GENERALIZED P-COLORINGS OF KNOTS

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ABSTRACT

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The concept of $p$-colorings was originally developed by R.H. Fox. Consideration of this knot invariant can range from the simple intuitive definitions to the more sophisticated. In this paper the notion of abstract colorings by the group $T_p \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes C_3$ is developed and explored. The abstract coloring by $T_p$ yields some close connections to the Alexander polynomial, and so Fox’s “free calculus” construction of the Alexander polynomial is closely examined as well. The advantage of this calculation is that it allows the Alexander polynomial to be easily constructed once one knows the knot group of a particular knot. Due to the way $C_3$ acts on $(\mathbb{Z}_p \times \mathbb{Z}_p)$ in $T_p$, calculating $T_p$-colorability amounts to some basic algebraic manipulation. This can also be combined with the linear algebra approach of calculating the exact number of $T_p$ colorings to yield a potentially stronger invariant.
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CHAPTER 1: INTRODUCTION TO KNOTS

1.1 KNOTS AND KNOT EQUIVALENCE

Humans have often utilized knots for various purposes: stabilizing a load on a length of rope, sealing bags, and many others. The widely used knots are even named – the trefoil knot and granny knot, for instance. In their own right, knots serve as an interesting (and useful!) object worthy of mathematical study. To begin such study, we must formalize the definition of a knot.

For the purposes of this thesis, the following definition is used:

Definition 1.1. A knot is any embedding $K : S^1 \rightarrow \mathbb{R}^3$.

Intuitively, a knot is a closed loop of string in Euclidean space. Though knots are three-dimensional objects and can be visualized, it is sometimes difficult or cumbersome to render complicated knots in three dimensions. We often resort to working with drawings which are “good enough” visualizations of knots.

Definition 1.2. A knot diagram is a projection of a knot $K$ to the plane.

That is, a knot diagram is a two-dimensional representation of a particular knot. In Figure 1.1 below we have a diagram representing one of the trefoil knots. It is a matter of convention that the breaks in the arcs represent an arc going underneath another arc (in other words, it is the underside of a crossing).

Some natural questions arise. When can we consider two knots to be the same? How do we distinguish two knots which are different? When is a knot the simplest? These notions shall be formalized shortly, but it makes intuitive sense that two knots are equivalent when they can be
stretched, tangled or untangled to match one another without cutting and gluing. The “simplest” knot should be one without any tangling whatsoever. It is referred to as the *unknot* – a circle without any kinks or crossings.

![Figure 1.1: A diagram of the trefoil knot.](image)

Our use of diagrams may introduce additional issues. It is possible that one knot may have many different-looking diagrams, such as in Figure 1.3. Is it enough to compare two particular diagrams of knots rather than the knots themselves? The answer is contained in the theorem

![Figure 1.2: A diagram representing the unknot.](image)

![Figure 1.3: Two diagrams of the unknot.](image)
below, which also formalizes the idea of knot equivalence.

**Theorem 1.3.** (Reidemeister’s Theorem) Two knots, $K$ and $K'$ with diagrams $D, D'$ are equivalent if and only if their diagrams are related by a finite sequence of intermediate diagrams such that each differs from its predecessor by one of the following four elementary operations. (Each operation is only applied to a specific area of the diagram.)

1. $(R_0)$ A planar isotopy.

2. $(R_1)$ “Un-kinking” an arc.

3. $(R_2)$ Pulling apart two arcs where one is completely above the other.

4. $(R_3)$ Pulling an arc over or under an intersection, provided it is completely above or below.
These elementary operations are known as the *Reidemeister moves*. The theorem says that if we can go from one knot diagram to another by a finite sequence of these Reidemeister moves, then both diagrams in fact represent the same knot. The next result is immediate:

**Proposition 1.4.** “Equivalence through a finite sequence of diagrams,” denoted as “∼,” is an equivalence relation.

*Proof.* Suppose $K$ is a knot, and $D$ is a diagram of $K$. Then $K ∼ K$ by the sequence $D = D_0 = D$; a sequence of length one. So ∼ is reflexive. We can see that ∼ is symmetric: suppose $K ∼ K'$ by the sequence $D = D_0, D_1, \ldots, D_n = D'$. Then we also have that $D' = D_n, D_{n-1}, \ldots, D_1, D_0 = D$, so that $K' ∼ K$ by a finite sequence of diagrams. Finally, suppose $K ∼ K'$ and $K' ∼ K''$; say that $K$ is related to $K'$ by the sequence $D = D_0, D_1, \ldots, D_n = D'$ and that $K'$ is related to $K''$ by the sequence $D' = D'_0, D'_1, \ldots, D'_m = D''$. Then the sequence $D = D_0, \ldots D_n, D'_0, D'_1, \ldots, D'_m = D''$ is a finite sequence relating $K$ and $K''$, so ∼ is transitive. 

We may expand our definition of diagrams to *oriented* diagrams, where there is an assignment of orientation to the knot. The directed arc segments must give a consistent orientation of the knot. Equivalence of oriented diagrams is defined analogously to equivalence of unoriented diagrams. We may also define “knots” with more than one piece of string:

**Definition 1.5.** A **link** is an embedding $L : (S^1)^n \to \mathbb{R}^3$ (where $(S^1)^n$ is the product of $S^1$ with itself $n$ times) in which each component’s embedding is disjoint from the others. Intuitively, it is a finite collection of mutually disjoint closed loops of string in $\mathbb{R}^3$. Each “loop” of string is referred to as a **component** of a link.

It is clear from the above definition that a knot is a one component link. Again, equivalence of links is defined similarly to equivalence of knots.

Armed with a formal notion of what it means for two knots (or links) to be equivalent, one can begin to study the objects seriously. However, it can often be difficult to distinguish two knots from one another – from our definition, one must show there exists no sequence of Reidemeister
moves relating the two diagrams. Indeed, it may happen that two very similar diagrams represent completely different knots!

![Figure 1.4: Diagrams of two different knots.](image)

In Figure 1.4, for example, the diagrams differ at only one crossing – however, the diagram on the left is equivalent to the unknot while the diagram on the right is not. To aid in the study of knots, we need tools that can distinguish two knots from one another. These tools must illustrate properties which are shared across every possible diagram of a particular knot so that one can truly observe a difference between two different knots. In particular, these properties must remain unchanged by any sequence of Reidemeister moves.

### 1.2 KNOT INVARIENTS

At first glance, one may be tempted to state that the number of crossings in a particular diagram is representative of the knot. However, Figure 1.3 presents a counterexample to that notion – the left diagram has four crossings, yet it represents the unknot, which is usually represented without crossings. Reidemeister move $R_1$ adds one crossing to the diagram, while $R_2$ subtracts two. Therefore crossing number in a diagram is not preserved by Reidemeister moves. However, the *minimal* crossing number taken over all possible diagrams of a knot must be unchanged by Reidemeister moves. This minimal crossing number is an example of what is referred to as a “knot invariant.”

**Definition 1.6.** A *knot invariant* is any property of knots respected by Reidemeister moves.
Specifically, suppose that a knot $K$ has a property $\rho$. If $K \sim K'$ implies that $K'$ also has property $\rho$ for all $K$ with property $\rho$, then $\rho$ is a knot invariant.

The main use of a knot invariant is to distinguish two knots which are different. Suppose that a knot $K$ has a particular knot invariant, and that $K'$ does not share this property. Then $K \nsim K'$. Two knots sharing a knot invariant property, however, does not imply the knots are equivalent.

The “values” which knot invariants take vary wildly – from integers to polynomial rings to groups and beyond. This thesis will be primarily concerned with a few specific knot invariants.

**Examples.**

1. The minimal crossing number, as mentioned above, is by definition a knot invariant.

2. The number of components in a particular link is a knot invariant – this is called the *linking number*. The fact that this is a knot invariant should be readily apparent; however, it is not an overly useful property for distinguishing links.

3. *Tricolorability* is another less obvious knot invariant, which is the ability to color arcs of a diagram such that the following are obeyed:

   (a) At least two, but no more than three colors are used to color the arcs.

   (b) Coloration remains constant from one intersection to the next (that is, arcs do not change color “mid” arc, only at intersections).

   (c) At each intersection, each of the incident arcs are either all the same or all different colors.

   For example, the trefoil knot is tricolorable while the unknot isn’t. A generalization of this particular invariant is studied in-depth later in the paper.

4. Various *knot polynomials* which are constructed in many different ways. Among them are the Jones and Alexander polynomials; the latter will be the subject of its own chapter.
One must think of knot invariants as tools in an arsenal to distinguish knots from one another. There are many different knot invariants, each with its own uses, advantages and disadvantages. So far there is no “perfect” knot invariant – that is, there is no simple and nontrivial invariant which completely distinguishes all nonequivalent knots. Discovering new invariants and new ways to construct old invariants is an area of serious study among knot theorists.

1.3 KNOT GROUPS

In topology, as in knot theory, one must consider properties which can distinguish various topological spaces. One of these properties is actually a group which gives information about the topological space. We will visit these concepts only on their surface; the importance of this section is the computation of a particular knot invariant called the knot group. To proceed, we recall some basic definitions from topology (I will reference James Munkres’ text for the most part [7]).

Definition 1.7. Let $X$ be a topological space. A path is a continuous mapping $f : [0, 1] \to X$. We call $f(0)$ the initial point of the path and $f(1)$ the final point of the path.

Definition 1.8. Two paths $f$ and $f'$ mapping into the topological space $X$ are said to be path homotopic if the following hold:

1. $f(0) = f'(0)$ and $f(1) = f'(1)$.

2. There exists some continuous map $F : I \times I \to X$ so that $F(s, 0) = f(s)$, $F(s, 1) = f'(s)$, $F(0, t) = f(0)$, and $F(1, t) = f(1)$.

In other words, paths which are homotopic to one another can be continuously deformed into one another while keeping the initial and final points of the path the same. Though it shall not be proven here, homotopy of paths is an equivalence relation – thus we consider the equivalence class of these paths under homotopy rather than individual paths. In addition, one may form a “product” of two paths by concatenating them and re-scaling the interval. That is, given two paths $f$ and $g$, ...
we define the path $f \ast g$ as:

\[
(f \ast g)(t) = \begin{cases} 
  f(2t) & \text{for } t \in [0, \frac{1}{2}], \\
  g(2t - 1) & \text{for } t \in [\frac{1}{2}, 1].
\end{cases}
\]

The operation $\ast$ on the equivalence class of paths has some nice properties: it is associative, there are identities (paths which do not change other paths), and inverses (paths with opposite direction).

**Definition 1.9.** Let $X$ be a topological space; let $x_0 \in X$. Any path in $X$ with initial and final point $x_0$ is called a loop; the set of path homotopy classes of loops based at $x_0$ with the operation $\ast$ is called the fundamental group of $X$ relative to $x_0$ and is written $\pi_1(X, x_0)$.

With the notion of a fundamental group, we may finally define a knot group. First, an observation: let $X$ be a path-connected topological space (a space in which any two points have a path connecting them). For any two points $x_0$ and $x'_0$ in $X$, $\pi_1(X, x_0) \cong \pi_1(X, x'_0)$. In such spaces, then, our choice of base point does not matter and we may simply refer the isomorphism class of these groups as $\pi_1(X)$. Recall that a knot $K$ is a projection of a circle into $\mathbb{R}^3$. We can make the following definition:

**Definition 1.10.** Let $K$ be a knot. The knot group of $K$, denoted as $\pi(K)$, is the fundamental group of the space $X = \mathbb{R}^3 \setminus K$.

Notice we did not make a mention to a base point for the fundamental group – by the above observation, we may choose whichever base point we like and the fundamental group will remain the same up to isomorphism (this is because $X$ as given above is path-connected). One can show (see for more details) that the complements of two equivalent knots, $K$ and $K'$, are homeomorphic, hence $\pi(K) \cong \pi(K')$. In other words, the (isomorphism class of the) knot group is an invariant for knots.

