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A multinomial lattice option pricing methodology for valuing risky ventures: Multiple sources of uncertainty

Kamrad, Bardia, Ph.D.

Case Western Reserve University, 1990

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A MULTINOMIAL LATTICE OPTION PRICING
METHODOLOGY FOR VALUING RISKY VENTURES:
MULTIPLE SOURCES OF UNCERTAINTY

BY

BARDIA KAMRAD

Submitted in partial fulfillment of the requirements
for the Degree of Doctor of Philosophy

Thesis Advisor: Professor Peter Ritchken

Department of Operations Research
Case Western Reserve University

August 1990
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GRADUATE STUDIES

We hereby approve the thesis of

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candidate for the Ph.D.

degree.*

Signed:

(Chairman)

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A Multinomial Lattice Option Pricing
Methodology for Valuing Risky Ventures;
Multiple Sources of Uncertainty

Abstract

by

Bardia Kamrad

The methodology for determining the value of a claim whose payoffs depend upon the stochastic prices of other assets is referred to as Contingent Claims Analysis (CCA).

This methodology has now become an industry standard for valuation of financial assets such as options, warrants, bonds, convertibles, and a host of other financial derivative securities.

In recent years, techniques of Contingent Claims Analysis (CCA) and stochastic control theory have also been used to value risky ventures characterized by significant operating flexibility.

While the advantages of these methods over alternative valuation approaches have been well documented, implementation problems have emerged, primarily due to the immense mathematical and computational complexity inherent in these approaches.

The objectives of this dissertation are twofold. The first part is concerned with development of new lattice based option pricing algorithms that account for multiple sources of uncertainty
and provide computational advantages when compared to existing models.

The second part of this thesis is concerned with development of arbitrage based models for valuing real claims. Specifically, by applying techniques of Contingent Claims Analysis (CCA), and stochastic control theory, these new lattice based option pricing algorithms will be generalized to provide a multinomial lattice framework for valuation of risky ventures.
TO MY PARENTS AND BROTHERS FOR THEIR EVERLASTING SUPPORT AND ENCOURAGEMENT.
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Finally, I would like to acknowledge the support and friendship of Patricia Carroll, a friend for the rest of my life.
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CHAPTER 1
INTRODUCTION

(1.1) Introduction

This dissertation has been organized in two distinct parts. The first part is concerned with development of alternative lattice based option pricing algorithms that account for multiple sources of price uncertainty. The second part is concerned with development of arbitrage based models for valuing real claims by utilizing the concepts of Contingent Claims Analysis and Stochastic Control Theory.

The technique for determining the value of a claim whose payoffs depend upon the prices of other assets is commonly referred to as Contingent Claims Analysis (CCA).

This methodology can be applied to the value of a call option in terms of the price of the underlying stock.¹ The technique can also be used to obtain the value of a mining venture in terms of the price of the output commodity, or in obtaining the flexibility

¹An option provides its owner with the right and without the obligation, to buy (a call option) or sell (a put option) a designate security (e.g. stock) by paying a prespecified amount (exercise or strike price) on a specified date (maturity or exercise date). An option is called an American option if it can be exercised before its maturity, if it can be exercised only at maturity it is know as a European Option.
value associated with a flexible production facility. The roots of contingent claims analysis can be traced to the theory of financial option pricing, and date back to the pioneering work of Bachelier (1900). It was not, however, until the early 1970s when Black and Scholes (1973) and Merton (1973) independently demonstrated what has now become a benchmark in academic papers and an industry standard for practitioners in the financial community. In their seminal paper, Black and Scholes made a major breakthrough by deriving a second degree partial differential equation that must be satisfied by the price of a European option dependent on the underlying stock. Black and Scholes succeeded in solving their differential equation to obtain exact formulas for the prices of European call and put options. However, their elegant formulation cannot yield an analytic solution to American options, and numerical procedures must be employed to obtain the option's value.

By accounting for an additional underlying source of variability, Stulz (1982) provided analytic solutions for European options on the minimum or maximum of two risky assets. The motivation for such a generalization stems from the fact that a wide variety of contingent claims of interest have payoff functions that depend on the prices of two risky assets. Examples of such claims include secured debt, foreign currency bonds, optional bonds, and certain compensation contracts. Later, Johnson (1987) extended Stulz's findings to valuation of European options with k sources of uncertainty (k ≥ 2).
As in the Black and Scholes option pricing model, these formulations cannot result in closed form solutions when applied to American options, and numerical techniques must be used to obtain the option's value.

The various approaches that have been suggested for approximating option prices when there is no analytic solution include, (1) simulation techniques, as considered by Boyle (1977), Johnson and Shanno (1985), and Hull and White (1987); (2) finite difference methods, as discussed by Schwartz (1977), Brennan and Schwartz (1979), and Courtadon (1982); and (3) lattice based techniques, as considered by Cox, Ross, and Rubinstein (1979), Boyle (1986, 1989), and Boyle, Evnine and Gibbs (1989).

All these numerical procedures have a dual objective of speed and accuracy in computation. In general greater accuracy often results in increased computation time. Lattice based techniques require the use of a risk neutral argument. By approximating the stochastic evolution of the underlying source of uncertainty, these techniques compute the option price as the discounted values of the expected option payoff. As such, the binomial lattice option pricing algorithm of Cox, Ross, and Rubinstein (1979) has marked the beginning of the multinomial lattice approach to option valuation and is quite attractive due to its intuitive appeal. The basic idea is to replace the lognormal distribution of stock prices by a two-point discrete time jump process over successively smaller time intervals. By increasing the number of steps, convergence to
the true option value is obtained. In addition, the binomial model provides a direct link between the approximating process and the replicating arbitrage strategy.

The objective of the first part of this dissertation is to develop alternative lattice based option pricing algorithms that account for multiple sources of uncertainty, and provide computational accuracy.

An interesting feature of the Black and Scholes analysis was that of a qualitative insight which has proven as significant as their quantitative model. Specifically, they showed that corporate liabilities, can be seen as combinations of simple option contracts. Accordingly, this perspective has established an integrated framework that provides the opportunity for the use of Contingent Claims Analysis (CCA) in valuing corporate securities in light of the structure of corporate liabilities.2

In recent years, techniques of CCA and stochastic control theory have been used to value risky ventures (real assets) that are characterized by significant operating flexibility and

2See, for instance; Merton (1974), Ingersol (1976), Galai and Masulis (1976), Black and Cox (1976), Brennan and Schwartz (1977), Merton (1977), and Jones, Mason, and Rosenfeld (1983).
strategic options. ³

In this context, operating flexibility reflects the set of operational options that are available to the decision maker. As such, these options account for the additional project value that is derived due to the flexibility to revise operating decisions in response to market changes. Examples of operating flexibility include the option to defer initial investment, alter input rates, expand or contract production rates, shut down temporarily, abandonment, alter mixes of input and outputs, or switch technologies.

On the other hand the strategic options associated with a project result from its interdependence with future and followup investments. For instance, by not investing in a small project, the potential future opportunities in larger follow on ventures may be closed off or at least delayed. The value of retaining options to these future projects is embedded into the value of these strategic options. While other approaches to capital budgeting are available, the motivation for use of CCA in project valuation stems from their inherent advantages over more traditional techniques such as the discounted cash flow methodology. Under discounted cash flow techniques, the expected pattern of future cash flows

³For a survey see Mason and Merton (1985) and Ritchken and Rabinowitz (1988).
over a prespecified life of an investment project is discounted at
a rate deemed appropriate to their risk, and the resulting present
value is compared to the cost of the project. The drawback in
applying this approach to projects such as natural resource based
ventures, or commodity based production contracts stems from a high
degree of uncertainty attached to the input/output prices. For
instance, Bodie and Rosansky (1980) report that the standard
deviation of annual changes in futures prices of silver, copper and
platinum are typically above 25%. In addition, the technique often
ignores the fact that managers respond to price variations by
adjusting decisions which impact future cash flows.

Proponents of discounted cash flow methods argue that
managerial responses to price uncertainty can be incorporated into
the analysis by using decision trees and simulation techniques.
But problems remain since future commodity price scenarios must be
forecasted and appropriate discount rates for the cash flows
identified.4

On the other hand, Contingent Claims Analysis provides several
advantages when applied to real asset valuation. First, the
methodology requires minimal data forecast. Specifically, the
valuation mechanism does not depend on forecasts of prices into the

4 For excellent reviews of discounted cash flow techniques
see, Mason and Herton (1985), Brennan and Schwartz (1984), Hayes
and Garvin (1982), Myers and Turnbull (1977), and Sick (1989).
future. Second, risk adjusted discount rates are not explicitly required. Third, the approach is not a static valuation procedure. That is, CCA is able to value the operating flexibility and strategic options within a single project. Fourth, CCA methods not only value projects, but also can be used to determine optimal policies for managing the venture. Fifth, CCA is able to distinguish between "good risk" and "bad risk". Specifically, the existence of real options (managerial control) in any project provides the decision makers with the tools necessary to adapt its future action such that the upside potential of projects can be improved while the downside losses can be limited. Viewed in this light, uncertainty, or large volatility can be advantageous since it expands the upside potential without damaging the downside. Finally, CCA techniques do not requisite preference or utility functions for valuation purposes. A major disadvantage of the CCA approach results from the immense mathematical and computational complexity inherent in the structure of the models which often make it difficult for an intuitive grasping. In addition, CCA cannot (currently) be used to value all classes of projects. Perhaps its best applications occur in situations where the major sources of uncertainty associated with future cash flows can be traced to one, two, or three primary sources.

Recent literature with application of CCA to capital budgeting problems includes articles by McDonald and Siegel (1985) in valuing projects when the firm retains the option to shut down production.

For the above mentioned articles, the general underlying assumptions used to explain price uncertainty imply that the project value evolves according to a diffusion process during the project's life. For the resulting valuation models, this implication leads to a second order partial differential equation governing the value of the project. These elegant formulations, however, usually require numerical methods to yield solutions. The standard approach to solving these problems is to approximate the partial differential equations by difference equations. An alternative approach would be to approximate the stochastic process driving the underlying source of uncertainty. As such, the objective of the second part of this dissertation is to provide a multinomial lattice framework for valuation of a fairly large class of risky ventures that are characterized by multiple sources of
uncertainty and contain operating flexibility. While this approach has been used to value financial options, to date no research has been conducted to apply this techniques to valuation issues in real assets.

(1.2) Research Objectives in Financial Option Pricing Models

In Chapter (2), we shall initially consider a multinomial option pricing algorithm that accounts for a single source of uncertainty. By assuming that this uncertainty is explained by a Geometric Wiener process, the resulting logarithmic return process will then be approximated by a discrete time trinomial jump process. This trinomial lattice is constructed in such a way that in each approximating interval the price variable could either increase in value, maintain its current value, or decrease in value. To obtain the jump probability terms, the moments of the approximating and the continuous processes must be equated. By appropriately generalizing this approximating process to reflect option properties, the option's value can then be obtained recursively.

To account for model accuracy as well as computational efficiency and complexity, the trinomial option pricing algorithm will be compared to the binomial lattice model of Cox, Ross, and Rubinstein (1979). It will be shown that the binomial model is indeed a special case of this algorithm, and that the trinomial
model consistently provides more accurate results.

Later, this algorithm will be generalized to account for an additional source of uncertainty. For the resulting five jump approximating process, the unique probability expressions will be obtained by equating the moments as well as the covariance terms of the true and the approximating process.

As in the trinomial algorithm, the option's value is obtained through a backward recursive approach by identifying the appropriate boundary conditions. The resulting five jump algorithm will be compared to the four jump option pricing model of Boyle, Evnine, and Gibbs (1989), where matters regarding accuracy, efficiency and computational complexity will be discussed in detail. In addition, we shall illustrate that the four jump option pricing model is too, a special case of the five jump algorithm.

Our approach will be extended to provide option pricing algorithms that account for \( k, k > 2 \), sources of variability, where closed form jump probability expressions will also be presented. As a special case, a model with three underlying sources of price variability will be considered.

(1.3) Research Objectives in Real Option Valuation Models

In the second part of this dissertation, our focus will be directed toward valuation of real assets using a multinomial lattice framework.

Toward this goal, Chapter (3) defines a risky venture claim
process where a formulation for the general valuation problem will be presented. By taking into account the fact that in risky ventures, the decision maker's actions influence payouts, the valuation model will be generalized to a problem in stochastic control, and solved through stochastic dynamic programming procedures. By using arbitrage strategies, we shall illustrate that for valuation purposes and under certain assumptions, the need for an exogenously furnished discount rate or utility function is eliminated, and the resulting models do not depend on preferences and aversions toward risk and reward.

Valuation of production contracts with multiple sources of price uncertainty is considered in Chapter (4). Specifically, our modelling efforts will be geared toward valuing contracts in which the manufacturer is required to deliver fixed quantities of a finished good according to a deterministic schedule. To allow for the stochastic variability in input and output prices, while accounting for inventory and production capacities, as well as the demand schedule, the investment decision will be modelled as a problem of stochastic control. Necessary assumptions will be made in order to justify an arbitrage based valuation approach. The resulting valuation problem is solved by a stochastic dynamic program defined on a multinomial lattice, the lattice representing the price of the underlying stochastic variables. Numerical examples and comparative statics for special case models will be presented in Chapter (5).
CHAPTER 2

MULTINOMIAL APPROXIMATING MODELS FOR OPTION VALUATION WITH k SOURCES OF UNCERTAINTY

(2.1) Introduction

A wide variety of contingent claims of interest in financial economics have a payoff function that depends on two or more underlying sources of uncertainty. Examples of such claims include, secured debt, foreign currency bonds, optional bonds, and certain compensation contracts.¹

In most cases involving options on several assets it is possible to derive the general partial differential equation which governs the option's value. This differential equation can be solved in some specialized situations. Stultz (1982) provides analytical formulas for European put and call options on the minimum or maximum of two risky assets, while Johnson (1987) generalizes these results to pricing of European options on the minimum or maximum of several assets. Margrabe (1982), Cheng (1987), and Hemler (1988) have also derived closed form solutions in terms of multivariate normal integrals for European options on the maximum or minimum of several assets.

¹For an excellent introduction to these contingent claims, see Ritchken (1987).
These solutions, however, cannot account for the early exercise feature of American style options, and for valuation purposes the resulting partial differential equation must be solved numerically. The various approaches that have been suggested for approximating the option prices when there is no closed form solution include, analytic approximation (of p.d.e.), numerical integration, simulation, finite difference methods, and lattice based approaches. Exhibit 1 provides a schematic classification for these approximating techniques.

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<td>Compound Option Approach</td>
<td>Geske and Johnson (1984)</td>
</tr>
<tr>
<td>Numerical Integration</td>
<td>Parkinson (1977)</td>
</tr>
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</table>

**Exhibit 1: Approximating Techniques for Option Valuation**
The objective of this chapter is to develop alternative multinomial lattice algorithms for valuing financial options where the payoff from the option depends on one or more underlying sources of uncertainty. In general, lattice based techniques provide option values by approximating the underlying stochastic processes of the state variables. The binomial lattice approach of Cox, Ross, and Rubinstein (1979), fits into this class, and is attractive because it explicitly establishes the arbitrage strategy that replicates the claim. However, providing a direct link between the approximating process and the arbitrage strategy is not essential. Specifically, if the market is complete and trading strategies allow no arbitrage opportunities, then an equivalent martingale measure exists that allows any contingent claim to be valued by computing an appropriate conditional expectation.\(^2\) If this conditional expectation is difficult to evaluate then approximating processes can be taken and numerical methods invoked. Recently, Boyle (1988), used a multiperiod trinomial process to approximate a Geometric Wiener process. In Boyle's model the lattice jump probabilities were obtained by equating the first two moments of the underlying lognormal distribution to those of the approximating distribution. To ensure all jump probabilities were

\(^2\)See Cox and Ross (1976) and Harrison and Pliska (1981) for discussions on this point.
nonnegative, Boyle introduced a stretch parameter, \( \lambda \), that had to be constrained. Given a feasible stretch parameter, the valuation of the conditional expectation is accomplished by backward recursion.

Boyle also establishes a five jump model to approximate a joint bivariate lognormal process and shows its use by valuing American style options that depend on prices of two state variables. Unfortunately, extending this procedure to three or more state variables is cumbersome because sets of parameters yielding nonnegative probabilities for the jumps have to be first obtained and the difficulty of selecting suitable parameters results in implementation problems. To overcome these problems Boyle, Evnine and Gibbs, (BEG) (1989), considered an alternative approximating procedure. Specifically, for the two state variable problem they used a 4 jump multiperiod lattice to approximate the logarithmic return process. By equating the moment generating function of the approximating distribution to the true normal moment generating function, in each time increment, a system of four equations in four unknowns were obtained. The unique probability expressions for their multinomial distribution was then used to value American style claims. The approach is quite elegant since it overcomes the problem of the earlier model and generalizes readily to \( k \) state variables.

In this chapter we develop alternative approximating techniques for valuing claims on one or more state variables.
Specifically, a multinomial approximating option model is developed that has advantages over existing models. Like the BEG model, this model is based on approximating the logarithmic returns process by a discrete multinomial lattice. Unlike the BEG model, however, the opportunity for horizontal jumps exist. It is shown that within this framework, the binomial option model appears as a special case of the one-state variable model, and the BEG model appears as a special case of two state model. In the following sections the one-state option pricing model is introduced and its performance is compared to the well known binomial model. Then the two-state model is developed and is compared to the (BEG) model. Later the k-state model is generated and exemplified by investigating a 3-state model. In each case, model properties are considered and analyzed while highlighting their significant advantages through computational comparisons. Throughout this chapter the setting of the Black and Scholes (1973) model is assumed.

(2.2) The Single State Model: Trinomial Process

Consider the asset price $S(t)$ at time $t$, which is assumed to be lognormally distributed. This implies that,

$$\frac{dS(t)}{S(t)} = \alpha dt + \sigma dZ(t)$$

with $\alpha$ and $\sigma$ as the instantaneous mean and volatility, and $dZ$ as a standard Wiener increment.

Let $\mathcal{P}(T) = \{0 = t_0 < t_1 < \ldots < t_n = T\}$ be a partition of the
interval \([0,T]\), where \(T\) is the option's time to maturity, with \(t_{\ell+1} - t_{\ell} = \Delta t = \frac{T}{n}, (\ell = 0,1,2,...,n-1)\). The lognormality assumption for the asset price implies that:

\[
\ln(S(t+\Delta t)) = \ln(S(t)) + \zeta(t)
\]

where \(\zeta(t) \sim N(\mu\Delta t, \sigma^2\Delta t)\), with \(\mu = r - \frac{\sigma^2}{2}\) as the drift of the risk neutralized process, \(r\), the riskless rate and \(\sigma\), the instantaneous volatility.

Let \(\zeta^a(t)\) be an approximating distribution for \(\zeta(t)\) over the interval \([t, t + \Delta t]\). \(\zeta^a(t)\) is a discrete random variable with the following distribution

\[
\zeta^a(t) = \begin{cases} 
  v & \text{w.p. } p_1 \\
  0 & \text{w.p. } p_2 \\
  -v & \text{w.p. } p_3 \\
\end{cases}
\]

with \(\sum_{j=1}^{3} p_j = 1\).

This is equivalent to approximating the lognormal distribution of asset price by a three point discrete jump process in an interval of length \(\Delta t\). That is:

\[
S(t) \xrightarrow{\text{w.p. } p_1} S(t)u \\
S(t) \xrightarrow{\text{w.p. } p_2} S(t) \\
S(t) \xrightarrow{\text{w.p. } p_3} S(t)d
\]

which indicates that in an interval of length \(\Delta t\), the asset price can either increase in value by an amount \(u\) (size of an up jump), or maintain its time \(t\) value (a horizontal jump), or decrease in value by an amount \(d\) (size of a down jump).
To allow for the occurrence of a horizontal jump in each time interval $\Delta t$, let $u = e^{\lambda \sigma \sqrt{\Delta t}}$ with $\lambda > 1$, where, for convenience the condition $ud = 1$ is imposed. The foregoing implies that $v = \xi u = \lambda \sigma \sqrt{\Delta t}$, $\lambda \geq 1$. To ensure convergence, the mean and variance of the approximating distribution are chosen to equal the mean and variance of $\zeta(t)$. Specifically, we have;

$$E\{\zeta(t)^2\} = v(p_1 - p_3) = \mu \Delta t \quad (3a)$$

$$\text{Var}\{\zeta(t)^2\} = v^2(p_1 + p_3) = \sigma^2 \Delta t + O(\Delta t) \quad (3b)$$

Substituting for $v = \lambda \sigma \sqrt{\Delta t}$, and for sufficiently small $\Delta t$, it follows that

$$\lambda \sigma \sqrt{\Delta t}(p_1 - p_3) = \mu \Delta t \quad \rightarrow \quad p_1 - p_3 = \frac{\mu \sqrt{\Delta t}}{\lambda \sigma} \quad (3c)$$

$$\lambda^2 \sigma^2 \Delta t(p_1 + p_3) = \sigma^2 \Delta t \quad \rightarrow \quad p_1 + p_3 = \frac{1}{\lambda^2} \quad (3d)$$

which can be solved in terms of $p_1$. Equations (3c), (3d) in conjunction with the property that $\sum_{j=1}^{3} p_j = 1$ yield

$$p_1 = \frac{1}{2\lambda^2} + \frac{\mu \sqrt{\Delta t}}{2\lambda \sigma}$$

$$p_2 = 1 - \frac{1}{\lambda^2} \quad (4)$$

$$p_3 = \frac{1}{2\lambda^2} - \frac{\mu \sqrt{\Delta t}}{2\lambda \sigma}$$

By choosing different values for $\lambda$, a range of values for the above probabilities can be obtained through which option valuation
can be accomplished. Note that if $\lambda = 1$, then $p_2 = 0$, and the expressions (4) collapse to the binomial model of Cox, Ross, and Rubinstein (1979). Note too, that unlike Boyle's trinomial procedure, any value for $\lambda, \lambda \geq 1$ yields a feasible set of probabilities.

Proof of Convergence for the Trinomial Model

Let $\Psi_A^{\Delta t}(\theta)$ and $\Psi_C^{\Delta t}(\theta)$ be the characteristic functions for the approximating and the normal distributions presented by (1) and (2) respectively. That is,

$$
\Psi_A^{\Delta t}(\theta) = E\left\{e^{i\theta C^{\Delta t}(t)}\right\} = p_1 e^{i\theta \Delta t} + p_2 e^{i\theta \Delta t} + p_3 e^{-i\theta \Delta t}
$$

$$
= p_1 e^{i\theta \Delta t} + p_2 e^{i\theta (\theta)} + p_3 e^{-i\theta \Delta t} = p_1 e^{i\theta (\lambda \sigma \sqrt{\Delta t})} + p_2 + p_3 e^{-i\theta \sigma \sqrt{\Delta t}}
$$

expansion of the exponential terms yields

$$
\Psi_A^{\Delta t}(\theta) = p_1 \left\{1 + i\theta \lambda \sigma \sqrt{\Delta t} + \frac{i^2}{2} \theta^2 \lambda^2 \sigma^2 \Delta t\right\}
$$

$$
+ p_2 + p_3 \left\{1 - i\theta \lambda \sigma \sqrt{\Delta t} + \frac{i^2}{2} \theta^2 \lambda^2 \sigma^2 \Delta t\right\} + O(\Delta t)
$$

$$
= (p_1 + p_2 + p_3) + i\theta (\lambda \sigma \sqrt{\Delta t})(p_1 - p_3)
$$

$$
- \frac{i^2}{2} \theta^2 (\lambda^2 \sigma^2 \Delta t)(p_1 + p_3) + O(\Delta t)
$$

$$
\Psi_A^{\Delta t}(\theta) = 1 + i\theta (\lambda \sigma \sqrt{\Delta t})(p_1 - p_3)
$$

$$
- \frac{i^2}{2} \theta^2 (\lambda^2 \sigma^2 \Delta t)(p_1 + p_3) + O(\Delta t)
$$

(I)
Similarly for the continuous distribution:

$$\psi^c_{\Delta t}(\theta) = E\left\{e^{i\theta \Delta t} \left( \frac{S(t + \Delta t)}{S(t)} \right) \right\} = E \left\{ e^{i\theta \xi(t)} \right\} = E \left\{ e^{i\theta (\mu \Delta t + \sigma \sqrt{\Delta t} Z)} \right\}$$

where $Z \sim N(0,1)$

$$\psi^c_{\Delta t}(\theta) = E \left\{ 1 + i\theta (\mu \Delta t + \sigma \sqrt{\Delta t} Z) + \frac{1}{2} \theta^2 (\mu^2 \Delta t + \sigma^2 \Delta t Z)^2 + O(\Delta t) \right\}$$

$$\psi^c_{\Delta t}(\theta) = 1 + i\theta (\mu \Delta t) - \frac{1}{2} \theta^2 (\sigma^2 \Delta t) + O(\Delta t) \quad \text{(II)}$$

Substituting (3e) and (3d) into (I) yields

$$\psi^a_{\Delta t}(\theta) = 1 + i\theta (\lambda \sigma \Delta t \left( \frac{\mu \sigma \Delta t}{\lambda^2} \right) - \frac{1}{2} \theta^2 (\lambda^2 \Delta t \left( \frac{1}{\lambda^2} \right) + O(\Delta t)$$

$$\psi^a_{\Delta t}(\theta) = 1 + i\theta (\mu \Delta t) - \frac{1}{2} \theta^2 (\sigma^2 \Delta t) + O(\Delta t) = \psi^c_{\Delta t}(\theta) \quad \forall \Delta t \leq T$$

**Closed Form Solution for Option Valuation**

**Using the Trinomial Model**

Although closed form solutions as provided here are not directly used for valuation purposes, they do, however, provide two important features. First, they can be used to show convergence in the option's price. Second, they provide the necessary intuition for coding of the algorithm.

By definition the asset value in each period of length $\Delta t$ can either increase by an amount $u$ with probability $p_1$, remain unchanged with probability $p_2$, or decrease by an amount $d$ with probability $p_3$ such that $ud = 1$. 
Let $S_0$ be the time zero value of the asset and define $S_{1j}$ as the asset price at time $t_1$, given that it has made $j$ jumps (up, horizontal, or down), where $j \in J(1) = \{-1, -1+1, \ldots, 0, \ldots, 1-1, 1\}$, $1 \leq n$.

Here, $j$ represents a given state of the process generated by the multiplicative trinomial lattice, while $J(1)$ defines the set of feasible realizations for $j$ at stage $t_1$.

Let $p^{(1)}_j$ be the probability that the price variable is in state $j$, $j \in J(1)$ at time $t_1$, that is, the 1 step probability of reaching state $j$, and for convenience let $p^{(1)}_j = p_j$, $j \in J(1) = \{-1,0,1\}$. That is,

$$P\{\tilde{S}_{(t_1)} = S_{1j}\} = p^{(1)}_j$$

and

$$P\{\tilde{S}_{(t_1)} = S_{11}\} = p_1, \ P\{\tilde{S}_{(t_1)} = S_{10}\} = p_2 = p_0, \ P\{\tilde{S}_{(t_1)} = S_{1,-1}\} = p_3 = p_{-1}$$

This convention is helpful in providing closed form results, and since $ud = 1$, then

$$S_{1j} = S_0(u)^j \quad \text{w.p.} \quad p^{(1)}_j \quad j \in J(1) \quad (5)$$

By assumption, after one period, the trinomial lattice process generates three possible states, and for each possible realization, one period hence, there exists three other possible realizations.

