bar{C} \) is the global orientation coordinate transformation matrix. Differentiating Eq. (3.71) with respect to time yields the expression of the global absolute acceleration vector or

\[
\ddot{\omega} = \dddot{\bar{C}} + \ddot{\bar{C}} \dot{\bar{C}}
\]  

(3.75)

where

\[
\ddot{\bar{C}} = \ddot{\bar{\omega} C} + \bar{C} \dddot{\bar{\omega}}
\]  

(3.76)

in which \( \bar{\omega} \) is a block-diagonal matrix with diagonal blocks consisting of the skew-symmetric matrices of absolute angular
velocities or,

\[
\tilde{\omega} = \text{Diag} \begin{bmatrix} 0_{3\times3} | \tilde{\omega}_b | \tilde{\omega}_{11} | \tilde{\omega}_{12} | \cdots | \tilde{\omega}_{61} | \tilde{\omega}_{62} \end{bmatrix}
\]  

(3.77)

Inserting Eqs. (3.71) and (3.75) into Eqs. (3.65) and (3.66) and premultiplying the result by \( \tilde{\mathbf{C}}^T \) yields

\[
\left( \tilde{\mathbf{C}}^T \tilde{\mathbf{M}} \tilde{\mathbf{C}} \right) \dot{\tilde{\omega}} = \tilde{\mathbf{Q}} + \mathbf{f}^*(\Omega, \mathbf{C}, \xi)
\]  

(3.78)

\[
\tilde{\mathbf{A}}^*_\ell \tilde{\mathbf{Q}} = -\mathbf{B}^*_\ell (\Omega, \mathbf{C}), \quad \forall \ell \in \mathcal{D}
\]  

(3.79)

where

\[
\tilde{\mathbf{Q}}^T = \left( \tilde{\mathbf{C}}^T \tilde{\mathbf{Q}} \right)^T = \begin{bmatrix} 0^T_{3\times3} | 0^T_{3\times3} | \mathbf{M}^T_{11} | \mathbf{M}^T_{12} | \cdots | \mathbf{M}^T_{61} | \mathbf{M}^T_{62} \end{bmatrix}
\]  

(3.80)

\[
\mathbf{f}^* = \tilde{\mathbf{C}}^T \left( \mathbf{f} - \tilde{\mathbf{M}} \tilde{\mathbf{C}} \tilde{\mathbf{Q}} \right)
\]  

(3.81)

\[
\mathbf{B}^*_\ell = \mathbf{B}^*_\ell + \tilde{\mathbf{A}}^*_\ell \tilde{\mathbf{Q}}
\]  

(3.82)

Equations (3.78) and (3.79) are the equations of motion and the constraint equations in terms of relative quasi-coordinates. \( \tilde{\mathbf{C}}^T \tilde{\mathbf{M}} \tilde{\mathbf{C}} \) is referred to as the inertia matrix in terms of relative quasi-coordinates.

3.7 FORMULATION OF EQUATIONS OF MOTION IN TERMS OF TRUE COORDINATES

Let \( \alpha^*_B \) denote the Euler angles (true coordinates) specifying the orientation of the B-frame relative to the N-frame, let \( \alpha^*_\ell_1 \) denote the Euler angles specifying the orientation of the \( \ell_1 \)-frame relative to the B-frame, and let \( \alpha^*_\ell_2 \) denote the Euler angles
specifying the orientation of the \( \ell_2 \)-frame relative to the \( \ell_1 \)-frame. \( \mathbf{a}_b \) is a vector of dimension 3 by 1. \( \mathbf{a}_{\ell_1} \) and \( \mathbf{a}_{\ell_2} \) are vectors of dimension \( d_1 \) by 1 and \( d_2 \) by 1, if the hip joint and the knee joint have \( d_1 \) and \( d_2 \) degrees of freedom, respectively. The relative angular velocities and the time derivatives of the Euler angles are related by transformation matrices, called velocity transformation matrices. The velocity transformation matrices are a trigonometric function of the Euler angles and are defined as follows:

\[
\Omega_b = D_b^T \mathbf{\dot{a}}_b \tag{3.83}
\]

\[
\Omega_{\ell_1} = D_{\ell_1}^T \mathbf{\dot{a}}_{\ell_1}, \quad \forall \ell \in \mathcal{L} \tag{3.84}
\]

\[
\Omega_{\ell_2} = D_{\ell_2}^T \mathbf{\dot{a}}_{\ell_2}, \quad \forall \ell \in \mathcal{L} \tag{3.85}
\]

where \( D_b, D_{\ell_1}, \) and \( D_{\ell_2} \) are velocity transformation matrices of dimension 3 by 3, 3 by \( d_1 \) and 3 by \( d_2 \), respectively. Defining

\[
\mathbf{\alpha}^T = \begin{bmatrix}
R_b^T & \mathbf{a}_b^T & \mathbf{a}_{\ell_1}^T & \mathbf{a}_{\ell_2}^T & \mathbf{a}_{61}^T & \mathbf{a}_{62}^T
\end{bmatrix}
\]

Eqs. (3.83) through (3.85) can also be expressed as

\[
\mathbf{\dot{\alpha}} = \mathbf{D} \mathbf{\dot{\alpha}} \tag{3.87}
\]

where

\[
\mathbf{D} = \text{Diag} \left[ \begin{bmatrix} 1_{3 \times 3} & D_b & D_{\ell_1} & D_{\ell_2} & D_{61} & D_{62} \end{bmatrix} \right]
\]

is a block diagonal matrix of dimension 42 by \( 6+6(d_1+d_2) \). Differentiating Eq. (3.87) with respect to time yields
\[
\ddot{\zeta} = \ddot{\beta}_{\zeta} + \ddot{\beta}_{\dot{\zeta}}
\]  

(3.89)

which defines the relationship between the relative angular accelerations and the second time derivatives of the Euler angles. Inserting Eqs. (3.87) and (3.89) into Eqs. (3.78) and (3.79) and premultiplying the result by \( \dot{D}^T \), also keeping in mind that the velocity transformation matrices are also trigonometric functions of the Euler angles, the equations of motion and constraint equations can be expressed as

\[
\begin{pmatrix}
\dot{D}^T \eta^{MC\dot{D}}
\end{pmatrix} \ddot{\zeta} = \ddot{\tau} + \dddot{f}^*(\dot{\zeta}, \ddot{\zeta}, \tau)
\]  

(3.90)

\[
A_{\ell} \dddot{C_{\dot{D}}} \zeta = -B_{\ell}^{**}(\dot{\zeta}, \ddot{\zeta}), \quad \forall \ell \in D
\]  

(3.91)

where

\[
\dddot{\tau} = \begin{bmatrix} \dot{D}^T \eta \end{bmatrix}^T = \begin{bmatrix} O^T & 0^T & \tau^T & \tau^T & \tau^T & \tau^T & \tau^T & \tau^T & \tau^T & \tau^T \\ 0_{3x1} & 0_{3x1} & 0_{3x1} & 0_{3x1} & 0_{3x1} & 0_{3x1} & 0_{3x1} & 0_{3x1} & 0_{3x1} & 0_{3x1} \end{bmatrix}
\]  

(3.92)

\[
\dddot{f}^* = \dot{D}^T \dddot{f}^* - \dddot{C_{\dot{MC}\dot{D}}} \dddot{\zeta}
\]  

(3.93)

\[
B_{\ell}^{**} = B_{\ell}^{**} + A_{\ell} \dddot{C_{\dot{D}}} \dddot{\zeta}
\]  

(3.94)

\( \tau_{\ell_1} \) and \( \tau_{\ell_2} \) are vectors of (true-coordinate) joint moments, which are of dimension \( d_1 \) by 1 and \( d_2 \) by 1, respectively. \( \dot{D}^T \eta^{MC\dot{D}} \) is referred to as the inertia matrix for true coordinates.

In the case of simulation of a dynamic system, displacements \( \zeta \), velocities \( \dot{\zeta} \) and drive joint moments \( \tau \) are given, and accelerations \( \dddot{\zeta} \) are to be solved for. Assume \( n \) legs are in contact with the
ground. In view of Eq. (3.90), there are \(6+6(d_1+d_2)\) scalar equations and \(6+6(d_1+d_2)+3n\) scalar unknowns, \(6+6(d_1+d_2)\) unknown accelerations and \(3n\) unknown constraint forces. Hence, in addition to Eq. (3.90), \(3n\) other equations are needed to solve for unknown accelerations. These \(3n\) equations are provided by the constraint equations which are given by Eq. (3.91).

In the case of control of a dynamic system, displacements \(\ddot{\mathbf{x}}\), velocities \(\dot{\mathbf{x}}\) and accelerations \(\ddot{\mathbf{x}}\) are given, and drive motor joint moments \(\tau\) are to be solved for. In this case, Eqs. (3.91) are trivially satisfied as they are geometric constraints. This turns out that only \(6+6(d_1+d_2)\) equations, provided by Eq. (3.90), are available. Also, there are \(6+6(d_1+d_2)+3n\) unknowns, \(6+6(d_1+d_2)\) drive joint moments and \(3n\) constraint forces. Therefore, in the control problem there are more unknowns than equations. Optimization may be used for the purpose of solving the control problem. One possible solution is specified by the minimization of a quadratic cost function defined as follows:

\[
\text{COST} = \sum_{\ell=\mathcal{E}} \tau_{\ell_1}^T W_{\ell_1} \tau_{\ell_1} + \sum_{\ell=\mathcal{E}} \tau_{\ell_2}^T W_{\ell_2} \tau_{\ell_2} + \sum_{\ell=\mathcal{D}} \tau_{\ell_3}^T W_{\ell_3} \tau_{\ell_3} \tag{3.95}
\]

where \(W_{\ell_1}\), \(W_{\ell_2}\) and \(W_{\ell_3}\) are positive-definite weighting matrices. This criterion effectively takes the bending loads on the legs into consideration and yields low active joint moments and low bending loads.

In an experimental approach, force plates may be used to measure ground reaction forces. A high-performance force plate is a
sensitive electronic scale that can measure not only the vertical component but also the horizontal and lateral components of the reactions between the subject's foot and the force plate. A typical high-performance force plate is a light, rigid platform suspended on a suitable arrangement of force transducers. Moreover, with high-speed motion pictures taken while the subject moves over a force plate, joint motions can be recorded using a three-dimensional motion analysis system. These joint angles are recorded over time and, hence, permit joint velocities and joint accelerations to be determined using numerical methods. Therefore, experiments can provide data including joint angles, joint velocities, joint accelerations and ground reaction forces, which can then be input to the equations of motion to obtain joint control moments.
CHAPTER FOUR
OPEN- AND CLOSED-CHAINED MULTIBODY SYSTEM
HEXAPOD MODEL II

4.1 INTRODUCTION

In the simulation of a dynamic system, the computational burden is mainly due to the evaluation of the equations of motion. If the dynamic system is constrained, the constraint equations are also included. Therefore, the real time simulation is difficult to achieve.

The motivation of this chapter is to construct a model which reduces the computational burden of the simulation yet preserves the important aspects of the model. The foundation of this simplified model is the assumption that the inertia of each leg is much less than the inertia of the central body. This assumption permits each stance leg to be treated as if it is in static equilibrium. This assumption also leads to the conclusion that the reactions at the hip joint of a leg which is in its return stroke (swing) are much less than the reactions at the hip joint of a leg in stance. Therefore, the reactions at the hip joint of the swinging leg do not contribute greatly to the dynamics of the system. Neglecting the inertial efforts of the legs on the body permits the integrated dynamic system to be uncoupled between the central body and the legs. Hence, the complex dynamic system is uncoupled into a set of smaller systems while the total number of equations remain the same.
This greatly reduces the computational burden.

