HYPERCYCLIC EXTENSIONS OF BOUNDED LINEAR OPERATORS

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ABSTRACT

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If $X$ is a topological vector space and $T : X \to X$ is a continuous linear operator, then $T$ is said to be hypercyclic when there is a vector $x$ in $X$ such that the set $\{T^n x : n \in \mathbb{N}\}$ is dense in $X$. If a hypercyclic operator has a dense set of periodic points it is said to be chaotic.

This paper is divided into five chapters. In the first chapter we introduce the hypercyclicity phenomenon. In the second chapter we study the range of a hypercyclic operator and we find hypercyclic vectors outside the range. We also study arithmetic means of hypercyclic operators and their convergence. The main result of this chapter is that for a chaotic operator it is possible to approximate its periodic points by a sequence of arithmetic means of the first iterates of the orbit of a hypercyclic vector. More precisely, if $z$ is a periodic point of multiplicity $\alpha$, that is $T^\alpha z = z$ then there exists a hypercyclic vector of $T$ such that $A_{n,\alpha} x = \frac{1}{n} (z + T^\alpha z + \cdots + T^{(n-1)\alpha} z)$ converges to the periodic point $z$. In the third chapter we show that for any given operator $T : M \to M$ on a closed subspace $M$ of a Hilbert space $H$ with infinite codimension it has an extension $A : H \to H$ that is chaotic. We conclude the chapter by observing that the traditional Rota model for operator theory can be put in the hypercyclicity setting. In the fourth chapter, we show that if $T$ is an operator on a closed subspace $M$ of a Hilbert space $H$, and $P : H \to M$ is the orthogonal projection onto $M$, then there is an operator $A : H \to H$ such that $PAP = T, PA^*P = T^*$ and both $A, A^*$ are hypercyclic.
I dedicate this dissertation to Georgia.
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>CHAPTER 1: Introduction</td>
<td>1</td>
</tr>
<tr>
<td>CHAPTER 2: Hypercyclic means</td>
<td>8</td>
</tr>
<tr>
<td>CHAPTER 3: Extensions to chaotic operators</td>
<td>21</td>
</tr>
<tr>
<td>CHAPTER 4: Compressions of hypercyclic operators</td>
<td>38</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>49</td>
</tr>
</tbody>
</table>
CHAPTER 1

Introduction

The invariant subspace problem is one of the most puzzling unsolved problems in operator theory. The problem is to determine whether every bounded linear operator $T : H \to H$ defined on an infinite dimensional Hilbert space $H$ has a non-trivial invariant subspace, that is a closed linear subspace $M$ of $H$ which is different from $0$ and $H$ such that $TM \subset M$.

The question has been open for over eighty years. There are, though, special cases for which the question has been answered in the case that the underlying space $H$ is a Banach space. In the case where the vector space is finite dimensional over the complex field, every operator has an eigenvector, so it has an invariant subspace. Also the spectral theorem proves that all normal operators (an operator that commutes with its adjoint is said to be normal) admit invariant subspaces. It is easy to see that if the vector space $H$ is not separable then the question can be answered in the affirmative; say, take any nonzero vector $x$, the closed linear subspace generated by $\{x, Tx, T^2x, \ldots\}$ is invariant under $T$ and it is different from $X$. Recall that a topological vector space is a vector space with a topology with respect to which the two vector space operations, namely vector addition and scalar multiplications are continuous. A Hilbert space is a topological vector space whose topology is induced by an inner product, which makes the space complete. A topological space is separable if it has a countable dense subset.
Definition 1. Let \( T : X \to X \) be an operator on a topological vector space \( X \) and let \( x \) be a vector in \( X \). Then we define the orbit of \( x \) under \( T \) to be the set \( \{ x, Tx, T^2x, T^3x, \ldots \} \) and we denote it by \( \text{orb}(x, T) \).

Suppose \( X \) is nonseparable. The orbit of any non-zero vector \( x \in X \) is an invariant subset of \( T \). It is clear that \( \overline{\text{span}(\text{orb}(x, T))} \), the closed subspace generated by the orbit of \( x \) under \( T \), is an invariant subspace of \( T \), in fact the smallest closed subspace containing the vector \( x \). Since the orbit is countable, the closed subspace generated by the orbit is separable and thus a nontrivial invariant subspace of \( X \).

Definition 2. Let \( T : X \to X \) be a continuous linear operator on a topological vector space \( X \) and \( x \) be a vector in \( X \). We call the vector \( x \) a cyclic vector for \( T \) if the linear span of its orbit under \( T \) is dense in \( X \); that is, if

\[
\overline{\text{span}(\text{orb}(x, T))} = X.
\]

The operator \( T \) is called a cyclic operator if it has a cyclic vector.

Definition 3. Let \( T : X \to X \) be a continuous linear operator on a topological vector space \( X \) over a scalar field \( \mathbb{F} \) and \( x \) be a vector in \( X \). We call the vector \( x \) a supercyclic vector for \( T \) if the set of the scalar multiples of the vectors in the orbit is dense in \( X \); that is, if

\[
\bigcup_{\lambda \in \mathbb{F}} \lambda \cdot \text{orb}(x, T) = X.
\]

The operator \( T \) is called a supercyclic operator if it has a supercyclic vector.

Definition 4. Let \( T : X \to X \) be a continuous linear operator on a topological vector space \( X \) and \( x \) a vector in \( X \). We call the vector \( x \) a hypercyclic vector for \( T \) if its orbit under \( T \) is dense in \( X \); that is, if

\[
\overline{\text{orb}(x, T)} = X.
\]

The operator \( T \) is called a hypercyclic operator if it has a hypercyclic vector.
Notice that if all vectors of a linear space $X$ are cyclic then $X$ does not have an invariant nontrivial subspace and the invariant subspace problem can be rephrased in terms of cyclic vectors. The answer to the invariant subspace problem is negative if and only if there is an operator for which all the nonzero vectors are cyclic. In the same way, a hypercyclic operator having all vectors hypercyclic if and only if it has no nontrivial invariant closed subset. This suggests that hypercyclicity theory is strongly connected to, if not generated by, the interest in solving the invariant subspace problem.

**Definition 5.** Let $T$ be a linear operator on a topological vector space $X$. The operator $T$ is called a chaotic operator if it is hypercyclic and it possesses a dense set of periodic vectors.

Chaotic operators suggest a connection to ergodic theory and an evolution of hypercyclicity into the theory of linear dynamical systems. Unfortunately we do not have enough space in the dissertation to give more details but we recommend [11] for further reading on the subject of chaotic dynamical systems.

Historically the term “hypercyclic” was first used in its modern acception by Beauzamy [4]. There are, though, examples of *avant la lettre* hypercyclic operators. In 1929 Birkhoff [7] essentially showed the hypercyclicity of the translation operator on the Fréchet space of entire functions $H(\mathbb{C}) = \{ f : \mathbb{C} \to \mathbb{C} : f \text{ is analytic} \}$. More precisely he showed the existence of an entire function $f$ with the property that the set $\{ f(z), f(z+1), f(z+2), f(z+3), \ldots \}$ is dense in the space of entire functions $H(\mathbb{C})$ endowed with the topology of uniform convergence on compact sets. The translation operator $T : H(\mathbb{C}) \to H(\mathbb{C})$, defined by

$$ T(f)(z) = f(z + 1) $$

was shown by Birkhoff to have a dense orbit and thus to be hypercyclic. More than twenty years later, in the same area of complex analysis MacLane [21] proved that there exists an entire function $f$ such that the set $\{ f, f', f'', \ldots \}$ is dense in $H(\mathbb{C})$. This proves that the
differentiation operator $D : H(\mathbb{C}) \to H(\mathbb{C})$ defined by

$$D(f)(z) = f'(z)$$

is hypercyclic.

