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Optimal group replacement policies

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OPTIMAL GROUP REPLACEMENT POLICIES

by

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Submitted in partial fulfillment of the requirements
for the Degree of Doctor of Philosophy

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OPTIMAL GROUP REPLACEMENT POLICIES

Abstract

by

ALI BENMERZOUGA

This work addresses a number of gaps in the literature of group repair and replacement policies for machines that are operating in parallel. Three new models will be introduced. The objective function, in all models, is to minimize the expected operating cost per unit time.

The first model allows the incorporation of a random repair time. For machines with i.i.d. exponentially distributed failure times, a procedure is provided for finding the optimal m-failure policy when repair time is assumed to be a random variable with known parameters.

In the second model the failure time parameters are no longer
known with certainty but statistical learning about the nature of
the failure time distribution is allowed. A new class of policies
is introduced for machines with i.i.d. exponentially distributed
failure times and a prior distribution (not necessarily of a
conjugate form) for the failure time parameter $\lambda$.

Finally, a new class of policies that is more adaptive than
the $m$-failure, $T$-age and $(m,T)$ policies will be introduced. This
approach applies to the case where the failure time parameter is
assumed known.
DEDICATION

To my wife
and
my daughters
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CHAPTER I

INTRODUCTION

AND

LITERATURE REVIEW
The focus of this work is on systems of N parallel machines, which are all subject to stochastic failures from the same distribution. Many group maintenance policies have appeared in the literature. Some of the most important policies are going to be discussed.

The first policy, called the \(m\)-failure policy, where the system is not replaced until the number of machines that have failed is at least \(n\) (\(m<N\)) (e.g., Assaf and Shanthikumar (1987), Nakagawa (1979), and Yasui (1988)).

The second policy, called the \(T\)-age policy, calls for group replacement when the system is of age \(T\) or when all machines have failed since the last maintenance, whichever occurs first (e.g., Barlow and Hunter (1960), and Okumoto and Elsayed (1983)).

The third policy, which is a combination of the above two policies, called the \((m,T)\) policy, where the system is maintained after the \(m\)th failure or after \(T\) units of time since the last maintenance, whichever occurs first (e.g., Ritchken and Tapiero (1985) and Ritchken and Wilson (1990)).

The work in this thesis addresses a number of gaps in the literature. The first one is the incorporation of a non-zero repair time. The second one is the statistical estimation problem, where the data obtained from the failure of machines is included in the decision making. The third one and the last, is allowing policies that are more adaptive than the \(m\), \(T\) and \((m,T)\) policies.

Chapter 2 contains the solution of the non-zero repair time model, where each failed machine takes \(R\) units of time to be
repaired, where R could be a deterministic positive value or a non-negative random variable. The model analyzed is a generalization of that considered by Assaf and Shanthikumar (1987). The failure times of n machines are i.i.d. exponential random variable with known parameter \( \lambda \). The behavior of the optimal policy as a function of the cost parameters is investigated. The cost function is shown to be unimodal and an easily implemented algorithm for finding optimal policies is developed.

In Chapter 3, a new approach is developed, where the parameters of the failure time distribution are no longer known with certainty, but statistical learning about the nature of the failure time distribution is allowed. This approach investigates the problem of integrating the statistical information obtained as the machines are operating with group replacement strategies. It is assumed that the failure times of each machine are i.i.d. exponential random variables with unknown parameter \( \lambda \). The objective function will be the expected cost per unit time. Statistical information in the form of a prior distribution over the parameter is provided by the supplier. It is not necessary that this prior be of conjugate form.

A new class of policies is introduced that calls for replacement whenever the posterior distribution of the failure time parameter indicates that the system is unreliable. It is shown that, for a fixed value of \( \lambda \), the preposterior likelihoods of the sufficient statistics are linear combination of "shifted"
gamma densities. Given the latter result it is shown that the preposterior probability of replacement occurring after the $k^{th}$ failure is the expectation with respect to the prior distribution of a polynomial in $\lambda$ times $\exp(-\lambda)$ to the power of a known constant. These results are used to arrive at expressions for the various expected values that only involve one dimensional integrals with straightforward regions of integration.

In Chapter 4, a general adaptive approach is introduced. This class of strategies includes the $m$-failure, $T$-age and $(m,T)$ policies as special cases. Assuming continuous inspection, an easily implemented replacement policy is developed that takes account of the actual state of the system at each stage.
CHAPTER II

m-FAILURE POLICIES WITH A NON-ZERO REPAIR TIME
2.1 INTRODUCTION AND MODEL DESCRIPTION

Consider \( n \) machines that are subject to random failures. The failure times are independent identically distributed exponential random variables with parameter \( \lambda \). Each failed machine may be repaired at anytime and is considered as good as new once it has been repaired. The problem is to determine when to start repairing machines. Unlike Assaf and Shanthikumar (1987) and Okumoto and Elsayed (1983), repair is not assumed to be instantaneous. Instead, each failed machine takes \( R \) units of time to be repaired, where \( R \) is a nonnegative random variable with finite mean and density \( f(\cdot) \). In practice this generalization is of economic importance. An optimal policy for the instantaneous repair case can be decidedly non-optimal when repair takes time. Of course, assuming that repair time is random introduces extra difficulties to the analysis. Specifically one has to account for the possibility that other machines may fail while a machine is being repaired. The challenge is to model the problem in such a way that the assumption of non-zero repair time does not overly complicate the solution procedure. Once repair has commenced, the repair crew will keep working until all \( n \) machines are working.

A fixed cost of \( c_0 \) is assumed. This cost could reflect, for instance, the paperwork and travel costs incurred in bringing a repair crew to the factory. The cost of repairing an individual machine consists of two components, one fixed and the other time related. Specifically, \( c_f \) is the fixed cost of repairing a machine
and includes such costs as the price of replacement parts, etc., while \( c_r \) denotes the cost per unit time of repairing a machine. This latter cost could be interpreted as a labor cost. Each machine that fails accumulates down time costs at a rate of \( c_d \) per unit time.

2.2 DERIVATION OF THE OBJECTIVE FUNCTION

Probabilities that govern the system:

The probability that an individual machine will break down during a fixed period of length \( r \) is given by \( 1 - e^{-r \lambda} \). Suppose that \( i \) machines are broken. Then the number not working at the end of the period of length \( R \) taken to fix the first machine equals:

\[
(1-1) + \text{(# Of remaining machines that fail during the period of length } R). \]

Hence \( p_{ij}(r) \), the probability that \( j \) machines are not working at the end of the period taken to fix the first machine given that the first machine will take \( r \) time units to fix, \( i \geq 1 \) are currently broken and repair is now starting, is given by

\[
p_{ij}(r) = \begin{cases} 
\binom{n-1}{j+1-i} e^{-(n-j-1)r \lambda} (1-e^{-r \lambda})^j, & \text{for } i-1 \leq j \leq n-1, \\
0 & \text{otherwise .} 
\end{cases} \quad (1)
\]
The expected repair time:

Let \( T_i \) denote the time required for all machines to be functioning given that \( i \geq 1 \) are currently broken and repair is about to commence.

**Proposition 2.1:**

Conditioning on the time taken to fix the first broken machine, the expected value of \( T_i \), \( i \geq 1 \), is given by

\[
E(T_i) = E(R) + \sum_{j=1}^{n-1} b_{ij} E(T_j),
\]

where

\[
b_{ij} = \int p_{ij}(r) f(r)dr \quad \text{and} \quad E(T_0) = 0.
\]

**Proof (see Appendix A.)**

The expected downtime:

Let \( D_i \) denote the total down time cost incurred until all machines are working if repair is about to commence and there are currently \( i \geq 1 \) broken machines. Suppose that the first machine takes \( r \) time units to repair. Then the down time cost during the period of length \( r \) consists of two components:

Downtime cost due to machines broken at the beginning of the period:

The downtime cost incurred from the \( i \) machines broken at the beginning of the period which is equal to:
\[ \frac{1c_d}{d} r \]  

PROPOSITION 2.2:

The downtime cost incurred from machines that break down within the next \( r \) time units given that \( (j-1+1) \) of the \((n-1)\) working machines break down is given by

\[ \frac{(j-1+1) c_d}{\lambda} \left[ \frac{r}{1 - e^{-\lambda r}} - \frac{1}{\lambda} \right] \]  

PROOF:

Using a basic probability concept we have the probability that the downtime of the \((j-1+1)\) failed machines is less than \( x \) given that \((T > 0, \) which is exponentially distributed with parameter \( \lambda )\) is less than \( r \) is given by

\[ \frac{(j-1+1) c_d}{\lambda} \int_0^r [ P \{ T < x | T < r \} ] \, dx = \]

\[ \frac{(j-1+1) c_d}{\lambda} \int_0^r \left[ \frac{1 - e^{-\lambda[r-x]}}{1 - e^{-\lambda r}} \right] \, dx = \]

\[ \frac{(j-1+1) c_d}{\lambda} \frac{1}{[1 - e^{-\lambda r}]} \int_0^r [1 - e^{-\lambda[r-x]}] \, dx = \]

\[ \frac{(j-1+1) c_d}{\lambda} \frac{1}{[1 - e^{-\lambda r}]} \left[ r - e^{-\lambda r} \int_0^r e^{\lambda x} \, dx \right] = \]

\[ \frac{(j-1+1) c_d}{\lambda} \frac{1}{[1 - e^{-\lambda r}]} \left[ r - e^{-\lambda r} \left[ (e^{\lambda r} - 1) / \lambda \right] \right] = \]

\[ \frac{(j-1+1) c_d}{\lambda} \left[ \frac{r - [(1 - e^{-\lambda r}) / \lambda]}{1 - e^{-\lambda r}} \right] = \]
\[(j-i+1) c_d \left[ \frac{r}{1-e^{-\lambda r}} - \frac{1}{\lambda} \right],\]

**PROPOSITION 2.3**

The expected value of \(D_i\), \(i \geq 1\), is given by

\[
E(D_i) = c_d [nE(R) + (n-1)\lambda^{-1}(E(e^{-\lambda R}) - 1)] + \sum_{j=1-1}^{n-1} b_{ij} E(D_j). \quad (5)
\]

**PROOF (see APPENDIX A.)**

Suppose a policy of starting repair when \(i\) units are broken is followed. Then the length of the renewal period is the time taken for all machines to be repaired once repair has been started added to the time for \(i\) machines to fail.

**The expected time to the \(i^{th}\) failure:**

The expected time to \(i\) failures is given by

\[
\beta(i) = \sum_{k=0}^{i-1} \frac{1}{(n-k)\lambda}. \quad (6)
\]

**The expected length of the renewal cycle:**

Hence the expected length of the renewal period is given by the following expression:

\[
\beta(i) + E(T_i). \quad (7)
\]

**The expected downtime while waiting for \(i\) machines to fail:**

The expected down time incurred while waiting for \(i\) machines to fail is given by:

\[
\gamma(i) = \sum_{k=0}^{i-1} \frac{k}{(n-k)\lambda}. \quad (8)
\]
The long run cost per unit time:

Using the renewal theorem, the long run cost per unit time of following the i-policy is given by:

$$K(i) = \frac{c_0 + E(D_i) + c_d \gamma(1) + [c_r + (c_r/E(R))]E(T_i)}{\beta(1) + E(T_i)}.$$  \hspace{1cm} (9)

Hence, the objective is to find $\min_i K(i)$ and the $i$ at which this occurs. The quantities in (9) can be found from expressions (1)-(8).

An important special case occurs when the repair time is deterministic. In this case, all of the previous integrals are degenerate and the coefficients of the linear equations are easily determined. As an illustration, the next example contains explicit formulas for the three machine case where the repair time is deterministic.

2.3 THE THREE MACHINE CASE

Suppose that $n=3$ and each repair takes a fixed $r$ time units for completion. Let:

$$p = e^{-r \lambda}$$

and

$$q = 1 - e^{-r \lambda}.$$  

The expected repair time values:

On solving the equations given in (1), the values of $E(T_i)$
are as follows:

\[ E(T_1) = (q^2+p) p^{-3} r, \]
\[ E(T_2) = (p^2+q^2+p) p^{-3} r, \]
\[ E(T_3) = (p^3+p^2+q^2+p) p^{-3} r. \]

The expected downtime values:

From expression (2), the values of \( E(D_1) \) are as follows:

\[ E(D_1) = (2p^2+3q^2+2pq+p)p^{-3} c_d r - (q^2+2p^2+2q^2)p^{-3} \lambda^{-1} c_d, \]
\[ E(D_2) = (5p^2+3q^2+2pq+p)p^{-3} c_d r - (q^2+3p^2+2pq^2)p^{-3} \lambda^{-1} c_d, \]
\[ E(D_3) = (3p^3+5p^2+3q^2+2pq+p)p^{-3} c_d r - (q^2+3p^2+2pq^2)p^{-3} \lambda^{-1} c_d. \]

The long run cost per unit time:

Using the above and some algebraic manipulation the values of \( K(1) \) are given by:

\[ K(1) = (\lambda c_0 + c_d + \phi(e^{3r\lambda} - e^{2r\lambda} + e^{r\lambda}) \{\frac{2}{6} + \lambda r(e^{3r\lambda} - e^{2r\lambda} + e^{r\lambda})\}^{-1} \]
\[ K(2) = (\lambda c_0 + \frac{5}{2} c_d + \phi(e^{3r\lambda} - e^{2r\lambda} + e^{r\lambda}) \{\frac{5}{6} + \lambda r(e^{3r\lambda} - e^{2r\lambda} + e^{r\lambda})\}^{-1} \]
\[ K(3) = (\lambda c_0 + \frac{11}{2} c_d + \phi(e^{3r\lambda} - e^{2r\lambda} + e^{r\lambda} + 1)) \{\frac{11}{6} + \lambda r(e^{3r\lambda} - e^{2r\lambda} + e^{r\lambda} + 1)\}^{-1}, \]

where

\[ \phi = r\lambda(3c_d + c_r) + \lambda c_f - c_d. \]

The optimal decision rule:

Assume that

\[ \min(K(1), K(2), K(3)) < 3c_d. \]
since otherwise it would not be profitable to operate the system.

By appropriately manipulating the \( K(i) \), the following decision rule can be obtained: wait for all machines to fail if

\[
c_d \leq \lambda(c_r + rc_r) + 3r\lambda c_0 \frac{2+2\lambda[E(T_3) - E(T_2)]}{11E(T_2) - 5E(T_3)};
\]

start repairing as soon as any machine fails if

\[
c_d > \lambda(c_r + rc_r) + 3r\lambda c_0 \frac{1+2\lambda[E(T_3) - E(T_1)]}{5E(T_1) - 2E(T_2)};
\]

start repairing as soon as exactly two machines fail if \( c_d \) does not satisfy any of the previous two inequalities.