Section 4.2 will discuss how a stance leg's equilibrium is dealt with. Also, the equations of motion will be formulated separately for the central body, the swinging leg and the stance leg, as discussed in Sections 4.3 to 4.5. Finally, in Section 4.6 an algorithm is introduced which reduces the computational burden in the dynamic simulation. It should be noted that each leg of the hexapod investigated in this chapter is assumed to have three degrees of freedom.

4.2 CALCULATION OF THE HIP JOINT FORCE AND MOMENT IN A STANCE LEG USING THE JACOBIAN

Consider one of the stance legs, which is viewed as a three-link manipulator as shown in Figure 4.1. Its base is the ground. The first link and the second link are the tibia and the femur of the leg, respectively. Also, the third link is the central body.

Because the foot is constrained to zero translational displacement but is still free to rotate, the first joint is treated as a ball joint which has three revolute degrees of freedom. The second joint is the knee joint which has \( n_2 \) degrees of freedom. The third joint is the hip joint which has \( n_3 \) degrees of freedom. Note that \( n_2 + n_3 = 3 \) because each leg is assumed to have three degrees of freedom.
Define the N-frame as an inertial coordinate system, the 1-frame as a coordinate system fixed to link 1 with its origin located at joint 1, the 2-frame as a coordinate system fixed to link 2 with its origin located at joint 2, the 3-frame as a coordinate system fixed to link 3 with its origin located at joint 3. Let \( \tau_3 \) denote the (true-coordinate) joint moment of joint 3, expressed with respect to the 3-frame, which is applied by link 2 and acting on link 3. \( \tau_2 \) denotes the (true-coordinate) joint moment of joint 2, expressed with respect to the 2-frame, which is applied by link 1 and acting on link 2. \( \tau_1 \), denotes the (true-coordinate) joint moment of joint 1, expressed with respect to the 1-frame, which is applied by the base and acting on link 1. \( \tau_1 \), \( \tau_2 \) and \( \tau_3 \) are vectors of dimension 3 by 1, \( n2 \) by 1 and \( n3 \) by 1, respectively. Note that \( \tau_1 \) does not exist in reality as the foot is only subjected to concentrated force applied by the ground. That is, \( \tau_1 = 0 \).

Because the inertia of a stance leg is neglected, the reaction force and moment between the body and the femur at the hip joint can be determined by means of the principle of virtual work. Work is a scalar and is independent of the coordinates chosen for its measure. Specifically, equating the work done in Cartesian space with the work done in joint space gives a method of formulating the hip joint reaction force and moment in terms of joint moments. The virtual work done in joint space is the summation of the work done by each joint moment or
\[ \delta W = \tau_1^T \delta \alpha_1 + \tau_2^T \delta \alpha_2 + \tau_3^T \delta \alpha_3 \]  

(4.1)

where \( \delta \alpha_i \) (i=1,2,3) is the virtual displacement of the set of Euler angles \( \alpha_i \) specifying the orientation of the i-frame relative to the (i-1)-frame.

Before formulating the virtual work in Cartesian space, consider the free body diagram of link 3 and define \( \vec{F} \) and \( \vec{M} \) as the external force vector and the external moment vector acting on link 3 at joint 3, respectively. \( \vec{F} \) is expressed with respect to the N-frame and \( \vec{M} \) is expressed with respect to the 3-frame. Both \( \vec{F} \) and \( \vec{M} \) are vectors of dimension 3 by 1. In addition, let \( \vec{r} \) denote the absolute position vector of joint 3 measured in the N-frame and let \( \delta \vec{\beta} \) denote the rotational virtual displacement in absolute quasi-coordinates associated with link 3 and expressed with respect to the 3-frame. The virtual work done in Cartesian space can be expressed as

\[ \delta W = \vec{F}^T \delta \vec{r} + \vec{M}^T \delta \vec{\beta} \]  

(4.2)

Equating Eq. (4.1) with Eq. (4.2) and rearranging the result yields the following:

\[
\begin{bmatrix}
\vec{F}^T \\
\vec{M}^T
\end{bmatrix}
\begin{bmatrix}
\delta \vec{r} \\
\delta \vec{\beta}
\end{bmatrix}
= 
\begin{bmatrix}
\tau_1^T \\
\tau_2^T \\
\tau_3^T
\end{bmatrix}
\begin{bmatrix}
\delta \alpha_1 \\
\delta \alpha_2 \\
\delta \alpha_3
\end{bmatrix}
\]

(4.3)
The absolute position vector of joint 3 measured in the N-frame can be expressed as
\[ \mathbf{r} = \mathbf{R} + \mathbf{C}_{W1} \mathbf{L}_{1} + \mathbf{C}_{W2} \mathbf{L}_{2} \]  
(4.4)
where \( \mathbf{R} \) is the position vector of joint 1 (the location of the foot), which is measured in the N-frame and is a constant vector. \( \mathbf{L}_{1} \) and \( \mathbf{L}_{2} \) are the length vectors from joint 1 to joint 2 and from joint 2 to joint 3, respectively. \( \mathbf{L}_{1} \) is expressed with respect to the 1-frame and \( \mathbf{L}_{2} \) is expressed with respect to the 2-frame.

Differentiating Eq. (4.4) with respect to time and then expressing the result in variational form yields the translational virtual displacement of joint 3 in terms of the rotational virtual displacements in absolute quasi-coordinates or
\[ \delta \mathbf{r} = \mathbf{C}_{W1} \mathbf{L}_{1}^T \delta \mathbf{\beta}_{1} + \mathbf{C}_{W2} \mathbf{L}_{2}^T \delta \mathbf{\beta}_{2} \]  
(4.5)

The relationship between the rotational virtual displacements in absolute quasi-coordinates and in relative quasi-coordinates can be expressed as
\[ \delta \mathbf{\beta}_{1} = \delta \mathbf{\xi}_{1} \]  
(4.6)
\[ \delta \mathbf{\beta}_{2} = \mathbf{C}_{W2} \mathbf{\xi}_{1} + \delta \mathbf{\xi}_{2} \]  
(4.7)
\[ \delta \mathbf{\beta}_{3} = \mathbf{C}_{W3} \mathbf{\xi}_{1} + \mathbf{C}_{W2} \mathbf{\xi}_{2} + \delta \mathbf{\xi}_{3} \]  
(4.8)

Substituting Eqs. (4.6) and (4.7) into Eq. (4.5) yields the translational virtual displacement of joint 3 in terms of the rotational virtual displacements in relative quasi-coordinates or
\[ \delta \mathbf{r} = \left( \mathbf{C}_{W1} \mathbf{L}_{1}^T + \mathbf{C}_{W2} \mathbf{L}_{2}^T \mathbf{C}_{W1} \right) \delta \mathbf{\xi}_{1} + \mathbf{C}_{W2} \mathbf{L}_{2}^T \delta \mathbf{\xi}_{2} \]  
(4.9)
Moreover, the relative angular velocity can be expressed as a dot product of a velocity transformation matrix and the time derivative of the set of Euler angles:

\[
\Omega_i = \frac{d\xi}{dt} = D_i \dot{\alpha}_i, \quad i = 1, 2, 3
\]  

(4.10)

where \( D_i \) is a transformation matrix which is a trigonometric function of \( \alpha_i \). Taking the variational form of Eq. (4.10), the rotational virtual displacements in relative quasi-coordinates can be expressed in terms of the rotational virtual displacement in Euler angles (joint angles) as

\[
\delta \xi = D_i \delta \alpha_i, \quad i = 1, 2, 3
\]  

(4.11)

Combining Eqs. (4.8) and (4.9) into a single matrix equation and then substituting Eq. (4.11) into the result, the relationship between the virtual displacements in Cartesian coordinates and the virtual displacements in joint coordinates can be expressed as

\[
\begin{bmatrix}
\delta r \\
\delta \theta \\
\delta \beta
\end{bmatrix} = J
\begin{bmatrix}
\delta \alpha_1 \\
\delta \alpha_2 \\
\delta \alpha_3
\end{bmatrix}
\]  

(4.12)

where \( J \) is the Jacobian matrix and is given by
\[ J = \begin{bmatrix}
    C_{\text{N1}} & C_{\text{N2}} & C_{\text{C1}} & C_{\text{C2}} & C_{\text{T}} & 0 \\
    0 & 0 & D_1 & 0 & 0 & 0 \\
    0 & 0 & 0 & D_2 & 0 & 0 \\
    C_{31} & C_{32} & 0 & 0 & 0 & D_3 \\
\end{bmatrix} \]

\[ = \begin{bmatrix}
    C_{\text{N1}} & C_{\text{N2}} & C_{\text{C1}} & C_{\text{C2}} & C_{\text{T}} & 0 \\
    0 & 0 & D_1 & 0 & 0 & 0 \\
    0 & 0 & 0 & D_2 & 0 & 0 \\
    C_{31} D_1 & C_{32} D_2 & 0 & 0 & 0 & D_3 \\
\end{bmatrix} \]

\[(4.13)\]

where \( I \) is a 3 by 3 identity matrix. Substituting Eq. (4.12) into Eq. (4.3) and considering the definition that a virtual displacement is an imagined, instantaneous and nonzero quantity, an equation describing the relationship between the hip joint reactions and the joint moments is formed as

\[ \begin{bmatrix} F^T \\ M^T \end{bmatrix} J = \begin{bmatrix} T_1^T \\ T_2^T \\ T_3^T \end{bmatrix} \]

\[(4.14)\]

where both \([F^T; M^T] \) and \([T_1^T; T_2^T; T_3^T] \) are of dimension 1 by 6 and, hence, \( J \) is a 6 by 6 square matrix.