Many of the problems explored in the later chapters of this paper involve homomorphisms from various groups to the knot groups. It is a useful invariant which seems difficult to compute, at least without the help of the following two theorems (which are not proved here).
Theorem 1.11. (Van Kampen’s Theorem) [8] Let \(X\) be a topological space containing subsets \(U, V\) such that \(U, V, W = U \cap V\) are all open and path-connected, and \(U \cup V = X\). Let \(x_0\) be a basepoint in \(W\). Let the fundamental groups of \(U, V, W\) be given by presentations:

\[
\pi_1(U, x_0) = \langle S_U : R_U \rangle, \quad \pi_1(V, x_0) = \langle S_V : R_V \rangle, \quad \pi_1(W, x_0) = \langle S_W : R_W \rangle.
\]

Consider the inclusions \(i_U : W \to U\), \(i_V : W \to V\) and their induced maps of fundamental groups \(i_*^U\) and \(i_*^V\). For each \(g \in S_W\), pick a word \(j_U(g) \in W(S_U)\) representing the element \(i_*^U(g)\), and a word \(j_V(g) \in W(S_V)\) representing the element \(i_*^V(g)\). Then \(\pi_1(X, x_0)\) has a presentation

\[
\langle S_U \cup S_V : R_U \cup R_V \cup \{j_U(g) = j_V(g) : \forall g \in S_W\} \rangle.
\]

This theorem describes the fundamental group of a topological space in terms of the fundamental groups of subspaces within it (which are possibly easier to compute or work with). If we consider some loop in \(U \cap V\), we may describe it as a word in the fundamental group of \(U\) AND as a word in the fundamental group of \(V\). Strictly speaking these are different words (their alphabets are different, after all) but they should represent the same element in the fundamental group of \(X\). So, we simply add a new relation for each generator in the fundamental group of \(U \cap V\) to equate the two.

Theorem 1.12. (The Wirtinger Presentation) [8] Let \(K\) be a knot. Take a diagram \(D\) of the knot and orient it. Label each arc \(a_1, a_2, \ldots, a_k\). At each crossing, we have two possibilities:

![Crossing Diagram]

On the left is a “positive” crossing and on the right is a “negative crossing.” Associate the relation
\(y^{-1}xy = z\) to the positive crossings, and \(yxy^{-1} = z\) to the negative crossings to obtain a set of relations \(R\); then \(\pi(K) \cong \langle a_1, a_2, \ldots, a_k : R \rangle\).

This theorem grants a straightforward computation of the knot group of a particular knot. Thus we have a clear depiction of the knot group and its relations – this means we may compute group homomorphisms between knot groups and other groups. In the next chapter, we will expand on this concept more explicitly.
CHAPTER 2: COLORINGS OF KNOTS

2.1 TRICOLORABILITY

In the section regarding invariants for knots, tricolorability was mentioned. Recall that a knot is tricolorable if, using at least two but no more than three colors, we can color each arc a single color with the intersections obeying:

On the left, the three incident arcs are all the same color. On the right, they are all different colors. Figure 2.1 is an example of a tricoloring of a knot (the labeling is more friendly to black-and-white printers).

![Figure 2.1: A tricoloring of the trefoil knot.](image)

The label “R” is for red, “B” is for blue, and “G” is for green; each arc is respectively colored with those three colors. Observe that at every intersection, each of the incident arcs is a different
color. Tricolorability is in some cases easily computable and is easy to understand.

**Proposition 2.1.** Tricolorability of a knot is a knot invariant.

*Proof.* Suppose that $K$ is a knot which is tricolorable. We must show that each of the Reidemeister moves respects the tricolorability of $K$; by induction the argument will extend to any knot equivalent to $K$. Clearly $R_0$ will not affect colorability as it does not add or subtract intersections. Then perform $R_1$ on a particular segment of $K$:

\[ \begin{array}{c}
  \text{R} \\
  \downarrow \\
  \text{R} \\
  \text{R}
\end{array} \]

$R_1$ adds an intersection, but notice that if we keep everything the same color as it was originally, then the incident arcs of the intersection are the same color. For the remaining two Reidemeister moves there are a few cases to consider. However, they are similar to one another. Suppose that we have two arcs which are different colors and we perform $R_2$:

\[ \begin{array}{c}
  \text{R} \\
  \text{B} \\
  \text{R}
\end{array} \]

To keep the rest of the diagram’s colors consistent, we are forced to keep the red and blue strands on the top and bottom; then, if we change the middle arc to green, the two intersections still satisfy tricolorability. Now, we examine $R_3$. Again, there are several cases to consider but they are similar
in nature to the one presented here:

![Color diagram]

To maintain consistency, the top two strands must be colored red and blue, and the bottom must be colored red and green. Additionally, the middle strand must be colored green – to maintain tricolorability we must force the three “bottom right” arcs to be green (monochrome intersections are allowed).

Now, we have that tricolorability of a knot is a knot invariant. This is useful for distinguishing some basic knots. For example, the unknot is not tricolorable (it has only trivial or monochrome colorings). This is an easy way to justify why the trefoil knot, which is relatively simple, is worth studying. Further, tricolorability may be an easy calculation for simple knots, but combinatorially speaking, the number of choices we have to make to reach a coloring will escalate very quickly. In addition, the invariant is too simple in the sense that a knot is either tricolorable or it is not. We may turn our attention to a refinement of this invariant where the number of tricolorings is considered rather than a knot’s ability to be colored. As we shall see, this new approach generalizes well.

### 2.2 COLORABILITY: THE LINEAR ALGEBRA APPROACH

Suppose that instead of labeling arcs by names of colors we let them correspond to numbers. Say that “R” is 0, “B” is 1, and “G” is 2. Recall that a coloring of an intersection is acceptable if all three incident arcs are the same, or they are all different. With the change from colors to numbers, what do the triples (0, 1, 2), (0, 0, 0), (1, 1, 1), and (2, 2, 2) have in common? Interestingly the sum of coordinates in each triple is a multiple of 3.

This phenomenon is not by chance or accident. Say that we have a knot with $n$ arcs and
$n$ intersections; if we label each arc by $x_1, x_2, \ldots, x_n$, then examining whether an intersection satisfies tricolorability boils down to finding solutions of the equation

$$x_i + x_j + x_k \equiv 0 \pmod{3},$$

for segments $i, j$ and $k$ incident to the intersection.

So, to get colorability of the knot, we obtain a system of equations with $n$ variables and $n$ equations. Note that there will always be 3 trivial solutions to this system, which correspond to coloring each arc the same color. For our purposes, tricolorability means that there exist *more* colorings than the trivial ones. Further, colorings always appear in triples (we may simply swap all three colors for each other and obtain a new coloring). So the total number of tricolorings of a knot is $3^k$ for some $k$. If $k > 1$ then the knot is tricolorable; however at this stage we have more information than we did in the first section. We know if a knot is tricolorable, but the *number* of tricolorings also holds significance. In this way, we may yet distinguish two knots that are tricolorable by illustrating the number of the tricolorings are different.

If we turn our attention back to linear systems of equations, we find that the solutions to the homogeneous system of equations form a vector space over $\mathbb{Z}_3$. If we call this vector space $N$, then we see that the number of tricolorings of a diagram is $3^{\dim N}$. Alternatively, we can compute a first minor of the matrix; if 3 divides the minor, then we can tricolor the diagram – after all, this implies that the minor does not have full rank (divisibility by 3 means it is 0 modulo 3) and the dimension of $N$ is at least 2.
**Example.** Suppose we have a knot diagram with each arc labeled:

![Knot Diagram](image)

From this diagram, we obtain the system of equations

\[
\begin{align*}
x_1 + x_3 + x_5 &\equiv 0 \\
x_1 + x_4 + x_6 &\equiv 0 \\
x_1 + x_3 + x_5 &\equiv 0 \\
x_1 + x_4 + x_6 &\equiv 0 \\
x_2 + x_3 + x_5 &\equiv 0 \\
x_2 + x_4 + x_6 &\equiv 0
\end{align*}
\]

with equivalence mod 3. Already, we can tell that the rank of the corresponding matrix is less than 6. The matrix itself is

\[
\begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
\]
Two pairs of rows are identical – with some elementary row operations the matrix becomes:

\[
\begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Clearly this matrix has rank at most 4; this means that the nullity of the matrix is at least 2. Hence, the diagram (and the knot) is tricolorable.

In the linear algebra context, tricolorability generalizes well to other colorings – we simply replace 3 with \( p \), where \( p \) is any prime number. Then every calculation is instead performed mod \( p \) instead of mod 3. However, we must make a small change to the system of equations. At every intersection we have the following situation:

Here we impose the condition that \( x_i - 2x_j + x_k \equiv 0 \mod p \). Again, solutions to the homogeneous equation will form a vector space, \( N \), over \( \mathbb{Z}_p \). Then the number of \( p \)-colorings will be \( p^{\dim N} \), and so if \( \dim N > 1 \) we have that the knot is \( p \)-colorable.
2.3 COLORABILITY: THE GROUP-THEORETIC APPROACH

Recall that for a knot $K$, the knot group, $\pi(K)$, is defined to be the fundamental group of the space $\mathbb{R}^3 \setminus K$. Due to the Wirtinger Presentation theorem, acquiring a set of generators and relations consists of labeling the arcs of a knot and assigning conjugation relations at each intersection. It is this “labeling” which is of particular interest. Our notion of coloring a knot diagram is intimately related with labeling the arcs a certain way (as demonstrated in the previous section).

Example.

Above, we have an oriented diagram of the trefoil knot with each arc labeled. Using Theorem 1.11, we obtain the following system of relations:

$$xyx^{-1} = z$$
$$zxz^{-1} = y$$
$$yzy^{-1} = x$$

Thus the presentation for the knot group of the trefoil as depicted above is

$$\langle x, y, z : yzy^{-1} = x, zxz^{-1} = y, xyx^{-1} = z \rangle.$$
relations for \( \pi(K) \) are of the same ilk. Indeed, this is consistent across all of the knot groups by Theorem 1.11. To begin our discussion of how coloring is related to the labelings of the knot, we must first recall some definitions from elementary abstract algebra (the text by Dummit and Foote [3] is an excellent resource).

**Definition 2.2.** Let \((G, \cdot)\) be a group, and suppose that \(x, y \in G\). The product \(yxy^{-1}\) is called the conjugation of \(x\) by \(y\). The elements \(x\) and \(y\) are said to be conjugates of one another if there exists some \(z \in G\) so that \(zxz^{-1} = y\).

**Definition 2.3.** Let \((G, \cdot)\) be a group, and suppose that \(a \in G\). The set

\[
\{ b \in G : b = cac^{-1} \text{ for some } c \in G \}
\]

is called the conjugacy class of \(a\).

It is immediate that if two elements are conjugate, they have the same conjugacy class. Therefore we may refer to conjugacy classes without referring to specific elements of the group. More compactly, the conjugacy classes of \(G\) can be thought of as the orbits of \(G\) acting on itself by conjugation.

By the way we determine knot groups, it should be clear that for any knot its set of generators will lie in the same conjugacy class. With this fact in mind, we may begin to label arcs of a diagram by elements of any abstract group we desire. We need only ensure that the labeling is consistent with the conjugation relations in the knot group. In other words, we need to preserve the operation of both groups, which amounts to calculating group homomorphisms between them.

**Definition 2.4.** Let \((G, \cdot)\) and \((H, \ast)\) be groups. We say that the map \(\varphi : G \to H\) is a group homomorphism if for any \(g_1, g_2 \in G\) we have that \(\varphi(g_1g_2) = \varphi(g_1) \ast \varphi(g_2)\).