For this example the decision rule indicates that the optimal policy is a nondecreasing function of \( c_0 \), \( c_r \), \( c_r \) and a nonincreasing function of \( c_d \). It will be shown in Section 2.5 that these statements remain true for the general problem.

2.4 THE UNIMODALITY OF THE OBJECTIVE FUNCTION

**Theorem 2.1**

The objective function \( K(i) \) is a unimodal function of \( i \).

**Proof:**

First some new notation will be introduced. Suppose there are exactly \( j+1 \) machines in the system and one of them has failed.
Define $E_j$ to be the expected time for all $j+1$ machines to be working if repair commences immediately. Then, the following relationship between the variable $T_k$ for the $n$-machine problem and the $E_j$ for the $(j+1)$-machine problems must hold:

$$E(T_k) = \sum_{j=1}^{k} E_{n-j}. \quad (10)$$

Define $C(i)$ to be the numerator in (5), i.e. $C(i)$ is the expected cost during a cycle if the $i$-policy is followed. Note that policy $i$ is better than policy $k$ if and only if

$$C(k)[\beta(k) + E(T_k)]^{-1} > C(i)[\beta(i) + E(T_1)]^{-1},$$

which can be rewritten as

$$C(k) - C(i) > K(i)[\beta(k) + E(T_k) - \beta(i) - E(T_1)]. \quad (11)$$

First it will be shown that if policy $i$ is better than policy $i+1$, then it must be better than all policies $k$ with $k \geq i+1$. The proof will be by induction. By assumption the statement is true for $k=i+1$. Assume that (7) is true for some $k=i+1$. The goal is to show that (11) is then true for $k+1$. Using the induction hypothesis:

$$C(k+1) - C(i) = [C(k+1) - C(k)] + [C(k) - C(i)]$$
$$> C(k+1) - C(k) + K(i)[\beta(k) + E(T_k) - \beta(1) - E(T_1)].$$
Comparing the right hand side of the above expression with the right hand side of (11) with \( k \) set equal to \( k+1 \), it is clear that the proof will be complete if it can be shown that

\[
C(k+1) - C(k) > K(1)[\beta(k+1) + E(T_{k+1}) - \beta(k) - E(T_k)]. \quad (12)
\]

If one waits for \( k+1 \) machines to fail rather than \( k \) before initiating repair, the extra costs consist of: the downtime costs incurred from \( k \) failed machines while waiting for another machine to fail; the downtime cost incurred from having one extra failed machine during the period \( T_k \); the total cost, excluding fixed cost, of bringing the system with one failed machine to a system with all \( n \) machines working. Therefore the left hand side of (12) can be written as

\[
C(k+1) - C(k) = k \ c_d (\beta(k+1) - \beta(k)) + c_d E(T_k) + [C(1) - c_0]. \quad (13)
\]

Now use (3), (11) and (13) to see that (12) is equivalent to

\[
(n-k)^{-1} \lambda^{-1}[\lambda c_d - K(1)] + c_d E(T_k) + C(1) - c_0 > K(1) E_{n-k-1}. \quad (14)
\]

Since policy 1 is assumed to be better than policy \( i+1 \), expression (11) is true when \( k \) is set equal to \( i+1 \). Comparing this with (12) it is clear that (12) and hence (14) are true for \( k=1 \). The left hand side of (14) increases in \( k \) while the right hand side decreases. Therefore (14) is true for all \( k \geq i+1 \) and the proof is
Now suppose that action 1 is better than action 1-1. The goal is to show that this shows that action 1 is better than action k for all k=1-1. Again the proof will be by induction. By assumption the statement is true for k=1-1. Assume it is true for some k=1-1. The goal is to show that (11) holds for k-1. Proceeding as before, the proof will be complete if it can be shown that

\[ C(k-1) - C(k) > K(i)[\beta(k-1) + E(T_{k-1})] - \beta(k) - E(T_k) \]

which can be written as

\[ \frac{(n-k+1)^{-1}}{\lambda^{-1}} [(k-1)c_d - K(i)] + c_d E(T_{k-1}) + C(1) - C_0 < K(i) E_{n-k}. \] (15)

By assumption, the above expression is satisfied at k=1. The proof is completed on noting that, as k decreases, the left hand side of (15) decreases while the right hand side increases.

From the above, if policy 1 is better than policies 1-1 and 1+1, it must be an optimal policy. Thus the function K(i) can turn no more than once.

2.5 BEHAVIOR OF OPTIMAL ACTIONS AS A FUNCTION OF THE COST PARAMETER

THEOREM 2.2

The optimal policy is a nondecreasing function of \( c_0, c_r, c_r \) and a nonincreasing function of \( c_d \).
**Proof:**

Let $K(1)$ denote the objective function when the cost parameters equal $c_0$, $c_d$, $c_r$ and $c_r$. Let $K_\delta(1)$ denote the cost function if the fixed cost $c_0$ increases to $c_0+\delta$ while all other costs are held fixed. Then, using (9),

$$K_\delta(1) = K(1) + \delta[\beta(1) + E(T_1)]^{-1}$$

(16)

Suppose $i^*$ is optimal for the problem with fixed cost $c_0$. Then, using (16), the fact that the last term on the right hand side of (16) is a decreasing function of $i$ and the assumption that $K(1) \geq K(1^*) \forall i$, shows that

$$K_\delta(1) \geq K_\delta(1^*) \text{ for all } i \leq i^*.$$

Thus the optimal policy when the fixed cost is $c_0+\delta$ can be no smaller than the optimal policy for the problem with fixed cost equal to $c_0$, i.e. the optimal policy is a nondecreasing function of $c_0$.

Now the relationship between optimal policies and the downtime cost $c_d$ will be investigated. The goal is to show that if policy $i-1$ is better than policy $i$ when the fixed cost is $c_d$, then it is also better when the fixed cost is $c_d+\delta$, where $\delta \geq 0$. This combined with the unimodality of the cost function will show that the optimal policy is a nonincreasing function of $c_d$. Policy
1-1 is better than policy i if and only if

\[
\frac{c_0 + \left[ c_r + \left( c_f / E(R) \right) \right] E(T_{i-1})}{\beta (i-1) + E(T_{i-1})} > \frac{c_0 + \left[ c_r + \left( c_f / E(R) \right) \right] E(T_i)}{\beta (1) + E(T_i)}
\]

\[
< c_d \left\{ \frac{E(X_i)}{\beta (1) + E(T_i)} - \frac{E(X_{i-1})}{\beta (i-1) + E(T_{i-1})} \right\},
\]  

(17)

where \( X_j \) denotes the sum of the downtimes for all machines during a cycle where the \( j \)-policy is followed. Assume that (17) is true.

The goal is to show that this implies that (17) must be true when \( c_d \) is replaced by \( c_d + \delta \), where \( \delta \geq 0 \). If the right hand side of (17) is negative then (17) must also hold true for all downtime cost rates between \( 0 \) and \( c_d \). This leads to a contradiction since policy 1-1 can never be better than policy i if the downtime cost rate is 0. Therefore the right hand side of (17) must be positive, which implies that (17) also holds whenever \( c_d \) is replaced by any larger value.

Now the effect of varying \( c_r \) and \( c_f \) will be considered. Policy i is at least as good as 1-1 if and only if

\[
\frac{c_0 + c_d E(X_{i-1})}{\beta (i-1) + E(T_{i-1})} < \frac{c_0 + c_d E(X_i)}{\beta (1) + E(T_i)}
\]

\[
= \left( c_r + \frac{c_f}{E(R)} \right) \left\{ \frac{E(T_i)}{\beta (1) + E(T_i)} - \frac{E(T_{i-1})}{\beta (i-1) + E(T_{i-1})} \right\}
\]

(18)

Assume that (18) is true. The goal is to show that (18) remains
true whenever $c_r$ and $c_f$ are replaced with larger values. If the right hand side of (18) is negative then the result clearly follows. So suppose the right hand side of (18) is positive. In that case, if (18) is not true when $c_r$ and $c_f$ are replaced by $c_r + \delta_1$ and $c_f + \delta_2$, for some nonnegative $\delta_1$ and $\delta_2$, then

$$\frac{c_0 + c_d E(x_{i-1})}{\beta(1-1)+E(T_{i-1})} - \frac{c_0 + c_d E(x_i)}{\beta(1)+E(T_i)}$$

$$< (x+y/E(R)) \left\{ \frac{E(T_i)}{\beta(1)+E(T_i)} - \frac{E(T_{i-1})}{\beta(1-1)+E(T_{i-1})} \right\}, \quad (19)$$

for all $x > c_r + \delta_1$ and for all $y > c_f + \delta_2$. This however is a contradiction since for sufficiently large repair costs it must always be the case that policy $i$ will be better than policy $i-1$. Thus the optimal action is a nondecreasing function of the parameters $c_r$ and $c_f$.

2.6 ALGORITHM FOR COMPUTING OPTIMAL POLICIES

PROPOSITION 2.4

(1) \[ E(T_n) = f_n^{-1} E(R) \sum_{i=1}^{n} (-1)^{n-i} f_i. \quad (20) \]

(11) \[ E(D_n) = f_n^{-1} c_d \sum_{i=1}^{n} (-1)^{n-i} f_i \{ nE(R) + \lambda^{-1}(n-1)[E(e^{-\lambda R}) - 1] \} \]

(11) Starting with $f_1 = 1$, the $f_i$'s ($i = 2, 3, \ldots, n$) are given by the following recursion:
\[ f_{k+1} = b_{k+1,k}^{-1} \left\{ (b_{k,k}-1)f_k - \sum_{i=1}^{k-1} (-1)^{k-1-i} b_{i,k} f_i \right\} \]  \hspace{1cm} (22)

**Proof** (see Appendix A.)

**Algorithm:**

**Step 1**

Compute \( E[R], E[e^{-RA}] \) and \( b_{ij} \).

**Step 2**

Set \( f_1 = 1 \) and compute \( f_2, \ldots, f_n \) from the following recursion:

\[ f_{k+1} = b_{k+1,k}^{-1} \left\{ (b_{k,k}-1)f_k - \sum_{i=1}^{k-1} (-1)^{k-1-i} b_{i,k} f_i \right\}. \]

Then

\[ E(T_n) = f_n^{-1} E(R) \sum_{i=1}^{n} (-1)^{n-1} f_i \]  \hspace{1cm} (23)

and

\[ E(D_n) = f_n^{-1} c_d \sum_{i=1}^{n} (-1)^{n-1} f_i \{ nE(R) + \lambda^{-1}(n-1)[E(e^{-\lambda R})-1] \}. \]  \hspace{1cm} (24)

**Step 3**

Using (2), (5), (6), (8) and (9) and the above expressions for \( E(T_n) \) and \( E(D_n) \), sequentially compute \( K(n), K(n-1), \ldots, \) until \( K(i-1) > K(i) \) for some \( i \). This value for \( i \) is optimal. (If \( K(i-1) \leq K(i) \) \( \forall i \), then the 1-policy is optimal.)

Expressions (23) and (24) above are simply the results of solving (2) and (5) for \( E(T_n) \) and \( E(D_n) \) by successively eliminating the equations for \( i=1, 2, \ldots \). From the unimodality result of Section 3 the above algorithm will always work. Each step of the algorithm
requires little extra computation since values of $E(D_i)$ and $E(T_i)$ will be retained from previous steps. An algorithm that does not search through policies according to their order will be inefficient since calculation of the cost function at a given policy requires knowledge of all expected downtimes and times to completion of repair for all policies greater than the one being considered.

2.7 EXAMPLE WITH GAMMA $(\alpha, \beta)$ REPAIR TIME DISTRIBUTION

Suppose that $R$ has a gamma distribution with parameters $\alpha$ and $\beta$, i.e. the density is given by

$$f(r) = \frac{\beta^\alpha}{\Gamma(\alpha)} r^{\alpha-1} e^{-\beta r}, \quad r > 0.$$

**PROPOSITION 2.5**

The $b_{ij}$'s are given by the following expressions:

$$b_{ij} = \frac{(n-1)!}{(n-j-1)!} \beta^\alpha \sum_{k=0}^{j+1-i} (-1)^k [(n+k-j-1)\lambda+\beta]^{-\alpha} [k! (j+1-i-k)!]^{-1},$$

and

$$E[e^{-R\lambda}] = \left( \frac{\beta}{\beta+\lambda} \right)^\alpha.$$

**PROOF (see APPENDIX A.)**

**Values of the parameters:**

Suppose the parameters take the following values:

$$c_0 = 20.00, \quad c_d = 0.87.$$
\( c_r = \$1.00, \)
\( c_r = \$2.00, \)
\( \lambda = 0.1, \)

and

\[ N = 10. \]

Assume that the time taken to repair a single machine is a gamma random variable with parameters \( \beta = 0.1 \) and \( \alpha \). In this case \( E(R) = 10\alpha. \)
GRAPH OF THE OPTIMAL ACTIONS

Average cost per unit time.

FIGURE 2.1
2.8 RESULTS AND CONCLUSION.

The lower curve of Figure 1 shows the cost associated with the optimal policy as a function of $E(R)$. If repair time were instantaneous, the optimal policy using Assaf and Shanthikumar's algorithm would be to repair the system when seven machines break down. The top curve in Figure 1 shows the cost per unit time associated with this policy as a function of $E(R)$. It is clear that the optimal policy computed using the algorithm in this section can result in significantly lower costs compared to the 7-policy.
2.9 APPENDIX

PROOF (of proposition 2.1)

The expected value of $T_{i}, i \geq 1$ is given by

$$E(T_{i}) = \left\{ \sum_{j=1-1}^{n-1} p_{ij}(r)(r + E(T_{j})) \right\} f(r)dr$$

which can be written as follows

$$= \sum_{j=1-1}^{n-1} \left\{ p_{ij}(r) r f(r) dr + \sum_{j=1-1}^{n-1} \left[ \int p_{ij}(r) f(r) dr \right] E(T_{j}) \right\}$$

where the first expression is just the expected value of the random variable $R$ noted by $E(R)$, and the second expression is the summation over all $E(T_{j})$ weighted by the $b_{ij}$'s.

