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Thermocapillary flows in an enclosure of unit order aspect ratio

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THERMOCAPILLARY FLOWS
IN AN ENCLOSURE OF UNIT ORDER ASPECT RATIO

by

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Submitted in partial fulfillment of the requirements
for the Degree of Doctor of Philosophy

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Date 6/8/90

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THERMOCAPILLARY FLOWS
IN AN ENCLOSURE OF UNIT ORDER ASPECT RATIO

Abstract

by

DIDI HU

Steady thermocapillary flows in a two-dimensional rectangular enclosure of unit order aspect ratio are studied for fluids of small, unit order, and large Prandtl numbers. Scaling analysis is applied to obtain velocity and length scales for different flow regimes along with the dimensionless parameters that define these regimes. The scaling analysis in this work is for the entire range of the Reynolds number and the influence of the flow on the driving force, which leads to a change in the thermal signature, is also incorporated. Numerical simulations are performed and agreements are found between the results from the scaling analysis and the results from the numerical simulations. The analysis also predicts the behavior of the thermocapillary flow in the range of Marangoni number at which present numerical schema are incapable of simulating.
ACKNOWLEDGEMENT

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NOMENCLATURE

A = \frac{H}{L} \quad \text{— aspect ratio}

c \quad \text{— specific heat}

Ca = \frac{\sigma \cdot \Delta T}{\sigma} \cdot A \quad \text{— capillary number}

F \quad \text{— free surface deformation}

f \quad \text{— dimensionless free surface deformation}

H \quad \text{— height of the enclosure}

L \quad \text{— length of the enclosure}

M \quad \text{— volume flow rate of the fluid at } \theta_s \text{ in the surface flow}

Ma = \frac{\sigma \cdot \Delta T \cdot H}{\alpha \cdot \mu} \quad \text{— Marangoni number}

Ma^* \quad \text{— modified Marangoni number}

P \quad \text{— pressure}

p^* \quad \text{— dimensionless pressure}

P^* \quad \text{— dimensionless pressure scale}

P_c = \frac{(A \cdot \sigma \cdot \Delta T)^2}{\nu \cdot \mu} \quad \text{— reference pressure}

Pr = \frac{\nu}{\sigma \cdot \Delta T \cdot H} \quad \text{— Prandtl number}

Re = \frac{\sigma \cdot \Delta T \cdot H}{\nu \cdot \mu} \quad \text{— Reynolds number}

Re^* \quad \text{— modified Reynolds number}

T \quad \text{— temperature}

t \quad \text{— dimensionless time}

T_h, T_c \quad \text{— imposed wall temperatures}
\( U, V \)  
- velocity components

\( u, v \)
- dimensionless velocity components

\( \epsilon, \sigma \)
- dimensionless velocity scales

subscript \( s \) for the surface flow

subscript \( d \) for the region of the
- surface flow with driving force

subscript \( r \) for the return flow

subscript \( h \) for the turning flow next
to the hot wall

subscript \( c \) for the turning flow next
to the cold wall

\[
\begin{align*}
U_c &= \frac{\sigma \cdot \Delta T}{\mu \cdot A} \\
V_c &= \frac{I \cdot \Delta T}{\mu \cdot A^2}
\end{align*}
\]
- reference velocity in the \( x \) direction

- reference velocity in the \( y \) direction

\( X, Y \)
- coordinates

\( \chi, \psi \)
- dimensionless coordinates

\( \alpha \)
- thermal diffusivity

\( \Delta T = T_h - T_c \)
- reference temperature

\( \delta \)
- dimensionless length scales

subscript \( v \) for the velocity field

subscript \( t \) for the temperature field

subscript \( h \) for next to the hot wall

subscript \( c \) for next to the cold wall

subscript \( d \) for the region of the
surface flow with driving force

subscript $s$ for the surface flow

$\mu$ — viscosity

$\nu$ — kinematic viscosity

$\theta$ — dimensionless temperature

$\theta^*$ — dimensionless temperature scale

subscript $s$ for the surface flow without driving force

subscript $r$ for the return flow

$\Delta\theta^*$ — dimensionless temperature difference scale

subscript $h$ for across the thermal boundary layer next to the hot wall

subscript $c$ for across the thermal boundary layer next to the cold wall

$\rho$ — density

$\sigma$ — coefficient of surface tension

$\sigma_T$ — surface tension temperature coefficient

$\tau$ — time

$\tau_c = \frac{\mu \cdot L}{\Lambda \cdot \sigma_T \cdot \Delta T}$ — reference time

xi
CHAPTER I

Introduction

Tangential to the interface of two immiscible fluids, there exists a force called surface tension due to the difference in the properties of the molecules. When the surface tension is non-uniform along the interface, i.e., when there is a surface tension gradient, the tangential stress balance on the interface requires that this surface tension gradient be balanced by the viscous stress. Since a velocity gradient is needed for a viscous stress, it follows that the fluids cannot be at rest and flow arises. This flow which is due to the surface tension gradient along an interface is called surface tension driven flow [1].

The magnitude of the surface tension on the interface between two immiscible fluids depends not only on the properties of the two fluids, but also on temperature and concentrations of different species. Hence any non-uniformity in these properties along the interface will cause a non-uniformity in the surface tension and a surface
tension gradient ensues. The flow generated by the surface tension gradient due to a temperature gradient is termed thermocapillary flow.

Thermocapillary flows are important in many industrial processes, such as flame spreading above a liquid pool [2], welding [3], and crystal growth [4,5], especially in a low-gravity environment [6,7,8]. Thermocapillary flows have been investigated experimentally [9,10,11,12], numerically [14,15,16,17], and analytically [13,18,19]. The effects of free surface deformation [20,21] and buoyancy [22] have also been investigated. Stability analysis has been applied to the problem [23,24,25].

Since the driving force of thermocapillary flows is the surface tension gradient induced by a surface temperature gradient, the temperature field is very important, which means that heat transfer is always important. But the temperature field itself depends on the flow field through convection. Therefore the basic equations for the flow field, the momentum conservation equations, and that for the temperature field, the energy conservation equation, are coupled through the convection terms in the energy
conservation equation and the tangential stress balance boundary condition on the surface on which the surface tension gradient exists. Also because thermocapillary flows depend on the surface tension gradient, there must be a free surface. Thus free surface deformation could also be important. From the above discussion we can conclude that analysis of thermocapillary flows is very complicated and difficult due to these characteristics.

These complications can sometimes be avoided by careful analysis. If free surface deformation and its influence on flow and temperature fields are negligible, free surface deformation can be neglected and the problem can be simplified to a non-deformable surface one. When through analysis it is shown that the conduction terms in the energy conservation equation dominate and the convection terms are negligible compared to the conduction terms, the basic equations are decoupled. But these simplifications must be preceded by careful analysis to obtain the proper dimensionless parameters signifying the relative importance of the terms and factors to be neglected and the conditions under which these simplifications are valid. For example, in the case of neglecting the convection terms in the energy
conservation equation the relative importance of the convection and conduction terms, the Peclet number, has to be evaluated correctly. These dimensionless parameters can only be obtained through a careful scaling analysis of the problem.

In experimental investigations of a complex problem, such as that of thermocapillary flows, the influences of other possible driving forces have to be considered to ensure that the dominant driving force is the surface tension gradient induced by a temperature gradient on the interface. To control the experimental conditions, relevant dimensionless parameters have to be derived and used to ensure the success of the experiment. Also very important is the knowledge of the scales of the quantities to be measured and the sizes of possible subregions, such as boundary layer thicknesses, so that proper instruments with required resolutions are utilized and no important features will be missed. These important dimensionless parameters can only be obtained through a careful scaling analysis of the problem.

When numerical simulations are attempted, one of the most important considerations is that the spatial and temporal
resolutions of the numerical scheme have to be selected so as to minimize the computing time and the computer storage and at the same time these resolutions must be sufficient to resolve the physics of the problem, while the numerical errors are limited to the extent that they are negligible compared to physical terms. To do this the possible boundary layer thicknesses and other length scales associated with flow subregions have to be obtained so that sufficient resolution is assured to resolve the profiles within these layers. With the determination of length scales in different regions of the physical domain grids of different sizes can be adopted in different regions according to the length scales obtained. These important dimensionless parameters, length scales, can only be obtained through a careful scaling analysis of the problem.

Scaling analysis was first applied to thermocapillary flows by Ostrach [26] and the influence of buoyancy was also evaluated. However, the possible change of the region on which the driving force acts due to the flow was not incorporated. Later the length scales for the regions on which the driving force acts was incorporated in the scaling analysis by Lai [20]. In [20] the length scales for the
regions with the driving force next to the hot and the cold walls were both interpreted as the thermal boundary layer thicknesses and this leads to the conclusion that these two length scales are the same due to the same balances utilized. But in our numerical analysis it is shown that there is a substantial difference in these two length scales, which will be shown later in Section 6.1. In the scaling analysis of Rivas [27] for low Prandtl number fluids, a heat flux boundary condition was introduced. With this boundary condition the length scale for the region with the driving force is fixed by the imposed heat flux, thus the change of the thermal signature was avoided. In this work we introduce the length scales for the regions with the driving force as unknowns and derive them from the scaling analysis so that the change of thermal signature due to the flow is incorporated in our analysis. Unlike the analysis in [20], we also treat the regions with the driving force next to the hot and the cold walls differently. Thus the difference in these two length scales are in agreement with our numerical analysis.

In this work scaling analysis will be applied to steady state thermocapillary flows with a non-deformable free
surface in a two-dimensional rectangular enclosure of unit order aspect ratio. Fluids of small, unit order, and large Prandtl numbers will all be considered. The analysis is for laminar flow.

In Chapter II, the problem is formulated mathematically and non-dimensionally. Chapter III gives a general description of the temperature and the velocity fields with the aid of some numerical simulations, and the terminologies and the symbols for the scales are presented. In Chapter IV scaling analysis is performed for small Prandtl number fluids and in Chapter V for fluids of unit order and large Prandtl numbers. In Chapter VI we will discuss the scales obtained in Chapters IV and V and compare the results of the scaling analysis to some results from our numerical analysis and the results of works by other investigators. Chapter VII contains concluding remarks.
CHAPTER II

Mathematical Formulation

![Diagram](image)

Fig.1

In this work we analyze the basic characteristics of thermocapillary flows. In order to eliminate the complications due to different geometrical configurations, a simple two-dimensional rectangular configuration of length $L$ and depth $H$ is chosen as shown in Fig.1, where $U$ and $V$ are the velocity components in the $X$ and $Y$ directions, respectively. $T_h$ and $T_c$ are the temperatures imposed on the walls at $X = 0$ and $X = L$, respectively. $Y = F(X,\tau)$ is
the free surface on which the driving force acts and \( \sigma \) is the coefficient of surface tension on the free surface. Here \( \tau \) is the temporal variable.

2.1 Basic Equations

The domain as shown in Fig.1 is filled with an incompressible Newtonian fluid of density \( \rho \), kinematic viscosity \( \nu \), and thermal diffusivity \( \alpha \). The free surface at \( Y = F(X, \tau) \) is the interface of this fluid with another fluid whose density, viscosity, and thermal conductivity are negligible compared to those of the fluid in the enclosure. The two fluids are immiscible. The viscous dissipation terms in the energy conservation equation are neglected, because the dimensionless parameter representing the ratio of the viscous dissipation terms to the other terms in the energy conservation equation can be shown to be small in the cases in which we are interested. This parameter can be easily derived with the scales obtained later in this work.

Furthermore, we assume that there is no body force acting on the fluid. With these assumptions the conservations of
mass, momenta in both $X$ and $Y$ directions, and energy are expressed in the following basic equations for the problem.

Continuity

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0$$  \hspace{1cm} (1.a)

Momentum in the $X$ direction

$$\frac{\partial U}{\partial \tau} + U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = - \frac{1}{\rho} \frac{\partial P}{\partial X} + \nu \left( \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} \right)$$  \hspace{1cm} (1.b)

Momentum in the $Y$ direction

$$\frac{\partial V}{\partial \tau} + U \frac{\partial V}{\partial X} + V \frac{\partial V}{\partial Y} = - \frac{1}{\rho} \frac{\partial P}{\partial Y} + \nu \left( \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} \right)$$  \hspace{1cm} (1.c)

Energy

$$\frac{\partial T}{\partial \tau} + U \frac{\partial T}{\partial X} + V \frac{\partial T}{\partial Y} = \kappa \left( \frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} \right)$$  \hspace{1cm} (1.d)

2.2 Boundary Conditions
2.2.1 On the hot wall ( $X = 0$ )

No slip and impermeability conditions are imposed on the velocities.

\[ U = 0 \]  \hspace{1cm} (2.a)

\[ V = 0 \]  \hspace{1cm} (2.b)

\[ T = T_h \]  \hspace{1cm} (2.c)

2.2.2 On the cold wall ( $X = L$ )

Similarly as in Section 2.2.1, we have

\[ U = 0 \]  \hspace{1cm} (3.a)

\[ V = 0 \]  \hspace{1cm} (3.b)

\[ T = T_c \]  \hspace{1cm} (3.c)

2.2.3 The symmetry conditions ( $Y = 0$ )
The boundary at \( Y = 0 \) is considered to be a line (or a plane) of symmetry. This way the real physical domain can be considered as composed of two of the rectangles in Fig.1 with their lines (or planes) of symmetry coincident at \( Y = 0 \).