At this moment, it is interesting to study the physical meaning of Eq. (4.14). Define \( M_{\text{i1}} \) (i=1,2,3) as the (quasi-coordinate) joint moment at joint i which is applied by link (i-1) and acting on link i. \( M_{\text{i1}} \) is expressed with respect to the i-frame and is always a vector of dimension 3 by 1. According the virtual work that the work done in Cartesian space must be equal to the work done in joint space, it follows that

\[ M_{\text{i1}}^T S_2 = T_{\text{i1}} S_{\text{x1}} \]

\[(4.15)\]
Inserting Eq. (4.11) into Eq. (4.15), the relationship between the (quasi-coordinate) joint moment and the (true-coordinate) joint moment can be expressed as

$$M^T_D = \tau^T_1$$

(4.16)

Decomposing both sides of Eq. (4.14) into 3 column blocks and substituting Eq. (4.16) into the result, Eq. (4.14) can be expressed as 3 simultaneous equations as follows:

$$\begin{bmatrix} \tau^T_1 & \begin{bmatrix} 0 \\ D_3 \end{bmatrix} \end{bmatrix} = M^T_D$$

(4.17)

$$\begin{bmatrix} \tau^T_1 & \begin{bmatrix} C_{12} L^T_{12} \\ C_{32} D_2 \end{bmatrix} \end{bmatrix} = M^T_{22}$$

(4.18)

$$\begin{bmatrix} \tau^T_1 & \begin{bmatrix} C_{11} L^T_{11} + C_{12} L^T_{21} \\ C_{31} D_1 \end{bmatrix} \end{bmatrix} = M^T_{11}$$

(4.19)

From Eq. (4.17) it follows that

$$M = M_3$$

(4.20)

Eq. (4.20) is a trivial condition, as $M$ and $M_3$ are the same vector defined with different symbols. Expanding Eq. (4.18) and then substituting Eq. (4.20) into the result, it follows that

$$M_{22} - C_{23} M_3 - L_{22} C_{23} F = 0$$

(4.21)
which describes the static equilibrium condition of link 2 in terms of the total moment about joint 2. Similarly, expanding Eq. (4.19) and then substituting Eq. (4.21) into the result yields the equation which describes the static equilibrium condition of link 1 in terms of the total moment about joint 1 or

\[
M_1 - C_{12}M_2 - \bar{C}_1C_1F = 0
\]

(4.22)

Because each leg has three degrees of freedom, the Jacobian matrix is a 6 by 6 square matrix. In this case, transposing both sides of Eq. (4.14) and then premultiplying both sides of the result by \(J^T\), an expression for the hip joint reaction force and moment in terms of joint moments can be expressed as

\[
\begin{bmatrix}
F_1 \\
F_2 \\
M_1 \\
M_2 \\
M_3
\end{bmatrix}
= J^{-T}
\begin{bmatrix}
\tau_1 \\
\tau_2 \\
\tau_3
\end{bmatrix}
\]

(4.23)

Neglecting the inertia of a stance leg also implies that the reaction forces at each joint of the leg must be the same. This can be seen clearly from the viewpoint of Newton's law, \(F=ma\). The constraint force caused by the ground and the joint forces of the hip joint and the knee joint must be equivalent, which is given by the force \(F\) in the left hand side of Eq. (4.23). Hence, the constraint forces can be determined from the input of the joint moments and will not be unknowns in the simulation problem.
4.3 DYNAMICS OF THE CENTRAL BODY

Consider the free body diagram of the central body. The active forces and moments on the central body of the hexapod include the reaction forces and moments at the six hip joints as well as the gravitational force. Using the definitions from Chapter Three and illustrated in Figure 3.1, the absolute position of a differential mass on the central body with respect to the N-frame is

\[ \dot{r}_B = R^B + C_{MB} \dot{p}_B \]  

(4.24)

Differentiating Eq. (4.24) with respect to time yields the absolute velocity of the differential mass \( \dot{r}_B \). The kinetic energy of the central body can be expressed as

\[ T_B = \frac{1}{2} \int \dot{r}_B^T \ddot{r}_B \, dm_B = \frac{1}{2} m_B \dot{R}^B \dot{R}^B + \frac{1}{2} \omega^T \omega^B \]  

(4.25)

The potential energy of the central body can be expressed as

\[ V_B = \int -g^{T} \dot{r}_B \, dm_B = -m_B g^{T} \dot{r}_B^B \]  

(4.26)

The virtual work, done by the reaction forces and the reaction moments at the six hip joints, can be expressed in the form:

\[ \delta W = \sum_{\ell=1}^{6} \left( -T^T e_\ell \delta p_\ell - \omega^{T} e_\ell \delta \beta_\ell \right) \]  

(4.27)

where \( \delta \beta_\ell \) is the rotational virtual displacement in absolute quasi-coordinates of the central body. \( \delta p_\ell \) is the translational virtual displacement of the hip joint \( \ell \). \( \delta \beta_\ell \) and \( \delta p_\ell \) are expressed with respect to the B-frame and the N-frame, respectively. As
defined in Chapter Three, \( F_{\ell_1} \) and \( M_{\ell_1} \) are the force and the moment applied by the central body on the femur. Hence, minus signs are required inside the summation term of Eq. (4.27). The position of the hip joint \( \ell \), with respect to the N-frame, is

\[
P_{-\ell} = R_{-\ell} + C_{NB-\ell}
\]

(4.28)

Differentiating Eq. (4.28) with respect to time and expressing the result in variational form, the translational virtual displacement of the hip joint can be expressed as

\[
\delta P_{-\ell} = \delta R_{-\ell} + C_{NB-\ell} \delta \delta
\]

(4.29)

Substituting Eq. (4.29) into Eq. (4.27), the virtual work can be expressed as

\[
\delta W = -\sum_{\ell=1}^{6} \left( C_{NL1-F_{\ell_1}} \right)^{\top} \delta R_{-\ell} - \sum_{\ell=1}^{6} \left( \tilde{H}_{\ell} C_{NL1-F_{\ell_1}} + C_{NL1-M_{\ell_1}} \right) \delta \delta
\]

(4.30)

Lagrange's equations of motion for the central body can be expressed as

\[
\frac{d}{dt} \left( \frac{\partial T_B}{\partial \dot{R}} \right) = \frac{\partial T_B}{\partial R} + \frac{\partial V_B}{\partial R} = -\sum_{\ell=1}^{6} C_{NL1-F_{\ell_1}}
\]

(4.31)

\[
\frac{d}{dt} \left( \frac{\partial T_B}{\partial \omega} \right) + \tilde{\omega} \frac{\partial T_B}{\partial \dot{\omega}} - \frac{\partial T_B}{\partial C_{NB}} + \frac{\partial V_B}{\partial C_{NB}} = -\sum_{\ell=1}^{6} \left( \tilde{H}_{\ell} C_{NL1-F_{\ell_1}} + C_{NL1-M_{\ell_1}} \right)
\]

(4.32)

Eq. (4.31) governs the translational motion of the central body while Eq. (4.32) governs its rotational motion. Substituting Eqs.
(4.25) and (4.26) into Eqs. (4.31) and (4.32) and rearranging the result, the equations of motion can be expressed as

\[
m_{\text{B-B}} \ddot{R}_{\text{B}} = m_{\text{B}} g - \sum_{l=1}^{6} C_{\text{B1}} F_{\text{L1}}
\]

\[
I_{\text{B-B}} \ddot{\omega}_{\text{B}} = -\ddot{\omega}_{\text{B}} I_{\text{B-B}} \omega - \sum_{l=1}^{6} \tilde{R}_{\text{B1}} F_{\text{L1}} - \sum_{l=1}^{6} C_{\text{B1}} M_{\text{L1}}
\]

(4.33)

(4.34)

The angular velocity of the central body can be related to the time derivative of the set of Euler angles specifying the orientation of the central body relative to the \(N\)-frame by a velocity transformation matrix or

\[
\omega_{\text{B}} = D_{\text{B}} \dot{\alpha}_{\text{B}}
\]

(4.35)

where \(D_{\text{B}}\) is always a square matrix of dimension 3 by 3 because a rigid body has three rotational degrees of freedom in three-dimensional space. Differentiating Eq. (4.35) with respect to time yields

\[
\dot{\omega}_{\text{B}} = D_{\text{B}} \dot{\alpha}_{\text{B}} + D_{\text{B}} \ddot{\alpha}_{\text{B}}
\]

(4.36)

Inserting Eqs. (4.35) and (4.36) into Eq. (4.34) and premultiplying the result by \(D_{\text{B}}^T\), the rotational equation of motion can be expressed in terms of Euler angles and their time derivatives as follows:
\[
\begin{array}{c}
\left( D^T B^{-1} A B \right) \ddot{a}_B = - D^T B^{-1} \left( \ddot{R}_B A_B + \left( D B^{-1} A B \right) ( D B^{-1} A B ) \right) + \sum_{\ell=1}^{6} \tilde{h}_\ell C_{B\ell1} F_{\ell1} \\
+ \sum_{\ell=1}^{6} C_{B\ell1} M_{\ell1}
\end{array}
\] (4.37)

Equations (4.33) and (4.37) are the equations of motion of the central body in terms of true coordinates and their time derivatives. In the simulation problem, \( R_B, \dot{R}_B, \alpha_B \) and \( \ddot{\alpha}_B \) are given by initial conditions, and \( F_{\ell1} \) and \( M_{\ell1} \) are inputs. Because of the assumption made in Section 4.1, that the reactions at the hip joint of a leg in its return stroke can be assumed to be much less than the reactions at the hip joint of a stance leg, \( F_{\ell1} \) and \( M_{\ell1} \) are set to zero for a leg in its return stroke and given by Eq. (4.23) for a stance leg.

4.4 DYNAMICS OF A LEG DURING ITS RETURN STROKE

The dynamics of a swinging leg can be formulated by viewing it as a two-link manipulator with a moving base. Let \( P_\ell \) denote the position vector of the hip joint with respect to the N-frame, which can be expressed in the form:

\[
P_\ell = R_B + C_{NB} \ell
\] (4.38)

Substituting Eq. (4.38) into Eqs. (3.2) and (3.3), the position vectors of differential masses on each leg segment can be expressed as
\[ r_{l1} = P_{l} + C_{nt1} \omega_{l1} \]  
\[ r_{l2} = P_{l} + C_{nt1} L_{l1} + C_{nt2} \omega_{l2} \]  
(4.39)  
(4.40)

Differentiating these position vectors with respect to time yields the velocities of the differential masses:

\[ \dot{r}_{l1} = \dot{P}_{l} + C_{nt1} \ddot{\omega}_{l1} \]  
\[ \dot{r}_{l2} = \dot{P}_{l} + C_{nt1} L_{l1} \dot{\omega}_{l1} + C_{nt2} \ddot{\omega}_{l2} \]  
(4.41)  
(4.42)

Hence, the kinetic energies of the leg segments can be expressed as

\[ T_{l1} = \frac{1}{2} \int \dot{r}_{l1} \cdot d m_{l1} \]  
\[ = \frac{1}{2} \left( m_{l1} \dot{P}_{l} \cdot \dot{r}_{l1} \right) + \frac{1}{2} \left( \omega_{l1} \cdot L_{l1} \omega_{l1} \right) + \dot{P}_{l}^{T} C_{nt1} \ddot{\omega}_{l1} \omega_{l1}^{T} S_{l1} \]  
(4.43)

\[ T_{l2} = \frac{1}{2} \int \dot{r}_{l2} \cdot d m_{l2} \]  
\[ = \frac{1}{2} \left( m_{l2} \dot{P}_{l} \cdot \dot{r}_{l2} \right) + \frac{1}{2} \left( m_{l2} L_{l1}^{T} \dot{\omega}_{l1} \omega_{l1} L_{l1} \right) + \frac{1}{2} \left( \omega_{l2} \cdot L_{l2} \omega_{l2} \right) + \dot{P}_{l}^{T} C_{nt2} \ddot{\omega}_{l2} \omega_{l2}^{T} S_{l2} \]  
(4.44)

Also, the potential energies of the leg segments can be expressed as

\[ V_{l1} = \int -g \cdot \dot{r}_{l1} \cdot d m_{l1} \]  
\[ = -m_{l1} g \cdot \dot{r}_{l1} - \dot{r}_{l1}^{T} C_{nt1} S_{l1} \]  
(4.45)
\[ V_{t_2} = \int -g^{T} \Gamma_{t_2} \cdot d\Gamma_{t_2} \]

\[ = -m_{t_2} g^{T} \mathcal{P}_{t} - m_{t_2} g^{T} \mathcal{C}_{t_1} \cdot \Gamma_{t_1} - g^{T} \mathcal{C}_{t_2} \cdot S_{t_2} \]  \hspace{1cm} (4.46)

The virtual work associated with a swinging leg can be expressed in the form:

\[ \delta W_{t} = \left( M_{t_1} - C_{t_1} \mathcal{M}_{t_2} \right)^{T} \delta \theta_{t_1} + \mathcal{M}_{t_2} \delta \theta_{t_2} + \left( \mathcal{C}_{t_1} \mathcal{F}_{t_1} \right)^{T} \delta \mathcal{P}_{t} \]  \hspace{1cm} (4.47)

where \( \delta \theta_{t_1} \) and \( \delta \theta_{t_2} \) are the rotational virtual displacements in absolute quasi-coordinates of the femur and the tibia, respectively. \( \delta \mathcal{P}_{t} \) is the translational virtual displacement of the hip joint.