Hence

$$E(T_{i}) = E(R) + \sum_{j=1-1}^{n-1} b_{ij} E(T_{j})$$

which is the claimed result. □

PROOF (of proposition 2.3)

The expected value of $D_{1}, i \geq 1$ is given by

$$E(D_{1}) = \left\{ \sum_{j=1-1}^{n-1} p_{ij}(r)[1_{c_{d}}r + (j-i+1)c_{d}\frac{r}{1-e^{-\lambda r}} - \frac{1}{\lambda}] + E(D_{i}) \right\} f(r)dr$$

$$= 1_{c_{d}} \sum_{j=1-1}^{n-1} \left\{ p_{ij}(r) r f(r) dr + \right\}$$

$$\left\{ \sum_{j=1-1}^{n-1} [j-i+1]p_{ij}(r)\left(\frac{r}{1-e^{-\lambda r}} - \frac{1}{\lambda}\right)f(r)dr + \right\}$$
\[
\sum_{j=1-1}^{n-1} \left[ \int p_{ij}(r) f(r) dr \right] E(D_j)
\]

Note that the down time costs incurred after the present period depend only on the number of failed machines and not the length of the first repair period. On noting that:

\[
\sum_{j=1-1}^{n-1} [j-i+1]p_{ij}(r) = (n-i)[1-e^{-\lambda r}]
\]

(which is just the mean of the binomial distribution with parameters \( n = n-i \) and \( p = 1-e^{-\lambda r} \).)

and

\[
\begin{align*}
&c_d \sum_{j=1-1}^{n-1} \left[ \int p_{ij}(r) [j-i+1] \left[ \frac{r}{1-e^{\lambda r}} - \frac{1}{\lambda} \right] f(r) dr = \\
&c_d \int (n-i)[1-e^{\lambda r}][\frac{r}{1-e^{\lambda r}} - \frac{1}{\lambda}]f(r) dr = \\
&c_d \int (n-i)[r - \lambda^{-1}(1-e^{\lambda r})]f(r) dr = \\
&c_d \int (n-i)[r - \lambda^{-1} + \lambda^{-1}e^{\lambda r}]f(r) dr = \\
&c_d(n-i)E(R) - c_d(n-i)\lambda^{-1} + c_d(n-i)\lambda^{-1} E(e^{-\lambda R})
\end{align*}
\]

\[
\begin{align*}
&c_d \sum_{j=1-1}^{n-1} \int p_{ij}(r) rf(r)dr = ic_dE(R)
\end{align*}
\]

collecting terms from a, b and c the following is obtained

\[
E(D_1) = c_d \left[ i E(R) + (n-i)E(R) - (n-i)\lambda^{-1} + (n-i)\lambda^{-1} E(e^{-\lambda R}) \right]
\]
\[ \sum_{j=1}^{n-1} b_{ij} E(D_j) \]

by simplifying the above the final expression for \( E(D_1) \) is as follows

\[ E(D_1) = n c_d E(R) - (n-1) \lambda^{-1} c_d [1 - E(e^{-\lambda R})] + \sum_{j=1}^{n-1} b_{ij} E(D_j) \quad \square \]

**Proof (of proposition 2.3)**

We know that:

\[ E(T_i) = E(R) + \sum_{j=1}^{n-1} b_{ij} E(T_j) i = 1, \ldots, n \]

\[ E(T_0) = 0. \]

writing the above system of equations more explicitly would give the following:

\[
\begin{align*}
E(T_1) &= E(R) + b_{11} E(T_1) + b_{12} E(T_2) + b_{13} E(T_3) + \ldots + b_{1n-1} E(T_{n-1}) + 0. E(T_n) \\
E(T_2) &= E(R) + b_{21} E(T_1) + b_{22} E(T_2) + b_{23} E(T_3) + \ldots + b_{2n-1} E(T_{n-1}) + 0. E(T_n) \\
& \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
E(T_{n-1}) &= E(R) + b_{n-1,n-1} E(T_{n-1}) + 0. E(T_n) \\
E(T_n) &= E(R) + b_{n,n-1} E(T_n) \\
\end{align*}
\]

(S1)

For a matter of simplicity have the following renaming of some parameters:

\[ E(T_i) = X_i i = 1, \ldots, n \]

and

\[ E(R) = Y_1. \]

by replacing and rearranging the above (S1) the following is
obtained:

\[(b_{11} - 1)X_1 + b_{12} X_2 + b_{13} X_3 + \ldots + b_{1,n-2} X_{n-2} + b_{1,n-1} X_{n-1} + 0 \cdot X_n + Y_1 = 0.\]

\[b_{21} X_1 + (b_{22} - 1)X_2 + b_{23} X_3 + \ldots + b_{2,n-2} X_{n-2} + b_{2,n-1} X_{n-1} + 0 \cdot X_n + Y_2 = 0.\]

\[b_{32} X_2 + (b_{33} - 1)X_3 + \ldots + b_{3,n-2} X_{n-2} + b_{3,n-1} X_{n-1} + 0 \cdot X_n + Y_3 = 0.\]

\[b_{43} X_3 + \ldots + b_{4,n-2} X_{n-2} + b_{4,n-1} X_{n-1} + 0 \cdot X_n + Y_4 = 0.\]

\[\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \]

\[b_{n-1,n-2} X_{n-2} + (b_{n-1,n-1} - 1)X_{n-1} + 0 \cdot X_n + Y_{n-1} = 0.\]

\[b_{n,n-1} X_{n-1} + 0 \cdot X_n + Y_n = 0.\]

\[(S_2).\]

by multiplying the first equation by \((-1)\) and the second equation by \(b_{21}^{-1}(b_{11} - 1)\) of \((S_2)\) and adding both of them the following is obtained:

a. The coefficient of \(X_1\) is zero, which means that the variable \(X_1\) does not exist.

b. By having \(f_1 = 1.\)

and \(f_2 = b_{21}^{-1}(b_{11} - 1)\)

the following result is obtained:

\[\left( f_2 (b_{22} - 1) - b_{12} f_1 \right) X_2 + \left( f_2 (b_{23} - b_{13}) \right) X_3 + \ldots + \left( f_2 b_{2,n-1} - f_1 b_{1,n-1} \right) X_{n-1} +
\]

\[0 \cdot X_n + \left[ f_2 Y_2 - f_1 Y_1 \right] = 0. \quad (R_1).\]

Now the same process is used the result given by \((R_1)\) and the third equation of \((S_2)\) are the new equation that we have to play with them in order to eliminate the variable \(X_2.\)

By multiplying \((R_1)\) by \((-1)\) and the third equation of \((S_2)\)
by \( f_3 = b_{32}^{-1}(f_2(b_{22} - 1) - b_{12}f_1) \)

\[
[f_3(b_{33} - 1) - b_{23}f_2 - b_{13}f_1]X_3 + \ldots + [f_3b_{3,n-1} - f_2b_{2,n-1} + f_1b_{1,n-1}]X_{n-1} + 0.X_n + [f_3Y_3 - f_2Y_2 + f_1Y_1] = 0. \quad (R_2).
\]

Note that \((R_2)\) can be written as follows:

\[
[f_3(b_{33} - 1) + \sum_{i=1}^{3} (-1)^{3-i-1}[ -f_{1i}b_{i3}]X_3 + \ldots + \left[3 \sum_{i=1}^{3} (-1)^{3-i-1}b_{i,n-1}f_i \right]X_{n-1} + 0.X_n + \left[3 \sum_{i=1}^{3} (-1)^{3-i-1}f_iY_i \right] = 0. \quad (R'_2).
\]

Let's assume this is true for \(k\) which means that:

\[
[f_k(b_{kk} - 1) + \sum_{i=1}^{k} (-1)^{k-i-1}[ -f_{1i}b_{i,k}]X_k + \ldots + \left[k \sum_{i=1}^{k} (-1)^{k-i-1}b_{i,n-1}f_i \right]X_{n-1} + 0.X_n + \left[k \sum_{i=1}^{k} (-1)^{k-i-1}f_iY_i \right] = 0. \quad (R_{k-1}).
\]

where

\( f_1 = 0. \)

and

\( f_j = b_{j, j-1}^{-1}\left[ (b_{j-1,j-1} - 1)f_{j-1} + \sum_{i=1}^{j-1} (-1)^{j-1-i}b_{i,j-1}f_i \right] \)

for \(j = 1, \ldots, k\)

Let's prove it's true for \(k+1\), in other word the result obtained by multiplying \((R_{k-1})\) by \((-1)\) and the \((k+1)\)nd equation of \((S_2)\) by the following:

\[
f_{k+1} = b_{k+1,k}^{-1}\left[ (b_{k+1,k} - 1)f_k - \sum_{i=1}^{k-1} (-1)^{k-1-i}b_{i,k}f_i \right]
\]

Note that the result of such addition eliminates the variable \(X_k\) and the following is obtained:
\[
\left[ (b_{k+1,k+1}^{-1} f_{k+1} - \sum_{i=1}^{k} (-1)^{k-1} b_{i+1,k+1} f_{i+1} \right] x_{k+1} + \ldots + \\
\left[ f_{k+1} b_{k+1,n-1} + \sum_{i=1}^{k} (-1)^{k-1} f_{i+1,n-1} \right] x_{n-1} + 0. x_n + \\
\left[ f_{k+1} y_{k+1} - \sum_{i=1}^{k} (-1)^{k-1} f_{i+1} y_{i+1} \right] = 0. \quad (R_k)
\]

by rearranging the third term of the lefthand side of \((R_k)\) the following is obtained:

\[
\left[ (b_{k+1,k+1}^{-1} f_{k+1} - \sum_{i=1}^{k} (-1)^{k-1} b_{i+1,k+1} f_{i+1} \right] x_{k+1} + \ldots + \\
\left[ f_{k+1} b_{k+1,n-1} + \sum_{i=1}^{k} (-1)^{k-1} f_{i+1,n-1} \right] x_{n-1} + 0. x_n + \\
\left[ \sum_{i=1}^{k} (-1)^{k-1} f_{i+1} y_{i+1} \right] = 0. \quad (R_k').
\]

Therefore to solve for \(x_n, x_{n-1}\) must be eliminated, and this can be done by looking to \((R_{n-1})\). In other words the index \(k\) must be replaced by the index \((n-1)\). Hence the following is obtained:

\[
\left[ (b_{n,n}^{-1} f_n - \sum_{i=1}^{n-1} (-1)^{n-1-i} b_{i+1,n} f_{i+1} \right] x_n + \\
\left[ \sum_{i=1}^{n} (-1)^{n-1} f_{i+1} y_{i+1} \right] = 0. \quad (R_{n-1}).
\]

\[x_n = \left[ (b_{n,n}^{-1} f_n - \sum_{i=1}^{n-1} (-1)^{n-1-i} b_{i+1,n} f_{i+1} \right]^{-1} \left[ \sum_{i=1}^{n} (-1)^{n-1} f_{i+1} y_{i+1} \right].\]

(i) To get the expression of \(E(T_n)\) just replace \(x_n\) by \(E(T_n)\) and \(y_{i+1}\) by \(E(R)\).

(ii) To get the expression of \(E(D_n)\) just replace \(x_n\) by \(E(D_n)\) and \(y_{i+1}\) by \(F(1)\).

(iii) The recursion of \(f_k\) is given by expression \((R_k)\). \(\Box\)
PROOF (of proposition 2.4):

Given that the repair time $R$ is gamma distributed with parameter $\alpha$ and $\beta$ which is given by

$$f(r) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} r^{\alpha-1} e^{-\beta r} & r > 0. \\ 0. & \text{otherwise.} \end{cases}$$

and we know that

$$b_{1j} = \int_{0}^{\infty} p_{1j}(r) f(r) \, dr$$

where

$$p_{1j}(r) = \begin{cases} \left(\frac{n-1}{j+1-1}\right) e^{-(n-j-1)r\lambda} (1-e^{-r\lambda})^{j+1-1} & 1-1 \leq j \leq n-1. \\ 0 & \text{otherwise.} \end{cases}$$

Hence bringing everything together the following is obtained

$$b_{1j} = \int_{0}^{\infty} \left(\frac{n-1}{j+1-1}\right) e^{-(n-j-1)r\lambda} (1-e^{-r\lambda})^{j+1-1} \frac{\beta^\alpha}{\Gamma(\alpha)} r^{\alpha-1} e^{-\beta r} \, dr$$

$$= \left(\frac{n-1}{j+1-1}\right) \frac{\beta^\alpha}{\Gamma(\alpha)} \int_{0}^{\infty} e^{-(n-j-1)r\lambda} (1-e^{-r\lambda})^{j+1-1} r^{\alpha-1} e^{-\beta r} \, dr$$

It is known for a fact that:

$$(1-x)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k$$

Hence

$$(1-e^{-r\lambda})^{j+1-1} = \sum_{k=0}^{j+1-1} (-1)^k \binom{j+1-1}{k} (e^{-r\lambda})^k$$
\[ j+1-1 \sum_{k=0}^{j+1-1} (-1)^k \binom{j+1-1}{k} e^{-r(k\lambda)} \]

Therefore the following is obtained:

\[ b_{ij} = \left( \frac{n-1}{j+1-1} \right) \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \sum_{k=0}^{j+1-1} (-1)^k \binom{j+1-1}{k} e^{-r(k\lambda)} e^{-(n-j-1)\lambda + \beta} \beta r \ dr \]

\[ = \left( \frac{n-1}{j+1-1} \right) \frac{\beta^\alpha}{\Gamma(\alpha)} \sum_{k=0}^{j+1-1} (-1)^k \binom{j+1-1}{k} \int_0^\infty r^{\alpha-1} e^{-(n+k-j-1)\lambda + \beta} \beta r \ dr \]

the integral of such expression is known to be a gamma distribution with parameter \( \alpha \) and \( \beta = [(n+k-j-1)\lambda + \beta] \) therefore the \( b_{ij} \)'s are equal to:

\[ = \left( \frac{n-1}{j+1-1} \right) \frac{\beta^\alpha}{\Gamma(\alpha)} \sum_{k=0}^{j+1-1} (-1)^k \binom{j+1-1}{k} \frac{\Gamma(\alpha)}{(n+k-j-1)\lambda + \beta} \]

by simplifying \( \Gamma(\alpha) \) in the numerator and denominator we obtain:

\[ = \left( \frac{n-1}{j+1-1} \right) \frac{\beta^\alpha}{\sum_{k=0}^{j+1-1} (-1)^k \binom{j+1-1}{k} \left[ \frac{\beta}{(n+k-j-1)\lambda + \beta} \right]^{\alpha}} \]

note that

\[ \binom{n-1}{j+1-1} \binom{j+1-1}{k} = \frac{(n-1)!}{(j+1-1)! (n-j-1)! (k)! (j+1-1-k)!} \]

also by simplifying the above the following is obtained:

\[ = \frac{(n-1)!}{(k)! (n-j-1)! (j+1-1-k)!} \]

Thus:
\[ b_{ij} = \begin{cases} \frac{(n-1)!}{(n-j-1)!} \sum_{k=0}^{j+1-1} (-1)^k \frac{1}{(k)!} \frac{1}{(j+1-1-k)!} \left[ \frac{\beta}{(n+k-j-1)\lambda + \beta} \right]^\alpha & \text{for } 1-1 < j < n-1 \\ 0 & \text{Otherwise.} \end{cases} \]

which gives the final result.