The requirement for symmetry is that this interface be a streamline and the derivatives of variables in the direction perpendicular to it be zero. Under this symmetry condition the domain of our problem can be viewed as half of a plane which is perpendicular to the walls and the line at \( Y = 0 \) as the center line in a half zone simulation of floating zone crystal growth.

\[
V = 0 \quad (4.a)
\]

\[
\frac{\partial u}{\partial y} = 0 \quad (4.b)
\]

\[
\frac{\partial T}{\partial y} = 0 \quad (4.c)
\]

\[
\frac{\partial p}{\partial y} = 0 \quad (4.d)
\]

2.2.4 On the free surface \( (Y = F(X,\tau)) \)
(1) Heat flux boundary condition

The assumption about the negligibility of the thermal conductivity of the surrounding fluid suggests that there is no heat transfer through the free surface.

\[ - \frac{\partial T}{\partial x} \frac{\partial F}{\partial x} \frac{1}{(1 + (\frac{\partial F}{\partial x})^2)^{1/2}} + \frac{\partial T}{\partial y} \frac{1}{(1 + (\frac{\partial F}{\partial x})^2)^{1/2}} = 0 \]

or

\[ \frac{\partial F}{\partial x} \frac{\partial T}{\partial x} = \frac{\partial T}{\partial y} \]  \hspace{1cm} (5.a)

(2) Kinematic boundary condition

Since the free surface is a streamline, we have the following condition for the free surface.

\[ \frac{\partial F}{\partial \tau} = V - U \frac{\partial F}{\partial x} \]  \hspace{1cm} (5.b)

(3) Normal force balance

The balance of the stresses normal to the free surface is

\[ p + \frac{2\mu}{1 + (\frac{\partial F}{\partial x})^2} \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) \frac{\partial F}{\partial x} \frac{\partial F}{\partial x}^2 - \frac{\partial V}{\partial \tau} \]
\[
\frac{\sigma \cdot \partial^2 F}{\partial x^2} = - \frac{\partial F}{\partial x} \frac{1}{(1 + (\frac{\partial F}{\partial x})^2)^{3/2}}
\] (5.c)

Here the pressure of the surrounding fluid is taken to be zero.

(4) Tangential force balance

The surface tension on the free surface is assumed to vary linearly with temperature.

\[
\sigma = \sigma_c + \frac{\partial \sigma}{\partial T} (T - T_c)
\]

or \[
\sigma = \sigma_c - \sigma_T (T - T_c)
\]

where \(\sigma_T\) is a constant and \(\sigma_c\) is the value of the surface tension at temperature \(T_c\). With this, the balance of the stresses tangential to the free surface is

\[
2\mu \cdot \left(\frac{\partial V}{\partial Y} - \frac{\partial U}{\partial X} \cdot \frac{\partial F}{\partial X} + \mu \cdot \left(\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X}\right) \cdot \left(1 - (\frac{\partial F}{\partial X})^2\right)\right)
\]

\[
= -\sigma_T \cdot \left(1 + (\frac{\partial F}{\partial X})^2\right)^{1/2} \cdot \left(\frac{\partial T}{\partial X} + \frac{\partial T}{\partial Y} \frac{\partial F}{\partial X}\right)
\] (5.d)
A detailed derivation of Eqs.(5) can be found in [20].

2.3 Global Analysis — based on the geometrical length scales

To non-dimensionalize the basic equations and the boundary conditions the following dimensionless quantities are introduced.

\[ x = \frac{X}{L} \quad (6.a) \]

\[ y = \frac{Y}{H} \quad (6.b) \]

\[ f = \frac{F}{H} \quad (6.c) \]

\[ u = \frac{U}{U_c} \quad (6.d) \]

\[ v = \frac{V}{V_c} \quad (6.e) \]

\[ p = \frac{P}{P_c} \quad (6.f) \]
\[
\theta = \frac{T - T_c}{\Delta T}
\]
\[
t = \frac{\tau}{\tau_c}
\]

with \( H, L, U_c, V_c, P_c, \tau_c \), and \( \Delta T \) as reference quantities.

To obtain the unknown reference quantities we choose \( \Delta T = T_h - T_c \) as imposed by the boundary conditions. Then the terms \( \mu \frac{\partial U}{\partial Y} \) and \( -\sigma_T \frac{\partial T}{\partial X} \) in the tangential force balance boundary condition on the free surface (driving force) are equated to give \( U_c \), the only two terms in the continuity equation are equated to give \( V_c \), the inertia terms \( \frac{\partial U}{\partial t} \) and \( U \cdot \frac{\partial U}{\partial X} \) in the momentum conservation equation in the \( X \) direction are equated to give the propagation reference time \( \tau_c \), and the pressure term and the inertia term \( U \cdot \frac{\partial U}{\partial X} \) in the momentum conservation equation in the \( X \) direction are equated to give \( P_c \). Here we are not concerned with the reference quantities for time and pressure, because we are concerned only with the steady-state situation and the mechanism of the driving force is the surface tension gradient through viscosity, which means that pressure is a passive force and
not crucial in obtaining other scales. We have

$$U_c = \frac{\sigma \cdot \Delta T}{\mu} \cdot A,$$  \hspace{1cm} (7.a)

$$V_c = \lambda \cdot U_c = \frac{\sigma \cdot \Delta T}{\mu} \cdot A^2,$$  \hspace{1cm} (7.b)

$$\tau_c = L/U_c = \frac{\mu \cdot L}{\sigma \cdot \Delta T \cdot A}$$  \hspace{1cm} (7.c)

$$P_c = \rho \cdot U_c^2 = (\sigma \cdot \Delta T \cdot A)^2/(\nu \mu)$$  \hspace{1cm} (7.d)

2.4 Dimensionless Basic Equations and Boundary Conditions

With the reference quantities Eqs.(7), the basic equations Eqs.(1) become

Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$  \hspace{1cm} (8.a)

Momentum in the x direction
\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} + \frac{1}{Re} \left( \frac{\partial^2 u}{\partial x^2} + \frac{1}{A^2} \frac{\partial^2 u}{\partial y^2} \right) \quad (8.b)
\]

Momentum in the \(y\) direction

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{1}{A^2} \frac{\partial p}{\partial y} + \frac{1}{Re} \left( \frac{\partial^2 v}{\partial x^2} + \frac{1}{A^2} \frac{\partial^2 v}{\partial y^2} \right) \quad (8.c)
\]

Energy

\[
\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = \frac{1}{Ma} \left( \frac{\partial^2 \theta}{\partial x^2} + \frac{1}{A^2} \frac{\partial^2 \theta}{\partial y^2} \right) \quad (8.d)
\]

The boundary conditions Eqs.(2) through Eqs.(5) become

At \( x = 0 \)

\[
u = 0\quad (9.a)
\]

\[
v = 0\quad (9.b)
\]

\[
\theta = 1\quad (9.c)
\]

At \( x = 1 \)
\[ u = 0 \quad \text{(10.a)} \]

\[ v = 0 \quad \text{(10.b)} \]

\[ \theta = 0 \quad \text{(10.c)} \]

At \( y = 0 \)

\[ v = 0 \quad \text{(11.a)} \]

\[ \frac{\partial u}{\partial y} = 0 \quad \text{(11.b)} \]

\[ \frac{\partial \theta}{\partial y} = 0 \quad \text{(11.c)} \]

\[ \frac{\partial p}{\partial y} = 0 \quad \text{(11.d)} \]

At \( y = f(x,t) \)

(1) Heat flux boundary condition

\[ A^2 \cdot \frac{\partial f}{\partial x} \cdot \frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial y} \quad \text{(12.a)} \]

(2) Kinematic boundary condition
\[
\frac{\partial f}{\partial t} = v - u \frac{\partial f}{\partial x}
\]  \hspace{1cm} (12.b)

(3) Normal force balance

\[
p + \frac{2}{Re} \frac{1}{1+A^2 \left( \frac{\partial f}{\partial x} \right)^2} \left( \frac{\partial u}{\partial y} + A^2 \frac{\partial v}{\partial x} \right) \frac{\partial f}{\partial x} - A^3 \frac{\partial u}{\partial x} \left( \frac{\partial f}{\partial x} \right)^2 - A \frac{\partial v}{\partial y} \right)
\]

\[
= \frac{\partial^2 f}{\partial x^2}
\]

\[
= \frac{A}{Ca \cdot Re} \left( 1 + A \left( \frac{\partial f}{\partial x} \right)^2 \right)^{3/2}
\]  \hspace{1cm} (12.c)

(4) Tangential force balance

\[
2 \cdot A^2 \left( \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) \frac{\partial f}{\partial x} + \left( \frac{\partial u}{\partial y} + A^2 \frac{\partial v}{\partial x} \right) \left( 1 - A^2 \left( \frac{\partial f}{\partial x} \right)^2 \right)
\]

\[
= -\left( 1 + A^2 \left( \frac{\partial f}{\partial x} \right)^2 \right)^{1/2} \left( \frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial \theta}{\partial y} \right)
\]  \hspace{1cm} (12.d)

The dimensionless parameters appearing in the dimensionless basic equations and boundary conditions are defined as

Aspect ratio \( A = \frac{H}{L} \)  \hspace{1cm} (13.a)
Prandtl number \[ Pr = \frac{\nu}{\alpha} \] (13.b)

Reynolds number \[ Re = \frac{UcL}{\nu} = \frac{\sigma T \cdot \Delta T \cdot H}{\nu \cdot \mu} \] (13.c)

Marangoni number \[ Ma = Re \cdot Pr = \frac{UcL}{\alpha} = \frac{\sigma T \cdot \Delta T \cdot H}{\alpha \cdot \mu} \] (13.d)

Capillary number \[ Ca = \mu \cdot Uc / \sigma = \frac{\sigma T \cdot \Delta T}{\sigma} \] (13.e)

2.5 Dimensionless Basic Equations and Boundary Conditions in the Case of Steady State and Non-deformable Free Surface with Unit Aspect Ratio

For steady state we have \[ \frac{\partial}{\partial t} = 0. \]

From Eq.(12.c) we can estimate the free surface deformation according to pressure variation. Eq.(12.c) can also be written as

\[
Ca \cdot \frac{1}{1 + A^2 \left( \frac{\partial f}{\partial x} \right)^2} \left( \frac{\partial u}{\partial y} + A^2 \frac{\partial v}{\partial x} \frac{\partial f}{\partial x} - A^3 \frac{\partial u}{\partial x} \left( \frac{\partial f}{\partial x} \right)^2 - A \frac{\partial v}{\partial y} \right)
\]
\[
\frac{\delta^2 f}{\delta x^2} = A \cdot \frac{\left(1 + A \cdot \left(\frac{\delta f}{\delta x}\right)^2\right)^{3/2}}{\left(1 + A \cdot \left(\frac{\delta f}{\delta x}\right)^2\right)^{3/2}} - Ca \cdot Re \cdot p
\]

Now we consider the terms on the right hand side of the above equation. The first term is the free surface deformation generated by the second term, the pressure term. If the pressure is normalized, the first term will be very small at \( Ca \cdot Re \ll 1 \). In this case the free surface deformation, \( f \), is not normalized. \( f \) must be very small compared to the geometrical length scales. Hence \( Ca \cdot Re \ll 1 \) can be taken as a first approximate condition under which the free surface deformation can be neglected.

Now the basic equations and boundary conditions Eqs.(8) through Eqs.(12) become

Continuity

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (14.a)
\]

Momentum in the \( x \) direction
\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \]  

(14.b)

**Momentum in the y direction**

\[ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \]  

(14.c)

**Energy**

\[ u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = \frac{1}{Ma} \left( \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) \]  

(14.d)

At \( x = 0 \)

\[ u = 0 \]  

(15.a)

\[ v = 0 \]  

(15.b)

\[ \theta = 1 \]  

(15.c)

At \( x = 1 \)

\[ u = 0 \]  

(16.a)
\[ v = 0 \quad (16.b) \]

\[ \theta = 0 \quad (16.c) \]

At \( y = 0 \)

\[ v = 0 \quad (17.a) \]

\[ \frac{\partial u}{\partial y} = 0 \quad (17.b) \]

\[ \frac{\partial \theta}{\partial y} = 0 \quad (17.c) \]

\[ \frac{\partial p}{\partial y} = 0 \quad (17.d) \]

At \( y = 1 \)

(1) Heat flux boundary condition

\[ \frac{\partial \theta}{\partial y} = 0 \quad (18.a) \]

(2) Kinematic boundary condition

\[ v = 0 \quad (18.b) \]
(3) Normal force balance

\[ p = 0 \]  \hspace{1cm} (18.c)

(4) Tangential force balance

\[ \frac{\partial u}{\partial y} = - \frac{\partial \theta}{\partial x} \]  \hspace{1cm} (18.d)

From Eqs. (14) we can see that the viscous terms in the momentum conservation equations and the conduction terms in the energy conservation equation are small compared to the inertia terms and the convection terms when \( \text{Re} \gg 1 \) and \( \text{Ma} \gg 1 \), respectively. However, these terms are of the highest order in the equations and if they are neglected, not all boundary conditions can be satisfied. This leads to the introduction of length scales other than the geometrical ones and flow subregions. With the introduction of those length scales the velocity scales also change. All these complicate the analysis. In the next few chapters, we will derive those length scales other than the geometrical ones and velocity scales by dividing the flow regimes according to the Prandtl number and the occurrences of viscous and thermal boundary
layers. In each situation we will make local balances and global balances to determine the scales and the conditions under which the derived scales are valid.

