STUDENT PERSONAL CONCEPT DEFINITION OF LIMITS AND ITS IMPACT ON FURTHER LEARNING OF MATHEMATICS

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ABSTRACT

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Calculus is an introductory course for most students in Science, Technology, Mathematics and Education (STEM). Limits are an essential part of the learning of calculus. It has been previously documented that students tend to struggle when first learning the topic of limits. This paper is an investigation into the personal concept definition of limits and how consistent a student’s personal concept definition to the formal definition of the limit after they have completed courses in introductory calculus. The study took place during a 15-week semester, with 28 students, who were taking either Calculus III or Ordinary Differential equations. These 14 male and 14 female participants attended the same Midwestern Public University. They were given a short, in-class, survey where they could demonstrate their concept definition and the operability of their concept definition as it pertains to limits and limit-based problems. Three categories were created to signify the students level of operability in solving limit-based problems based on their responses to the survey. The categories were Low, Mid, and High Scoring, which demonstrated an inoperable, partially operable, or fully operable concept definition, respectively. The majority of students fell in to Low and Mid Scoring categories, indicating their lack of operability in personal concept definition as it pertains to limits. This study suggests that students in mathematics should be encouraged to develop their conceptual understanding and move past procedural knowledge as a way of mastering a mathematical topic, such as limits.
Dedicated to my wife, whom I could not have done this without.
ACKNOWLEDGMENTS

I would like to acknowledge . . .

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CHAPTER 1 INTRODUCTION

Calculus is viewed as an introductory course for students majoring in engineering, economics, statistics, and other natural sciences to prepare them for the level of mathematics necessary to deeply study these subjects (Zollman, 2014). As an undergraduate student, the researcher had personal experience with having a personal definition of the limit that differed from the formal definition, which in turn affected the ability of the researcher to fully grasp the formal definition when encountered in a Real Analysis course. As a student of advanced mathematics, not having a connection between how a concept is formally defined, how to properly apply that concept when solving problems, and not seeing the connections between various topics, such as limits and continuity, lead to difficult transition to formal mathematics and formal mathematical thinking. Understanding the concept of limits is vital for advanced mathematical thinking, and limits are the foundation of calculus-based mathematics. Hence, we see the need to investigate the level of understanding that students have pertaining to the concept of limits in mathematics courses where having a robust understanding of calculus is vital to their success in that particular course.

The formal definition of the limit, how a mathematician would define a limit, and a students personal definition of the limit, how the student defines a limit, often differ to varying degrees. It has been previously shown (Przenioslo, 2004) that students in calculus who have a formal and operable concept definition of limits are better suited to solve limit-based problems in math. Similarly, according to Roh (2010), without having an operable and formal concept definition of limits, students struggle when studying limit-based mathematics. There is a lack of research, however, on students concept definition of limits after they have completed introductory calculus. After first learning the concept of limits in an introductory calculus course, students often do not see a formal definition of limit again until Real Analysis. In math courses, which use calculus-based mathematics during this time, there is not a heavy emphasis on the formal (epsilon-delta) definition of the limit. Without continuing to allow students to make the connection among their personal concept definition, the formal concept definition, and how their concept definition allows them to
solve limit-based problems, advanced students in math may not possess or see the need for a deep conceptual understanding of other mathematical topics. There is a lack of research, however, on the concept definition of students that have completed courses in introductory calculus, but are continuing to study mathematics in courses that use calculus.

Additionally, the formal definition of continuity has ties with the formal definition of limit in concept as well as a common use of symbols (epsilon-delta). Research has shown undergraduate students troubles in combining their informal and formal understandings about the concept of continuity (Bezuidenhout, 2001; Tall & Vinner, 1981; Vinner, 1992; Williams, 1991). Findings also show that students have trouble when combining the idea of continuity with specific functions (Vinner, 1992; Wilson 1994). So, there is also a need to investigate how students relate the concept of limits and continuity in their mathematical learning.