To calculate homomorphisms between knot groups and other abstract groups, we need ways to precisely write them down. We then require some basic results from the theory of free groups (these results can also be found in [3]). Knowing some basic terminology from free groups goes
a long way to improve one’s understanding of group presentations (which are explored more in-depth in Chapter 3). In addition, these elementary results will explicitly show the correspondence between coloring a knot $K$ by a group $G$ and homomorphisms from $\pi(K)$ to $G$.

**Definition 2.5.** Let $S$ be a set. Then we call the **free group on** $S$ the set of all words with letters in $S$, with the operation being concatenation of words (defining $xx^{-1}$ to be the empty word for all $x$ in $F(S)$). The free group on $S$ will be denoted $F(S)$.

Notice that any group $G$ can be presented so that $G = \langle S : R \rangle$, a system of generators with relations. It is therefore natural to refer to free groups because they satisfy the following universal property: for any set $S$ and group $G$, any set map $\varphi : S \to G$ induces a unique homomorphism of groups $\Phi : F(S) \to G$ so that

$$
\begin{array}{c}
S \\
\downarrow \varphi \\
F(S) \downarrow \Phi \\
G
\end{array}
$$

is a commutative diagram, where $\iota : S \to F(S)$ is the natural inclusion map. In other words, this means that for any $s \in S$ we have that $\varphi(s) = \Phi \circ \iota(s)$. The uniqueness of the map $\Phi$ means that set maps between $S$ and $G$ are in bijective correspondence with homomorphisms between $F(S)$ and $G$. Then, we may translate this notion from free groups to homomorphisms between abstract groups with the following result.

**Lemma 2.6.** Let $H = \langle S : R \rangle$ and $G$ be groups. Then there is a bijective correspondence between homomorphisms $f : H \to G$ and set maps $\varphi : S \to G$ whose associated $\Phi$ homomorphisms satisfy $\Phi(r_1) = \Phi(r_2)$ for any relations $r_1 = r_2$ in $R$.

**Proof.** Suppose $f : H \to G$ is a group homomorphism. This induces a map of sets $\varphi_f : S \to G$ so that $\varphi_f(s) = f(\bar{s})$, where $\bar{s}$ is the equivalence class of $s$ under the relations in $H$. By the universal property of the free group we get a unique homomorphism of groups $\Phi_f : F(S) \to G$. 
Suppose \( r_1 = r_2 \) is a relation in \( H \). Then we have

\[
\Phi_f(r_1) = f(\bar{r}_1) = f(\bar{r}_2) \quad (f \text{ is a homomorphism}) = \Phi_f(r_2).
\]

So, every homomorphism admits a unique set mapping with the desired property. Conversely, let \( \varphi : S \to G \) be a set map and \( \Phi : F(S) \to G \) be its induced homomorphism so that \( \Phi(r_1) = \Phi(r_2) \) for any relation \( r_1 = r_2 \) in \( R \). Similarly, let \( \bar{\varphi} : S \to H \) be the natural map (which takes generators to generators) and \( \bar{\Phi} : F(S) \to H \) be the induced map. Observe that since \( \bar{\Phi} \) is clearly surjective we have that \( F(S)/\ker \bar{\Phi} \cong H \). By the universal property of quotient groups we have that there is a unique homomorphism \( f_\Phi \) so that

\[
\begin{array}{ccc}
F(S) & \xrightarrow{\pi} & F(S)/\ker \bar{\Phi} \cong H \\
\Phi & \downarrow & \downarrow f_\Phi \\
& G & \\
\end{array}
\]

is a commutative diagram. Hence every set map induces a unique homomorphism between \( H \) and \( G \). The two operations are mutually inverse to one another, and hence there is a bijection between the two. \( \square \)

Recall that coloring a knot with \( p \) colors amounts to labelling arcs of the diagram with numbers so that these numbers obey a certain condition. We may now define an analogous statement involving labelling a knot with group elements:

**Definition 2.7.** Let \( K \) be a knot and \( G \) be any group. Then a **coloring** of \( K \) by \( G \) is any homomorphism from \( \pi(K) \) to \( G \); a **trivial coloring** of \( K \) by \( G \) is a homomorphism \( \varphi : \pi(K) \to G \) so that \( \varphi(x_i) = g \) for all the generators \( x_i \) of \( \pi(K) \). We say that a knot is **colorable by** \( G \) if there exist nontrivial colorings of \( K \) by \( G \).
If all we are doing is determining whether or not a knot is colorable by a group $G$, then we must count homomorphisms from $\pi(K)$ to $G$. The above lemma says that it suffices to say where the generators of $\pi(K)$ go and to check that the conjugation relations are satisfied. Additionally, the lemma implies that $\text{Hom}(H, G)$, the set of homomorphisms between $H$ and $G$ is finite if $H$ has finitely many generators and $G$ is finite.

The conjugation relations in a knot group ensure that the labellings by elements of $G$ must all be in the same conjugacy class. Let $K$ be a knot, $G$ be a group, and $C$ be a conjugacy class in $G$. We may define a new invariant $\lambda(K, G, C)$ – the number of labellings of a knot by elements of a particular conjugacy class. Certainly, $\lambda(K, G, C)$ will be an invariant of knots – this is because Reidemeister moves preserve the isomorphism class of the knot group. As such, these Reidemeister moves will not disturb the homomorphisms from $\pi(K)$ to $G$. As mentioned previously, this invariant can be seen as a generalization of the coloring of a knot. Indeed, tricolorability of a knot is a special case of $\lambda(K, G, C)$.

**Proposition 2.8.** Let $K$ be any knot and consider $S_3$, the group of permutations on three letters. If $C = \{(1, 2), (1, 3), (2, 3)\}$ is the conjugacy class of the transpositions (2-cycles in $S_3$) then tricolorability of $K$ is determined by $\lambda(K, S_3, C)$.

**Proof.** Let $R = (1, 2)$, $B = (1, 3)$ and $G = (2, 3)$. Notice that self-conjugation fixes each element (that is, $RRR^{-1} = R$, and so on). Conjugating either $R, B$ or $G$ by different elements of $C$ produces the third element, i.e.:

\[
RBR^{-1} = G = R^{-1}BR \\
RGR^{-1} = B = R^{-1}GR \\
BRB^{-1} = G = B^{-1}RB \\
BGB^{-1} = R = B^{-1}GB
\]

and so on. Thus to obey conjugation relations at every crossing, each of the incident arcs must be labeled by the same element or each must be labeled with a different element – this is precisely the
As it turns out, the $p$-colorability of a knot can be determined in a similar way:

**Theorem 2.9.** Let $K$ be a knot and let $p \geq 3$ be a prime. Consider $D_{2p}$, the symmetry group of a regular $p$-gon in the Euclidean plane. Let $C$ be the conjugacy class of reflections in $D_{2p}$. Then the number of $p$-colorings of the knot $K$ is determined by $\lambda(K, D_{2p}, C)$.

**Proof.** To begin, a small lemma:

**Lemma 2.10.** Let $\{R_0, R_2, \ldots, R_{p-1}\}$ be the conjugacy class of reflections in $D_{2p}$, where $R_k$ is a reflection over the line making an angle $\pi k/p$ with the positive $x$-axis. Then $R_i \cdot R_j \cdot R_i = R_{2i-j} \pmod p$.

**Proof.** Recall that $D_{2p}$ has the presentation

$$\langle \rho, \sigma : \rho^p = \sigma^2 = 1, \sigma \rho = \rho^{-1} \sigma \rangle$$

where $\sigma = R_0$ and $\rho$ is the counterclockwise rotation by $2\pi/p$. We may uniquely express any element in $D_{2p}$ as $\sigma^k \rho^i$ for $k \in \{0, 1\}$ and $i \in \{0, 1, \ldots, p-1\}$. In particular, $R_i = \sigma \rho^i$. Then, we have

$$R_i R_j R_i = (\sigma \rho^i)(\sigma \rho^j)(\sigma \rho^i)$$

$$= \sigma \sigma \rho^{-i} \rho^j \sigma \rho^i$$

$$= \rho^{j-i} \sigma \rho^i$$

$$= \sigma \rho^{i-j} \rho^i$$

$$= \sigma \rho^{2i-j}$$

$$= R_{2i-j}.$$
the knot admits the labelings by elements of \( C \)

\[
\begin{array}{ccc}
  z & y & z \\
  \downarrow & \uparrow & \downarrow \\
  x & y & x \\
\end{array}
\]

at each crossing, where \( x, y, z \) are integers between 0 and \( p - 1 \); the labeling “\( i \)” denotes the reflection \( R_i \) for \( i \in \{0, 1, 2, \ldots, p-1\} \). To satisfy the relations given in the Wirtinger presentation, we must have that

\[ R_y R_x R_y = R_z = R_{2y-x}. \]

In particular, we have that \( z = 2y - x \), or that \( x - 2y + z \equiv 0 \) (mod \( p \)). This is precisely the relation we require for an intersection of \( K \) to be \( p \)-colored. Conversely, any equation determining \( p \)-colorability will likewise admit a relation among reflections in \( D_{2p} \). Hence constructing any homomorphism \( \varphi : \pi(K) \to D_{2p} \) is equivalent to finding a \( p \)-coloring of \( K \).

All methods of computing colorability are intimately related to one another – from the basic, qualitative definition to the more sophisticated group-theoretic definition. We shall explore colorings with groups (and the invariants spawned from them) in the subsequent chapters.
CHAPTER 3: THE ALEXANDER POLYNOMIAL

3.1 PRESENTATIONS AND PRESENTATION EQUIVALENCE

The construction of the Alexander polynomial for knots that we present here differs from Alexander’s. It will rely heavily on group and ring theory. To prove essential facts about the construction of the Alexander polynomial (such as its invariance), we need to give firm definitions of what it means for a group to be defined by presentations.

We have stated earlier (using facts from [8]) that the knot group is an invariant for knots – that is, two equivalent knots have isomorphic fundamental groups. The knot groups are essentially easy to calculate by presentations thanks to the Wirtinger presentation, but determining whether two groups are isomorphic based on presentation alone remains an immensely difficult question to answer. For example, consider the presentations

\[
\langle x, y : x^2 = y, xy = 1 \rangle \text{ and } \langle x : x^3 = 1 \rangle.
\]

The first presentation involves two generators and two relations, while the second has one generator and one relation. However, both groups are in fact isomorphic. This is a simple example, but nonetheless illustrates how these systems of generators can be complicated (especially after a sequence of Reidemeister moves adding more crossings and relations). We are then required to formalize some notions from group theory before proceeding. In this chapter we use materials and theory from Fox [4] unless otherwise specified.

**Definition 3.1.** [4] Let \( S \) be a set. A **presentation** is a pair of objects \( S \) and \( R \), denoted by \( (S : R) \), where \( R \) is a set of words in \( F(S) \).
Definition 3.2. [4] Let $S$ be a set. We say that the presentation of a group, denoted as $\langle S : R \rangle$, is the quotient group $F(S)/\bar{R}$, where the set $\bar{R}$ is the consequence of all relators $r$ in $F(S)$ (that is, $\bar{R}$ is the smallest normal subgroup of $F(S)$ containing $R$). We say that the presentation is a presentation of $G$, where $G$ is any group, if there is some isomorphism from $\langle S : R \rangle$ onto $G$.

Definition 3.3. [4] We say that a presentation $\langle S : R \rangle$ is finitely generated if $S$ is finite, finitely related if $R$ is finite, and finitely presented if it is both finitely generated and related.

Though these definitions seem highly abstract, they allow us to be more precise which will aid us greatly when it comes to proving the invariance of Fox’s construction of the Alexander polynomial. Bear in mind that the purpose of this section is to give (at least, in theory) a solution to the problem of determining when two group presentations actually represent isomorphic groups.