The basic equations, Eqs.(14), and the boundary conditions, Eqs.(15) through Eqs.(18), will be the working equations and boundary conditions in the next four chapters as we investigate the velocity and temperature fields and derive the scales for the steady state and non-deformable free surface situation.
CHAPTER III

A General Description of Thermocapillary Flow

In An Enclosure of Unit Order Aspect Ratio

In this chapter we describe some general features of thermocapillary flow in an enclosure of unit order aspect ratio with the help of some results from numerical simulations. These results are numerical solutions of the steady-state Navier-Stokes equations coupled with the energy conservation equation and the non-deformable free surface boundary condition imposed on the free surface. The equations and boundary conditions are given by Eqs.(14) through Eqs.(18). The numerical scheme used is the SIMPLE algorithm given by Patankar [28]. Similar numerical simulations were performed by Zebib et al. [14]. However the boundary conditions imposed on the boundary at $y = 0$ are different between the simulations in [14] and those given by Eqs.(12). Also the amount of results given in [14] is limited and we would like to have as much numerical results as possible for comparison with our scaling results.
The general description in this chapter is valid for small, unit order, and large Prandtl number fluids.

3.1 The Velocity Field

The surface tension gradient due to the temperature gradient on the free surface \((y = 1)\) drives the fluid near the free surface through viscosity from the hot wall \((x = 0)\) towards the cold wall \((x = 1)\), which is the direct effect of the tangential force balance boundary condition given by Eq.(18.d). Fig.2 shows the streamline pattern of the basic flow field. This flow just under the free surface and driven by the surface tension gradient is termed the surface flow. When the flow reaches the vicinity of the cold wall, it is deflected by the cold wall as in the case of a stagnation flow. This turning flow, which can be approximated by a stagnation flow, will be called the turning flow next to the cold wall. After the flow is turned by the cold wall, the direction of the flow is towards the hot wall because of mass conservation in the enclosure, which requires that the amount of fluid driven in the surface flow is equal to that in this flow towards the hot wall. This flow towards
the hot wall will be termed the return flow. As in the case of the turning flow next to the cold wall, the hot wall turns the return flow and this turning flow is called the turning flow next to the hot wall.

3.1.1 $\text{Re} \leq 1$

In the case of $\text{Re} \leq 1$, the length scales for the velocity field are the geometric lengths used as reference lengths in Chapter II. No viscous boundary layers exist in this case. The basic flow pattern can be seen in Fig.2. The streamline pattern in Fig.2 shows a circulating flow with length scales of unit order. For fluid of $\text{Pr} \gg 1$ thermal boundary layers will appear even though there are no viscous boundary layers. In this case the length scales of the temperature field have to be taken into account. Section 5.2 treats this case in detail.

3.1.2 $\text{Re} \gg 1$

When the Reynolds number is much larger than unity, inertia terms dominate in Eq. (14.b) and Eq. (14.c). However, the viscous terms are the highest order derivatives and when
neglected, not all boundary conditions can be satisfied. This gives rise to the existence of viscous boundary layers next to the walls.

Also important is the viscous term \( \frac{1}{Re} \frac{\partial^2 u}{\partial y^2} \) of Eq.(14.b) in the surface flow, because it is related to the driving force for the bulk of the fluid in the surface flow. Therefore there must be a layer under the free surface, in which \( \frac{1}{Re} \frac{\partial^2 u}{\partial y^2} \) is as important as the inertia terms. Fig.3 shows the scales for the velocity field.

![Fig.3]

The flow is divided into four regions as indicated before: the surface flow, the turning flow next to the cold
wall, the return flow, and the turning flow next to the hot wall. Since the Marangoni number can also be much larger than unity in this case, we assume an S-shaped profile for temperature distribution on the free surface. The reason for this assumption will be explained later in Section 3.2.2. With this assumption a length scale $\delta_{td}$ for the temperature distribution on the free surface appears as an unknown. When the velocity field is considered, $\delta_{td}$ represents the region of the surface flow with driving force. Outside this region the driving force is negligible compared to that inside this region because of small temperature gradient. With the introduction of this length scale, $\delta_{td}$, dividing the surface flow into two regions distinguishable from each other by the presence or absence of driving force, we introduce two different length scales $\delta_{vd}$, $\delta_{vs}$ and velocity scales $u_1$, $u_s$ as the average thicknesses and velocities in the $x$ direction in the two subregions of the surface flow.

Let $\delta_{vc}$ and $v_{c}$ be the viscous boundary thickness and the velocity scale in the $y$ direction in the turning flow next to the cold wall, respectively. Let $\delta_{vh}$ and $v_{h}$ be the viscous boundary layer thickness and the velocity scale in the $y$ direction in the turning flow next to the hot wall,
respectively. \( u_r^* \) denotes the velocity scale of the return flow in the \( x \) direction. These scales will be derived later in Chapters IV and V for fluids of \( \text{Pr} \ll 1 \) and \( \text{Pr} \gg 1 \), respectively.

3.2 The Temperature Field

In our problem a constant temperature, \( T_h^* \), is imposed on the wall at \( x = 0 \) and a constant temperature, \( T_c^* \), is imposed on the wall at \( x = 1 \) as shown in Fig.1. On the free surface at \( y = 1 \) and on the bottom \( y = 0 \), heat flux across the boundaries is zero. Since \( T_h^* > T_c^* \), heat will be conducted from the hot wall (\( x = 0 \)) to the fluid next to the hot wall; conducted and convected in the fluid towards the cold wall (\( x = 1 \)); then conducted from the fluid to the cold wall; finally some of the heat will be convected towards the hot wall in the return flow.

3.2.1 \( \text{Ma} \ll 1 \)

When \( \text{Ma} \ll 1 \), conduction dominates in the heat transfer in the enclosure and temperature decreases linearly from the
hot wall to the cold wall. This can be observed from the isotherm pattern and the free surface temperature distribution in Fig.4.

Upon increasing $Ma$ convection becomes more and more important. At moderate $Ma$ more heat will be transferred in the surface flow towards the cold wall due to the influence of convection than in the return flow, because in the return flow the convection is in an opposing direction to conduction. In the return flow conduction is in the $x$ direction from the hot wall to the cold wall, while convection in the return flow is towards the hot wall. Hence the isotherms are distorted from their conduction pattern as shown in Fig.5. However, convection does not dominate in the enclosure and the geometric lengths still serve as the length scales for the temperature field.

3.2.2 $Ma > 1$

In this case convection heat transfer dominates in the enclosure. However, the conduction terms in Eq.(14.d) are the highest order derivatives and cannot be neglected. Otherwise, not all boundary conditions can be satisfied.
This leads to the occurrence of thermal boundary layers next to the walls.

With the existence of thermal boundary layers, the length scales for the temperature field are no longer of unit order. Fig.6 exhibits the temperature distribution on the free surface. It shows that there is a large temperature gradient next to the cold wall, which is due to the thermal boundary layer next to the cold wall. Also important to notice is that the temperature gradient in the middle portion becomes quite small. Fig.6 shows an S-shaped profile for the temperature distribution on the free surface. The S-shaped profile signifies a change of thermal signature. The S-shape profile has also been observed and indicated by other investigators [9,13,20]. Based on this observation we characterize the free surface temperature distribution by two length scales $\delta_{td}$ and $\delta_{te}$, where $\delta_{te}$ is the thermal boundary layer thickness next to the cold wall and $\delta_{td}$ defines the region next to the hot wall, outside which the driving force due to temperature gradient is negligible compared to that inside. Thus the change of the thermal signature is included in our analysis.
At $Ma > 1$, thermal boundary layers will appear next to the walls. The length and temperature scales at $Ma > 1$ then can be depicted as those in Fig. 7 where $\delta_{th}$ and $\delta_{tc}$ are the thicknesses of the thermal boundary layers next to the hot and cold walls, respectively. $\delta_{ts}$ is the thermal boundary thickness under the free surface in the region indicated by $\delta_{td}$. $\Delta\theta^*_h$ and $\Delta\theta^*_c$ are the temperature scales corresponding to the temperature differences across the thermal boundary layers next to the hot and the cold walls, respectively.

![Diagram of thermal boundary layers](image)

Fig. 7

In Chapters IV and V the scales depicted in Fig. 7 will be derived for fluids of $Pr < 1$ and $Pr \geq 1$, respectively.
CHAPTER IV

Small Prandtl Number Case

In this chapter we analyze and derive the scales for fluids of Prandtl number much less than unity. Because viscous and thermal boundary layers appear when $Re \gg 1$ and $Ma \gg 1$, respectively. The flow will undergo through three stages as $Re$ increases from zero to infinity: flow with no boundary layers exist, flow with viscous boundary layers and but no thermal boundary layers, and flow with both viscous and thermal boundary layers.

4.1 Flow With No Boundary Layers

In this case there is neither a viscous nor a thermal boundary layer. Consequently, the length scales for the velocity and the temperature fields are just those geometrical lengths used as reference lengths in Chapter II. The dimensionless length scales are unity for both velocity and temperature fields. The dimensionless velocity scale in
the x direction, \( u^* \), can be obtained through the balance of the two terms in Eq.(18.d), because this is the boundary condition through which the flow is driven.

\[
\frac{u^*}{1} = \frac{1}{1}
\]

Hence

\[ u^* = 1 \]

The condition for the above scale to be valid is that \( Re \leq 1 \) and \( Ma \leq 1 \). Because \( Ma = Re \cdot Pr \) and \( Pr < 1 \), \( Re \leq 1 \) insures that \( Ma \leq 1 \) is also true. Hence we need only one condition for this case, which is

\[ Re \leq 1 \]

4.2 Flow With Viscous Boundary Layers and No Thermal Boundary Layers

When there are no thermal boundary layers, the dimensionless length scales for the temperature field are
unity in both $x$ and $y$ directions. The velocity field in this case is just that described in Section 3.1.2 with $\delta_{td} = 1$. Fig.8 shows the scales for the velocity field.

![Diagram showing velocity scales]

Here $\delta_{vh}$ and $\delta_{vc}$ are the thicknesses of viscous boundary layers next to the hot and the cold walls, respectively. $\delta_{vs}$ and $u_s^*$ are the length scale in the $y$ direction and the velocity scale in the $x$ direction for the surface flow, respectively. $v_h^*$ and $v_c^*$ are the velocity scales in the $y$ direction in the turning flow next to the hot and the cold walls, respectively. $u_r^*$ is the velocity scale in the $x$ direction in the return flow.

4.2.1 Determination of the scales
(1) \( u_s^* \) and \( \delta_{vs}^* \)

The balance of the only two terms in the driving force boundary condition, Eq.(18.d), is

\[
\frac{u_s^*}{\delta_{vs}^*} = \frac{1}{1}
\]

(19.a)

As explained in Section 3.1.2, the viscous term, \( \frac{1}{Re} \frac{\partial^2 u}{\partial y^2} \), is as important as the inertia term, \( u_s^* \frac{\partial u}{\partial x} \), in the momentum conservation equation in the x direction. The balance of these two terms in Eq.(14.b) is

\[
\frac{u_s^*}{1} = \frac{1}{Re} \frac{u_s^*}{\delta_{vs}^*}
\]

(19.b)

(2) \( v_c^* \) and \( \delta_{vc}^* \)

As described in Section 3.1, the turning flow next to the cold wall can be considered as a stagnation flow, in which the velocity scales in both directions, \( u_s^* \) and \( v_c^* \), are the same. Hence
\[ v_c^* = u_s^* \]  

(19.c)

The inertia term, \( v \cdot \frac{\partial v}{\partial y} \), and the viscous term, \( \frac{1}{Re} \frac{\partial^2 v}{\partial x^2} \), in the momentum conservation equation in the y direction, Eq. (14.c), are balanced for the viscous boundary layer next to the cold wall in the turning flow next to the cold wall. The length scale in the y direction for this balance comes from its connection with the surface flow and is \( \delta_{vs}^* \).

\[ \frac{v_c^*}{\delta_{vs}^*} = \frac{1}{Re} \frac{v_c^*}{\delta_{vc}^*} \]  

(19.d)

\( u_r^* \)

The mass balance between the surface and the return flows is

\[ u_r^* \cdot 1 = u_s^* \cdot \delta_{vs}^* \]  

(19.e)

\( v_h^* \) and \( \delta_{vh}^* \)

Similar to the situation in the turning flow next to the
cold wall, for the turning flow next to the hot wall we have the velocity scale in the y direction to be the same as that in the x direction of the return flow.

\[ v_h^* = u_r^* \quad (19.f) \]

The length scale in the y direction for the balance of the inertia term, \( v \frac{\partial v}{\partial y} \), and the viscous term, \( \frac{1}{Re} \frac{\partial^2 v}{\partial x^2} \), of Eq. (14.c) is from the length scale of the return flow in the y direction which is unity.

\[ v_h^* \frac{v_h^*}{1} = \frac{1}{Re} \frac{v_h^*}{\delta_{vh}^2} \quad (19.g) \]

Solving Eqs. (19) we obtain the scales for this case.

\[ u_s^* = v_c^* = \delta_{vs} = \frac{1}{Re^{1/3}} \quad (20.a) \]

\[ u_r^* = v_h^* = \frac{1}{Re^{2/3}} \quad (20.b) \]

\[ \delta_{vc} = \frac{1}{Re^{1/2}} \quad (20.c) \]
\[ \delta_{wh} = \frac{1}{Re^{1/6}} \quad (20.d) \]

4.2.2 The conditions for this case

Because the velocity scales are not of unit order any more, Ma does not necessarily represent the ratio of convection to conduction. In our analysis there are no thermal boundary layers and the condition for this assumption has to be re-evaluated in view of the change of the velocity scales.