Having derived the virtual work expressions, the Lagrange's equations for a swinging leg can be expressed in the form:

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\omega}_{t_1}} \right) - \frac{\partial T}{\partial \omega_{t_1}} + \frac{\partial V}{\partial \omega_{t_1}} = \mathcal{C}_{t_1} \mathcal{F}_{t_1} \]  \hspace{1cm} (4.48)

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\omega}_{t_2}} \right) + \dot{\omega}_{t_1} \frac{\partial T}{\partial \omega_{t_1}} - \frac{\partial T}{\partial \mathcal{C}_{t_1}} + \frac{\partial V}{\partial \mathcal{C}_{t_1}} = \mathcal{M}_{t_1} - C_{t_1} \mathcal{M}_{t_2} \]  \hspace{1cm} (4.49)

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\omega}_{t_2}} \right) + \dot{\omega}_{t_2} \frac{\partial T}{\partial \omega_{t_2}} - \frac{\partial T}{\partial \mathcal{C}_{t_2}} + \frac{\partial V}{\partial \mathcal{C}_{t_2}} = \mathcal{M}_{t_2} \]  \hspace{1cm} (4.50)

where

\[ T = T_{t_1} + T_{t_2} \]  \hspace{1cm} (4.51)

\[ V = V_{t_1} + V_{t_2} \]  \hspace{1cm} (4.52)

Eq. (4.48) governs the translational motion of the hip joint, while Eqs. (4.49) and (4.50) govern the rotational motion of the femur and the tibia, respectively. Substituting Eqs. (4.43) through (4.46)
into Eqs. (4.48) through (4.50) and expressing the result in matrix form yields

\[
\mathbf{M}(z_0, \alpha_{\ell_1}, \alpha_{\ell_2}) = \mathbf{F}_1(a_0, \alpha_{\ell_1}, \alpha_{\ell_2}, \omega_{\ell_1}, \omega_{\ell_2}) + \begin{bmatrix}
\begin{array}{c}
\tilde{F}_1 \\
\omega_{\ell_1} \\
\omega_{\ell_2}
\end{array}
\end{bmatrix}
\begin{bmatrix}
\begin{array}{c}
\mathbf{C}_{\omega_{\ell_1}} \\
\mathbf{C}_{\omega_{\ell_1}} \\
\mathbf{C}_{\omega_{\ell_2}}
\end{array}
\end{bmatrix}
\begin{bmatrix}
\begin{array}{c}
\mathbf{F}_{\ell_1} \\
\mathbf{M}_{\ell_1} - \mathbf{C}_{\ell_1} \mathbf{M}_{\ell_2} \\
\mathbf{M}_{\ell_2}
\end{array}
\end{bmatrix}
\]

(4.53)

where

\[
\mathbf{M} = \begin{bmatrix}
\begin{array}{ccc}
(m_{\ell_1} + m_{\ell_2}) & \mathbf{C}_{\omega_{\ell_1}} & \mathbf{C}_{\omega_{\ell_1}} \\
\mathbf{C}_{\omega_{\ell_1}} & \mathbf{I}_{\ell_1} + m_{\ell_2} \mathbf{L}_{\ell_1} & \mathbf{L}_{\ell_1} \\
\mathbf{C}_{\omega_{\ell_1}} & \mathbf{L}_{\ell_1} & \mathbf{I}_{\ell_2}
\end{array}
\end{bmatrix}
\]

(4.54)

\[
\mathbf{F}_1 = -\begin{bmatrix}
\begin{array}{c}
\mathbf{C}_{\omega_{\ell_1}} \mathbf{L}_{\ell_1} \mathbf{S}_{\ell_1} + m_{\ell_2} \mathbf{C}_{\omega_{\ell_1}} \mathbf{L}_{\ell_1} + \mathbf{C}_{\omega_{\ell_2}} \mathbf{L}_{\ell_2} \mathbf{L}_{\ell_2}
\end{array}
\end{bmatrix}
\]

(4.55)

The relationship between the absolute angular velocity and the relative angular velocity as shown in Eqs. (3.68) and (3.69) can be
expressed in the form:

\[
\begin{bmatrix}
\dot{\mathbf{p}}_l \\
\omega_{l1} \\
\omega_{l2}
\end{bmatrix} = \mathbf{C} \begin{bmatrix}
\dot{\mathbf{p}}_l \\
\Omega_{l1} \\
\Omega_{l2}
\end{bmatrix} + \begin{bmatrix}
0 \\
C_{\ell_1B-B} \\
C_{\ell_2B-B}
\end{bmatrix}
\]  \quad (4.56)

where

\[
\mathbf{C} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & C_{\ell_2\Omega_{l1}} & 1
\end{bmatrix}
\]  \quad (4.57)

Substituting Eq. (4.56) and its time derivative into Eq. (4.53) and premultiplying the result by \(\mathbf{C}^T\), it follows that

\[
\mathbf{C}^T \mathbf{w}_C \mathbf{e}_{\mathbf{c}} = \mathbf{F}_2 \begin{bmatrix}
\mathbf{p}_l \\
\Omega_{l1} \\
\Omega_{l2}
\end{bmatrix} = \mathbf{F}_2 \begin{bmatrix}
\alpha_8\dot{\mathbf{e}}_{\mathbf{c}} \\
\alpha_{\ell_1}\dot{\mathbf{e}}_{\mathbf{c}} \\
\alpha_{\ell_2}\dot{\mathbf{e}}_{\mathbf{c}}
\end{bmatrix} + \begin{bmatrix}
\mathbf{C}_{\Omega_{l1}\ell_1} \mathbf{F}_2 \\
\mathbf{M}_{\Omega_{l1}} \\
\mathbf{M}_{\Omega_{l2}}
\end{bmatrix}
\]  \quad (4.58)

where

\[
\mathbf{F}_2 = \mathbf{C}^T \mathbf{e}_{\mathbf{c}} \begin{bmatrix}
\mathbf{I} & -\mathbf{M} \\
-\mathbf{M}^T & \mathbf{I}
\end{bmatrix}
\]  \quad (4.59)
Also, the relationship between the relative angular velocity and the
time derivative of the set of Euler angles as shown in Eqs. (3.84)
and (3.85) can be expressed in the form:

\[
\begin{bmatrix}
\dot{\alpha}_{l1} \\
\dot{\alpha}_{l2} \\
\end{bmatrix} = \mathbf{D}
\begin{bmatrix}
\dot{\alpha}_{l1} \\
\dot{\alpha}_{l2} \\
\end{bmatrix}
\]  

(4.60)

where

\[
\mathbf{D} =
\begin{bmatrix}
\mathbf{I} & 0 & 0 \\
0 & D_{l1} & 0 \\
0 & 0 & D_{l2}
\end{bmatrix}
\]  

(4.61)

where \(\mathbf{I}\) is a 3 by 3 identity matrix. Substituting Eqs. (4.60) and
(3.83) and their time derivative into Eq. (4.58) and premultiplying
the result by \(\mathbf{D}^T\), it follows that

\[
\mathbf{D}^T \mathbf{C}^T \mathbf{MCD} \begin{bmatrix}
\dot{\alpha}_{l1} \\
\dot{\alpha}_{l2} \\
\end{bmatrix} = \mathbf{F}_3(a_{l1}, a_{l2}, \dot{a}_{l1}, \dot{a}_{l2}, \dot{a}_{l1}, \dot{a}_{l2}) + \begin{bmatrix}
C_{N\ell_1} F_{\ell_1} \\
\tau_{l1} \\
\tau_{l2}
\end{bmatrix}
\]  

(4.62)

where
\[
F_3 = D^T \left( F_2 - C^{\text{MC}} \begin{bmatrix}
\dot{z}_{\ell_1} \\
\dot{z}_{\ell_2}
\end{bmatrix} \right)
\]

(4.53)

Differentiating Eq. (4.38) with respect to time twice and substituting Eq. (3.83) into the result yields an expression for the acceleration of the hip joint,

\[
\ddot{z}_{\ell} = \ddot{z}_B + C_{NB} \dot{\omega}_B H_{\ell} + C_{NB} \ddot{\omega}_B H_{\ell}
\]

\[
\dot{\ddot{z}}_{\ell} = \dddot{z}_B + C_{NB} \left( D \ddot{\alpha}_B + \dot{D} \dddot{\alpha}_B \right) H_{\ell} + C_{NB} \left( D \dddot{\alpha}_B \right) H_{\ell}
\]

(4.54)

Equation (4.62) is the equation of motion of a swinging leg in terms of true coordinates and their time derivatives. Note that \(z_B\), \(\dot{z}_B\), and \(\ddot{z}_B\) are included in the equation of motion because the dynamics of the swinging leg is coupled with the dynamics of the central body. Also, \(\dot{z}_B\), \(\ddot{z}_B\), and \(\dddot{z}_B\) can be fully determined from the simulation of the central body, as described in Section 4.3. This requires the evaluation of Eqs. (4.33) and (4.37) before the evaluation of Eq. (4.62). Therefore, for the simulation of the swinging leg, Eq. (4.62) consists of 3 vector equations and 3 vector unknowns: \(F_{\ell_1}\) in the first row of the right hand side, \(\ddot{z}_{\ell_1}\) and \(\ddot{z}_{\ell_2}\) in the second and the third rows of the left hand side. Hence, Eq. (4.62) can be solved numerically.
4.5 KINEMATICS OF LEGS IN STANCE

When the motion of the central body is fully described, as is described in Section 4.3, the motion of the legs in stance can be studied by investigating their kinematics. The constraint caused by their contact with the ground can be expressed in the form:

\[ R_B + C_{NB-t} + C_{nt1-t1} + C_{nt2-t2} = \text{constant} \]  \hspace{1cm} (4.65)

Differentiating Eq. (4.65) with respect to time yields

\[ \dot{R}_B + C_{NB-t} \dot{\omega}_B + C_{nt1-t1} \dot{\omega}_1 + C_{nt2-t2} \dot{\omega}_2 = 0 \]  \hspace{1cm} (4.66)

Substituting Eqs. (3.67), (3.68), (3.69), (3.83), (3.84) and (3.85) into Eq. (4.66) and rearranging the result, it follows that

\[
\begin{bmatrix}
\dot{\alpha}_{\ell_1} \\
\dot{\alpha}_{\ell_2} \\
-\dot{\alpha}_B
\end{bmatrix}
= -B(\alpha_B, \alpha_{\ell_1}, \alpha_{\ell_2}, \dot{\alpha}_B)
\]  \hspace{1cm} (4.67)

where

\[
A = \begin{bmatrix}
(C_{nt1-t1} + C_{nt2-t2} D_{\ell_1} | C_{nt2-t2} D_{\ell_2})
\end{bmatrix}
\]  \hspace{1cm} (4.68)

\[
B = \dot{R}_B + \begin{bmatrix}
C_{NB-t} \dot{\omega}_B + C_{nt1-t1} \dot{\omega}_1 + C_{nt2-t2} \dot{\omega}_2
\end{bmatrix}
\]  \hspace{1cm} (4.69)

\(A\) is a square matrix of dimension 3 by 3 and \(B\) is a vector of dimension 3 by 1. Hence, premultiplying both sides of Eq. (4.67) by \(A^{-1}\) yields

\[
\begin{bmatrix}
\dot{\alpha}_{\ell_1} \\
\dot{\alpha}_{\ell_2} \\
-\dot{\alpha}_B
\end{bmatrix}
= -A^{-1}B
\]  \hspace{1cm} (4.70)
This differential equation can then be solved numerically, given initial conditions, to obtain $\mathcal{g}_{t_1}$ and $\mathcal{g}_{t_2}$ at the end of each time step. After the differential equation is solved, the new values of $\mathcal{g}_{t_1}$ and $\mathcal{g}_{t_2}$ are substituted back into Eq. (4.70), to obtain the new values of $\hat{\mathcal{g}}_{t_1}$ and $\hat{\mathcal{g}}_{t_2}$. Similar to the swinging leg solution discussed in the previous section, it should be noted that Eqs. (4.33) and (4.37) must be evaluated before solving Eq. (4.70).