Definition 3.4. [4] Let $\langle S : R \rangle$ and $\langle T : Q \rangle$ be presentations. A homomorphism $f : F(S) \rightarrow F(T)$ is said to be a presentation map, denoted $f : \langle S : R \rangle \rightarrow \langle T : Q \rangle$, if $f(R) \subseteq Q$.

We may define composition of two presentation mappings in the obvious way – this composition behaves nicely, as it is associative and there are identity mappings. One may think of the collection of presentations together with presentation maps as a category, if he or she is inclined to think of mathematics in a categorical fashion.

The presentation maps will naturally induce group homomorphisms between presentations of groups – that is, if $f : \langle S : R \rangle \rightarrow \langle T : Q \rangle$ is a presentation map, there exists a unique group homomorphism $f' : \langle S : R \rangle \rightarrow \langle T : Q \rangle$ so that

$\begin{array}{ccc}
F(S) & \xrightarrow{f} & F(T) \\
\pi & & \pi \\
\langle S : R \rangle & \xrightarrow{f'} & \langle T : Q \rangle
\end{array}$

is a commutative diagram. Additionally, up to a certain equivalence relation, the converse is also true – that is, any homomorphism between $\langle S : R \rangle$ and $\langle T : Q \rangle$ will admit a homomorphism from
F(S) to F(T). Hence we may refer to presentation maps as homomorphisms between presentations of groups or by their definition.

By now, it may seem that we have gotten away from the core issue – when do two group presentations actually represent the same group? Our notion of mappings between presentations differs from the idea of a simple group homomorphism, and so we must additionally define a presentation equivalence.

**Definition 3.5.** [4] Let \((S : R)\) and \((T : Q)\) be presentations. The presentations are said to be **presentation equivalent** if there exists a pair of mappings \(f : (S : R) \to (T : Q)\) and \(g : (T : Q) \to (S : R)\) so that \(f' \circ g' = id\) and \(g' \circ f' = id\). The pair (or either one separately) is called a **presentation equivalence**.

We get the following result immediately:

**Proposition 3.6.** [4] Two presentations \((S : R)\) and \((T : Q)\) are presentation equivalent if and only if \(\langle S : R \rangle \cong \langle T : Q \rangle\).

**Proof.** Suppose that the pair \(f, g\) is a presentation equivalence. Then we have that \(f' \circ g' = id\) and \(g' \circ f' = id\) so that \(f'\) and \(g'\) are mutual inverses; hence \(f'\) is an isomorphism (with \((f')^{-1} = g'\)). Conversely if \(f' : \langle S : R \rangle \to \langle T : Q \rangle\) is an isomorphism then we have another homomorphism \((f')^{-1}\) satisfying the desired property, and \((f')^{-1}\) will lift to a presentation map. \(\square\)

The general idea becomes thus: we can take a presentation for a group and alter it while keeping everything presentation equivalent. Then at each stage or tiny alteration, the underlying groups will clearly be isomorphic to one another. What results is a very evident parallel between group theory’s presentation equivalence and knot theory’s knot equivalence. It is the following very fundamental theorem which draws this strong connection.

**Theorem 3.7.** [4] (The Tietze Transformation Theorem) Suppose that \((S : R)\) and \((T : Q)\) are finitely presented and presentation equivalent with \(f : (S : R) \to (T : Q)\) being the presentation equivalence. Then there exists a finite sequence of elementary presentation equivalences \(T_1, T_2, \ldots, T_n\) so that \(f = T_1T_2\ldots T_n\), where each \(T_i\) is one of the following:
1. (Adding a superfluous relator) \( T_1 : (S : R) \to (S : R') \), where \( R' = R \cup \{r\} \) and \( r \in \bar{R} \).

2. (Subtracting a superfluous relator) \( T_2 : (S : R) \to (S : R') \) where \( R' = R \setminus \{r\} \) and \( r \in \bar{R} \).

3. (Adding a superfluous generator) \( T_3 : (S : R) \to (S' : R') \) where \( S' = S \cup \{x\} \), and \( x \) is any symbol which is not in \( S \); \( R' = R \cup \{wx^{-1}\} \), where \( w \) may be any word in \( F(S) \).

4. (Subtracting a superfluous generator) \( T_4 : (S : R) \to (S' : R') \) where \( S' = S \setminus \{x\} \); \( R' = R \cup \{wx^{-1}\} \), where \( wx^{-1} \in R, w \in F(S') \), and no other members of \( R' \) involve \( x \).

It is clear that \( T_1 \) and \( T_2 \) are mutual inverses, as are \((T_3)' \) and \((T_4)' \), so each of these elementary “moves” is in fact a presentation equivalence. The Tietze Theorem naturally brings our attention to Reidemeister’s Theorem. The two are somewhat analogous in that Reidemeister’s Theorem guarantees any two equivalent knots will be represented by diagrams separated by a finite sequence of Reidemeister moves, while Tietze’s Theorem says any two isomorphic groups will have presentations separated by a finite sequence of Tietze transformations. A question arises naturally – are there so-called “presentation invariants” for groups, that is, properties which are invariant under the Tietze transformations? Because we already know that any knot can be in turn finitely presented as a group, any presentation invariants would also have to be invariants for knots. This shall be explored subsequently.

### 3.2 FOX DERIVATIVES AND ELEMENTARY IDEALS

We wish to come up with the powerful knot polynomials (in particular, the Alexander polynomial). What follows develops some of the theory of Fox’s construction of the Alexander polynomial and proof of its invariance. There is quite a bit of heavy machinery at work here (as foreshadowed by the first section in the chapter), but regardless of the complexity of the theory surrounding this version of the Alexander polynomial, its computation is extremely straightforward in practice.

We adopt Ralph H. Fox’s approach here by constructing a powerful presentation invariant via what is known as the Fox derivative (Fox’s original name for it was free calculus). It is a way to define a calculus-type structure on any group (we may assume finitely presented).
Definition 3.8. Suppose that $G$ is finitely presented and that $\mathbb{Z}[G]$ is the group ring with integer coefficients. The Fox derivative will be any linear map $D : \mathbb{Z}[G] \to \mathbb{Z}[G]$ obeying $D(gh) = D(g) + g \cdot D(h)$. This map is uniquely determined by $G$ and hence it acts uniquely with the generating set of $G$ to yield partial derivatives. In other words, suppose that $G$ consists of words in symbols $x_1, x_2, \ldots, x_n$ and their inverses. Then we may calculate the partial Fox derivatives $\frac{\partial}{\partial x_i}$ by

1. $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$

2. $\frac{\partial}{\partial x_i}(1) = 0$

3. $\frac{\partial}{\partial x_i}(x_i^{-1}) = -x_i^{-1}$

4. $\frac{\partial}{\partial x_i}(w_1 \cdot w_2) = \frac{\partial}{\partial x_i}(w_1) + w_1 \frac{\partial}{\partial x_i}(w_2)$

where $\delta_{ij}$ is the Kronecker delta and $w_1, w_2$ are any words in $G$.

Existence and uniqueness will not be proven here (see Fox [4] for the full details). However, the entire Fox derivative may be described in terms of partial derivatives and hence we may only consider them.

Example. Consider the word $w = xyx^{-1}z^{-1}$.

$$\frac{\partial w}{\partial x} = 1 + x \left( \frac{\partial}{\partial x}(yx^{-1}z^{-1}) \right) = 1 + x \left( 0 + y \left( \frac{\partial}{\partial x}(x^{-1}z^{-1}) \right) \right) = 1 + xy \left( -x^{-1} + x^{-1} \frac{\partial}{\partial x}z^{-1} \right) = 1 - xyx^{-1}.$$ 

The algebra involved in computing the partial Fox derivatives, as shown above, is nothing complicated. This makes any construction based upon these derivatives very calculable. Bearing in mind the similarities between the partial Fox derivatives and the typical partial derivative, we can define the Jacobian matrix of the group presentation. Suppose $G$ has presentation $\langle x_1, x_2, \ldots, x_n :$
Then the $m \times n$ Jacobian matrix of the presentation is given by $[m_{ij}]$ where $m_{ij} = \frac{\partial r_i}{\partial x_j}$. If we keep our end goal (a polynomial in one variable) in mind, we must also pass each entry of the Jacobian through the abelianizer.

**Definition 3.9.** [3] Let $G$ be any group and $[G,G]$ its commutator subgroup. Then the quotient group $G/[G,G]$ is called the **abelianization of $G$**, and the natural projection map associated to it is the **abelianizer**.

**Definition 3.10.** [4] Let $G$ be any group and let $[m_{ij}]$ be the Jacobian of the group’s presentation. We define the **Alexander matrix** of the group $G$ to be the matrix $A = [a_{ij}]$, where $a_{ij} = ab(m_{ij})$, the image under the abelianizer.

By now the attentive reader may have noticed that the Fox derivatives assume some sort of ordering among generators and relators which is not a necessary component of a group’s presentation. To compensate for the arbitrary ordering of generators and relators, we then define a new equivalence relation.

**Definition 3.11.** [4] Let $A$ and $B$ be two matrices with entries in some arbitrary commutative, unital ring. We say that $A$ and $B$ are equivalent to one another, denoted $A \equiv B$, if there exists some finite sequence of matrices $A = A_1, A_2, \ldots, A_n = B$ so that each successive term in the sequence is obtained via one of the following elementary operations:

1. Permuting rows or columns.

2. Adjoining a row of zeros.

3. Adding a linear combination of rows to another row.

4. Adding a linear combination of columns to another column.

5. Adjoining a new row and column to the matrix such that each entry in the new row and column are zero, except their intersection, which is $1$ (the unit of the ring).
It is an easy exercise to verify that “≡” is a true equivalence relation. If we were to change the ordering of generators and relators and obtain two different Alexander matrices we see that they will be considered equivalent anyway (it amounts to a permutation of rows and columns).

As in vector calculus the determinant of the Jacobian matrix is important – in fact, it is what we will use eventually to define the Alexander polynomial (and the other knot polynomials). The definition relies upon a generalization called the elementary ideals of rings.

**Definition 3.12.** Suppose that \( R \) is any commutative ring with unity and let \( A \) be an \( m \times n \) matrix with entries in \( R \). Then we may define the \( k \)-th elementary ideal of \( A \), \( E_k(A) \), as:

1. If \( 0 < n - k \leq m \), then \( E_k(A) \) is the ideal generated by the determinants of all \((n - k) \times (n - k)\) submatrices of \( A \).
2. If \( n - k > m \) then \( E_k(A) = 0 \).
3. If \( n - k \leq 0 \) then \( E_k(A) = R \).

It is worth noting that our Alexander matrices satisfy the above definition, as each entry within an Alexander matrix is the image under the abelianizer’s natural extension to \( \mathbb{Z}[G] \), hence the ring is commutative with 1. Additionally, these elementary ideals are in fact an ascending chain of ideals. As one may suspect, equivalent Alexander matrices will also yield the same chain of ideals.

Again it may seem that we have gotten away from our goal by introducing the elementary ideals. However, it is an important diversion – this is because the elementary ideals are the theoretical underpinnings of the knot polynomials. So here we adopt the same approach as Fox: we first prove that a presentation yields a particular chain of elementary ideals, then show the presentation invariance of the elementary ideals, which will in turn imply the knot invariance of the knot polynomials (in particular, the Alexander polynomial).

Recall that an arbitrary presentation mapping \( f \) will in turn induce a homomorphism \( f' \) between presentations for the underlying groups. Naturally, composing these homomorphisms with the abelianizer will yield a homomorphism \( f'' \) between the abelianized groups. Perhaps most importantly, if \( f, g \) is a presentation equivalence, the pair \( f'', g'' \) will also be an equivalence of groups.
Lemma 3.13. [4] If the pair \( f, g \) is a presentation equivalence, then \( f'' \) and \( g'' \) are isomorphisms onto the underlying abelianized groups and are mutually inverse to one another.