This implies that in time period two there are nine potential
states for the price movement. However, due to the symmetry and path independence of the process, at each stage a number of potential realizations generated by the process will be simultaneously identical in value. Let \( \mathcal{L}_1(k) \) represent the number of distinct states generated at time (stage) \( t_1 \) by a multinomial lattice involving \( k \) sources of uncertainty, then for \( k = 1 \) the following proposition is provided.

**Proposition (1)**

For a discrete three jump (trinomial lattice) process that approximates a lognormally distributed random variable, the number of distinct states, \( \mathcal{L}_1(1) \), generated by the process at the \( i \)th stage is

\[
\mathcal{L}_1(1) = 2i + 1 \quad 0 \leq i \leq n
\]  

(6)

**Proof**

Since \( \mathcal{L}_0(1) = 1 \) and by definition \( \mathcal{L}_1(1) = 3 \), then \( \mathcal{L}_1(1) = \mathcal{L}_0(1) + 2 \). To show \( \mathcal{L}_{i+1}(1) = 2i + 3 \), assume \( \mathcal{L}_1(1) = 2i + 1 \).

Let \( J(1) = \{ -1, -1+1, \ldots, 0, \ldots, 1-1, 1 \} \) represent the set of feasible states of the process at time \( t_1 \) \( \{J(1) \) contains \( 2i+1 \) elements by assumption\). At time \( t_1 \), consider the state variable value \( S_{1,j} \), \( j \in J(1) \) where by definition:

1. \( \forall -1 < j < 1, S_{1,j+1} = uS_{1,j} \) and \( S_{1,j-1} = dS_{1,j} \), with \( ud = 1 \)
(11) $\forall \ j \in J(1)$ we have

$$P_{i+1,j+1} = uS_{i,j}$$

$$S_{i,j} \rightarrow P_{i+1,j} = S_{i,j}$$

$$S_{i+1,j-1} = dS_{i,j}$$

Clearly, $\forall \ -1 < k < 1$, $\exists \ S_{i+1,k} \ni S_{i+1,k} = S_{i,j}$ with $j \in J(1)$. This implies that $J(1+1) = J(1) + \{-(1+1), (1+1)\}$, that is, $P_{i+1,1} = P_{i,1} + 2 = 21+3$ \text{QED}

In order to value an option by using a trinomial model, first consider an European call option. Let $C_0$ be the current price of a call option with strike $X$, and let $C_{i,j}$ represent the call price when the underlying asset value is $S_{i,j}$.

At its maturity, the call value $\tilde{C}_n$ can take any of the potential $P_n(1)$ values where

$$C_{n,j} = \max\left\{S_{n,j} - X, 0\right\} = \max\left\{S_0u^j - X, 0\right\}$$ (7)

with $j \in J(n) = \{-n, -n+1, \ldots, 0, \ldots, n-1, n\}$, and $t_n = T$.

The risk neutralized process suggests that $C_0 = \mathbb{E}\left[\tilde{C}_n\right]e^{-rT}$ where $\mathbb{E}\left[\tilde{C}_n\right]$ is given by

$$\mathbb{E}\left[\tilde{C}_n\right] = \sum_{j=-n}^{n} p_j^{(n)} \cdot C_{n,j}$$ (8)
with $C_{nj}$ as defined by (7) and $\forall 1 \leq n$;

$$
\begin{align*}
   p^{(1)}_j &= \begin{cases} 
   \left( \frac{p_{j/1}}{p_{j/1}} \right)^1 & \text{if } j = \pm 1 \\
   1 \sum_{k=-1}^{1} p_k \cdot p^{(1-1)}_{j-k} & \text{otherwise}
   \end{cases} \\
   &\text{where } \left( \frac{p_{1-1}}{p_{1-1}} \right) = 0 \text{ if } |a| > (1-1)

   \text{with } a \in \{j-1, j, j+1\}
\end{align*}
$$

Expression (9) captures the path independence of the process by taking into account the number of times a given state has been used in computing the expected value. For valuation purposes, it is best to use a one period call value and work recursively backward in order to obtain $C_0$. The computational complexity and storage requirements associated with expressions (8) and (9) suggest a backward recursion approach. Accordingly, the one period call value is obtained by

$$
C_{1j} = \frac{p_1 \cdot C_{1+1,j+1} + P_0 \cdot C_{1+1,j} + P_{-1} \cdot C_{1+1,j-1}}{e^{r\Delta t}} \quad \forall 1 < n
$$

and

$$
C_{nj} = \text{Max}\left\{S_0 u^j - X, 0\right\} \quad j \in J(n)
$$

The above expressions can be easily modified to account for European put as well as American style options, where probability terms $p_1$, $p_0$, and $p_{-1}$ are obtained from equations (4). Recall that by convention $P_0 = p_2$, $P_{-1} = p_3$. 
Computational Findings

Table (1) illustrates the difference between true (Black-Scholes) and computed European call prices for different trinomial specifications for in, at, and, out-the-money contracts. As can be seen from Table (1), given any number of iterations, trinomial models out perform binomial models. Figure (1) compares the speed of convergence for an at the money European call option for the case $\lambda = 1.22474$, (which corresponds to $p_2 = 1/3$) and $\lambda = 1$ (which corresponds to $p_2 = 0$ or the binomial model).

Figure (1) shows that the results for the trinomial model after 15 iterations are comparable to the results of the binomial model after 55 iterations. The results for in and out the money options are somewhat similar and are shown in Figures (2.1) and (2.2) respectively.

Of course a trinomial iteration is computationally more expensive, so unless the convergence rate is significantly more rapid, the advantage disappears. Let $N_k(n)$ be the total number of states generated in $n$ iterations for a model with $k$ state variables, that is, $N_k(n) = \sum_{i=0}^{n} \mathcal{Z}_1(k)$ where $\mathcal{Z}_1(k)$ represents the number of distinct states generated at time $t_i$ by a multinomial lattice involving $k$ sources of uncertainty\textsuperscript{3}. Then

\textsuperscript{3}For the binomial model $\mathcal{Z}_1(1) = (i + 1), i \leq n.$
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Table (1): Difference Between True and Computed Option Prices

Case parameters: \( S(0) = 40.0; \) \( r = 0.20; \) \( r = 4.879\%; \)
\( T = 7 \) months (.58833 years).
FIGURE 1: CONVERGENCE RATES OF TRINOMIAL AND BINOMIAL MODELS FOR AN AT-THE-MONEY CALL OPTION
FIGURE 2.1: CONVERGENCE RATES OF TRINOMIAL AND BINOMIAL MODELS FOR AN IN-THE-MONEY CALL OPTION
FIGURE 2.2: CONVERGENCE RATES OF TRINOMIAL AND BINOMIAL MODELS FOR AN OUT-THE-MONEY CALL OPTION
\[ N_i(n) = \begin{cases} \sum_{i=0}^{n} (2i+1) = (n+1)^2 & \text{for a trinomial model} \\ \sum_{i=0}^{n} (i+1) = (n+1)(n+2)/2 & \text{for a binomial model} \end{cases} \]

Except for the last period, where option values are determined by their boundary conditions, each state in the trinomial model requires 3 multiplications and 2 additions, while the binomial model requires 2 multiplications and 1 addition.

Hence, the total number of multiplications and additions are:

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</tbody>
</table>

Clearly a trinomial model with \( n \) steps requires less computational work (measured in terms of multiplications and additions) than a binomial model with 2\( n \) steps. Table (2) compares the accuracy of the two models for pricing at-the-money options on this work adjusted basis.\(^4\) In all cases the error of the trinomial model was about half the size of the binomial error. The trinomial algorithm as developed in this section is shown to be an efficient valuation technique that provides significant computational advantages, where the complexity of both algorithms shown to be of the order \( O(n^2) \). The issue of dividends in using this model is addressed in Appendix 2A.

\(^4\)For an additional example, see Appendix 2B.
<table>
<thead>
<tr>
<th>n iterations on a trinomial</th>
<th>Error on trinomial with n iterations</th>
<th>Error on binomial with 2n iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>.013</td>
<td>.027</td>
</tr>
<tr>
<td>20</td>
<td>.005</td>
<td>.011</td>
</tr>
<tr>
<td>30</td>
<td>.002</td>
<td>.006</td>
</tr>
<tr>
<td>40</td>
<td>.001</td>
<td>.004</td>
</tr>
<tr>
<td>50</td>
<td>.000</td>
<td>.002</td>
</tr>
</tbody>
</table>

Table (2): Comparison of Errors Between the Trinomial Model Using n Iterations and the Binomial Model with 2n Iterations (table (1) case parameters are used).

*The total amount of multiplications and additions on a binomial with 2n iterations exceeds that of the trinomial with n iterations.
(2.3) MODELS WITH TWO SOURCES OF UNCERTAINTY: 5-Jump Process

In this section a multinomial lattice framework is developed that accounts for two source of variability, with opportunity for horizontal jumps.

Let \( \{S_1(t), S_2(t)\} \) define the asset price pair at time \( t \). Assume the joint density of the two underlying securities is bivariate lognormal. Let \( \mu_i = r - \sigma_i^2/2 \) and \( \sigma_i^2 \) be the instantaneous mean and variance for asset \( i \), \( (i = 1, 2) \) and let \( \rho \) be the correlation coefficient. As before, for each asset we have over \([t, t + \Delta t]\),

\[
\ln S_i(t + \Delta t) = \ln S_i(t) + \zeta_i(t) \quad i = 1, 2
\]

where \( \zeta_i(t) \) is a normal random variable with mean \( \mu_i \Delta t \) and variance \( \sigma_i^2 \Delta t \). The instantaneous correlation between \( \zeta_1(t) \) and \( \zeta_2(t) \) is \( \rho \).

The joint normal random variables \( \{\zeta_1(t), \zeta_2(t)\} \) will be approximated by a pair of multinomial discrete random variable having the following distribution:

<table>
<thead>
<tr>
<th>( \zeta_1^a(t) )</th>
<th>( \zeta_2^a(t) )</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_1 )</td>
<td>( v_2 )</td>
<td>( p_1 )</td>
</tr>
<tr>
<td>( v_1 )</td>
<td>( -v_2 )</td>
<td>( p_2 )</td>
</tr>
<tr>
<td>( -v_1 )</td>
<td>( v_2 )</td>
<td>( p_3 )</td>
</tr>
<tr>
<td>( -v_1 )</td>
<td>( -v_2 )</td>
<td>( p_4 )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>( p_5 )</td>
</tr>
</tbody>
</table>

(11)

where \( v_i = \lambda_i \sigma_i \sqrt{\Delta t}, \ (i = 1, 2) \), with \( \sum_{j=1}^{5} p_j = 1 \).

This is equivalent to approximating the joint distribution of two lognormally distributed random variables by a
five point discrete jump process in an interval of length \( \Delta t \). That is:

\[
\begin{align*}
\{S_1(t), S_2(t)\} & \quad \rightarrow \{S_1(t)u_1, S_2(t)u_2\} \quad \text{w.p.} \quad p_1 \\
& \rightarrow \{S_1(t)u_1, S_2(t)d_2\} \quad \text{w.p.} \quad p_2 \\
& \rightarrow \{S_1(t), S_2(t)\} \quad \text{w.p.} \quad p_3 \\
& \rightarrow \{S_1(t)d_1, S_2(t)d_2\} \quad \text{w.p.} \quad p_4 \\
& \rightarrow \{S_1(t)d_1, S_2(t)u_2\} \quad \text{w.p.} \quad p_5
\end{align*}
\]

with \( u_1 = e^{\lambda_1 \sigma_1 \sqrt{\Delta t}} \), \( u_1 d_1 = 1 \), and \( \lambda_1 > 1 \) \( (i = 1, 2) \). Thus, \( \nu_1 = \lambda u_1 \), with \( u_1 \) as the size of an up jump for asset \( i = 1, 2 \).

In order to ensure the convergence of the approximating distribution to the true distribution as \( \Delta t \to 0 \) the first two moments of the approximating distribution are set equal to the true moments of the continuous distribution. Specifically, from (11) we have:

\[
\begin{align*}
E(\xi_1^a(t)) &= \nu_1 (p_1 + p_2 - p_3 - p_4) = \mu_1 \Delta t \quad (12a) \\
E(\xi_2^a(t)) &= \nu_2 (p_1 - p_2 - p_3 + p_4) = \mu_2 \Delta t \quad (12b) \\
\text{Var}(\xi_1^a(t)) &= \nu_1^2 (p_1 + p_2 + p_3 + p_4) = \sigma_1^2 \Delta t + O(\Delta t) \quad (12c) \\
\text{Var}(\xi_2^a(t)) &= \nu_2^2 (p_1 + p_2 + p_3 + p_4) = \sigma_2^2 \Delta t + O(\Delta t) \quad (12d)
\end{align*}
\]

In addition to the equality of the two moments, the covariance terms must also be equal. This can be done by equating the expected value of the product of the two variables which yields

\[
E(\xi_1^a(t)\xi_2^a(t)) = \nu_1 \nu_2 (p_1 - p_2 + p_3 - p_4) = \sigma_1 \sigma_2 \rho \Delta t + O(\Delta t) \quad (12e)
\]

Substituting \( \nu_1 = \lambda_1 \sigma_1 \sqrt{\Delta t} \) \( (i = 1, 2) \), and for sufficiently small \( \Delta t \), the above equations simplify to
\begin{align}
p_1 + p_2 - p_3 - p_4 &= \frac{\mu_1 \sqrt{\Delta t}}{\lambda_1 \sigma_1} \tag{13a} \\
p_1 - p_2 - p_3 + p_4 &= \frac{\mu_2 \sqrt{\Delta t}}{\lambda_2 \sigma_2} \tag{13b} \\
p_1 + p_2 + p_3 + p_4 &= \frac{1}{\lambda_1^2} \tag{13c} \\
p_1 + p_2 + p_3 + p_4 &= \frac{1}{\lambda_2^2} \tag{13d} \\
p_1 - p_2 + p_3 - p_4 &= \frac{\rho}{\lambda_1 \lambda_2} \tag{13e}
\end{align}

Equations (13c) and (13d) imply that \( \lambda_1 = \lambda_2 = \lambda \) with \( \lambda \geq 1 \). The resulting four equations can then be solved in terms of \( \lambda \) to yield unique expressions for \( p_1, p_2, p_3, \) and \( p_4 \). Since \( \sum_{j=1}^{5} p_j = 1 \), then the solution is:

\begin{align}
p_1 &= \frac{1}{4} \left( \frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda} \left( \frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2} \right) + \frac{\rho}{\lambda^2} \right) \tag{14a} \\
p_2 &= \frac{1}{4} \left( \frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda} \left( \frac{\mu_1}{\sigma_1} - \frac{\mu_2}{\sigma_2} \right) - \frac{\rho}{\lambda^2} \right) \tag{14b} \\
p_3 &= \frac{1}{4} \left( \frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda} \left( \frac{-\mu_1}{\sigma_1} + \frac{-\mu_2}{\sigma_2} \right) + \frac{\rho}{\lambda^2} \right) \tag{14c} \\
p_4 &= \frac{1}{4} \left( \frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda} \left( \frac{-\mu_1}{\sigma_1} - \frac{-\mu_2}{\sigma_2} \right) - \frac{\rho}{\lambda^2} \right) \tag{14d} \\
p_5 &= 1 - \frac{1}{\lambda^2} \tag{14e}
\end{align}

Unlike Boyle's 5 jump model, the search for a set of
nonnegative probability values is not necessary. Indeed any value of \( \lambda, \lambda \geq 1 \), will suffice. Note too that if \( \lambda = 1 \), \( p_5 = 0 \), and the overall approximation process reduces to the four jump process obtained by the BEG model.

Proof of Convergence for the Five Jump Model

Let \( \psi^A_{\Delta t}(\theta_1, \theta_2) \) and \( \psi^C_{\Delta t}(\theta_1, \theta_2) \) be the characteristic functions for the approximating and the joint normal distributions respectively. For the approximating distribution

\[
\psi^A_{\Delta t}(\theta_1, \theta_2) = E\left\{ e^{i\theta \zeta_1^A(t)} + i\theta \zeta_2^A(t) \right\} = p_1 e^{i(\theta v_1 + \theta v_2)} \\
+ p_2 e^{i(\theta v_1 - \theta v_2)} + p_3 e^{-i(\theta v_1 + \theta v_2)} \\
+ p_4 e^{i(-\theta v_1 + \theta v_2)} + p_5
\]

\[
= p_1 \left\{ 1 + 4\Delta t(\theta_1 \lambda_{\sigma_1} + \theta_2 \lambda_{\sigma_2}) + \frac{1}{2} \Delta t(\theta_1 \lambda_{\sigma_1}^2 + \theta_2 \lambda_{\sigma_2}^2) + O(\Delta t) \right\} + \\
+ p_2 \left\{ 1 + 4\Delta t(\theta_1 \lambda_{\sigma_1} - \theta_2 \lambda_{\sigma_2}) + \frac{1}{2} \Delta t(\theta_1 \lambda_{\sigma_1}^2 - \theta_2 \lambda_{\sigma_2}^2) + O(\Delta t) \right\} + \\
+ p_3 \left\{ 1 - 4\Delta t(\theta_1 \lambda_{\sigma_1} + \theta_2 \lambda_{\sigma_2}) + \frac{1}{2} \Delta t(\theta_1 \lambda_{\sigma_1}^2 + \theta_2 \lambda_{\sigma_2}^2) + O(\Delta t) \right\} + \\
+ p_4 \left\{ 1 + 4\Delta t(-\theta_1 \lambda_{\sigma_1} + \theta_2 \lambda_{\sigma_2}) + \frac{1}{2} \Delta t(-\theta_1 \lambda_{\sigma_1}^2 + \theta_2 \lambda_{\sigma_2}^2) + O(\Delta t) \right\} + p_5
\]

\[
\psi^A_{\Delta t}(\theta_1, \theta_2) = 1 + \lambda 4\Delta t \left\{ p_1 (\theta_1 \sigma_1 + \theta_2 \sigma_2) + p_2 (\theta_1 \sigma_1 - \theta_2 \sigma_2) \\
- p_3 (\theta_1 \sigma_1 + \theta_2 \sigma_2) + p_4 (-\theta_1 \sigma_1 + \theta_2 \sigma_2) \right\} - \frac{\lambda^2}{2} \left\{ p_1 (\theta_1 \sigma_1 + \theta_2 \sigma_2)^2 \\
+ p_2 (\theta_1 \sigma_1 - \theta_2 \sigma_2)^2 + p_3 (\theta_1 \sigma_1 + \theta_2 \sigma_2)^2 + p_4 (-\theta_1 \sigma_1 + \theta_2 \sigma_2)^2 \right\} + O(\Delta t)
\]
which upon simplification yields

$$\psi^A_{\Delta t}(\theta_1, \theta_2) = 1 + \lambda \ln \sqrt{\Delta t} \left( \theta_1 \sigma_1 (p_1 + p_2 - p_3 + p_4) + \theta_2 \sigma_2 (p_3 - p_2 + p_1 + p_4) \right) -$$

$$\lambda^2 \frac{\Delta t}{2} \left( (p_1 + p_2 + p_3 + p_4) \left[ (\theta_1 \sigma_1)^2 + (\theta_2 \sigma_2)^2 \right] + (p_1 - p_2 + p_3 - p_4) 2 \theta_1 \theta_2 \sigma_1 \sigma_2 \right) + O(\Delta t)$$  \hspace{1cm} (III)

Similarly for the continuous distribution

$$\psi^C_{\Delta t}(\theta_1, \theta_2) = E \left\{ e^{i \theta_1 \ln \left( \frac{\tilde{S}(t + \Delta t)}{S(t)} \right) + i \theta_2 \ln \left( \frac{\tilde{S}(t + \Delta t)}{S(t)} \right)} \right\}$$

$$= E \left\{ e^{i \theta_1 \zeta_1(t) + i \theta_2 \zeta_2(t)} \right\}$$

$$= E \left\{ e^{i \theta_1 (\mu_1 \Delta t + \sigma_1 \sqrt{\Delta t} Z_1) + i \theta_2 (\mu_2 \Delta t + \sigma_2 \sqrt{\Delta t} Z_2)} \right\}$$

where $Z_j \sim N(0,1)$, $j = 1,2$, and $E(Z_1 Z_2) = \rho$. Using Taylor’s series expansion and after simplifying

$$\psi^C_{\Delta t}(\theta_1, \theta_2) = 1 + i \Delta t (\theta_1 \mu_1 + \theta_2 \mu_2) - \frac{1}{2} \left\{ \theta_1^2 \Delta t (\mu_1 \Delta t + \sigma_1^2) + \theta_2^2 \Delta t (\mu_2 \Delta t + \sigma_2^2) + 2 \theta_1 \theta_2 \Delta t (\mu_1 \mu_2 \Delta t + \sigma_1 \sigma_2 \rho) \right\} + O(\Delta t)$$

$$\psi^C_{\Delta t}(\theta_1, \theta_2) = 1 + i \Delta t (\theta_1 \mu_1 + \theta_2 \mu_2) \frac{\Delta t}{2} \left\{ \theta_1^2 \sigma_1^2 + \theta_2^2 \sigma_2^2 + 2 \theta_1 \theta_2 \sigma_1 \sigma_2 \rho \right\} + O(\Delta t)$$  \hspace{1cm} (IV)

Substituting equations (13a, b, c, and e) into expression III with the knowledge that $\lambda_1 = \lambda_2 = \lambda \geq 1$ results
\[ \psi^A_{\Delta t}(\theta_1, \theta_2) = 1 + i\lambda \Delta t \left( \theta_1 \sigma_1 \left( \frac{\mu_1 \sqrt{\Delta t}}{\lambda \sigma_1} \right) + \theta_2 \sigma_2 \left( \frac{\mu_2 \sqrt{\Delta t}}{\lambda \sigma_2} \right) \right) - \frac{\lambda^2 \Delta t}{2} \left[ \frac{1}{\lambda^2} \left( \theta_1 \sigma_1 \right)^2 + \left( \theta_2 \sigma_2 \right)^2 \right] + \frac{\rho}{\lambda^2} \theta_1 \theta_2 \sigma_1 \sigma_2 + O(\Delta t) \]

Simplification results in

\[ \psi^A_{\Delta t}(\theta_1, \theta_2) = 1 + i\Delta t (\theta_1 \mu_1 + \theta_2 \mu_2) - \frac{\Delta t}{2} \left( \theta_1^2 \sigma_1^2 + \theta_2^2 \sigma_2^2 + 2 \theta_1 \theta_2 \sigma_1 \sigma_2 \rho \right) + O(\Delta t) \]

\[ = \psi^C_{\Delta t}(\theta_1, \theta_2) \quad \forall \Delta t \leq T \]

Closed Form Solution for Option Valuation Using a 5-Jump Process

As in the trinomial model, the closed form solutions presented here serve to provide; (i) a basis for convergence in option's prices and (ii) an intuitive basis for coding the algorithm.

Without loss of generality let \( S_{i,j,k} \in \mathbb{R}^2 \) define the joint price of the two assets at time \( t_1 \), given that the first asset has made \( j \) jumps and the second asset has made \( k \) jumps where \( j \in \{ J(1), J'(1) \} \) and \( k \in \{ K(1), K'(1) \} \). Here, \( (j,k) \) represent a given state of the process while \( \{ J(1), J'(1) \} \), and \( \{ K(1), K'(1) \} \) are the set of feasible realizations for \( j \) and \( k \) at the \( i \)th stage of the process.

Let \( S_{0,0} = (S_1(t_0), S_2(t_0)) \) be the time zero value of the price pair where, after an elapsed time of length \( \Delta t \) the process
generates five potential realizations for the variables in
question, that is;

\[
\begin{align*}
S_{1,1,1} &= (S_1(t_0)u_1, S_2(t_0)u_2) \\
S_{1,1,-1} &= (S_1(t_0)u_1, S_2(t_0)d_2) \\
S_{1,-1,-1} &= (S_1(t_0)d_1, S_2(t_0)d_2) \\
S_{1,-1,1} &= (S_1(t_0)d_1, S_2(t_0)u_2) \\
S_{1,0,0} &= (S_1(t_0), S_2(t_0))
\end{align*}
\]

w.p. \( p_1 = p_{1,1} \)

w.p. \( p_2 = p_{1,-1} \)

w.p. \( p_3 = p_{-1,-1} \)

w.p. \( p_4 = p_{-1,1} \)

w.p. \( p_5 = p_{0,0} \)

where \( \sum_{a=1}^5 p_a = 1 \), and \( u_1 \) is the size of an up jump for asset 1, \( d_1 \) is
the size of a down jump, with \( u_1 d_1 = 1 \) \((i = 1,2)\). As before, the
above convention is helpful in providing closed form solutions.

Define \( p_{j,k}^{(1)} \) as the probability that the price pair is in state

\((j,k)\) at time \( t_1 \) and, for convenience let \( p_{j,k}^{(1)} = p_{j,k} \). \((j,k) \in
\{-1,0,1\}\) which implies that for \( i = 0,1,2,\ldots,n;\)

\[
P\{\tilde{S}(t_1) = S_{1,j,k} \} = p_{j,k}^{(1)} \quad j \in \{J(1), J'(1)\}, k \in \{K(1), K'(1)\}
\]

In this light the price vector at time \( t_1 \) is given by

\[
S_{1,j,k} = (S_1(t_0)u_1^j, S_2(t_0)u_2^k) \quad \text{w.p. } p_{j,k}^{(1)}
\]

with \( j \in \{J(1), J'(1)\}, k \in \{K(1), K'(1)\} \) where \( \forall i \leq n, \) the set of
feasible realizations are;

\[
J(1) = K(1) = \{-1, (-1+2), (-1+4), \ldots, (i-4), (i-2), i\}
\]

\[
J'(1) = K'(1) = \{(-1+1), (-1+3), \ldots, (i-3), (i-1)\}
\]

As in the single variable case, the multinomial lattice obtained by
a 5 jump process is characterized by symmetry and path independency
which results in the simultaneous embedding of the states of the
process. Accordingly, the following proposition is provided.

**Proposition (2)**

For a discrete 5 jump process that approximates two lognormally distributed random variables, the number of distinct states, $\mathcal{L}_1(2)$, generated by the process at the $i^{th}$ stage is

$$\mathcal{L}_1(2) = (i+1)^2 + i^2 \quad 0 \leq i \leq n \quad (16)$$

**Proof**

By definition $\mathcal{L}_1(2) = 5$. To show $\mathcal{L}_{i+1}(2) = (i+2)^2 + (i+1)^2$, assume $\mathcal{L}_1(2) = (i+1)^2 + i^2$. Let $(j,k)$ represent a state of the process at time $t_1$ with $j \in \{J(1), J'(1)\}$ and $k \in \{K(1), K'(1)\}$.

At time $t_1$, let $J(1) = K(1) = \{-1, -i+2, \ldots, 1-2, 1\}$, $J'(1) = K'(1) = \{-i+1, -i+3, \ldots, 1-3, 1-1\}$ be the set of feasible realizations for $(j,k)$. Note that both $J(1)$ and $K(1)$ contain $(i+1)$ elements each, so when $(j,k) \in \{J(1), K(1)\}$ there are exactly $(i+1)^2$ distinct realizations for $(j,k)$. Similarly, when $(j,k) \in \{J'(1), K'(1)\}$ each set $J'(1)$ and $K'(1)$ contains 1 elements, as a result of which, $i^2$ distinct realizations are noted, thus confirming $\mathcal{L}_1(2) = (i+1)^2 + i^2$. At time $t_1$ let;

(I) $(j,k) \in \{J'(1), K'(1)\}$ where after an elapsed time of $\Delta t$, i.e. at time $t_{i+1}$ we have $J'(i+1) = K'(i+1) = \{-1, -i+2, \ldots, 1-2, i\}$ with each set containing exactly $(i+1)$ elements, hence generating $(i+1)^2$ states.