The other expression that needs to be proved is \( E(e^{-\lambda R}) \)

\[ E(e^{-\lambda R}) = \int_0^\infty e^{-\lambda r} \frac{\beta^\alpha}{\Gamma(\alpha)} r^{\alpha-1} e^{-\beta r} \, dr \]

\[ = \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} r^{\alpha-1} e^{-[\beta+\lambda]r} \, dr \]

\[ = \left( \frac{\beta}{\beta+\lambda} \right)^\alpha \]
2.10 REFERENCES


CHAPTER III

BAYESIAN GROUP REPLACEMENT POLICIES
3.1 INTRODUCTION AND MODEL DESCRIPTION

Consider a parallel system of n machines whose failure times are independent identically distributed random variables. If n=1, an m-failure policy is defined to be the strategy where the current system is replaced after it has failed for the m\textsuperscript{th} time. For n>1 define an m-failure policy to be the strategy that calls for replacement of all n machines as soon as the m\textsuperscript{th} failure occurs. These policies have received much attention in the literature. Generally it is assumed that failed machines incur downtime costs and a fixed cost applies whenever a system is replaced. Salvage values and repair and replacement costs may also be involved. All parameters of the failure time distribution are generally assumed to be known. This, combined with the assumption that the decision maker will want to operate the system over a long horizon, has made minimising the average cost per unit time the preferred objective in much of the literature. For examples of this approach see Rade (1976), Nakagawa (1979), Okumoto and Elsayed (1983), Assaf and Shanthikumar (1987), and Wilson and Benmerroug (1990). Many authors have also considered T-age policies which calls for replacing the system after T time units have elapsed (see, e.g., Barlow and Hunter (1960), Boland (1982), Boland and Proschan (1982)). A combination policy which calls for replacement at time T or after the system has experienced m failures has been analysed by Nakagawa (1983), Yeh (1988), and Ritchken and Wilson (1990).
A common feature in the work mentioned above is the assumption that the parameters of the failure time distribution are known with certainty. No statistical learning about the nature of the failure time distribution is allowed. Suppose for instance that n=50 and a 40-failure policy is being used. This policy treats the situation where all machines work perfectly for year and 40 fail on day 366 as equivalent to the situation where 39 machines fail on the first day of operation and the next failure occurs on day 366. Most decision makers would want to treat these two situations in a different manner. This chapter investigates the problem of integrating the statistical information obtained as the machines are operating with group replacement strategies. The policies mentioned above have the advantage of being reasonably intuitive and easy to implement. Given the practical nature of this problem, any class of policies that utilise the failure time data should also have these properties. Such a class of policies will be introduced in Section 2. Other approaches to incorporating statistical estimation in reliability problems have generally not taken explicit account of the cost minimisation aspects of the problem. For instance, there are Bayesian approaches for estimating the reliability of multicomponent systems (see, e.g., Martz, Waller and Fickas (1988)). Systems of components can often be analysed using queueing theory methodology. McGrath, Gross and Singpurwalla (1987) provide an analysis of queueing systems that incorporates estimation of the underlying parameters.
In this chapter it will be assumed that the failure times of each machine are independent identically distributed random variables. This chapter considers the case where the failure time distributions are i.i.d. exponential random variables with unknown parameter $\lambda$. Optimality results and algorithms have only recently appeared for the case of known $\lambda$ — see Assaf and Shanthikumar (1987). So this seems an appropriate assumption with which to start an investigation of Bayesian policies. The objective function, as for the papers above, will be the expected cost per unit time.

Assume that $n$ machines are operating in parallel and that their failure times are independent identically distributed exponential random variables with parameter $\lambda$. Assume further that these $n$ machines are all bought at one time from the manufacturer. The manufacturer provides quality control information about the machines. In practice, it is rarely the case that precise values of the failure time distribution are known. Instead the supplier provides statistical information about the lifetime. In particular, assume that $\pi(\lambda)$ is the density function for the distribution of parameter values from batch to batch. This type of assumption would be particularly appropriate for the purchase of items such as electronic components. Since the quality control standards of the supplying company are assumed constant, the same density function $\pi(\lambda)$ applies whenever machines are purchased.

Each time a batch of $n$ machines is purchased, a fixed cost of
$c_0$ is incurred. This fixed cost includes the cost of buying $n$ new machines and other costs, such as paper work etc., involved in the purchase. The salvage value of $i$ functioning but used machines is given by the function $c_s(i)$. Each failed machine is assumed to have zero salvage value. Each failed machine incurs a downtime cost of $c_d$ per unit time.

Section 2 introduces a new class of policies that calls for replacement whenever the posterior distribution of the failure time parameter indicates that the system is unreliable. It is shown how these policies can be characterised in terms of the sufficient statistics. The problem is formally defined in Section 3. The approach taken here has the advantage of being straightforward and intuitive. However, the approach in this section is only useful for very small values of $n$ due to the complexity of the multidimensional integrals that are involved. The case of two machines with a gamma prior is considered in Section 4. Here the decision rule takes a particularly appealing form: replace both components if the first failure occurs before a certain time; otherwise wait for both to fail. Section 5 contains the results needed to extend the approach to larger values of $n$. It is shown that, for a fixed value of $\lambda$, the preposterior likelihoods of the sufficient statistics are linear combinations of "shifted" gamma densities. Building on this result it is shown that the preposterior probability of replacement occurring after the $k^{th}$ failure is the expectation with respect to the prior distribution of a polynomial in $\lambda$ times
exp(-λ) to the power of a known constant. These results are used in Section 6 to arrive at expressions for the various expected values that only involve one dimensional integrals with straightforward regions of integration.

3.2 POLICIES BASED ON SUFFICIENT STATISTICS

Let \( \tau_i \) denote the time of the \( i^{th} \) machine to fail. Suppose the \( i^{th} \) machine has just failed. Then the likelihood function for the data is given by

\[
\lambda^i e^{-\lambda \tau_i - \lambda \sum_{j=1}^{i-1} \tau_j} e^{-\lambda (n-i) \tau_i} \quad (1)
\]

Thus, at the time of the \( i^{th} \) failure,

\[
s_i = (n-i+1) \tau_i + \sum_{j=1}^{i-1} \tau_j \quad (2)
\]

is a sufficient statistic for the parameter \( \lambda \). Thus, this quantity and the fact that \( m \) machines have failed are the crucial quantities on which decisions should be based. The smaller the value of the sufficient statistic, the more evidence there is that the system is unreliable. The posterior mean for \( \lambda \) at the time of the \( i^{th} \) failure is a function of the sufficient statistic. The larger this posterior mean, the more evidence that the system is
unreliable. Thus, it makes sense to replace the system whenever this quantity is large.

For a given \( t = 0 \) and \( m \in \{1, \ldots, n\} \), define the \((t, m)\) policy as follows: for \(1 \leq i < m\) replace the system at the \( i^{th}\) failure if \( E[\lambda | s_i] > t\); otherwise replace the system at the \( m^{th}\) failure. On putting \( t = \infty\), the class of \((t, m)\) policies is seen to contain the class of \(m\)-failure policies as a special case. Besides being more general than \(m\)-failure policies, \((t, m)\) policies incorporate the decision maker's subjective feelings about the reliability of the system in a very natural and intuitive way.

Suppose that the prior density function is a point mass at a specific \( \lambda \) value. In this case, the parameter of the failure time distribution is assumed to be known with absolute certainty. Thus, from the theory of Markov decision processes, the optimal \((t, m)\) policy occurs when \( t = \infty\). This stationary policy case is the one considered by Assaf and Shanthikumar (1987). For non-degenerate prior distributions, the optimal \((t, m)\) policy can occur at a non-zero value of \( t\).

The following theorem provides an alternative description for the class of \((t, m)\) policies. This new description makes \((t, m)\) policies more amenable to mathematical analysis.

**Theorem 3.1**

For each \((t, m)\) policy there exists a nondecreasing sequence of numbers \(\{a_i(t)\}_{i=1}^{m-1}\) such that the \((t, m)\) policy is equivalent to the following decision rule: for \(1 \leq i < m\) replace the system at the
ith failure if \( s_i < a_i(t) \); otherwise replace the system at the \( m \)th failure.

Proof:

Define the sequence \( \{b_i(t)\}_{i=1}^{m-1} \) by

\[
b_i(t) = \inf\{s: E[\lambda|s_1=s] \leq \int_0^t \frac{\lambda^{i+1}e^{-\lambda s} \pi(\lambda)d\lambda}{\int_0^\infty \lambda e^{-\lambda s} \pi(\lambda)d\lambda} \leq t\}.
\]

The first part of the proof will be to show that

\[E[\lambda|s_1=s] \leq t \text{ whenever } s \geq b_i(t).\]

Note first that \( E[\lambda|s_1=s] \leq t \) if and only if \( D(s) \geq 0 \) where \( D(s) \) is defined by

\[D(s) = \int_0^\infty \lambda^{i+1}e^{-\lambda s} (t-\lambda) \pi(\lambda)d\lambda.\]

On differentiating the above with respect to \( s \), splitting up the resulting integral, bounding \( \lambda \) and recalling the definition of \( D(s) \), the following is obtained:

\[
\frac{d}{ds}(D(s)) = -\int_0^t (\lambda^{i+1}e^{-\lambda s} (t-\lambda) \pi(\lambda)d\lambda + \int_t^\infty (\lambda^{i+1}e^{-\lambda s} (\lambda-t) \pi(\lambda)d\lambda
\]

\[
= -t \int_0^t \lambda^{i+1}e^{-\lambda s} (t-\lambda) \pi(\lambda)d\lambda + t \int_t^\infty \lambda^{i+1}e^{-\lambda s} (\lambda-t) \pi(\lambda)d\lambda
\]

\[= -t D(s).\]
Hence,
\[
\frac{d}{ds} \left( e^{ts}D(s) \right) = 0,
\]
which, after integrating between \( b_1(t) \) and \( s \), implies that \( D(s) \) satisfies \( \exp(ts)D(s) = \exp(tb_1(t))D(b_1(t)) + (s-b_1(t)) \). Therefore \( D(s) \geq 0 \) when \( s \geq b_1(t) \) since, by definition of \( b_1(t) \), \( D(b_1(t)) \) is nonnegative.

If the \( b_i(t) \) are nondecreasing in \( i \), set \( a_i(t) = b_i(t) \) and the theorem is proved. Now suppose that \( b_i(t) > b_{i+1}(t) \) for some \( i \). Then
\[
s_i \geq b_i(t) \Rightarrow s_{i+1} \geq b_{i+1}(t),
\]
since \( s_i \) is a nondecreasing function of \( i \). Thus, in this case, \( b_{i+1}(t) \) can be redefined to take the value \( b_i(t) \) without any loss of generality. In other words, defining the \( a_i(t) \) by the recursive procedure
\[
a_i(t) = \begin{cases} 
    b_i(t) & \text{for } i=1 \\
    \max\{a_{i-1}(t), b_i(t)\} & \text{for } 2 \leq i < m
\end{cases}
\]
produces a nondecreasing sequence and a rule whose outcome is always equivalent to that of the \((t,m)\) policy. \( \square \)

**Example 1.**

Assume that \( \pi(\lambda) \) is the gamma density with the parameters \( \alpha \).
and $\beta$, i.e.

$$\pi(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta \lambda}, \ x \geq 0.$$ 

Varying $\alpha$ and $\beta$ produces a large family of curves that can model many different shapes of prior density curve. At the time of the $1^{\text{st}}$ failure, the posterior distribution for $\lambda$ is gamma with parameters $\alpha+1$ and $\beta+s_1$. Thus, $E[\lambda|s_1]>t$ if and only if $\frac{\alpha+1}{\beta+s_1}>t$.

Hence the $(t,m)$ policy calls for replacement at the first failure where $s_1 < a_1(t) = \frac{\alpha+1}{t} - \beta$ or at the $m^{\text{th}}$ failure, whichever occurs first.

### 3.3 EXPECTED COST ASSOCIATED WITH $(t,m)$ POLICY

This section contains a formalisation of the optimisation problem. The results will be used in the next section to analyse the case where there are two components.

The joint density function of $(\tau_1, \ldots, \tau_k)$:

The joint density function of $(\tau_1, \ldots, \tau_k)$ is given by

$$f_k(x_1, \ldots, x_k; \lambda) = \frac{n!}{(n-k)!} \lambda^{k-1} e^{-\lambda\{(n-k+1)x_1 + \sum_{i=1}^{k-1} x_i\}}.$$  \hspace{1cm} (3)
where $x_1 \leq x_2 \leq \ldots \leq x_k$. Suppose the $(t,m)$ policy is being used.

**Definition of the region $A_k$:**

Let $A_k$, $k < m$, denote the set of failure times which leads to replacement at the $k$th failure, i.e.

$$A_k = \left\{ (x_1, x_2, \ldots, x_k) : x_1 \leq x_2 \leq \ldots \leq x_k, \quad (n-j+1)x_j + \sum_{i=1}^{j-1} x_i \geq a_j(t) \right\} \quad \forall 1 \leq j < k; \quad (n-k+1)x_k + \sum_{j=1}^{k-1} x_j < a_k(t) \right\}.$$  \quad (4)

**Definition of the region $B_m$:**

Define $B_m$ to be the set of values where the $m$ part of the $(t,m)$-policy is invoked, i.e.

$$B_m = \{ (x_1, x_2, \ldots, x_m) : x_1 \leq x_2 \leq \ldots \leq x_m$$

and $(n-k+1)x_k + \sum_{i=1}^{k-1} x_i \geq a_k(t), \quad \forall 1 \leq k < m \}$$ \quad (5)

**The expected length of a cycle:**

Assume for the moment that $\lambda$ is fixed. Then the expected length of a cycle when the $(t,m)$-policy is used is given by
\[ T(t, m; \lambda) = \int_{x_1}^{x_m} \cdots \int_{x_1}^{x_m} f(x_1, \ldots, x_m; \lambda) dx_1 \cdots dx_m \]
\[ + \sum_{k=1}^{m-1} \int_{x_1}^{x_k} \cdots \int_{x_1}^{x_k} f_k(x_1, \ldots, x_k; \lambda) dx_1 \cdots dx_k. \]  
(6)

The expected salvage value:

The salvage value in a cycle depends on the number of working components when group replacement is made. Thus, the expected salvage value is given by

\[ S(t, m; \lambda) = \int_{x_1}^{x_m} \cdots \int_{x_1}^{x_m} f(x_1, \ldots, x_m; \lambda) dx_1 \cdots dx_m \]
\[ + \sum_{k=1}^{m-1} \int_{x_1}^{x_k} \cdots \int_{x_1}^{x_k} f_k(x_1, \ldots, x_k; \lambda) dx_1 \cdots dx_k. \]  
(7)

The expected downtime costs:

The expected downtime costs during a cycle are given by

\[ D(t, m; \lambda) = \int_{x_1}^{x_m} \cdots \int_{x_1}^{x_m} c_d (x_1 + \cdots + x_m) f(x_1, \ldots, x_m; \lambda) dx_1 \cdots dx_m \]
\[ + \sum_{k=1}^{m-1} \int_{x_1}^{x_k} \cdots \int_{x_1}^{x_k} c_d (x_1 + \cdots + x_k) f_k(x_1, \ldots, x_k; \lambda) dx_1 \cdots dx_k. \]  
(8)

The expected cost per unit time:

The above expressions are computed assuming that the parameter of the failure time distribution is fixed at a particular \( \lambda \). Since the actual value of \( \lambda \) is unknown, the
functions S, D and T must be integrated with respect to the prior distribution. The process is renewed each time a batch of n components is purchased. Thus, from the Renewal Theorem, the limiting value of the long run average cost per unit time = (expected cost per cycle) + (expected cycle length). Thus, the expected cost per unit time when the \((t,m)\)-policy is used is given by

\[
C(t,m) = \frac{c_0 \int S(t,m;\lambda)\pi(\lambda)d\lambda + \int D(t,m;\lambda)\pi(\lambda)d\lambda}{\int T(t,m;\lambda)\pi(\lambda)d\lambda},
\]  

(9)

and the goal is to find the \((t,m)\) policy at which the above expression is minimized.