Most research on students conceptual understanding of limits, prior to the learning of Real Analysis, is done in introductory calculus courses, either the first or second semester (Bezuidenhout, 2001; Oehrtman, 2009; Szydlik, 2000; Williams, 1991). This research does not give insight in to how students concept definition of limits may warp or fade after first learning the topic. Also, other research on students understanding of limits, aside from research performed on introductory calculus students, is often on those achieving degrees in pure mathematics or mathematics education (Cory & Garofalo, 2011; Moore, 1994; Weber, 2005). These students have had more of an opportunity to study proof-based mathematics, thus seeing the need for a deeper conceptual understanding than their non-math majoring peers. Gaining insight into the conceptual understanding of non-math majoring students, after completing introductory calculus, should still be of interest to math education researchers for a few reasons. One reason being that these students will likely never take a course in Real Analysis, hence they are less likely to encounter the formal definition of limit again in their mathematical course work. Additionally, Calculus I or II is not the terminal math course for many students majoring in Science, Technology, Education and Mathematics (STEM). Since these students may continue to use calculus-based mathematics in further courses, and limits are the bedrock of calculus, there is an interest in gaining insight in to the conceptual
understanding of limits amongst them. So, the purpose of this study, then, is to gain insight into the concept definition of students regarding limits from demographics of students not previously or fully investigated, having completed introductory calculus, but still continuing their mathematics education.
CHAPTER 2 LITERATURE REVIEW

It is well documented that calculus students have difficulty in understanding the concept of limits (Tall & Vinner, 1981; Williams, 1991), and what understanding students do have is often incomplete pieces of the idea (Cornu, 1981). Students difficulties with problems involving limits also persist in mathematics courses beyond Calculus I and II (Moore, 1994; Roh, 2010; Weber, 2005). Students also use their concept image, whether correct or incorrect, to think about continuity and continuous functions (Tall & Vinner, 1981). Undergraduate students, however, have an inadequate understanding of the connections between limits and continuity (Bezuidenhout, 2001; Williams, 1991). Thus, there is a need to discover to what degree students troubles with limits impacts their conceptual understanding in mathematics courses later in their academic career. Additionally, it should be of interest to mathematics educators to what degree students concept definition of limits coincides with the formal definition, even after they have completed their coursework in introductory calculus. A theoretical framework is necessary to fully examine and categorize the conceptual understanding of limits in students enrolled in calculus-based mathematics courses. This study utilizes Tall and Vinners (1981) idea of concept image and concept definition. Tall and Vinners scheme provides a practical system to examine the use of definitions in order to solve advanced problems in mathematics.

Concept image, as described by Tall and Vinner (1981) is, the total cognitive structure associated with a concept, which includes all mental pictures, properties, and processes (p. 152). Concept image then, with regards to limits, alludes to all of the examples, theorems, solution methods, problem-solving procedures, and metaphors associated with limits. Following the definition of concept image, Tall and Vinner (1981) define concept definition as, a form of words used to specify that concept (p. 152). In the case of limits, the personal concept definition of a student refers to the body of words and symbols that the student uses in their description of limits or what a limit is. Meanwhile the formal concept definition of limits comes from Cauchys (1821) original definition, translated from words to symbols in the (epsilon delta) definition.
As described by Bills and Tall (1998), a mathematical definition or theorem is said to be *formally operable* if, (an) individual is able to use it in creating or (meaningfully) reproducing a formal argument (p. 104). For students studying advanced mathematics, in order to have a formal definition entrenched in their concept image one must have the means to do more than simply recite a definition. Students need to be able to apply the definition to construct a formal and well-thought mathematical argument.

In order to fully understand a topic in advanced mathematics, a student must have a formal and operable definition of the concept or idea (Edwards & Ward, 2008). Edwards and Ward (2004) also note that undergraduate students do not use definitions in the way that a mathematician may use them, even if the student can correctly state the definition and explain it. Edwards and Ward (2008) point out that, some undergraduates with advanced mathematical thinking and decent, sometimes excellent, grades do not completely understand the nature and role of mathematical definitions (p. 227). Hence there is a need to examine how consistent a students personal concept definition is to the actual concept definition in mathematics. It can be hypothesized that the formal concept definition of limit is a portion of the problem-solving process for students in advanced mathematics, however, there is reason to believe that this definition is not fully functional or developed for many students. These students, having completed introductory calculus, but still enrolled in advanced mathematics courses, are still an important body of students to research. If a student in this demographic has inconsistencies in their concept definition of limits, it could affect their ability to continue to deeply study mathematics.

This study seeks insight in to how students view the concept of limits in mathematics courses after calculus I, and II, and how their perception affects their later performance in mathematics courses. In order to gain this insight, the following research questions are posed:

1) To what degree are students personal concept definition of limits consistent with the formal concept definition?
2) To what degree are students personal concept definition of limits operable in solving limit problems?
CHAPTER 3 METHODOLOGY

In this paper, we present an investigation into students conceptual understanding of limits, a topic first learned in Calculus I, and how their understanding may affect further performance in mathematics. The study takes place in a 15-week semester at a large, Midwestern, public state university. All student participants in the study were enrolled in Calculus III or Ordinary Differential Equations. Of the 28 participants in the study, 14 were female and 14 were male.