(II) $(j,k) \in \{J(1), K'(1)\}$ where after a single period of length
Δt, or at time $t_{i+1}$, $(j,k) \in \{J(i+1), K(i+1)\}$ with each set containing exactly $(i+2)$ elements. As such, $J(i+1)$ and $K(i+1)$ will generate $(i+2)^2$ distinct states.

From (I) and (II), $L_{1+i}^2(2) = (i+2)^2 + (i+1)^2$ QED

In order to value an option using a $5$-jump model let $C_0$ be the current price of an option with strike $X$, and define $C_{i,j,k}$ as the option price when the underlying asset price is given by $S_{i,j,k} \in \mathbb{R}^2$. At maturity the call value $\tilde{C}_n$ can take any of the potential $L_n(2)$ values, where $\mathcal{V}(j,k)$, $C_{n,j,k}$ is given by

$$C_{n,j,k} = \mathcal{F}(S_{10}^j, S_{20}^k), X, n) \quad (17)$$

with $j \in \{J(n), J'(n)\}$, $k \in \{K(n), K'(n)\}$. The risk neutralized process implies

$$C_0 = \mathbb{E}(\tilde{C}_n) \cdot e^{-rT}$$

where

$$\mathbb{E}(\tilde{C}_n) = \sum_{j \in J(n)} \sum_{k \in K(n)} \mathbb{P}_{j,k}^{(n)} \cdot C_{n,j,k} + \sum_{j \in J'(n)} \sum_{k \in K'(n)} \mathbb{P}_{j,k}^{(n)} \cdot C_{n,j,k} \quad (18)$$

with $C_{n,j,k}$ as defined by (17), and the set of feasible realization at maturity are

$J(n) = K(n) = \{-n, -(n+2), -(n+4), \ldots, (n-4), (n-2), n\}$

$J'(n) = K'(n) = \{-(n+1), -(n+3), \ldots, (n-3), (n-1)\}$

and for every $i$, $i = 0, 1, 2, \ldots, n$;
\[
\begin{align*}
P_{jk}^{(1)} &= \begin{cases} 
\{p_{j+1,k+1}\}^i & \text{if } j, k = \pm 1 \\
\sum_{\ell \in \mathbb{J}(1)} \sum_{m \in \mathbb{K}(1)} p_{\ell,m} \cdot p_{j-\ell,k-m}^{(1-1)} + p_{0,0}^{(1-1)} & \text{otherwise}
\end{cases} \\
\text{where } p_{a,b}^{(i-1)} = 0 \text{ if } |a|, |b| > (i-1) \\
\text{with } a \in \{j-1, j, j+1\} \text{ and } b \in \{k-1, k, k+1\}
\end{align*}
\]

In general, for valuation purposes, it is best to compute the one period call value and work recursively backward to obtain the option value. The one period call value, \( V1 < n \) is

\[
C_{1,j,k} = \frac{(p_{1,1} \cdot C_{i+1,j+1,k+1}) + (p_{1,-1} \cdot C_{i+1,j+1,k-1}) + (p_{-1,-1} \cdot C_{i+1,j-1,k-1})}{e^{r\Delta t}} \\
+ \frac{(p_{-1,1} \cdot C_{i+1,j-1,k+1}) + (p_{0,0} \cdot C_{i+1,j,k})}{e^{r\Delta t}}
\]

(20)

where for \( i = n \), \( C_{n,j,k} \) can be obtained from equation (17).

The above expressions can be easily modified to account for European put as well as American Options.

**Computational Findings**

To examine the efficiency of a 5-jump model we investigate how well it approximates the price of a call option on the maximum of two securities. Such options have been well studied and Stulz (1982) has provided analytical solutions for them. Table (3) shows the difference between the true option prices and the computed option price for in, at, and out-the-money contracts, for different \( \lambda \) values, and for different numbers of iterations.
Sources of Uncertainty
A Multinomial Lattice with Two Underlying
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<th>( \beta )</th>
<th>( n = \text{number of iterations} )</th>
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</tr>
<tr>
<td>1.41421</td>
<td>0.500</td>
<td>0.078</td>
<td>0.037</td>
</tr>
<tr>
<td>1.29100</td>
<td>0.400</td>
<td>0.053</td>
<td>0.026</td>
</tr>
<tr>
<td>1.22474</td>
<td>0.333</td>
<td>0.037</td>
<td>0.019</td>
</tr>
<tr>
<td>1.11803</td>
<td>0.200</td>
<td>0.026</td>
<td>0.013</td>
</tr>
<tr>
<td>1.05409</td>
<td>0.100</td>
<td>0.025</td>
<td>0.009</td>
</tr>
<tr>
<td>1.000</td>
<td>0.000</td>
<td>0.087</td>
<td>0.044</td>
</tr>
<tr>
<td>45.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.41421</td>
<td>0.500</td>
<td>0.003</td>
<td>-0.001</td>
</tr>
<tr>
<td>1.29100</td>
<td>0.400</td>
<td>-0.006</td>
<td>-0.001</td>
</tr>
<tr>
<td>1.22474</td>
<td>0.333</td>
<td>-0.009</td>
<td>0.013</td>
</tr>
<tr>
<td>1.11803</td>
<td>0.200</td>
<td>-0.020</td>
<td>0.012</td>
</tr>
<tr>
<td>1.05409</td>
<td>0.100</td>
<td>-0.021</td>
<td>0.000</td>
</tr>
<tr>
<td>1.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.006</td>
</tr>
</tbody>
</table>

Table (3): Difference Between True and Computed Option Prices on the Maximum of Two Assets.

Case parameters: \( S_1(0) = S_2(0) = 40.0; \sigma_1 = 0.20, \sigma_2 = 0.30 \)
\( \rho = 0.50; \ r = 4.879%; \ T = 7 \text{months} \) (0.5833 years)
Notice that for values of \( \lambda \) that force \( p_5 \) to be near 1/5, the results are quite satisfactory, while for very small (\( p_5 = 0 \)) and large (\( p_5 = 0.50 \)) values of \( p_5 \) the residual terms are large.

Figure (3) compares the rates of convergence of the 5 jump model with \( \lambda = 1.11803 \) (corresponding to \( p_5 = 1/5 \)) to the case where \( \lambda = 1 \) (which corresponds to \( p_5 = 0 \), or the BEG model) for an at-the-money European call option on the maximum of two securities.

As can be seen, the results are somewhat similar to those of Figure 1. That is the speed of convergence is smoother and more rapid for the 5 jump model. Figures (4.1) and (4.2) show that similar results hold for in and out-the-money options.

For approximating models involving two state variables the total number of states generated by the process after \( n \) iterations is \( N_2(n) = \sum_{i=0}^{n} \mathcal{P}_i(2) \), that is

\[
N_2(n) = \begin{cases} \sum_{i=0}^{n} (i+1)^2 = \frac{(4n^3 + 12n^2 + 14n + 6)}{6} & \text{for a 5 jump model} \\ \sum_{i=0}^{n} (i+1)^2 = \frac{(2n^3 + 9n^2 + 13n + 6)}{6} & \text{for a 4 jump model} \end{cases}
\]

Except for the last period, where the option values are governed by the boundary conditions, each state in a 5 jump model requires five multiplications and four additions, while the 4 jump model requires one less multiplication and one less addition per

---

\( ^5 \) For the 4-jump model \( \mathcal{P}_i(2) = (i+1)^2, 1 \leq n. \)
FIGURE 3: CONVERGENCE RATES OF 5-JUMP AND 4-JUMP MODELS FOR AN AT-MONEY CALL OPTION ON THE MAXIMUM OF TWO ASSETS
Figure 4.1: Convergence Rates of 5 Jump and 4 Jump Models for an In-the-Money Call Option on the Maximum of Two Assets
FIGURE 4.2: CONVERGENCE RATES OF 5 JUMP AND 4 JUMP MODELS FOR AN OUT-THE-MONEY CALL OPTION ON THE MAXIMUM OF TWO ASSETS
given state. Therefore, the total number of multiplications and additions are:

<table>
<thead>
<tr>
<th>MODEL</th>
<th>No. of MULTIPLICATIONS</th>
<th>No. of ADDITIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 jump</td>
<td>( \frac{5}{3}(2n^3 + n) )</td>
<td>( \frac{4}{3}(2n^3 + n) )</td>
</tr>
<tr>
<td>4 jump</td>
<td>( \frac{2}{3}(2n^3 + 3n^2 + n) )</td>
<td>( \frac{1}{2}(2n^3 + 3n^2 + n) )</td>
</tr>
</tbody>
</table>

Table (4) compares the size of errors of a 4 jump model to a 5 jump model, where, in each case the work done by the 4 jump model (measured in terms of multiplications and additions) dominates that of a 5 jump model.

In each case, the size of the error in the 4 jump model is approximately double that of the 5 jump model. The 5 jump models developed to approximate the logarithmic return process here, avoid the problems of the 5 jump models that approximate the lognormally distributed prices of Boyle (1988) and, like the BEG model, can be readily extended to \( k \) sources of uncertainty.

This section has shown that the 5 jump algorithm provides computational advantages when compared to the four jump model as developed by BEG (1989). The computation complexity for each model is shown to be in the order of \( O(n^3) \).
### Table (4): Comparison of Errors Between the 5 Jump Model Using n Iterations and 4 Jump Model Using n' Iterations

<table>
<thead>
<tr>
<th>n iterations on the 5 jump</th>
<th>n' iterations on the 4 jump</th>
<th>5 jump with n iterations</th>
<th>4 jump with n' iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>15</td>
<td>.026</td>
<td>-.045</td>
</tr>
<tr>
<td>20</td>
<td>30</td>
<td>.013</td>
<td>.030</td>
</tr>
<tr>
<td>30</td>
<td>45</td>
<td>.009</td>
<td>-.015</td>
</tr>
<tr>
<td>40</td>
<td>55</td>
<td>.007</td>
<td>-.012</td>
</tr>
<tr>
<td>50</td>
<td>70</td>
<td>.005</td>
<td>.013</td>
</tr>
</tbody>
</table>

The total amount of multiplications and additions on a 4 jump model with n' iterations exceeds that of a 5 jump model with n iterations.
(2.4) Models with k Sources of Uncertainty

Without loss of generality, it is assumed that the joint density of the asset prices follows a multivariate lognormal distribution. Let $\mu_i = r - \sigma_i^2/2$, and $\sigma_i^2$ be the instantaneous mean and variance respectively and let $\rho_{ij}$ be the correlation between the logarithmic return processes $i$ and $j$. $1, j = (1, 2, \ldots, k)$. When the approximation process includes the opportunity for horizontal jumps there exists $2^k + 1$ probability expressions such that:

$$\sum_{l=1}^{2^k+1} p_l = 1 \quad (21)$$

In order to ensure the existence of horizontal jumps set $v_i = \lambda_i \sigma_i \sqrt{\Delta t}$, with $\lambda_i \geq 1$. Necessary conditions for convergence require the equality of the means, variances, and the pairwise covariance terms, where, the variance equality equations provide the opportunity for setting $\lambda_i = \lambda \geq 1$ ($i = 1, \ldots, k$). With $k$ underlying sources of variability, our approximating process requires $2^k + 1$ jumps in an interval of length $\Delta t$. Let $m$ define a given state of the process after an elapsed time of $\Delta t$, with $p_m$ representing the probability of state $m$, ($m = 1, 2, \ldots, 2^k+1$). Let $p_{k+1}$ represent the probability of a horizontal jump, $k \geq 2$. That is, the probability that the prices at time $t$, remains unchanged at

---

6 For instance, when $k = 2$ and $m = 1$, $p_m$ represents the probability of state 1 in which both assets have an up jump in value over the internal $[t, t+\Delta t]$, see Section III.
time \((t + \Delta t)\). For the remaining \(2^k\) probability terms we can show that for \(m = 1, 2, \ldots, 2^k\) and \(k \geq 2, \lambda > 1\);

\[
p_m = \frac{1}{2^k} \left\{ \frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda} \sum_{i=1}^{k} x_{1m} \left( \frac{\mu_i}{\sigma_i} \right) + \frac{1}{\lambda^2} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} P_{ij} x_{1j} x_{1m}(\rho_{ij}) \right\}
\]

(22)

where

\[
x_{1m} = \begin{cases} 
1, & \text{if asset } i \text{ has an up jump in state } m \\
-1, & \text{if asset } i \text{ has a down jump in state } m
\end{cases}
\]

and

\[
x_{1j} = \begin{cases} 
1, & \text{if assets } i \text{ and } j \text{ have jumps in the same direction in state } m \\
-1, & \text{if assets } i \text{ and } j \text{ have jumps in opposite direction in state } m
\end{cases}
\]

Given equation (21), the probability expression for the horizontal jump, \(p_{2^k+1}\)

\[
p_{2^k+1} = 1 - \frac{1}{\lambda^2}
\]

(23)

When \(\lambda = 1\), the probability of a horizontal jump is zero, and equation (22) reduces to the BEG model [1989].

Further, with \(k \geq 1\) sources of uncertainty, the total number of states generated by an approximating multiprocess process at time \(t_1\), \((i \leq n)\) is;

\[
\mathbb{N}_i(k) = \begin{cases} 
(i+1)^k + 1^k & \text{for models with horizontal jumps} \\
(i+1)^k & \text{for models without horizontal jumps}
\end{cases}
\]

As an example, of equations (22) and (23) consider the case of
three sources of uncertainty. The approximating multivariate
distribution \( \xi_i(t) \), \( i = 1, 2, 3 \) is then given by

<table>
<thead>
<tr>
<th>( \xi_1(t) )</th>
<th>( \xi_2(t) )</th>
<th>( \xi_3(t) )</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_1 )</td>
<td>( v_2 )</td>
<td>( v_3 )</td>
<td>( p_1 )</td>
</tr>
<tr>
<td>( v_1 )</td>
<td>( v_2 )</td>
<td>( -v_3 )</td>
<td>( p_2 )</td>
</tr>
<tr>
<td>( v_1 )</td>
<td>( -v_2 )</td>
<td>( v_3 )</td>
<td>( p_3 )</td>
</tr>
<tr>
<td>( v_1 )</td>
<td>( -v_2 )</td>
<td>( -v_3 )</td>
<td>( p_4 )</td>
</tr>
<tr>
<td>( -v_1 )</td>
<td>( v_2 )</td>
<td>( v_3 )</td>
<td>( p_5 )</td>
</tr>
<tr>
<td>( -v_1 )</td>
<td>( v_2 )</td>
<td>( -v_3 )</td>
<td>( p_6 )</td>
</tr>
<tr>
<td>( -v_1 )</td>
<td>( -v_2 )</td>
<td>( v_3 )</td>
<td>( p_7 )</td>
</tr>
<tr>
<td>( -v_1 )</td>
<td>( -v_2 )</td>
<td>( -v_3 )</td>
<td>( p_8 )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( p_9 )</td>
</tr>
</tbody>
</table>

Note that in this example, \( m = 1 \) refers to the first state, in
which all three variables have an up jump with probability \( p_1 \) while
\( m = 7 \) identifies the state in which the first two variables have a
down jump while the third variable follows an up jump with
probability \( p_7 \).

From equation (22) the resulting expressions for \( p_m \) (\( m = 1, \ldots, 8 \)) reduce to

\[
p_1 = \frac{1}{8} \left( \frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda} \left( \frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2} + \frac{\mu_3}{\sigma_3} \right) + \frac{\rho_{12} + \rho_{13} + \rho_{23}}{\lambda^2} \right)
\]

\[
p_2 = \frac{1}{8} \left( \frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda} \left( \frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2} - \frac{\mu_3}{\sigma_3} \right) + \frac{\rho_{12} - \rho_{13} - \rho_{23}}{\lambda^2} \right)
\]

\[
p_3 = \frac{1}{8} \left( \frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda} \left( \frac{\mu_1}{\sigma_1} - \frac{\mu_2}{\sigma_2} + \frac{\mu_3}{\sigma_3} \right) + \frac{-\rho_{12} + \rho_{13} - \rho_{23}}{\lambda^2} \right)
\]
\[ P_4 = \frac{1}{8} \left( \frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda} \left( \frac{\mu_1}{\sigma_1^2} - \frac{\mu_2}{\sigma_2^2} - \frac{\mu_3}{\sigma_3^2} \right) + \frac{-\rho_{12} - \rho_{13} + \rho_{23}}{\lambda^2} \right) \]

\[ P_5 = \frac{1}{8} \left( \frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda} \left( \frac{\mu_1}{\sigma_1^2} + \frac{\mu_2}{\sigma_2^2} + \frac{\mu_3}{\sigma_3^2} \right) + \frac{-\rho_{12} - \rho_{13} + \rho_{23}}{\lambda^2} \right) \]

\[ P_6 = \frac{1}{8} \left( \frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda} \left( \frac{\mu_1}{\sigma_1^2} + \frac{\mu_2}{\sigma_2^2} - \frac{\mu_3}{\sigma_3^2} \right) + \frac{-\rho_{12} + \rho_{13} - \rho_{23}}{\lambda^2} \right) \]

\[ P_7 = \frac{1}{8} \left( \frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda} \left( \frac{\mu_1}{\sigma_1^2} - \frac{\mu_2}{\sigma_2^2} + \frac{\mu_3}{\sigma_3^2} \right) + \frac{\rho_{12} - \rho_{13} - \rho_{23}}{\lambda^2} \right) \]

\[ P_8 = \frac{1}{8} \left( \frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda} \left( \frac{\mu_1}{\sigma_1^2} - \frac{\mu_2}{\sigma_2^2} - \frac{\mu_3}{\sigma_3^2} \right) + \frac{\rho_{12} + \rho_{13} + \rho_{23}}{\lambda^2} \right) \]

and from equation (23) we obtain

\[ P_9 = 1 - \frac{1}{\lambda^2} \]

When \( \lambda = 1 \), \( P_9 \) becomes zero and the resulting model reduces to the BEG model with three state variables.

(2.5) Conclusion

The valuation of contingent claims whose values depend on multiple sources of uncertainty is an important problem in financial economics. Since numerical methods for valuing such claims can be computationally expensive, the need for efficient algorithms is clear. Approximating the underlying risk neutralized stochastic process by a multinomial lattice is known to produce good results.
An issue that is raised is what order multinomial process should be used to approximate joint normally distributed logarithmic returns. This chapter has provided new multinomial models that include, as special cases, previously developed models. These new multinomial option pricing models are shown to have superior computational advantages over the existing models.
Appendix (2A)

Option Pricing with Dividends

Trinomial Model

Consider an asset which pays a constant yield \( g \) on each ex-dividend date. By definition, at the ex-dividend date, \( t_\ell \), the asset price will drop by \( S_\ell \cdot g \). Define \( \phi_\ell \) such that:

\[
\phi_\ell = \begin{cases} 
1 & \text{if there is no dividend in period } t_\ell \\
1 - g & \text{if there is dividend in period } t_\ell 
\end{cases}
\]

Then after \( t_\ell \) periods the asset price \( S_\ell \) can take on any of the \((2\ell + 1)\) values

\[
S_{\ell, j} = S_0 u^j \prod_{i=1}^{\ell} \phi_i \\
j \in J(\ell)
\]

Let \( a(\ell) \) be the number of ex-dividend dates in the first \( t_\ell \) periods, i.e.

\[
\prod_{i=1}^{\ell} \phi_i = (1-g)^{a(\ell)} \\
\forall \ell, j \in J(\ell)
\]

Suppose that this option was an European call option, then by invoking the risk neutrality argument

\[
C_0 = \frac{E(\tilde{C})}{e^{rT}}
\]

and using equation (8)

\[
C_0 = \sum_{j=-n}^{n} p_j^{(n)} \left( \max \{ S_\ell - X, 0 \} \right) e^{-rT}
\]

\[
C_0 = \sum_{j=-n}^{n} p_j^{(n)} \left( \max \{ S_0 u^j (1 - g)^{a(n)} - X, 0 \} \right) e^{-rT} \\
j \in J(n) \quad (24)
\]
with \( p_j^{(n)} \) as defined earlier. Equation (24) provides an adjustment to the asset price in order to reflect the dividend payments.

However, for the American option early exercise is a possibility. To see this, suppose that a dividend occurs in the last period. Consider state \( j \) at time \( t_{n-1} \), chosen in such a way that regardless of the last price movement the call expires in the money, that is;

\[
\begin{align*}
S_{n-1,j} &= uS_{n-1,j} (1-g) \\
C_{n-1,j} &= uS_{n-1,j} (1-g) - X \\
S_{n,j} &= S_{n-1,j} (1-g) \\
C_{n,j} &= S_{n-1,j} (1-g) - X \\
S_{n,j-1} &= dS_{n-1,j} (1-g) \\
C_{n,j-1} &= dS_{n-1,j} (1-g) - X
\end{align*}
\]

The value of the call unexercised is then given by:

\[
C_{n-1,j} = \frac{p_1 C_{n,j+1} + p_0 C_{n,j} + p_{-1} C_{n,j-1}}{e^{r\Delta t}}
\]

\[
p_1 (uS_{n-1,j} (1-g) - X) + p_0 (S_{n-1,j} (1-g) - X) + p_{-1} (dS_{n-1,j} (1-g) - X)
\]

\[
= \frac{e^{r\Delta t}}{}
\]

\[
C_{n-1,j} = \frac{S_{n-1,j} (1-g) (u_1 + p_0 + dp_{-1}) - X}{e^{r\Delta t}}
\]

Since \( \{u_1 + p_0 + dp_{-1}\}/e^{r\Delta t} = 1 \), then\(^7\)

\[
C_{n-1,j} = S_{n-1,j} (1-g) - \frac{X}{e^{r\Delta t}}
\]

On the other hand, the value of the call exercised is its intrinsic value, \( S_{n-1,j} - X \). Since early exercise is only appropriate if the

\[\text{7The equality is obtained by substituting } u = e^{\lambda \sigma \sqrt{\Delta t}}, \ d = \frac{1}{u}, \text{ and expanding through Taylor's series.}\]
value of the call exercised exceeds the value of the call unexercised, then an early exercise is appropriate if:

\[ S_{n-1,j} - X > S_{n-1,j} (1 - g) - \frac{X}{e^{r \Delta t}} \]

or equivalently if

\[ S_{n-1,j}(g) > X - \frac{X}{e^{r \Delta t}} \] (26)

which implies that the early exercise is only valid if the dividend amount \( S_{n-1,j}(g) \) exceeds the foregone interest on the strike, \( X - \frac{X}{e^{r \Delta t}} \). Note too, that when \( g = 0 \), early exercise is never appropriate.

In order to account for the early exercise feature when dividends are a factor the recursive call pricing equation (10) has to be modified, where this modified recursion, \( \forall i \leq n - 1, j \in J(i) \) is given by:

\[
C_{i,j} = \max \left\{ S_0^i (1-g)^{a(i)} - X, \frac{\left( P_1^* C_{i+1,j+1} + P_0^* C_{i+1,j} + P_{-1}^* C_{i+1,j-1} \right)}{e^{r \Delta t}} \right\}
\] (27)

where \( C_{i+1,j+1}, C_{i+1,j} \) and \( C_{i+1,j-1} \) are obtained from earlier recursions.
Appendix (2B)

Comparison of Errors Between the Trinomial and the Binomial Models for an At-the-Money Call (European) Option

<table>
<thead>
<tr>
<th>Trinomial n-iterations</th>
<th>Binomial 2n-iterations</th>
<th>Error on Trinomial</th>
<th>Error on Binomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>10</td>
<td>.025</td>
<td>.048</td>
</tr>
<tr>
<td>10</td>
<td>20</td>
<td>.014</td>
<td>.026</td>
</tr>
<tr>
<td>15</td>
<td>30</td>
<td>.011</td>
<td>.018</td>
</tr>
<tr>
<td>20</td>
<td>40</td>
<td>.009</td>
<td>.014</td>
</tr>
<tr>
<td>25</td>
<td>50</td>
<td>.007</td>
<td>.012</td>
</tr>
<tr>
<td>30</td>
<td>60</td>
<td>.007</td>
<td>.010</td>
</tr>
<tr>
<td>35</td>
<td>70</td>
<td>.006</td>
<td>.009</td>
</tr>
</tbody>
</table>

Case parameters: \( \sigma = .20, \ r = .04879, \ S = 40.0 \)

\( X = 40.0, \ T = 4 \) months = .333 years

\( \lambda = 1.22474 \)
CHAPTER 3

A CONTINGENT CLAIMS MODEL FOR VALUING REAL ASSETS

(3.1) Introduction

In recent years techniques of continuous time arbitrage and stochastic control theory have been used to value risky ventures characterized by significant operating flexibility and strategic options.

Among the most noted are the works of McDonald and Siegel (1983) who considered the option to defer or initiate investment in a project. Paddock, Siegel and Smith (1988) value various aspects of developing petroleum leases by considering an undeveloped reserve as an option to acquire developed reserves. Brennan and Schwartz (1985) value production flexibility in natural resources. McDonald and Siegel (1983) examine projects in which the firm maintains the option to shut down production, and Majd and Pindyck (1987) value the flexibility of delaying activities in projects that produce no cash flows until completed. Myers and Majd (1983) examine options associated with abandonment of projects.

For the above mentioned articles, the general underlying assumption is that the project value evolves according to a diffusion process during the project’s life. For the resulting models, this assumption leads to a second order partial differential equation (p.d.e.) governing the value of the project.
subject to a set of boundary conditions.

These elegant formulations, however, require sophisticated numerical methods to be used in order to obtain solutions. The standard approach to solving these problems is to approximate the partial differential equations by difference equations. An alternative approach would be to approximate the stochastic process driving the underlying source of uncertainty. While this approach has been used to value financial options, to date little research has been conducted to apply this approach to valuation issues in risky ventures characterized with many real options.¹ As such the purpose of this chapter is to provide a simple framework for the valuation of a fairly large class of risky ventures that use a CCA approach.

Initially the framework that is used is the binomial based CCA approach. Later, the concepts of this approach will be generalized to account for the trinomial model as developed in Chapter (2). It should remain clear that the binomial and the trinomial framework reflect a single source of uncertainty. The valuation of risky ventures with multiple sources of uncertainty is addressed in the closing sections of this chapter.