In his detailed and almost exhaustive survey on universal families and hypercyclic operators [15] Grosse-Erdmann credited Rolewicz to be the initiator of the general theory of hypercyclicity. In 1969 Rolewicz [23] produced a very simple example of a hypercyclic operator, the backward shift on the Banach space $\ell^p$ where $p \geq 1$. If $B : \ell^p \to \ell^p$ is the backward shift defined by

$$B(a_0, a_1, a_2, \ldots) = (a_1, a_2, a_3, \ldots),$$

then the operator $\lambda B$ is hypercyclic whenever $|\lambda| > 1$. The magnitude of the scalar $\lambda$ needs to be strictly greater than one because the backward shift is a contraction and so any vector’s orbit under $B$ would be bounded and hence not hypercyclic.

One of the most useful results in hypercyclicity is the so called Kitai’s Hypercyclicity Criterion due to Kitai [17] and it was rediscovered a few years later in much greater generality by Gethner and Shapiro [13]. Later Bés and Peris [6] weakened the sufficient condition of the criterion and this is the form in which we choose to state it.

**Theorem 6. The Hypercyclicity Criterion.** Let $X$ be a separable $F$-space and $T$ be a continuous linear operator on $X$. Suppose that there are dense subsets $X_0$ and $Y_0$ of $X$, an increasing sequence $(n_k)$ of positive integers and (possibly nonlinear and discontinuous) mappings $S_{n_k} : Y_0 \to X$ such that

1. for every $x \in X_0$, $T^{n_k}x \to 0$

2. for every $y \in Y_0$, $S_{n_k}y \to 0$

3. for every $y \in Y_0$, $(T^{n_k} \circ S_{n_k})y \to y$.

Then the operator $T$ is hypercyclic.
We should note that this condition is only sufficient. M. De La Rosa and C. Read [10] showed that there exist hypercyclic Banach space operators which do not satisfy the Hypercyclicity Criterion. Recently, F. Bayart and E. Matheron [3] proved that such operators can be constructed on a large class of Banach spaces, including $c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$. Another amazing result in hypercyclicity due to Read [22] states the existence of a hypercyclic operator on the Banach space $\ell^1$ for which every nonzero vector is hypercyclic. This means Read’s result solved the invariant subset problem for $\ell^1$ in the negative.

It is very interesting to note that hypercyclicity appears only in the setting of infinite dimensional spaces. Rolewicz [23] showed that no linear operator on a finite dimensional space is hypercyclic. One would think that bizarre objects such as hypercyclic operators must be very rare. It turns out that this is actually false. In fact, Ansari [2] and Luis Bernal-González [5] showed that every separable infinite dimensional Banach space carries a hypercyclic operator. On the other hand chaotic operators are not as common. Bonet, Martinez-Gimenez and Peris showed that there exists a Banach space which admits no chaotic operator [8].

The set of hypercyclic vectors for a hypercyclic operator is residual and every vector is the sum of two hypercyclic vectors [15]. We remind here that a residual set is the complement of a meager set, or equivalently, the intersection of countably many sets with with dense interiors. Also, an $F$-space is a topological vector space $X$ whose topology is defined by a translation invariant metric $d$ and such that $(X,d)$ is complete.

**Proposition 7.** Let $X$ be a separable $F$-space.

1. If $T$ is a hypercyclic operator on $X$, then every vector in $X$ is the sum of two hypercyclic vectors for $T$.
2. If $T_n (n \in \mathbb{N})$ are hypercyclic operators on $X$, then there exists a vector in $X$ that is hypercyclic for each $T_n$. In fact, the set of common hypercyclic vectors is a dense $G_\delta$ subset of $X$ and thus residual.
In operator theory it is always interesting to find bounded linear extensions $A : X \to X$ that preserve or improve the properties of the original operator $T : M \to M$, where $M$ is a closed linear subspace of $X$. The fact that it is always possible to extend a bounded linear functional $\lambda : M \to \mathbb{C}$ to $f : X \to \mathbb{C}$ with $\|f\| = \|\lambda\|$ was proven by Hahn in 1927; see [19]. This is the well known Hahn-Banach theorem. A subnormal operator is defined to be an operator with a normal extension. However, a subnormal operator can never be hypercyclic; see [17]. This also means that none of the techniques in the theory of subnormal operators can be used in hypercyclicity. In this dissertation we show that any bounded linear operator $T : M \to M$ defined on a separable, infinite dimensional Banach space $M$ can be extended to a chaotic operator $A : X \to X$, where $M$ is a closed subspace of $X$. As we pointed out every Banach space admits a hypercyclic operator, but not every Banach space admits a chaotic operator; see [2], [8]. Our result shows that, in fact, every operator can be extended to be chaotic.

Sophie Grivaux [14] proved a similar result which we state here. Let $X$ be any Banach space and $T$ is a bounded operator on $X$. An extension $(\tilde{X}, \tilde{T})$ of the pair $(X, T)$ consists of a Banach space $\tilde{X}$ in which $X$ embeds isometrically through an isometry $i$ and a bounded operator $\tilde{T}$ on $\tilde{X}$ such that $\tilde{T} \circ i = i \circ T$ on $X$. When $X$ is separable, it is additionally required that $\tilde{X}$ be separable. We say that $(\tilde{X}, \tilde{T})$ is a topologically transitive extension of $(X, T)$ when $\tilde{T}$ is topologically transitive on $\tilde{X}$, i.e. for every pair $(\tilde{U}, \tilde{V})$ of non-empty open subsets of $\tilde{X}$ there exists an integer $n$ such that $\tilde{T}^n(\tilde{U}) \cap \tilde{V}$ is nonempty. Grivaux shows that any such pair $(X, T)$ admits a topologically transitive extension $(\tilde{X}, \tilde{T})$, and that when $H$ is a Hilbert space, $(H, T)$ admits a topologically transitive extension $(\tilde{H}, \tilde{T})$ where $\tilde{H}$ is also a Hilbert space. Grivaux shows that these extensions are indeed chaotic. In the setting of a Hilbert space Feldman [12] showed that there is a continuous linear operator $T$ on a separable Hilbert space such that if $f : X \to X$ is a continuous function on a compact metric space $X$, then there exists an invariant closed set $K$ for $T$ such that $T|K$ is topologically conjugate to $f$. 
In this dissertation we extend an operator $T$ defined on a subspace $M$ of infinite codimension in a Hilbert space $H$ to a chaotic operator defined on $H$ that has a hypercyclic subspace, that is an infinite dimensional closed subspace consisting entirely of hypercyclic vectors, except the zero vector. In the same setting of a Hilbert space $H$ with a subspace $M$ having infinite codimension, we show that for every operator $T$ defined on $M$, there exist an operator $A$ defined on $H$ that is hypercyclic with a hypercyclic adjoint $A^*$, so that the compression of $A$ to $M$ is $T$. That is, $PAP = T$, where $P : H \to H$ is the orthogonal projection onto $M$.

In this dissertation, we also make connection to a very famous theorem in ergodic theory which is known as von Neumann’s mean ergodic theorem [18].

**Theorem 8.** If $T$ is a contraction in a Hilbert space $H$, and $P$ the projection on $F = \{ g \in H : Tg = g \}$, then $A_n f = n^{-1} \sum_{k=1}^{n-1} T^k f$ converges in norm to $P f$ for $f \in H$ as $n$ goes to $\infty$.