3.1 Data Collection Procedure

There are two types of data collected in this study. First, is a survey with various questions involving limits (Appendix A). The purpose of the survey was to allow students to communicate their personal concept definition in written form as well as demonstrate the operability of their concept definition with respect to limit-based problems in mathematics. This survey was given out at the beginning of a typical class period, with the first fifteen minutes reserved for its voluntary completion. Second, interview data was also collected by conducting one-on-one interviews with three students. The interview participants elaborated on their answers to the survey via a basic interview protocol (Appendix B). The interviews were conducted so that the students could verbally expand on their written work, and further elaborate on their personal concept definition of limits. Of the twenty-eight participants, seven volunteered to be interviewed in regards to their responses on the in-class survey. Of the seven volunteers, the three interviewees were selected based on how well each individual represented a particular level of operability of their personal concept definition, and belonging to the Low, Mid, and High Scoring students.

3.2 Survey Design

The survey consisted of two short-answer questions, one multiple-part question in which they were directed to respond in an always, sometimes, or never true format, and three graphical questions all in involving limits (see Appendix A). The questions were chosen so that the students could demonstrate their knowledge of limits in calculations, graphical observation, and display their writ-
ten personal concept definition. Two questions were open-ended questions, placed at the beginning and end of the survey respectively, in order to not influence the answers to either question, as well as to best display the participants personal concept definition.

3.3 Interview Design

The three interview participants were given the same interview protocol (Appendix B). All face-to-face interviews were audio recorded. During the interviews, each student tended to maintain and display confidence in their written responses to the survey, rather than providing any further mathematical insight in to their responses. As such, only the subsections of the interviews where participants added to their written personal concept definition were transcribed and analyzed to better understand their personal concept definition. These portions of the interviews were selected as an addition to their concept definition based on if the student incorrectly explained, or elaborated on their reasoning behind incorrect responses to survey questions, as well as elaborated on their perceived need to have a personal concept definition consistent with the formal definition of limits. These portions were analyzed to see whether or not the students felt it necessary to have a concept definition consistent with the formal definition, and not simply an operable concept definition. All students were confident in their responses and chose not to make any changes to any portion of the written survey.

3.4 Survey Question Choice and Analysis Techniques

The first survey question (Q1) asked students to describe the meaning of a limit with the intention being that students would be less likely to be influenced by any of the language on the rest of the survey. Thereby, examining students personal concept definitions of limit without being influenced by the formal definition of limit, which appears in part (v) of Q2. Q1 also intentionally had informal phrasing, like mathematical meaning in hopes of extracting different language and responses than Q6, where the mathematical notation elicited more formal responses when pilot testing the survey. Q1 also attempted to illicit responses using metaphors, specific examples, or graphs of functions pertaining to the students concept definition. Responses to Q1 were analyzed
for mathematical coherency and categorized for whether they chose to use a dynamic metaphor or some other description of the limit.

The final survey question (Q6) was posed so that students could attempt to formalize their personal concept definition of the limit of a function. The question also allows the researcher to measure to what degree their personal concept definition coincides with the formal definition of a limit, often not encountered until a course in Real Analysis. This question had more precise mathematical phrasing than Q1, where there was expected to be a more formal response. This question was not analyzed thematically, as the responses by student participant were nearly identical; a translation of the mathematical phrase seen in Q6 in to words.

The second survey question (Q2) was a multiple-choice question (parts i-v) modeled after a study (Bezuidenhout, 2001) in first year calculus students conceptualization of limits and continuity. The question was reformatted to have students respond to each individual answer in the form of always, sometimes, or never true statements. Q2 was used to investigate the relationship between students personal concept definition of limits and other mathematical concepts, such as continuity. The choice in the variety of questions was so that the students could elaborate on their personal concept definition of limits as it pertains to continuity and their collective use of precise mathematical phrases or language, which students may not typically use. It was also of interest to see whether or not the students would answer in a logically consistent manner from the multiple-choice question, Q2, to the graphical questions Q3-Q5. Additionally, Q2 was scored out of five possible points, (+1) for a correct response and (0) for a blank or incorrect response. This point system was used to quantify students personal concept definition as being operable or inoperable concept definition with regards to limits.