Since inertia and viscous terms are balanced in the surface flow, from \( Pr < 1 \) we have

\[
\begin{align*}
\frac{\text{convection}}{\text{conduction}} &= \frac{\text{inertia}}{\text{viscous}} \cdot Pr < 1
\end{align*}
\]

We can infer that conduction is always important in the surface flow. Hence there will be no thermal boundary layer in the surface flow.

In the return flow the ratio of the convection terms to the conduction terms, the modified Marangoni number, \( Ma^* \), is
$$Ma^* = Ma \cdot u_r = Pr \cdot Re^{1/3}$$  \hspace{1cm} (21)

For there to be no thermal boundary layers, we must have $Ma^* \leq 1$.

In the analysis of this case there are viscous boundary layers. Hence the thicknesses of viscous boundary layers must be much less than unity. In this case there are three such thicknesses, $\delta_{vs}$, $\delta_{vh}$, and $\delta_{vc}$. As Re increases, the viscous boundary layer next to the cold wall appears first, then that under the free surface, and finally that next to the hot wall. Each of them being much smaller than unity can be used as a condition. The most strict one will be the appearance of the viscous boundary layer next to the hot wall, which is $Re^{1/6} \gg 1$. In this surface tension gradient driven flow we can regard the length scale for the surface flow to be important. Then we must have $\delta_{vs} \ll 1$.

From the above analysis the conditions under which the scales of Eqs. (20) are valid are

$$Pr \cdot Re^{1/3} \leq 1$$  \hspace{1cm} (22.a)
and

\[ Re^{1/3} \gg 1 \quad (22.b) \]

4.3 Flow With Both Viscous and Thermal Boundary Layers

For \( Re \gg 1 \) the ratio of the convection terms to the conduction terms in the energy conservation equation, Eq.(14.d), is the modified Marangoni number derived in Section 4.2.2.

\[ Ma^* = Pr \cdot Re^{1/3} \]

When \( Re^{1/3} \gg Pr^{-1} \), we have \( Ma^* \gg 1 \), which means that heat convection dominates and thermal boundary layers will appear. The occurrence of thermal boundary layers also influences the velocity field because of the introduction of scales for the temperature distribution on the free surface, which directly controls the driving force. Consequently, the scales for this case are the combination of those described in Sections 3.1.2 and 3.2.2. The following figure, Fig.9, shows the scales for this case.
In Fig. 9 $\delta_{vd}$, $\delta_{td}$, and $u_d^*$ are the viscous boundary layer thickness, the thermal boundary layer thickness, and the velocity scale in the x direction in the region of the surface flow with the driving force, respectively. The length scale in the x direction for this region is $\delta_{td}$. $\delta_{vs}$ and $u_s^*$ are the length scale in the y direction and the velocity scale in the x direction for the region of the surface flow without driving force, respectively. $\delta_{vc}$, $\delta_{tc}$, and $v_c^*$ are the viscous boundary layer thickness, the thermal boundary layer thickness, and the velocity scale in the y direction in the turning flow next to the cold wall,
respectively. \( \delta_{vh} \), \( \delta_{th} \), and \( v^*_h \) are the viscous boundary layer thickness, the thermal boundary layer thickness, and the velocity scale in the y direction in the turning flow next to the hot wall, respectively. \( \Delta \theta^*_h \) and \( \Delta \theta^*_c \) are the scales for the temperature differences across the thermal boundary layers next to the hot and the cold walls, respectively. Finally, \( u^*_r \) denotes the velocity scale in the x direction in the return flow.

4.3.1 Determination of the scales

(1) \( u^*_s \) and \( \delta^*_v \)

![Diagram](image)

Fig.10

We draw a control volume as shown in Fig.10 in which the top interface of the control volume coincides with the free
surface, the left interface coincides with the hot wall, the bottom interface is the boundary between the surface flow and the return flow, and the right interface is in the region without driving force.

In Fig. 10 \( F_{\text{shear}} \) is the total shear force exerted on the bottom interface of the control volume and \( \Delta p \) is the average pressure difference between the left and the right interfaces of the control volume. The momentum balance for this control volume in the \( x \) direction requires that the total force exerted on the control volume in the \( x \) direction is equal to the difference of the momenta of the fluid flowing into and out of the control volume in the \( x \) direction. The amount of fluid flowing through this control volume is \( u_s \cdot \delta_{vs} \) and the momentum increase in the \( x \) direction is then \( Re \cdot \delta_{vs} \cdot u_s^2 \).

The total forces on the control volume are \( \frac{\Delta \theta_s}{\delta_{td}} \cdot \delta_{td} \), which is the driving force and derived from Eq. (18.d), \( \Delta p \cdot \delta_{vs} \), and \( F_{\text{shear}} \). Among these three forces, \( \frac{\Delta \theta_s}{\delta_{td}} \cdot \delta_{td} \) is the dominate one, since it is the only driving force. Therefore the momentum balance is
\[
\frac{\Delta \theta^{\ast}}{\delta_{td}} \cdot \delta_{td} = \text{Re} \cdot \delta_{vs} \cdot u_{s}^{\ast 2}
\]  

(23.a)

As we explained in Section 3.1.2 the viscous term, \( \frac{1}{\text{Re}} \frac{\partial^2 u}{\partial y^2} \), in Eq.(14.b) is always important in the surface flow, the balance of it with the leading inertia term, \( u \cdot \frac{\partial u}{\partial x} \), gives

\[
u_{s} \cdot \frac{u_{s}^{\ast}}{1} = \frac{1}{\text{Re}} \frac{u_{s}^{\ast}}{\delta_{vs}^2}
\]

(23.b)

(2) \( \delta_{vc}, \delta_{tc} \) and \( v_{c}^{\ast} \)

The same balances in the turning flow next to the cold wall as in Section 4.2.1 and the balance of the convection term, \( v \cdot \frac{\partial \theta}{\partial y} \), and the conduction term, \( \frac{1}{\text{Ma}} \frac{\partial^2 \theta}{\partial x^2} \), of Eq.(14.d) in the thermal boundary layer next to the cold wall are

\[
u_{c}^{\ast} = u_{s}^{\ast}
\]

(23.c)

\[
\nu_{c}^{\ast} \cdot \frac{v_{c}^{\ast}}{\delta_{vs}} = \frac{1}{\text{Re}} \frac{v_{c}^{\ast}}{\delta_{vc}^2}
\]

(23.d)
\[ v \cdot \frac{\Delta \theta \circ}{\Delta \theta \circ_c} = \frac{1}{Ma} \cdot \frac{\Delta \theta \circ}{\delta_{tc}^2} \quad (23.e) \]

(3) \( u_r \)

The mass balance between the surface flow and the return flow is

\[ u_r \cdot 1 = u_s \cdot \delta_{vs} \quad (23.f) \]

(4) \( \delta_{vh}, \delta_{th}, \text{ and } v_h \)

Because the driving force is confined to the region with length scale \( \delta_{td} \), the low pressure created by the driving force in this region directs the fluid from the return flow mostly towards this region with driving force. Just outside this region the thickness of the viscous boundary layer under the free surface grows rapidly. Further towards the cold wall the growth of the thickness of this viscous boundary layer is very small due to the negligible driving force exerted on this part of the free surface. The mass balance of the return flow and that towards the region with driving
force is

\[ \frac{v^*}{h} \cdot \delta_{td} = \frac{u^*}{r} \cdot 1 \quad (23.g) \]

For the viscous boundary layer next to the hot wall we have the following inertia and viscous term balance which is the same as the one in Section 4.2.1.

\[ \frac{v^*}{h} \cdot \frac{1}{1} = \frac{1}{Re} \cdot \frac{v^*}{h} \delta_{\nu h}^2 \quad (23.h) \]

With the same length scale in the y direction, unity, for this turning flow next to the hot wall, the balance of the convection term, \( v \cdot \frac{\partial \theta}{\partial y} \), and the conduction term, \( \frac{1}{Ma} \cdot \frac{\partial^2 \theta}{\partial x^2} \), of Eq.(14.d) for the thermal boundary layer next to the hot wall is

\[ \frac{v^*}{h} \cdot \frac{1}{1} = \frac{1}{Ma} \cdot \frac{\Delta \theta_h^*}{\delta_{th}^2} \quad (23.1) \]

(5) \( \delta_{vzd}, \delta_{tsd} \) and \( u^* \)

These scales are in the region of the surface flow with
the driving force. The driving force boundary condition, Eq.(18.d), requires the balance of its only two terms, since this is the mechanism through which the flow is driven.

$$\frac{\Delta \theta_h}{\delta_{td}} = \frac{u_d}{\delta_{vsd}}$$  \hspace{1cm} (23.j)

The viscous boundary layer thickness is obtained by the balance of the viscous term, $\frac{1}{Re} \frac{\partial^2 u}{\partial y^2}$, and the inertia term, $u \frac{\partial u}{\partial x}$, of Eq.(14.d).

$$\frac{u_d}{\delta_{td}} = \frac{1}{Re} \frac{u_d}{\delta_{vsd}}$$  \hspace{1cm} (23.k)

Since $Pr < 1$, the thickness of the thermal boundary layer is greater than the thickness of the viscous boundary layer in the region of the surface flow with driving force. In the thermal boundary layer the flow is mainly towards the viscous boundary layer in the surface flow with the driving force due to the entrainment effect of the driving force on the viscous boundary layer. Let the velocity scale in the $y$ direction in the thermal boundary layer be $v_d^*$ as shown in Fig.11.
The mass balance of the fluid flowing through the thermal boundary layer into the viscous boundary layer and the fluid flowing out of the viscous boundary layer is

\[ v_d^* \cdot \delta_{td} = \delta_{vd} \cdot u_d^* \]  \hspace{1cm} (23.1)

With velocity scale \( v_d^* \), the balance of the convection term, \( v \cdot \frac{\partial \theta}{\partial y} \), and the conduction term, \( \frac{1}{Ma} \frac{\delta^2 \theta}{\delta y^2} \), is

\[ v_d^* \frac{\Delta \theta_h^*}{\delta_{tsd}} = \frac{1}{Ma} \frac{\Delta \theta_h}{\delta_{tsd}^2} \]  \hspace{1cm} (23.m)

(6) \( \Delta \theta_h^* \), \( \Delta \theta_c^* \), and \( \delta_{td} \)

Since the total dimensionless temperature difference from
the hot wall to the cold wall is unity, we have

$$\Delta \theta^*_{h} + \Delta \theta^*_{c} = 1 \quad (23.n)$$

Analogous to the balance of mass among the flows in different regions, a global heat transfer balance is necessary to link the temperature fields in different regions. The total heat conduction across the thermal boundary layer next to the hot wall, the total heat convection in the bulk of the fluid towards the cold wall, and the total heat conduction across the thermal boundary layer next to the cold wall must be the same. The total heat convection in the flow towards the cold wall can be the net heat convection across any constant $\times$ section in the flow field. If we choose the net heat convection in the region of the surface flow with the driving force, we have the following global heat transfer balances.

$$\frac{\Delta \theta^*_{h}}{\delta_{th}} = Ma^* \Delta \theta^*_{h} \cdot \frac{\delta_{vd} \cdot u^*}{\delta_{tc} \cdot v_w} = \frac{\Delta \theta^*_{c}}{\delta_{tc}} \quad (23.o)$$