In this dissertation we replace the contraction in the above theorem by a hypercyclic operator which has a dense orbit; also instead of fixed points we look at periodic points. We also look at the range of a hypercyclic operator and we construct a hypercyclic operator such that not all of its hypercyclic vectors are in its range.
CHAPTER 2

Hypercyclic means

An operator $T$ is called Cesaro hypercyclic provided there exists a vector $x \in X$ so that its Cesaro orbit $\{\frac{1}{n}(I + T + T^2 + \cdots + T^{n-1})x : n = 1, 2, \ldots\}$ under $T$ is dense in $X$. F. León-Saavedra showed that $T$ is Cesaro hypercyclic if and only if there exists a vector $y \in X$ such that the sequence $\{\frac{T^n}{n}y : n = 1, 2, \ldots\}$ is dense in $X$, see [20]. In this chapter, however, we obtain results motivated by von Neumann’s mean ergodic theorem by studying arithmetic means $\frac{1}{n}(I + T + T^2 + \cdots + T^{n-1})$ of hypercyclic operators and their convergence. We prefer to call these arithmetic means of hypercyclic operators, simply, hypercyclic means. The main result of this chapter establishes an approximation of the periodic points of a chaotic operator with hypercyclic vectors by the way of hypercyclic means.

Definition 9. Let $T$ be a bounded linear operator defined on a separable, infinite dimensional Banach space. If there is a vector $x$ whose orbit $\{T^n x : n \geq 0\}$ is dense in $X$, then we call $T$ a hypercyclic operator and $x$ a hypercyclic vector for $T$.

We can rephrase the definition in terms of invariant subsets for $T$; that is a vector $x$ is a hypercyclic vector if and only if the smallest closed invariant subset containing $x$ is the whole space $X$. We show two easy ways in which a vector $x$ fails to be hypercyclic.

Remark 10. Let $X$ be a separable infinite dimensional Banach space and $T : X \to X$ be a bounded linear operator with $\|T\| > 1$. If there is a vector $x$ in $X$ and an increasing sequence
(n_k) of positive integers such that \( \|T^{n_k}x\| \leq \|T\|^{n_k-n_{k+1}} \) then the vector \( x \) is not hypercyclic.

**Proof.** Let \( m \in \mathbb{N} \) be arbitrarily fixed. Then there is \( k \) such that \( n_k \leq m < n_{k+1} \) and thus

\[
\|T^m x\| = \|T^{m-n_k}T^{n_k}x\| \\
\leq \|T\|^{m-n_k}\|T^{n_k}x\| \\
< \|T\|^{n_{k+1}-n_k}\|T^{n_k}x\| \\
\leq 1.
\]

So the orbit of \( x \) under \( T \) is bounded and hence \( x \) cannot be hypercyclic.

\[\square\]

**Remark 11.** Let \( X \) be a separable Banach space, and \( T : X \to X \) be a bounded linear operator with \( \|T\| \geq 1 \). Let \( x \) be vector in \( X \) and \( (n_k) \) be an increasing sequence of positive integers such that \( M := \sup\{n_{k+1} - n_k : k \in \mathbb{N}\} < \infty \) and \( (T^{n_k}x) \) converges to the zero vector. Then the vector \( x \) is not hypercyclic.

**Proof.** Without loss of generality we may assume that \( \|T^{n_k}x\| < 1 \) for every \( k \). Let \( m \in \mathbb{N} \) be arbitrarily fixed. Then there is an integer \( k \) such that \( n_k \leq m < n_{k+1} \) and we have

\[
\|T^m x\| = \|T^{m-n_k}T^{n_k}x\| \\
\leq \|T\|^{m-n_k}\|T^{n_k}x\| \\
\leq \|T\|^M\|T^{n_k}x\| \\
< \|T\|^M,
\]

and thus the orbit of \( x \) is bounded.

\[\square\]

The range of a hypercyclic operator is a dense subset of the whole Banach space \( X \); in fact the orbit of every hypercyclic vector is dense. When a hypercyclic operator \( T \) is not onto it is simple to see that it has a hypercyclic vector \( x \) that is not in the range of \( T \). To see that let \( HC(T) \) be the set of all hypercyclic vectors for \( T \). By Proposition
\[ HC(T) + HC(T) = X. \] Hence if we assume \( HC(T) \subseteq \text{ran}T \) then it would follow that \( X = HC(T) + HC(T) \subseteq \text{ran}T + \text{ran}T = \text{ran}T \), and thus \( T \) is onto. In the following, we provide a construction of such a hypercyclic vector that is not in the range of \( T \), to give an illustration of how such a vector can arise.

**Proposition 12.** There is a hypercyclic operator \( T \) such that not all of its hypercyclic vectors are in its range.

**Proof.** We construct a hypercyclic unilateral weighted backward shift \( T \) defined on \( \ell^2(\mathbb{N}) \) together with a hypercyclic vector \( x \) which is not in the range of \( T \). Let \( D = \{d_n : n \in \mathbb{N}\} \) be a countable dense subset of \( \ell^2 \) formed by vectors with finitely many nonzero entries. For each \( d_n \), we write \( d_n = \sum_{k=0}^{l_n} \hat{d}_n(k)e_k \), where \( \hat{d}_n(l_n) \neq 0 \) and \( \{e_k : k = 0, 1, 2, \ldots\} \) is the canonical basis in \( \ell^2 \). A linear operator \( T : \ell^2 \to \ell^2 \) is a unilateral backward weighted shift with the sequence of bounded weights \( (w_n) \) if \( Te_n = w_ne_{n-1} \) for all \( n \geq 1 \) and \( Te_0 = 0 \). For this we define inductively a sequence \( (m_k) \) of increasing positive integers and a sequence \( (a_k) \) of real numbers. Let \( m_1 \) be a positive integer such that \( \|d_1\| < \frac{1}{2} \) and let \( a_1 = \frac{|\hat{d}_1(l_1)|}{2^{m_1}} > 0 \). Inductively we define \( m_{k+1} \) for \( k \geq 1 \), such that

\[
\frac{\|d_{k+1}\|}{a_1 \cdots a_k 2^{m_{k+1}-k-m_k}} < \frac{1}{2^{k+1}}, \tag{2.1}
\]

and

\[
m_{k+1} > 2(m_k + l_k + k) \tag{2.2}
\]

and define

\[
a_{k+1} = \frac{|\hat{d}_{k+1}(l_{k+1})|}{a_1 \cdots a_k 2^{m_{k+1}-k}} > 0.
\]

Note that each \( a_k \) satisfies \( a_k < 1 \) by \([2.1]\). Now, using the sequence \( (a_k) \) we define a bounded sequence of weights \( (w_j) \) by \( w_j = a_k \) if \( j = m_k + l_k + 1 \) and \( w_j = 2 \) otherwise. Let \( T \) be the unilateral backward weighted shift defined by these weights, and let \( S : \ell^2 \to \ell^2 \) be the possibly unbounded unilateral forward shift defined by the sequence of weights \( (w_k^{-1}) \); that
is, $Se_k = w_{k+1}^{-1}e_{k+1}$, whenever $k \geq 0$.

We define a vector in $\ell^2$, using the dense subset $D$ and the sequences $(a_k)$ and $(n_k)$ by

$$x = \sum_{k=1}^{\infty} \sum_{j=0}^{l_k} \frac{\hat{d}_k(j)}{a_1 \cdots a_{k-1} 2^{m_k-(k-1)}} e_{m_k+j}$$

Note that $m_k + l_k < m_{k+1}$ for each $k$ by [2.2] and hence by [2.1]

$$\|x\|^2 \leq \sum_{k=1}^{\infty} \left[ \frac{\|d_k\|}{a_1 \cdots a_{k-1} 2^{m_k-(k-1)}} \right]^2 \leq \sum_{k=1}^{\infty} \left( \frac{1}{2^k} \right)^2 < \infty,$$

and so $x \in \ell^2$. By definition, for each $k$, we have that

$$T^{m_k} e_{m_k+j} = (w_{m_k+j} \cdots w_{j+2} w_{j+1}) e_j.$$

Using the definition of the weight sequence $(w_j)$, one can check that if $0 \leq j \leq l_k$, then

$$w_{m_k+j} \cdots w_{j+2} w_{j+1} = a_1 \cdots a_{k-1} 2^{m_k-(k-1)}.$$

It follows that

$$T^{m_k} \left( \sum_{j=0}^{l_k} \frac{\hat{d}_k(j)}{a_1 \cdots a_{k-1} 2^{m_k-(k+1)}} e_{m_k+j} \right) = \sum_{j=0}^{l_k} \hat{d}_k(j) e_j = d_k.$$