Survey questions 3 through 5 (Q3, Q4, Q5) were questions involving graphs, where students needed to say whether a limit exists given the graphical information. They were then asked to elaborate if possible. The reasoning for separate graphs was to have functions with different types of discontinuities. Because the formal definition of limit is an abstraction, graphical questions were posed to aide in determining which portion of the formal definition is not consistent with the stu-
dents personal concept definition. These questions were also given points for correct or incorrect responses. This scoring system aided in the categorization of students operability of their personal concept definition when solving limit-based problems by graphing.

Q2, Q3, Q4, and Q5 were scored based on accuracy, in order to classify students into the following three categories: Low-Level Scoring, Mid-Level Scoring, and High-Level Scoring students. Q2 was worth a total of five points and Q3, Q4, and Q5 were worth one point each. There was a total of eight possible points between Q2-Q5. The classification in to these groups was as follows: (0-2 possible points) Low Level Scoring, (3-6) Mid-Level Scoring, (7-8) High-Level Scoring. These categories also apply to students operability of their personal concept definition; Low-Level Scoring students are said to have an inoperable concept definition, Mid-Level Scoring students are said to have a partially operable concept definition, and High-Level Scoring students are said to have a fully operable concept definition when solving limit-based problems.

In order to sort the results of the survey and to select students for further participation, the surveys were analyzed for inconsistencies between their written personal concept definition and later answers to graphical questions. Additionally, interview participants were selected based on how well their responses represented the responses of students in all three previously described categories: Low, Mid, and High Scoring students. The three interviewees written responses were analyzed in order to be able to make more broad conclusions about the entire body of participants, without the need to interview each student individually.
CHAPTER 4  RESULTS

There were several patterns worth mentioning in regards to students personal concept definition of limits (RQ1) and their ability to solve limit-based problems in mathematics (RQ2). The following is an investigation of both the quantitative and qualitative data from the study. This examination includes an analysis of responses to Q1, a distribution of students scores for Q2 through Q5, an observational analysis on written responses, an examination of the interviews conducted with students regarding their written responses, and other general comments.

4.1 Quantitative Results

To better understand how consistent students were when applying their concept definition on the various parts of the survey, we analyzed and categorized responses to Q1 and we analyzed the distribution of correct and incorrect responses to Q2 through Q5.

4.1.1 Results of Q1: Students Description of the Limit

Q1 was first analyzed for if the students description of the limit was mathematically coherent. Furthermore, to characterize the nature of the correct and incorrect responses, students descriptions fell into two categories: (i) using a dynamic process by incorporating words such as approach or phrases such as getting closer and closer; and (ii) other mathematical concepts - such as graphs or examples of specific functions. The results are shown in Table 4.1.

Table 4.1: Students’ Answers to Question 1

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<th>Answer</th>
<th>Correct</th>
<th>Incorrect</th>
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<tr>
<td>Used a dynamic idea</td>
<td>21</td>
<td>2</td>
</tr>
<tr>
<td>Other</td>
<td>2</td>
<td>3</td>
</tr>
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4.1.2 Results of Q2: Consistency of the Students Concept Definition and the Formal Definition of Limits

Question 2 was scored as correct and incorrect, along with analyzing the frequency of specific responses to the always, sometimes, or never portion. Figure 1 is a breakdown of correct and incorrect responses to individual components of Question 2. It is observed here that the majority of students (23/28) incorrectly or Did Not Answer (DNA) Part (ii) of Question 2. This result suggests that there is an inconsistency between students personal concept definition and the formal concept definition of limit. Since most students incorrectly answered the portions of Q2, where the given choice was expressed in mathematical symbols and notation, we see the discrepancy between students concept definition and the formal concept definition. A similar comparison was made with Q2 Part (v), where thirteen (13/28) students responded incorrectly or did not respond. This add further evidence to our observation, that students have difficulty connecting the formal definition with their personal concept definition. Figure 2 shows the breakdown of the total points for each individual student. With the following breakdown of the total points for each individual student.
Of all the participants, the mean score for Question 2 was 2.14 out of 5 possible points with the median score being 2 out of 5. These data emphasize the overreliance of continuous functions as metaphors for limits of functions. These results also show the inoperability of many students concept definition in regards to Q2, with either average being below a passing on a traditional A-F grading scale. Along with being scored as correct and incorrect for each individual response, Q2 was also analyzed for the frequency of particular responses. Table 4.2 shows the breakdown of the number of students that answered a certain way for Q2. The fifth column of Table 4.2, Incorrect DNA denotes the number of students who either did not answer or answered in the form of a question mark. With parts (i), (iii), and (iv) of Q2, we see the reliance upon the use of continuous functions in order for students to describe a limit, with the majority of the incorrect responders believing that a limit only exists provided that a function is continuous or defined at that choice.
of x. Particularly, with student responses to Q2 part (iii) with (12/28) explicitly stating that the function must be continuous in order for the limit to exist. The researcher notes that five students did not respond to part (ii) of Q2 and six students did not respond to part (v) of Q2. All five (5/5) students who did not respond to Part (ii) also did not respond to part (v). Part (ii) was the choice that displayed the definition of a limit equaling infinity. Part (v) was the choice that displayed the formal definition of limit with epsilon and delta replaced with s and r. Four of the five (4/5) students who did not respond parts (ii) and (v) of Q2 were categorized as Low Scoring based on all their survey responses; their personal concept definition was inoperable in solving limit-based problems. The remaining student (1/5) was a Mid-Level Scoring student. The results for parts (ii) and (v) of Q2 suggest that most students are not fully exposed to the formal definition of limit in this way in an introductory course in calculus.