Solving Eqs.(23), we obtain the following scales.
\begin{align}
\dot{u}_s &= \dot{v}_c = \frac{1}{\text{Pr}^{2/11} \text{Re}^{13/33}} \quad (24.a) \\
\dot{u}_d &= \frac{1}{\text{Re}^{1/2}} \quad (24.b) \\
\dot{u}_r &= \frac{1}{\text{Pr}^{1/11} \text{Re}^{23/33}} \quad (24.c) \\
\dot{v}_h &= \frac{\text{Pr}^{5/11}}{\text{Re}^{17/33}} \quad (24.d) \\
\dot{v}_d &= \frac{\text{Pr}^{3/11}}{\text{Re}^{19/33}} \quad (24.e) \\
\delta_{vs} &= \frac{\text{Pr}^{1/11}}{\text{Re}^{10/33}} \quad (24.f) \\
\delta_{vsd} &= \frac{1}{\text{Pr}^{3/11} \text{Re}^{14/33}} \quad (24.g) \\
\delta_{vh} &= \frac{1}{\text{Pr}^{5/22} \text{Re}^{8/33}} \quad (24.h) \\
\delta_{vc} &= \frac{\text{Pr}^{3/22}}{\text{Re}^{5/11}} \quad (24.i) \\
\delta_{td} &= \frac{1}{\text{Pr}^{6/11} \text{Re}^{2/11}} \quad (24.j)
\end{align}
\[ \delta_{td} = \frac{1}{\Pr^{14/11} \cdot Re^{14/33}} \]  
(24.k)

\[ \delta_{th} = \frac{1}{\Pr^{8/11} \cdot Re^{8/33}} \]  
(24.l)

\[ \delta_{tc} = \frac{1}{\Pr^{4/11} \cdot Re^{5/11}} \]  
(24.m)

\[ \Delta \theta_h^* = \frac{1}{\Pr^{3/11} \cdot Re^{1/11}} \]  
(24.n)

\[ \Delta \theta_c^* = 1 \]  
(24.o)

Here the scale for the temperature difference across the thermal boundary layer next to the hot wall can be approximated by unity, because in almost all situations encountered it is not the case that \( \Pr^{3/11} \cdot Re^{1/11} > 1 \). With this approximation we have the following scales for this case.

\[ u_s^* = v_c^* = \frac{1}{Re^{1/3}} \]  
(25.a)

\[ u_d^* = \frac{Pr^{1/5}}{Re^{4/15}} \]  
(25.b)
\[ u_r^* = \frac{1}{Re^{2/3}} \quad (25.c) \]

\[ v_h^* = \frac{Pr^{3/5}}{Re^{7/15}} \quad (25.d) \]

\[ v_d^* = \frac{Pr^{2/5}}{Re^{8/15}} \quad (25.e) \]

\[ \delta_{v_s} = \frac{1}{Re^{1/2}} \quad (25.f) \]

\[ \delta_{v_{sd}} = \frac{1}{Pr^{2/5}Re^{7/15}} \quad (25.g) \]

\[ \delta_{v_h} = \frac{1}{Pr^{3/10}Re^{4/15}} \quad (25.h) \]

\[ \delta_{v_c} = \frac{1}{Re^{1/2}} \quad (25.i) \]

\[ \delta_{t_d} = \frac{1}{Pr^{3/5}Re^{1/5}} \quad (25.j) \]

\[ \delta_{t_{sd}} = \frac{1}{Pr^{7/5}Re^{7/15}} \quad (25.k) \]

\[ \delta_{t_h} = \frac{1}{Pr^{4/5}Re^{4/15}} \quad (25.l) \]
\[ \delta_c = \frac{1}{Ma^{1/2}} \quad (25.m) \]

\[ \Delta \theta^* = \Delta \theta^* = 1 \quad (25.n) \]

4.3.2 The conditions for this case

For the scales of Eqs.(24) and Eqs.(25) to be valid both viscous and thermal boundary layers have to be present in the flow. The existence of the viscous boundary layer of the surface flow is guaranteed by the existence of the thermal boundary layers for \( \Pr \approx 1 \), which can be insured by the condition that the modified Marangoni number given by Eq.(21) is larger than unity. That is

\[ \Pr \cdot Re^{1/3} > 1 \quad (26.a) \]

The condition for the scales given by Eqs.(24) and Eqs.(25) to be valid is the inequality of Eq.(26.a). In addition to Eq.(26.a) for the scales given by Eqs.(25) to be valid, \( \Delta \theta^* \) have to be on the order of unity. Hence we have another condition for the scales of Eqs.(25), which is

\[ \Pr^{3/11} Re^{1/11} \leq 1. \quad (26.b) \]
CHAPTER V

Unit Order And Large Prandtl Number Case

In this chapter the scales for both the velocity and the temperature fields are derived for fluids of unit order or large Prandtl numbers, namely, $Pr \approx 1$. Because the appearances of thermal boundary layers and viscous boundary layers are due to $Ma > 1$ and $Re > 1$, respectively, the flow can be divided into the following regimes in the increasing order of $Re$: flow with no boundary layers, flow with thermal boundary layers but no viscous boundary layers, which is only applicable for the case of $Pr > 1$, since $Ma = Re\cdot Pr$, and flow with both thermal and viscous boundary layers.

In the case of $Pr > 1$, the velocity scales for the thermal boundary layers next to the walls are not the same as the velocity scales outside, because the velocity scales we derive are the scales for the maximum velocities. The difference in the length scales for the viscous and the thermal boundary layers requires a modification of the
velocity scales inside thermal boundary layers. Velocity scales in thermal boundary layers are approximated by

\[
\text{velocity scale} \approx \frac{\text{thermal boundary layer thickness}}{\text{length scale for the velocity field}}
\]

The length scales for the velocity field in the above formula are the viscous boundary layer thicknesses in the case of flow with viscous boundary layers and unity in the case of no viscous boundary layers.

For fluids of unit order Prandtl number the above formula is also applicable since the thicknesses of the thermal boundary layer and the viscous boundary layer are of the same order and the above formula contains no modification of the velocity scale.

5.1 Flow With No Boundary Layers

The treatment of this case is the same as that of the situation in Section 4.1 where no boundary layers exist and fluid has \( \text{Pr} \ll 1 \). The dimensionless length scales for both the velocity and the temperature fields are of unit order.
The velocity scale in the x direction, \( u^* \), can be obtained from the balance of the two terms in Eq. (18.d), which is the driving force boundary condition.

\[
\frac{u^*}{1} = \frac{1}{1}
\]

We get

\[ u^* = 1 \]

The conditions for the above scale to be valid are that \( Re \leq 1 \) and \( Ma \leq 1 \). Since \( Ma = Re \cdot Pr \) and \( Pr \geq 1 \), \( Ma \leq 1 \) guarantees that \( Re \leq 1 \). Therefore the only condition for this case is

\[ Ma \leq 1 \]

5.2 Flow With Thermal Boundary Layers and NoViscous Boundary Layers

This case is for \( Ma > 1 \) and \( Re \leq 1 \). From \( Ma = Re \cdot Pr \) it is apparent that this situation can only occur when \( Pr > \)
1.

The temperature field of this case is just that described in Section 3.2.2. Because of the appearance of the length scale $\delta_{td}$ next to the hot wall for the free surface temperature distribution the length scale for the velocity scale inside the region of the surface flow with driving force is not unity and let this length scale be $\delta_v$. $u^*$ and $v^*$ are the velocity scales in the $x$ and $y$ directions, respectively.

![Diagram](image)

**Fig.12**

In Fig.12 $\delta_{tc}$ is the thermal boundary layer thickness in the turning flow next to the cold wall, $\delta_{th}$ is the thermal boundary layer thickness in the turning flow next to the hot
wall, and $\delta_{ts}$ is the thermal boundary layer thickness in the region of the surface flow with driving force, respectively. $\Delta\theta_h$ and $\Delta\theta_c$ are the scales for the temperature differences across the thermal boundary layers next to the hot and the cold walls, respectively.

5.2.1 Determination of the scales

(1) $u^*, v^*$, and $\delta_v$

Here the balance of the two terms in the driving force boundary condition, Eq.(18.d), gives

$$\frac{u^*}{\delta_v} = \frac{\Delta\theta_h}{\delta_{td}} \quad (27.a)$$

The balance of the only two terms in the continuity equation, Eq.(14.a) is

$$\frac{u^*}{l} = \frac{v^*}{l} \quad (27.b)$$

Because the driving force is confined to the region with length scale $\delta_{td}$ and in this region the flow is viscous, the
length scale in both \( x \) and \( y \) directions will be the same for this region. Hence

\[
\delta_v = \delta_{td}
\]  
(27.c)

(2) \( \delta_{th} \) and \( \delta_{tc} \)

Similar to the situation in Section 4.3 the convection-conduction balances in the thermal boundary layers next to the walls are

\[
\frac{\delta_{th} \cdot \Delta \theta_h^*}{1} \cdot \frac{1}{\frac{1}{1}} = \frac{1}{\text{Re} \cdot \text{Pr} \cdot \frac{\Delta \theta_h^*}{\delta_{th}^2}}
\]  
(27.d)

\[
\frac{\delta_{tc} \cdot \Delta \theta_c^*}{1} \cdot \frac{1}{\frac{1}{1}} = \frac{1}{\text{Re} \cdot \text{Pr} \cdot \frac{\Delta \theta_c^*}{\delta_{tc}^2}}
\]  
(27.e)

Here we notice the modification of the velocity scale in the thermal boundary layers.

(3) \( \delta_{ts} \)

In the thermal boundary layer in the region of the
surface flow with driving force there is no need for the modification of the velocity scale, because velocity is maximum on the free surface. The convection and conduction balance in this thermal boundary layer is also similar to the situation in Section 4.3.

\[
\frac{\Delta \theta^*}{u} \cdot \frac{\Delta \theta^*}{\delta_{td}} = \frac{1}{\text{Re} \cdot \text{Pr}} \cdot \frac{\Delta \theta^*}{\delta_{ts}} \quad (27.f)
\]

\[
\delta_{td}, \Delta \theta^*_h, \text{ and } \Delta \theta^*_c
\]

(4) Similar global heat transfer balances to those in Section 4.3 are

\[
\frac{\Delta \theta^*_h}{\delta_{th}} = \text{Ma} \cdot \text{Ma}^* \cdot \delta_{ts} \cdot u^* = \frac{\Delta \theta^*_c}{\delta_{tc}} \quad (27.g)
\]

Solving Eqs. (27), we obtain the scales for this case.

\[
\delta_{td} = \delta_v = \frac{1}{\text{Ma}^{1/3}} \quad (28.a)
\]

\[
\delta_{ts} = \frac{1}{\text{Ma}^{2/3}} \quad (28.b)
\]
\[ \delta_{tc} = \delta_{th} = \frac{1}{Ma^{1/3}} \]  
(28.c)

\[ u^* = v^* = 1 \]  
(28.d)

\[ \Delta\theta_h = \Delta\theta_c = 1 \]  
(28.e)

5.2.2 The conditions for this case

For the thermal boundary layers to exist the derived thicknesses of the thermal boundary layers must be much less than unity. If we take the thermal boundary layer in the region of the surface flow with driving force as the criteria, this condition is \( \delta_{ts} < 1 \). We have

\[ Ma^{2/3} > 1 \]  
(29.a)

The Reynolds number representing the ratio of the inertia terms to the viscous terms cannot be much larger than unity for there can be no viscous boundary layers.

\[ Re \leq 1 \]  
(29.b)
The inequalities of Eqs. (29) are the conditions for this case. It is quite obvious that for the two conditions of Eqs. (29) to be met simultaneously the Prandtl number, Pr, must be much greater than unity.

5.3 Flow With Both Thermal and Viscous Boundary Layers

When Re is much greater than unity both thermal and viscous boundary layers will appear because Ma = Re·Pr. In this case the velocity and the temperature fields are those described in Sections 3.1.2 and 3.2.2, respectively. The scales for this case are shown in Fig.13.

In Fig.13 $\delta_{td}$, $\delta_{vd}$, and $u^*_d$ are the thermal boundary layer thickness, the viscous boundary layer thickness, and the velocity scale in the x direction in the region of the surface flow with driving force, respectively. The length scale in the x direction for this region of the surface flow with driving force is denoted by $\delta_{td}$. $\delta_v$ and $u^*_x$ are the viscous boundary layer thickness and the velocity scale in the x direction for the region of the surface flow without driving force, respectively. $\delta_{tc}$, $\delta_{vc}$, and $v^*_c$ are the thermal...
boundary layer thickness, the viscous boundary layer thickness, and the velocity scale in the \( y \) direction in the turning flow next to the cold wall, respectively. \( u^* \) is the velocity scale in the \( x \) direction in the return flow. \( \delta_{th} \), \( \delta_{vh} \), and \( v^*_h \) are the thermal boundary layer thickness, the viscous boundary layer thickness, and the velocity scale in the \( y \) direction in the turning flow next to the hot wall, respectively. Finally, \( \Delta \Theta^*_h \) and \( \Delta \Theta^*_c \) represent the scales for the temperature differences across the thermal boundary layers next to the hot and the cold walls, respectively.

![Diagram of boundary layers](image)

**Fig. 13**

In this case of both thermal and viscous boundary layers
the length scales for the velocity field used in the modification of the velocity scales in the thermal boundary layers are the corresponding viscous boundary layer thicknesses as mentioned in the beginning of this chapter.

5.3.1 Determination of the scales

(1) $\delta_{vs}$ and $u_s^*$

Here again as in (1) of Section 4.3.1 we draw a control volume as shown in Fig.14. The balance of the driving force on and the momentum increase of fluid through the control volume is

![Diagram](image-url)
\[ \frac{\Delta \theta}{\delta_{td}} \cdot \delta_{td} = u_s^* \cdot \delta_{vs} \cdot \text{Re} \]

(30.a)

The balance of the inertia term, \( u \cdot \frac{\partial u}{\partial x} \), and the viscous term, \( \frac{1}{\text{Re}} \cdot \frac{\partial^2 u}{\partial y^2} \), for the boundary layer of the surface flow in the region without driving force is

\[ \frac{u_s^*}{u_s^* - 1} = \frac{1}{\text{Re}} \cdot \frac{u_s^*}{\delta_{vs}^2} \]

(30.b)

Here the length scale in the \( x \) direction is unity.