Thus using [2.1] we get,

$$\|T^{m_k} x - d_k\|^2 \leq \sum_{i=k+1}^{\infty} \left( \frac{\|d_i\|}{a_1 \cdots a_{i-1} 2^{m_i-m_k-(i-1)}} \right)^2 \leq \sum_{i=k+1}^{\infty} \left( \frac{1}{2^i} \right)^2 \leq \frac{1}{3} \cdot \frac{1}{4^k} < \frac{1}{4^k} \quad (2.3)$$
To show that the orbit of \( x \) under \( T \) is dense in \( \ell^2 \), let \( y \) be a vector in \( \ell^2 \) and let \( \epsilon > 0 \). There is a positive integer \( N \) such that if \( k > N \) then

\[
\frac{1}{4^k} < \frac{\epsilon}{2},
\]

and there is \( k > N \) such that

\[
\|d_k - y\| < \frac{\epsilon}{2}.
\]

The triangle inequality together with (2.3), (2.4) and (2.5) prove that the vector \( x \) has a dense orbit and thus hypercyclic. We will show that \( x \) is not in the range of \( T \). Notice that for each \( k \)

\[
S(e_{m_k+l_k}) = w_{m_k+l_k+1}^{-1}e_{m_k+l_k+1} = a_k^{-1}e_{m_k+l_k+1}
\]

and hence

\[
\|S(x)\|^2 \geq \sum_{k=1}^{\infty} \left[ a_k^{-1} \frac{|d_k(l_k)|}{a_1 \cdots a_{k-1}2^{m_k-(k-1)}} \right]^2 = \sum_{k=1}^{\infty} 1 = \infty.
\]

Suppose that there were a vector \( y \) in \( \ell^2 \) such that \( Ty = x \) it would follow that \( y = Sx \) and the norm of \( y \) would be infinite. We have constructed a vector \( x \) that is hypercyclic for \( T \) but it is not in its range.

We define the arithmetic mean of a hypercyclic operator in the manner suggested by the ergodic means.

**Definition 13.** Let \( X \) be a Banach space and let \( T \in B(X) \), we define the \( n \)-th arithmetic mean of \( T \) to be the operator \( A_n = \frac{1}{n}(I + T + T^2 + \cdots + T^{n-1}) \) and for a hypercyclic vector \( x \) we call \( A_n x \) the hypercyclic mean, when the hypercyclic operator \( T \) is understood.

We state the simplest version of the Hypercyclicity Criterion due to Kitai [17] that we are using in the rest of this chapter. This version is definitely simpler than the one we stated in Theorem [6], but it already provides all we need in this chapter.

**Theorem 14.** Kitai’s Hypercyclicity Criterion. Let \( T \) be a bounded linear operator on
a separable infinite dimensional Banach space $X$. Suppose $D$ is a countable dense subset of $X$ and $S$ is a right inverse for $T$ such that:

1. $T^n \to 0$ pointwise on $D$
2. $S^n \to 0$ pointwise on $D$

then the operator $T$ is hypercyclic.

The next proposition is inspired by von Neumann’s ergodic theorem.

**Theorem 15.** If $T$ is a contraction in a Hilbert space $H$, and $P$ the projection on $F = \{ g \in H : Tg = g \}$, then $A_n f = n^{-1} \sum_{k=1}^{n-1} T^k f$ converges in norm to $Pf$ for $f \in H$ as $n$ goes to $\infty$.

We replace the contraction by a hypercyclic operator, which can never be a contraction since its orbit is a dense subset. We prove the result for periodic points, but notice that a fixed point is a particular kind of periodic point.

**Proposition 16.** Let $T$ be a bounded linear operator on a separable infinite dimensional Banach space $X$ that satisfies the Hypercyclicity Criterion and let $\alpha$ be a positive integer. Then there is a hypercyclic vector $x$ in $X$ such that $A_{n,\alpha} x = \frac{1}{n} (x + T^\alpha x + T^{2\alpha} x + \cdots + T^{(n-1)\alpha} x)$ converges to the zero vector in $X$.

**Proof.** Let $\{d_1, d_2, d_3, \cdots, d_n, \cdots \}$ be an enumeration of the dense subset $D$. We define inductively a sequence $(b_n)$ of positive integer multiples of $\alpha$, $b_n = \alpha c_n$, where $c_n$ is an integer. Let $b_1$ be such that $\|S^m d_1\| < \frac{1}{2}$ and $\|T^m d_1\| < \frac{1}{2}$ for all $m \geq b_1$. Let $b_2$ be such that $b_2 > 2b_1$, $\|S^m d_2\| < \frac{1}{2^2}$ and $\|T^m d_1\| < \frac{1}{2}$ and $\|T^m d_2\| < \frac{1}{2^2}$ for all $m \geq b_2$.

Suppose we have constructed $b_1, b_2, \cdots, b_{k-1}$. We define $b_k$ to be large enough such that the following conditions are satisfied:

\[ b_k > 2b_{k-1}, \text{ for all } k > 2, \] (2.6)
\[ \|S^m d_i\| < \frac{1}{2^k}, \text{ for all } m \geq b_k - b_{k-1} \text{ and all } i \leq k + 1, \quad (2.7) \]

\[ \|T^m d_i\| < \frac{1}{2^k}, \text{ for all } m \geq b_k - b_{k-1}, \text{ and all } i \leq k, \quad (2.8) \]

\[ \frac{1}{c_k} \left( \|S^{b_{k-1}} d_i\| + \cdots + \|d_i\| + \|T^a d_i\| + \cdots + \|T^{b_{k-1}} d_i\| \right) < \frac{1}{2^k}, \text{ for all } i \leq k + 1. \quad (2.9) \]

We define a vector \( x \) by
\[ x = \sum_{i=1}^{\infty} S^{b_i} d_i. \]
Observe that \( x \) is well defined since by (2.7), \( \|x\| = \| \sum_{i=1}^{\infty} S^{b_i} d_i \| < \sum_{i=1}^{\infty} \frac{1}{2^i} = 1. \)

To show that the vector \( x \) is hypercyclic let \( y \) be a vector in \( X \) and let \( \epsilon > 0 \). Since \( D \) is a dense subset of \( X \) there is a positive integer \( k \) such that \( \|y - d_k\| < \frac{\epsilon}{2} \) and \( \frac{k}{2^k} < \frac{\epsilon}{2} \). On the other hand

\[ \|T^{b_k} x - d_k\| = \|T^{b_k} \sum_{i=1}^{\infty} S^{b_i} d_i - d_k\| \]
\[ = \| \sum_{i=1}^{\infty} T^{b_k} S^{b_i} d_i - d_k \| \]
\[ = \|T^{b_k - b_1} d_1 + \cdots + T^{b_k - b_{k-1}} d_{k-1} + S^{b_{k+1} - b_k} d_{k+1} + \cdots \|, \quad (2.10) \]

and using the triangle inequality and inequality (2.6) we get

\[ \|T^{b_k} x - d_k\| \leq \| \sum_{i=1}^{k-1} T^{b_k - b_i} d_i \| + \| \sum_{i=k+1}^{\infty} S^{b_i - b_k} d_i \| \]
\[ < (k - 1) \frac{1}{2^k} + \sum_{i=k+1}^{\infty} \frac{1}{2^i} \]
\[ = \frac{k}{2^k}. \quad (2.11) \]
It is clear now that $T^{b_n}x$ approximates the vector $y$ since

$$\|T^{b_n}x - y\| < \|T^{b_n}x - d_k\| + \|d_k - y\| < \epsilon$$

and thus $x$ is a hypercyclic vector.