4.6 GLOBAL DYNAMICS

Based on the uncoupled dynamics of the central body and individual legs developed in the previous sections, an algorithm is proposed for the purpose of simulating the motion of the hexapod. This algorithm consists of four steps as follows:

Step 1

Given: $\mathbf{R}_b$, $\mathbf{R}_b$, $\mathcal{g}_{t_1}$, $\mathcal{g}_{t_2}$, $\hat{\mathbf{R}}_b$, $\hat{\mathbf{R}}_b$, $\hat{\mathcal{g}}_{t_1}$ and $\hat{\mathcal{g}}_{t_2}$ ($t=1,2,3,4,5,6$) at time $t$.

Step 2

Calculate the reaction force and moment between the body and the femur of the stance leg by means of Eq. (4.23).

Step 3

Construct a set of ordinary differential equations defined as follows:
\[
\begin{bmatrix}
\dot{R}_B \\
\dot{a}_B \\
\dot{a}_{11} \\
\dot{a}_{12}
\end{bmatrix}
= \begin{bmatrix}
\dot{R}_B \\
\dot{a}_B \\
\dot{a}_{11} \\
\dot{a}_{12}
\end{bmatrix}
\]

Given by Eq. (4.33).

\[
\begin{bmatrix}
\dot{R}_B \\
\dot{a}_B \\
\dot{a}_{11} \\
\dot{a}_{12}
\end{bmatrix}
= \begin{bmatrix}
\dot{R}_B \\
\dot{a}_B \\
\dot{a}_{11} \\
\dot{a}_{12}
\end{bmatrix}
\]

Given by Eq. (4.37).

\[
\begin{bmatrix}
\dot{R}_B \\
\dot{a}_B \\
\dot{a}_{11} \\
\dot{a}_{12}
\end{bmatrix}
= \begin{bmatrix}
\dot{R}_B \\
\dot{a}_B \\
\dot{a}_{11} \\
\dot{a}_{12}
\end{bmatrix}
\]

Set to 0 for stancing leg.

Repeated for each leg.

\[
\begin{bmatrix}
\dot{R}_B \\
\dot{a}_B \\
\dot{a}_{11} \\
\dot{a}_{12}
\end{bmatrix}
= \begin{bmatrix}
\dot{R}_B \\
\dot{a}_B \\
\dot{a}_{11} \\
\dot{a}_{12}
\end{bmatrix}
\]

Given by Eq. (4.62) for swinging leg.

\[
\begin{bmatrix}
\dot{R}_B \\
\dot{a}_B \\
\dot{a}_{11} \\
\dot{a}_{12}
\end{bmatrix}
= \begin{bmatrix}
\dot{R}_B \\
\dot{a}_B \\
\dot{a}_{11} \\
\dot{a}_{12}
\end{bmatrix}
\]

Set to 0 for stancing leg.

Repeated for each leg.

\[
\begin{bmatrix}
\dot{R}_B \\
\dot{a}_B \\
\dot{a}_{11} \\
\dot{a}_{12}
\end{bmatrix}
= \begin{bmatrix}
\dot{R}_B \\
\dot{a}_B \\
\dot{a}_{11} \\
\dot{a}_{12}
\end{bmatrix}
\]

Then, solve the initial-value problem posed by this ordinary differential matrix equation. This step will yield solutions for all unknowns at time \(t+\Delta t\) except \(\dot{a}_{t_1}\) and \(\dot{a}_{t_2}\) of the stance leg. It should be noted that in evaluating Eq. (4.71) the central body's motion must be solved before the legs. This is because of the assumption that the inertia of each leg is much less than the inertia of the central body. Therefore, the dynamics of each leg is coupled with the dynamics of the central body while the central body's motion is assumed to be uncoupled from the motion of the legs.
Step 4

Substitute the solution from step 3 into Eq. (4.70) to determine $\ddot{z}_{L_1}$ and $\ddot{z}_{L_2}$ of the stance leg.

4.7 COMPARISON BETWEEN MODEL I AND MODEL II

In comparing the complete model in Chapter Three with the simplified model in this chapter, there are two major differences:

First, in view of Eqs. (3.90) and (4.71), the complete model is a set of equations linear in terms of accelerations whereas the simplified model consists of several small sets of equations linear in terms of accelerations. Therefore, in performing dynamic simulation, the complete model needs one matrix inverse and the simplified model needs several small matrix inverses. However, the inertia matrix of the complete model can be inverted by using tridiagonal system method which takes only $O(N)$ operations as compared with $O(N^3)$ operations associated with the Gaussian elimination method. Therefore, there will be little difference in the CPU time required for matrix inverse between the complete model and the simplified model.

Second, the simplified model has simple equations of motion because the coupling terms that relate the dynamics of the legs to the dynamics of the central body are neglected. Therefore, the simplified model has the advantage of easy coding and can reduce CPU time in performing dynamic simulation.
CHAPTER FIVE
SIMULATION AND CONTROL

5.1 INTRODUCTION

In the study of the control of locomotion, the walking gait is often known while the joint moments and ground reaction forces are the unknowns of interest. The walking gait is described in terms of the joint angles and their time derivatives, which are then input to the equations of motion to obtain the joint moments and ground reaction forces. However, in Chapter Three, it has been shown that in the general case the dynamic model including constraints has more unknowns than equations because the ground reaction forces are redundant constraint forces. Therefore, some supplementary assumptions or experimental data are necessary in order to determine the joint moments and ground reaction forces for a given gait pattern. For example, a cost function may be defined as a quadratic summation of joint moments and ground reaction forces, and then minimized to obtain optimal solutions. In this chapter, an alternative approach is proposed. A dynamic simulation with feedback control and with the given gait as the desired motion is performed. The joint moments and ground reaction forces are then extracted from the results of the simulation.

Sections 5.2 and 5.3 discuss the feedback controller and the simulation procedures. Numerical examples and discussions follow in Sections 5.4 and 5.5.
5.2 FEEDBACK CONTROL

In a typical control strategy, an output quantity is measured and then compared with a desired value. This resulting error is then used to correct the system's output. This concept is called feedback or closed-loop control. Figure 5.1 shows the block diagram of a closed-loop control strategy associated with a manipulator. The desired joint trajectories are input to the joint moment calculation module (off-line) which outputs the desired feedforward joint moments for each joint for each simulation time step. The feedforward joint moments are then input to the dynamic simulation module the output of which is the simulated manipulator joint motion, constituting the open-loop controller. In the mean time, the simulated joint motion is compared with the desired value. This resulting error is then input to a feedback controller module which outputs feedback joint moments. These feedback joint moments are in turn fed back into the dynamic simulation module for the purpose of correcting deviations from the desired joint trajectories. This feedback loop constitutes the closed-loop controller.

The above control strategy can also be explained in terms of a perturbation scheme. It consists of two parts: a zero-order open-loop feedforward controller and a first-order closed-loop feedback controller to stabilize the dynamic system and eliminate small deviations from the desired trajectories.

Consider the dynamic system consisting of a manipulator with rigid links. The equations of motion can be expressed in the general
form as follows:

$$M^*(\ddot{\alpha}) + \dot{C}^*(\alpha, \dot{\alpha}) + \dot{V}^*(\alpha) = \tau \quad (5.1)$$

where $M^*$ is an $n$ by $n$ inertia matrix, assuming the dynamic system has $n$ degrees of freedom. $\dot{C}^*$ is an $n$ by 1 vector of Coriolis and centrifugal terms. $\dot{V}^*$ is an $n$ by 1 vector of gravitational terms. Also, $\alpha$ and $\tau$ are $n$ by 1 vectors of joint angles and joint moments, respectively.

By applying a first-order perturbation method, the original joint angles are perturbed into zero-order terms and first-order terms. The zero-order terms represent desired joint trajectories. Also, the first-order terms represent relatively small deviations from desired values. Therefore, the perturbed joint angles can be expressed as follows:

$$\alpha = \alpha_0 + \alpha_1 \quad (5.2)$$

where subscripts "0" and "1" denote the zero-order terms and the first-order terms, respectively. Accordingly, the joint moments are also perturbed as follows:

$$\tau = \tau_0 + \tau_1 \quad (5.3)$$

where the zero-order terms, $\tau_0$, represent feedforward joint moments and the first-order terms, $\tau_1$, represent feedback joint moments, respectively. The substitution of Eqs. (5.2) and (5.3) into Eq. (5.1) yields perturbed equations of motion. Assuming that first-order terms are small enough so that a linear approximation is accurate and neglecting terms of order higher than first order, the
perturbed equations of motion can be separated into two parts: zero-order equations and first-order equations, which are expressed in the general form as follows:

\[
M^* (\alpha_0) \dddot{\alpha}_0 + C^* (\alpha_0, \dot{\alpha}_0) + V^* (\alpha_0) = \tau_0
\]  

(5.4)

\[
M^* (\alpha_0) \dddot{\alpha}_1 + G^* (\alpha_0, \dot{\alpha}_0) \dot{\alpha}_1 + K^* (\alpha_0, \dot{\alpha}_0, \ddot{\alpha}_0) \alpha_1 = \tau_1
\]  

(5.5)

Equation (5.4) is the zero-order equation which appear the same as Eq. (5.1) with a "0" subscript on the joint variables. Eq. (5.5) is the first-order equation in which \(M^*\) is an inertia matrix which is the same as the inertia matrix of the zero-order equations and \(G^*\) and \(K^*\) are gyroscopic and stiffness matrices, respectively. Here, it should be noted that, if the zero-order motion is known, the first-order equations of motion are linear with known time-varying coefficients.

Recall the discussion at the end of Chapter Three. When legs are in stance, closed kinematic loops have to be taken into consideration. This leads to an infinite number of solutions for the feedforward joint moment calculation. Hence, it is convenient to neglect feedforward joint moments and use pure closed-loop control only. In this case, the output of the dynamic simulation provides ground reaction forces and active joint moments. Figure 5.2 shows the block diagram of the proposed control strategy. Comparing with Figure 5.1, the only difference is the removal of the open-loop controller.
Because no feedforward joint moments are input to the dynamic simulation module, the zero-order terms of the joint moments of Eq. (5.3) are null. Hence, the closed-loop, first-order equations of motion associated with this feedback controller take the following form:

\[
\mathbf{M}^*(\mathbf{q}_0)\ddot{\mathbf{q}} - \mathbf{G}^*(\mathbf{q}_0, \dot{\mathbf{q}}_0)\ddot{\mathbf{q}} + \mathbf{K}^*(\mathbf{q}_0, \dot{\mathbf{q}}_0, \ddot{\mathbf{q}}_0)\mathbf{q} = \mathbf{\tau}_1 + \mathbf{\tau}_d \tag{5.6}
\]

which appear almost the same as Eq. (5.5) with one more term, \(\mathbf{\tau}_d\). 