(2) \( \delta_{tc}, \delta_{wc}, \) and \( v_c^* \)

The turning flow next to the cold wall is considered as a stagnation flow and explained in (2) of Section 4.2.1. Hence

\[ v_c^* = u_s^* \]

(30.c)

The balance of the inertia term, \( v \cdot \frac{\partial v}{\partial x} \), and the viscous term, \( \frac{1}{\text{Re}} \cdot \frac{\partial^2 v}{\partial x^2} \), of Eq.(14.c) for the viscous boundary layer next to the cold wall is
\[ v^* \cdot \frac{v_c}{\delta_{vs}} = \frac{1}{\text{Re}} \cdot \frac{v_c}{\delta_{vc}} \]  

(30.d)

The balance of the convection term, \( v^* \frac{\partial \theta}{\partial y} \), and the conduction term, \( \frac{1}{\text{Ma}} \cdot \frac{\delta^2 \theta}{\delta x^2} \), of Eq.(14.d) with the modification of the velocity scale described at the beginning of this chapter is

\[ \frac{\delta_{tc}}{\delta_{vc}} \cdot v^* \frac{\Delta \theta_c}{\delta_{vs}} = \frac{1}{\text{Ma}} \cdot \frac{\Delta \theta_c}{\delta_{tc}^2} \]  

(30.e)

(3) \( u_r^* \)

To determine the velocity scale in the x direction in the return flow, the mass balance between the surface flow and the return flow is used.

\[ u_r^* \cdot 1 = u_s^* \cdot \frac{\delta}{\delta_{vs}} \]  

(30.f)

(4) \( \delta_{th}, \delta_{vh} \) and \( v_h^* \)

The fluid from the return flow is mostly driven towards the region of the surface flow with driving force and the
mass balance between this flow towards the region of the surface flow with driving force and the return flow is

\[ \frac{v^*}{h} \cdot \delta_{td} = \frac{u^*}{r} \cdot 1 \]  

(30.g)

The inertia term, \( v^* \frac{\partial v}{\partial y} \), and the viscous term, \( \frac{1}{Re} \frac{\partial^2 v}{\partial x^2} \), of Eq. (14.c) are balanced in the viscous boundary layer next to the hot wall.

\[ \frac{v^*}{h} \cdot \frac{v^*}{h} = \frac{1}{Re} \frac{v^*}{v_h} \]  

(30.h)

With the modification of the velocity scale explained at the beginning of this chapter the convection term, \( v^* \frac{\partial \theta}{\partial y} \), and the conduction term, \( \frac{1}{Ma} \frac{\partial^2 \theta}{\partial x^2} \), of Eq. (14.d) are balanced in the thermal boundary layer next to the hot wall.

\[ \frac{\delta_{th}}{\delta_{vh}} \cdot v^* \cdot \Delta \theta = \frac{1}{Ma} \frac{\Delta \theta}{\delta_{th}} \]  

(30.1)

(5) \( \delta_{tsd} \), \( \delta_{vd} \), and \( u^* \)

Here we have similar balances to those in (5) of Section
4.3.1 for this region of the surface flow with driving force.

\[ \frac{u_d}{\delta_{vsd}} = \frac{\Delta \theta_h}{\delta_{td}} \]  \hspace{1cm} (30.j)

\[ u_d \cdot \frac{u_d}{\delta_{td}} = \frac{1}{Re} \cdot \frac{u_d}{\delta_{vsd}} \]  \hspace{1cm} (30.k)

\[ u_d \cdot \frac{\Delta \theta_h}{\delta_{td}} = \frac{1}{Ma} \cdot \frac{\Delta \theta_h}{\delta_{tsd}} \]  \hspace{1cm} (30.l)

(6) \Delta \theta_h, \Delta \theta_c, \delta_{td}

Here the similar global heat transfer balances to those in (6) of Section 4.3.1 are used.

\[ \frac{\Delta \theta_h}{\delta_{th}} = u_d \cdot \Delta \theta_h \cdot \delta_{tsd} \cdot Ma = \frac{\Delta \theta_c}{\delta_{tc} \cdot \delta_{vs}} \]  \hspace{1cm} (30.m)

Solving Eqs.(30), we obtain the scales for this case.

\[ \delta_{td} = \frac{1}{Pr^{2/11}Re^{2/11}} \]  \hspace{1cm} (31.a)

\[ \delta_{th} = \frac{1}{Pr^{9/33}Re^{8/33}} \]  \hspace{1cm} (31.b)
\[ \delta_{tc} = \frac{1}{\text{Pr}^{19/66} \text{Re}^{5/11}} \]  
(31.c)

\[ \delta_{tsd} = \frac{1}{\text{Pr}^{13/22} \text{Re}^{14/33}} \]  
(31.d)

\[ \delta_{vsd} = \frac{\text{Pr}^{1/33}}{\text{Re}^{10/33}} \]  
(31.e)

\[ \delta_{vsh} = \frac{1}{\text{Pr}^{1/11} \text{Re}^{14/33}} \]  
(31.f)

\[ \delta_{vh} = \frac{1}{\text{Pr}^{5/66} \text{Re}^{8/33}} \]  
(31.g)

\[ \delta_{vc} = \frac{\text{Pr}^{1/22}}{\text{Re}^{5/11}} \]  
(31.h)

\[ u_s = v_c = \frac{1}{\text{Pr}^{2/33} \text{Re}^{13/33}} \]  
(31.i)

\[ u_d = \frac{1}{\text{Re}^{1/3}} \]  
(31.j)

\[ u_r = \frac{1}{\text{Pr}^{1/32} \text{Re}^{23/33}} \]  
(31.k)

\[ v_h = \frac{\text{Pr}^{5/33}}{\text{Re}^{17/33}} \]  
(31.l)
\[ \Delta \theta^*_h = \frac{1}{Ma^{1/11}} \]  

(31.m)

\[ \Delta \theta^*_c = 1 \]  

(31.n)

The scale for the temperature difference across the thermal boundary layer next to the hot wall \( \Delta \theta^*_h \) can be approximated by unity, because in most situations it is not the case that \( Ma^{1/11} \gg 1 \). With this approximation the scales for this case are

\[ \delta_{td} = \frac{1}{Ma^{1/5}} \]  

(32.a)

\[ \delta_{th} = \frac{1}{Pr^{13/30}Re^{4/15}} \]  

(32.b)

\[ \delta_{tc} = \frac{1}{Pr^{1/3}Re^{1/2}} \]  

(32.c)

\[ \delta_{tsd} = \frac{1}{Pr^{19/30}Re^{7/15}} \]  

(32.d)

\[ \delta_{ws} = \frac{1}{Re^{1/3}} \]  

(32.e)
\[ \delta_{v_{sd}} = \frac{1}{Pr^{2/15}Re^{7/15}} \]  (32.f)

\[ \delta_{v_{h}} = \frac{1}{Pr^{1/10}Re^{4/15}} \]  (32.g)

\[ \delta_{v_{c}} = \frac{1}{Re^{1/2}} \]  (32.h)

\[ \frac{u^*}{v^*} = \frac{1}{Re^{1/3}} \]  (32.1)

\[ u^*_d = \frac{Pr^{1/15}}{Re^{4/15}} \]  (32.j)

\[ u^*_r = \frac{1}{Re^{2/3}} \]  (32.k)

\[ v^*_h = \frac{Pr^{1/5}}{Re^{7/15}} \]  (32.1)

\[ \Delta \theta^*_h = \Delta \theta^*_c = 1 \]  (32.m)

5.3.2 The condition for this case

The condition for the above obtained scales to be valid
is that both thermal and viscous boundary layers exist. Since \( \text{Pr} \approx 1 \) in this case, the existence of viscous boundary layers guarantees the existence of thermal boundary layers. Therefore the condition is that viscous boundary layers exist. Here we have four viscous boundary layer thicknesses, \( \delta_{vs}, \delta_{vsd}, \delta_{vh}, \) and \( \delta_{vc} \). The viscous boundary layer next to the cold wall will appear first and the appearance of the viscous boundary layer next hot wall gives the most strict condition. In this thrmocapillary flow if we choose the viscous boundary layer of the surface flow in the region without driving force as the representative one, we have the following condition.

\[
\delta_{vs} = \frac{\text{Pr}^{1/33}}{\text{Re}^{10/33}} \ll 1
\]  \hspace{1cm} (33)

If we consider the situation where the scales given by Eqs.(32) are valid, we have the following condition when it is not the case that \( \text{Ma}^{1/11} > 1 \).

\[
\delta_{vs} = \frac{1}{\text{Re}^{1/3}} \ll 1
\]  \hspace{1cm} (34.a)

and

\[
\text{Ma}^{1/11} \leq 1
\]  \hspace{1cm} (34.b)
CHAPTER VI

Discussion

In this chapter the temperature and velocity fields are discussed in view of the scales obtained. Also the results from the scaling analysis are discussed in comparison to the numerical simulations and the results from works of other investigators. The numerical simulations are the solutions of Eqs. (14) with boundary conditions Eqs. (15) through (18) and the numerical scheme is the SIMPLE algorithm given by Patankar [28]. At the end of this chapter the results from the scaling analysis are summarized.

6.1 The Temperature Field

When there are no thermal boundary layers, the length scales for the temperature field are unity and the temperature field is just that described in Section 3.2.1.

In our scaling analysis we have obtained the thermal
boundary layer thicknesses next to the hot and the cold walls, $\delta_{th}$ and $\delta_{tc}$. The length scales in the $y$ direction of these two boundary layers are of order 1 and $\delta_{\infty}$, because of the stagnation flow approximation in which the length scales of the turning flows are the same in both directions. This can be confirmed qualitatively by the isotherms presented in Fig.15 for $Pr = 10$. Also in the case of $Pr \gg 1$, the scales for the thicknesses of these two thermal boundary layers are $\delta_{th} = \frac{1}{Pr^{3/30}Re^{4/15}}$ and $\delta_{tc} = \frac{1}{Pr^{1/2}Re^{1/2}}$ as given by Eqs.(32), and as Re increases, $\delta_{tc}$ decreases faster than $\delta_{th}$. The isotherm of Fig.16 shows that the thermal boundary layer next to the cold wall is thinner than that next to the hot wall.

The S-shaped profile for the temperature distribution on the free surface was observed for $Ma \gg 1$ and discussed by Chun [9], Fu and Ostrach [13], and Lai [20]. In the scaling analysis we have obtained the length scales for the temperature distribution on the free surface, $\delta_{td}$ and $\delta_{tc}$ for the portions of the temperature distribution next to the hot wall and the thermal boundary layer next to the cold wall with driving force. The temperature gradient in the mid portion is very small and the driving force generated by it
is negligible compared to that in the region defined by $\delta_{td}$. Fig. 16 is the free surface temperature distribution at $Pr = 10$, $Re = 1000$ and $Re = 10000$. It clearly shows the S-shaped characteristic and that the region with the driving force next to the hot wall is larger than that next to the cold wall. This confirms our assertion in Chapter I that the two regions with the driving force cannot be interpreted the same way. For $Pr \approx 1$, our scaling analysis gives $\delta_{td} = \frac{1}{Ma^{1/5}}$ and $\delta_{tc} = \frac{1}{Pr^{1/3}Re^{1/2}}$ given by Eqs. (32). The ratio of $\delta_{td}$ to $\delta_{tc}$ is $Pr^{2/15}Re^{3/10}$, which suggests that $\delta_{tc}$ is much smaller than $\delta_{td}$ at $Re \gg 1$. This is also shown by Fig. 16. Also we notice that $\delta_{td}$ decreases quite slowly with increasing $Ma$, which is observed in Fig. 16.

The length scale $\delta_{td}$ for low Prandtl number fluids is $\frac{1}{Pr^{3/5}Re^{1/5}}$ and this shows that the appearance of the S-shaped free surface temperature distribution comes at a much higher $Re$ in the case of low Prandtl number than in the case of high Prandtl number. In $Pr \approx 1$ case, the larger the Prandtl number, the smaller $\delta_{td}$ for a given Reynolds number. Hence the S-shaped free surface temperature distribution can only be shown for large Prandtl number.
situations in our numerical simulations due to the difficulty encountered in the numerical simulations of large Reynolds number flows.

In our scaling analysis the determination of $\delta_{td}$ depends on the global heat transfer balances. Because these balances are not local, the length scales in the direction perpendicular to the two-dimensional plane under consideration has to be incorporated. In a two-dimensional situation the length scale in that third direction can be taken as unity. However, if cylindrical or other geometrical coordinate systems are used, we have to take into account different length scales in the third direction for the different terms in the global heat transfer balances. Also in our analysis both the entire hot and the entire cold walls are kept at constant temperatures. When other heating methods are incorporated, the global heat transfer balances will be different, too. Knowing the dependency of $\delta_{td}$ on the global heat transfer balances and the dependency of the global heat transfer balances on the heating method and the geometry, we can conclude that the length scale $\delta_{td}$ characterizing the part of the S-shaped free surface temperature distribution next to the hot wall, depends on the
heating method and the geometry.