3.4 THE TWO COMPONENT CASE WITH A GAMMA PRIOR

Now the case of two components where \(\pi(\lambda)\) is gamma density with parameters \(\alpha\) and \(\beta\) will be analysed in detail. As will be seen the policy based on sufficient statistics has a particularly appealing form for this case. A \((t,1)\) policy is simply a 1-failure policy. A \((t,2)\) policy calls for group replacement at the second failure unless \(s_1 = 2\tau_1 < \frac{z+1}{t} - \beta\), in which case replacement is made at the first failure. Thus, in this case, the class of all \((t,m)\) policies is equivalent to the class of policies "replace the system at the first failure if \(\tau_1 < z\), otherwise replace at the second failure" where \(z \in [0,\infty]\).
(Putting \( z=\omega \) produces the 1-failure policy, \( z=0 \) gives the 2-failure policy and \( z=\frac{\alpha+1}{2t} \) produces the \((t,2)\) policy.)

The joint density function of \((x_1, x_2)\):

\[ f_2(x_1, x_2) = 2 \lambda^2 e^{-\lambda(x_1 + x_2)} \]  

(10)

Suppose a \((t,2)\) policy is being used.

Region \( A_1 \):

The set of values where batch replacement is made at the first failure is given by

\[ A_1 = \{ x_1 : x_1 < \frac{\alpha+1}{2t} \} \]  

(11)

Region \( B_2 \):

The set of values where group replacement is made at the second failure is given by

\[ B_2 = \{ (x_1, x_2) : x_1 \leq x_2 \text{ and } x_1 \geq \frac{\alpha+1}{2t} \} \]  

(12)

The expected length of a cycle:

The expected length of a cycle when \( \lambda \) is the parameter of the failure time distribution is given by
\[
T(t, 2; \lambda) = \int_{z}^{\infty} \int_{z}^{x_{2}} x_{2} f_{2}(x_{1}, x_{2}) dx_{1} dx_{2} + \int_{0}^{z} x_{2} f_{1}(x_{1}) dx_{1},
\]

where \[z = \frac{\alpha+1}{2t} - \frac{\beta}{2} \cdot \]

Thus, the expected length of the cycle is given by

\[
\int T(t, 2; \lambda) \pi(\lambda) d\lambda = \int (z+\frac{3}{2\lambda}) e^{-2\lambda z} \pi(\lambda) d\lambda + \int \left\{ \frac{1}{2\lambda} - (z+\frac{1}{2\lambda}) e^{-2\lambda z} \right\} \pi(\lambda) d\lambda
\]

\[= \beta(2\alpha-2)^{-1} + (\alpha-1)^{-1} \beta^\alpha / (\beta+2z)^{\alpha+1}. \tag{13} \]

The expected salvage value during a cycle:

The expected salvage value during a cycle is given by

\[
E[S(t, 2; \lambda)] = \mathbb{E} \left[ c_s(1) \int_{0}^{z} f_{1}(x_{1}) dx_{1} \right]
\]

\[= c_s(1) \mathbb{E} \left[ 1 - e^{-2\lambda z} \right]
\]

\[= c_s(1) \left[ 1 - \left( \frac{\beta}{(\beta+2z)} \right)^\alpha \right]. \tag{14} \]

The expected downtime cost during a cycle:

The expected downtime cost in a cycle is given by

\[
E[D(t, 2; \lambda)] = \mathbb{E} \left[ \int_{z}^{\infty} \int_{z}^{x_{2}} c_d(x_{2} - x_{1}) f_{2}(x_{1}, x_{2}) dx_{1} dx_{2} \right]
\]

\[= c_d \mathbb{E} \left[ \lambda^{-1} e^{-2\lambda z} \right]
\]
\begin{equation}
= c_d \beta^\alpha (\alpha - 1)^{-1} / (\beta + 2z)^{\alpha - 1}.
\end{equation}

The expected cost per unit time:

Thus, the expected cost per unit time when the \((t,2)\) policy is used is given by

\begin{equation}
C(t;m) = \{(c_0 - c_1(1))(\alpha - 1)(\beta + 2z)^\alpha + c_d \beta^\alpha (\beta + 2z) + (\alpha - 1)c_1(1)\beta^\alpha \}
\{\beta^\alpha (\beta + 2z) + 0.5\beta(\beta + 2z)^{\alpha - 1}\}^{-1}.
\end{equation}

The optimal expected cost per unit time:

Upon differentiating the above with respect to \(z\), equating to 0 and simplifying, the optimal \(z\) can be seen to be a solution to the equation

\begin{equation}
A \left[ \frac{\beta + 2z}{\beta} \right]^\alpha - \alpha \left[ \frac{\beta + 2z}{\beta} \right]^{\alpha - 1} - 2 = 0,
\end{equation}

where

\begin{equation}
A = \{2(\alpha - 1)(c_0 - c_1(1)) - c_d \beta \} / c_1(1).
\end{equation}

If \(A = 0\), then the optimal \(z\) is equal to infinity, i.e. the 1-failure policy is optimal. If \(A > 0\), then the optimal \(z\) is finite and non-zero. In this latter case, the optimal \(z\) can be found from (16) or (17), using any standard numerical technique since these expressions are functions of only one unknown.
Example 2.

Define the parameters as follows:

\( a = 5. \)

\( \beta = 50. \)

\( c_0 = $14. \)

\( c_s(1) = $1. \)

and

\( c_d = $2. \)

Putting these values in expression (18), a positive value for \( A \) is obtained which is equal to 4. Expression (17) becomes

\[
4 \left[ \frac{50 + 2z}{50} \right]^5 - 5 \left[ \frac{50 + 2z}{50} \right]^4 - 2 = 0
\]

whose only solution is at \( z = 9.64 \). Thus, if the first breakdown occurs before time 9.64, the system is immediately replaced. If the first breakdown occurs after time 9.64, the system is replaced at the time of the second failure. The expected cost per unit time associated with this policy is $2.072.

3.5 DERIVATION OF PREPOSTERIOR DISTRIBUTIONS AND PROBABILITIES

The analysis in the previous section was possible since the limits of integration for the functions \( T(t,2;\lambda) \), \( D(t,2;\lambda) \) and \( S(t,2;\lambda) \) were straightforward to obtain. The same analysis is much more difficult even for moderately larger values of \( n \). This
is because the regions of integration rapidly become very complex.
The remainder of the chapter is devoted to developing procedures
which overcome these difficulties.

Throughout this section assume that $n, m$ and $t$ are fixed. In
order to simplify the notation, the argument $t$ will be suppressed
when referring to the quantity $a_i(t)$. For $k < m$, define $A_k$ to be
the event $\bigcap_{i=1}^{k} \{s_i \geq a_i\}$, i.e. $A_k$ will occur if and only if group
replacement occurs beyond the time of the $k^{th}$ failure. At any
given failure, determination of posterior distributions is an easy
task since all failure times are known and all transitions (given
$\lambda$) are Markovian. However, at the beginning of a cycle,
determination of the preposterior distributions is not so simple.
For instance, suppose a posterior distribution conditioned on the
event $A_i$ is required. It is not sufficient to only condition on
the time of the $i^{th}$ failure: not knowing the precise values of
$\tau_i, \ldots, \tau_{i-1}$ means that all of the events $\{s_j \geq a_j\}_{j=1}^{i}$ have an
influence on the posterior distribution. From Theorem 1,
attention can be restricted to the random variables $s_j$. For fixed
$\lambda$, Theorem 2 provides the distribution of $s_j$ given the event $A_j$
while Corollary 1 provides the distribution of $s_j$ given the event
$A_{j-1}$. The preposterior distributions can then easily be obtained
by integrating these expressions with respect to the prior
density. Theorem 3 shows that the probability of the event $A_j$ is
the expectation with respect to the prior density of a known
polynomial in $\lambda$ times $\exp(-\lambda a_j)$. This is the result that will be
used in section 6 to provide an efficient procedure for the
calculation of the cost associated with the \((t, m)\) policy.

**Definition of the function \(\phi_{j, k}(z)\):**

Define a function \(\phi_{j, k}(\cdot)\) as follows:

\[
\phi_{j, k}(z) = \sum_{i=0}^{j} \left( (\lambda z)^{i+1} - (\lambda a_k)^{i+1} + (i+1)(\lambda a_k)^{i+1} \right) / (i+1)!
\]

(20)

**Definition of the sequence \(C_{j, k}(\lambda)\):**

Now define a sequence of polynomials in \(\lambda\), \(C_{j, k}(\lambda)\), by the following recursive procedure:

\[
C_{1, k}(\lambda) = 1
\]

\[
C_{j, k}(\lambda) = \begin{cases} 
[D_k(\lambda)]^{-1} C_{j-1, k-1}(\lambda) \text{ for } j \geq 2, \ k \geq 2 \\
- [D_k(\lambda)]^{-1} \sum_{i=1}^{k-1} C_{i, k-1}(\lambda) \frac{(\lambda a_{k-1})^i}{i!} \text{ for } j = 1,
\end{cases}
\]

(21)

where

\[
D_k(\lambda) = \sum_{j=1}^{k-1} C_{j, k-1}(\lambda) \phi_{j-1, k-1}(a_k).
\]

(22)

Before proceeding to Theorem 2, a technical lemma, whose proof is deferred to the appendix, is required.

**Lemma 3.1.**

Assume that \(z = a_{k+1}^j, j \geq 0\) and \(k \geq 1\). Then the following
relationships are true.

\[
(a) \int_{0}^{a_{k+1}} \int_{a_{k+1}}^{z} x^{j-1} e^{-\lambda (x+y)} \, dx \, dy + \int_{a_{k+1}}^{z} \int_{a_{k+1}}^{z-y} x^{j-1} e^{-\lambda (x+y)} \, dx \, dy
\]

\[= \Gamma(j) \lambda^{-(j+1)} \left\{ e^{-\lambda a_{k+1}} \phi_{j-1,k}(a_{k+1}) - e^{-\lambda z} \phi_{j-1,k}(z) \right\}
\]

\[
(b) \int_{0}^{a_{k+1}} \int_{a_{k+1}}^{\infty} x^{j-1} e^{-\lambda (x+y)} \, dx \, dy + \int_{a_{k+1}}^{\infty} \int_{a_{k+1}}^{\infty} x^{j-1} e^{-\lambda (x+y)} \, dx \, dy
\]

\[= \Gamma(j) \lambda^{-(j+1)} \lambda a_{k+1} \phi_{j-1,k}(a_{k+1}).
\]

**THEOREM 3.2**

Let \( g_{k}(z;\lambda) \), \( 1 \leq k \leq m \) denote the conditional density function of \( s_{k} \) given that \( \lambda \) is the failure time parameter and the event \( A_{k} \) has occurred. Then

\[
g_{k}(z;\lambda) = \sum_{j=1}^{k} \lambda a_{k} \phi_{j,k}(\lambda) h_{j}(z;\lambda) \quad \text{for} \quad z \geq a_{k},
\]

where

\[
h_{j}(z;\lambda) = \frac{\lambda^{j}}{\Gamma(j)} z^{j-1} e^{-\lambda z} \quad \text{for} \quad z > 0.
\]

**Proof:**

The proof will be by induction on \( k \). Let \( P_{\lambda}(\cdot) \) denote the
probability operator when the failure time parameter is fixed at the specific value \( \lambda \). For \( k=1 \) and \( z\geq a_1 \),

\[
P_{\lambda} [s \leq z|s \geq a_1] = P_{\lambda} [n\tau \leq z|n\tau \geq a_1]
\]

\[
= 1 - e^{-\lambda(z-a_1)}
\]

(since \( n\tau \) is an exponential random variable with parameter \( \lambda \)) which, upon differentiation, provides the required result for \( g_1(z;\lambda) \).

Now assume that the theorem is true for \( k=1 \). It will be shown that this implies the theorem is true when \( k \) is replaced by \( k+1 \). Let \( Y_k \) denote the time between the \( k^{th} \) and the \((k+1)^{st}\) failure. Then, \((n-k)Y_k\) is an exponential random variable with parameter equal to \( \lambda \), is independent of \( s_k \) and

\[
s_{k+1} - s_k = (n-k)[\tau_{k+1} - \tau_k]
\]

\[
= (n-k) Y_k.
\]

Thus, using the previous expression and the fact that \( A_{k+1} = (s_{k+1} \geq a_{k+1}) \cap A_k \), the conditional distribution function of \( s_{k+1} \) can be written as

\[
P_{\lambda} [s \leq z|A_{k+1}] = P_{\lambda} [s + (n-k)Y_k \leq z|A_{k+1}]
\]
\[ P_{\lambda_{k+1}}[n_k \leq s_k + (n-k)Y_k \leq z | A_k] = \frac{P_{\lambda_k}[(n-k)Y_k \geq a_{k+1} | A_k]}{P_{\lambda_k}[(n-k)Y_k \geq a_{k+1} | A_k]} , \]  

(24)

where \( z = a_{k+1} \). First an expression for the denominator and then one for the numerator in (24) will be obtained. Then it will be shown that the derivative of the above distribution function is equal to \( g_{k+1}(z; \lambda) \).