The remainder of this chapter is organized as follows. The next section defines a simple static claim and reviews the primary

¹See Ritchken and Rabinowitz (1988), and Trigeorgis (1987).
idea of CCA. Section (3.3) defines a risky venture claim process and provides a formulation of the general valuation problem in the traditional sense as a stochastic dynamic program. By traditional approach, it is meant that certain parameters of the problem are assumed to be exogenously provided, for instance, the discount rate and the probability values. As a special case, the stochastic dynamic program is defined on a binomial lattice, the lattice representing the "price" of the underlying stochastic variable. Later, in section (3.5) the binomial option pricing model is generalized by taking into account the fact that for risky ventures, managerial actions may influence payouts. The resulting stochastic control problem is solved in a special way yielding a risky venture valuation model. This new model can be viewed as a corrected form of the traditional stochastic dynamic program where the appropriate discount rate and probability assignments over the lattice need not be exogenously specified. In section (3.6), the trinomial risky venture valuation model is provided. The topic of convenience yield and its impact on the valuation process is reviewed in Section (3.7). Section (3.8) provides a brief conclusion to the chapter, while the extension of these models to situations involving multiple sources of uncertainty can be found in the accompanying appendices.
(3.2) Simple, Static Claims

A simple claim is characterized by a cash flow which is received up to some horizon \( T \), called the maturity date, and a random quantity which is received at that time. The cash flow at time \( t \) is denoted by \( f(S,t) \), while the sum received at maturity is \( H(S,T) \). Here, \( S(t) \) represents the underlying stochastic process upon which all cash flows are contingent. The pair \( \{ f(S,t), H(S,T) \} \) represents a simple claim on \( S(t) \).

An example of a simple claim is an European stock option. Here the underlying source of uncertainty, \( S(t) \), represents the stock price at time \( t \). An European option provides the owner with the right to buy the stock for a fixed price \( X \) at time \( T \). As such it represents a simple claim with \( f(S,t) = 0 \), and \( H(S,T) = \max \{ S(T)-X,0 \} \).

The value of a simple claim at time zero is a number representing the price for which it is worth buying the claim \( \{ f(S,t), H(S,T) \} \). One possible value could be obtained by computing the expected present value of the cash flows i.e.

\[
V_0 = E \left\{ \int_0^T e^{-\rho t} f(S,t) dt + e^{-\rho T} H(S,T) \right\}
\]

(1)

where \( \rho \) is a discount rate chosen according to the risk of the cash flow. If individual preferences and aversions to risk are to be explicitly modeled then equation (1) is replaced by an expected utility functional form. Although appealing this latter approach is extremely difficult to implement since decision makers can
rarely capture their preferences to risk and reward over time in functional form.

In many instances the value of the claim may be directly available. For example the value of an European stock option may be directly available from an option exchange. Indeed, whenever there is a market for the claim \( \{f(S,t), H(S,T)\} \) then, regardless of risk and reward preferences, an objective value (i.e. price) will be available. The actual valuation problem thus arises for nonmarketable simple claims.

One approach to solving this valuation problem is to see whether the claim which presents itself for valuation can somehow be duplicated in the financial markets. Specifically, if a trading strategy can be constructed in the financial markets such that all cash inflows and outflows of the resulting portfolio duplicate the inflows and outflows associated with the simple claim, then the current value of the claim should equal the current value of the replicating strategy.\(^2\) This arbitrage based approach was first used by Black and Scholes (1973) and Merton (1973) to value stock options under the assumption that prices behaved like diffusions.

\(^2\)More formally we say that two portfolios are equivalent if provided they are not altered until time \( T \) they produce identical cash flows up to maturity, \( T \), and at that time have the same terminal payoffs. It is assumed that the price system allows no arbitrage opportunities and thus equivalent portfolios must command the same market price. Two dynamic trading strategies are said to be equivalent if their corresponding portfolios are equivalent at any time.
Harrison and Pliska (1981), using Martingale concepts, generalized the type of stochastic processes involved in the modeling of the prices which create the underlying sources of uncertainty. The importance of binomial representations of price movements was highlighted by Cox, Ross and Rubinstein (1979). In what follows we assume the underlying source of uncertainty consists of a stochastic process which we call a price. For example, the variable could be the output prices of a commodity such as gold. For simplicity we assume that units of this commodity can be purchased and sold in competitive markets.

In general, assume the price $S(t)$ follows a diffusion (Ito stochastic differential) process of the form

$$dS = \alpha(S,t) \, dt + \sigma(S,t) dZ$$

where $\alpha(S,t)$ and $\sigma(S,t)$ are the instantaneous mean and volatility and $dZ$ is a standard Wiener increment. A most common assumption for prices to follow, in efficient market literature, is the Geometric Wiener Process where $\alpha(S,t) = \alpha S(t)$ and $\sigma(S,t) = \sigma S(t)$.\(^3\)

Discrete time approximations can be constructed for Ito processes. In particular, if the price $S(t)$ follows a Geometric Wiener process, a very efficient binomial or trinomial lattice (see Chapter 2) can be constructed.

\(^3\)For a discussion of Ito processes, and binomial approximations to certain Ito processes, see; Cox and Rubinstein (1985) and Cox and Miller (1965).
To approximate the evolution of the stochastic process $S(t)$ by a binomial process we consider the interval $[0,T]$, into $n$ equi-distant increments of width $\Delta t$. Let $P(T)$ represent this partition with

$$P(T) = \{0 = t_0 < t_1 < t_2 \ldots < t_n = T\}$$

The partition is defined fine enough such that in each period $[t_i, t_{i+1}]$ the underlying stochastic variable can only change to one of two possible values. As the partition $P(T)$ becomes finer and finer, then, by suitably controlling the binomial parameters, the process can be made to converge onto well known diffusion and jump processes.

Let $S_{ij}$ be the price at time $t_i$ (period $i$) given that $j$ upward movements have occurred. We assume each upward movement increases the price by $(u-1)$ 100% while, a downward movement decreases the price to $100d\%$ of its value. With $S_0$ representing the time zero price we have

$$S_{ij} = S_0 u^j d^{1-j} \quad j=0,1,2,\ldots,i \quad i=1,2,\ldots,n \quad \text{(2)}$$

$$\text{Prob}(S(t_i) = S_{ij}) = p_{ij} = {i \choose j} p^j (1-p)^{i-j} \quad j=0,1,2,\ldots,i \quad i=1,2,\ldots,n \quad \text{(3)}$$

where $p$ is the probability of an upward movement in any period. Then, following equation (1) the value of the claim, $V_0$, is given by:

$$V_0 = \left[ \sum_{i=1}^{n} \sum_{j=0}^{i} f_{ij} p_{ij} (1+r^\ast)^i + H_{n,j} p_{nj} (1+r^\ast)^n \right]$$

$$\text{(4)}$$
where \( f_{ij} \) is the cash flow received in period \( i \), under state \( j \), and \( H_{nj} \) is the terminal value in period \( n \), under state \( j \). Here \( 1/(1+r^e) \) represents the single period risk adjusted discount factor. Equation (4) is just the discrete version of (1), and converges to (1) as the partition becomes increasingly more dense.

Let \( V_{ij} \) be the value of the claim in node \((i,j)\) after the benefit has been received. Then the value of the claim in period \( i \) is related to the value of the claim in period \( i+1 \) through the relationship.

\[
V_{ij} = p \frac{V_{i+1,j+1} + f_{i+1,j+1}}{(1+r^e)} + (1-p) \frac{V_{i+1,j} + f_{i+1,j}}{(1+r^e)} \tag{5}
\]

The value \( V_0 \) can then be simply obtained through backward recursion where the initial constraint \( V_{nj} = H_{nj} \) is given.

(3.3) Replication of Simple Claims By Dynamic Trading Strategies

Cox, Ross and Rubinstein show how the cash flow and terminal values associated with the static claim can be replicated by an equivalent dynamic trading strategy, whose market value is known. Specifically, consider the cash flow and values associated with the claim in period \( i \) and \( i+1 \).

\[
\begin{align*}
V_{ij} &\rightarrow V_{i+1,j+1} + f_{i+1,j+1} \\
V_{i+1,j} + f_{i+1,j} &\leftarrow V_{i+1,j} + f_{i+1,j}
\end{align*}
\]

The dynamic trading strategy is designed such that the cash flow it
produces in each period and the terminal values are the same as that of the simple claim. Let \( R_{i,j} \) be the value of the replicating portfolio at node \( i,j \) and let \( \eta_{i,j} \) be the amount withdrawn (cash flow) from the portfolio at node \( i,j \).

\[
R_{i+1,j+1} + \eta_{i+1,j+1} \\
R_{i,j} \quad \leftarrow \quad R_{i+1,j} + \eta_{i+1,j}
\]

For the trading strategy to replicate the payouts of the claim we require \( \eta_{i,j} = f_{i,j} \) for all nodes \((i,j)\) and we also require the terminal values to be the same, i.e. \( R_{n,j} = H_{n,j} \). To achieve this, the replicating portfolio consists of dynamically adjusting the amount of leverage in the underlying commodity upon which the price depends. Specifically, at node \((i,j)\) assume the replicating portfolio consists of \( Q_{i,j} \) units of the commodity partially financed by borrowing \( B_{i,j} \) dollars at the risk free rate \( r_f \). That is:

\[
R_{i,j} = Q_{i,j} S_{i,j} - B_{i,j} \tag{6}
\]

The end of period value of this portfolio is shown below:

\[
\delta Q_{i,j} S_{i+1,j+1} - (1+r_f)B_{i,j} \\
R_{i,j} \quad \leftarrow \quad \delta Q_{i,j} S_{i+1,j} - (1+r_f)B_{i,j}
\]

where \( \delta \) is chosen to reflect the cost of carrying the underlying commodity one period and for simplicity is assumed to be a fraction of the spot price (\( 0 < \delta \leq 1 \)).
To replicate the cash flow of the venture we require

\[ \eta_{i+1, j+1} = f_{i+1, j+1} \]

\[ \eta_{i+1, j} = f_{i+1, j} \]  

Let \( R_{i+1, j+1} \) (\( R_{i+1, j} \)) be the value of the replicating portfolio once the cash flows have been removed. Then from equation (7) we have:

\[ R_{i+1, j+1} = \delta Q \eta_{i+1, j+1} - (1+r_f)B_{i+1, j} - f_{i+1, j+1} \]  

(9a)

and

\[ R_{i+1, j} = \delta Q \eta_{i+1, j} - (1+r_f)B_{i+1, j} - f_{i+1, j} \]  

(9b)

Now if \( i+1 = n \) the replicating values \( R_{n+1, j+1} \) and \( R_{n+1, j} \) must equal the boundary condition \( (h_{nj} = h_{nj}) \). Hence for this case the above two equations can be solved for \( Q_{ij} \) and \( B_{ij} \) yielding

\[ Q_{ij} = \frac{\left( R_{i+1, j+1} + f_{i+1, j+1} \right) - \left( R_{i+1, j} + f_{i+1, j} \right)}{\delta \left( S_{i+1, j+1} - S_{i+1, j} \right)} \]  

(10a)

\[ B_{ij} = \frac{\left( R_{i+1, j+1} + f_{i+1, j+1} \right) S_{i+1, j+1} - \left( R_{i+1, j} + f_{i+1, j} \right) S_{i+1, j+1}}{(1+r_f) \left( S_{i+1, j+1} - S_{i+1, j} \right)} \]  

(10b)

Substituting equations (10a) and (10b) into (7) and rearranging yields the following recursive equation for \( R_{ij} \):

\[ R_{ij} = \frac{\theta \left[ R_{i+1, j+1} + f_{i+1, j+1} \right] + (1-\theta) \left[ R_{i+1, j} + f_{i+1, j} \right]}{(1+r_f)} \]  

(11)

with \[ \theta = \frac{(1+r_f) - \delta d}{\delta (u-d)} \]  

(12)
Through the backward recursive application of equation (11) we can obtain the current value of a dynamic trading strategy that produces the same cash flows as the simple claim and terminates with the same value as the claim. The dynamic trading strategy consists of purchasing $Q_{1j}$ units of the underlying commodity partially financed by borrowing $B_{1j}$ dollars at the risk free rate if node $(1,j)$ is reached. In node $(1,j)$ a dividend of $\eta_{1j} = f_{1j}$ is paid out.

Since the dynamic trading strategy produces identical cash flows as the simple claim, to avoid arbitrage opportunities the market value of the simple claim, $V_0$, should equal the initial market value of the dynamic trading strategy, $R_0$.

Note that by comparing equation (11) with equation (5) it can be seen that the value of $p$ is replaced by $\theta$ and the value of $r^*$ is replaced by $r_f$. Thus, corresponding to equation (4), the arbitrage free price of a simple claim, $R_0$, is given by

$$R_0 = \sum_{i=1}^{n} \left[ \sum_{j=0}^{1} f_{ij} \theta_j \frac{1}{(1+r_f)^i} + H_{nj} \theta_j \frac{1}{(1+r_f)^n} \right]$$ (13)

where

$$\theta_{1j} = \binom{i}{j} \theta^j (1-\theta)^{i-j} \quad j=1,\ldots,n; i=1,\ldots,n$$ (14)

The binomial stock option pricing model discussed by Cox, Ross and Rubinstein is a special case of this equation with $f_{1j} = 0$ and

$$H_{nj} = \max[S_n - X, 0]$$ where $X$ is a constant.

The above claims are termed simple claims for two reasons. First their value derives solely from the underlying exogenously provided stochastic process of prices. Second, the decision maker
(1.e. the holder of the claim) cannot influence the outcome of the payouts. This clearly is the case for European options where the decision maker (holder) cannot influence the outcome of the terminal payout \( H(S,T) \).

We now turn to risky ventures where managerial decisions impact cash flows.

### 3.4 Risky Ventures

We define a risky venture as a cash flow process up to maturity \( T \), together with a random quantity received at maturity. Unlike simple claims where the cash flows depend solely on an underlying stochastic process \( S(t) \), here the cash flows also depend on the setting of a managerial control variable, called a technology. Let \( f(S,x,t;j) \) represent the cash flow rate at time \( t \) when the underlying price is \( S(t) \), the state of the venture is \( x(t) \) and the current technology in place is \( j \). Let \( H(S,x,T;j) \) represent the random quantity received at maturity.

Switches in technology occur at discrete times over the interval \([0,T]\). Let \( F(t) = \{0=t_0 < t_1 < t_2 \ldots < t_n = T\} \) where \( t_i \) is a switching time, \( i < n \). The cost of switching from technology \( k \) to

---

\(^4\)For American style options, \( f(S,t) = 0 \) \( 0 \leq t \leq \tau \) and the terminal value is \( H(S,\tau) = \max(S(\tau) - X, 0) \) for \( 0 \leq \tau \leq T \), where \( \tau \) is a controlled stopping time. Here, the form of \( H(\cdot) \) remains the same, but the decision maker can influence the timing.
technology \( t \) at time \( t_1 \) is \( \pi_1(k, t) \).

Let \( x(t_1) \) be the state of the venture at the beginning of period 1, (i.e. at time \( t_1 \)). Let \( d_1 \) be the technology selected for use in period \([t_1, t_{i+1}]\). The choice of the technology at time \( t_1 \) may be restricted by the current technology in place, \( d_{i-1} \), and the state of the system \( x(t_1) \). Let \( G(S(t_1), x(t_1), t_1; d_{i-1}) \) represent the set of feasible selections for \( d_1 \).

Define an admissible control \( \alpha \) to be a sequence of feasible technologies used in successive periods. That is \( \alpha \in \mathbb{R}^{n+1} \) with \( \alpha = (d_0, d_1, d_2, \ldots, d_n) \) where \( d_i \in G(S(t_i), x(t_i), t_i; d_{i-1}) \) for \( i=1, 2, \ldots, n \) and \( d_0 \) is the current technology in place. Let \( \mathcal{A}^d_0 \) represent the set of admissible controls with initial setting \( d_0 \), i.e. \( \mathcal{A}^d_0 = \{\alpha | \alpha \) is an admissible control with initial technology \( d_0 \} \).

To obtain the value of the risky venture, conditional on a predetermined \( \alpha \in \mathcal{A}^d_0 \) we have:

\[
V^\alpha_0(S(t_0), x(t_0); d_0) = E \left\{ \sum_{i=1}^{n} \epsilon^{t_i} e^{-\rho t_i} f(S, x, t; d_i) dt \right\} - e^{-\rho t_1} \pi_1(d_{i-1}, d_1) + e^{-\rho t_n} H(S, x, T; d_n) \}
\]

(15)

As in the case of equation (1), individual risk and reward preferences and aversions could be dealt with if the intertemporal utility function for the individual decision maker was known. Moreover, in equation (15), if the risks change according to decisions on technologies then the discount rate \( \rho \) may be a
function of technology. i.e. \( \rho = \rho(j) \). Let

\[
J_0(S(t_0), x(t_0); d_0) = \sup_{\alpha \in \mathcal{A}} \left\{ V_0^\alpha(S(t_0), x(t_0); d_0) \right\}
\]  \hspace{1cm} (16)

That is \( J_0(\cdot) \) is the value associated with the optimal admissible control, given the initial underlying stochastic variable is \( S(t_0) \), the current state of venture is \( x(t_0) \) and the current control is \( d_0 \).

The usual procedure used to compute \( J_0(\cdot) \) is dynamic programming. To achieve this let \( J_i(S(t_i), x(t_i); k) \) be the maximum total expected value of the venture at time \( t_i \) given the underlying stochastic variable at time \( t_i \) is \( S(t_i) \), the state variable \( x(t_i) \), and the technology in place exiting period \( i-1 \) is \( k \). Then the Bellman equation for time periods \( i=0,1,2,\ldots,n-1 \), is

\[
J_i(S(t_i), x(t_i); k) = \max_{\ell} \left[ \mathbb{E} \left( \int_{t_i}^{t_{i+1}} f(S(t), x(t); \ell) \, e^{-\rho(t-t_i)} \, dt \right) \right.

+ e^{-\rho(t_{i+1}-t_i)} J_{i+1} \left( S(t_{i+1}, x(t_{i+1}); \ell) \right)

\left. + r_i(k, \ell) \right]
\]  \hspace{1cm} (17)

subject to

\[
\ell \in G(S(t_i), x(t_i); k)
\]

\[
x(t_{i+1}) = g(x(t_i); \ell)
\]

\[
s(t_{i+1}) = \zeta(S(t_i))
\]

The first constraint, \( G(S(t_i), x(t_i); k) \), represents the set of feasible selections which depends on the state variable \( x \) at time \( t_i \) and the technology that is currently in place. The second constraint, \( x(t_{i+1}) = g(x(t_i); \ell) \), shows that the state variable at time \( t_{i+1} \), depends on the state variable at time \( t_i \) and the
technology used during time period \([t_{i1}, t_{i+1}].\) Finally, the last constraint indicates that the stochastic variable at time \(t_{i+1}\) depends on its value at time \(t_i.\) The above dynamic program is completely defined once the boundary conditions at time \(T\) are established. These are:

\[
J_n[S(t_n), x(t_n); k] = H(S(t_n), x(t_n); k)
\]

where the terminal function \(H()\) was defined earlier.

We now consider the valuation problem under the assumption that prices of the underlying commodity can be approximated by a binomial process.

Let \(J_{i,j}(x_i; k)\) be the maximum total expected value of the venture at time \(t_i\) given the price is \(S_{ij},\) the state \(x_i,\) the entering technology, \(k\) and optimal decisions are to be followed. Let \(J_{i,j}(x_i; k)\) be the total expected value of the venture at node \((i,j)\) given the entering technology is \(k,\) and technology \(\ell\) is chosen to be used in \([t_{i1}, t_{i+1}]\) with optimal decisions made thereafter. The cash flows received in period \(i+1,\) and the value of the project in the next period are shown below.

\[
J_{i,j}^{\ell}(x_i; k) \left\{ \begin{array}{l}
f_{i+1, j+1}(\ell) + J_{i+1, j+1}(x_{i+1}; \ell) \\
f_{i+1, j}(\ell) + J_{i+1, j}(x_{i+1}; \ell)
\end{array} \right.
\]

where \(x_{i+1} = g(x_i; \ell)\) and \(f_{i,j}(\ell) = f(S_{ij}, \ell).\)

Following equation (17) we have for \(j=1,2, \ldots, 1;\) and
\[ J_{ij}(x_{1};k) = \max_{\ell \in G_{ij}(x_{1};k)} \left\{ J_{ij}^{\ell}(x_{1};k) - \pi_{1}(k,\ell) \right\} \]  

where \( G_{ij}(x_{1};k) \) is the set of feasible switches that can be made from node \((1,j)\) with:

\[
J_{ij}^{\ell}(x_{1};k) = p\left[ \frac{f_{i+1,j+1}(\ell)+J_{i+1,j+1}(x_{i+1};\ell)}{(1+r^*)} \right] \\
+ (1-p)\left[ \frac{f_{i+1,j}(\ell)+J_{i+1,j}(x_{i+1};\ell)}{(1+r^*)} \right]
\]

Where \( 1/(1+r^*) \) is the single period risk adjusted discount rate and \( p \), as before, is the probability of an up movement in prices.

The boundary condition is given as:

\[ J_{nj}(x_{n};k) = H_{nj}(x_{n};k) \]

Equations (20), (21), and (22) define a dynamic program which allows the value of the risky venture to be computed. The dynamic program can be viewed as a simple extension of equation (5) when, managerial actions can affect the outcomes. As with equation (5), the resulting value that we obtain depends on probabilities and on preferences.
(3.5) A Contingent Claims Valuation Model for Risky Ventures

In this section we show how arbitrage arguments can be used to value risky ventures. As such, this section generalizes the procedure used in section 3.

Let $R_0(\alpha)$ be the current market value of a dynamic trading strategy that produces the exact same pattern of cash flows and terminal values as the risky venture under admissible control $\alpha \in A^d_0$. Then clearly the arbitrage free price of this venture restricted to control $\alpha$, i.e. $V_0^\alpha(\cdot)$ is $R_0(\alpha)$. If all the cash flows of all admissible controls can be replicated in financial markets then the arbitrage free value of the venture should equal the value of the equivalent dynamic trading strategy that has maximum market value. That is:

$$J_0(S(t_0), x(t_0); d_0) = \text{Sup}_{\alpha \in A^d_0} \{R_0(\alpha)\} = R_0$$

Rather than searching through admissible controls $\alpha \in A^d_0$ to find $J_0(\cdot)$ directly, an alternative approach is to search through all equivalent dynamic trading strategies and find the strategy that maximizes market value.

To construct the optimal dynamic trading strategy consider an arbitrary node $(i, j)$ entered into in state $x_i$ using technology $k$. If a switch is made to technology $\ell$ a cost of $\pi_i(k, \ell)$ is incurred and future cash flows in period $i+1$ are either $f_{i+1,j}^{1+1}(\ell)$ or $f_{i+1,j}^{1+1}(\ell)$ depending on which state occurs.

Let $R_{i,j}(x_i; k)$ be the value of the replicating portfolio at
node \((i, j)\) that replicates the cash flow of the risky venture, given the risky venture enters node \((i, j)\) in state \(x_1\) using technology \(k\) and optimal decisions are made thereafter. Equivalently, \(R_{ij}(x_1; k)\) is the value of the replicating portfolio at node \((i, j)\) that corresponds to some admissible control having the property that of all feasible admissible replicating strategies, this one has maximum market value. Then

\[
R_{ij}(x_1; k) = \max_{\ell \in G_{ij}(x_1; k)} \left\{ R_{ij}^\ell(x_1; k) - \pi_{ij}(k, \ell) \right\}
\]

where \(R_{ij}^\ell(x_1; k)\) is the value of the trading strategy that replicates the cash flows of the venture that enters node \((i, j)\) in state \(x_1\), switches technology from \(k\) to \(\ell\) for use from time \(t_1\) to time, \(t_{i+1}\) and then replicates the cash flows of the optimally managed venture thereafter. The value and cash flow of this replicating strategy from node \((i, j)\) are shown below:

\[
R_{ij}^\ell(x_1; k) = \begin{cases} 
\eta_{i+1, j+1}(\ell) + R_{i+1, j+1}(x_{i+1}; \ell) \\
\eta_{i+1, j}(\ell) + R_{i+1, j}(x_{i+1}; \ell)
\end{cases}
\]

where \(x_{i+1} = g(x_1; \ell)\).

Following (7) the portfolio that duplicates these cash flows consists of \(Q_{ij}^\ell(x_1; k)\) units of the underlying commodity partially financed by borrowing \(B_{ij}^\ell(x_1; k)\) dollars. The value of this portfolio from node \((i, j)\) is shown below
\[ R_{1j}^j(x_i;k) = Q_{1j}^j(x_i;k)S_{1j}^j - B_{1j}^j(x_i;k) - (1+r_f)B_{1j}^j(x_{i+1};k) \]

\[ Q_{1j}^j(x_i;k)S_{1j}^j - (1+r_f)B_{1j}^j(x_i;k) \]

\[ \eta_{1+j_1, j+1}^j(\ell) = f_{1+j_1, j+1}^j(\ell) \]

\[ \eta_{1+j_1, j}^j(\ell) = f_{1+j_1, j}^j(\ell) \]

The replicating portfolio is chosen such that the cash flows (withdrawals of the replicating portfolio) are equal to the cash flows of the venture. Hence we require

\[ R_{nj}^j(x_n) = H_{nj}^j(x_n; \ell) \quad j = 0, 1, \ldots, n \]

Hence for \( i = n-1 \) we require

\[ R_{1+j_1, j+1}^j(x_{i+1}; \ell) = B_{1+j_1, j+1}^j(x_{i+1}; k) - f_{1+j_1, j+1}^j(\ell) \]

(28a)

\[ R_{1+j_1, j}^j(x_{i+1}; \ell) = B_{1+j_1, j}^j(x_{i+1}; k) - f_{1+j_1, j}^j(\ell) \] (28b)

Solving these two equations for \( Q_{1j}^j(x_i;k) \) and \( B_{1j}^j(x_i;k) \) yields:
\[ Q_{ij}^{l}(x_i; k) = \frac{[R_{i+1, j+1}(x_{i+1}; l) + f_{i+1, j+1}(l)]}{\delta [S_{i+1, j+1} - S_{i+1, j}]} \]  
\[ - \frac{[R_{i+1, j}(x_{i+1}; l) + f_{i+1, j}(l)]}{\delta [S_{i+1, j+1} - S_{i+1, j}]} \]  
\[ B_{ij}^{l}(x_i; k) = \frac{[R_{i+1, j+1}(x_{i+1}; l) + f_{i+1, j+1}(l)] S_{i+1, j}}{[S_{i+1, j+1} - S_{i+1, j}] (1+r_f)} \]  
\[ - \frac{[R_{i+1, j}(x_{i+1}; l) + f_{i+1, j}(l)] S_{i+1, j+1}}{[S_{i+1, j+1} - S_{i+1, j}] (1+r_f)} \]

Substituting (29a) into the expression for \( R_{ij}^{l}(x_i; k) \), we obtain, upon simplification

\[ R_{ij}^{l}(x_i; k) = \theta \left[ \frac{[R_{i+1, j+1}(x_{i+1}; l) + f_{i+1, j+1}(l)]}{(1+r_f)} \right] \]  
\[ + (1-\theta) \left[ \frac{[R_{i+1, j}(x_{i+1}; l) + f_{i+1, j}(l)]}{(1+r_f)} \right] \]

\[ \theta = \frac{(1+r_f) - \delta d}{\delta (u-d)} \]  

To avoid riskless arbitrage it must follow that the value of the venture at node \((i, j)\) after a switch to \(l\) is made must equal \( R_{ij}^{l}(x_i; k) \). Moreover, the value of the optimally managed venture given node \((i, j)\) is entered with state \(x_i\) using technology \(k\).
follows immediately, and is:

\[ R_{ij}(x_i; k) = \max_{\ell \in G(x_i; k)} \left\{ r^\ell_{ij}(x_i; k) - \pi_i(k, \ell) \right\} \]  

(32)

Equations (30) and (32) together with the boundary conditions (27) define a backward dynamic programming recursion through which a dynamic trading strategy that replicates the payouts of an optimally managed project value can be obtained. Notice that in comparing this dynamic program with (9) and (10) the probability of an upward movement is again replaced by the value of and the risk adjusted factor \( r^* \) is replaced by the value \( r_r^* \). This dynamic program can be viewed as a corrected version of the valuation equation given in (20) and (21) which does not require the subjective inputs on \( p \) and \( r^* \) to be exogenously provided. The above procedure is best illustrated by an example.