6.2 The Velocity Field

In our analysis the determination of the velocity scale of the surface flow is very important, because it is the direct consequence of the driving force. The S-shaped temperature distribution on the free surface for $Ma^{1/5} > 1$ is taken into account and only negligible driving force is present on the mid-portion of the free surface when the length scales for the two regions of the free surface next to the walls are much smaller than unity. When $Re \leq 1$, the aforementioned velocity scale is of unit order. At $Re > 1$, the velocity scale is $\frac{1}{Re^{1/3}}$ with the following conditions for the four cases with viscous boundary layers. In our conditions the thermal or viscous boundary layers in the surface flow are taken to be the representative ones for the reasons explained in Sections 4.2.2 and 5.2.2, respectively.

1. $Pr < 1, \, Re^{1/3} > 1, \, \text{and} \, Pr \cdot Re^{1/3} \leq 1$

2. $Pr < 1, \, Re^{1/3} > 1, \, Pr \cdot Re^{1/3} > 1$,
and $Pr^{3/11} Re^{1/11} \leq 1$

(3) $Pr > 1$, $Pr^{3/2} Re^{1/2} > 1$, $Re^{1/2} \leq 1$, and $Ma^{1/19} \leq 1$

(4) $Pr \geq 1$, $Re^{1/3} > 1$, and $Ma^{1/11} \leq 1$

Fig. 17 shows that numerical results of the free surface velocity in the $x$ direction in comparison with the scaling analysis result of $\frac{1}{Re^{1/3}}$. With increasing $Re$, the agreement becomes better between the scaling result and the numerical results. This is to be expected, since the boundary layer approach is expected to render good results only when the thicknesses of the boundary layers are much smaller than unity, which in turn means a very large $Re$ flow. The trend of the numerical data also approach that of the scaling analysis. This confirms the validity of the velocity scale of the surface flow.

It is very difficult to compare the scaling results for the viscous boundary layer thicknesses, because the boundaries of the viscous boundary layers are not readily identifiable and are not necessarily the zero-velocity lines.
The difficulty also arises from the fact that we are not able to simulate numerically flows when $\text{Re}^{1/3}$ is much larger than unity, because the viscous boundary layer thickness of the surface flow is on the order of $\frac{1}{\text{Re}^{1/3}}$ and in order to identify this surface viscous boundary layer clearly we need the condition of $\text{Re}^{1/3} > 1$ in our simulations.

6.3 Comparisons With Other Works

When no viscous boundary layer exists, the velocity scale for the surface flow is of unit order, which is the direct result of the balance of the two terms in the driving force boundary condition Eq.(18.d). In the case of flow with viscous boundary layers, the thickness of the surface flow and the velocity scale in the $x$ direction for the surface flow are both $\frac{1}{\text{Re}^{1/3}}$ under the conditions $\text{Pr}^{2/11} \text{Re}^{1/11} \leq 1$ for the case of $\text{Pr} < 1$ and $\text{Pr} \cdot \text{Re}^{1/3} > 1$, $\text{Ma}^{1/19} \leq 1$ for the case of $\text{Pr} > 1$, $\text{Pr}^{3/2} \text{Re}^{1/2} > 1$, and $\text{Re}^{1/2} \leq 1$, and $\text{Ma}^{1/11} \leq 1$ for the case of $\text{Pr} \geq 1$ and $\text{Re}^{1/3} > 1$.

This result agrees with Ostrach's analysis [26], even
though Ostrach's analysis does not consider the influence of the S-shaped profile for the free surface temperature distribution.

In the analysis of low Prandtl number flow by Rivas and Ostrach [27], the length scale for the temperature distribution on the free surface is fixed due to the imposed heat flux boundary condition on the free surface with a fixed length scale. Thus we can only compare Rivas' results with our case of \( Pr < 1 \) when there are no thermal boundary layers. In this case our results for the thickness of the surface flow, the velocity scale for the surface flow, and the velocity scale for the return flow when there are viscous boundary layers agree with Rivas' results. However, the length scales for the boundary layers next to the walls are not given in Rivas' work. Consequently, no comparison can be made for these boundary layers.

In the analysis of \( A < 1 \) flows by Strani et al. [29], it is mentioned that there is no viscous boundary layer. This is only true for fixed \( Re \) and \( A \to 0 \) as is mentioned in [29]. In our case the aspect ratio \( A \) is fixed with unit order and \( Re \) increases, hence boundary layers will eventually
appear when Re is sufficiently large.

The analysis by Zebib et al. [14] for Pr = 1 assumes that the $\delta_{td}$ in our analysis is independent of Re. This leads to disagreement between the results of [14] and ours. In [14] this assumption is mainly supported by the observation of the numerical data for Pr = 1 which does not show a substantial decrease in $\delta_{td}$ with increasing Re and no explanation is given for the assumption. Therefore we think this assumption is not well justified. For Pr = 1 the result of our scaling analysis for $\delta_{td}$ is

$$\delta_{td} = \frac{1}{Ma^{1/5}}$$

in the range of $Ma^{1/11}$ not much greater than unity, which is satisfied in all the numerical simulations in [14] and our numerical simulations. The Ma in the numerical results of [14] is not large enough so that $Ma^{1/5} \gg 1$. Thus this change of thermal signature, which occurs when $Ma^{1/5} \gg 1$, cannot be observed in the results of [14]. Therefore our analysis is capable of predicting the behavior of thermocapillary flows in the range of Ma at which present numerical analysis is incapable of predicting.
The method of deriving $\delta_{td}$ discussed in Section 6.1 is strongly dependent on the geometry and the heating method. Accordingly, we suggest that the length scale next to the hot wall for the S-shaped free surface temperature distribution has to be derived separately for different geometries and heating methods, and in the case of differentially heated two-dimensional enclosure with unit order aspect ratio, $\delta_{td}$ decreases very slowly with increasing Re and its order is given in Chapters IV and V for different flow parameter ranges.

6.4 Summary of The Results from the Scaling Analysis

Here we present a summary of the scaling results in the order of increasing Reynolds number along with the conditions under which these results are valid. All equality and inequality signs should be referred to as "on the order of". The scaling results are for two-dimensional thermocapillary flow in an enclosure of unit order aspect ratio with non-deformable free surface and differentially heated walls. The precise problem is defined in Chapter II by Eqs.(14)
through Eqs. (18). The definitions of the scales are given in Chapters IV and V.

6.4.1 $Pr < 1$

Here we identify three flow regimes:

- no boundary layers
- viscous boundary layers and no thermal boundary layers
- both viscous and thermal boundary layers

(1) Flow with no boundary layers

In this case the flow is of viscous type and heat is transferred from the hot wall to the cold wall through the fluid mainly by conduction.

The condition for this case is

$$Re = 1$$

(35)

The scales for the surface flow, $u^*_s$, and the return flow, $u^*_r$, in the $x$ direction are
\[ u^* = u_r^* = 1 \]  

(36)

(2) Flow with viscous boundary layers and no thermal boundary layers

In this case the flow is of boundary layer type and viscous boundary layers appear but no thermal boundary layers. Convection does not dominate in heat transfer.

The definitions of the scales are given in Section 4.2 and the conditions under which the results here are valid are

\[ \text{Re}^{1/3} > 1 \]  

(37.a)

and

\[ \text{Pr} \cdot \text{Re}^{1/3} \leq 1 \]  

(37.b)

The results for this case are

\[ \frac{u^*}{c} = \frac{v^*}{c} = \frac{\delta^*}{c} = \frac{1}{\text{Re}^{1/3}} \]  

(38.a)
\[ u^* = v^*_h = \frac{1}{Re^{2/3}} \] (38.b)

\[ \delta v_c = \frac{1}{Re^{1/2}} \] (38.c)

\[ \delta v_h = \frac{1}{Re^{1/6}} \] (38.d)

(3) Flow with both viscous and thermal boundary layers

In this case both the flow and temperature fields are of boundary layer type. Convection is the dominating heat transfer mode in the flow.

The scales for this case are defined in Section 4.3 and the conditions under which the results are valid are

\[ Pr \cdot Re^{1/3} \gg 1 \] (39.a)

and

\[ Pr^{2/11} Re^{1/11} \leq 1 \] (39.b)

The results are
\[ u^* = v_c^* = \frac{1}{Re^{1/3}} \]  
\[ u_d^* = \frac{Pr^{1/5}}{Re^{4/15}} \]  
\[ u_r^* = \frac{1}{Re^{2/3}} \]  
\[ v_h^* = \frac{Pr^{3/5}}{Re^{7/15}} \]  
\[ v_d^* = \frac{Pr^{2/5}}{Re^{8/15}} \]  
\[ \delta_{vs} = \frac{1}{Re^{1/3}} \]  
\[ \delta_{vd} = \frac{1}{Pr^{2/5}Re^{7/15}} \]  
\[ \delta_{vh} = \frac{1}{Pr^{3/10}Re^{4/15}} \]  
\[ \delta_{vc} = \frac{1}{Re^{1/2}} \]  
\[ \delta_{td} = \frac{1}{Pr^{3/5}Re^{1/5}} \]
\[ \delta_{\text{td}} = \frac{1}{\Pr^{7/5} \text{Re}^{7/15}} \]  
\[ \delta_{\text{th}} = \frac{1}{\Pr^{4/5} \text{Re}^{4/15}} \]  
\[ \delta_{\text{tc}} = \frac{1}{\text{Ma}^{1/2}} \]  
\[ \Delta \theta_{h} = \Delta \theta_{c} = 1 \]

Here we notice that the length scale for the driving force region next to the hot wall is on the order of \( \frac{1}{\Pr^{3/5} \text{Re}^{1/5}} \). Hence the S-shaped free surface temperature distribution appears when \((\text{Pr} \cdot \text{Re}^{1/3})^{3/5} \gg 1\). From the above summary we can see that for fluids of \( \text{Pr} < 1 \), \( \text{Pr} \cdot \text{Re}^{1/3} \) is a very important parameter since it serves as the identifying parameter for different flow types.

6.4.2 Pr = 1

The results for \( \text{Pr} = 1 \) are derived in Chapter V and there are two flow regimes
no boundary layers
both thermal and viscous boundary layers

(1) Flow with no boundary layers

In this case the flow is of viscous type and heat is transferred mainly by conduction.

The condition for this case is

\[ \text{Re} \leq 1 \] (41)

The scales for the surface flow, \( u_s^* \), and the return flow, \( u_r^* \), in the x direction are

\[ u_s^* = u_r^* = 1 \] (42)

(2) Flow with both thermal and viscous boundary layers

In this case both the flow and temperature fields are of boundary layer type and convection dominates.
The definitions for the scales here are given in Section 5.3 and the conditions under which the following results are valid are

\[ \text{Re}^{1/3} > 1 \quad (43.a) \]

and

\[ \text{Re}^{1/11} \leq 1 \quad (43.b) \]

The results from the scaling analysis are

\[ \delta_{td} = \frac{1}{\text{Re}^{1/5}} \quad (44.a) \]

\[ \delta_{th} = \frac{1}{\text{Re}^{4/15}} \quad (44.b) \]

\[ \delta_{tc} = \frac{1}{\text{Re}^{1/2}} \quad (44.c) \]

\[ \delta_{tsd} = \frac{1}{\text{Re}^{7/15}} \quad (44.d) \]

\[ \delta_{vs} = \frac{1}{\text{Re}^{1/3}} \quad (44.e) \]
\[ \delta_{wd} = \frac{1}{\text{Re}^{7/15}} \] (44.f)

\[ \delta_{wh} = \frac{1}{\text{Re}^{4/15}} \] (44.g)

\[ \delta_{uc} = \frac{1}{\text{Re}^{1/2}} \] (44.h)

\[ \frac{u^*}{v^*} = \frac{1}{\text{Re}^{1/3}} \] (44.i)

\[ u_{d}^* = \frac{1}{\text{Re}^{4/15}} \] (44.j)

\[ u_{r}^* = \frac{1}{\text{Re}^{2/3}} \] (44.k)

\[ v_{h}^* = \frac{1}{\text{Re}^{7/15}} \] (44.l)

\[ \Delta\theta_{h}^* = \Delta\theta_{c}^* = 1 \] (44.m)

With fluids of \( \text{Pr} = 1 \), only \( \text{Re} \) appears in the scales.

In this \( \text{Pr} = 1 \) situation, the S-shaped free surface
temperature distribution appears when \( \text{Re}^{1/5} > 1 \).

6.4.3 \( \text{Pr} > 1 \)

For \( \text{Pr} > 1 \) three flow regimes are identified as follows:

- no boundary layers
- thermal boundary layers and no viscous boundary layers
- both thermal and viscous boundary layers

(1) Flow with no boundary layers

In this case the flow is of viscous type and heat is transferred mainly by conduction.