Note that the distribution of \( Y_k \) conditioned on the event \( A_k \) is the same as the unconditional distribution of \( Y_k \). This and the fact that \( Y_k \) is independent of \( s_k \) implies that the denominator of (24) can be written as follows:

\[
P_{\lambda_k}[(n-k)Y_k \geq a_{k+1} | A_k] = \int_0^{a_{k+1} - a_k} \int_{a_{k+1} - y}^{\infty} g_k(x; \lambda) \lambda e^{-\lambda y} dx dy \]

\[+ \int_{a_{k+1} - a_k}^{\infty} \int_{a_k}^{\infty} g_k(x; \lambda) \lambda e^{-\lambda y} dx dy .\]

Use the induction hypothesis to obtain

\[
P_{\lambda_k}[(n-k)Y_k \geq a_{k+1} | A_k] = \sum_{j=1}^{k} C_{j,k}(\lambda) \frac{\lambda^{j+1}}{\Gamma(j)} e^{-\lambda a_k} \int_0^{a_{k+1} - a_k} \int_{a_{k+1} - y}^{\infty} x^{j-1} e^{-\lambda(x+y)} dx dy + \]

\[\sum_{j=1}^{k} C_{j,k}(\lambda) \frac{\lambda^{j+1}}{\Gamma(j)} e^{-\lambda a_k} \int_{a_{k+1} - a_k}^{\infty} \int_{a_k}^{\infty} x^{j-1} e^{-\lambda(x+y)} dx dy .\]
Now use part (b) of Lemma 3.1, simplify and obtain

\[ P_\lambda[s_k+(n-k)Y_k \geq a_{k+1} | A_k] = \sum_{j=1}^{k} C_{j,k}(\lambda) e^{\lambda_{a_{k+1}}} \phi_{j-1,k}(a_{k+1}). \]  

(25)

Now the numerator of (24) will be evaluated in a similar way.

Again, since the distribution of \((n-k)Y_k\) is exponential with parameter \(\lambda\) and \(Y_k\) is independent of \(s_k\),

\[ P_\lambda[a_{k+1} \leq s_k+(n-k)Y_k \leq z | A_k] = \int_{0}^{a_{k+1}-a_k} \int_{z-y}^{\infty} g_k(x; \lambda) \lambda e^{-\lambda y} dx dy \]

\[ + \int_{a_{k+1}-a_k}^{z-a_k} \int_{z-y}^{\infty} g_k(x; \lambda) \lambda e^{-\lambda y} dx dy. \]

Use the induction hypothesis and part (a) of Lemma 3.2 to obtain

\[ P_\lambda[a_{k+1} \leq s_k+(n-k)Y_k \leq z | A_k] \]

\[ = \sum_{j=1}^{k} C_{j,k}(\lambda) e^{\lambda_{a_{k+1}}} \left\{ e^{-\lambda a_1} \phi_{j-1,k}(a_{k+1}) - e^{-\lambda z} \phi_{j-1,k}(z) \right\}. \]

(26)

Divide (25) into (26) and use (22) to obtain
\[ P_{\lambda} [s_{k+1} \leq z | A_{k+1}] = 1 - \left[D_{k+1}(\lambda)\right]^{-1} e^{\lambda a_{k+1}} \sum_{j=1}^{k} C_{j,k}(\lambda) e^{-\lambda z} \phi_{j-1,k}(z), \]

which on differentiating becomes

\[ g_{k+1}(z; \lambda) = \sum_{j=1}^{k} \left[D_{k+1}(\lambda)\right]^{-1} C_{j,k}(\lambda) e^{-\lambda(z-a_{k+1})} \phi_{j-1,k}(z) \]

\[ - \sum_{j=1}^{k} \left[D_{k+1}(\lambda)\right]^{-1} C_{j,k}(\lambda) e^{-\lambda(z-a_{k+1})} \sum_{i=0}^{j-1} \frac{(i+1)! \lambda(\lambda z)^i}{(i+1)!} \cdot \]

Use the definition of \( \phi_{j-1,k}(z) \), gather terms, simplify and recall expression (21) to obtain

\[ g_{k+1}(z; \lambda) = \]

\[ \sum_{j=1}^{k} \left[D_{k+1}(\lambda)\right]^{-1} C_{j,k}(\lambda) e^{-\lambda(z-a_{k+1})} \left\{ \sum_{i=0}^{j-1} \frac{(\lambda a_{k})^i}{i!} - \sum_{i=0}^{j-1} \frac{(\lambda a_{k})^{i+1}}{(i+1)!} \right\} + \]

\[ \sum_{j=1}^{k} \left[D_{k+1}(\lambda)\right]^{-1} C_{j,k}(\lambda) e^{-\lambda(z-a_{k+1})} \left\{ \sum_{i=0}^{j-1} \frac{\lambda(\lambda z)^{i+1}}{(i+1)!} - \sum_{i=0}^{j-1} \frac{\lambda(\lambda z)^i}{i!} \right\} \]

\[ = \sum_{j=1}^{k} \left[D_{k+1}(\lambda)\right]^{-1} C_{j,k}(\lambda) e^{-\lambda(z-a_{k+1})} \left\{ 1 - \frac{\lambda a_{k}}{j!} \right\} \]

\[ + \sum_{j=1}^{k} \left[D_{k+1}(\lambda)\right]^{-1} C_{j,k}(\lambda) e^{-\lambda(z-a_{k+1})} \left\{ \frac{\lambda(\lambda z)^j}{j!} - \lambda \right\} \]
\[= - \left[ D_{k+1}(\lambda) \right]^{-1} \sum_{j=1}^{k} \frac{\lambda^j}{j!} a^j_k C_{j,k}(\lambda) h_1(z;\lambda) \]

\[+ \sum_{j=2}^{k+1} \left[ D_{k+1}(\lambda) \right]^{-1} \frac{\lambda^j}{j!} a^j_{k+1} C_{j-1,k}(\lambda) h_j(z;\lambda) \]

\[= C_{1,k+1}(\lambda) h_1(z-a_{k+1};\lambda) + \sum_{j=2}^{k+1} \frac{\lambda^j}{j!} a^j_{k+1} C_{j,k+1}(\lambda) h_j(z;\lambda) , \]

which is the required form for \( g_{k+1}(z) \). \( \square \)

**COROLLARY 3.1**

Set \( a_0 \equiv 0, D_0(\lambda) \equiv D_1(\lambda) \equiv 1 \). Then, for \( k \geq 0 \),

\[
P_\lambda [ s_{k+1} \equiv a_{k+1} | A_k ] = e^{\frac{\lambda(a_{k+1}-a_{k+1})}{k+1}} \frac{\lambda(a_{k+1}-a_{k+1})}{k+1} \frac{\lambda(a_{k+1}-a_{k+1})}{k+1} .
\]

**Proof:**

From Theorem 2 the right hand side of (23) is the density function of \( s_k \) given that the event \( A_k \) has occurred. Repeat the algebra given in the third paragraph of the theorem and arrive at the result given by (25):

\[
P_\lambda [ s_{k+1} \equiv a_{k+1} | A_k ] = \sum_{j=1}^{k} C_{j,k}(\lambda) e^{\frac{\lambda(a_{k+1}-a_{k+1})}{j-1,k+1}} \frac{\lambda(a_{k+1}-a_{k+1})}{j-1,k+1} .
\]

The result follows by applying (22). \( \square \)

The results of **Theorem 3.2** and the **corollary 3.1** involve the
polynomials \( C_{j,k}(\lambda) \). Eventually an integration will be performed with \( \lambda \) as the variable. Thus a result showing precisely how the \( \lambda \)'s enter into the problem is required. Towards this end define a sequence of constants \( \{c_j\} \) by the following recursive procedure

\[
c_0 = 1 \\
c_j = -\sum_{i=0}^{j-1} c_i \frac{(a_j)^{j-i}}{(j-i)!} \quad \text{for } j=1,2,\ldots
\]  

(27)

The following lemma, whose proof can be found in the appendix, provides relationships between the polynomials \( C_{j,k}(\lambda) \) and the constants \( c_j \). This Lemma 3.2 will be used in the proof of Theorem 3.3.

**Lemma 3.2**

(a) \( C_{k-1,k}(\lambda) = c_j \lambda^j \prod_{j=1}^{k} [D_j(\lambda)]^{-1} \), for \( j=0,1,\ldots,k-1 \).

(b) \[ \sum_{j=1}^{k} C_{j,k}(\lambda) = \left\{ \prod_{j=1}^{k} [D_j(\lambda)]^{-1} \right\} \left\{ \sum_{j=0}^{k-1} c_j \lambda^j \right\}, \]

for \( i=1,2,\ldots,k \).

Let \( p_k(\lambda) \) denote the probability that group replacement occurs later than the \( k^{th} \) failure if the parameter of the failure time distribution is known to equal \( \lambda \). Note that, for \( k<m \),
\[ p_k(\lambda) = \prod_{j=1}^{k} P_{\lambda}[s_j \neq a_j | A_{j-1}], \]

where the subscript on the probability operator indicates that the parameter of the failure time distribution is assumed to equal a fixed value \( \lambda \). The term inside the product sign above is simply the probability that group replacement is not made at the \( j^{th} \) failure given that it has not been made at any previous failure.

Theorem 3 shows that \( p_k(\lambda) \) can be expressed as a linear combination of terms of the form \( \lambda^j e^{-\lambda a_k} \), where all coefficients are known or easily calculated using (27).

**THEOREM 3.3**

For \( 1 \leq k < m \), and failure time parameter equal to \( \lambda \), the probability of replacing after the \( k^{th} \) failure is given by

\[ p_k(\lambda) = e^{-\lambda a_k} \sum_{j=0}^{k-1} d_j \lambda^j, \]

where

\[
d_j = \begin{cases} 
1 & \text{for } j=0 \\
\sum_{i=0}^{j} \frac{(a_k)^{j-1}}{(j-1)!} & \text{for } j=1,2,\ldots,k-2 \\
k-2 \frac{(a_k)^{k-1} - (a_{k-1})^{k-1}}{(k-1)!(k-1-1)!} & \text{for } j=k-1 .
\end{cases}
\]

**Proof:**

From Corollary 1 and the definition of \( D_k(\lambda) \),
\[ p_k(\lambda) = \prod_{i=0}^{k} P[s_i \geq a_i | A_{i-1}] \]

\[ = \prod_{i=0}^{k} e^{-\lambda(a_{i-1} - a_i)} D_i(\lambda) \]

\[ = e^{-\lambda a_k} \left\{ \prod_{i=1}^{k-1} D_i(\lambda) \right\} \sum_{j=1}^{k-1} C_{j,k-1}(\lambda) \phi_{j-1,k-1}(a_k) \]

Use the definition of \( \phi_{j,k}(\cdot) \) and interchange the order of summation to obtain

\[ p_k(\lambda) = \]

\[ e^{-\lambda a_k} \left\{ \prod_{i=1}^{k-1} D_i(\lambda) \right\} \sum_{i=0}^{k-2} \frac{(\lambda a_k)^{i+1} - (\lambda a_{k-1})^{i+1} + (i+1)(\lambda a_{k-1})^i}{(i+1)!} \sum_{j=1}^{k-1} C_{j,k-1}(\lambda), \]

which, on applying part (b) of Lemma 3.2, becomes

\[ p_k(\lambda) = e^{-\lambda a_k} \left\{ \sum_{i=0}^{k-2} \frac{(\lambda a_k)^{i+1}}{(i+1)!} \sum_{j=1}^{k-1} c_{j-1} \lambda^{j-1} \right\} \]

\[ - \sum_{i=0}^{k-2} \frac{(\lambda a_{k-1})^{i+1}}{(i+1)!} \sum_{j=1}^{k-1} c_{j-1} \lambda^{j-1} + \sum_{i=0}^{k-2} \frac{(\lambda a_{k-1})^i}{i!} \sum_{j=1}^{k-1} c_{j-1} \lambda^{j-1} \right\}. \]
On gathering terms involving powers of \( \lambda \) the above reduces to

\[
p_k(\lambda) = e^{-\lambda a_k} \sum_{j=0}^{k-2} \lambda^j \left\{ \sum_{i=0}^{j} c_i \frac{(a_k)^{j-1}}{(j-1)!} \right\}
+ e^{-\lambda a_k} \lambda^{k-1} \left\{ \sum_{i=0}^{k-2} c_i \frac{(a_k)^{k-1-1} - (a_{k-1})^{k-1-1} - (a_k)^{k-1-1}}{(k-1-1)!} \right\},
\]

the required expression. \( \Box \)

3.6 Procedure for Large \( n \)

In this section it is shown how the probabilities obtained in Section 5 can be used to determine an expression for the cost associated with the \((t, m)\) policy.

Suppose that group replacement is not made at the \( i^{th} \) failure. Then there are \( n-1 \) functioning machines and the time to the next failure is an exponential random variable with parameter \((n-1)\lambda\). Thus, the expected time from the \( i^{th} \) to the \((i+1)^{st}\) failure is given by \(((n-1)\lambda)^{-1}\) and the expected downtime cost incurred over this period is given by \(1((n-1)\lambda)^{-1}\). Then:

The expected Length of the cycle:

The expected length of the cycle is given by the following:
The expected downtime costs:

The expected downtime costs during a cycle are given by the following:

$$E[D(t,m;\lambda)] = E\left[\sum_{i=1}^{m} \frac{(i-1)p_{i-1}(\lambda)}{(n-i+1)\lambda}\right]$$  \hspace{1cm} (29)

The expected salvage value:

The expected salvage value during a cycle is given by the following:

$$E[S(t,m;\lambda)] = E\left[\sum_{i=1}^{m} c_{i}(n-i)(p_{i-1}(\lambda)-p_{i}(\lambda))\right].$$  \hspace{1cm} (30)

where the expectations are taken with respect to the prior distribution.

From Theorem 3, each of expressions (28) and (29) can be written as the sum of single variable integrals where the integrands are of the form $\pi(\lambda)\lambda^{-1}$ times a known polynomial in $\lambda^{-1}$. A similar statement can be made about expression (30). Thus, calculation of the above expected values is relatively straightforward using one dimensional numerical
integration procedures. This contrasts favourably with the procedure of Section 3 where multiple integrals with complex regions of integration are involved.