(3.5.1) Example

To illustrate the procedure, we consider valuing a hypothetical risky venture which we refer to as a gold mine. Given a finite known amount of nonrenewable resource, fixed production and extraction costs, and known switching costs associated with altering production amounts, the problem is to establish an optimal extraction schedule that responds to the stochastic selling price in a manner so as to maximize the current value of the mine.

For illustrative purposes we consider a 3 period problem and assume that the underlying price of gold in each period follows a
multiplicative binomial process with \( u = 1.50 \), \( d = 0.80 \). The risk free rate of interest perceived is 10\%. The binomial lattice for the price of finished commodity is shown in Exhibit (1).

\[
\begin{align*}
S_0 &= 10 \\
S_{10} &= 8 \\
S_{11} &= 15 \\
S_{20} &= 6.40 \\
S_{21} &= 12.0 \\
S_{30} &= 5.12 \\
S_{31} &= 9.60 \\
S_{32} &= 18.0 \\
S_{33} &= 33.75
\end{align*}
\]

**EXHIBIT 1**: Binomial lattice for price movement 

\((u = 1.50, \ d = .80)\)

The total amount of resource that the mine is capable of producing is 4 units of finished product. In each period the firm has the option of producing 0 units at a cost of \( C(0) = \$1.00 \), 1 unit at a cost of \( C(1) = \$4.00 \), or 2 units at a cost of \( C(2) = \$9.00 \). The switching costs of production are: \( \pi_i(k, \ell) = |k-\ell| \) where \( k \) and \( \ell \in \{0,1,2\} \). Production initiated in period 1 is completed at the beginning of period 1+1 and immediately sold into the marketplace. The benefit function is \( f_{1j}(\ell) = \ell s_{1j} - C(\ell) \). The state variable is the amount of resource remaining. The salvage value of the mine is given by \( h_{3j} = \frac{1}{2}x_3 s_{3j} \), where \( x_3 \) is the amount of resource that remains in period 3.

The dynamic programming recursion equation for this example is
given by: $R_{1j}(x_1;k)$

$$= \max_{l \in G(x;k)} \left\{ \frac{0.43 \left( R_{i+1,j+1}(x_1-l;l) + ts_{i+1,j+1} - c(l) \right)}{1.10} + \frac{0.57 \left( R_{i+1,j}(x_1-l;l) + ts_{i+1,j} - c(l) \right)}{1.10} - \left| k-l \right| \right\}$$

where $G(x;k)$ is the set of integers from 0 to $Z$, with $Z = \max \{2, x\}$, $\theta = 0.43$ and the boundary conditions are: $R_{jj} = H_{jj} = \frac{1}{2} x_{jj}$.

Exhibit (2) shows the set of optimal decisions for the entire valuation process. The optimal production decisions over all time periods are shown on the arcs of the lattice. The cash flows and switching costs are also indicated. Exhibit 3 shows the optimal dynamic trading strategy. For gold the carrying cost is negligible and $\delta$ is assumed to be $1$.

At time zero, the value of the project equals the value of a dynamic trading strategy, which is $25.10. According to the production schedule a switching cost of $1.00 must be incurred. Hence, for the trading strategy to produce an equivalent cash flow $1 must be deposited into the account. The resulting value of the account is now $26.10. The funds in this account are used to purchase 3.83 units of gold for a cost of $38.30. This requires borrowing $12.20 at the risk free rate.

After one period the value of the portfolio is $44.02 in the upstate or $17.23 in the downstate. Assume the downstate occurs. In the first period 1 unit was produced and this unit sells for $8.
Since the optimal production schedule in period 2 is 1 unit no switching cost is incurred and hence the net outflow in this state is $4. Hence the replicating portfolio releases $4 reducing its value to $13.23. This $13.23 is rearranged in a portfolio of 2.58 units of gold partially financed by $7.41.

The value of this account in the next period is either $22.82 or $8.36. Assume the upstate occurs. From exhibit 2 the net outflow at this node is 8-1 = $7. Hence the trading strategy releases $7, reducing value to $15.82. This portfolio is again rearranged as shown in exhibit 3. In the final period this account will be valued at either $27.0 in the upstate or $10.2 in the downstate. Note that these numbers exactly match the net outflows (cash flows plus terminal values) of the project.

The example shows that the cash flows and terminal values associated with the given production schedule can be matched by the given dynamic trading strategy. To avoid arbitrage opportunities, the value of the gold mine which operates under this schedule must equal $25.10. All other feasible production schedules can have their cash flows and terminal values replicated by other trading strategies. In all instances, however, the current values of these other trading strategies are less than $25.10.

We note the observation of Cox, Ross, and Rubinstein (1979) that slight modifications of their binomial lattice can be used to value an option on a stock paying a continuous dividend when the instantaneous dividend yield, say b, is constant.
As such, if \( \delta \), the cost of carrying the underlying commodity for one period can be viewed as a constant proportion of the spot price, then it becomes necessary to replace \( r_f \) by \( (r_f + \delta) \). As a result, both equations (12) and (31) will be of the form:

\[
\theta = \frac{e^{(r+\delta)\Delta t} - d}{u - d}
\]

(33)
(3.6) A Trinomial Contingent Claims Model for Valuing Risky Ventures

This section provides a generalization of the binomial Contingent Claims model for valuing risky ventures. Specifically, if the market is complete and trading strategies allow no arbitrage opportunities, then an equivalent martingale measure exists that allows any contingent claim to be valued by computing an appropriate conditional expectation.

Without loss of generality, let \( R_{ij}(x; k) \) be defined as the maximum total expected value of the venture at time \( t_i \) given the price is \( S_{ij} \), the state of the venture is \( x_i \), the entering technology, \( k \) and optimal decisions are to be followed. Accordingly, \( R_{ij}^\ell(x; k) \) is defined as the total expected value of the venture at node \((1, j)\) given the entering technology is \( k \), and technology \( \ell \) is chosen to be used in \([t_i, t_{i+1}]\) with optimal decisions made thereafter. Given a trinomial specification the cash flows received in period \((i+1)\), and the value of the project in the next period are shown below:

\[
R_{ij}^\ell(x; k) \rightarrow f_{i+1, j+1}(\ell) + R_{i+1, j+1}(x_{i+1}; \ell) \\
\rightarrow f_{i+1, j}(\ell) + R_{i+1, j}(x_{i+1}; \ell) \\
\rightarrow f_{i+1, j-1}(\ell) + R_{i+1, j-1}(x_{i+1}; \ell)
\]

(34)

with \( x_{i+1} = g(x_i; \ell) \), where \( g(x_i; \ell) \) indicates that the state variable \( x(t_{i+1}) \) depends on its current value \( x(t_i) \) and the technology used during time period \([t_i, t_{i+1}]\). As before, \( f_{i, j}(\ell) = f(S_{ij}, \ell) \).
Following equation (17), we have for \( j \in J(1) \) and \( 1 \leq n - 1 \),

\[
R_{1j}(x_1; k) = \max_{\ell \in G_{i1j}(x_1; k)} \left\{ R_{1j}^\ell(x_1; k) - \pi_1(k, \ell) \right\}
\]

(35)

where \( G_{i1j}(x_1; k) \) represents the set of feasible technology switches that can be made from node \((1, 1)\), and \( \pi_1(\cdot) \) is the cost associated with a switch at time \( t_1 \), with,

\[
\begin{align*}
R_{1j}(x_1; k) &= \frac{p_{11} \left\{ f_{1+1, j+1}(\ell) + R_{1+1, j+1}(x_{1+1}; \ell) \right\}}{e^{r \Delta t}} \\
&+ \frac{p_{00} \left\{ f_{1+1, j}(\ell) + R_{1+1, j}(x_{1+1}; \ell) \right\}}{e^{r \Delta t}} \\
&+ \frac{p_{-11} \left\{ f_{1+1, j-1}(\ell) + R_{1+1, j-1}(x_{1+1}; \ell) \right\}}{e^{r \Delta t}}
\end{align*}
\]

(36)

where \( \frac{1}{e^{r \Delta t}} \) is a single period discount factor, and the trinomial jump probabilities are obtained from chapter (2). For \( i = n \) the boundary condition is given as;

\[
R_{nj}(x_n; k) = H_{nj}(x_n; k)
\]

(37)

Equations (35), (36) and (37) define a dynamic program on a trinomial lattice which allows the value of a risky venture to be computed. In the preceding, if the storage costs are a constant proportion of the spot price, then it becomes necessary for the valuation process to reflect the cost of carrying the commodity.
Let $\delta$ represent such a cost. To incorporate $\delta$ into the valuation scheme, $r$ must be replaced by $(r+\delta)$ in order to obtain the jump probability values, namely, $p_1$, $p_0$, and $p_{-1}$. To do so, simply set $\mu = (r+\delta) - \sigma^2/2$ instead of $\mu = r - \sigma^2/2$ which was the case for the trinomial model presented in chapter (2).

(3.7) **Convenience Yield**

In the previous sections we assumed that the underlying source of uncertainty was a traded security. As such, a traded security describes a traded asset that is held solely for investment purposes by a significant number of investors. Stocks, bonds, gold and silver are all traded securities with specific markets in which they are traded. However, most commodities are not traded securities.

Had our analysis focused on copper instead, which is held in inventory not by individual investors but by manufacturers of copper based products who maintain a commercial interest in the metal, then the benefit of having an inventory of copper on hand should have been accounted for in the overall valuation scheme.

Such inventories, like inventories of any raw material are held because they allow production to proceed without interruptions caused by shortages of raw materials. Often inventories will continue to be held even if the spot price is expected to decline, where, the decline in inventory value is offset by the convenience of having the inventory on hand.
This benefit, from having an inventory on hand, is known as the "Convenience Yield". Accordingly, the marginal convenience yield is the benefit yielded by the marginal unit of inventory net of any costs of physical storage.\(^5\) In general, when an underlying source of uncertainty is the price of a traded security it's expected growth rate and the individual risk and reward preferences and aversions are irrelevant with regard to valuation of derivative securities. For instance, in our binomial risky venture example, the project can be viewed as a derivative security (a claim) whose value is dependent on the price of a traded asset, namely, gold. However, when the underlying variable is not the price of a traded security, risk and reward preferences and aversions become important factors and a parameter commonly known as the market price of risk enters into the pricing of derivative securities.\(^6\)

Nonetheless, risk neutral valuation agreement can be extended to account for these situations through the use of Commodity Futures Prices.\(^7\)

Specifically, if \(\delta\) is a storage cost expressed as a constant proportion of the spot price, then the futures price of the commodity, \(F = F(S, t)\) is given by

\(^5\)See Brennan (1958).

\(^6\)That is for valuation purposes the growth rate and the market price of risk would have to be estimated.

\(^7\)See Chapters 2 and 7 of Hull (1989) for a detailed discussion.
\[ F = S_t e^{(r+\delta-K)(T-t)} \]

where \( S_t \) is the spot price of the commodity at time \( t \), \( r \) is the constant short term interest rate (risk free), \( T \) is the maturity of the futures contract, and \( K \) is the convenience yield which can be viewed as a nonobservable continuous dividend yield. The implications of the above equality are quite comforting. The equation shows that the commodity can be viewed as a traded security which pays a continuous dividend yield of magnitude \((K-\delta)\). As such, the growth rate and hence the market price of risk for the commodity need not be estimated. Given the existence of a futures contract for the commodity in question, its futures price \( F \) can be readily obtained from quoted futures prices. Since \( \delta \) is known, then the value of \( K \) which makes the above equality true is the convenience yield for the duration of the futures contract.

To properly account for convenience yield and storage cost, in our previous models we proceed as follows:

1. For the binomial model adjust the risk neutralized probability, \( \theta \), to:

\[
\theta = \frac{e^{(r+\delta-K)\Delta t} - \delta}{(u-d)}
\]

(38)

2. For the trinomial model compute the implied risk neutralized probabilities by using:

\[
\mu = (r+\delta-K) - \sigma^2/2
\]

(39)

In either model the discount factor remains as \( e^{r\Delta t} \).
(3.8) Conclusion

Valuing natural resource or commodity based production ventures which contain operating flexibility and strategic options is best accomplished using a contingent claims approach. This chapter has shown how multinomial models can be applied to obtain an arbitrage free value for risky ventures (projects).

While other approaches are available, the advantages of CCA techniques presented in this chapter are significant.

First, the methodology is not static. It provides the opportunity to account for uncertainty during the valuation process, without requiring subjective inputs on probabilities and on the risk adjusted discount rates. Second, the approach provides more than a valuation; it provides a guideline for selecting optimal actions in response to market conditions. In the following chapter we present new valuation models that provide an opportunity for valuing projects whose outputs cannot be regarded as traded securities, nor there may exist futures contracts in the output products.
Appendix (3A)

Risky Venture Valuation: Multiple Sources of Uncertainty

This section provides a generalization of the framework used in the previous sections to account for multiple sources of uncertainty. As before a risky venture is defined as a cash flow process up to maturity time $T$, together with a random quantity received at maturity. Unlike simple static claims, where the cash flows depend solely on an underlying stochastic process, $S(t)$, here the cash flows also depend on the setting of a (managerial) control variable, called technology.

Let $S(t) \in \mathbb{R}^0$ represents the price of the underlying technology, where associated with each choice of technology there exists a finite number of modes of operation, where

$$\frac{dS(t)}{S(t)} = \alpha dt + \sigma dZ(t).$$

Initially we assume that in each period of operation only one technology and its respective mode of operation can be employed. At the end of this period the management has the flexibility to either maintain the current technology and switch to a new mode of operation, or switch to a new technology and a corresponding mode, all depending on the optimal (value maximizing) decision.

Let $P(T)$ be a partition of $[0,T]$. Switching is assumed to

---

8Technology refers to a product or an asset or commodity, while its respective mode reflects the amount or the quantity of technology.
occur at discrete times over \([0,T]\) with each \(t_i, 1 < n\), as a potential switching time.

Define switching as:

1. **Technology (inter) switch**: the decision to switch among \(Q\) (classes) types of technology—e.g. products, assets, commodities, etc.

2. **Mode (intra) switch**: the decision to change input (output) level or quantity within a prespecified technology.

Let \(f(S,x,t; q(m))\) represent the cash flow rate at time \(t\), given that the \(m\)th mode of operation under the \(q\)th technology is in place, the underlying price is \(S(t)\), and the state of venture is \(x(t)\), with both \(S(t)\) and \(x(t)\) corresponding to the \(q\)th technology \(q = \{1,2, \ldots, Q\}\) and \(m = \{0,1,\ldots,M\}\).

Let \(H(S,x,T; q(m))\) be the random quantity received at \(T\), and define \(\pi_i(q(m'),q(m))\) as the cost of an intra (modal) switch at time \(t_i, 1 < n\). That is, a switch from mode \(m'\) to mode \(m\) under the \(q\)th technology \(q = \{1,\ldots, Q\}\) with \(m', m = \{0, 1,\ldots, M\}\). Let \(\Delta_i(q', q)\) be the cost associated with an inter (technology) switch at time \(t_i, 1 < n\) with \(q, q' = \{1,\ldots, Q\}\). It is assumed that \(\pi_i(q(m),q(m)) = 0\), and \(\Delta_i(q, q) = 0\). Let \(x(t_i)\) represent the state of the venture at time \(t_i\), and let \(D_i(d_i)\) define the mode, \(d_i\), corresponding to technology \(D_i\) that is selected for use in period \([t_i, t_{i+1}].\)

The choice of technology and mode at time \(t_i\) may be restricted by the current technology and mode in place, i.e., \(D_{i-1}(d_{i-1})\), and
the overall state of the system. As such, let \( X(t_1) \in \mathbb{R}^0 \) with
\[
X(t_1) = \{ x_1(t_1), \ldots, x_q(t_1) \}
\]
represent the state of the system at
time \( t_1 \) for all technologies. Let
\[
G(\tilde{S}(t_1), X(t_1); \tilde{d}_{l-1}(d_{l-1}))
\]
represent the set of feasible selections for \( d_l(d_l) \) with \( \tilde{S}(t_1) \in \mathbb{R}^0 \)
where \( \tilde{S}(t_1) = \{ S_1(t_1), S_2(t_1), \ldots, S_Q(t_1) \} \) reflects the underlying
price vector at time \( t_1 \) pertaining to the \( Q \) different technologies
and \( \tilde{d}_{l-1}(d_{l-1}) \in \mathbb{R}^0 \) as the vector of technologies and respective
modes of operation at time \( t_{1-1} \). Define an admissible control, \( \tilde{\alpha} \in \mathbb{R}^{Q \times (n+1)} \) as a sequence of feasible technologies and corresponding
modes of operation for use in successive periods, that is;
\[ \tilde{\alpha} = \left\{ \tilde{d}_0(d_0), \tilde{d}_1(d_1), \ldots, \tilde{d}_n(d_n) \right\} \tag{40} \]
where \( \forall l \leq n, \; \tilde{d}_l(d_l) \in G(\tilde{S}(t_1), X(t_1); \tilde{d}_{l-1}(d_{l-1})) \) with \( \tilde{d}_l(d_l) \in \mathbb{R}^0 \).
It is possible to view \( \tilde{\alpha} \) as a \( Q \times (n+1) \) matrix with the rows as the
set of available technologies and the columns as time epochs from
which feasible switches can be made from. That is,
\[
\tilde{\alpha} = \begin{bmatrix}
1(d_0) & 1(d_1) & \cdots & 1(d_l) & \cdots & 1(d_{n-1}) & 1(d_n) \\
2(d_0) & 2(d_1) & \cdots & 2(d_l) & \cdots & 2(d_{n-1}) & 2(d_n) \\
\vdots & \vdots & & \vdots & & \vdots & \\
q(d_0) & q(d_1) & \cdots & q(d_l) & \cdots & q(d_{n-1}) & q(d_n) \\
\vdots & \vdots & & \vdots & & \vdots & \\
Q(d_0) & Q(d_1) & \cdots & Q(d_l) & \cdots & Q(d_{n-1}) & Q(d_n)
\end{bmatrix}
\]
Let $A^D_0$ represent the set of admissible controls with the initial setting $\bar{D}_0(d_0)$. Since, by assumption in each period only one technology can be employed, then $\forall \bar{\alpha}, \exists D_1(d_1) \ni d_1 \neq 0$ with $D_1(d_1) \in \bar{D}_1(d_1), 1 \leq n$. Thus,

$A^D_0 = \{\bar{\alpha}: \bar{\alpha} \text{ is an admissible control with initial setting} \}
D_0(d_0) \in \bar{D}_0(d_0), d_0 \neq 0\}.

To obtain the value of a risky venture conditional on a predetermined $\bar{\alpha} \in A^D_0$ we have;

$$V_0^{\bar{\alpha}}(S(t_0), X(t_0); \bar{D}_0(d_0)) = E \left\{ \sum_{i=1}^{n} \left[ e^{-rT_i} f(S, t, x; D_1(d_1)) dt - e^{-rT_i} \Delta_i (D_{i-1}, D_1) - e^{-rT_i} \chi_i (D_{i-1}, D_1) \right] + e^{-rT_n} H(S, x, T; D_n(d_n)) \right\}$$

(41)

The individual risk and reward preferences and aversions could be dealt with if the intertemporal utility function for the individual decision maker was known. In addition, in the above equation (41) the discount rate $r^*$ which is chosen according to the risk of the cash flows may be a function of the technology and mode employed, i.e., $r^* = r^*(D(d))$. Let

$$J_0^{\bar{\alpha}}(S(t_0), X(t_0); \bar{D}_0(d_0)) = \sup_{\bar{\alpha} \in A^D_0} \left\{ V_0^{\bar{\alpha}}(S(t_0), X(t_0); \bar{D}_0(d_0)) \right\}$$

(42)
That is, \( J_0(t) \in \mathbb{R}^1 \) is the value associated with the optimal admissible control given that the initial underlying stochastic price vector is \( \bar{S}(t_0) \in \mathbb{R}^Q \), the time zero state of the venture is \( X(t_0) \in \mathbb{R}^Q \) corresponding to \( \bar{d}_0(d_0) \). The approach for computing \( J_0(t) \) is dynamic programming. Toward this end let

\[
J_1(t_1, X(t_1); q(m)) \text{ be the maximum total expected value of the venture at time } t_1 \text{ given that mode } m \text{ under technology } q \text{ was in place during } [t_{i-1}, t_1], q = \{1, 2, \ldots, Q\}, m = \{0, 1, \ldots, M\}, \text{ the underlying stochastic vector is } \bar{S}(t_1), \text{ and the state variable at time } t_1 \text{ is } X(t_1), \text{ and optional decisions are to be followed. The Bellman equation for every } q, q' = \{1, \ldots, Q\}, \text{ and every } m, \ell = \{0, 1, \ldots, M\} \text{ and } t_1 < n \text{ is:}
\]

\[
J_1(t_1, X(t_1); q(m)) = \max_{q'} \left\{ \max_{\ell|q'} \left\{ \text{Max} \int_{t_1}^{t_{i+1}} e^{-r(t-t_1)} f(S(t), X(t); q'') dt + e^{-r(t_{i+1}-t_1)} J_{i+1}(\bar{S}(t_{i+1}), X(t_{i+1}); q'') \right\} \right. \\
- \Delta_i(q, q') - \pi_i(q(m), q(\ell)) \right\} 
\]  

(43)
s.t. 

(1) \( q'(\ell) \in G\{\hat{S}(t_1), X(t_1); q(m)\} \)

(1i) \( X(t_{1+1}) = g\{X(t_1); q'(\ell)\} \)

(1ii) \( \hat{S}(t_{1+1}) = \zeta\{\hat{S}(t_1)\} \)

(1iii) \( \Delta_i(q, q') = 0 \) if \( q = q' \)

(1v) \( \pi_1(q(m), q(\ell)) = 0 \) if \( m = \ell \)

The first constraint represents the set of feasible selections, while the second constraint identifies the time \( t_{1+1} \) state of the venture as a function of the current state and the value maximizing choice of technology and mode of operation during the next period. Dependence of the future price vector on its current value is reflected by constraint (1ii). Constraints (1iv) and (1v) are self explanatory.

The above dynamic program is completely defined once the boundary conditions as time \( t_n = T \) are established. These conditions for every \( q \in \{1, 2, \ldots, Q\} \) and every feasible \( m \) given \( q \) are:

\[
J_n\{\hat{S}(t_n), X(t_n); q(m)\} = H_n\{\hat{S}(t_n), X(t_n); q(m)\} \tag{44}
\]

Thus far we have assumed that in each period a single technology can be employed. However, this need not be the case. Suppose that in each period we have the capability of employing a subset of the available technologies simultaneously, namely, those technologies
that result in value maximization. Let \( q^* \leq Q \) be the maximum number of technologies that can be employed simultaneously in each period of operation and define \( \bar{q}(\bar{m}) \in R_+^0 \) as the vector of technologies and their corresponding modes of operation.

Without loss of generality, let \( J_1(\bar{S}(t_1), X(t_1); \bar{q}(\bar{m})) \in R^1 \) represent the maximum total expected value of the venture at time \( t_1 \), given that \( \bar{q}(\bar{m}) \in R_+^0 \) was in place during time period \([t_{i-1}, t_1]\). The Bellman equation for every \( t_1 \leq n-1 \), and for every feasible vector \( \bar{q}(\bar{m}) \in R_+^0 \) and \( \bar{q}'(\bar{\ell}) \in R_+^0 \) is given by:

\[
J_1(\bar{S}(t_1), X(t_1); \bar{q}(\bar{m})) = \max_{\bar{q}'} \left\{ \max_{\bar{\ell}} \left\{ E \left( \int_{t_1}^{t_1+1} e^{-r(t-t_1)} f(\bar{S}(t), \bar{X}(t); \bar{q}'(\bar{\ell})) dt + e^{-r(t_1+1-t_1)} J_1(\bar{S}(t_1), X(t_1); \bar{q}'(\bar{\ell})) \right) \right\} \right.
\]

\[
- \Delta_1(\bar{q}, \bar{q}') - \pi_1(\bar{q}(\bar{m}), \bar{q}(\bar{\ell})) \right\} \right) \right) \right) \right) \right) \right) \right) \right)
\]

s.t.

(1) \( \bar{q}'(\bar{\ell}) = G(\bar{S}(t_1), X(t_1); \bar{q}(\bar{m})) \)

(11) \( X(t_{i+1}) = g(X(t_i); q'(\bar{\ell})) \)

(iii) \( \bar{S}(t_{i+1}) = \zeta(\bar{S}(t_i)) \)

(iv) \( \Delta_1(\bar{q}, \bar{q}') = 0 \) if \( \bar{q} = \bar{q}' \) with \( \bar{q}, \bar{q}' \in R_+^0 \)

(v) \( \pi_1(\bar{q}(\bar{m}), \bar{q}(\bar{\ell})) = 0 \) if \( \bar{m} = \bar{\ell} \) with \( \bar{q}(\bar{m}), \bar{q}(\bar{\ell}) \in R_+^0 \)
Equation (45) is a simple extension of equation (43). Here, the function \( f(\tilde{S}(t), \tilde{X}(t); \tilde{q}'(\tilde{e})) \) represents the cash flow rate from employing the vector \( \tilde{q}'(\tilde{e}) \) and can be obtained by:

\[
f(\tilde{S}(t), \tilde{X}(t); \tilde{q}'(\tilde{e})) = \sum_{q'(e) \in \tilde{q}'(\tilde{e})} f(S(t), x(t); q'(e))
\]  

(46)

The boundary conditions for time \( t_n \) and for every feasible choice of \( \tilde{q}(\tilde{m}) \in \mathbb{R}^q \) are:

\[
J_n(\tilde{S}(t_n), X(t_n); \tilde{q}(\tilde{m})) = H_n(\tilde{S}(t_n), X(t_n); \tilde{q}(\tilde{m}))
\]  

(47)

Equations (43), (44), (45), and (47) define dynamic programs that allow the value of the risky venture to be computed. Note that in these equations, the resulting values depend on probabilities and on preferences which must be provided exogenously.

In the following appendices we show how arbitrage arguments can be used to value this risky venture. Specifically, the next section generalizes the procedures that were used in section (2.6) of this chapter where the subject inputs on the probabilities and \( r^* \) are not required.
Appendix (3B)

Valuation with Multinomial Lattices

We now consider the valuation problem under the assumption that the underlying commodity price, $\tilde{S}(t) \in \mathbb{R}^2$, can be approximated by a multinomial lattice. For convenience we assume $Q = 2$, and develop the valuation model by imposing the condition that in each period the venture can only produce one item, that is, only one technology can be employed. In addition, we assume that the price process is given by

$$dS_t = \alpha_t S_t dt + \sigma_t S_t dZ_t(t) \quad (t = 1, 2) \quad (48)$$

where $dZ_1(t) \cdot dZ_2(t) = \rho dt$. Equations (48) are then approximated by a five jump process as developed in Chapter (2). Accordingly, let $S_{i, j, k} \in \mathbb{R}^2$ represent the time $t_i$ value of the price pair given that the first commodity's price is in state $j$, and the second in state $k$. From Chapter (2) have:

$$S_{i, j, k} = \begin{pmatrix} S_1(t_0)u_1^j & S_2(t_0)u_2^k \end{pmatrix}$$

with $j \in \{J(1), J'(1)\}$, $k \in \{K(1), K'(1)\}$

where $J(1) = K(1) = \{-1, -1+2, \ldots, 1-2, 1\}$

and $J'(1) = K'(1) = \{-1+1, -1+3, \ldots, 1-3, 1-1\}$

represent the set of feasible realizations for $(j,k)$ at time $t_1$.