To show that the hypercyclic mean $A_{n,\alpha}x$ converges to the zero vector in $X$, let $n$ be a positive integer. Then there is a positive integer $k$ such that $b_k \leq (n-1)\alpha < b_{k+1}$ and

$$A_{n,\alpha}x = \frac{1}{n}(x + T^{\alpha}x + T^{2\alpha}x + \ldots + T^{(n-1)\alpha}x)$$

$$= \frac{1}{n} \left( \sum_{i=1}^{\infty} S^{b_i}d_i + T^{\alpha} \sum_{i=1}^{\infty} S^{b_i}d_i + \ldots + T^{(n-1)\alpha} \sum_{i=1}^{\infty} S^{b_i}d_i \right)$$

$$= \frac{1}{n} \sum_{i=1}^{\infty} (S^{b_i}d_i + T^{\alpha} S^{b_i}d_i + \ldots + T^{(n-1)\alpha} S^{b_i}d_i)$$

$$= \frac{1}{n} \sum_{i=1}^{k-1} (S^{b_i}d_i + S^{b_i-\alpha}d_i + \ldots + d_i + T^{\alpha}d_i + \ldots + T^{(n-1)\alpha-b_i}d_i)$$

$$\quad + \frac{1}{n} (S^{b_k}d_k + S^{b_k-\alpha}d_k + \ldots + d_k + T^{\alpha}d_k + \ldots + T^{(n-1)\alpha-b_k}d_k)$$

$$\quad + \frac{1}{n} \sum_{i=k+1}^{\infty} (S^{b_i}d_i + S^{b_i-\alpha}d_i + \ldots + S^{b_i-(n-1)\alpha}d_i).$$

We approximate each of the three summands above separately. We use conditions (2.9)
and (2.8) and the fact that $b_k \geq 2b_{k-1}$ to approximate the first summand:

\[
\frac{1}{n} \left| \sum_{i=1}^{k-1} (S_{b_i} d_i + S_{b_i}^{b_i-\alpha} d_i + \cdots + d_i + T^\alpha d_i + \cdots + T^{b_i+\alpha} d_i + \cdots + T^{(n-1)\alpha-b_i}) d_i \right|
\]

\[
\leq \frac{1}{n} \left| \sum_{i=1}^{k-1} (S_{b_i} d_i + S_{b_k}^{b_k-\alpha} d_i + \cdots + d_i + T^\alpha d_i + \cdots + T^{b_k} d_i) \right|
\]

\[
+ \frac{1}{n} \left| \sum_{i=1}^{k-1} T^{b_k-1+\alpha} d_i + \cdots + T^{(n-1)\alpha-b_k} d_i \right|
\]

\[
< \frac{1}{2^k} + \frac{n}{n2^k}
\]

\[
= \frac{1}{2^k} + \frac{1}{2^k}
\]

\[
= \frac{3}{2^k}.
\]

(2.13)

For the second summand we use the conditions (2.7), (2.8) and (2.9) to get:

\[
\frac{1}{n} \left| S_{b_k} d_k + S_{b_k}^{b_k-\alpha} d_k + \cdots + d_k + \cdots + T^{(n-1)\alpha-b_k} d_k \right|
\]

\[
\leq \frac{1}{n} \left| S_{b_k} d_k + S_{b_k}^{b_k-\alpha} d_k + \cdots + S_{b_k}^{b_k-1+\alpha} d_k \right|
\]

\[
+ \frac{1}{n} \left| S_{b_k-1} d_k + S_{b_k-1}^{b_k-1-\alpha} d_k + \cdots + d_k + T^\alpha d_k + \cdots + T^{b_k-1} d_k \right|
\]

\[
+ \frac{1}{n} \left| T^{b_k-1+\alpha} d_k + T^{b_k-1+2\alpha} d_k + \cdots + T^{(n-1)\alpha-b_k} d_k \right|
\]

\[
< \frac{1}{n} \cdot \frac{1}{2^k} + \frac{1}{n} \cdot \frac{1}{2^k} + \frac{1}{n} \cdot \frac{1}{2^k}
\]

\[
\leq \frac{3}{2^k}.
\]

(2.14)
We use condition (2.7) to approximate the third summand:

\[
\frac{1}{n} \| \sum_{i=k+2}^{\infty} (S_{b_i}d_i + S_{b_{i-1}-\alpha}d_i + \cdots + S_{b_{i-(n-1)}\alpha}d_i) \|
\]

\[
< \frac{1}{n} \cdot n \cdot \sum_{i=k+1}^{\infty} \frac{1}{2^i}
\]

\[
= \frac{1}{2^{k-1}}.
\]

(2.15)

We use these three approximations and (2.12) to show that \( A_{n,\alpha}x \) converges to zero.

\[
\|A_{n,\alpha}x\| < \frac{1}{2^{k-1}} + \frac{3}{2^k} + \frac{1}{2^{k-1}}
\]

\[
= \frac{7}{2^k}.
\]

(2.16)

We let \( n \) go to infinity and thus we let \( k \) go to infinity and from (2.16) we get the desired result \( A_{n,\alpha}x \to \infty \).

Next we give a counterexample, that is an example of a hypercyclic operator, twice the unilateral backward shift, and a hypercyclic vector for which the hypercyclic mean does not converge.

**Example 17.** Let \( T = 2B \) defined on \( \ell^2 \), where \( B \) is the unilateral backward shift. There exists a hypercyclic vector \( x \) in \( \ell^2 \) such that its hypercyclic means \( A_{n,1}x \) do not converge to 0.

**Proof.** Let \( D \) be the set of all sequences in \( \ell^2 \) with finite number of nonzero entries all of which are rational. Since \( D \) is countable, we let \( D = \{x_1, x_2, \ldots\} \) and observe that \( D \) is a dense set of vectors in \( \ell^2 \). We will denote by \( \delta \) the length of a vector \( x \) in \( \ell^2 \); that is, if \( x = (\hat{x}(0), \hat{x}(1), \ldots) \) and \( j \) is the largest integer so that \( \hat{x}(j) \neq 0 \), then the length \( \delta \) of \( x \) is said to be \( \delta = j + 1 \). Choose a sequence of integers \( (a_n) \) such that:

\[
\frac{\|x_n\|}{2^{a_n}} \leq \frac{1}{2^n}, \quad \text{for all } n \in \mathbb{N}.
\]
Let $F : \ell^2 \to \ell^2$ be the unilateral forward shift and $S = \frac{1}{2} F$. Define $b_n$ by $b_n = a_1 + \delta_1 + \cdots + a_{n-1} + \delta_{n-1} + a_n$ and let $x = \sum_{n=1}^{\infty} S^{b_n} x_n$. Then

$$\|x\| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$$

and $x$ is a well defined vector in $\ell^2$. Notice that if $T = 2B$, then $TS$ is the identity and $T^{b_n} x_i = 0$ for all $i \leq n - 1$. Hence

$$\|T^{b_n} x - x_n\| = \|x_n + \sum_{i=n+1}^{\infty} S^{b_i-b_n} x_i - x_n\| \leq \sum_{i=n+1}^{\infty} \frac{\|x_i\|}{2^{a_i}} \leq \sum_{i=n+1}^{\infty} \frac{1}{2^{2^i}} = \frac{1}{2^n}$$

and so the vector $x$ is hypercyclic.

Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a permutation and $(\xi_n)$ and $(\alpha_n)$ be the sequences given by $\xi_n = x_{\sigma(n)}$ and $\alpha_n = a_{\sigma(n)}$. If we define $\beta_n = \alpha_1 + \delta_{\sigma(1)} + \cdots + \alpha_{n-1} + \delta_{\sigma(n-1)} + \alpha_n$ then the vector $\xi = \sum_{n=1}^{\infty} S^{\beta_n} \xi_n$ is still hypercyclic. Indeed since

$$\sum_{n=1}^{\infty} \left( \frac{\|x_{n1}\|}{2^{a_{n1}}} \right)^2 = \sum_{n=1}^{\infty} \left( \frac{\|\xi_n\|}{2^{\alpha_{n1}}} \right)^2$$

the vector $\xi$ is hypercyclic. The sequence $(x_n)$ has a subsequence $(n_k)$ in $\mathbb{N}$ and a subsequence $(x_{n_k})$ with $x_{n_k} = (x^{1}_{n_k}, 0, 0, \cdots)$ and $2^{n_k-1} < \|x_{n_k}\| \leq 2^{n_k}$, and $a_{n_k} = 2n_k$ for all $k$ and also $n_1 \neq 1$. We construct a new sequence $(y_n)$ obtained by permuting the terms of the sequence $(x_n)$. Define $y_1 = x_1$. For all $j \notin \{n_k : k \in \mathbb{N}\}$ define $y_j = x_l$ and $\alpha_j = a_l$, where $l = \min\{i \in \mathbb{N} : x_i \notin \{y_1, \cdots, y_{j-1}\}\}$ and this condition makes our new sequence $(y_j)$ a permutation of our original sequence $(x_i)$. Define $y_{n_j} = x_{n_{k_j}}$, where $k_j \in \{n_i : i \in \mathbb{N}\}$ is such that $a_{n_{k_j}} \geq \alpha_1 + \delta_1 + \cdots + \alpha_{n_{j-1}} + \delta_{n_{j-1}}$, and $|y^1_1 + y^1_2 + \cdots + y^1_{n_j}| \geq |y^1_{n_j}|$ for all $j \in \mathbb{N}$. 
Also define $\alpha_{n_j} = a_{nk_j}$. We get that for any $j$ there is a $k_j$ such that

$$\frac{\|y_{n_j}\|}{\beta_{n_j}} \geq \frac{\|y_{n_j}\|}{2\alpha_{n_j}} = \frac{\|x_{nk_j}\|}{2a_{nk_j}} \geq \frac{2^{nk_j}}{8n_k} > 1, \text{ for all } n_k \geq 6.$$

Now the vector $y$ is hypercyclic in $\ell^2$ and for any $j$ large we have

$$\frac{|<y + \cdots + T^{\beta_{n_j}}y, e_1>|}{\beta_{n_j}} \geq \frac{\|y_{n_j}\|}{\beta_{n_j}} = \frac{\|y_{n_j}\|}{\beta_{n_j}} > 1.$$

So $\frac{\|y + \cdots + T^{\beta_{n_j}}y\|}{\beta_{n_j}}$ cannot converge to 0 and we constructed a hypercyclic vector without $y$ such that its hypercyclic means $A_n y$ do not converge to zero.

\[\square\]

**Proposition 18.** Let $x$ be a vector in a Banach space $X$ and $T \in B(X)$. If $A_{n,\alpha} x$ converges to a vector $a$ in $X$, then $a$ is a periodic point of multiplicity $\alpha$, that is $T^\alpha a = a$.

**Proof.** Suppose $A_{n,\alpha}(x)$ converges to $a$, then $T^\alpha(A_{n,\alpha}x)$ converges to $T^\alpha a$. On the other hand we have:

$$T^\alpha(A_{n,\alpha}x) = A_{n,\alpha}T^\alpha x$$

$$= \frac{1}{n}(T^\alpha x + \cdots + T^{n\alpha} x)$$

$$= -\frac{x}{n} + \frac{1}{n}(x + T^\alpha x + \cdots + T^{n\alpha} x)$$

$$= -\frac{x}{n} + \frac{n+1}{n}A_{n+1,\alpha}x$$

$$\rightarrow 0 + 1a$$

$$= a$$

Hence, $T^\alpha a = a$.

\[\square\]

Now we have all the tools necessary to state and prove the main result of this chapter. Chaotic operators satisfy the Hypercyclicity Criterion, see [6]. Here, we impose an extra condition such that they satisfy Kitai’s version of the criterion stated above. We prove that
for any periodic point of a chaotic operator there is hypercyclic vector whose hypercyclic mean converges to this periodic point.

**Theorem 19. Approximation of a periodic point by a hypercyclic vector** Let $X$ be a Banach space, and $T : X \to X$ be a chaotic operator that satisfies the Kitai’s Hypercyclicity Criterion. If $z$ is a periodic point of multiplicity $\alpha$, that is $T^\alpha z = z$ then there exists a hypercyclic vector of $T$ such that $A_{n,\alpha}x$ converges to the periodic point $z$.

**Proof.** First note that $z$ is a fixed point vector for $A_{n,\alpha}$ since

$$A_{n,\alpha}z = \frac{1}{n}(z + T^\alpha z + \cdots + T^{(n-1)\alpha} z) = \frac{n z}{n} = z.$$ 

Also note that by Proposition [16] there exists a hypercyclic vector $y$ such that $A_{n,\alpha}y \to 0$.

We define the vector $x$ by $x = z - y$. Then we have that

$$A_{n,\alpha}x = A_{n,\alpha}(z - y) = A_{n,\alpha}z - A_{n,\alpha}y = z - A_{n,\alpha}y.$$ 

Since $A_{n,\alpha}y$ converges to 0, $A_{n,\alpha}x$ must converge to $z$.

To show that $x$ is a hypercyclic vector for $T$ we first use a result of Ansari [11] that $T^\alpha$ is hypercyclic whenever $T$ is. Let $f$ be a vector in $X$ and let $\epsilon > 0$. Since $z - f \in X$ and since $y$ is hypercyclic there exists $(n_k)$ so that $\|T^{\alpha n_k}y - (z - f)\| < \epsilon$. Now we have that

$$\|T^{\alpha n_k} x - f\| = \|T^{\alpha n_k} (z - y) - f\| = \|T^{\alpha n_k} y - (z - f)\| < \epsilon$$

and we show that we can obtain a periodic point by using a hypercyclic vector.

We have seen some properties of hypercyclic vectors and periodic points of a hypercyclic operator. These two sets of vectors are seemingly to have totally totally different properties, but nonetheless, they can co-exist as interesting properties of an operator; each of these two sets of vectors is dense. **Theorem 19** provided an interesting way of approximating vectors in one set by means of vectors in another.
CHAPTER 3

Extensions to chaotic operators

In this chapter we prove some results concerning extensions of bounded linear operators to hypercyclic operators. First we show that any bounded linear operator can be extended to a chaotic operator. Grivaux proved a somewhat similar result; that is, any bounded linear operator on a Banach space has a topologically transitive extension, see [14]. We start with a uniformly bounded sequence of operators \((T_j)_j\) on a Banach space \(X\). We construct a larger Banach space \(\tilde{X}\) which contains \(X\) as a complemented subspace, say \(\tilde{X} = X \oplus Y\), where \(Y\) is a closed subspace of \(X\) and we denote by \(\tilde{T}_j : \tilde{X} \to \tilde{X}\), the operator obtained by extending \(T_j : X \to X\) to \(\tilde{X}\) by defining it to be identically zero on \(Y\). We show that there exists and an operator \(\tilde{T}\) on \(\tilde{X}\) such that the operator obtained by adding \(\tilde{T}\) to any \(\tilde{T}_j\) is a chaotic operator on \(X\).