4.1.3 Results of Q3-Q5: Graphical Questions

The results of the graphical questions 3-5 are displayed in Figure 3 below. There were two common themes among written explanations of incorrect answers to Q3. Most participants incorrectly stated the limit did not exist, or that the limit was equal to the defined function value. Interestingly, 42% (6/14) of incorrect responders to Q3 were able to correctly answer Q4, both of which were discontinuous graphs. It is also noteworthy that of the fourteen incorrect responders to Q3, 57% (8/14) had the incorrect but logically consistent response of always true to Q2 - part
(i), the function being defined when \( x = 4 \). A similar comparison was made between incorrect responses to Q3 and responses to Q2 part (iii). Here the researcher notes that of the fourteen incorrect responders to Q3, 50\% (7/14) had the incorrect but logically consistent responses of always true to Q2 part (iii), which stated that the function must always be continuous at \( x = 4 \). Therefore, these results could suggest that students have an overreliance on the use of continuous functions when describing limits.

4.1.4 Operability of Students Personal Concept Definition

Students were also categorized in to three, predetermined categories depending on their scores on the survey Questions 2-5. The results were as follows: (0-2 points) Low Level Scoring nine students; (3-6 points) Mid-Level Scoring - thirteen students; (7-8 points) High-Level Scoring five students. It is worth noting that the majority of students (22/28) students were placed in to the low or mid-level of operability and achievement. Similarly, one student (1/28) scored a score of zero, a completely inoperable concept definition with respect to limits, and three students (3/28) scored 8 possible points, a fully operable concept definition. There were four students (4/28) who correctly responded to Q2 part (ii) and Q2 part (v), with three of these students in the High-Level Scoring category, with the remaining student in the Mid-Level Scoring Category. From this we see that knowing the formal definitions of limit was a good predictor (3/4) of having a highly operable concept definition. One student with a fully operable concept definition came from the Calculus III course while the remaining two students were from Ordinary Differential Equations.

4.2 Qualitative Results

With the exception of students who demonstrated a concept definition which is fully operable, the remaining participants gave logically inconsistent answers. For example, one participant gave the logically incorrect response to Question 3, no (the limit does not exist) since its not continuous. Their response to the next question, on the other hand, which was also a discontinuous graph, was, yes [the limit does exist] but it must be evaluated at both sides. This example is representative of the type of inconsistent answers that a large percentage of the Low and Mid-Level Scoring students
provided. This type of inconsistency supports the notion that many of the participants may be confusing limits with continuity.

Additionally, many students were able to mention various characteristics of limits, such as the left-hand and right-hand limits being equal in order for the limit to exist, yet many were unable to correctly apply these facts to solving problems involving limits. When students gave graphical examples to describe limits, they were either continuous functions or discontinuous functions with a single point of discontinuity (jump, removable, or an asymptote). One student said that, A limit is the area under a curve.

4.3 Interview Data and Analysis

The various responses above are investigated further by analyzing the specific answers given by individual participants. Three participants were interviewed to further investigate their written responses and allow them to expand on their work. The three participants, Student A, Student B, and Student C, represent the habits and tendencies in students written responses to the limit-based survey questions. Students A, B, and C were each representative of the general student body of participants, identified as Low-Level, Mid-Level, and High-Level Scoring students respectively.