\(\mathbf{\tau}_d\) is a vector of disturbance moments which are functions of the zero-order motion and can be expressed as

\[
\mathbf{\tau}_d = -\mathbf{M}^*(\mathbf{q}_0)\ddot{\mathbf{q}}_0 - \mathbf{C}^*(\mathbf{q}_0, \dot{\mathbf{q}}_0) - \mathbf{V}^*(\mathbf{q}_0) \tag{5.7}
\]

Appearance of zero-order terms in first-order equations means that the first-order motion is excited by the zero-order motion. In other words, the dynamic system could be unstable.

Generally, robotic manipulators have their actuators and sensors installed at the joints permitting decentralized collocated vibration control. The collocated, proportional-derivative (PD) feedback control can effectively provide variable-rate active springs and dampers at the joints. In this case, the feedback joint moments can be expressed as

\[
\mathbf{\tau}_1 = -Z_p \mathbf{\dot{q}} - Z_b \ddot{\mathbf{q}} \tag{5.8}
\]

where \(Z_p\) and \(Z_b\) are positive-definite gain matrices, which could be time-varying. Substituting Eq. (5.8) into Eq. (5.6), the closed-loop first-order dynamic system may be expressed as follows:
\[ M_{0}\ddot{\alpha} + (G_{0} + Z_{D})\dot{\alpha} + (K_{0} + Z_{P})\alpha = \tau \]

(5.9)

where subscript "0" denotes a function of zero-order terms. That is, \( M_{0} = M(\alpha_{0}) \), \( G_{0} = G(\alpha_{0}, \dot{\alpha}_{0}) \) and \( K_{0} = K(\alpha_{0}, \dot{\alpha}_{0}, \ddot{\alpha}_{0}) \).

When this closed-loop controller is applied to the first-order system, the manipulator acts as a flexible structure with natural modes of vibration. In keeping with the concept of Uniform Damping Control, the active stiffness is chosen to be proportional to the inertia matrix. This is physically reasonable because larger inertias require larger forces for the same acceleration response. The active damping matrix is chosen to be proportional to the inertia matrix also. Therefore, the gain matrices can be defined as follows:

\[ Z_{P} = z_{P} M_{0}^{*} \]

(5.10)

\[ Z_{D} = z_{D} M_{0}^{*} \]

(5.11)

where \( z_{P} \) and \( z_{D} \) are positive scalar proportionality constants. Moreover, note that, if the manipulator is static, the zero-order velocities and accelerations, \( \dot{\alpha}_{0} \) and \( \ddot{\alpha}_{0} \), become null. Hence, \( C_{0} \) and \( G_{0} \) and \( K_{0} \) become null. Consequently, substituting Eqs. (5.10) and (5.11) into Eq. (5.9), the first-order closed-loop dynamic system consisting of a static manipulator is expressed as follows:

\[ \ddot{\alpha}_{0} + z_{D} \dot{\alpha}_{0} + z_{P} \alpha_{0} = -M_{0}^{-1} \tau \]

(5.12)

The proportionality constants, \( z_{P} \) and \( z_{D} \), can be chosen based on desired system performance.
5.3 SIMULATION PROCEDURES

From the mathematical viewpoint, a dynamic simulation problem is an initial-value problem i.e. ordinary differential equations which are defined by the dynamic model.

The simulation problem is as follows: Given initial conditions at time $t_1$, solve the ordinary differential equations and obtain new values at time $t_{1+1}$. The resulting new values are then used as the initial values for the next time step $[t_{1+1}, t_{1+2}]$. The initial-value problem is solved repeatedly until the end of the simulation.

In the case where the dynamic model does not have kinematic constraints, the simulation procedure is straightforward because the system equations of the dynamic model do not change from time step to time step during the simulation. However, in the case of simulating hexapod walking machine, closed-loop kinematic constraints appear in the dynamic model. The constraints change when any leg switches from its return stroke to its drive stroke or vice versa.

For the dynamic model developed in Chapter Three, the system equations consists of the equations of motion and the constraint equations, Eqs. (3.90) and (3.91). The equations of motion are always the same for each time step but the constraint equations will change when any leg is changing phase. For the simplified dynamic model developed in Chapter Four, the global equation defined by Eq. (4.71) constitutes the set of ordinary differential equations. It
can be seen that the part of Eq. (4.71), which corresponds to the
degrees of freedom of the leg undergoing the change of phase between
recover and drive, will have different definitions. Therefore, in
using either dynamic model, the existence of constraints imposes a
problem which needs to be dealt with in order to perform simulations
of the hexapod. The problem is: how to decide the time at which a
leg is about to change from its recover phase to its drive phase or
vice versa. The following is a step-by-step description of the
simulation procedure for using the simplified dynamic model:

**Step 1**

Given: Initial conditions of the system state at time $t_1$, which
include:

- Position and Euler angles of the body and their time
derivatives.
- Joint angles of hip joints and knee joints and their time
derivatives.
- Status of each leg, either recover or drive phase.

**Step 2**

Given: Feedforward joint moments. As is mentioned in the
previous section, the control strategy consists of a feedback
controller only. Hence, feedforward joint moments are eventually
set to zero.

**Step 3**

Compare actual values of the system state with the desired
values. Based on the resulting errors, calculate the feedback joint
moments using the following equations:

\[
\begin{bmatrix}
\mathcal{I}_{\ell_1} \\
\mathcal{I}_{\ell_2}
\end{bmatrix} = z_p \hat{I}_\ell \begin{bmatrix}
\dot{a}_{\ell_1} - \dot{a}_{\ell_2} \\
\dot{a}_{\ell_2} - \dot{a}_{\ell_1}
\end{bmatrix} + z_d \hat{I}_\ell \begin{bmatrix}
\ddot{a}_{\ell_1} - \ddot{a}_{\ell_1} \\
\ddot{a}_{\ell_2} - \ddot{a}_{\ell_2}
\end{bmatrix}
\]  

(5.13)

where the superscript "\*" denotes desired values, \( z_p \) and \( z_d \) are proportional and derivative control gains, and \( \hat{I}_\ell \) is a diagonal matrix which is equal to the lower-right diagonal elements of \( \mathbf{D}^T \mathbf{C}^T \mathbf{W} \mathbf{C} \mathbf{D} \) in Eq. (4.62) associated with the degrees of freedom \( a_{\ell_1} \) and \( a_{\ell_2} \).

**Step 4**

Sum the feedforward joint moments of Step 2 and the feedback joint moments of Step 3. The results are the applied joint moments.

**Step 5**

Calculate hip joint forces and moments in the stance legs using the Jacobian defined in Eq. (4.14).

---

**Step 6**

Check the hip joint forces of Step 5. Because the inertias of stance legs are neglected in calculating the hip joint forces, the ground reaction forces should be equal to the hip joint forces. Therefore, if the foot to ground contact force acts in tension, the phase of the corresponding leg must switch to recover. This is because the ground surface to foot contact force can only act in compression.
Step 7

Set the hip joint forces and moments of all the swinging legs to zero. This follows from the assumption that the inertias of swinging legs are small enough that the body's dynamics is not affected by the swinging legs' dynamics.

Step 8

Setup the simulation problem as an initial-value (time=$t_1$) problem of ordinary differential equations defined by Eq. (4.71). Implement a numerical scheme to solve the initial-value problem and then obtain the new system state at time $t_{1+1}$.

Step 9

Check the new foot position at time $t_{1+1}$ of the swing leg which is initially in contact with the ground and switched from the drive phase to the recover phase in Step 6. If its new foot position is below the ground surface, switch the phase of the this leg back to drive, abandon the new system state and redo Steps 8 and 9. This cycle consisting of Steps 8 and 9 is repeated until the check in Step 9 is passed. This simulation loop compensates for modeling errors resulting from the simplifications in the dynamic model. Neglecting the stance leg's inertia implies that the ground reaction force is equal to the hip joint force. This assumption is reasonable when the stance leg is carrying the body's weight. But, in the transition region in which a stance leg is about to raise and, hence, is unloaded, the task of supporting the body is transferred to other legs that are in contact with the ground.
Therefore, the hip joint force grows smaller and the relative contribution of inertia to the hip joint force grows larger in this transition region. That is, the hip joint force of an unloaded stance leg could have a component pointing downward due to the leg's weight while the ground reaction force still has an upward component in order to support the stance leg's weight. There is a discontinuity in the model during the transition.

**Step 10**

Use the new system state at time $t_{i+1}$ as the initial conditions of the next time step $[t_{i+1}, t_{i+2}]$. Repeat Steps 1 to 10 until the end of the simulation period.

5.4 **EXAMPLE THREE**

The dynamic model considered in this example is a hexapod walking machine that walks on a smooth horizontal surface, as shown in Figure 5.3. It consists of the central body and six legs. Each leg is of the same construction with two segments and two revolute joints. The central body has six degrees of freedom: three for translation and three for rotation. The hip joint, connecting the central body and the femur, has two degrees of freedom. The first degree of freedom allows the femur to swing forward and backward in the horizontal plane. The second degree of freedom permits the femur to swing in the vertical plane and raise or lower the tibia. The knee joint between the two segments of the leg has one degree of freedom whose axis of rotation is parallel to the axis of the second
degree of freedom of the hip joint. Therefore, the dynamic model has a total number of 24 degrees of freedom.

Four different coordinate systems are introduced and defined as follows:

**N-frame**

This frame is an inertia coordinate system with $x$- and $y$-axes attached to the ground surface, and $z$-axis pointing upward from the ground surface. The $x$-axis is in the direction of the hexapod's motion.

**B-frame**

This frame is fixed to the central body with its origin located at the center of mass of the central body. Its $x$-axis lies along the longitudinal axis of symmetry of the central body. The $y$-axis marks the lateral axis of symmetry of the central body. The $x$-, $y$- and $z$-axes are directed toward to the front-, the left-hand side and upward from the central body, respectively.

**t1-frame**

This frame is fixed to the femur with its origin located at the hip joint. The $y$-axis is along the straight line between the hip joint and the knee joint. The orientation of the $t1$-frame is coincident with the orientation of the $B$-frame when the femur is extended perpendicularly from the central body in the horizontal plane, as shown in Figure 5.4.
$\ell_2$-frame

This frame is fixed to the tibia with its origin located at the knee joint. The $y$-axis is along the straight line between the knee joint and the foot. The orientation of the $\ell_2$-frame is coincident with the orientation of the $\ell_1$-frame when the two segments are aligned, as shown in Figure 5.4.

Because the dynamic system has 24 degrees of freedom, 24 generalized (true) coordinates are required to specify the configuration of the dynamic system. They are divided into four groups and defined as follows:

$R_{B}^{\mathbb{R}}$

A 3 by 1 column matrix defining the position of the $B$-frame's origin relative to the $N$-frame.