Without loss of generality let $R_{i, j, k}(\tilde{x}_i; q(m))$ be the maximum total expected value of the venture at time $t_i$, given that the
price is \( S_{i,j,k} \) the state of the system is \( \bar{x}_1 = (x_{11}, x_{21}) \) and mode \( m \in \{0,1,\ldots,M\} \) under the \( q^{th} \) technology, \( q \in \{1,2\} \) was in place during period \([t_{i-1}, t_i]\), and optimal decisions are to be followed.

Let \( R_{i,j,k}^{q'(m')}(\bar{x}_i; q(m)) \) be the total expected value of the venture from node \((i,j,k)\), given the entering technology and node is \( q(m) \), and node \( m' = \{0,1,\ldots,M\} \) under technology \( q' = \{1,2\} \) is chosen for use during period \([t_i, t_{i+1}] \) with optimal decisions thereafter.

The cash flows received, and the value of the project in the next period are shown below.

\[
\begin{align*}
R_{i,j,k}^{q'(m')}(\bar{x}_i; q(m)) &
= f_{i+1,j+1,k+1}(q'(m')) + R_{i+1,j+1,k+1}(\bar{x}_{i+1}; q'(m')) \\
&
= f_{i+1,j+1,k-1}(q'(m')) + R_{i+1,j+1,k-1}(\bar{x}_{i+1}; q'(m')) \\
&
= f_{i+1,j,k}(q'(m')) + R_{i+1,j,k}(\bar{x}_{i+1}; q'(m')) \\
&
= f_{i+1,j-1,k-1}(q'(m')) + R_{i+1,j-1,k-1}(\bar{x}_{i+1}; q'(m')) \\
&
= f_{i+1,j-1,k+1}(q'(m')) + R_{i+1,j-1,k+1}(\bar{x}_{i+1}; q'(m'))
\end{align*}
\]

where \( \bar{x}_{i+1} = g(\bar{x}_i, q'(m')) \) and \( f_{i,j,k}(q'(m')) = f(S_{i,j,k}; q'(m')) \).
Following equation (43) we have for every \( i \leq n - 1 \), and \( j \in \{J(1), J'(1)\} \), \( k \in \{K(1), K'(1)\} \).

\[
R_{i,j,k}(\bar{x}_i, q(m)) = \max_{q'} \left\{ \max_{q''} \left\{ \max_{q'''} \left\{ R_{i',j',k'}(\bar{x}_i'; q(m)) - \Delta_i(q, q') - \Delta_j(q', q'''(m)) \right\} \right\} \right\}
\]

(50)

where \( q'(m') \in G_i,j,k(\bar{x}_i, q(m)) \) and \( G_i,j,k(\cdot) \) as the set of feasible switches that can be made from node \((i,j,k)\) and;

\[
R_{i',j',k'}^q(\bar{x}_i'; q(m)) = e^{-\Delta t} \sum_{\ell_1} \sum_{\ell_2} p_{\ell_1, \ell_2} \left[ R_{i'+1,j-\ell_1',k-\ell_2'}(\bar{x}_{i'+1}; q'(m')) + f_{i'+1,j-\ell_1',k-\ell_2'}(q'(m')) \right]
\]

(51)

where \( \ell_1 \in \{J(1), J'(1)\} \) and \( \ell_2 \in \{K(1), K'(1)\} \) and \( p_{\ell_1, \ell_2} \) as defined in Chapter (2) and \( r \) as the riskless rate of return. The boundary conditions for \( i = n \), and \( j \in \{J(n), J'(n)\} \), \( k \in \{K(n), K'(n)\} \) and for every feasible \( q(m) \) are:

\[
R_{n,j,k}(\bar{x}_i, q(m)) = H_{n,j,k}(\bar{x}_i', q(m))
\]

(52)

Equations (50), (51) and (52) define a dynamic program which allows the value of the risky venture to be computed. This dynamic program, in contrast to the valuation equations given by (43) and (44), does not require the subjective inputs on the probabilities and discount rates. In addition, equations (49), (50), (51), and (52) can be easily modified to account for simultaneous production of all technologies.
Chapter 4

A Multinomial Lattice Approach for Valuing Risky Ventures:

Multiple Sources of Price Uncertainty

(4.1) Introduction

The main thrust of this chapter is concerned with the development of an arbitrage based approach for valuing contracts where the output may not be a traded security and where futures markets may not exist for the output.

Specifically, a methodology is developed for valuing contracts which require the manufacturer to deliver fixed quantities of finished goods according to a deterministic delivery schedule. While the manufacturer has capacity constraints, a variety of feasible schedules exist that allow the demand schedule to be met. Given that the raw material (input) costs and finished good (output) prices fluctuate over time and conversion costs depend on the production rates and switching costs, the problem is to establish an optimal inventory building policy such that the demand is met.

The model for valuing the above contract treats both the input and the output prices as stochastic variables. The model also takes explicit account of managerial control over the output rates, a time dependent variable which responds to the input and the output price uncertainty, as well as the deterministic demand.
The approach used here is based on the construction of a self-financing portfolio of traded securities whose cash flows replicate the cash flows of the contract in question.\(^1\) The idea of a replicating self-financing portfolio is similar to the Black and Scholes (1973) and Merton (1973) option pricing models.

For valuation purposes, we assume the existence of futures market in the input commodities, and that the convenience yields can be expressed as a constant proportion of the current spot prices for the inputs\(^2\). In order to justify an arbitrage based approach when in fact the output is not a traded security, we assume the existence of a traded portfolio whose value is correlated (less than one) with the output price. This traded portfolio may consist of commodities and securities whose aggregate value closely reflect that of the output price.

Quigg (1989) uses this idea to derive an option based model for valuing real estate where, the building on the land is not a traded security. The previous assumptions, together with the constancy of the riskless rate of return and the construction of a replicating self-financing portfolio, provide two important advantages with regard to the valuation process.

\(^1\)See Harrison and Kreps (1979), and Brennan and Schwartz (1985).

First, the need for estimating the expected rate of return for both the input and output price processes is eliminated. Second, the process does not require an exogenously provided discount rate.

To account for the dependence of the output rate on the stochastic nature of the input and output prices, as well as the deterministic demand rate, the valuation process is modeled as a problem in stochastic control. Recent literature with application of stochastic optimal control theory to investment decision problems include the works of Constantinides (1978), Constantinides and Richard (1978), Dothan and Williams (1980), Pindyck (1980), Brennan and Schwartz (1982a,b), Brennan and Schwartz (1985), Constantinides (1985), and Morck, Schwartz, and Stangeland (1990).

Like the natural resources investment model of Brennan and Schwartz (1985), the models presented in this chapter provide a framework for considering whether, when, and how much to produce a given product, and addresses the concerns of a financial analyst regarding the valuation of such a contract.

The remainder of this chapter is organized as follows. The assumptions and notation for model specification are listed in section (4.2). In the following section, a classical approach to the valuation problem is provided, where the necessary inputs for such an approach to valuation would have to be exogenously provided. Our purpose in formulating the problem in the traditional operations research sense is to indicate that the proposed arbitrage based valuation models require less input
parameter estimations and do not require a risk adjusted discount rate, and are independent of preferences or utility functions.

Section (4.4) develops an arbitrage valuation model, which involves a partial differential equation for each production period, inventory and production constraints, and appropriate boundary conditions. This partial differential equation governs the value of the contract in every production period under an output policy \( \psi \). Since there is no analytic solution to the p.d.e.s, they must be solved numerically in each period in order to obtain the time zero value of the contract along with optimal decisions to be followed. Section (4.5) provides a numerical approximation approach for solving the model that is presented in section (4.4).

Specifically, by superimposing the control problem on a multinomial lattice that represents the stochastic evolution of the input and the output prices, a recursive procedure is developed which can be solved efficiently. Section (4.6) summarizes the chapter findings. Numerical examples and comparative statics for section (4.5) will be the topic of discussion in Chapter (5).
(4.2) **Assumptions and Notation**

Consider a manufacturing firm that produces a product which requires two distinct commodities as inputs.³ The firm contemplates entering into a contract where upon commitment, it must supply this product at equal time intervals over a prespecified horizon of length \([0,T]\).

Let \(P(T) = \{0 = t_0 < t_1 < \ldots < t_n = T\}\) represent a partition of \([0,T]\) with \(t_{i+1} - t_i = T/n, \ i = 0,1,\ldots,n-1\). The demand at time \(t_i\) is \(D(t_i), \ i = 1,\ldots,n\). The firm produces \(q(t_i)\) units over an interval of length \([t_i, t_{i+1}]\). The raw material for production in \([t_i, t_{i+1}]\) is purchased at time \(t_i\) at prices determined in competitive markets. The output in any period \([t_i, t_{i+1}]\) is assumed to be a variable with an upper bound of \(Q\), that is, \(0 \leq q(t_i) \leq Q\).

In meeting its contractual obligations, \(D(t_i)\), the firm also maintains an inventory of its output, \(I(t_i)\), with \(0 \leq I(t_i) \leq W\), where \(W\) is the inventory capacity. Given the above setting, the firm’s objective is to establish an optimal production/inventory schedule that meets the delivery requirements in light of the input and output price uncertainty.

To value this contract using an arbitrage based framework, we proceed with the following assumptions.

---

³For instance, special metal alloys for aerospace industry.
(1) The commodity (input) price processes are:

\[
\frac{dP_j(t)}{P_j(t)} = \alpha_j \, dt + \sigma_j \, dZ_j(t) \quad j = 1, 2
\]

(1)

with \(\alpha_j\) and \(\sigma_j\) as the instantaneous mean and volatility for commodity \(j\), and \(dZ_j(t)\) representing an increment in the Standard Wiener process, \((j = 1, 2)\). The instantaneous correlation coefficient \(\rho_{12}\) is given by

\[
dZ_1(t) \cdot dZ_2(t) = \rho_{12} \, dt
\]

(2)

In addition, we assume futures markets in the raw material commodities. As such, let \(F_j(P_j(t_1); (T-t_1))\) represent the futures price for the \(j^{th}\) input at time \(t_1\), \((1 = 0, 1, \ldots, n), \ (j = 1, 2)\). Following Ross (1978), we have

\[
F_j(P_j(t_1), (T-t_1)) = P_j(t_1) \cdot e^{(r-K_j)(T-t_1)} \quad (j = 1, 2)
\]

(3)

with \(K_j\) representing the constant convenience yield (net of storage costs) proportional to the current spot price, and \(r\) as the constant riskless rate of return over \([0, T]\). Using Ito's lemma, and equations (1) and (2), it can be shown that the instantaneous change in the futures price for input commodity \(j\) can be expressed in terms of the instantaneous change in the spot price of commodity \(j\) and its corresponding convenience yield \(K_j\), \(j = 1, 2\). This relationship will be used later to derive the arbitrage model.
(2) The output (finished good) price process is also assumed to follow a Geometric Wiener process and is given by

\[
\frac{dS(t)}{S(t)} = \alpha_s \, dt + \sigma_s \, dZ_s(t) \quad (4)
\]

where \( \alpha_s \) and \( \sigma_s \) are the drift and the volatility parameters of the output price respectively, and \( Z_s(t) \), as the Standard Wiener process, with

\[
dZ_s(t) \cdot dZ_j(t) = \rho_{sj} \, dt \quad (j = 1,2) \quad (5)
\]

Here, \( \rho_{sj} \) represents the constant correlation coefficient among the input/output prices.

(3) Since the output is not a traded security, we assume the existence of a traded portfolio whose value, \( M(t) \) is partially correlated with the output price \( S(t) \). This traded portfolio may consist of commodities and securities whose aggregate value, in a frictionless market, closely reflect that of the output price. The price process for this portfolio is given by

\[
\frac{dM(t)}{M(t)} = \alpha_m \, dt + \sigma_m \, dZ_m(t) \quad (6)
\]

where \( \alpha_m \) and \( \sigma_m \) are the drift and volatility parameters, and \( dZ_m(t) \), is an increment in the Standard Wiener process.

Following Capozza and Sick (1988), decomposition of the stochastic component of the output price into both the systematic and unsystematic risk elements result in

\[
dZ_s(t) = \rho_{sm} \, dZ_m(t) + \rho_{sm} \, dZ_m(t) \quad (7)
\]
with \( \rho_{SM} \) signifying the constant correlation between the traded portfolio and the output price, and \( dZ_H \) as the unsystematic (diversifiable) risk.\(^4\) In addition, by imposing the condition that

\[
\text{Cov}(dZ_m(t), dZ_H(t)) = 0
\]

(8)
equation (4) can be written as:

\[
\frac{dS(t)}{S(t)} = \alpha S dt + \sigma_S \rho_{SM} dZ_m(t) + \sigma_S \rho_{SH} dZ_H(t)
\]

(9)

(4) To ensure that the output price has an expected return that solely depends on its correlation with the traded portfolio, \( M \), we assume that the traded portfolio can be regarded as a completely diversified market portfolio. In other words, portfolio \( M \) represents a large portion of the priced risk elements, which in turn implies that:

\[
\alpha_S = (r - \eta) + \beta_S(\alpha - r) = (r - \eta) + \left( \frac{\rho_{SM}}{\sigma_M} \cdot \sigma_S \right) \theta
\]

(10)

and

\[
\alpha_M = r + \theta
\]

(11)

In the above equation (10), \( \eta \) represents the benefit or the "convenience yield" from having the finished product. This

\(^4\)As in the Capital Asset Pricing Model (CAPM), where the total risk can be broken down into systematic (market attributed) and unsystematic (diversifiable) components.
convenience yield may be thought of as the value of being able to benefit from temporary local shortages of the product through the ownership of the physical product. In addition, \( \eta \) is assumed to be a constant proportion of spot price, with \( r \) as defined earlier, \( \theta \) as the risk premium, and \( \beta_s \) as the output price’s volatility relative to the traded portfolio \( M \).

The derivation of equation (10) is a straightforward task and is shown in Appendix (4B).

In addition to the above assumptions, let \( I(t_1) \), \( 1 = 0, 1, 2, \ldots, n \) represent the net inventory level at time \( t_1 \) where

\[
I(t_1) = I(t_{1-1}) + q(t_{1-1}) - D(t_1)
\]  

(12)

and \( I(t_0) = I(t_n) = 0 \).

Define \( K(q(t_1)) \) as the cost of producing \( q(t_1) \) units in period \( [t_1, t_{1+1}] \). It is assumed that this cost is incurred at the end of the period, that is, at time \( t_{1+1} \).

Let \( R(q(t_1)) \in \mathbb{R}^2 \) represent the raw material requirement vector for production at rate \( q(t_1) \) with \( R(\cdot) = (R_1, R_2) \). The procurement cost function \( \mathcal{A}(R(\cdot)) \) is observed at time \( t_1 \) and is given by:

\[
\mathcal{A}(R(q(t_1))) = \sum_{j=1}^{2} P_j(t_1)R_j  
\]

(13)

\( ^5 \)For an excellent discussion of equation (10), see McDonald and Siegel (1985).
The inventory holding cost rate is defined by \( h(I(t_1)) \) and is incurred at time \( t_1 \). To account for the cost in output rate variations, the switching cost is defined by \( w_1(q(t_{i-1}), q(t_1)) \). This cost reflects the dollar charge for changing the production rate from \( q(t_{i-1}) \) to \( q(t_1) \) and is incurred at time \( t_1 \). Production and inventory capacity constraints remain as defined, that is:

\[
0 \leq q(t_1) \leq Q \\
0 \leq I(t_1) \leq W
\]

with \( q(t_n) = 0 \), and \( I(t_0) = I(t_n) = 0 \). In addition, we assume that no shortages or backlogging are permitted.

Let \( f(P, S, I, t_1; q) \) represent the cash flow rate over the time period \([t_1, t_{i+1}]\), given that \( q(t_1) \) units are to be produced over this period, the time \( t_1 \) level of inventory is \( I(t_1) \), the input price vector is \( P = (P_1(t_1), P_2(t_1)) \), and the output price is \( S(t_1) \), with \( P \) and \( S \) as defined before. Then, for every \( t_i, 1 = 0, 1, \ldots, n \), the cash flow in period \([t_1, t_{i+1}]\) is

\[
f(P, S, I, t_1; q) = S(t_1) \cdot D(t_1) - \left\{ A(R(q(t_1))) + K(q(t_{i-1})) + h(I(t_1)) \right\}
\]

We next formulate the valuation problem as a traditional maximization problem, where the solution depends on parameter estimations and preference functions.
(4.3) Traditional Approach to the Valuation Problem

Let \( C(\vec{P}, S, I, t; q(t_1)) \) represent the time \( t_1 \) value of the contract given that the input price vector at time \( t_1 \) is \( \vec{P} = (P_1(t_1), P_2(t_1)) \), the output price is \( S(t_1) \), the net inventory level is \( I(t_1) \), and the control setting (output rate) over time period \([t_1, t_{i+1}]\) is \( q(t_i) \), \( i = 0, 1, \ldots, n \). The time zero value of this contract, conditional on a predetermined feasible \( q(t_1) \) is then obtained from:

\[
C(\vec{P}, S, 0, t_0; q(t_0)) = E\left\{ \max_{q(t_1)} \left\{ \int_{t_{i-1}}^{t_i} e^{-ut} f(\vec{P}, S, I, t; q(t)) dt \right\} \right\} - e^{-ut_1} \pi_1(q(t_1), q(t_1)) \right\} \right\} 
\]

(15)

s.t.

\[
\frac{dP_j(t)}{P_j(t)} = \alpha_j dt + \sigma_j dZ_j(t) \quad j = 1, 2 \quad (15a)
\]

\[
\frac{dS(t)}{S(t)} = \alpha_s dt + \sigma_s dZ_s(t) \quad (15b)
\]

\[
I(t_1) = I(t_{i-1}) + q(t_{i-1}) - D(t_1) \quad 0 < i < n \quad (15c)
\]

\[
I(t_0) = I(t_n) = 0 \quad (15d)
\]

\[
q(t_n) = 0 \quad (15e)
\]

In the above equation (15) risk and reward preferences and aversions could be dealt with if the intertemporal utility function for the individual decision maker was known. In addition, if risks change according to the changes in the output rates, then the
exogenously furnished discount rate \( u \) may be a function of the mode of production, that is, \( u = u(q(t_1)) \). For valuation purposes, the model also requires the forecasts of the expected future price scenarios, which in turn necessitates the estimation of the drift parameters in equations (15a) and (15b) respectively.

Using arbitrage pricing methodology and stochastic optimal control the next section provides a valuation model where the objective function is free of preferences, and the answer does not depend on the drift terms.

(4.4) **Arbitrage Valuation Model**

In deriving an arbitrage based valuation model the assumptions (1) - (4) of section (4.2) are used. Specifically, for this valuation model we consider a hedged portfolio consisting of long positions in the contract, and the futures contracts in the input commodities together with a short position in the traded portfolio \( M \). The portfolio position quantities are chosen in such a way that the return from this hedged portfolio is nonstochastic. In order to avoid riskless arbitrage opportunities, it must be that the return from the hedged portfolio is equal to the riskless rate of return. This relationship is then used to obtain a model for valuing the contract. Toward this goal we proceed as follows where for specific model derivation the interested reader is referred to Appendix (4A).

Define \( C_1(\tilde{P}, S, I(t_1); q(t_{1-1})) \) as the maximum value of the
contract at time \( t_1 \), given that the input price vector is \( \bar{P} \), the
output price is \( S(t_1) \), the net inventory level is \( I(t_1) \), and the
production rate over \([t_{i-1}, t_1]\) is \( q(t_{i-1}) \) and an optimal production
schedule is to be followed thereafter.

Consider a hedged portfolio, \( V \), over the time interval
\([t_i, t_{i+1}]\) consisting of a long position in the contract, along with
\( y_1 \) units short in the traded portfolio \( M \), \( y_2 \) units long in the
futures contracts in the first commodity, and \( y_3 \) units long in the
futures contracts in the second commodity,\(^6\) where

\[
y_1 = \frac{\partial C}{\partial S} \frac{S^r_s}{M^r_m} \frac{S^m}{M^m} \quad (16)
\]

\[
y_2 = -\frac{\partial C}{\partial P_1} \quad \text{and} \quad y_2 > 0 \quad (17)
\]

\[
y_3 = -\frac{\partial C}{\partial P_2} \quad \text{and} \quad y_3 > 0 \quad (18)
\]

The time \( t_1 \) value of this portfolio, \( V(t_1) \), is:

\[
V(t_1) = C_1(\bar{P}, S, I(t_1); q(t_{i-1})) - y_1 M(t_1) \quad (19)
\]

with \( M(t) \) as defined by equation (6).

The instantaneous change in the value of this portfolio, \( dV \),

\(^6\)Note that \( \frac{\partial C}{\partial P_1} < 0 \), and \( \frac{\partial C}{\partial P_2} < 0 \).
at time $t, t_1 < t < t_{1+1}$ is given by

$$dV = dC - y_1 dM + y_2 dF_1 + y_3 dF_2$$  \hspace{1cm} (20)$$

where, from Ito's lemma, the instantaneous change in the contract's value, $dC$, is

$$dC = \left\{ \frac{\partial C}{\partial P_1} dP_1 + \frac{\partial C}{\partial P_2} dP_2 + \frac{\partial C}{\partial S} dS + \frac{\partial C}{\partial t} dt ight.$$ 

$$+ \frac{1}{2} \left[ \frac{\partial^2 C}{\partial P_1^2} (dP_1)^2 + \frac{\partial^2 C}{\partial P_2^2} (dP_2)^2 + \frac{\partial^2 C}{\partial S^2} (dS)^2 ight]$$ 

$$+ 2 \frac{\partial^2 C}{\partial P_1 \partial P_2} (dP_1)(dP_2) \right\}$$  \hspace{1cm} (21)$$

Substituting equations (1), (4), (21) and the instantaneous change in the futures price $dF_j, j = 1, 2$, into equation (20) for a fixed $q(t)$ with $t_1 < t < t_{1+1}$ we obtain:

$$dV = \left\{ \frac{\partial C}{\partial S} S_0 \alpha_S + \frac{\partial C}{\partial P_1} \rho_{P_1} \alpha_{P_1} + \frac{\partial C}{\partial P_2} \rho_{P_2} \alpha_{P_2} + \frac{\partial C}{\partial t} ight.$$ 

$$+ \frac{1}{2} \left[ \frac{\partial^2 C}{\partial S^2} S_0^2 \left( \rho_{S_0}^2 + \rho_{SM}^2 \right) + \frac{\partial^2 C}{\partial P_1^2} P_{10}^2 \sigma_{P_1}^2 + \frac{\partial^2 C}{\partial P_2^2} P_{20}^2 \sigma_{P_2}^2 \right]$$ 

$$+ \frac{\partial C}{\partial S} S_0 \rho_{SM} dZ_S - \frac{\partial C}{\partial S} S_0 \rho_{SM} \frac{\alpha_S}{\sigma_S} dt 
- \frac{\partial C}{\partial P_1} (\kappa_1 - \kappa_2)$$ 

$$- \frac{\partial C}{\partial P_2} (\kappa_2 - \kappa_2) \right\}$$  \hspace{1cm} (22)$$

From equation (7) we have:

$$E(dZ_S)^2 = E(\rho_{SM} dZ_S + \rho_{SM} dZ_S)^2$$
\[ dt = (\rho_{sa}^2 + \rho_{sn}^2)dt + 2\rho_{sa} \rho_{sn} \text{E}(dZ_a \cdot dZ_n). \]

From equation (8), \( \text{Cov}(dZ_a, dZ_n) = 0 \), and since by definition
\[ (dZ_a)(dZ_n) = \rho_{sn} \cdot dt, \]
then it follows that
\[ \rho_{sa}^2 + \rho_{sn}^2 = 1 \]  \hspace{1cm} (23)

In addition, by assuming that the risk \( dZ_n \) is fully diversifiable and, hence, not priced, the expected total return on this portfolio, \( E(dV) \), should then be the riskless rate of return, that is, for \( t_1 < t < t_{1+1} \):
\[ E(dV) = r(C(\bar{P}, S, I(t); q(t))) - y_{1} M(t))dt \]  \hspace{1cm} (24)

Substituting for \( \rho_{sa}^2 + \rho_{sn}^2 = 1 \) in equation (22) and by passing the expectation operator through the resulting expression, that is then equated to the right hand side of equation (24), we obtain, after simplifying, the following partial differential equation:

\[
\frac{\partial C}{\partial S}(S(r - \eta) + \frac{\partial C}{\partial P_1} P_1 (r - \kappa_1) + \frac{\partial C}{\partial P_2} P_2 (r - \kappa_2) + \frac{\partial C}{\partial t}) \\
+ \frac{1}{2} \left\{ \frac{\partial^2 C}{\partial S^2} S^2 \sigma_s^2 + \frac{\partial^2 C}{\partial P_1^2} P_1^2 \sigma_{11}^2 + \frac{\partial^2 C}{\partial P_2^2} P_2^2 \sigma_{22}^2 \right\} \\
+ \frac{\partial^2 C}{\partial S \partial P_1} PS \sigma_s \sigma_{s1} \rho_{s1} + \frac{\partial^2 C}{\partial S \partial P_2} PS \sigma_s \sigma_{s2} \rho_{s2} + \frac{\partial^2 C}{\partial P_1 \partial P_2} P_1 P_2 \sigma_{12}^2 \rho_{12} \\
- rC = 0 
\]  \hspace{1cm} (25)

Equation (25) governs the value of the contract for \( t_1 < t < t_{1+1} \). To obtain the time zero value of the contract we
use a backward recursion in such a way that at each decision point (stage) \( t_i \), \( i = 0, 1, \ldots, n \), we adjust the value of the contract by accounting for the cash flows. Specifically, at time \( t_n \), we have:

\[
C_n(\bar{P}, S, I(t_n); q(t_{n-1})) = S(t_n) \cdot D(t_n) + \pi_n(q(t_{n-1}), q(t_n)) - K(q(t_{n-1}))
\]

with

\[
I(t_n) = q(t_n) = 0
\]

and

\[
\psi_n(\bar{P}, S, I; q) = 0
\]

where \( \pi_n(q(t_{n-1}), 0) \) can be viewed as a close down cost, while \( \psi_n(\cdot) \) reflects the optimal production policy for the period starting at time \( t_n \). Consider now the time period \([t_{n-1}, t_n] \). Given \( \bar{P}, \) and \( S \) at time \( t_{n-1} \), together with \( q(t_{n-2}) \) and \( I(t_{n-1}) \), the solution to the p.d.e. (25) subject to the boundary condition equation (26) and the relevant production and inventory constraints is given by:

\[
C_{n-1}(\bar{P}, S, I(t_{n-1}); q(t_{n-2})) = \max_{q(t_{n-1})} \left\{ \frac{e^{-r(t_{n-1})}}{C_n(\bar{P}, S, I(t_n); q(t_{n-1}))} \right\}
\]

where

\[
C_n(\bar{P}, S, I(t_n); q(t_{n-1})) = \left\{ S(t_n) \cdot D(t_n) - \left( R(q(t_{n-1})) + K(q(t_{n-1})) \right) + h(I(t_{n-1})) \right\} - \pi_{n-1}(q(t_{n-2}), q(t_{n-1}))
\]
s.t.

\[
0 \leq q(t_{n-1}) \leq Q \\
0 \leq I(t_{n-1}) \leq W \\
I(t_{n-1}) + q(t_{n-1}) = D(t_n)
\]

where

\[
\psi_{n-1}(\bar{P}, S, I(t_{n-1}); q(t_{n-2})) = q^*(t_{n-1})
\]

and \( \tilde{E} \) defines the expectation operator with respect to

\[
\frac{d\bar{P}_j(t)}{\bar{P}_j(t)} = \alpha_j \, dt + \sigma_j \, dZ_j(t) \quad j = 1, 2
\]

\[
\frac{dS(t)}{S(t)} = \alpha_s \, dt + \sigma_s \, dZ_s(t)
\]

The above equation (27) is a direct consequence of the Feynmann-Kac results in which the solution of certain partial differential equations are shown to be equivalent to the solution of an appropriately adjusted expectation.\(^7\)

\[\text{In our case, this appropriately adjusted expectation is obtained from a risk neutralized probability distribution, where the drift parameters } \alpha_j \text{ in equation (1) are replaced by } (r - \kappa_j), \ j = 1, 2, \text{ and the drift parameter } \alpha_s \text{ in equation (4) is replaced by } (r - \eta).\]

\[\text{For the Feynmann-Kac results, see Duffie (1988).}\]
In equation (27), \( \psi_{n-1}(\cdot) \) represents the optimal feasible production policy for time period \([t_{n-1}, t_n]\). Accordingly, the boundary condition for time period \([t_{n-2}, t_{n-1}]\) has been developed.