Proof. We know that \( f'' \) and \( g'' \) are both homomorphisms. Now, since \( f, g \) is a presentation equivalence we have that \( f' \circ g' = id \) and \( g' \circ f' = id \). Observe that

\[
id = f' \circ g' = (f \circ g)'
\]
\[
id = (f' \circ g')' = f'' \circ g''.
\]

In other words, because the composition \( f' \circ g' \) is the identity, its natural extension to the abelianized groups must also be the identity, in turn implying \( f'' \circ g'' \) is the identity. Similarly, \( g'' \circ f'' = id \).

In addition, we need the following lemmas.

Lemma 3.14. [4] Let \( G = \langle x_1, x_2, \ldots, x_k : r_1, r_2, \ldots, r_m \rangle \) be any group and \( \frac{\partial}{\partial x_j} : \mathbb{Z}[G] \to \mathbb{Z}[G] \) be the partial Fox derivative with respect to \( x_j \). Define the element

\[
\frac{g^n - 1}{g - 1} = \begin{cases} 
0, & \text{if } n = 0, \\
\sum_{i=0}^{n-1} g^i, & \text{if } n > 0, \\
-\sum_{i=n}^{-1} g^i, & \text{if } n < 0.
\end{cases}
\]

Then we have that \( \frac{\partial}{\partial x_j}(g^n) = \frac{g^n - 1}{g - 1} \cdot \frac{\partial}{\partial x_j}(g) \).

Proof. We proceed by induction – first on positive \( n \), and then on negative \( n \). Here we may assume \( g \) is a word which is made of at least one \( x_j \) or else the partial derivative is zero and satisfies the lemma trivially. Note that if \( n = 0 \) then we have that \( \frac{\partial}{\partial x_j}(1) = 0 \), and if \( n = 1 \) we have that
\( \frac{\partial}{\partial x_j}(g) = 1 \cdot \frac{\partial}{\partial x_j}(g) = \frac{\partial}{\partial x_j}(g) \). Now suppose that the above holds for positive \( n \). We then get

\[
\frac{\partial}{\partial x_j}(g^{n+1}) = \frac{\partial}{\partial x_j}(g^n) + g^n \frac{\partial}{\partial x_j}(g) = \frac{g^n - 1}{g - 1} \frac{\partial}{\partial x_j}(g) + g^n \frac{\partial}{\partial x_j}(g) = \sum_{i=1}^{n-1} g^i \frac{\partial}{\partial x_j}(g) + g^n \frac{\partial}{\partial x_j}(g) = \sum_{i=1}^{n} g^i \frac{\partial}{\partial x_j}(g) = g^{n+1} - 1 \frac{\partial}{\partial x_j}(g).
\]

Next, suppose \( n \) is negative. Observe that if \( n = -1 \) we get that \( \frac{\partial}{\partial x_j}(g^{-1}) = -g^{-1} \frac{\partial}{\partial x_j}(g) \) by definition. Now suppose \( n < -1 \). Then

\[
\frac{\partial}{\partial x_j}(g^{n-1}) = \frac{\partial}{\partial x_j}(g^n) + g^n \frac{\partial}{\partial x_j}(g^{-1}) = \frac{g^n - 1}{g - 1} \frac{\partial}{\partial x_j}(g) + g^n \frac{\partial}{\partial x_j}(g^{-1}) = \sum_{i=n}^{-1} g^i \frac{\partial}{\partial x_j}(g) + g^{n-1} \frac{\partial}{\partial x_j}(g) = \sum_{i=n-1}^{-1} g^i \frac{\partial}{\partial x_j}(g) = g^{n-1} - 1 \frac{\partial}{\partial x_j}(g).
\]

\[\square\]

**Lemma 3.15.** [4] Let \( \varphi : R \rightarrow R' \) be an arbitrary ring homomorphism where \( R \) and \( R' \) are commutative rings with 1. If \( A = [a_{ij}] \) is a matrix with entries in \( R \), define

\[ \varphi(A) = [\varphi(a_{ij})]. \]

If \( \varphi \) is onto, then \( \varphi(E_k(A)) = E_k(\varphi(A)) \), where \( E_k \) represents the \( k \)-th elementary ideal.
Proof. Let \( \varphi \) be onto. Naturally, \( \varphi(0) = 0 \), so that \( \varphi(E_k(A)) = E_k(\varphi(A)) \) when \( n - k \) exceeds \( m \). Additionally if \( n - k \leq 0 \) we need that \( \varphi(E_k(A)) = E_k(\varphi(A)) = R' \), which follows because \( \varphi \) is onto. Now suppose that \( 0 < n - k \leq m \). The image of the set of determinants of all \( n - k \times n - k \) submatrices of \( A \) is exactly the set of determinants of all \( n - k \times n - k \) submatrices of \( \varphi(A) \). It suffices to show that the image of a finitely generated ideal is the ideal generated by the images of the respective generators. Specifically, let \( I \) be the ideal generated by \( r_1, r_2, \ldots, r_l \), and \( I' \) be the ideal generated by \( \varphi(r_1), \varphi(r_2), \ldots, \varphi(r_l) \). But \( \varphi \) is a surjective ring homomorphism, and we must have that \( \varphi(I) = I' \).

We are now able to prove the big theorem of the section, from which the invariance of the Alexander polynomial will follow. The argument here proceeds like Fox’s.

**Theorem 3.16.** ([4] Invariance of the Elementary Ideals) Let \((S : R)\) and \((T : Q)\) be two equivalent presentations with \( f \) the presentation equivalence between them. Then the \( k \)-th elementary ideal of \((S : R)\) is mapped onto the \( k \)-th elementary ideal of \((T : Q)\) by \( f'' \).

**Proof.** In light of the Tietze Transformation Theorem and Lemma 3.13, it suffices to show that Tietze transformations \( T1 \) and \( T3 \) yield the same chain of elementary ideals. So, consider \( T1 : (S : R) \rightarrow (S : R \cup w) \) with \( S = \{x_1, x_2, \ldots, x_n\} \) and \( R = \{r_1, r_2, \ldots, r_m\} \), and \( w \in \bar{R} \). Additionally, let \( \pi \) denote the natural projection from \( F(S) \) to \( \langle S : R \rangle \). Because \( w \in \bar{R} \) we have that

\[
w = \prod_{k=1}^{l} v_k r_i^\alpha v_k^{-1},
\]

where \( v_k \) are words with letters in \( S \). Now,

\[
\frac{\partial w}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \prod_{k=1}^{l} v_k r_i^\alpha v_k^{-1} \right)
= \frac{\partial}{\partial x_j} (v_1 r_1^\alpha v_1^{-1}) + (v_1 r_1^\alpha v_1^{-1}) \frac{\partial}{\partial x_j} (v_2 r_2^\alpha v_2^{-1}) + \cdots + \left( \prod_{k=1}^{l-1} v_k r_i^\alpha v_k^{-1} \right) \frac{\partial}{\partial x_j} (v_l r_l^\alpha v_l^{-1}).
\]

We now consider \( \pi \)'s natural extension as a ring homomorphism – the fact that \( \pi(r_i) = 1 \) for all \( i \)
gives us

\[ \pi \left( \frac{\partial w}{\partial x_j} \right) = \sum_{k=1}^{l} \pi \left( \frac{\partial}{\partial x_j} (v_k r_{ik}^{\alpha_k} v_k^{-1}) \right). \]

By Lemma 3.14, we have that

\[
\frac{\partial}{\partial x_j} (v_k r_{ik}^{\alpha_k} v_k^{-1}) = \frac{\partial}{\partial x_j} (v_k) + v_k \frac{\partial}{\partial x_j} (r_{ik}^{\alpha_k} v_k^{-1}) \\
= \frac{\partial}{\partial x_j} (v_k) + v_k \frac{r_{ik}^{\alpha_k} - 1}{r_{ik} - 1} \frac{\partial}{\partial x_j} (r_{ik}) + v_k r_{ik}^{\alpha_k} \frac{\partial}{\partial x_j} (v_k^{-1}) \\
= \frac{\partial}{\partial x_j} (v_k) + v_k \frac{r_{ik}^{\alpha_k} - 1}{r_{ik} - 1} \frac{\partial}{\partial x_j} (r_{ik}) - (v_k r_{ik}^{\alpha_k} v_k^{-1}) \frac{\partial}{\partial x_j} (v_k).
\]

From the definition of the element \( \frac{r_{ik}^{\alpha_k} - 1}{r_{ik} - 1} \) observe that

\[ \pi \left( \frac{r_{ik}^{\alpha_k} - 1}{r_{ik} - 1} \right) = \alpha_k. \]

Thus we have that

\[ \pi \left( \frac{\partial}{\partial x_j} (v_k r_{ik}^{\alpha_k} v_k^{-1}) \right) = \alpha_k \pi (v_k) \pi \left( \frac{\partial r_{ik}}{\partial x_j} \right). \]

Notice on the right-hand side the element \( \pi \left( \frac{\partial r_{ik}}{\partial x_j} \right) \) is precisely form of the entries of the Fox partial derivative Jacobian matrix. Thus, we apply the abelianizer \( \text{ab} \) to both sides:

\[
(ab \circ \pi) \left( \frac{\partial w}{\partial x_j} \right) = \sum_{k=1}^{l} ab(\alpha_k \pi (v_k)) \cdot (ab \circ \pi) \left( \frac{\partial r_{ik}}{\partial x_j} \right)
\]

Note that elements of the type \( ab(\alpha_k \pi (v_k)) \) are constants. This says that the Alexander matrix of \( (S : R \cup w) \) is exactly the same as the Alexander matrix for \( (S : R) \) with an additional row that is a linear combination of the others (which means that the Alexander matrix is equivalent under the previously mentioned equivalence relation). This yields the same chain of elementary ideals.

Now, we consider the Tietze transformation \( T3 \). Specifically, \( T3 \) takes \( (S : R) \) to \( (S \cup \{x\} : R \cup \{xw^{-1}\}) \), where \( x \) is a symbol not in \( S \) and \( w \) is a word in \( F(S) \). Let \( G_1 = \langle S : R \rangle \) and \( G_2 = \langle S \cup \{x\} : R \cup \{xw^{-1}\} \rangle \), with \( \pi_1 \) and \( \pi_2 \) being the respective projection maps. Denote the
Alexander matrices for \( G_1 \) and \( G_2 \) by \( A_1 = [a_{ij}^{(1)}] \) and \( A_2 = [a_{ij}^{(2)}] \). By definition, we have

\[
a_{ij}^{(1)} = (ab \circ \pi_1) \left( \frac{\partial r_i}{\partial x_j} \right).
\]

Because \( T3 : F(S) \to F(S \cup \{x\}) \) is the inclusion presentation map, we have that

\[
T3''(a_{ij}^{(1)}) = (T3'' \circ ab \circ \pi_1) \left( \frac{\partial r_i}{\partial x_j} \right) = (ab \circ \pi_2 \circ T3) \left( \frac{\partial r_i}{\partial x_j} \right) = (ab \circ \pi_2) \left( \frac{\partial r_i}{\partial x_j} \right) = a_{ij}^{(2)}.
\]

Because the symbol \( x \) is not in \( S \), we have that \( \frac{\partial r_i}{\partial x} = 0 \) for all \( i \) and that \( \frac{\partial}{\partial x}(xw^{-1}) = 1 \). Thus,

\[
A_2 = \begin{bmatrix}
T3''(A_1) & 0 \\
(ab \circ \pi_2) \left( \frac{\partial}{\partial x_1}xw^{-1} \right) & \cdots & (ab \circ \pi_2) \left( \frac{\partial}{\partial x_n}xw^{-1} \right) & 1
\end{bmatrix}
\]

By adding linear combinations of the last column to the others we see that

\[
A_2 \equiv \begin{bmatrix}
T3''(A_1) & 0 \\
0 & 1
\end{bmatrix}
\]

which means that \( T3''(A_1) \equiv A_2 \). Because \( T3 \) is a presentation equivalence, Lemma 3.13 shows that \( T3'' \) is an isomorphism – in particular, it is onto. Thus by Lemma 3.15 we see

\[
E_k(A_2) = E_k(T3''(A_1)) = T3''(E_k(A_1)),
\]

and the elementary ideals are preserved up to isomorphism.