4.3.1 Interview Results for Student A

Student A Low-Level Scoring Student. Student As written response to the survey items demonstrated his lack of understanding of the topic of limits. When describing the limit, his representation was graphical and relied on continuous functions, though he did mention that, both sides of the graph need to converge to the same point in order for the limit to exist. Student As personal concept definition is largely inoperable, getting every response to Question 2 incorrect. The responses to Question 2 show an over-reliance on and use of continuous functions when describing limits. Student As difference in reasoning for Question 3 and 4 make apparent the inconsistency between this students personal concept definition and the formal definition. That is, in Question 3, the student incorrectly stated that the limit is the defined function value, whereas in Question 4 the student was able to correctly state that the limit was equal to one, the value that both sides
of the function approach. When asked how the student was able to succeed in calculus without having a formal concept of the limit, he responded: *I think that knowing definitions in math is only important for people who study engineering or some other advanced science. I only need to be able to solve problems, where as they need to know the theory behind it.* Student A’s response here is especially telling of the attitude that students have towards math, being largely procedural in nature, with conceptual understanding reserved for more advanced students.

4.3.2 Interview Results for Student B

Student B Mid-Level Scoring Student. Student B’s response to the written component of the survey demonstrated a partially operable personal concept definition with traces or fragments of the formal definition, demonstrated by the following. His response to Q1 of the written survey was, *For any (epsilon) you demand of me, I can give you a (delta) to which lim f(x) (delta) is less than (epsilon).* It is clear that Student B views limits as an active process, with a particular choice in (epsilon) forcing the value of (delta). His attempt at defining the limit is scattered portions of the formal definition. His responses to Q2 demonstrate the partial operability of his concept definition, correctly getting three out of five responses correct. Though it is worth mentioning that Student B made the incorrect claim in Q2 that f is continuous when x = 4 is an always true statement. Student B was able to correctly solve all graphing problems surrounding limits. Thus, even with a personal concept definition largely inconsistent with the formal concept definition, Student B’s concept definition was operable in solving the graphical portion of the survey. Student B was a good representation of the Mid-Scoring group for several reasons, including that his personal concept definition did contain bits and pieces of the formal definition, a common theme among this range of student. Though this personal concept definition was skewed, it did not fully hinder the Mid-Scoring students ability to solve limit-based problems.

4.3.3 Interview Results for Student C

Student C High-Level Scoring Student. Student C had a highly operable concept definition and one most consistent with the formal concept definition. The following is how Student C described
the mathematical meaning of the limit: *What a function appears to converge to. At a specific point, the function does not necessarily have to be defined there. If the left-hand limit = right hand limit, then the \( LHL = RHL = Limit \)* His personal concept definition, though missing some logical quantifiers and containing a few terms that were not well defined, was fully operable in solving limit-based problems. When asked to elaborate on their description of the limit, Student C was able to properly sketch (epsilon) and (delta) strips. Student C also chose to place appears in quotations, emphasizing that the limit is not necessarily the defined function value. Student C also demonstrated on Question 2 of the written survey that their personal concept definition is indeed operable, by accurately answering all portions of the question. During the interview Student C was even able to explain why portion (ii) of Question 2 was never true, demonstrating a clear connection between symbols, terminology and important mathematical ideas. Student C had a robust variety of examples to justify their ideas on limits, not limited to continuous functions, numerical evidence, or graphs. Student C's personal concept definition of a limit, the role in which it played in his concept image, and the operability of his concept definition, was an exception to the rule of participants.
CHAPTER 5 DISCUSSION AND IMPLICATIONS

This study was designed to gain insight into students personal concept definition of limits, and how that concept definition impacts their performance in limit-based problems and mathematics as a whole. As demonstrated through the study, after first learning limits and given time to disengage from the material, many students do not have a formal concept definition of limit consistent with the formal definition. Additionally, through the study, it was found that the majority of the participants personal concept definition of limits was inoperable in solving problems involving limits, along with not agreeing with the formal definition of the limit. This discrepancy between students personal concept definition and the formal concept definition of limit is likely the source of their inoperable problem-solving skills when it comes to limits. This discovery leads one to question whether or not these participants are performing mathematical problem solving or imploring problem-solving techniques in their individual fields of study, given their prevalent inconsistent application of the limit definition on this focused survey.

With the exception of a few students (Student C included), most participants seemingly did not have both a formal and operable concept definition of limit entrenched in their personal concept definition of limit. The observation made here is consistent with other research on students understanding of limits (Williams, 1991; Roh, 2010). From this we can begin to draw the conclusion that students have trouble in understanding the concept of limit and this lack of understanding hinders their ability to solve limit-based problems in mathematics. It was also discovered that, when encouraged to describe the concept of a limit in their own words or to solve problems involving limits, most participants relied heavily on the use of continuous functions, with many students stating that a function needs to be continuous in order for the limit to exist. To the student, continuous functions appear to be more accessible both in procedure and concept than discontinuous functions when solving problems involving limits. The reliance upon continuous functions could possibly be due to the fact that students are less likely to encounter discontinuous functions when studying various scientific disciplines than they are in advanced classes studying theoretical mathematics.
When they come across such discontinuous functions, the types of discontinuities are often limited to jump or removable discontinuity (e.g., a finite number of discontinuities) or a vertical asymptote.