In general, for valuation purposes, we have for every period \([t_i, t_{i+1}]\), \(i = 0, 1, \ldots, n-1\), the following Bellman equation:

\[
C_i(\bar{F}, S, I(t_i); q(t_{i-1})) = \\
\text{Max}_{q(t_i)} \left\{ \tilde{E} \left\{ S(t_i)D(t_i) - \left\{ A(R(q(t_i))) + K(q(t_{i-1})) \right\} \right\} \\
+ h(I(t_i)) - \pi_1(q(t_{i-1}), q(t_i)) \\
+ e^{-r(t_{i+1} - t_i)} C_{i+1}(\bar{F}, S, I(t_{i+1}); q(t_i)) \right\} \right\} \right\}
\]

s.t.

\[
0 \leq q(t_i) \leq Q \tag{28a}
\]
\[
0 \leq I(t_i) \leq W \tag{28b}
\]
\[
I(t_i) = I(t_{i-1}) + q(t_{i-1}) - D(t_i) \geq 0 \tag{28c}
\]

where

\[
\psi_i(\bar{F}, S, I(t_i); q(t_{i-1})) = q^*(t_i) \tag{29}
\]

By applying the above backward recursion, the time zero value of the contract \( C_0(\bar{F}, S, 0; 0) \) can be obtained through solving the p.d.e. (25) at each stage. Unlike the traditional approach as depicted by equation (15), the above arbitrage valuation framework requires significantly less parameter estimation. Since in
general the p.d.e. (25) does not yield an analytic solution, in the following section we provide a numerical approximation approach for solving the above model. Specifically, by superimposing the control problem on a multinomial lattice representing the stochastic evolution of the input and the output prices a recursive valuation model is developed which can be solved efficiently. The next section also provides various model derivatives as special cases.

(4.5) **Multinomial Lattice Approach**

Without loss of generality, the arbitrage valuation equation (25) suggests that equations (1) and (4) can be replaced by:

\[
\frac{dP_j(t)}{P_j(t)} = (r - \kappa_j)dt + \sigma_j dZ_j(t) \quad j = 1, 2
\]

(30)

and

\[
\frac{dS(t)}{S(t)} = (r - \eta)dt + \sigma_S dZ_S(t)
\]

(31)

respectively, with equations (2) and (5) remaining as before. The above price processes can then be approximated by a nine jump lattice model as developed in Chapter (2). To obtain the value of the contract using a multinomial lattice framework, let 

\[S_{i,j,k,l} \in \mathbb{R}^3\]

represent the price vector at time \(t_i\), such that: the first input price is in state \(j\), the second input price is in state \(k\), and the output price in state \(l\). That is, \(S_{i,j,k,l}\) is the time \(t_i\) representation of the input/output prices with
\[ S_{i,j,k,\ell} = \left\{ P_1(t_i), P_2(t_i), S(t_i) \right\} = \left\{ P_1(t_0)u_1^j, P_2(t_0)u_2^k, S(t_0)u_3^\ell \right\} \] (32)

Here, \((j,k,\ell) \in \{B(1), B'(1)\}\) with \(B(1)\) and \(B'(1)\) as the set of feasible realizations for \((j,k,\ell)\) at time \(t_i\), where for every \(i = 0,1,2,\ldots,n\)

\(B(1) = \{-1, -1+2, \ldots, i-2, 1\}\) and

\(B'(1) = \{-1+1, -1+3, \ldots, i-3, i-1\}\).

The up jump magnitudes, \(u_m = e^{\lambda \Delta t/\Delta t} = \lambda + 1\) (\(m = 1,2,3\)) and

\[ \Delta t = \frac{T}{n} = [t_i, t_{i+1}], i = 0,1,\ldots,n-1. \]

While maintaining our previous assumptions, let

\(C_{i,j,k,\ell}(I_1; q_{i-1})\) represent the maximum total expected value of the contract given the price vector is \(S_{i,j,k,\ell} \in \mathbb{R}^3\), the net inventory at time \(t_i\) is \(I_1\), and \(q_{i-1}\) units were produced during the time period \([t_{i-1}, t_i]\) and optimal decisions are to be followed thereafter.

Define \(C_{i,j,k,\ell}(I_1; q_{i-1})\) as the total expected value of the contract form node \((i,j,k,\ell)\) given the net inventory level at time \(t_i\) is \(I_1\), and \(q_{i-1}\) units were produced during \([t_{i-1}, t_i]\) while \(q_i\) units are scheduled for production during period \([t_i, t_{i+1}]\).

Given node \((i,j,k,\ell)\), and that an optimal production policy is to be followed from \([t_i, t_{i+1}], \ldots, [t_{n-1}, t_n]\), the decision \(q_i\)
results in a profit level over the last \((n-1)\) periods of magnitude,

\[
E[q_1^{q_1} \mid (I_{1+1}^{q_1}; q_{1-1}) + e^{-\Delta t} \left[ \frac{\tilde{c}_{1+1}^{q_1}(I_{1+1}^{q_1}; q_1)}{(j,k,\ell)} \right]}
\]

(33)

with \((\tilde{c}_{1+1}^{q_1}(I_{1+1}^{q_1}; q_1))\) as the maximum total value of the contract at time \(t_{i+1}'\), given node \((i,j,k,\ell)\) was realized at time \(t_i\), where

\[
E[\tilde{c}_{1+1}^{q_1}(I_{1+1}^{q_1}; q_1) \mid (j,k,\ell)] = p_1 \cdot c_{1+1,j+1,k+1,\ell+1}(I_{1+1}^{q_1}; q_1)
+ p_2 \cdot c_{1+1,j+1,k+1,\ell-1}(I_{1+1}^{q_1}; q_1)
+ p_3 \cdot c_{1+1,j+1,k-1,\ell+1}(I_{1+1}^{q_1}; q_1)
+ p_4 \cdot c_{1+1,j+1,k-1,\ell-1}(I_{1+1}^{q_1}; q_1)
+ p_5 \cdot c_{1+1,j-1,k+1,\ell+1}(I_{1+1}^{q_1}; q_1)
+ p_6 \cdot c_{1+1,j-1,k+1,\ell-1}(I_{1+1}^{q_1}; q_1)
+ p_7 \cdot c_{1+1,j-1,k-1,\ell+1}(I_{1+1}^{q_1}; q_1)
+ p_8 \cdot c_{1+1,j-1,k-1,\ell-1}(I_{1+1}^{q_1}; q_1)
+ p_9 \cdot c_{1+1,j,k,\ell}(I_{1+1}^{q_1}; q_1)
\]

(34)

with \(p_m, m = 1, \ldots, 9\) as provided by equation (2.22). In addition, the total value of the contract from node \((i,j,k,\ell)\) is given by:

\[
q_1^{q_1} \cdot c_{i,j,k,\ell}(I_{1}; q_{1-1}) = S(t_0) u_3^D(t_1) - \left\{ P_1(t_0) u_1^R + P_2(t_0) u_2^R \right\}
+ K(q_{1-1}) + h(I_{1}) + \pi_1(q_{1-1}, q_1)
\]

(35)
with \( R_1, R_2, K(\cdot), \) and \( h(\cdot) \) as defined previously in section (2).
Equations (33), (34), and (35) can be used to obtain a backward recursion for valuing the contract. Specifically, for valuation purposes the Bellman equation for every \( i = 0, 1, 2, \ldots, n-1 \) and for every feasible node \((i, j, k, \ell)\) with \((j, k, \ell) \in \{B(1), B'(1)\}\) is given by

\[
C_{i,j,k,\ell}(I_1; q_{i-1}) = \max_{q_1} \left\{ E \left\{ C_{i,j,k,\ell}(I_1; q_{i-1}) \right. \right.
\left. + \left[ e^{-r \Delta t} \tilde{C}_{i+1}(I_{i+1}; q_i) \right] | (j, k, \ell) \right\} \right\}
\]

s.t.

\[
\begin{align*}
0 & \leq q_1 \leq Q \\
0 & \leq I_1 \leq W \\
I_1 &= I_{i-1} + q_{i-1} - D(t_i) \geq 0
\end{align*}
\]

and

\[
\Psi_{i,j,k,\ell}(I_1; q_{i-1}) = q_1^*
\]

Here, \( \Psi_{i,j,k,\ell}(\cdot) \) represents the optimal production policy for period \([t_i, t_{i+1}]\) given that node \((i, j, k, \ell)\) was realized at time \( t_i \).

The recursion equation (36) is completely defined once the boundary conditions at time \( t_n = T \) are specified. These conditions for every feasible node \((n, j, k, \ell)\) with \((j, k, \ell) \in \{B(n), B'(n)\}\) are

\[
C_{n,j,k,\ell}(I_n; q_{n-1}) = \left\{ S(t_0)u_3 T_n(t_n) - K(q_{n-1}) - \pi_n(q_{n-1}, 0) \right\}
\]

with \( I_n = q_n = 0 \) and \( \Psi_{n,j,k,\ell}(\cdot) = 0 \). Equations (36), (37), and
(38) define a recursive valuation model through which the time zero value of the contract can be obtained. As such, these equations provide an efficient discrete time approximation to the valuation model provided by equations (25), (26), and (28). In what follows, we consider special cases of this approach.

(4.5.1) **Trinomial Lattice Approach with Input Price Uncertainty**

In this section we assume a single source of uncertainty in the input price. The output price is assumed to be known and contractually fixed at a level of \( \delta(t_i) \) dollars per unit demand satisfied. The input price process for this case is given by

\[
\frac{dP(t)}{P(t)} = (r - \kappa)dt + \sigma dZ(t)
\]  

(39)

For valuation purposes, the trinomial model of Chapter (2) is used to approximate the above price process. Accordingly, let \( P_{1j} \in \mathbb{R}^1 \) represent the input price at time \( t_1 \) given state \( j \) has been realized. That is,

\[
P_{1j} = P(t_0)u^j
\]  

(40)

with \( i = 0, 1, \ldots, n \), and \( j \in \{-1, -1+1, \ldots, 1-1, 1\} \) and \( u = e^{\lambda \sqrt{\Delta t}} \), \( \lambda \geq 1 \).

The total expected value of the contract from node \((1, j)\) given that the net inventory level at time \( t_1 \) is \( I_1 \), and \( q_{i-1} \) units were produced during \([t_{i-1}, t_i]\) and \( q_i \) units are scheduled for production
during period \([t_i, t_{i+1}]\) is obtained from:

\[
C_{1j}(I_1; q_{i-1}) = \delta(t_i) D(t_i) - \left\{ P(t_0) u_i^j R + K(q_{i-1}) + h(I_1) + \pi_i(q_{i-1}, q_i) \right\}
\]

\[(41)\]

Following the format of equation (36), and, for every feasible node \((i, j)\) with \(i \leq n-1\), and \(j \in \{-1, \ldots, 1\}\), the maximum total expected value of the contract is given by:

\[
C_{1j}(I_1; q_{i-1}) = \max_{q_1} \left\{ \mathbb{E} \left\{ C_{1j}(I_1; q_{i-1}) \right\} \right. \\
+ \sum_{\ell = -1}^{1} p_{\ell} \left. \cdot C_{1,i+1,j+\ell}(I_{i+1}; q_1) \right\} \\
\] \[(42)\]

s.t.

\[
0 \leq q_i \leq Q \\
0 \leq I_1 \leq W \\
I_1 = I_{i-1} + q_{i-1} - D(t_i) \geq 0
\]

and

\[
\psi_{1j}(I_1; q_{i-1}) = q^*_1
\]

In the above equation (41), the probability terms \(p_{\ell}, \ell = (-1, 0, 1)\) are obtained from equation (2.4). In addition, for valuation purposes the boundary conditions for time \(t_n\) must be specified. These conditions for every feasible node \((n, j)\) with \(j \in \{-n, \ldots, n\}\) are:

\[
C_{n,j}(I_n; q_{n-1}) = \delta(t_n) D(t_n) - K(q_{n-1}) - \pi_n(q_{n-1}, 0)
\]

\[(43)\]

with \(q_n = I_n = 0\), where the optimal production policy for period
[t_n, t_{n+1}], is given by \( \psi_{n,j} (I_n, q_{n-1}) = 0. \)

The multinomial lattice valuation approach presented in this section can be used to solve a variety of investment decision problems in an efficient manner. Note too, that this model is also a special case of the trinomial valuation model that was considered in Chapter (3), with the source of uncertainty as the output price.

(4.5.2) **Models with Two Sources of Uncertainty**

In this section we consider two multinomial lattice based valuation models as special cases of the model provided by equations (36) and (37).

For the first model, we relax the assumption of a stochastic output price and consider valuing a contract in which the sales price is contractually fixed at a level of $s$ dollars per unit. The original assumption regarding the input price uncertainty as depicted by equation (30) remains intact. For the second model, we assume a single source of uncertainty in the input price while maintaining the original assumption of a stochastic output price as given by equation (31). For valuation purposes, the five jump process of Chapter (2) is employed.

(1) **Stochastic Input Prices, Deterministic Output Price**

Let \( s_{t_1, j, k} \in \mathbb{R}^2 \) represent the time \( t_1 \) input price vector given that the first input price is in state \( j \) and the second input price in state \( k \). That is:
\[ S_{i,j,k} = \{ P_1(t_0)u^1_1, P_2(t_0)u^2_2 \} \]  
with \((j,k) \in \{ B(1), B'(1) \}\) where for every \(i = 0, 1, 2, \ldots, n,\)

\[ B(1) = \{-1, -1+2, \ldots, 1-2, 1\} \quad \text{and} \quad B'(1) = \{-1+1, -1+3, \ldots, 1-3, 1-1\}\]

\[ \lambda = \frac{1}{\Delta t}, \quad \lambda \geq 1 \quad \text{and} \quad m = 1, 2.\]

Let \( \delta(t_i) \) represent the unit sales price at time \( t_i, \ i > 0.\)

Without loss of generality, for every feasible realization of \((i,j,k)\) we have:

\[ C_{i,j,k}(I_1; q_{i-1}) = \delta(t_i)D(t_i) - \left( P_1(t_0)u^1_1 + P_2(t_0)u^2_2 \right) \]

\[ + K(q_{i-1}) + h(I_i) + \kappa_1(q_{i-1}, q_1) \]  

(45)

with

\[ E\left( \tilde{C}_{i+1}(I_{i+1}; q_1) \right| (j,k) = \begin{array}{l}
p_1 \cdot C_{i+1, j+1, k+1}(I_{i+1}; q_1) \\ + p_2 \cdot C_{i+1, j+1, k-1}(I_{i+1}; q_1) \\ + p_3 \cdot C_{i+1, j-1, k-1}(I_{i+1}; q_1) \\ + p_4 \cdot C_{i+1, j-1, k+1}(I_{i+1}; q_1) \\ + p_5 \cdot C_{i+1, j, k}(I_{i+1}; q_1) \end{array} \]  

(46)

where \( p_m, m = 1, \ldots, 5 \) can be obtained from equations (2.14a-e).

Given equations (45) and (46), the valuation equation for every \(i = 0, 1, \ldots, n-1\) and every feasible node \((i,j,k)\) with \((j,k) \in \{ B(1), B'(1) \}\) is given by
\[ C_{i,j,k}(I_1;q_{i-1}) = \max_{q_1} \left\{ E \left\{ C_{i,j,k}(I_1;q_{i-1}) \right. \right. \\
= e^{-r\Delta t} \left\{ \tilde{C}_{i+1}(I_1+q_1)(j,k) \right\} \left\} \right\} \]  
\text{subject to:} \\
0 \leq q_1 \leq Q \\
0 \leq I_1 \leq W \\
I_1 = I_{i-1} + q_{i-1} - D(t) \geq 0
\]  
where \( \phi_{i,j,k}(I_1;q_{i-1}) = q_1 \) defines the optimal operating policy for time period \([t_1, t_{i+1}]\) given the realization of node \((i,j,k)\).

The boundary conditions for time \(t_n\), and for every feasible node \((n,j,k)\) with \((j,k) \in \{B(n), B'(n)\}\) are:

\[ C_{n,j,k}(I_n;q_{n-1}) = \delta(t_n) D(t_n) - K(q_{n-1}) - \pi_n(q_{n-1}, 0) \]  
with \( I_n = q_n = 0 \), and \( \phi_{n,j,k}(I_n;q_{n-1}) = 0 \). Equations (47) and (48) provide a dynamic programming model defined on five jump lattice process that leads to the valuation of the contract as described.

(ii) Single Stochastic Input Price, Single Stochastic Output Price

Here, \( S_{i,j,k} \in \mathbb{R}^2 \) represents the price vector at time \( t_1 \), given that the input price is in state \( j \), and the output price in state \( k \), that is

\[ S_{i,j,k} = \{ P(t_0)u^i_1, S(t_0)u^k_2 \} \]

with \((j,k) \in \{B(1), B'(1)\}\) and \(B(1), B'(1)\) as defined earlier. It follows that for every feasible node \((i,j,k), i = 0, 1, \ldots, n-1\)
\[ C_{1,j,k}(I_1; q_{i-1}) = S(t_0)u_2^k \cdot D(t_1) - \left\{ P(t_0)u_1^k R + K(q_{i-1}) + h(I_1) + \pi_1(q_{i-1}, q_i) \right\} \]  

(50)

with \( E(\hat{C}_{1,j,k}(I_1; q_i) | (j,k)) \) as given by equation (46) although it varies in the context according to our current analysis. Equations (46) and (50) can then be used to derive a dynamic programming lattice valuation model.

Specifically, for \( i = 0, 1, \ldots, n-1 \) and \( (j,k) \in \{B(i), B'(i)\} \) we have

\[ C_{1,j,k}(I_1; q_{i-1}) = \max_{q_1} \left\{ E \left( C_{1,j,k}(I_1; q_{i-1}^*) \right. \\
\left. + e^{-\Delta t} \left( \hat{C}_{1,j,k}(I_1; q_i^*) | (j,k) \right) \right) \right\} \]  

(51)

s.t.

\[ 0 \leq q_i \leq Q \]
\[ 0 \leq I_1 \leq W \]
\[ I_1 = I_{i-1} + q_{i-1} - D(t_1) \geq 0 \]

and

\[ \psi_{i,j,k}(I_1; q_{i-1}) = q_1^* \]

where \( \psi_{i,j,k}(\cdot) \) represents the optimal production policy for period \([t_1, t_{i+1}]\) given node \((i,j,k)\) was realized at time \(t_1\).

To complete the above recursion, for \( i = n \), and for every node \((n,j,k)\) with \((j,k) \in \{B(n), B'(n)\}\) the boundary conditions are:

\[ C_{n,j,k}(I_n; q_{n-1}) = S(t_0)u_2^k D(t_n) - K(q_{n-1}) - \pi_n(q_{n-1}; 0) \]  

(52)

with \( I_n = q_n = 0 \), and \( \psi_{n,j,k}(I_n; q_{n-1}) = 0 \).
(4.6) Conclusion

Valuing claims characterized by multiple sources of uncertainty and which contain operating flexibility are important facets of investment decision analysis. Using an arbitrage based framework and stochastic control theory, this chapter has shown how an efficient multinomial lattice model can be developed to obtain an arbitrage free value of a contract where the output is not a traded security.

While other valuation alternatives are available, the methodology provided here has several advantages. First, the approach provides more than valuation; it provides an optimal guideline for selecting actions in response to market conditions. Second, the methodology is independent of preferences and aversions toward risk and reward. As such, the approach does not depend on an exogenously furnished utility function or a risk adjusted discount factor. Third, the model requires significantly less parameter estimates when compared to more traditional models. Fourth, the model can easily and effectively be adjusted to account for special case situations as shown in the final section of this chapter.

In addition, note that the lattice based models presented in Chapter (3) are special cases of the models considered in Chapter (4).
Appendix (4A)

In this Appendix, we derive the partial differential equation (4.25) as presented in section (4.3).

From equations (1), (3), (4), and (6) we have respectively

\[
\frac{d\tilde{P}_j}{\tilde{P}_j} = \alpha_j \, dt + \sigma_j \, dZ_j \quad (j = 1, 2) \quad (A1)
\]

\[
F_j(\tilde{P}_j, (T-t)) = \tilde{P}_j e^{(r-\kappa)(T-t)} \quad (j = 1, 2) \quad (A2)
\]

\[
\frac{dS}{S} = \alpha_s \, dt + \sigma_s \, dZ_s = \alpha_s \, dt + \sigma_s \rho_s \, dZ_m + \sigma_s \rho_s \rho_m \, dZ_N \quad (A3)
\]

The value of the traded portfolio is given by \( V = C - y_1 M \) where \( C = C(\tilde{P}, S, I; q) \) with \( \tilde{P} = (P_1', P_2') \). From equation (20)

\[
dV = dC - y_1 dM + y_2 dF_1 + y_3 dF_2 \quad (A4)
\]

where the instantaneous change \( dC \) is given by equation (21). In addition, from equation (6)

\[
\frac{dM}{M} = \alpha_m \, dt + \sigma_m \, dZ_m
\]

It follows from Ito's lemma that:

\[
(1) \quad dF = \frac{\partial F}{\partial \tilde{P}} \, d\tilde{P} + \frac{\partial F}{\partial t} \, dt + \frac{1}{2} \frac{\partial^2 F}{\partial \tilde{P}^2} (d\tilde{P}^2)
\]

where

\[(A2) \rightarrow \frac{\partial F}{\partial \tilde{P}} = e^{(r-\kappa)(T-t)} \] and \( \frac{\partial F}{\partial t} = -(r-\kappa)Pe^{(r-\kappa)(T-t)} \]

upon substitution and simplification we obtain for \( j = 1, 2 \):

\[
dF_j = \frac{\partial F}{\partial \tilde{P}_j} \tilde{P}_j \{\alpha_j - r + \kappa_j\} dt + \frac{\partial F}{\partial j} \tilde{P}_j \sigma_j \, dZ_j \quad (A5)
\]

Given equations (A1), (A3), (A4), (20) and (A5), it follows after simplification that:
\[ dV = \left\{ \frac{\partial C}{\partial S} S_a + \frac{\partial C}{\partial P_1} P_1 \alpha_1 + \frac{\partial C}{\partial P_2} P_2 \alpha_2 + \frac{\partial C}{\partial t} \right\} \\
+ \frac{1}{2} \left\{ \frac{\partial^2 C}{\partial S^2} S^2 \sigma_s^2 (\rho_{s1}^2 + \rho_{s2}^2) + \frac{\partial^2 C}{\partial P_1^2} P_1^2 \sigma_{11}^2 + \frac{\partial^2 C}{\partial P_2^2} P_2^2 \sigma_{22}^2 \right\} \\
+ \frac{\partial^2 C}{\partial S \partial P_1} S P_1 \sigma_s \rho_{s1} + \frac{\partial^2 C}{\partial S \partial P_2} S P_2 \sigma_s \rho_{s2} \\
+ \frac{\partial^2 C}{\partial P_1 \partial P_2} P_1 P_2 \sigma_1 \sigma_2 \rho_{12} \right\} dt + \frac{\partial C}{\partial S} S \sigma_s \left\{ \rho_{s1} dZ_1 + \rho_{s2} dZ_2 \right\} \\
+ \frac{\partial C}{\partial P_1} P_1 \sigma_1 dZ_1 + \frac{\partial C}{\partial P_2} P_2 \sigma_2 dZ_2 - y_1 \left\{ \frac{\partial F}{\partial P_1} P_1 \sigma_1 dZ_1 \right\} \\
+ y_2 \left\{ \left\{ \frac{\partial F}{\partial P_1} P_1 (\alpha_1 - \gamma + \kappa_1) \right\} dt + \frac{\partial F}{\partial P_1} P_1 \sigma_1 dZ_1 \right\} \\
+ y_3 \left\{ \left\{ \frac{\partial F}{\partial P_2} P_2 (\alpha_2 - \gamma + \kappa_2) \right\} dt + \frac{\partial F}{\partial P_2} P_2 \sigma_2 dZ_2 \right\} \]

Choose \( y_1, y_2 \) and \( y_3 \) in such a way that the risk elements \( dZ_1, dZ_2 \), and \( dZ_2 \) are eliminated. Without loss of generality we have:

\[ y_1 = \frac{\partial C}{\partial S} S \sigma_s \rho_{s1} \]  \hspace{1cm} (A6)

\[ y_2 = - \frac{\partial C}{\partial P_1} \]  \hspace{1cm} (A7)

\[ y_3 = - \frac{\partial C}{\partial P_2} \]  \hspace{1cm} (A8)

Substituting (A6), (A7), and (A8) in the above p.d.e., and simplifying the results yields
\[ dV = \left\{ \frac{\partial C}{\partial S} S(\alpha_s - \frac{\sigma_s \rho_s}{\sigma_s}) + \frac{\partial C}{\partial P_1} P_1 (r - \kappa_1) + \frac{\partial C}{\partial P_2} P_2 (r - \kappa_2) + \frac{\partial C}{\partial t} \right\} dt \]

\[ + \frac{\partial^2 C}{\partial S^2} \left( \frac{S^2 \sigma_s^2}{S^2} \right) + \frac{\partial^2 C}{\partial P_1^2} \left( \frac{P_1^2 \sigma_1^2}{P_1^2} \right) + \frac{\partial^2 C}{\partial P_2^2} \left( \frac{P_2^2 \sigma_2^2}{P_2^2} \right) \]

\[ + \frac{\partial^2 C}{\partial S \partial P_1} \left( \frac{S \sigma_s \rho_s}{S \sigma_s} \right) + \frac{\partial^2 C}{\partial S \partial P_2} \left( \frac{S \sigma_s \rho_s}{S \sigma_s} \right) \]

\[ + \frac{\partial^2 C}{\partial P_1 \partial P_2} \left( \frac{P_1 \sigma_1 \rho_1}{P_1 \sigma_1} \right) + \frac{\partial^2 C}{\partial S \partial \kappa_1} \left( \frac{S \kappa_1}{S \kappa_1} \right) + \frac{\partial^2 C}{\partial S \partial \kappa_2} \left( \frac{S \kappa_2}{S \kappa_2} \right) \]

\( (A9) \]

By assumption \( dZ \) is not priced, in addition from equation (10) we have

\[ \alpha_s = (r - \eta) + \beta_s (\alpha_s - r) = (r - \eta) + \left( \frac{\rho_s}{\sigma_s} \cdot \sigma_s \right) \theta \]

with \( \theta = \alpha_s - r \) as in equation (11).