$\alpha_{B}^{\mathbb{A}}$

A 3 by 1 column matrix of $z$-$y$-$x$ Euler angles specifying the spatial orientation of the $B$-frame relative to the $N$-frame. These Euler angles are described as follows:

- Heading angle, by which a rotation about the $z$-axis of the $N$-frame results in the first intermediate frame.
- Attitude angle, by which a rotation about the $y$-axis of the first intermediate frame results in the second intermediate frame.
- Bank angle, by which a rotation about the $x$-axis of the second intermediate frame results in the $B$-frame.
These three angles are also called Yaw, Pitch and Roll angles which are most often used for specifying the orientation of vehicles on or near the surface of the earth.

\[ \alpha_{t1} \]

A 2 by 1 column matrix of two joint angles associated with the degrees of freedom of the hip joint. These angular rotations are described as follows:

- First hip joint angle, by which a rotation about the z-axis of the B-frame results in an intermediate frame.
- Second hip joint angle, by which a rotation about the x-axis of the intermediate frame results in the \( t1 \)-frame.

\[ \alpha_{t2} \]

A 1 by 1 column matrix associated with the degree of freedom of the knee joint. This angular rotation about the x-axis of the \( t1 \)-frame results in the \( t2 \)-frame.

Figure 5.3 shows the relative positions of individual legs. The locations of the hip joints on one side of the body form a straight line which is parallel to the longitudinal axis of symmetry of the central body. The middle leg’s hip joint is located on the lateral axis of symmetry of the body, and the front and rear legs’ hip joints are located symmetrically with respect to the lateral axis of symmetry of the body. In addition, each contralateral pair of hip joints is symmetric with respect to the longitudinal axis of symmetry of the body. All six legs are constructed with the same
mechanical design, as shown schematically in Figure 5.4.

Dimensions and mechanical properties of the body and the individual legs are listed as follows:

**Central body**

- Mass = 2.28 Kg.

\[
\begin{bmatrix}
0.018 & 0 & 0 \\
0 & 0.039 & 0 \\
0 & 0 & 0.095
\end{bmatrix}
\text{Kg} \cdot \text{m}^2.
\]

**Femur**

- Length = 0.2 m.
- Mass = 0.219 Kg.
- Center of mass = 0.1 m from hip joint.

\[
\begin{bmatrix}
0.0029 & 0 & 0 \\
0 & 0.000044 & 0 \\
0 & 0 & 0.0029
\end{bmatrix}
\text{Kg} \cdot \text{m}^2.
\]

**Tibia**

- Length = 0.25 m.
- Mass = 0.075 Kg.
- Center of mass = 0.125 m from knee joint.

\[
\begin{bmatrix}
0.0016 & 0 & 0 \\
0 & 0.000015 & 0 \\
0 & 0 & 0.0016
\end{bmatrix}
\text{Kg} \cdot \text{m}^2.
\]

where the center of mass of each leg segment is located along the line of its length vector, and the moments of inertia of individual
bodies are expressed with respect to their corresponding body-fixed frames.

The hexapod is designed to walk on the horizontal surface. Each leg cycle consists of a 1-sec recovery phase and a 2-sec drive phase, as shown in Figure 5.32. The forward swing angle limit is set to 0.5404 rad and the backward swing angle limit is set to 0.5404 rad. The span between footfalls of contralateral pairs of legs is set to 0.38 m. Hence, viewing the machine from the rear, Figure 5.3, the femur is raised 60° from the horizontal plane, the tibia is vertical, and the center of mass of the central body is 0.07679 m above the ground. The motion of the central body is desired to be along a straight line with a constant velocity and a constant orientation parallel to the ground surface.

In the numerical simulation, the time step size is set to 0.002 sec. Feedback control gains are set as follows:

\[
Z_p = \begin{cases} 
2500 & \text{for recovery phase.} \\
5625 & \text{for drive phase.}
\end{cases}
\]

\[
Z_d = \begin{cases} 
100 & \text{for recovery phase.} \\
150 & \text{for drive phase.}
\end{cases}
\]

Results are shown in Figures 5.6 to 5.32, in which \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) are joint angles defined in Figure 5.5 and given by

\[
\begin{bmatrix}
\alpha_1 \\
-\alpha_2 \\
-\alpha_3
\end{bmatrix} = 
\begin{bmatrix}
\alpha_{l1} \\
\alpha_{l2}
\end{bmatrix}
\text{ for legs on the right-hand side of the body.}
\]
\[
\begin{bmatrix}
-\alpha_1 \\
\alpha_2 \\
-\alpha_3
\end{bmatrix} = \begin{bmatrix}
\alpha_1 \\
-l_1 \\
\alpha_2 \\
-l_2
\end{bmatrix}
\]
for legs on the left-hand side of the body.

\(\tau_1, \tau_2\) and \(\tau_3\) are motor torques associated with the joint degrees of freedom, \(\alpha_1, \alpha_2\) and \(\alpha_3\), respectively. \(\phi_1, \phi_2\) and \(\phi_3\) are Yaw, Pitch and Roll angles associated with the central body.

Figures 5.6 to 5.11 show time histories of the motor torques and Figures 5.12 to 5.14 show the time histories of the ground reaction forces of individual legs. Figure 5.15 shows a time history of the summation of ground reaction forces. From these results, several conclusions regarding the dynamics of stance legs can be made as follows:

**Motor torque \(\tau_1\) related to x-component of ground reaction force**

The front pair of legs exhibits positive values of \(\tau_1\) in stance, which means that this motor torque intends to swing the front leg forward, opposite the direction of motion of the front leg. The foot of a front leg is hence subjected to a negative ground reaction force in the x-direction, which decelerates the body. Inversely, the rear pair of legs exhibits negative values of \(\tau_1\) in stance, hence, the motor torque is in the same direction as the leg is moving. Therefore, the foot of a rear leg is subjected to a positive ground reaction force in the x-direction, which accelerates the body. For the middle leg pair, the ground reaction force is negative at the beginning and then positive after the
middle point of the drive phase. Hence, the middle leg decelerates the body in the first half of its drive phase and then accelerates the body in the second half of the drive phase. Note that the right-hand side middle leg has its x-component ground reaction force biased to a positive value. This causes the central body to turn toward the left side.

**Motor torque \( \tau_z \) related to z-component of ground reaction force**

All six legs exhibit negative values of motor torque \( \tau_z \). This is expected because the task of these motors is to support the weight of the body. The front legs show increasing magnitude, the rear legs show decreasing magnitude and the middle legs show a combination of first increasing magnitude and then decreasing magnitude during their respective drive phases. This reflects the fact that a leg nearer the center of mass of the body carries more body weight than a leg farther away from the center of mass of the body. During the drive phase the front leg moves toward the mass center, the rear leg moves away from the mass center, while the middle leg moves toward the mass center at first and then away from the mass center. This phenomenon is mirrored in the z-component ground reaction force patterns.

**Motor torque \( \tau_y \) related to y-component of ground reaction force**

The motor torque \( \tau_y \) of each of the legs displays a positive value. This corresponds to the observed ground reaction forces in the y-direction: legs on the right-hand side of the body display positive values and legs on the left-hand side of the body display
negative values of y-component ground reaction forces.

**Summation of ground reaction forces**

The summation of the ground reaction forces of all the stance legs shows that the x- and y- components have zero averages. This is necessary to satisfy the prescribed motion of the body: constant velocity along a straight line. In addition, as is expected, the average of the z-component is equal to the weight of the body.

Figures 5.16 through 5.21 show the desired and simulated time histories of the joint angles. Also, the corresponding time histories of the joint angle errors are shown in Figures 5.22 to 5.27. Note that the errors in joint angle $\alpha_2$ have positive values and the errors in joint angle $\alpha_3$ have negative values, implying that the center of mass of the central body is lower than the desired height from the ground surface. This is to be expected for PD control. As seen in Eq. (5.12), the first-order system response is affected by the zero-order gravitational term.

Figures 5.28 through 5.31 show the desired and simulated motions of the body. The altitude of the center of mass is found to be lower than the desired value. Increasing the control gains makes the joints more stiff and reduces this error. High joint stiffness causes the hexapod to bounce off of the ground during locomotion with all six legs leaving the ground surface. The feedback controller is based on the joint variables and their time derivatives. Errors in the body's motion are not fed back to the controller. Hence, the feedback controller can only maintain the
body horizontal as desired and does not compensate for deviations of speed and direction of the body. The pitch and roll angles, $\phi_2$ and $\phi_3$, are then expected to remain zero as the robot walks, as shown in Figure 5.30. Also, the errors in the position of the center of mass in the x- and y-direction are found to grow.

Figure 5.32 shows a comparison of the simulated walking gait with the desired walking gait. This comparison reveals only very small differences.

5.5 EXAMPLE FOUR

Consider the same hexapod walking machine of Example Three. The desired walking gait is adjusted so that each leg cycle consists of a 1-sec recovery phase and a 1-sec drive phase. This walking gait is known as alternative tripod gait and is shown in Figure 5.59. Figures 5.33 to 5.59 show numerical results with the same time step size and control gains of Example Three. The qualitative results are nearly identical to those for Example Three. Each pair of legs show a unique ground reaction force pattern. The front leg decelerates the body, the rear leg accelerates the body, whereas the middle leg both decelerates and accelerates the body.

All simulations of Examples Three and Four were performed on a Sun Sparc 1+ workstation. Programs are written in C and compiled using cc with the -04 option. Also, the numerical scheme for solving the initial-value ordinary differential equations is the fourth-order Runge-Kutta integration method.
CHAPTER SIX
SUMMARY AND CONCLUSIONS

In Chapter Two, a locally optimal trajectory management method has been shown to reduce the base reactions of kinematically redundant manipulators while the end effector follows a prescribed path. In this approach the end-effector path is used to eliminate the nonredundant variables from the optimization problem. The redundant variables are expressed in terms of a Rayleigh-Ritz expansion over discrete time intervals throughout the manipulation time period. An optimization scheme determines the Rayleigh-Ritz coefficients which minimize the base reactions over each discrete time step.

In the two examples presented in Chapter Two, it is shown that care must be taken in choosing the cost function and the Rayleigh-Ritz expansion. If the reactions are minimized at the end of each discrete time step, the base reactions may fluctuate between discrete time steps. In the case that only one Rayleigh-Ritz coefficient is used, the base reactions do not fluctuate greatly due to the low order polynomial of the Rayleigh-Ritz expansion. However, the joint motions do not reach their objectives at the end of the manipulation. The null joint motion condition at the end of the manipulation is also not met if the cost function is defined as the sum of the base reactions at the beginning and at the end of each discrete time step. If the joint velocities are
unconditionally forced to full stop at the last time step, the base reactions display a relatively large peak at the last time step.

A modified cost function for the locally optimal trajectory management approach is proposed, which includes the kinetic energy along with the base reactions in a weighted and scaled sum. This modification shows excellent performance. The base reactions do not fluctuate greatly and the joint velocities reach their objectives at the end of the manipulation. It is noted that the CPU time is more than the manipulation time, so that real-time control does not appear possible. However, in the optimization scheme, the gradient of the cost function is computed by using the finite-difference method. If the gradient is computed off-line symbolically, the CPU time may be reduced greatly. Hence, real-time base reaction control may be possible. As a matter of fact, the modified locally optimal trajectory management method is capable of greatly reducing the base reactions of kinematically redundant manipulators. This method should prove valuable for the operation of manipulators in micro-gravity laboratories. In addition, the optimization technique is valid for joint management of kinematically redundant manipulators using other cost functions. An example of another such problem is the control of one leg of a walking machine.