It was also observed in the study that students rely heavily on the use of metaphors when describing the concept of limits. One student said that the limit is, where the graph intends to go. Many of these metaphors have been observed in prior studies involving students, such as by Oehrtman (2009). No definitive conclusion can be made on whether or not these metaphors are beneficial to students in solving limit-based problems.

We do see, however, that Student C possesses a personal concept definition which is operable in solving problems involving limits, as well as bits and pieces of the formal definition of limit entrenched in their personal concept definition. Student Cs approach to the survey questions may offer insight into how students can approach problems involving limits in their perspective fields of study, as well as student understanding of limits in general.

Based on the results of the study the researcher has a few suggestions for instructors of introductory calculus courses while teaching the topic of limits. One suggestion would be to incorporate activities such as Rohs (2010) epsilon-strip activity to aide in pictorially describing the limit. Student C was able properly draw epsilon-delta strips, hence we can conclude that having this ability as a part of his concept image was beneficial in his concept definitions being operable when solving problems. Additionally, the researcher suggests spending some portion of the time spent on learning limits to work with the formal definition of limit as described by Cauchy, even completing (epsilon-delta) proofs of limits of simple functions, such as first-degree polynomials. Additionally, due to students disconnection between limits and continuity, the researcher also suggests formally learning the definition of continuity, perhaps even proving some simple first-degree polynomials are continuous using the formal definition. This would allow the student to see the symbolic connection between limits and continuity, as well as an important application of limits.
5.1 Generalizability of the Study and Future Research

The study performed was an observation and investigation into students responses to limit-based questions in mathematics, as such there are three primary limitations to the study and the ensuing results. First, the results were established from a small number of students (twenty-eight), from two different mathematics courses (Calculus III and Ordinary Differential Equations), at the same Midwestern University. As such, the results may not be reflective of all students who study mathematics throughout the country. One possible solution to this is to perform a study spanning several semesters, across a range of universities in the United States. Another proposed solution to improve the study is to involve more participants, from all majors of Science, Technology, Education and Mathematics (STEM) - beginning when the students are in Calculus I, when students are first learning the concept of limits, and then tracking their performance until their completion of an upper-level math class, such as Ordinary Differential Equations or Linear Algebra. Second, the questionnaire included only a small variety of questions, with only five questions total, hence a somewhat limited lens in to their understanding of the topic of limits. One proposed improvement for the questionnaire would be for it to have a broader set of questions with varying levels of difficulty to create a bigger picture in to students understanding of limits. Additionally, some of the questions had to do with the concept of continuity, which may have influenced students thoughts on limits. Investigating the concept definition and concept image as it pertains to continuity could be an area of future research. Third, due to the small sample of students and nature of the study, statistical analysis of the results would be inappropriate. A portion of the results was qualitative in nature stemming from the face-to-face interviews; analysis of the interviews was subjective. Moreover, though gender was not a factor in the decision of whom to have face-to-face interviews, it is worth mentioning that Student A, Student B and Student C were all males, possibly skewing the results and their generalizability. Future studies may, therefore, benefit from having male and female interview participants.

The study began with the intent to also categorize and examine whether or not students personal concept definition had an impact on their performance in math classes after Calculus I & II.
This study did not, however, attain data sufficient to draw conclusions on this topic. This is an area of interest and possible future research.

Additionally, attitudes towards mathematics are varied from student to student. The researcher would be interested to discover the relationship between students attitudes towards mathematics and how that affects their learning various mathematical concepts.
CHAPTER 6  CONCLUSION

The intent of the study was to discover whether or not students personal concept definition was both operable in solving problems involving limits, as well as consistent with the formal concept definition when solving limit-based problems. The participants of the study were asked to answer formal and informal problems involving limits. The researcher investigated students mathematical ways of thinking about limits through a survey containing open-ended, graphical, and short-answer questions. Although the researcher was not specifically interested in whether or not students could recall the formal (epsilon-delta) definition of the limit, it was of interest as to whether or not fragments or entire portions of this definition were entrenched within students personal concept definition.