The condition for this case is

\[
\text{Ma} \leq 1 \tag{45}
\]

The scales for the surface flow, \( u^*_{s} \), and the return flow, \( u^*_{r} \), in the x direction are

\[
u^*_{s} = u^*_{r} = 1 \tag{46}\]
(2) Flow with thermal boundary layers and no viscous boundary layers

In this case there are thermal boundary layers, while there are no viscous boundary layers. Hence the flow is of viscous type and convection dominates.

The definitions for the scales in this case are given in Section 5.2 and the conditions under which the following results for the scales are valid are

\[ Ma^{2/3} > 1 \]  \hspace{1cm} (47.a)

and

\[ Re \leq 1 \]  \hspace{1cm} (47.b)

The results are

\[ \delta_{td} = \delta_{v} = \frac{1}{Ma^{1/3}} \]  \hspace{1cm} (48.a)
\[
\delta_{ts} = \frac{1}{\text{Ma}^{2/3}} \quad (48.b)
\]
\[
\delta_{tc} = \delta_{th} = \frac{1}{\text{Ma}^{1/3}} \quad (48.c)
\]
\[
u^* = v^* = 1 \quad (48.d)
\]
\[
\Delta \theta_h^* = \Delta \theta_c^* = 1 \quad (48.e)
\]

(3) Flow with both thermal and viscous boundary layers

In this case both the flow and temperature fields are of boundary layer type and convection dominates.

The definitions for the scales in this case are given in Section 5.3 and the conditions under which the results are valid are

\[
\text{Re}^{1/3} > 1 \quad (49.a)
\]

and

\[
\text{Ma}^{1/11} \leq 1 \quad (49.b)
\]
The results are

\[ \delta_{td} = \frac{1}{Ma^{1/5}} \]  \hspace{2cm} (50.a)

\[ \delta_{th} = \frac{1}{Pr^{13/30}Re^{4/15}} \]  \hspace{2cm} (50.b)

\[ \delta_{tc} = \frac{1}{Pr^{1/3}Re^{1/2}} \]  \hspace{2cm} (50.c)

\[ \delta_{td} = \frac{1}{Pr^{19/30}Re^{7/15}} \]  \hspace{2cm} (50.d)

\[ \delta_{vs} = \frac{1}{Re^{1/3}} \]  \hspace{2cm} (50.e)

\[ \delta_{vsd} = \frac{1}{Pr^{2/15}Re^{7/15}} \]  \hspace{2cm} (50.f)

\[ \delta_{vh} = \frac{1}{Pr^{1/10}Re^{4/15}} \]  \hspace{2cm} (50.g)

\[ \delta_{vc} = \frac{1}{Re^{1/2}} \]  \hspace{2cm} (50.h)
\[ u_s^* = v_c^* = \frac{1}{\text{Re}^{1/3}} \]  
(50.1)

\[ u_d^* = \frac{\text{Pr}^{1/15}}{\text{Re}^{4/15}} \]  
(50.2)

\[ u_r^* = \frac{1}{\text{Re}^{2/3}} \]  
(50.3)

\[ v_h^* = \frac{\text{Pr}^{1/5}}{\text{Re}^{7/15}} \]  
(50.4)

\[ \Delta \theta_h = \Delta \theta_c^* = 1 \]  
(50.5)

In this case \( \text{Ma}^{1/5} > 1 \) signifies the appearance of the S-shaped free surface temperature distribution.
CHAPTER VII

Concluding Remarks

In this work we have analyzed the problem of thermocapillary flow in an enclosure of unit order aspect ratio with differentially heated side walls, identified the flow regimes, and derived the scales for the different flow regimes. In our scaling analysis we have considered the coupling of the temperature and the flow fields through the consideration of length scales for the temperature distribution on the free surface. The roles played by the different regions in the domain are also analyzed and their influences on each other are incorporated. Numerical simulations are also performed and agreements are found between the results of the scaling analysis and the results of the numerical simulations.

The results of our analysis indicate that the flow is mainly driven by the temperature gradient next to the hot wall and the driving force from the temperature gradient next to the cold wall does not contribute significantly to the
main flow when the free surface temperature gradient exhibits an S-shaped profile. The results of our analysis also shows that the boundary layer thicknesses on different boundaries are of different orders and the orders of these thicknesses are obtained through our analysis.

The possible change of the thermal signature is incorporated in the analysis and the results indicate that at very large Ma the free surface temperature distribution exhibits an S-shaped profile signifying a change of thermal signature. The analysis also shows that present numerical schema are incapable of realistically simulating the thermocapillary flow in the range of Ma in which the change of the thermal signature occurs. Therefore our analysis is capable of predicting the behavior of the thermocapillary flow in the parametric range that present numerical schema are incapable of predicting.
REFERENCES


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APPENDIX A

Temperature Scales at Large Ma

In the present analysis, the scales of the temperature differences, \( \Delta \theta_h^* \) and \( \Delta \theta_c^* \), are used when there are thermal boundary layers. In some situations these two scales can not adequately represent all temperature scales and a more detailed analysis of the temperature scales are needed.

Free Surface Temperature Distribution

![Diagram of temperature distribution](image)

Fig. 18

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When $Ma$ is much larger than unity, thermal boundary layers will appear. This introduces not only length scales for the temperature field, but also temperature scales, such as the temperature scale for the return flow, $\theta_r^*$, and the temperature scale for the region of the surface flow without driving force, $\theta_s^*$. Fig.18 is a sketch of the temperature scales introduced.

In the scaling analysis of the previous chapters, the relation $\Delta \theta_h^* + \Delta \theta_c^* = 1$ suggests the assumption of $\theta_s^* = \theta_r^*$, since

\[ \Delta \theta_h^* = 1 - \theta_r^* \quad \text{and} \quad \Delta \theta_c^* = \theta_s^*. \]

The heat convected in the surface flow towards the cold wall is partly conducted to the cold wall and partly convected back towards the hot wall in the return flow. With this heat transfer balance we can derive the scale for $\theta_s^* - \theta_r^*$. Let the volume flow rate for the fluid with temperature scale $\theta_s^*$ in the surface flow be $M$. Now we treat the three cases where there are thermal boundary layers.
A.1 Pr $\ll 1$ and flow with both thermal and viscous boundary layers

In this case the scales are given by Eqs. (25). The above mentioned heat transfer balance is

$$\frac{\partial \theta^*}{\partial t_c} s v_s = (\theta^*_s - \theta^*_r) \cdot M_a \cdot M$$

where $M = u_s \cdot s v_s$. Solving this relation based on Eqs. (25), we obtain

$$\theta^*_s - \theta^*_r = \frac{1}{Pr^{1/2} Re^{1/6}}$$

From this result, the condition for $\theta^*_s = \theta^*_r$ is $Pr^{1/2} Re^{1/6} \gg 1$.

A.2 Pr $\gg 1$ and flow with thermal boundary layers and no viscous boundary layers

The scales for this case are derived in Section 5.2 and
given by Eqs. (28). In this case not all the fluid in the
surface flow is at \( \theta_s^* \), because of the existence of the
thermal boundary layer in the surface flow. The thickness of
the thermal boundary layer in the region of the surface flow
without driving force, \( \delta_t \), can be found from the following
convection-conduction balance.

\[
\frac{u^* \Delta \theta^*}{1} = \frac{1}{\frac{\Delta \theta^*}{\delta_t^2}} \frac{Ma^*}{\delta_t}
\]

Here \( \Delta \theta^* \) is the scale for the temperature difference in this
region. From this relation, we get

\[
\delta_t = \frac{1}{Ma^{1/2}}
\]

The heat transfer balance in this case is

\[
\frac{\theta^*}{\delta_{tc}} = \frac{(\theta^* - \theta_r^*)}{Ma \cdot M}
\]

where \( M = \delta_t^* u^* \). Solving the above relation based on
Eqs. (28), we obtain
\[ \theta_s^* - \theta_r^* = \frac{1}{Ma^{1/6}} \]

In this case the condition for \( \theta_s^* = \theta_r^* \) is \( Ma^{1/6} \gg 1 \).

A.3 \( Pr \approx 1 \) and flow with both thermal and viscous boundary layers

Similar to the situation in Section A.2 only the fluid in the thermal boundary layer in the surface flow is at \( \theta_s^* \). Let \( \delta_t \) be the thickness of the thermal boundary layer in the region of the surface flow without driving force. The convection-conduction balance in this region is

\[ u_s^* \frac{\Delta \theta^*}{l} = \frac{1}{Ma} \frac{\Delta \theta^*}{\delta_t^2} \]

Here \( \Delta \theta^* \) is the scale for the temperature difference in this region. From this relation, we get

\[ \delta_t = \frac{1}{Pr^{1/2} Re^{1/3}} \]

Then the heat transfer balance is
\[ \frac{\theta_s^* - \theta_r^*}{\delta_{tc} v_s} = (\theta_s^* - \theta_r^*) \cdot Ma \cdot M \]

where \( M = u_s^* \delta_t \). Solving it with Eqs.(32), we obtain

\[ \frac{\theta_s^* - \theta_r^*}{\delta_{tc} v_s} = \frac{1}{Ma^{1/6}} \]

Hence the condition for \( \theta_s^* = \theta_r^* \) is \( Ma^{1/6} \gg 1 \).

A.4 Validity of the scaling analysis

In our scaling analysis in the previous chapters the scales for the temperature differences, \( \Delta \theta_h^* \) and \( \Delta \theta_c^* \), can be expressed in terms of the temperature scales \( \theta_s^* \) and \( \theta_r^* \) as

\[ \Delta \theta_h^* = 1 - \theta_r^* \quad \text{and} \quad \Delta \theta_c^* = \theta_s^* \]

respectively. The temperature difference scales do not influence the local balances, because they are eliminated in the same balances. There are only three places where the temperature difference scales play an important role. In the
relation $\Delta \theta^*_{h} + \Delta \theta^*_{c} = 1$, the temperature difference scale, $\Delta \theta^*_{h}$, in the convection heat transfer in the global heat transfer balance and the driving force boundary condition balance from Eq.(18.d).

(1) $\Delta \theta^*_{h} + \Delta \theta^*_{c} = 1$

This relation can be expressed as

$$\Delta \theta^*_{h} + \Delta \theta^*_{c} = 1 + (\theta^*_s - \theta^*_r)$$

When the condition for $\theta^*_s = \theta^*_r$ is met, the approximation $\Delta \theta^*_{h} + \Delta \theta^*_{c} = 1$ is obviously valid. When the condition is not met, $\theta^*_s - \theta^*_r$ is at most on the order of unity from the results in Sections A.1, A.2, and A.3. Then the order of $1 + (\theta^*_s - \theta^*_r)$ is still unity. Hence the relation $\Delta \theta^*_{h} + \Delta \theta^*_{c} = 1$ is valid, since we are only concerned with the orders of the scales in the scaling analysis.

(2) $\Delta \theta^*_{h}$ in the convection heat transfer in the global heat transfer balance

In the global heat transfer balance, $\Delta \theta^*_{h}$ is used to
represent the temperature difference between the middle of the region of the surface flow with driving force and the return flow. More accurately, it should be

\[ \Delta \theta^* \approx \frac{1+\theta^*}{2} - \theta_r^* \]

This expression can be rearranged as

\[ \Delta \theta^* = \frac{1}{2} \cdot \Delta \theta_h^* + \frac{1}{2} \cdot (\theta_s^* - \theta_r^*) \]

Since \( \Delta \theta_h^* \) is of unit order, \( \Delta \theta^* \) is also of unit order when the condition for \( \theta_s^* = \theta_r^* \) is met. However, the order of \( \theta_s^* - \theta_r^* \) is never larger than that of \( \Delta \theta_h^* \), which can be easily verified by the results in Sections A.1, A.2, and A.3. Hence the leading term in the above expression for \( \Delta \theta^* \) is always \( \frac{1}{2} \cdot \Delta \theta_h^* \) which is of the same order as \( \Delta \theta_h^* \). Therefore the \( \Delta \theta_h^* \) used in the convection heat transfer in the global heat transfer balances is valid in the scaling analysis where only the orders of the scales are important.

(3) \( \Delta \theta_h^* \) in the driving force boundary condition, Eq. (18.d), balance
Similar to the situation in (2) of Section A.4, the temperature difference scale in the driving force boundary condition from Eq.(18.d) should be $1-\theta_s^*$. This can be expressed as

$$1 - \theta_s^* = \Delta \theta_h^* - (\theta_s^* - \theta_r^*)$$

For the same reason as given in (2) of Section A.4, the use of $\Delta \theta_h^*$ in the driving force boundary condition balance is always valid in the scaling analysis.
Streamline pattern for $Pr = 1$ and $Re = 1$

Figure 2
Isotherm pattern and free surface temperature distribution for $Pr = 1$ and $Re = 1$

Figure 4
Isotherm pattern for $Pr = 1$ and $Re = 1000$

Figure 5
Free surface temperature distribution for

\[ \text{Pr} = 100 \quad \text{and} \quad \text{Re} = 100 \]

Figure 6
Isotherm pattern for $Pr = 100$ and $Re = 100$

Figure 15
Free surface temperature distribution for

\[ \text{Pr} = 10 \text{ with } \text{Re} = 1000 \text{ and } \text{Re} = 10000 \]

Figure 16
Comparison of the free surface velocity from the numerical simulations with the scaling analysis

Figure 17