**Example 3.**

Suppose that \( n=4 \) and that the prior density over \( \lambda \) is gamma with parameters \( \alpha \) and \( \beta \). Then, from Example 1, \( a_i = \frac{\alpha + 1}{t_i} - \beta \) for \( i=1,2,3 \). Expressions (28), (29) and (30) assume that a particular \((t,m)\) policy is being used. Suppose the expected cost of a \((t,4)\) policy is desired. Using the explicit expressions given for the \( a_i \), Theorem 3 and simplifying, the following expressions for \( p_{1}(\lambda) \) can be obtained:

\[
p_{1}(\lambda) = \begin{cases} 
1 & \text{for } i=0 \\
-\lambda_{1} & \text{for } i=1 \\
e^{-\lambda_{2}} & \text{for } i=2 \\
[(1+t^{-1}\lambda)e^{-\lambda_{3}}] & \text{for } i=3 \\
0 & \text{otherwise.}
\end{cases}
\]  

Using the above and expressions (28), (29) and (30), the expected cost per unit time associated with any \((t,4)\) policy can be obtained.

For \((t,3)\) policies set

\[
p_{3}(\lambda) = 0
\]

and use the
\( p_0(\lambda), p_1(\lambda) \) and \( p_2(\lambda) \) defined in (31).

For \((t,2)\) policies set

\[
\begin{align*}
p_2(\lambda) &= 0 \\
p_3(\lambda) &= 0
\end{align*}
\]

and read

\[
p_0(\lambda) \quad \text{and} \quad p_1(\lambda) \quad \text{from (31)}.
\]

For the \((t,1)\) policies set

\[
\begin{align*}
p_1(\lambda) &= 0 \\
p_2(\lambda) &= 0 \\
p_3(\lambda) &= 0 \\
p_4(\lambda) &= 0
\end{align*}
\]

and

\[
p_0(\lambda) = 1.
\]

Once the expected cost per unit time of a \((t,m)\) policy has been computed, the incremental effort involved in computing the cost of the \((t,m-1)\) policy is minimal.

For the following particular case where

\[
\begin{align*}
\alpha &= 5 \\
\beta &= 4.5 \\
c_0 &= \$20
\end{align*}
\]
$$c_d = \$5$$

and

$$c_s(1) = 51.$$ 

Figure 1 contains a plot of $$\min_{1 \leq m \leq 4} C(t, m)$$ against $$t$$. Figure 2 contains a plot of $$C(t, 1), C(t, 2), C(t, 3)$$ and $$C(t, 4)$$ against $$t$$.

The optimal policy for this problem occurred at

$$t^* = .899$$

and

$$m = 3.$$ 

The associated expected cost per unit time is:

$$C(t^*, 3) = \$16.67.$$
Average cost per unit time.

FIGURE 3.1

(α_m)-policies.
3.7 CONCLUSION

Research on group maintenance policies has generally assumed that the parameters of the failure time distribution are known. In practice this is rarely the case. A more common situation is that where the decision maker is given a mean value and assumes that this is the fixed value of the parameter. This chapter has considered the problem of integrating the estimation of the underlying failure time parameters with group replacement methodology. An intuitively appealing class of group replacement policies has been introduced. Numerically tractable forms of the appropriate preposterior distributions have been derived. An important future research task is the extension of these results to failure time distributions other than the exponential. The challenge will be to find replacement policies that adequately incorporate the statistical information obtained by operating the machines, are intuitively acceptable, easy to implement and are mathematically tractable.
3.8 APPENDIX

Proof of Lemma 3.1:

First note that successive applications of integration by parts produces the following formula:

\[
\int_a^b x^{j-1} e^{-\lambda x} \, dx = \lambda^{-1} \Gamma(j) \left\{ e^{-\lambda a} \sum_{i=0}^{j-1} \frac{(\lambda a)^i}{i!} - e^{-\lambda b} \sum_{i=0}^{j-1} \frac{(\lambda b)^i}{i!} \right\}.
\]

Use the above to obtain

\[
\int_0^\infty \int_{x=a}^{x=a+k-1} x^{j-1} e^{-\lambda(x+y)} \, dx \, dy + \int_0^{\infty} \int_{x=a}^{x=a+k-1} x^{j-1} e^{-\lambda(x+y)} \, dx \, dy
\]

\[
= \lambda^{-1} \Gamma(j) \left\{ e^{-\lambda a} \sum_{i=0}^{j-1} \frac{(\lambda a)^i}{i!} - \int_0^{\infty} \frac{\lambda(a+y)^i}{i!} \, dy - e^{-\lambda z} \sum_{i=0}^{j-1} \frac{\lambda(z-y)^i}{i!} \, dy \right\}
\]

\[
+ e^{-\lambda a} \sum_{i=0}^{j-1} \frac{(\lambda a)^i}{i!} e^{-\lambda y} \, dy - e^{-\lambda z} \sum_{i=0}^{j-1} \frac{\lambda(z-y)^i}{i!} \, dy \right\}.
\]

Part (a) of the lemma follows by performing the integrations on the right hand side of the above expression, simplifying, gathering terms and recalling the definition of the function
\( \phi_{j-1,k}(\cdot) \). Part (b) of the lemma follows from part (a) by letting \( z \to \infty \).

**Proof of Lemma 3.2:**

The proof will be by induction on \( j \). Apply the recursion (21) \( k-1 \) times to see that

\[
C_{k,k}(\lambda) = [D_k(\lambda) D_{k-1}(\lambda) \ldots D_2(\lambda)]^{-1} C_{1,1}(\lambda).
\]

Thus, (a) is true for \( j=0 \) since \( c_0 = 1, D_1(\lambda) = 1, \) and \( C_{1,1}(\lambda) = 1 \). Now assume that

\[
C_{k-1,k}(\lambda) = c_1 \lambda^1 \prod_{\ell=1}^{k} [D_{\ell}(\lambda)]^{-1}, \text{ for } i=0,1,\ldots,j, \quad (A1)
\]

for some \( j=k-2 \). The goal is to show that the above remains true at \( j+1 \). Apply the recursion (21) \( (k-j-1) \) times to the left hand side of (A1) to obtain

\[
C_{j+1,j+1}(\lambda) \prod_{\ell=j+2}^{k} [D_{\ell}(\lambda)]^{-1} = c_1 \lambda^1 \prod_{\ell=1}^{k} [D_{\ell}(\lambda)]^{-1}, \text{ for } i=0,1,\ldots,j,
\]

which implies that

\[
C_{j+1,j+1}(\lambda) = c_1 \lambda^1 \prod_{\ell=1}^{j+1} [D_{\ell}(\lambda)]^{-1} \text{ for } i=0,1,\ldots,j. \quad (A2)
\]

Again using (21), \( C_{k-(j+1),k}(\lambda) \) can be written as
\[ C_{k-(j+1),k}(\lambda) = \left\{ \prod_{\ell=j+3}^{k} [D_\ell(\lambda)]^{-1} \right\} C_{1,j+2}(\lambda) \]

\[ = -\left\{ \prod_{\ell=j+3}^{k} [D_\ell(\lambda)]^{-1} \right\} [D_{j+2}(\lambda)]^{-1} \sum_{\ell=1}^{j+1} C_{\ell,j+1}(\lambda) \frac{(\lambda a_{j+1})^\ell}{\ell!} . \]

Now use (A2) to provide an expression for the \( C_{\ell,j+1}(\lambda) \) in the above sum to obtain

\[ C_{k-(j+1),k}(\lambda) = \]

\[ -\left\{ \prod_{\ell=j+2}^{k} [D_\ell(\lambda)]^{-1} \right\} \sum_{\ell=1}^{j+1} \left\{ \prod_{r=1}^{\ell} [D_r(\lambda)]^{-1} \right\} c_{j+1-\ell} \lambda^{j+1-\ell} \frac{(\lambda a_{j+1})^\ell}{\ell!} \]

\[ = \left\{ \prod_{\ell=1}^{k} [D_\ell(\lambda)]^{-1} \right\} \lambda^{j+1} \left\{ -\sum_{i=0}^{j} c_i \frac{(a_{j+1})^{j+1-i}}{(j+1-i)!} \right\} \]

\[ = \left\{ \prod_{\ell=1}^{k} [D_\ell(\lambda)]^{-1} \right\} \lambda^{j+1} c_{j+1} \]

the required result. Thus part (a) is true. Part (b) follows by applying (a) to the terms inside the summation \( \sum_{j=1}^{k} C_{j,k}(\lambda) \).
3.9 REFERENCES


Yeh, L., "A Note on the Optimal Replacement Time of Damaged
CHAPTER IV

A GENERAL ADAPTIVE GROUP MAINTENANCE POLICY
4.1. INTRODUCTION AND MODEL DESCRIPTION

Suppose that \( n \) components are operating in parallel. A loss of \( c_d \) per unit time is incurred for each failed component. The cost of replacing a broken component with a new component is denoted by \( c_b \), while the cost of servicing a working component to be "as good as new" is given by \( c_r \). A fixed cost of \( c_0 \) is incurred each time replacement or servicing of components occurs. In this chapter group replacement and servicing strategies are considered. The failure times of the components are assumed to be independent identically distributed random variables with increasing hazard rate and density function \( f(\cdot) \).

A number of group replacement policies have appeared in the literature. An \( m \)-failure policy calls for group replacement at the \( m^{th} \) failure (see Assaf and Shantikumar (1987), Nakagawa (1983), Park (1979)). T-age group replacement policies call for group replacement whenever the system reaches age \( T \) (see, e.g., Okumoto and Elsayed (1983)). Ritchken and Wilson (1990) introduce a class of policies, called \((m,T)\) group maintenance policies, that call for group replacement at the \( m^{th} \) failure or time \( T \) whichever occurs first. Wilson and Benmerzouga (1991) consider the case where the underlying failure distribution is exponential with an unknown parameter that must be estimated from the operation of the components.

The \( m \)-failure, T-age, and \((m,T)\) group replacement policies have one managerially unpalatable drawback: they ignore what
happens during the cycle until the $m^{th}$ failure or time $T$ has occurred. For instance, consider a problem where 100 components are operating in parallel and the optimal $m$-failure policy is to replace whenever the $51^{st}$ component breaks down. Consider the following two cases: the first 51 components to break down fail one year after being put into operation; 50 components fail on the second day of operation and the $51^{st}$ fails one year after the start of the cycle. The $51$-failure policy treats the above two cases in exactly the same manner - group replacement is performed after one year of operation. However, most decision makers would want to treat the above two situations in a completely different manner. Of course, the analysis that would have provided the $51$-failure policy as the optimal $m$-failure policy is based on average cost per unit time over an infinite horizon. Even though the cost of carrying 50 non-functioning components for one year might be very large, the probability of such an event occurring would be so low that this case would be essentially disregarded in the average per unit time analysis. But events of low probability can occur. While it might be reasonable to discount them a-priori, it would be foolish to ignore them if they actually occur. Assuming continuous inspection, this chapter provides an easily implemented replacement policy that takes account of the actual state of the system at each stage.

Define a decision rule $\delta$ to be a function from $\{(i,x) : 0 \leq i \leq n, 0 \leq x < \infty\}$ into the closed interval $[0,1]$, where $\delta(i,x)$ denotes the probability with which group replacement will be made if
exactly 1 components are broken and it has been x time units since the last group replacement. This class of strategies includes the m-failure, T-age and (m,T) policies as special cases. An m-failure policy is obtained by requiring \( \delta(i,x) \) to equal 1 if \( i \geq m \) and to equal 0 otherwise. A T-age replacement policy is obtained by setting \( \delta(i,x) \) equal to 1 for \( x \geq T \) and equal to 0 otherwise. An (m,T) policy is obtained by defining \( \delta \) to satisfy

\[
\delta(i,x) = \begin{cases} 
1 & \text{if } i \geq m \\
1 & \text{if } x \geq T \\
0 & \text{otherwise}
\end{cases}
\]

Implementation of a policy \( \delta \) as defined above only requires knowledge of the current state \((i,x)\). Thus no extra information other than normally available when using an m-failure policy is required. However, finding the optimal m-failure policy is relatively straightforward. Finding a policy \( \delta \) that is optimal or \( \epsilon \)-optimal is not quite so straightforward. However, once such a policy has been found implementation is not much harder than it would be for m-failure, T-age, or (m,T) policies.

Section 2 provides a procedure for finding an optimal replacement policy when decisions can only be made at times \( x_1 < x_2 < \ldots < x_m \). As \( x_m \to \infty \) and \( \max_j |x_{j+1} - x_j| \to 0 \), the cost associated with the policy computed using this assumption converges to the cost of the optimal policy for the case where decisions are allowed at anytime. Section 3 contains a procedure that will
produce lower bounds on the optimal cost. In section 4 it is shown how the results of sections 2 and 3 can be used to find $\epsilon$-optimal policies for the general case.

**Notation**

$c_0$  
The fixed cost of replacing the system (excluding the cost of buying new components or replacing old components to "as good as new").

$c_d$  
The downtime cost incurred by one failed component for one unit of time.

$c_b$  
The cost of buying a new component.

$c_r$  
The cost of replacing a working component with a new component (this can be interpreted as the cost of servicing a working component to "as good as new"; alternatively $c_b - c_r$ can be interpreted as the trade-in value for a working component).

$X$  
Time to failure of one component.

$f(x)$  
Density function of the random variable $X$.

$F_j(x)$  
Conditional distribution function of the time to failure of one component given that it has survived to time $x_j$;  
$F_j(x) = P[X \leq x | X > x_j]$.  

$(i, x_j)$  
The state of the process if it has been $x_j$ time units since the last group replacement and currently $i$ components are broken.

$P_{k,j-1}^{i,j}$  
The probability that exactly $i$ components are broken at time $x_j$ given that $k \leq 1$ were broken at time $x_{j-1}$.
The probability that during a cycle exactly 1 components are broken at time $x_j$ and the decision not to replace the system is made.

The probability that during a cycle exactly 1 components are broken at time $x_j$ and the decision to replace the system is made.

The immediate expected cost of taking action $a$ ($a=1$ if system is replaced and $a=0$ if system is not replaced) if it has been $x_j$ time units since the last replacement and exactly 1 components are now broken.

4.2. ALGORITHMIC PROCEDURE FOR RESTRICTED STATE SPACE

In this section, the case where the state of the system is only observed at times $x_1 < x_2 < \ldots < x_n$ is considered. A procedure for finding an optimal policy for this restricted problem will be provided. As the maximum distance between the $x_j$ goes to 0 and $x_m \to \infty$, the expected cost per unit time for the strategy that optimises the restricted problem will converge to the optimal expected cost per unit time for the general problem. The linear programming approach adopted is similar to that described in Heyman and Sobel (1984).

The process is defined to be in state $(i, x_j)$ if the current time is given by $x_j$ and exactly 1 components are broken.