The above shows that the elementary ideals are an invariant of presentations. This is enough
to say that elementary ideals can also be considered as invariants of knots, however these ideals are more unwieldy to use than the knot polynomials themselves. In the next section, we finally construct the Alexander polynomial.

3.3 KNOT POLYNOMIALS

Our discussion of the elementary ideals moved away from the knot groups to a more general setting. Here, we finally move back to knots and develop a well known and useful class of knot invariants – the knot polynomials.

In the particular case of knot groups, the abelianizations are exceedingly less complicated. Recall that, due to the Wirtinger presentation, the relators in a knot group take the form $x_i x_j x^{-1}_i x_k^{-1}$ from generators $x_1, x_2, \ldots, x_n$. Abelianizing such a group means to add the commutator relator $[x_i, x_j]$. Then the above conjugation becomes $x_j x^{-1}_k$. That is to say, the abelianized knot group has presentation

$$\langle x_1, x_2, \ldots, x_n : x_i x_j^{-1} \text{ for all } i \neq j \rangle.$$ 

The group as presented above is, of course, $\mathbb{Z}$ – so that the abelianized knot group is always infinite cyclic. This version of $\mathbb{Z}$ is often written $\langle t \rangle$ so that it is written multiplicatively rather than additively. Then, we may think of the ring $\mathbb{Z}[t]$ (the group ring) as Laurent polynomials in the variable $t$ with integer coefficients.

Recall that the elementary ideals are generated by determinants of the submatrices of the Alexander matrices. These generators (in the case of knot groups) will be such Laurent polynomials in one variable.

**Definition 3.17.** Let $\pi(K)$ be any knot group. The $k$-th knot polynomial, $\Delta_k(t)$, is the generator of the smallest principal ideal containing the $k$-th elementary ideal of the Alexander matrix of $\pi(K)$. The first knot polynomial, $\Delta_1(t)$, is called the **Alexander polynomial** of $K$ and is written as $\Delta(t)$.

Said differently, the $k$-th knot polynomial is the G.C.D. of the generators of the $k$-th elementary
ideal (which themselves are determinants of submatrices). Furthermore, each of these polynomials exists and is unique up to $\pm t^n$ (this is due to the fact that the only units in $\mathbb{Z}[t]$ are powers of $t$). The Alexander polynomial is particularly easy to calculate due to this fact. For any knot $K$, we can summarize the procedure as:

1. Determine the presentation of the knot group, $\pi(K)$.

2. Use Fox derivatives to calculate the Alexander matrix of $\pi(K)$.

3. Replace all variables with $t$.

4. Eliminate one row and one column.

5. Take the determinant of the remaining submatrix.

There is, as we have seen, quite a bit of technical detail which makes the procedure work so simply. For all intents and purposes, however, this construction provides an extremely simple way of calculating the Alexander polynomial in a purely algebraic way. The knot-invariance of these polynomials shall be shown shortly. Additionally, as mentioned previously, this method of constructing the Alexander polynomial differs from Alexander’s own but produces the same polynomial (up to, perhaps, powers of $t$). I shall not prove this result. Fox’s construction enjoys the benefit of a purely algebraic approach which feels closer in nature to group-theoretic methods of coloring knots.

Like all other knot invariants, the Alexander polynomial is only good for identifying when two knots aren’t the same. Two knots yielding the same knot polynomial still does not offer us any useful information in distinguishing the two. The Alexander polynomial is powerful and easily calculated, but also has its limits. Here I will again be adopting Fox’s approach by showing the invariance of all knot polynomials, in particular implying the invariance of the Alexander polynomial. This extremely important fact follows from the invariance of the elementary ideals that spawn the knot polynomials.
Theorem 3.18. [4] (Invariance of the Knot Polynomials) Let $\pi(K)$ and $\pi(\tilde{K})$ be two isomorphic knot groups with $f$ being the isomorphism between them. Then the $k$-th knot polynomial of $\pi(K)$ is mapped onto a unit-multiple of the $k$-th knot polynomial of $\pi(\tilde{K})$ by the induced isomorphism between $\text{ab}(\pi(K))$ and $\text{ab}(\pi(\tilde{K}))$.

Proof. Let $f''$ represent the isomorphism’s linear extension to the group rings from the abelianized knot groups (this is consistent with the previous notation). Let $(\Delta_k)$ and $(\tilde{\Delta}_k)$ be the principal ideals generated by $\Delta_k$ and $\tilde{\Delta}_k$, the $k$-th knot polynomials of $\pi(K)$ and $\pi(\tilde{K})$, respectively. Similarly, let $E_k$ and $\tilde{E}_k$ be the $k$-th elementary ideals. Due to Theorem 3.16, we have

$$f''(E_k) = \tilde{E}_k.$$ 

We also have, by definition, that $E_k \subseteq (\Delta_k)$ and $\tilde{E}_k \subseteq (\tilde{\Delta}_k)$. This in turn implies that $f''(E_k) \subseteq f''((\Delta_k))$, but $f''(E_k) = \tilde{E}_k$. By minimality of $(\tilde{\Delta}_k)$, we may conclude that

$$(\tilde{\Delta}_k) \subseteq f''((\Delta_k)).$$

If $g$ is the inverse of $f$, by the same argument we have that

$$(\Delta_k) \subseteq g''((\tilde{\Delta}_k)).$$

Hence $f''((\Delta_k)) = (\tilde{\Delta}_k)$. Because the isomorphic image of a principal ideal is principal, we actually have that

$$(f''(\Delta_k)) = (\tilde{\Delta}_k)$$

Thus $f''(\Delta_k)$ and $\tilde{\Delta}_k$ must be unit multiples of each other. 

The theorem above implies directly that the knot polynomials are an invariant of presentations,
which is good enough to be an invariant of knots:

Nonequal Alexander polynomials $\Rightarrow$ Nonisomorphic knot groups $\Rightarrow$ Different knot types

Due to the fact that Alexander polynomials are only unique up to units it may be difficult to tell when two polynomials are associates in $\mathbb{Z}[t]$. So often these polynomials are normalized to have only positive exponents and positive constant term.

**Example 1.**

![Image of unknot](attachment:image1.png)

As one might expect, the unknot has a very simple knot group, $\langle x \rangle$. This group is isomorphic to the presentation $\langle x : 1 \rangle$ (so we have a $1 \times 1$ matrix). The Alexander matrix is $[0]$. In this case, for $k \geq 1$, $E_k$ is defined to be $\mathbb{Z}[t]$, so in fact we must have that $\Delta(t) = 1$.

**Example 2.**

![Image of trefoil knot](attachment:image2.png)
We have already determined that the trefoil’s knot group has the presentation

\[ \langle x, y, z : yzy^{-1} = x, zzx^{-1} = y, xyx^{-1} = z \rangle. \]

The relators associated to the relations are \( yzy^{-1}x^{-1}, zzx^{-1}y^{-1}, \) and \( xyx^{-1}z^{-1} \). In a previous example it was shown that \( \frac{\partial}{\partial x}(xyx^{-1}z^{-1}) = 1 - xyx^{-1} \). We take the Fox partial derivatives of each relator with respect to each generator and arrange them in the matrix

\[
\begin{bmatrix}
1 - xyx^{-1} & z & -yzy^{-1}x^{-1} \\
x & -zxx^{-1}y^{-1} & 1 - yzy^{-1} \\
-xyx^{-1}z^{-1} & 1 - zxx^{-1} & y
\end{bmatrix}.
\]

After applying the abelianizer, this becomes

\[
\begin{bmatrix}
1 - t & t & -1 \\
t & -1 & 1 - t \\
-1 & 1 - t & t
\end{bmatrix}.
\]

Finally, we cut out one row and column and take the determinant. This yields our normalized Alexander polynomial:

\[ \Delta(t) = t^2 - t + 1. \]

It is clear that \( \Delta(t) \) is not an associate of the constant polynomial, and so this gives us even more evidence that the trefoil and unknot are different knots.
CHAPTER 4: $T_p$ COLORINGS OF KNOTS

4.1 MOTIVATION

We have explored several concepts of knot theory and group theory in depth. The remainder of this paper focuses on developing a knot coloring by a particular group. We examine the following exercise from Justin Roberts’ Knots Knots [8]:

Compute the number of labellings of the trefoil knot by 3-cycles from $S_4$, the permutation group on four letters.

Recall that cycle types are preserved under conjugation – this exercise asks for the number of homomorphisms from $S_4$ to $\pi(K)$, where $K$ is the trefoil knot, within the conjugacy class of 3-cycles.

As in many areas of mathematics, there exists a brute force calculation and a more elegant solution. The problem is quite a basic exercise in solving group-theoretic equations, but underneath lies some interesting theory that warrants further exploration.

\[
\langle x, y, z : yzy^{-1} = x, zz^{-1} = y, xyx^{-1} = z \rangle
\]
We have become quite familiar with the trefoil knot and its presentation via the Wirtinger Presentation. Now, the conjugacy class of 3-cycles from $S_4$ is composed of the following permutations:

\[
\begin{align*}
(1, 2, 3) & \quad (1, 3, 2) \\
(1, 2, 4) & \quad (1, 4, 2) \\
(1, 3, 4) & \quad (1, 4, 3) \\
(2, 3, 4) & \quad (2, 4, 3)
\end{align*}
\]

Naturally, there are the 8 trivial colorings – simply choose $x$, $y$, and $z$ to be labelled by the same permutation. The next step is to assume that $x$ is labelled by a particular permutation. For example, let $x = (1, 2, 3)$. We must then check which triples satisfy the conjugation relations in $\pi(K)$. The third element, $z$, will be determined by our choice for $y$ (for indeed, this element’s label is determined by the other relations being satisfied). Thus we check each of the remaining 7 permutations in the conjugacy class and see if any contradictions arise.

I shall omit the choices which lead to inconsistent labellings. After choosing $x$ to be labelled by $(1, 2, 3)$, we get (the first line serves to calculate $z$ and the remaining two illustrate that the
relations are satisfied consistently):

\[ y = (1, 3, 4) \implies z = (1, 4, 3)(1, 2, 3)(1, 3, 4) = (1, 4, 2) \]
\[ (1, 2, 4)(1, 3, 4)(1, 4, 2) = (1, 2, 3) \]
\[ (1, 3, 2)(1, 4, 2)(1, 2, 3) = (1, 3, 4) \]

\[ y = (1, 4, 2) \implies z = (1, 2, 4)(1, 2, 3)(1, 4, 2) = (2, 4, 3) \]
\[ (2, 3, 4)(1, 4, 2)(2, 4, 3) = (1, 2, 3) \]
\[ (1, 3, 2)(2, 4, 3)(1, 2, 3) = (1, 4, 2) \]

\[ y = (2, 4, 3) \implies z = (2, 3, 4)(1, 2, 3)(2, 4, 3) = (1, 3, 4) \]
\[ (1, 4, 3)(2, 4, 3)(1, 3, 4) = (1, 2, 3) \]
\[ (1, 3, 2)(1, 3, 4)(1, 2, 3) = (2, 4, 3) \]

Hence the triples \(((1, 2, 3), (1, 3, 4), (1, 4, 2)), ((1, 2, 3), (1, 4, 2), (2, 4, 3))\) and \(((1, 2, 3), (2, 4, 3), (1, 3, 4))\) offer the only consistent labellings with \(x\) fixed as \((1, 2, 3)\). Determining the remaining possible labellings proceeds similarly: fix an \(x\), choose a \(y\), check that the labelling is consistent.