It is vital for students studying Science, Technology, Education and Mathematics (STEM) subjects to have a conceptual understanding of various mathematical topics, especially topics in calculus, a required course for most STEM majors. Students understanding must be past procedural knowledge and should have some trace of the formal definition of these various topics within their personal concept definition. Having a formal and operable knowledge of a mathematical topic, such as limits in calculus, allows a student to better problem solve in their respective disciplines.

As it has been previously shown (Tall, 1990; Vinner, 1983), students who study mathematics at the university level possess the ability to solve procedural-based problems, but struggle when confronted with conceptual-based problems. Typical students of mathematics latch on to topics that they excel in and shy away from topics that give them trouble. Their overall view towards the need to understand mathematics is less than optimal, with most students believing that the ability to solve a limited number of problems on a given topic to be sufficient in understanding that topic or concept, as demonstrated by Student A. Advanced students, like Student C, have the ability to work with deep concepts in math by using a variety of examples consistent with the formal definition.

Though the issue of students conceptual understanding of mathematical topics, specifically
limits, is a problematic area of study for many, there are a few suggestions to aid in student understanding. Students need to make various connections between graphical and numerical representations of limits while relating them back to the formal definition of a limit. Additionally, students need to be able to take their metaphors and informal representations of limits and connect them to the formal concept definition, not using these metaphors as an endpoint of mathematical knowledge.

Overall, the participants survey responses indicated that they had personal concept definitions inconsistent (to various degrees) with the formal concept definition of the limit. This inconsistency made their concept definition inoperable (to various degrees) in solving limit-based problems in mathematics. Limits are a problematic topic for students, but improving activities and instruction can enhance their knowledge and concept definition. These improvements in student understanding could have dramatic implications in the performance of students in mathematics courses past calculus and in their own academic disciplines.
BIBLIOGRAPHY


APPENDIX A  IN-CLASS SURVEY

(1) Describe the mathematical meaning of a Limit
In your description you may use any collection of: graphs, symbols, tables or written explanations

(2) Given:

\[ \lim_{{x \to a}} f(x) = 2 \]

State whether the following statements are true: always, sometimes or never

- \( f \) is defined at \( x = 4 \)
- For every \( M \geq 2 \) there is a positive number \( N \) so that whenever \( 0 < |x - 4| < N \), then \( f(x) > M \)
- \( f \) is continuous at \( x = 4 \)
- \( f(4) = 2 \)
- For every positive number \( r \) there is a positive number \( s \) so that whenever \( 0 < |x - 4| < s \), then \( |f(x) - 2| < r \)

(3) Does the Limit exist as \( x \) approaches -1?
If yes, what is the value of the Limit?
Briefly justify your answer
(4) Does the Limit exist as $x$ approaches 0?
If yes, what is the value of the Limit?
Briefly justify your answer

(5) Does the Limit exist as $x$ approaches 1?
If yes, what is the value of the Limit?
Briefly justify your answer

(6) Describe what it means when:
\[ \lim_{{x \to a}} f(x) = L. \]
APPENDIX B  INTERVIEW PROTOCOL

Student Interview Outline

In Regards to Survey Question (1)
a) Can you think of any other words, examples, or metaphors to help you describe the concept of a limit?
b) During your studies in mathematics, are there any specific instances where it was helpful for your learning to have or use a formal definition of limit?
   (i) if so, tell me a bit more how that helped you
   (ii) if not, how did you get around it, in Calculus?
c) What other mathematical topics come to mind when you think about Limits?

In Regards to Survey Question (2)
Let’s take a look over the directions and your response to question two:
Noticing one given answer describes \( f(x) \) being continues when \( x = 4 \)
a) How would you describe any connections between limits and continuity?
b) In your own words, how would you describe what it means for a function to be continuous?
c) Given the formal definition of the limit at part(5) of Question 2, can you attempt to formalize the definition of continuity? Perhaps using limits?
d) Given the limit exists for \( f(x) \), is it required that 4 be in the domain of \( f \)?

In Regards to Questions 3-5 (Graphs)
Now that we’ve had some time to discuss your answers to Q1 and Q2, look over your answers to Q3-Q5 and see if there’s any changes you would like to me.

a) Did you make any changes? Why or Why not?
b) Must the function be defined at \( x \) in order for the limit to exist?
c) Is the function value always equal to the limit?
d) What if the left hand limit exists?
e) What if the right hand limit exists?

**In Regards to Question 6**

a) Given your answer to Question 1 and Question 6, can you compare and contrast your answers?