A randomised policy $\delta$ is a function from $\{(i, x_j) : 0 \leq i \leq n, 1 \leq j \leq m\}$ to the interval $[0, 1]$. Here $\delta(i, x_j) \in [0, 1]$ is to be interpreted as the probability with which the decision maker will
perform group replacement. For purposes of mathematical convenience, the analysis that follows will be for this general class of randomised strategies. However, the solution procedure will always produce a deterministic policy.

A cycle is defined to be the period between group replacements. From the Renewal Theorem, the limiting value for the expected cost per unit time is the expected cost per cycle divided by the expected length of a cycle. Assume that the randomised policy \( \delta \) is to be used. Set \( \delta(i, x_a) = 1 \) for all \( i \) (i.e. no matter how many components are working, replacement is required at time \( x_a \)). Let \( p_{1j0}(\delta) \) denote the probability that the state \((1, x_j)\) will be visited during the cycle and the decision not to replace the system at that time is made. Let \( p_{1j1}(\delta) \) denote the probability that the process will visit the state \((1, x_j)\) and the decision to replace all components is made. Let \( c_{ij_a}, a \in \{0, 1\} \), denote the expected cost incurred while the process moves from state \((1, x_j)\) to the next state if the process is currently in state \((1, x_j)\) and action \( a \) is taken. The expected cost per cycle can be written as \( \sum_{i, j_a} c_{ij_a} p_{ij_a}(\delta) \). Let \( \ell_{ij_a} \) be the transition time from state \((i, x_j)\) to the next state when action \( a \) is taken, i.e.

\[
\ell_{ij_a} = \begin{cases} 
  x_{j+1} - x_j & \text{for } a = 0 \\
  0 & \text{for } a = 1.
\end{cases}
\]  

(1)

Thus the expected cost per unit time from following the policy \( \delta \)
is given by \( \sum_{i,j,a} c_{ija} p_{ija}(\delta) \sum_{i,j,a} \ell_{ija} p_{ija} \). In what follows, an expression for \( c_{ija} \) will be provided. Then it will be shown how an optimal \( \delta \) can be found from a linear programming formulation of the problem.

Determination of \( c_{ija} \)

Assume that the current state of the system is \((i,x_j)\). If the action is to perform group replacement then the cost consists of two components: the cost of purchasing replacement components for those that have failed and the cost of performing maintenance on the components that are still working. Thus \( c_{ij} = c_0 + c_c + (n-1)c_r \). Now suppose that replacement is not performed at state \((i,x_j)\). The immediate expected cost of continuing at state \((i,x_j)\) is given by

\[
c_{ijo} = ic_d(x_{j+1} - x_j) + (n-1)c_d \{ x_{j+1} - E[\min(x_{j+1}, x)|X > x_j] \}
\]

\[
= ic_d(x_{j+1} - x_j) + (n-1)c_d \int_{x_j}^{x_{j+1}} F_j(x)dx,
\]

where the first term is the downtime cost of the \( i \) components that are broken at time \( x_j \) and the second term is the expected downtime cost for components that fall between \( x_j \) and \( x_{j+1} \). Thus, the costs \( c_{ija} \) can be written as follows:
\[ c_{1ja} = \begin{cases} 
  c_0 + 1c_b + (n-1) c_r & \text{for } a = 1 \\
  1c_d (x_{j+1} - x_j) + (n-1)c_d \int_{x_j}^{x_{j+1}} F_j(x) \, dx & \text{for } a = 0.
\] 

\textbf{LP Formulation}

The expected length, \( E[\tau_\delta] \), of the renewal cycle when policy \( \delta \) is used is given by

\[ E[\tau_\delta] = \sum_{i,j,a} \ell_{i ja} p_{i ja}(\delta). \]

The expected cost per unit time equals

\[ \sum_{i,j,a} c_{i ja} p_{i ja}/E[\tau_\delta] = \sum_{i,j,a} c_{i ja} f_{i ja} \text{ where } f_{i ja} = p_{i ja}/E[\tau_\delta]. \]

The quantities \( f_{i ja}(\delta) \) must satisfy a number of constraints. Since \( p_{i ja}(\delta) \geq 0 \) and \( \sum_{i,j,a} \ell_{i ja} p_{i ja}(\delta) = E[\tau_\delta] \), the quantities \( f_{i ja}(\delta) \) must satisfy

\[ f_{i ja}(\delta) \geq 0 \quad \forall \ i,j,a. \]  

and

\[ \sum_{i,j,a} \ell_{i ja} f_{i ja}(\delta) = 1. \]  

Define \( p_{k, j-1}^{1, J} \) to be the probability that exactly \( i \) components are broken at time \( x_j \) given that \( k \leq i \) were broken at time \( x_{j-1} \), i.e.

\[ p_{k, j-1}^{1, J} = \binom{n-k}{1-k} \{ P(X \leq x_{j+1} | X \geq x_j) \}^{i-k} \{ 1 - P(X \leq x_{j+1} | X \geq x_j) \}^{n-i} \]

The probability that the process visits state \((1, x_j)\) is given by
\[ p_{1jo}(\delta) + p_{1j1}(\delta). \]

The process can only be in this state if \( k = 1 \) components were broken at time \( x_{j-1} \), action \( a = 0 \) was taken, and exactly \( i - k \) components broke down during the period from time \( x_{j-1} \) to \( x_j \).

Thus, the quantities \( p_{1ja}(\delta) \) must satisfy

\[ p_{1jo}(\delta) + p_{1j1}(\delta) = \sum_{k \leq 1} p^{1,j}_{k,j-1} p_{1(j-1)0}(\delta). \]

On dividing the above expression by \( E[\tau_\delta] \), the quantities \( f_{1ja}(\delta) \) as seen to satisfy

\[ f_{1jo}(\delta) + f_{1j1}(\delta) = \sum_{k \leq 1} p^{1,j}_{k,j-1} f_{1jo}(\delta). \quad (6) \]

Hence, to summarise, for any given decision rule \( \delta \), the expected cost per unit time equals \( \sum_{i,j,a} c_{i,j,a} f_{ija}(\delta) \) where the quantities \( f_{ija}(\delta) \) satisfy the constraints (3), (4) and (6).

Conversely, suppose quantities \( y_{ija} \) are given that satisfy (3), (4) and (6) with each \( f_{ija}(\delta) \) replaced by \( y_{ija} \). Then define a decision rule by

\[ \delta(i, x_j) = y_{1jo}(y_{1jo} + y_{1j1})^{-1}. \]

Then it can be seen that the average cost per unit time associated with this rule equals \( \sum_{i,j,a} c_{i,j,a} y_{ija}. \)
Thus, for the state space \( \{(i,x_j) : 1 \leq i \leq n, 1 \leq j \leq m\} \), one need only find the quantities \( y_{ija} \) that minimise

\[
\sum_{i,j,a} c_{ija} y_{ija} \quad (7)
\]

subject to the constraints

\[
y_{ija} \geq 0 \quad \forall \ i,j,a \quad (8)
\]

\[
\sum_{i,j,a} \ell_{ija} y_{ija} = 1 \quad (9)
\]

\[
y_{ij0} + y_{ij1} = \sum_{k \leq 1} p_{k,j-1} y_{i(j-1)0} \quad \forall \ i,j \quad (10)
\]

where expressions for \( \ell_{ija}, c_{ija} \) and \( p_{k,j-1} \) are given by (1), (2) and (5) respectively. The above is simply a linear programming problem and can be solved using any of the standard linear programming software packages.

5.3. LOWER BOUNDS FOR GLOBAL OPTIMUM

In the previous section it was assumed that the state space is given by \( \sum =\{(i,x_j) : 0 \leq i \leq n, 1 \leq j \leq m\} \), where \( \{x_j\} \) is some given partition. A more general problem is to allow the state space to be \( \{(i, x) : 0 \leq i \leq n, 0 \leq x < x_\infty\} \). Assume that \( x_\infty \) is chosen so large that replacement must occur before \( x_\infty \) (the assumption of an increasing hazard rate implies that such an \( x_\infty \) exists). For suitably chosen partitions \( \{x_j\} \), the optimal solution obtained
using $\sum$ should be very close to the optimal solution for the more
general problem. The purpose of this section is to obtain a bound
on the difference between the optimal solution using $\sum$ and the
optimal solution using the general state space $\{(i, x) : 0 \leq i \leq n,$
$0 \leq x \leq x^*\}$. If, for a given partition, the bound is too high then a
finer partition should be chosen.

As before, the analysis will be for randomised strategies.
An optimal solution, of course, will always turn out to be a
non-randomised strategy. In this context a randomised strategy $\delta$
is a function from $\{(i, x) : 0 \leq i \leq n, 0 \leq x \leq x^*\}$ to the closed interval
$[0, 1]$, with $\delta(i, x)$ being the probability that group replacement
will be performed when the system is in state $(i, x)$.

Let $\delta^*$ denote an optimal strategy for the case where
decisions can be made at anytime $x=x^*$. Define a strategy $\delta_*$ on
the state space $\Sigma$ as follows: Group replacement using strategy $\delta_*$
occurrs at time $x_j$ if and only if a decision maker who had been
using $\delta^*$ would have replaced the system at time $x \in (x_{j-1}, x_j]$; set
$\delta_*(i, x_j) = 1$ for all $i$.

The expected length of each renewal cycle is longer when
using $\delta_*$ than $\delta^*$. This is because $\delta_*$ defers group replacement to
the first time $x_j$ that comes after the strategy $\delta^*$ would have
replaced the components. However, the expected cost over a cycle
using $\delta_*$ is larger than that using $\delta^*$. This increase in expected
cost arises from two sources: extra downtime cost and the cost
increment $c_b - c_r$, incurred for each extra failed component.
Suppose that, using $\delta_*$, group replacement is required at $(i, x_j)$. 

In this case, the extra downtime cost can be no more than $1c_d(x_j-x_{j-1})$. Redefine the costs of replacement to be $c'_{ij1}$ where

$$c'_{ij1} = c_0 + c + (n-1)c - 1c_d(x_j-x_{j-1}).$$  \hspace{1cm} (11)

The expected extra replacement costs using $\delta_*$ instead of $\delta^*$ cannot exceed $n(c_b - c)\max_{k\leq k+1} P[x < X < x_{k+1}]$ since $P[x < X < x_{k+1}]$ is an upper bound on the expected number of extra failures between times $x_k$ and $x_{k+1}$. Redefine the cost $c_{000}$ to be $c'_{000}$ where

$$c'_{000} = c_{000} - n(c_b - c)\max_{k\leq k+1} P[x < X < x_{k+1}].$$ \hspace{1cm} (12)

Suppose that the cost of $\delta_*$ is evaluated using the modified cost structure given above. Use of $c'_{000}$ instead of $c_{000}$ has the effect of subtracting the constant $n(c_b - c)\max_{k\leq k+1} P[x < X < x_{k+1}]$ from the true expected cost of the cycle while use of $c'_{ij1}$ instead of $c_{ij1}$ compensates for the extra downtime cost incurred by using $\delta_*$. Thus the expected cost per unit time using the strategy $\delta_*$ and the modified cost function is smaller than the expected cost (using the true costs) per unit time associated with the optimal strategy $\delta^*$. Thus the expected cost per unit time associated with the strategy that minimises (7) subject to the constraints (8), (9), and (10) where the $c_{ij1}$ are replaced by the $c'_{ij1}$ and $c_{000}$ by $c'_{000}$ (see (11) and (12), respectively) is smaller than the expected cost per unit time using $\delta^*$. 


4.4. Finding \( \varepsilon \)-Optimal Strategies

A measure of how effective it is to replace the general state space \( \{(i, x) : 0 \leq i \leq n, 0 \leq x < x_m^a \} \) with the approximating state space \( \{(i, x_j) : 0 \leq i \leq n, x_1^a < \ldots < x_m^a \} \) is provided by the following procedure. First, minimise (7) subject to (8), (9) and (10) where the costs \( c_{ij} \) are defined in (2). Then minimise (7) subject to (8), (9) and (10) where the costs \( c_{000} \) and \( c_{111} \) are replaced by \( c'_{000} \) and \( c'_{111} \). The difference between these two expected costs per unit time is an upper bound on the savings that can be achieved by looking at finer partitions. As \( \max_j|x_j-x_{j-1}| \rightarrow 0 \), the expected cost per unit time obtained by the linear programming procedure of the previous sections converges to the expected cost per unit time associated with \( \delta^* \).

Example

Suppose that \( x \) is a gamma random variable with parameters \( \alpha = 5 \) and \( \beta = 4.5 \), i.e. \( f(x) = (4.5)^5 x^4 e^{-(4.5)x} (4!)^{-1} \). Suppose the costs are as follows: 
\[ c_0 = 15, \]
\[ c_b = 7, \]
\[ c_r = 3, \]
\[ c_d = 15. \]

Figure 4 contains a graphical representation of the optimal policy. The solid lines show where replacement should be performed. For example, suppose the second machine fails at time 0.4. Then replacement should not immediately be performed.
Instead, replacement should be scheduled for time 0.9275 or when the third machine fails, whichever occurs first. For convenience, the partition for this example was equally spaced in increments of 0.0175 between times 0 and 3.5 (Note that the probability that any machine will survive past time 3.5 equals 0.0019). The optimal value, using the procedure of section 3, is 41.9498. The lower bound is 41.1428. Thus the policy found from the linear programming formulation can be no more than 41.9498 - 41.1428 = 0.807 units away from the optimal policy when no partition is used. Table 1 contains the values of the objective function for different equally spaced partitions of the interval 0 to 3.5. As can be seen, even moderately sized partitions produced reasonable results.
A General Adaptive Replacement Policy

FIGURE 4.1
4.5. CONCLUSION

This chapter introduces a group replacement policy that takes account of the actual state of the system at each stage. This general approach cannot have a higher expected cost per unit time than the m-failure, T-age, or (m,T) policies. The general approach is no harder to implement than any of these policies and has the great advantage of being intuitively much more acceptable. The drawback of the procedure is that more computational effort is required to find good policies. The adaptive policy allows the reliability engineer to efficiently use the information provided by failing components. This will result in a reduction in the variability of costs among cycles.
4.6 REFERENCES


TABLE 1

Results for varying partition sizes

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<thead>
<tr>
<th>Step size</th>
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<th>Upper bound using §2</th>
<th>Lower bound using §3</th>
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<td>.07</td>
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<td>200</td>
<td>41.950</td>
<td>41.143</td>
</tr>
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</table>
Chapter 2.


Chapter 3.


McGrath, M., Gross, D. and Singpurwalla, N., "A Subjective Bayesian Approach to the Theory of Queues II - Inference and Information in m/m/1 Queues", *Queueing Systems*, 1, 335-353, (1987).


Chapter 4.