To spare the reader some tedious calculations, I will make an observation: there are 3 consistent, non-trivial labellings after fixing the element \(x\). Hence there are the 8 trivial colorings along with \(3 \cdot 8\) non-trivial ones to yield a total of 32 colorings. The solution is believable but not very enlightening or interesting. I mentioned earlier that a more elegant solution to this problem exists. The existence of this alternate solution is not obvious, but lead to the inspiration for this research paper. As we shall see, it is closely related to the topics covered in the earlier chapters.
4.2 THE GROUP $T_p$

Here we introduce a group that is possibly unfamiliar to the reader.

**Definition 4.1.** Define $T_p$, where $p$ is any prime, to be the group given by the presentation

$$\langle a, b : a^3 = 1, b^3 = 1, (ab)^3 = 1, (ab^2)^p = 1 \rangle.$$  

The group is actually finite, and can be shown to be isomorphic to the group $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes C_3$, where $C_3$ represents the cyclic group of order 3 (our convention here is that $\mathbb{Z}_p$ is an additive group whereas $C_3$ is multiplicative). The homomorphism determining the action of elements of $C_3$ on $\mathbb{Z}_p \times \mathbb{Z}_p$ is defined as follows. Let $C_3 = \langle \varepsilon \rangle$. Define

$$\varphi(\varepsilon) : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p \times \mathbb{Z}_p$$

as

$$\varphi(\varepsilon)(x, y) = (-x - y, x).$$

Note that the automorphism $\varphi(\varepsilon)$ is clearly of order three. The convention from here on out will be to let elements of $T_p$ be of the form $((x, y), \varepsilon^i)$, where $x, y \in \mathbb{Z}_p$. This imposes the following multiplication structure on $T_p$:

$$((x, y), \varepsilon^i)((v, w), \varepsilon^j) = ((x, y) + (\varphi(\varepsilon))^i(v, w), \varepsilon^{i+j}).$$

The multiplication gives rise to the useful calculation that $((x, y), \varepsilon)^{-1} = ((-y, x+y), \varepsilon^2)$. At first glance the connection between $T_p$ and the question posed in the previous section does not seem apparent. However, the connection becomes readily visible once the following fact is established:

**Lemma 4.2.** When $p = 2$, $T_p \cong A_4$, the alternating group on four letters.

**Proof.** Consider the 3-cycles $\alpha = (1, 3, 2)$ and $\beta = (2, 3, 4)$. Clearly, $\alpha^3 = \beta^3 = (1)$. Further, observe that $\alpha \beta = (1, 3, 2)(2, 3, 4) = (1, 3, 4)$, so that $(\alpha \beta)^3 = (1)$. Finally, note that $\alpha \beta^2$ is the permutation $(1, 3)(2, 4)$, so that $(\alpha \beta^2)^2 = (1)$. Hence there is a unique homomorphism from $T_2$ to $A_4$ mapping $a$ to $\alpha$ and $b$ to $\beta$. This homomorphism is clearly surjective and injective, hence an
Because $T_2$ is in the same isomorphism class as $A_4$ they exhibit the same conjugacy class structure. In other words, $T_2$ has four different conjugacy classes of sizes 1, 3, 4, and 4. It is the conjugacy classes of $T_p$ that are of particular interest, because if our aim (as the chapter title suggests) is to color knots with the group $T_p$, then we examine labellings within a particular conjugacy class of $T_p$.

If our convention is to exhibit elements of $T_p$ as having the form $((x,y), \varepsilon^i)$, then it may come as no surprise that the conjugacy classes of $T_p$ are determined by which powers of $\varepsilon$ appear in the element. For instance, suppose that we consider the elements $((x,y), \varepsilon)$. If we conjugate by elements with the same power of $\varepsilon$ we get that

$$(v, w, \varepsilon)^{-1}((x, y), \varepsilon)((v, w), \varepsilon) = ((-w, v + w), \varepsilon^2)((x, y), \varepsilon)((v, w), \varepsilon) = ((v - w + y, v + 2w - x - y), \varepsilon).$$

We see that the above element is of the form $((x,y), \varepsilon)$. This happens similarly if you conjugate by elements with $\varepsilon^2$ or $\varepsilon^0$. This illustrates that the power of $\varepsilon$ is constant in a particular conjugacy class. Additionally, one can show that the element $((0,0), \varepsilon)$ is conjugate to $((x,y), \varepsilon)$ for any $(x,y) \in \mathbb{Z}_p \times \mathbb{Z}_p$ when $p \neq 3$. By the above calculation conjugation by $((v,w), \varepsilon)$ will yield the element $((v - w, v + 2w), \varepsilon)$. We can write any pair $(x,y)$ by choosing $w$ such that $3w \equiv y - x$ modulo $p$ (this is why we require $p \neq 3$). In particular this implies $C = \{((x,y), \varepsilon) : (x,y) \in \mathbb{Z}_p \times \mathbb{Z}_p \}$ is a true conjugacy class when $p \neq 3$.

### 4.3 $T_p$ COLORINGS

Recall that the invariant $\lambda(K, G, C)$ is the number of homomorphisms from the knot group of $K$ to a particular conjugacy class $C$ of $G$. In light of our new group’s definition, $T_p$ will be the group of interest, with the conjugacy class $C = \{((x,y), \varepsilon) : (x,y) \in \mathbb{Z}_p \times \mathbb{Z}_p \}$. For ease of notation, I will only be labelling arcs of a knot with ordered pairs rather than triples – the choice of $C$ lets
one assume the final coordinate is \( \varepsilon \). It will be useful to determine exactly what conjugation does to a particular element, given a positive or negative crossing.

Recall that we must have \( z = y^{-1}xy \) for the positive crossing (left) and \( z = yxy^{-1} \) for the negative crossing (right). Let \( y = ((y_1, y_2), \varepsilon) \) and \( x = ((x_1, x_2), \varepsilon) \). For the positive crossing, we get that

\[
z = y^{-1}xy \\
= ((-y_2, y_1 + y_2), \varepsilon^2)((x_1, x_2), \varepsilon)((y_1, y_2), \varepsilon) \\
= ((y_1 - y_2 + x_2, y_1 + 2y_2 - x_1 - x_2), \varepsilon).
\]

For a negative crossing the calculation becomes

\[
z = yxy^{-1} \\
= ((y_1, y_2), \varepsilon)((x_1, x_2), \varepsilon)((-y_2, y_1 + y_2), \varepsilon^2) \\
= ((2y_1 + y_2 - x_1 - x_2, y_2 - y_1 + x_1), \varepsilon).
\]

Upon determination of \( z \) the calculation reduces to a system of \( 2n \) equations with \( 2n \) variables, where \( n \) is the number of arcs in the knot. As with the previous \( p \) colorings, we carry out solving the system modulo \( p \).
Example 1.

Let $p = 2$. We return to the trefoil knot. Label each arc with a pair $(x_1, x_2)$, $(x_3, x_4)$ or $(x_5, x_6)$.

To satisfy the conjugation relations, we get the system

\begin{align*}
    x_5 &= x_3 - x_4 + x_2 \\
    x_6 &= -x_1 - x_2 + x_3 + 2x_4 \\
    x_1 &= x_3 + x_6 + x_4 \\
    x_2 &= x_5 + 2x_6 - x_3 - x_4 \\
    x_3 &= x_1 + x_2 + x_6 \\
    x_4 &= x_1 + 2x_2 - x_5 + x_6
\end{align*}

The system is extremely symmetric. Modulo $p$, after a few elementary row operations the matrix representing this system becomes

\[
\begin{bmatrix}
    1 & 1 & 1 & 0 & 0 & 1 \\
    0 & 1 & 1 & 1 & 1 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

The nullity of the matrix is 4, hence there are $2^4 = 16$ $T_2$ colorings (from the perspective of the conjugacy class $C$).
Notice that we could have easily described a $T_2$ coloring from the perspective of the conjugacy class containing elements with $\varepsilon$ power 2; we would discover that the number of colorings from that conjugacy class is also 16. We return to the motivating question:

*Compute the number of labellings of the trefoil knot by 3-cycles from $S_4$, the permutation group on four letters.*

The 3-cycles from $S_4$ are even permutations and hence contained in the alternating subgroup $A_4$. In $A_4$ there are two conjugacy classes of size 4 containing the 3-cycles. Determining the number of $T_2$ colorings from $C$ answers half of the question – computing the number of $T_2$ colorings from the other conjugacy class answers the second half. Hence there are exactly $16 + 16 = 32$ labellings by 3-cycles of $S_4$.

**Example 2.**

Let $p=2$.

Let $K$ be the figure-eight knot with the labelling shown above. After picking an orientation we get
a system of 8 equations with 8 variables:

\[
\begin{align*}
x_3 &= 2x_1 + x_2 - x_7 - x_8 \\
x_4 &= x_2 - x_1 + x_7 \\
x_5 &= x_1 - x_2 + x_8 \\
x_6 &= x_1 + 2x_2 - x_7 - x_8 \\
x_1 &= x_5 - x_6 + x_4 \\
x_2 &= x_5 + 2x_6 - x_3 - x_4 \\
x_7 &= x_3 - x_4 + x_6 \\
x_8 &= x_3 + 2x_4 - x_5 - x_6 \\
\end{align*}
\]

The matrix of this system has nullity of 4, and so there are a total of 16 \(T_2\) colorings. This particular invariant would not distinguish the figure-eight knot and trefoil knot, however, this result provides a distinction from the \(p\)-colorings: the figure-eight knot is not tricolorable but it is \(T_2\) colorable as mentioned. This distinguishes the two invariants!

### 4.4 \(T_p\) AND THE ALEXANDER POLYNOMIAL

As a conclusion, we examine a surprising connection between the \(T_p\) coloring invariant and the Alexander polynomial. Consider a positive crossing which has relation \(z = y^{-1}xy\). An earlier calculation showed that, to satisfy relations in \(T_p\), we need that

\[
(z_1, z_2) = (y_1 - y_2 + x_2, y_1 + 2y_2 - x_1 - x_2).
\]

If we consider these as column vectors we get the matrix equation

\[
\begin{bmatrix}
    z_1 \\
    z_2
\end{bmatrix} =
\begin{bmatrix}
    1 & -1 \\
    1 & 2
\end{bmatrix}
\begin{bmatrix}
    y_1 \\
    y_2
\end{bmatrix} +
\begin{bmatrix}
    0 & 1 \\
    -1 & -1
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix}.
\]
Define

\[ T = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}. \]

Then, the equation becomes

\[ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = (I - T) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \]

This can be used as an easier way to compute the relation at each crossing. Now, consider the trefoil knot. The matrix that decided upon its \( T_p \) colorability for some \( p \) prime, was

\[ \begin{bmatrix} -1 & 0 & 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & -1 & 1 & 2 \\ 1 & -1 & -1 & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 & -1 & -1 \\ 0 & 1 & 1 & -1 & -1 & 0 \\ -1 & -1 & 1 & 2 & 0 & -1 \end{bmatrix}. \]

If we “block” the matrix with \( T \) from before, it becomes

\[ \begin{bmatrix} -I & T & I - T \\ I - T & -I & T \\ T & I - T & -I \end{bmatrix}. \]

Recall that in calculating the Alexander polynomial we deleted a row and column and then took the determinant of the matrix that remained. We wish to do something similar here. It is not an identical process because we cannot simply delete a row and column and take the usual determinant. Instead, we will have to delete a *blocked* row and *blocked* column. If we consider the entries
of the matrix in a ring $k[T]$, then we may do an analogous determinant calculation:

$$\begin{bmatrix} -I & T \\ I - T & -I \end{bmatrix} \rightarrow I - T(I - T)$$

$$= I - T + T^2,$$

which is very similar to the Alexander polynomial of the trefoil, $\Delta(t) = t^2 - t + 1$. If we now consider the negative crossings, we get

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= (I - T^{-1}) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + T^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Indeed, nonzero entries in the Alexander matrix are all either $t$, $1 - t$, $t^{-1}$, $1 - t^{-1}$, or 1, thanks to the Fox derivatives. These are of course very similar to the entries in this blocked $T_p$ matrix. This is very similar to a result in [10].