**Theorem 20. Simultaneous hypercyclic extension**

Let \(X\) be an infinite dimensional separable Banach space. Given a uniformly bounded sequence \((T_j)_j\) of linear operators in \(B(X)\), there exists an infinite dimensional separable Banach space \(\tilde{X}\) and a bounded linear operator on \(\tilde{T} : \tilde{X} \to \tilde{X}\) such that the following are true:

1. \(X\) is a complemented subspace of \(\tilde{X}\)

2. operators \(\tilde{T} + \tilde{T}_j\) have a common hypercyclic vector
3. \((\tilde{T} + \tilde{T}_j)|X = \tilde{T}_j|X\) for each \(j\)

4. \(\tilde{T} + \tilde{T}_j\) is a chaotic operator for each \(j\).

**Proof.** We define the vector space \(\tilde{X}\) to be the direct sum of infinite many copies of \(X\); that is, we let each \(X_i = X\) and let

\[
\tilde{X} = \bigoplus_{i=0}^{\infty} X_i = \{ x \in \prod_{i=0}^{\infty} X_i : \|x\| = \sum_{i=0}^{\infty} \|\hat{x}(i)\| < \infty \},
\]

where we denote \(x = (\hat{x}(0), \hat{x}(1), \ldots)\) with each \(\hat{x}(i) \in X_i = X\). The vector space \(\tilde{X}\) is a Banach space with the norm given by \(\|x\| = \sum_{i=0}^{\infty} \|\hat{x}(i)\|\), for all \(x \in \tilde{X}\). For simplicity we use the same notation for the two different norms since it is obvious from the context to which one we refer.

Now we define a positive weight \(a\) by

\[
a = 3(\sup(\|T_j\|) + 1)
\]

and the operator \(\tilde{T} : \tilde{X} \to \tilde{X}\) by

\[
\tilde{T}(\hat{x}(0), \hat{x}(1), \hat{x}(2), \hat{x}(3), \ldots) = (a\hat{x}(1), a\hat{x}(2), a\hat{x}(3), \ldots),
\]

for any \(x = (\hat{x}(0), \hat{x}(1), \ldots) \in \tilde{X}\). Clearly \(\tilde{T}\) is bounded with \(\|\tilde{T}\| = a\). From now on we identify \(\tilde{X}/X\) with the closed subspace of \(\tilde{X}\) that consists of vectors \(x\) in \(\tilde{X}\) of the form \(x = (0, \hat{x}(1), \hat{x}(2), \ldots)\). We extend \(T_j : X \to X\) trivially to an operator \(\tilde{T}_j : \tilde{X} \to \tilde{X}\) given by

\[
\tilde{T}_j x = \begin{cases} 
T_j x, & \text{whenever } x \in X \\
0, & \text{whenever } x \in \tilde{X}/X.
\end{cases}
\]

By taking \(X_0\) to be the original given Banach space \(X\), we see condition (3) is satisfied and since \(\tilde{T}|X_0 = 0\) it is clear that the above extension of \(T_j\) on \(X\) to the whole Banach space \(\tilde{X}\)
remains a bounded linear operator on $\tilde{X}$. The operator $\tilde{T}$ on $\tilde{X}$ has a norm $a$ because for any $x = (\hat{x}(0), \hat{x}(1), \ldots)$ in $\tilde{X}$, we have

$$
\|\tilde{T}x\| = \|\tilde{T}(\hat{x}(0), \hat{x}(1), \ldots)\|
= \|a\hat{x}(1), a\hat{x}(2), \ldots\|
= a\|\hat{x}(1), \hat{x}(2), \ldots\|
\leq \|x\|,
$$

furthermore for any $x = (0, \hat{x}(1), \hat{x}(2), \ldots)$ in $\tilde{X}/X$

$$
\|\tilde{T}x\| = \|\tilde{T}(0, \hat{x}(1), \hat{x}(2), \ldots)\|
= \|a\hat{x}(1), a\hat{x}(2), \ldots\|
= a\|\hat{x}(1), \hat{x}(2), \ldots\|
= \|x\|.
$$

Thus $\tilde{T} + \tilde{T}_j$ is also a bounded linear operator for each $j$.

We remark here that, we can have a better uniform estimate for the norm of $\|\tilde{T} + \tilde{T}_j\|$. We claim that $\|\tilde{T} + \tilde{T}_j\| = a$ for each $j$. To prove that, we let $x = (\hat{x}(0), \hat{x}(1), \hat{x}(2), \ldots)$ be a vector in $\tilde{X}$, then

$$
\|\tilde{T} + \tilde{T}_j\| = \|(T_j\hat{x}(0) + a\hat{x}(1), a\hat{x}(2), a\hat{x}(3))\|
= \|T_j\hat{x}(0) + a\hat{x}(1)\| + \sum_{j=2}^{\infty} a\|\hat{x}(j)\|.
$$
Using the triangle inequality we get

\[
\|\tilde{T} + \tilde{T}_j\| \leq \|T_j\| \|\hat{x}(0)\| + a \|\hat{x}(1)\| + a \sum_{j=2}^{\infty} \|\hat{x}(j)\| \\
\leq \|T_j\| \|\hat{x}(0)\| + a \|\hat{x}(1)\| + \sum_{j=2}^{\infty} \|\hat{x}(j)\| \\
\leq a \|\hat{x}(0)\| + a \|\hat{x}(1)\| + \sum_{j=2}^{\infty} a \|\hat{x}(j)\| \\
\leq a \sum_{j=0}^{\infty} \|x\|.
\]

Thus \(\|\tilde{T} + \tilde{T}_j\| \leq a\) and since \(\|(\tilde{T} + \tilde{T}_j)(0, \hat{x}(1), 0, 0, 0, \ldots)\| = a \|\hat{x}(1)\|\) we conclude that \(\|\tilde{T} + \tilde{T}_j\| = a\).

It is easy to see that conditions (1) and (3) are satisfied by the way we constructed the Banach space \(\tilde{X}\) and the operator \(\tilde{T}\). A vector \(x = (\hat{x}(0), \hat{x}(1), \hat{x}(2), \ldots)\) is said to have finite length if there is a nonnegative integer \(K\) such that \(\hat{x}(j) = 0\), whenever \(j \geq K\). The smallest such integer \(K\) is called the length of the vector \(x\).

Fixing a countable dense subset \(\Delta\) of \(X\) that contains the origin, we let \(D\) be the countable subset of \(\tilde{X}\) consisting of vectors \(x = (\hat{x}(0), \hat{x}(1), \hat{x}(2), \ldots)\) of finite length and with each component \(\hat{x}(j) \in \Delta\). We now show that \(D\) is a dense subset of the Banach space \(\tilde{X}\). For that \(D\) is dense let \(x \in X\) and let \(\epsilon > 0\). Then since \(\|x\| < \infty\) there is a positive integer \(k\) such that \(\sum_{i=0}^{\infty} \|\hat{x}(i)\| < \frac{\epsilon}{2}\). For every \(i\) there exists a \(y_i \in \Delta\) such that \(\|y_i - \hat{x}(i)\| < \frac{\epsilon}{2(k+1)}\).

We define a vector \(d = (\bar{d}(0), \bar{d}(1), \bar{d}(2), \ldots)\) in \(D\) by \(\bar{d}(i) = y_i\) for \(i \leq k\) and \(\bar{d}(i) = 0\) for \(i > k\). Thus we found a vector in \(D\) that approximates \(x\), indeed:

\[
\|x - d\| \leq \sum_{i=0}^{k} \|y_i - \hat{x}_i\| + \sum_{i=0}^{\infty} \|\hat{x}(i)\| \\
\leq \frac{(k+1)\epsilon}{2(k+1)} + \frac{\epsilon}{2} = \epsilon
\]

(3.1)
and $D$ is a dense subset of $\tilde{X}$. Let $D = \{d_1, d_2, d_3, \ldots\}$ be an enumeration of $D$.

Consider the countable set of all possible ordered pairs $\{(\tilde{T}_i, d_j) : i, j \geq 1\}$ and let $\{p_1, p_2, p_3, \ldots\}$ be an enumeration of all such ordered pairs with $p_1 = (\tilde{T}_1, d_1)$. Let $l_k$ be the length of the vector in the second entry of the ordered pair $p_k$. For each pair $p_k$ we will construct a vector $x_k$ and a positive integer $n_k$.