Substituting for \( \alpha_s \) in (A9) and by taking the resulting expressions' expected value we obtain

\[ E(dV) = \left\{ \frac{\partial C}{\partial S} S \left( (r - \eta) - \frac{\sigma_s \rho_s}{\sigma_s} \right) + \frac{\partial C}{\partial P_1} P_1 (r - \kappa_1) \right\} \]

\[ + \frac{\partial C}{\partial P_2} P_2 (r - \kappa_2) + \frac{\partial C}{\partial t} \]

\[ + \frac{1}{2} \left\{ \frac{\partial^2 C}{\partial S^2} \left( \frac{S^2 \sigma_s^2}{S^2} \right) + \frac{\partial^2 C}{\partial P_1^2} \left( \frac{P_1^2 \sigma_1^2}{P_1^2} \right) + \frac{\partial^2 C}{\partial P_2^2} \left( \frac{P_2^2 \sigma_2^2}{P_2^2} \right) \right\} \]

\[ + \frac{\partial^2 C}{\partial S \partial P_1} \left( \frac{S \sigma_s \rho_s}{S \sigma_s} \right) + \frac{\partial^2 C}{\partial S \partial P_2} \left( \frac{S \sigma_s \rho_s}{S \sigma_s} \right) \]

\[ + \frac{\partial^2 C}{\partial P_1 \partial P_2} \left( \frac{P_1 \sigma_1 \rho_1}{P_1 \sigma_1} \right) \] \( dt \)

\( (A10) \]

The above expression is free of any risk elements. This in turn implies that the expected return on \( V \) is the riskless rate of return, that is,
\[ E(dV) = r(C - y_lM)dt = \left\{ rC - r \frac{\partial C}{\partial S} S \sigma_s \rho_s \frac{\partial \rho_s}{\partial s} \right\} dt \]  

(A11)

As such, (A10) and (A11) are equal, which in turn yields

\[ \frac{\partial C}{\partial S} S(r - \eta) + \frac{\partial C}{\partial P_1} P_1(r - \kappa_1) + \frac{\partial C}{\partial P_2} P_2(r - \kappa_2) + \frac{\partial C}{\partial t} \]

\[ + \frac{1}{2} \left( \frac{\partial^2 C}{\partial S^2} S^2 \sigma^2_s + \frac{\partial^2 C}{\partial P_1^2} P_1^2 \sigma^2_1 + \frac{\partial^2 C}{\partial P_2^2} P_2^2 \sigma^2_2 \right) \]  

(A12)

\[ + \frac{\partial^2 C}{\partial S \partial P_1} S \sigma_s \sigma_1 \rho_s + \frac{\partial^2 C}{\partial S \partial P_2} S \sigma_s \sigma_2 \rho_s \]

\[ + \frac{\partial^2 C}{\partial P_1 \partial P_2} P_1 P_2 \sigma_1 \sigma_2 \rho_{12} - rC = 0 \]

which is precisely equation (25).
Appendix (4B)

Let $\theta$ be the risk premium. Since $M$ is the market portfolio, then from (6) we have:

$$E\left( \frac{dM}{M} \right) = \alpha_s = r + \theta$$  \hspace{1cm} (B1)

By definition:

$$E\left( \frac{dS}{S} \right) = (r - \eta) + \left( E\left( \frac{dM}{M} \right) - r \right) \beta_s = \alpha_s$$

where

$$\beta_s = \frac{\text{Cov}(dS/S, dM/M)}{\text{Var}(dM/M)} = \frac{\rho_{sM} \sigma_s \sigma_M}{\sigma_M^2} = \rho_{sM} \frac{\sigma_s}{\sigma_M}$$

hence

$$E\left( \frac{dS}{S} \right) = (r - \eta) + (r + \theta - r) \frac{\rho_{sM}}{\sigma_s} \sigma_s$$

$$= (r - \eta) + \left( \frac{\rho_{sM}}{\sigma_s} \sigma_s \right) \theta = \alpha_s$$  \hspace{1cm} (B2)

which is equation (10).
CHAPTER 5
NUMERICAL ISSUES AND COMPARATIVE STATICS

(5.1) Introduction

The purpose of this chapter is to illustrate the numerical nature of the lattice based valuation models that were developed in Chapter (4).

Based on stylized examples, we consider various valuation issues for supply contracts in which the input and/or the output prices are uncertain. To account for the operating flexibility aspects in valuation, we allow an abandonment option whenever the value of the contract becomes negative. Specifically, in situations where the expected operating costs exceed the expected benefits from satisfying the contract's terms, the supplier can costlessly abandon the project.

For each valuation example, comparative statics will be provided to account for model sensitivity with respect to its input parameters. Throughout this chapter, we assume the setting of Chapter (4) notation and models.

(5.2) Example I

In this example, we consider a supply contract in which the finished product requires a single commodity input. The commodity input price process is given by:

\[
\frac{dP(t)}{P(t)} = (r - \kappa)dt + \sigma dZ(t)
\]
while the output price is assumed to be known and contractually fixed. The contract's duration is one year, and the demand is met according to a bimonthly schedule. The following table provides the relevant information.

<table>
<thead>
<tr>
<th>Case Parameters for Examples (I)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T = 1.0$ year, $n = 6$, $\lambda = 1.22474$</td>
</tr>
<tr>
<td>Demand schedule: $D_1 = 4$, $D_2 = 9$, $D_3 = 7$, $D_4 = 5$, $D_5 = 1$, $D_6 = 9$</td>
</tr>
<tr>
<td>Production capacity: $Q = 7$ units per period</td>
</tr>
<tr>
<td>Inventory capacity: $W = 5$ units</td>
</tr>
<tr>
<td>Raw material requirement per unit of finished product: $R(q) = 5$ units</td>
</tr>
<tr>
<td>Production cost: $K(q) = 5 + 1.67q + 1.25q^2$</td>
</tr>
<tr>
<td>Switching cost: $\pi = 0.5(q_{t-1} - q_t)^2$</td>
</tr>
<tr>
<td>Inventory holding cost: $h(I) = $1.50$ per unit</td>
</tr>
<tr>
<td>Initial raw material cost: $P(0) = $10.0$</td>
</tr>
<tr>
<td>Sales price per unit: $\delta = $65.0$</td>
</tr>
<tr>
<td>Convenience yield: $\kappa = 5%$ per annum</td>
</tr>
<tr>
<td>Price volatility: $\sigma = 35%$ per annum</td>
</tr>
<tr>
<td>Interest rate: $r = 10%$ continuously compounded (annual)</td>
</tr>
</tbody>
</table>

For valuation purposes, the trinomial lattice model as provided by equations (4.42) and (4.43) is employed. The time zero value of this contract and the optimal production and inventory policy for the above case parameters are provided in Table (1).
Table (1): Optimal Production and Inventory Policy, Example (I)

\[ C_0^+ = $88.77 \]
\[ C_0^- = $65.42 \]

<table>
<thead>
<tr>
<th>Demand:</th>
<th>4</th>
<th>9</th>
<th>7</th>
<th>5</th>
<th>1</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Production:</td>
<td>6</td>
<td>7</td>
<td>7</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Inventory:</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>Time:</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

Here \( C_0^+ \) and \( C_0^- \) represent the contract's value with and without an abandonment option respectively. Note that the abandonment option for this choice of input parameters results in a 26.3% increase in the contract's value. To account for more accurate results, the partition size was increased and the resulting values were then compared to those obtained by using a binomial valuation model. These results are provided in Table (2), where for either model the above optimal policy remained unchanged.

Table (2): Convergence results for the Trinomial and Binomial Models

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{Model} & n & 6 & 12 & 18 & 24 & 30 & 36 \\
\hline
\text{Trinomial} & 88.77 & 89.52 & 88.89 & 90.0 & 90.10 & 90.07 \\
\text{Binomial} & 90.88 & 87.70 & 88.97 & 89.50 & 90.21 & 90.40 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{Model} & n & 42 & 48 & 54 & 60 & 66 & 72 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{Trinomial} & 90.02 & 90.04 & 90.20 & 90.30 & 90.25 & 90.10 \\
\text{Binomial} & 89.96 & 89.45 & 89.94 & 90.10 & 90.0 & 90.40 \\
\hline
\end{array}
\]
Sensitivity analysis with respect to input price volatility \((\sigma)\) and convenience yield \((\kappa)\) were also considered, where in each case the project's value was found to be an increasing function of the two parameters. To provide some insight for this behavior, note that increasing the volatility causes the input prices to increase in the upside and to decrease in the downside. The effect of increasing input prices on the contract's value is limited due to the existence of an abandonment option. As such, this blocking effect, together with lower input prices in the downside result in an increase in the value of the contract. On the other hand, increasing the convenience yield causes the drift parameter of the risk neutralized process to decrease in a linear fashion. Accordingly, given our choice of input parameters, the contract's value will increase linearly. Figures (1) and (2) depict these results for a partition size of \(n = 30\).
FIGURE 1: CONTRACT VALUE vs. VOLATILITY
FIGURE 2: CONTRACT VALUE vs. CONVENIENCE YIELD
(5.3) **Example II**

In this example we consider a supply contract in which the finished product requires two distinct commodities as inputs. The input commodity price processes are:

\[
\frac{dP_j(t)}{P_j(t)} = (r - \kappa_j)dt + \sigma_j dZ_j(t) \quad j = 1, 2
\]

with \(dZ_1(t) \cdot dZ_2(t) = \rho_{12} dt\). As in the previous examples, the output price is assumed to be known and contractually fixed. In this example, the contract's duration is one year, however, the demand schedule is met every three months. The relevant information for this example are presented in the following table.

<table>
<thead>
<tr>
<th>Case Parameters for Examples (II)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>T</strong> = 1.0 year, <strong>n</strong> = 4, ( \lambda = 1.11803 )</td>
</tr>
<tr>
<td>Demand schedule: ( D_1 = 3, D_2 = 5, D_3 = 1, D_4 = 8 )</td>
</tr>
<tr>
<td>Production capacity: ( Q = 5 ) units per period</td>
</tr>
<tr>
<td>Inventory capacity: ( W = 4 ) units</td>
</tr>
<tr>
<td>Raw material requirements per unit of finished product: ( R(q) = (3, 4) )</td>
</tr>
<tr>
<td>Production cost: ( K(q) = 10 + 2.50q + 3.30q^2 )</td>
</tr>
<tr>
<td>Switching cost: ( \kappa = .50(q_{1-1} - q_1)^2 )</td>
</tr>
<tr>
<td>Inventory holding cost: ( h(I) = $1.50 ) per unit</td>
</tr>
<tr>
<td>Initial raw material cost: ( \bar{P} = (P_1(0), P_2(0)) = (10, 15) )</td>
</tr>
<tr>
<td>Sales price per unit: ( \delta = $120.0 )</td>
</tr>
<tr>
<td>Convenience yield: ( (\kappa_1, \kappa_2) = (3%, 5%) ) per annum</td>
</tr>
<tr>
<td>Price volatility: ( (\sigma_1, \sigma_2) = (30%, 35%) ) per annum</td>
</tr>
<tr>
<td>Correlation Coefficient: ( \rho_{12} = .50 )</td>
</tr>
<tr>
<td>Interest rate: ( r = 10% ) per annum</td>
</tr>
</tbody>
</table>
For valuation purposes the five jump lattice model as provided by equations (4.47) and (4.48) is used. The time zero value of this contract along with the optimal production and inventory policy for the above case parameters are provided in Table (3). As before, the abandonment option is accounted for, where $C_0^+$ and $C_0^-$ represent the contract’s value with and without an abandonment option respectively.

Table (3): Optimal Production and Inventory Policy, Example (II)

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>5</th>
<th>1</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Demand:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Production:</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>Inventory:</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>Time:</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

For this choice of case parameters, the abandonment option resulted in an increase of approximately 48% in the contract’s value. The effect of increasing the partition size on the contract’s value is shown in Table (4), where in each case the above optimal policy remained unchanged. Sensitivity analysis with respect to the input price volatilities ($\sigma_1, \sigma_2$) and the convenience yields ($\kappa_1, \kappa_2$) resulted in an increase in the contract value, where the justification for these findings are the same as the previous
example's reasoning.

Table (4): Convergence Results, Example (II)

<table>
<thead>
<tr>
<th>n</th>
<th>$C_0^t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>55.60</td>
</tr>
<tr>
<td>8</td>
<td>55.08</td>
</tr>
<tr>
<td>12</td>
<td>54.75</td>
</tr>
<tr>
<td>16</td>
<td>54.81</td>
</tr>
<tr>
<td>20</td>
<td>54.77</td>
</tr>
</tbody>
</table>

In addition, sensitivity analysis with respect to an increasing correlation coefficient among the input prices resulted in a linear increase in the contract's value. By increasing the correlation coefficient, $\rho_{12}$, we observed that the jump probability values either increased linearly, or decreased linearly, depending on the jump directions.

Specifically, when $\rho_{12}$ increases, the probability that both input prices having jumps in the same direction also increases, while the probability that both input prices having jumps in opposite directions decreases. If the jumps that are in the same direction pertain to a simultaneous increase in both input prices, then the contract value decreases. However, the abandonment option will serve to nullify this effect. If the jumps that are in the same direction pertain to a simultaneous decrease in input prices, then the contract value increases. Since the jumps in opposite directions have decreasing probability values, they do not impact the value of the contract as strongly as the other two jump types.
do. Figure (3) provides the results of sensitivity analysis with respect to the input correlation coefficient using a partition of size $n = 12$. 
FIGURE 3: CONTRACT VALUE vs. CORRELATION COEFFICIENT (INPUT PRICES)
(5.4) Example (III)

In this final example we consider a situation in which the output requires a single commodity input. In addition we assume that both the input and the output prices follow a geometric Wiener process. Using our Chapter (4) result, the input and the output price processes are given by

\[
\frac{dP(t)}{P(t)} = (r - \kappa)dt + \sigma_1 dZ_1(t)
\]

\[
\frac{dS(t)}{S(t)} = r dt + \sigma_5 dZ_5(t)
\]

respectively, with \( dZ_1(t)dZ_5(t) = \rho_{s1} dt \). In this case the contract's duration is assumed to be one year, where the output is delivered according to a trimonthly schedule. The relevant information for this example is provided in the following table.

<table>
<thead>
<tr>
<th>Case Parameters for Example (III)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T = 1.0 year, n = 4, ( \lambda = 1.11803 )</td>
</tr>
<tr>
<td>Demand schedule: ( D_1 = 3, D_2 = 5, D_3 = 1, D_4 = 8 )</td>
</tr>
<tr>
<td>Production capacity: ( Q = 5 ) units per period</td>
</tr>
<tr>
<td>Inventory capacity: ( W = 4 ) units</td>
</tr>
<tr>
<td>Raw material requirements per unit of finished product: ( R(q) = 2 )</td>
</tr>
<tr>
<td>Production cost: ( K(q) = 5 + 1.67q + 1.25q^2 )</td>
</tr>
<tr>
<td>Switching cost: ( \kappa = 0.50(q_{i-1} - q_i)^2 )</td>
</tr>
<tr>
<td>Inventory holding cost: ( h(I) = 1.50 ) per unit</td>
</tr>
<tr>
<td>Initial raw material cost: ( (P(0), S(0)) = (10, 22) )</td>
</tr>
<tr>
<td>Convenience yield: ( \kappa = 5% ) per annum</td>
</tr>
<tr>
<td>Price volatility: ( (\sigma_1, \sigma_5) = (35%, 25%) ) per annum</td>
</tr>
<tr>
<td>Correlation Coefficient: ( \rho_{15} = 0.50 )</td>
</tr>
<tr>
<td>Interest rate: ( r = 10% ) per annum</td>
</tr>
</tbody>
</table>
For valuation purposes, the five jump lattice model as provided by equation (4.51) and (4.52) is applied. The time zero value of the contract and the optimal production and inventory policy for the above case parameters are provided in Table (5). In this example, the abandonment option did not provide a noticeable additional value. However, by decreasing the output price, for instance by $2.0, the abandonment option's value resulted in an increase of approximately 8% in the contract's value. The abandonment option value also becomes more pronounced when the correlation coefficient between the input and the output price decreases. This is intuitively appealing since lower correlation coefficient values imply higher risks.

Table (5): Optimal Production and Inventory Policy, Example (III)

\[ C_0^* = \$57.97 \]

<table>
<thead>
<tr>
<th>Demand:</th>
<th>3</th>
<th>5</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Production:</td>
<td>3</td>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inventory:</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>Time:</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

The effect of increasing the partition size is also considered and the relevant information for this parameter are presented in Table (6).
Table (6): Convergence Results, Example (III)

<table>
<thead>
<tr>
<th>n</th>
<th>$C_0^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>57.97</td>
</tr>
<tr>
<td>8</td>
<td>58.29</td>
</tr>
<tr>
<td>12</td>
<td>58.23</td>
</tr>
<tr>
<td>16</td>
<td>58.26</td>
</tr>
<tr>
<td>20</td>
<td>58.29</td>
</tr>
</tbody>
</table>

Sensitivity analysis with respect to an increasing correlation coefficient between the input and the output prices resulted in a decrease in the contract's value as shown in Figure (4).

As the correlation coefficient increases the uncertainty in the contract's value is reduced due to the fact that increases in the input price are offset by the corresponding increases in the input price. Accordingly, the value of an abandonment option diminishes with increasing correlation coefficient values. On the other hand, when the correlation coefficient values are small, the uncertainty in the contract's value increases and the abandonment option becomes more valuable since it serves to offset the impact of higher input prices.

To provide additional insight for this example, sensitivity analysis with respect to the convenience yield ($\kappa$), the output price volatility ($\sigma_v$), and the input price volatility ($\sigma_1$) were also considered.

Increasing the convenience yield causes a systematic decrease in the drift parameter of the risk neutralized input process, which
in turn results in higher contract values as shown in Figure (5).

The impact of increasing output price volatility on the contract's value is twofold. It increases the upside potential for output prices, which in turn results in higher contract values. It also results in lower output prices in the downside, which in turn causes the contract value to decrease. However, the existence of an abandonment option serves to block the downside risk. As such, increases in output price volatility result in increased contract values as illustrated in Figure (6).

The opposite of this statement holds for increases in the input price volatility, $\sigma_i$. Specifically, when $\sigma_i$ increases, the input prices increase in the upside, thus resulting in lower contract values. However, this impact is offset due to the abandonment option. On the other hand, increases in $\sigma_i$ also result in lower input prices in the downside which in turn result in higher contract values as shown by Figure (7).
FIGURE 4: CONTRACT VALUE vs. CORRELATION COEFFICIENT
(INPUT / OUTPUT PRICES)
FIGURE 5: CONTRACT VALUE vs. CONVENIENCE YIELD
FIGURE 6: CONTRACT VALUE vs. OUTPUT PRICE VOLATILITY
FIGURE 7: CONTRACT VALUE vs. INPUT PRICE VOLATILITY
(5.5) Conclusion

By providing stylized examples, we have shown in this chapter how lattice based valuation techniques can be employed to value assets whose cash flows depend on highly variable input and/or output prices.

In addition to providing an interesting set of empirical results for further research, the valuation framework presented here should serve as a useful instrument for analysis of capital budgeting decisions in situations where the distribution of future cash flows is not given exogenously but must be determined by future management decisions.
CHAPTER 6
CONCLUSIONS AND FUTURE RESEARCH

6.1 Summary and Conclusions

This dissertation has been organized in two distinct parts. The first part is concerned with development of alternative lattice based option pricing algorithms that account for multiple sources of uncertainty.

The second part is concerned with development of arbitrage based models for valuing real claims by utilizing the concepts of contingent claims analysis and stochastic optimal control theory. In the first part, we initially considered an option pricing algorithm with a single source of uncertainty. By assuming that this uncertainty is explained by a Geometric Wiener process, the resulting logarithmic return process was approximated by a discrete trinomial lattice. This trinomial lattice was constructed in such a way that in each approximating interval the price variable could either move up in value (an up jump), maintain its current value (horizontal jump), or move down in value (a down jump). The resulting trinomial algorithm was then compared to the well known binomial lattice algorithm of Cox, Ross, and Rubinstein (1979). Case examples indicate that convergence to the true option prices (Black and Scholes prices) were significantly more rapid with the trinomial model. In addition, by adjusting the computational effort for the two models, so that for a prespecified partition of
the maturity interval the total number of multiplications and
additions would be equivalent, we showed that for at-the-money
options the error generated by the trinomial algorithm was about
half the size of the binomial error. A most interesting feature of
our trinomial algorithm is that it contains the binomial option
pricing model as a special case. Specifically, when the
probability for horizontal jumps is set to zero, the trinomial
model is reduced to the binomial model. By accounting for an
additional source of uncertainty, our trinomial algorithm was
generalized to a five jump approximating process.

The motivation for developing such a model stems from the fact
that a wide variety of contingent claims of interest in financial
economics have a payoff function that includes the payoffs of a put
or a call option on the minimum or maximum of the prices of two
risky assets. Examples of such claims include, secured debt,
foreign currency bonds, optional bonds and certain compensation
contracts.

In our five jump option pricing algorithm, the unique probability
terms were obtained by equating the first two moments, as well as
the covariance terms of the true and the approximating processes.
The five jump process was then compared to the four jump process of
Boyle, Evnine, and Gibbs (1989). The examples considered indicate
that convergence to the true option value (Stultz prices) were
significantly more rapid when utilizing the five jump algorithm.
Furthermore, by equating the computational effort of the two
algorithms, we showed that for at-the-money options the errors generated by a five jump model were approximately half the size of the four jump model errors. In addition, we also showed that the four jump model of BEG (1989) is indeed a special case of our five jump option pricing model.

Our five jump model with two sources of uncertainty was then generalized to a model with \( k \), \( k \geq 2 \), underlying sources of variability. We showed that for multinomial lattice models with \( k \) state variables and the opportunity for horizontal jumps there are \( 2^k + 1 \) probability terms. For this generalization, closed form probability expressions were also provided. To exemplify our findings with \( k \) state variables, we next considered a model with three underlying sources of uncertainty. An issue of concern is that what order multinomial process should be used to approximate the joint normally distributed logarithmic returns. Accordingly, the first part of this dissertation has demonstrated that algorithms that account for the opportunity of horizontal jumps offer significant computational advantages.

In the second part of this thesis, our focus was directed toward valuation of real assets using a multinomial lattice framework. While this approach has been used to value financial options, to date no research has been conducted to apply this approach to valuation issues in risky ventures that are characterized by operating flexibility and strategic options.

Toward this goal, in chapter (3) we defined a risky venture
claim process and provided a formulation of the general valuation problem using a binomial lattice framework by assuming that the price variable was lognormally distributed. This valuation model was later generalized by taking into account the fact that for risky ventures (real assets), managerial actions may influence payouts. The resulting stochastic control problem was then solved using a stochastic dynamic program yielding a risky venture valuation model. Concurrently, we showed how an equivalent dynamic trading strategy can be explicitly constructed that replicates the binomial risky venture claim.

In addition, we noted that providing a direct link between the valuation process and the arbitrage strategy is not essential so long as the market is complete and trading strategies allow no arbitrage opportunities. Accordingly, our binomial risky venture valuation model was generalized to a trinomial model. Moreover, by measuring convenience yields from observed futures prices, we illustrated how the probability terms for either the trinomial or the binomial risky venture model could easily be adjusted to allow an arbitrage free valuation.

To account for multiple sources of uncertainty, in chapter (4) we considered a valuation problem where both the input and the output prices followed a joint lognormal distribution. Specifically, our modelling efforts were geared toward valuing a supply contract in which the manufacturer (supplier) was required to deliver fixed quantities of a finished good according to a
deterministic demand schedule. Accordingly, for valuation purposes we were required to establish an optimal inventory building policy such that the demand was met, while production and inventory capacity constraints remained intact. In valuing this investment decision, we assumed futures markets in the input commodities, while maintaining that the output was neither a traded security nor there existed a futures market in the output. In order to justify an arbitrage based approach for valuation, when in fact the output is not a traded security we assumed the existence of a traded portfolio whose value closely (correlated) reflected that of the output price.

These assumptions, together with the construction of a requisite replicating self-financing portfolio provided two important advantages with regard to modelling. First, the need for estimating the drift components in both the input and output price processes were eliminated. Second, for valuation purposes we did not require an exogenously provided discount rate. In addition, to account for the dependence of the output rate on the stochastic nature of the input and output prices, as well as the deterministic demand rate the valuation process was modelled as a problem in stochastic optimal control.

We showed that our resulting arbitrage valuation approach lends itself to a second degree partial differential equation that governs the contract value during every period of the valuation process. The time zero value of this contract, along with optimal
decisions to be followed were then obtained through a backward recursion by solving the p.d.e. at every stage of the process. In general, our resulting p.d.e. does not yield an analytic solution and has to be solved numerically. As such, a numerical approximation approach for solving this arbitrage based valuation model was provided. Specifically, by superimposing our control problem on a multinomial lattice representing the stochastic evolution of the input/output prices, we developed a stochastic dynamic programming model which can be solved efficiently. In addition, we considered various model derivatives as special cases of our multinomial lattice valuation model, where numerical examples and comparative statics for these special case models were presented in chapter (5).

(6.2) Future Research

In developing our multinomial option pricing models we assumed that the underlying source(s) of variability could be explained by a Geometric Wiener process.

An interesting extension of our lattice option pricing approach would be to consider other realistic alternative stochastic processes for the underlying uncertainty. For instance, multinomial approximations to the Ornstein-Uhlenbeck (mean reverting) and the Constant Elasticity Variance (CEV) processes for asset prices would be of interest. These multinomial lattices would also reflect an interesting extension for the risky venture
valuation models.

Another interesting consideration is that of stochastic interest rates. This is especially important for the risky venture valuation models since, in general, real assets are long lived and in fact some are perpetual in nature. Although several continuous time models for interest rate uncertainties have been considered, lattice approximating models with opportunity for horizontal jumps are not yet available.

Specific extensions of our risky venture valuation models as provided in chapters (3) and (4) include incorporating and obtaining the value associated with:

(1) option to shutdown temporarily and consequently reopen.

(11) option to defer commitment to the contract until some future time.

Further applications of these models to the valuation of flexibility in other related areas such as real estate, research and development, and energy problems would also be of interest.

In addition, in these models we assumed that our production facilities were not subject to deterioration. In reality, productive machinery deteriorate in a random fashion and maintenance policies must be established to allow for a smooth production. Accordingly, our valuation models could be generalized to account for a maintenance policy which results in maximization of the stream of cash flows generated by the production facility.
Another interesting addition to the risky venture valuation models would be to solve the resulting differential equations by finite difference methods. As such, this effort will serve to provide a comparative basis for the results that were presented in Chapter (5).
REFERENCES


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