Lagrange's equations for quasi-coordinates provide a straightforward method of formulating the equations of motion of multibody dynamic systems when the energy expressions are in matrix form as explicit functions of angular velocities and orientation
coordinate transformation matrices. This method is suitable for symbolic computation and permits the dynamic analysis of relatively complex system. In Chapter Three, the equations of motion were derived for a hexapod walking system which may represent a robot or an insect. Equations of motion in terms of true coordinates were obtained by introducing transformation matrices into the equations of motion in terms of quasi-coordinates. Only the definitions of the transformation matrices need modification because the matrix form of the equations of motion does not change for different models which have different number of degrees of freedom associated with each of the joints.

The equations of motion of a hexapod were shown to have a greater number of unknowns than equations when displacements, velocities and accelerations are given and joint torques are the desired quantities. This problem can be solved analytically by the optimal force distribution method which minimizes a cost function which is defined as a quadratic summation of the active joint moments and the ground reaction forces. This leads to an optimal solution minimizing bending loads on the leg segments. In experimental approach, force plates and a data acquisition system record the ground reaction forces over time. Also, synchronized photos taken by a high-speed camera and 3-dimensional motion analysis provide information about displacements from which velocities and accelerations are obtained using numerical analysis. These experimental datum are input to the equations of motion to
obtain the joint torques.

A new strategy of simulating hexapod dynamic systems is devised, which is valid for cases in which the inertia of the individual legs is much less than the inertia of the body and is then neglected during the drive phase. Neglecting a stance leg's inertia permits the stance leg to be treated as if it is in static equilibrium. In this way, the ground reaction forces, which are equal to the hip joint forces because the inertia of the leg is neglected, are related to the active joint moments. The effects of the leg's inertia on the body are assumed negligible. Hence, the central body's motion is independent of the motion of a leg which is in its recovery phase. When a leg is in its recovery phase it is treated as an open-chained manipulator with a moving base. The equations of motion derived in Chapter Three are partitioned into parts associated with the central body and the individual legs, respectively, while the total number of the equations of motion does not change. The arithmetic computational load associated with the simplified model of Chapter Four is shown to be less than that associated with the complete model of Chapter Three.

A strategy for dynamic simulation of a hexapod is devised for the purpose of studying the mechanics of locomotion. The simulation scheme uses only closed-loop feedback control and no open-loop feedforward control. The feedback controller is a classic proportional-derivative controller and the control gains are chosen based on the desired performance of the first-order equations of
motion of the perturbed dynamic system. The desired walking gait pattern including joint angles and their derivatives are input to the dynamic simulation module. The joint torques and the ground reaction forces are extracted from the results of the dynamic simulation.

In the example of Chapter Five, it is found that the horizontal component of the ground reaction force of each leg fluctuates, even when the center of mass of the hexapod undergoes constant velocity. The front legs generate deceleration forces, acting like a brake. The rear legs apply acceleration forces, acting to increase the speed of the center of mass. The middle legs generate deceleration forces in the first part of their stance phase and then acceleration forces in the last half of their stance phase.

The simplified hexapod model is more computationally efficient for computer simulation. But, it is only useful for a hexapod with three degrees of freedom for each leg. When the number of the leg’s degrees of freedom is not equal to three, the Jacobian matrix will not be square and, hence, cannot be inverted. In order to extend the model to a generalized case, the Jacobian approach should be replaced by a Newton-Euler method.

PD control provides good performance for vibration control of a multibody system and has the advantage of straightforward implementation. In future work, body attitude control and coordination between legs should be included.
Dynamic simulation with pure closed-loop control permits the study of legged locomotion mechanics. Care must be taken in choosing the feedback control strategy because the solution of the joint torques is not unique for a given system state. Therefore, different control strategies could lead to different results for the same prescribed walking gait.
Figure 2.1 Generalized open-chained robotic manipulator.
Figure 2.2  Example one:
A 3-link 3-d.o.f. planar manipulator.
Figure 2.3  Example one:  
End-effector trajectories.
Figure 2.4  Example one:
Base reactions for case 1.
Figure 2.5  Example one:  
Joint trajectories for case 1.
Figure 2.6  Example one:
Time-lapse plot of manipulation for case 1.
Figure 2.7  Example one:
Base reactions for case 2.
Figure 2.8  Example one:
Joint trajectories for case 2.
Figure 2.9   Example one:
Time-lapse plot of manipulation for case 2.
Figure 2.10 Example one:
Base reactions for case 3.
Figure 2.11 Example one:
Joint trajectories for case 3.
Figure 2.12 Example one:
Time-lapse plot of manipulation for case 3.
Figure 2.13  Example one:
Base reactions for case 4.
Figure 2.14 Example one:
Joint trajectories for case 4.
Figure 2.15 Example one:
Time-lapse plot of manipulation for case 4.
Figure 2.16 Example one:
Base reactions for case 5.
Figure 2.17 Example one:
Joint trajectories for case 5.
Figure 2.18 Example one:
Time-lapse plot of manipulation for case 5.
Figure 2.19 Example one:
Base reactions for case 6.
Figure 2.20 Example one:
Joint trajectories for case 6.
Figure 2.21 Example one:
Time-lapse plot of manipulation for case 6.
Figure 2.22 Example two:
A 2-link 4-d.o.f. 3-dimensional robotic manipulator.
Figure 2.23  Example two:
End-effector trajectories.
Figure 2.24 Example two:
Base reactions for case 1.
Figure 2.25 Example two:
Joint trajectories for case 1.
Figure 2.26 Example two:
Base reactions for case 2.
Figure 2.27 Example two:
Joint trajectories for case 2.
Figure 2.28  Example two:
Base reactions for case 3.
Figure 2.29 Example two: Joint trajectories for case 3.
Figure 2.30  Example two:  
Base reactions for case 4.
Figure 2.31 Example two: 
Joint trajectories for case 4.
Figure 2.32  Example two:  
Base reactions for case 5.
Figure 2.33  Example two:  
Joint trajectories for case 5.
Figure 2.34  Example two:
Base reactions for case 6.
Figure 2.35  Example two:  
Joint trajectories for case 6.
Figure 3.1 Hexapod mechanical model.
Figure 4.1 Generalized 3-link robotic manipulator.
Figure 5.1  Block diagram of control strategy with both feedforward and feedback.
Figure 5.2  Block diagram of control strategy with feedback only.
Figure 5.3 Example three: Dimensions of hexapod.
Figure 5.4  Example three:
Conceptual construction of leg joints.
Figure 5.5  Example three:
Definition of joint angles.
Figure 5.6  Example three:
Motor torques of right-front leg.
Figure 5.7  Example three:
Motor torques of right-middle leg.
Figure 5.8  Example three:
Motor torques of right-rear leg.

\[ \tau_1 \text{ (N.m)} \]
\[ \tau_2 \text{ (N.m)} \]
\[ \tau_3 \text{ (N.m)} \]
\[ \tau_4 \text{ (N.m)} \]

Time (sec)
Figure 5.9  Example three:  
Motor torques of left-front leg.
Figure 5.10 Example three:
Motor torques of left-middle leg.
Figure 5.11 Example three: Motor torques of left-rear leg.
Figure 5.12  Example three:
Longitudinal ground reaction forces.
Figure 5.13  Example three:
Lateral ground reaction forces.
Figure 5.14 Example three:
Vertical ground reaction forces.
Figure 5.15 Example three: Summation of ground reaction forces.
Figure 5.16 Example three:
Simulated joint angles of right-front leg.
Figure 5.17 Example three: 
Simulated joint angles of right-middle leg.
Figure 5.18 Example three: Simulated joint angles of right-rear leg.

\[ \alpha_1 (\text{rad}) \]

\[ \alpha_2 (\text{rad}) \]

\[ \alpha_3 (\text{rad}) \]

Time (sec)
Figure 5.19  Example three:
Simulated joint angles of left-front leg.
Figure 5.20 Example three:
Simulated joint angles of left-middle leg.
Figure 5.21 Example three:
Simulated joint angles of left-rear leg.
Figure 5.22 Example three: Joint angle errors of right-front leg.
Figure 5.23  Example three:
Joint angle errors of right-middle leg.
Figure 5.24  Example three:
Joint angle errors of right-rear leg.
Figure 5.25  Example three:
Joint angle errors of left-front leg.
Figure 5.26 Example three:
Joint angle errors of left-middle leg.
Figure 5.27 Example three:
Joint angle errors of left-rear leg.
Figure 5.28 Example three:
Simulated position of central body.
Figure 5.29 Example three:
Position errors of central body.
Figure 5.30 Example three:
Simulated orientation of central body.
Figure 5.31 Example three:
Orientation errors of central body.
Figure 5.32 Example three: Simulated gait pattern.
Figure 5.33 Example four:
Motor torques of right-front leg.
Figure 5.34 Example four:
Motor torques of right-middle leg.
Figure 5.35 Example four:
Motor torques of right-rear leg.
Figure 5.36 Example four:
Motor torques of left-front leg.

\[ \tau_1 \text{ (N\cdot m)} \]

\[ \tau_2 \text{ (N\cdot m)} \]

\[ \tau_3 \text{ (N\cdot m)} \]

Time (sec)
Figure 5.37 Example four:
Motor torques of left-middle leg.
Figure 5.38  Example four:
Motor torques of left-rear leg.

\[
\begin{align*}
\tau_1 (\text{N.m}) & \\
\tau_2 (\text{N.m}) & \\
\tau_3 (\text{N.m}) & \\
\end{align*}
\]

Time (sec)
Figure 5.39  Example four:
Longitudinal ground reaction forces.
Figure 5.40  Example four:
Lateral ground reaction forces.
Figure 5.41  Example four:
Vertical ground reaction forces.
Figure 5.42 Example four:
Summation of ground reaction forces.
Figure 5.43 Example four:
Simulated joint angles of right-front leg.
Figure 5.44 Example four:
Simulated joint angles of right-middle leg.
Figure 5.45 Example four:
Simulated joint angles of right-rear leg.

\[ \alpha_1, \alpha_2, \alpha_3 \text{ (rad)} \]

Time (sec)
Figure 5.46 Example four:
Simulated joint angles of left-front leg.

\[ \alpha_1 \text{ (rad)} \]
\[ \alpha_2 \text{ (rad)} \]
\[ \alpha_3 \text{ (rad)} \]

Desired ---
Simulated •••••

Time (sec)
Figure 5.47  Example four:
Simulated joint angles of left-middle leg.
Figure 5.48  Example four:
Simulated joint angles of left-rear leg.
Figure 5.49 Example four:
Joint angle errors of right-front leg.
Figure 5.50  Example four:
Joint angle errors of right-middle leg.
Figure 5.51 Example four:
Joint angle errors of right-rear leg.
Figure 5.52  Example four:
Joint angle errors of left-front leg.
Figure 5.53 Example four:
Joint angle errors of left-middle leg.
Figure 5.54 Example four:
Joint angle errors of left-rear leg.
Figure 5.55  Example four:
Simulated position of central body.
Figure 5.56  Example four:
Position errors of central body.
Figure 5.57  Example four:
Simulated orientation of central body.
Figure 5.58 Example four:
Orientation errors of central body.
Figure 5.59  Example four:
Simulated gait pattern.

Desired

Simulated

Time (sec)
REFERENCES