We return, for the moment, to the Alexander polynomial $\Delta(t)$:

**Lemma 4.3.** Let $K$ be a knot and let $\Delta_K(t)$ be its Alexander polynomial. Then $p | \Delta_K(-1)$ if and only if the knot is $p$-colorable.

**Proof.** Suppose that $p | \Delta_K(-1)$. Let $A(t)$ be the Alexander matrix for $K$. Further, let $C^+$ and $C^-$ represent positive and negative crossings, respectively. Each crossing will correspond to a row in the Alexander matrix, and the row will contain each of the following exactly once:

<table>
<thead>
<tr>
<th>$C^+$</th>
<th>$C^-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$t$</td>
<td>$t^{-1}$</td>
</tr>
<tr>
<td>$1 - t$</td>
<td>$1 - t^{-1}$</td>
</tr>
</tbody>
</table>

All of the remaining entries in the row are 0. If we evaluate each of these expressions at $t = -1$, we get entries $-1$ and 2. In other words, $A(-1)$ is exactly the $p$-colorability matrix. Because
\( p|\Delta_K(-1) \), we have that its rank is strictly less than \( n - 1 \) and hence \( K \) is \( p \)-colorable. Now, suppose that \( K \) is \( p \)-colorable. Because of the above discussion, \( p \)-colorability of \( K \) implies that the determinant of the \((n - 1) \times (n - 1)\) principal minor is 0 modulo \( p \). In particular, we must have that \( p|\Delta_K(-1) \).

The lemma above offers an easy way to compute the \( p \)-colorability of a knot if one happens to know its Alexander polynomial. This is particularly useful for classes of knots in which the form of the Alexander polynomial is known for the entire class (such as the torus knots, mentioned below). We wish for a similar result regarding \( T_p \)-colorability.

First, notice that

\[
\begin{bmatrix}
  0 & 1 \\
-1 & -1
\end{bmatrix} = U \begin{bmatrix}
  \varepsilon & 0 \\
0 & \varepsilon^2
\end{bmatrix} U^{-1}
\]

for some invertible matrix \( U \), where \( \varepsilon \) is a primitive third root of unity. Now, let \( k \) be a field. If we consider a matrix \( A(t) \in M_n(k[t]) \), there is a natural homomorphism from \( M_n(k[t]) \) to \( M_{2n}(k) \), in the following way: let \( T \) be a \( 2 \times 2 \) matrix with entries in \( k \). If we map \( t \) to \( T \) and 1 to \( I \) (in other words, we map \( A(t) \) to \( A(T) \)), where \( I \) is the \( 2 \times 2 \) identity matrix, we get a blocked matrix with entries in \( R \). Further, now suppose that \( T \) is diagonalizable with

\[
T = U \begin{bmatrix}
a & 0 \\
0 & b
\end{bmatrix} U^{-1}
\]

for some invertible matrices \( U \) and \( U^{-1} \). This means that

\[
A(T) = \begin{bmatrix}
U & U & & \\
& U & & \\
& & \ddots & \\
& & & U
\end{bmatrix} \begin{bmatrix}
U^{-1} & \\
& U^{-1} \\
& & \ddots
\end{bmatrix} \begin{bmatrix}
U^{-1} \\
& U^{-1} \\
& & \ddots
\end{bmatrix}
\]
where $M$ is a matrix with blocks of the form

\[
\begin{bmatrix}
a & 0 \\
0 & b
\end{bmatrix}.
\]

**Lemma 4.4.** If we let $\det A(t) = \Delta(t)$, then $\det A(T) = (\Delta(a))(\Delta(b))$.

**Proof.** We can turn $A(T)$ into a blocked matrix of the following form:

\[
\begin{bmatrix}
A(a) & 0 \\
0 & A(b)
\end{bmatrix}.
\]

To see this, we perform a sequence of row swaps followed by column swaps. Let $A(t) = A = [A_{ij}]$ and $A(T) = A' = [A'_{ij}]$. If $A_{ij} = \sum_{k=1}^{n} \lambda_k t^k$, then after mapping $A$ to $A'$, $A_{ij}$ becomes a diagonal matrix of the form

\[
\begin{bmatrix}
\sum_{k=1}^{n} \lambda_k a^k & 0 \\
0 & \sum_{k=1}^{n} \lambda_k b^k
\end{bmatrix}.
\]

Then, observe that $A'_{ij}$ is an expression involving $a$ if $i$ and $j$ are odd, while it is an expression involving $b$ if $i$ and $j$ are both even, and 0 otherwise. Perform row swaps so that the first $n$ rows are all the original odd rows and the remaining $n$ rows are even. Then, do column swaps in the same order. In other words,
Doing these swaps results in the blocked matrix

\[
\begin{bmatrix}
A(a) & 0 \\
0 & A(b)
\end{bmatrix}.
\]

As determinants are invariant under row and column operations, we have that \(\det A(T) = \Delta(a)\Delta(b)\).

This allows us to return to our matrix \(T\) as it pertains to the group \(T_p\):

**Proposition 4.5.** Let \(K\) be a knot and let \(\Delta_K(t)\) be its Alexander polynomial. Then \(K\) is \(T_p\)-colorable if and only if \(p|\Delta_K(\varepsilon)\Delta_K(\varepsilon^2)\).

**Proof.** First suppose that \(K\) is \(T_p\)-colorable; then the determinant of the \((n-2) \times (n-2)\) principal minor of the \(T_p\)-colorability matrix is 0 modulo \(p\). Let \(A(t)\) be the Alexander matrix of \(K\). We have that \(A(T)\) is exactly the \(T_p\)-colorability matrix of \(K\) However, by Lemma 4.4 the determinant of an \((n-2) \times (n-2)\) minor is exactly \(\Delta_K(\varepsilon)\Delta_K(\varepsilon^2)\). This completes the first implication. Now, suppose that \(p|\Delta_K(\varepsilon)\Delta_K(\varepsilon^2)\). Then we have that the determinant of the \((n-2) \times (n-2)\) of the matrix

\[
\begin{bmatrix}
A(\varepsilon) & 0 \\
0 & A(\varepsilon^2)
\end{bmatrix}
\]
is 0 modulo \( p \). However, this determinant is preserved by similarity, and hence by Lemma 4.4 this is exactly the determinant of the \((n - 2) \times (n - 2)\) minor of \(A(T)\), the \(T_p\)-colorability matrix.

**Example 1.** We know the trefoil knot’s Alexander polynomial to be \( t^2 - t + 1 \). If we evaluate the expression for \(T_p\)-colorability we get:

\[
(\Delta(\varepsilon))(\Delta(\varepsilon^2)) = (\varepsilon^2 - \varepsilon + 1)(\varepsilon^4 - \varepsilon^2 + 1)
\]

\[
= (\varepsilon^2 - \varepsilon + 1)(\varepsilon - \varepsilon^2 + 1)
\]

\[
= \varepsilon^3 - \varepsilon^2 + \varepsilon - \varepsilon^4 + \varepsilon^3 - \varepsilon^2 + \varepsilon^2 - \varepsilon + 1
\]

\[
= 3 - \varepsilon^2 - \varepsilon
\]

\[
= 4 - (1 + \varepsilon + \varepsilon^2).
\]

In \( \mathbb{C} \), we have the condition that \( 1 + \varepsilon + \varepsilon^2 = 0 \). Hence 2 divides \((\Delta(\varepsilon))(\Delta(\varepsilon^2))\) and we have that the trefoil is \(T_2\)-colorable (which we already knew). However, this calculation also yields another bit of information: the trefoil knot is not \(T_p\)-colorable for any \( p \neq 2 \).

**Example 2.** Suppose that \( K = 6_2 \), one of the knots with six crossings. According to [2], the Alexander polynomial of \( K \) is

\[
\Delta(t) = t^4 - 3t^3 + 3t^2 - 3t + 1.
\]
If we proceed as in example 1, we get

\[
(\Delta(\varepsilon))(\Delta(\varepsilon^2)) = (\varepsilon^4 - 3\varepsilon^3 + 3\varepsilon^2 - 3\varepsilon + 1)(\varepsilon^8 - 3\varepsilon^6 + 3\varepsilon^4 - 3\varepsilon^2 + 1)
\]
\[
= (\varepsilon - 3 + 3\varepsilon^2 - 3\varepsilon + 1)(\varepsilon^2 - 3 + 3\varepsilon - 3\varepsilon^2 + 1)
\]
\[
= (3\varepsilon^2 - 2\varepsilon - 2)(-2\varepsilon^2 + 3\varepsilon - 2)
\]
\[
= 17 - 8\varepsilon^2 - 8\varepsilon
\]
\[
= 25 - 8\varepsilon^2 - 8\varepsilon
\]
\[
= 25.
\]

Thus $6_2$ is only colorable with $T_p$ when $p = 5$.

**Example 3.** Consider $K = 10_{124}$. By [2] this has Alexander polynomial

\[
\Delta_K(t) = t^8 - t^7 + t^5 - t^4 + t^3 - t + 1.
\]

If we evaluate the polynomial at $(-1)$ we get 1. Thus, $K$ is not $p$-colorable for any prime $p$. However, $\Delta_K(\varepsilon)\Delta_K(\varepsilon^2) = 25$ and hence $K$ is $T_p$ colorable for $p = 5$.

**Example 4.** The torus knots $T_{m,n}$ can be interpreted as the closure of a braid with $m$ strands twisted $n$ times. In [2] it is shown that the Alexander polynomial of the $T_{m,n}$ torus knot is

\[
\Delta_{T_{m,n}}(t) = \frac{(t^{mn} - 1)(t - 1)}{(t^m - 1)(t^n - 1)}.
\]

In addition, if $m = 2$, we get that the Alexander polynomial is

\[
\Delta_{T_{2,n}}(t) = \frac{t^n + 1}{t + 1} = \sum_{i=1}^{n} (-1)^{i-1}t^{i-1}.
\]
Suppose now that \( n > 3 \) and \( 3 \nmid n \). If \( n \) is 2 modulo 3, then we get that

\[
\Delta_{T_2,n}(\varepsilon) = \frac{\varepsilon^{3k+2} + 1}{\varepsilon + 1} = \frac{\varepsilon^2 + 1}{\varepsilon + 1}.
\]

Similarly, \( \Delta_{T_2,n}(\varepsilon^2) = \frac{\varepsilon + 1}{\varepsilon^2 + 1} \). If \( n \) is 1 modulo 3, we get

\[
\Delta_{T_2,n}(\varepsilon) = \frac{\varepsilon^{3k+1} + 1}{\varepsilon + 1} = \frac{\varepsilon + 1}{\varepsilon + 1} = 1.
\]

Likewise, \( \Delta_{T_2,n}(\varepsilon^2) = 1 \). In either case, the product \( \Delta_{T_2,n}(\varepsilon)\Delta_{T_2,n}(\varepsilon^2) = 1 \). Thus we have that \( T_{2,n} \) (where \( n > 3 \) is not divisible by 3) is not \( T_p \)-colorable for any prime \( p \).

One may refer to [2] for a very comprehensive table that contains the Alexander polynomials for all knots up to 10 crossings. The approach of testing \( T_p \)-colorability in this manner introduces a weakness – it will not distinguish knots which have the same Alexander polynomial. However, if we know two knots are \( T_p \)-colorable, we may also calculate the number of \( T_p \) colorings using the first method. The coupling of these two techniques yield a knot invariant potentially even stronger than the Alexander polynomial.
BIBLIOGRAPHY