First let $n_1 = 1$ and $x_1 = (0, a^{-1}\hat{d}_1(0), a^{-1}\hat{d}_1(1), \ldots, a^{-1}\hat{d}_1(l_1), 0, 0, \ldots)$ and observe that $\tilde{T}x_1 = d_1$.

Suppose we have constructed $x_j$ and $n_j$ for all $j \leq k$. Now we construct $n_{k+1}$ and $x_{k+1}$ corresponding to the pair $p_{k+1} = (\tilde{T}_u, d_v)$, where $d_v$ has length $l_{k+1}$.

Let $n_{k+1}$ be large enough such that:

$$\frac{\||d_v||}{a^{nk+1}} + \frac{||\hat{x}_k(0)||}{3^{nk+1}} + \frac{||\hat{x}_k(1)||}{3^{nk+1-1}} + \cdots + \frac{||\hat{x}_k(n_k + l_k)||}{3^{nk+1-(nk-l_k)}} < (\frac{4}{3})^{-n_k-1} \frac{1}{2k+1},$$

(3.2)

which is possible because $n_{k+1}$ appears on the left-hand side and $n_k$ appears on the right-hand side. Now we construct $x_{k+1}$ such that the following conditions are satisfied:

$$\hat{x}_{k+1}(i) = \hat{x}_k(i), \quad \text{whenever} \quad 0 \leq i < n_{k+1}$$

$$\hat{x}_{k+1}(n_{k+1}) = \frac{1}{a^{nk+1}} \left[ \hat{d}_v(0) - (T_{u}^{nk+1} \hat{x}_k(0) + aT_{u}^{nk+1-1} \hat{x}_k(1) + \cdots + a^{nk+l_k} T_{u}^{nk+1-(nk+l_k)} \hat{x}_k(n_k + l_k)) \right]$$

$$= \frac{1}{a^{nk+1}} \left[ \hat{d}_v(0) - (T_{u}^{nk+1} \hat{x}_{k+1}(0) + aT_{u}^{nk+1} \hat{x}_{k+1}(1) + \cdots + a^{nk+l_k} T_{u}^{nk+1-(nk+l_k)} \hat{x}_{k+1}(n_k + l_k)) \right]$$

(3.3)

$$\hat{x}_{k+1}(n_{k+1} + i) = \frac{\hat{d}_v(i)}{a^{nk+1}}, \quad \text{whenever} \quad 1 \leq i \leq l_{k+1}$$

$$\hat{x}_{k+1}(i) = 0, \quad \text{whenever} \quad i > n_{k+1} + l_{k+1}$$
Now we define
\[ x = x_1 + \sum_{k=1}^{\infty} (x_{k+1} - x_k), \]
and notice that for each \( m \in \mathbb{N} \),
\[ x = x_m + \sum_{k=m}^{\infty} (x_{k+1} - x_k). \]

To show that \( x \) is well-defined we observe that by (3.2):
\[
\|x_{k+1} - x_k\| \leq \frac{1}{a^{n_k+1}} \left( \|\hat{d}_v(0)\| + \|T_u^{n_k+1} \hat{x}(0)\| + a \|T_u^{n_k+1-1} \hat{x}(1)\| + \cdots \right.
\]
\[
+ a^{n_k+l_k} \|T_u^{n_k+1-(n_k+l_k)} \hat{x}(n_k + l_k)\|
\]
\[
+ \|\hat{d}_v(1)\| + \|\hat{d}_v(2)\| + \cdots + \|\hat{d}_v(l_{k+1})\|
\]
\[
\leq \frac{\|d_v\|}{a^{n_k+1}} + \frac{\|T_u\|^{n_k+1} \|\hat{x}_k(0)\|}{3(\|T_u\| + 1)^{n_k+1}} + \frac{\|T_u\|^{n_k+1-1} \|\hat{x}_k(1)\|}{3(\|T_u\| + 1)^{n_k+1-1}} + \cdots
\]
\[
(3.4)
\]
\[
\leq \frac{\|d_v\|}{a^{n_k+1}} + \frac{\|\hat{x}_k(0)\|}{3^{n_k+1}} + \frac{\|\hat{x}_k(1)\|}{3^{n_k+1-1}} + \cdots + \frac{\|\hat{x}_k(n_k + l_k)\|}{3^{n_k+1-(n_k+l_k)}}
\]
\[
< \frac{1}{2k+1} \left( \frac{4}{3} a \right)^{-n_k-1}
\]
and the sum in the definition of \( x \) converges. Hence \( x \) is well-defined.

We now show that for any triple \( (k, i, j) \), or in other words a pair \( p_k = (\tilde{T}_i, d_j) \) we have that
\[ (\tilde{T} + \tilde{T}_i)^{n_k} x_k = d_j \]
Indeed, using the definition of the vector \( x \), the definitions of \( T \) and \( \tilde{T}_i \) and also the fact that \( \hat{x}_k(i) = \hat{x}_{k-1}(i) \), whenever \( 0 \leq i < n_k \), and \( \hat{x}_k(i) = 0 \) whenever \( i > n_k + l_k \), a short
computation yields:

\[
(\hat{T} + \hat{T}_i)^{n_k}x_k = (T_i^{n_k}x_k(0) + aT_i^{n_k-1}x_k(1) + \cdots + a^{n_k-1}T_i\hat{x}_k(n_k - 1) + a^{n_k}\hat{x}_k(n_k),
\]

\[
a^{n_k}\hat{x}_k(n_k + 1), \ldots, a^{n_k}\hat{x}_k(n_k + l_k), 0, 0, \ldots
\]

\[
= (T_i^{n_k}x_k(0) + aT_i^{n_k-1}x_k(1) + \cdots + a^{n_k-1}T_i\hat{x}_k(n_k - 1)
\]

\[
+ a^{n_k-l}(T_i^{n_k-l}\hat{x}_k(n_k-l))\hat{x}_k(n_k-l-1)]d_j(0), \ldots, d_j(l_k))
\]

\[
= d_j.
\]

To show that \(x\) is a common hypercyclic vector we fix \(\hat{T}_{i_0}\). Let \(\epsilon > 0\) and \(y \in \hat{X}\). Then there exists a \(K\) such that for any \(k > K\) we have: \(\frac{1}{2^k} < \frac{\epsilon}{2}\). Now we consider the infinite family of pairs \(p_k = (\hat{T}_{i_0}, d_{j_0})\) with \(k > K\) and choose \(p_k\) such that: \(\|d_{j_0} - y\| < \frac{\epsilon}{2}\).

Since:

\[
\|(\hat{T}_{i_0} + \hat{T})^{n_k}x_k - d_{j_0}\| \leq \|(\hat{T}_{i_0} + \hat{T})^{n_k}x_k - d_{j_0}\| + \|(\hat{T}_{i_0} + \hat{T})^{n_k}\sum_{s=k}^{\infty}(x_{s+1} - x_s)\|
\]

\[
\leq 0 + \|(\hat{T}_{i_0} + \hat{T})^{n_k}\sum_{s=k}^{\infty}(x_{s+1} - x_s)\|
\]

\[
\leq \left(\left(\frac{a}{3} - 1\right) + a\right)^{n_k} \sum_{s=k}^{\infty} \|x_{s+1} - x_s\|
\]

\[
\leq \left(\left(\frac{4}{3}a\right)^{n_k} \sum_{s=k}^{\infty} \frac{1}{2^{s+1}}\left(\frac{4}{3}a\right)^{-n_s-1}\right)
\]

\[
= \frac{1}{2^k}
\]

\[
< \frac{\epsilon}{2}
\]

and

\[
\|(\hat{T}_{i_0} + \hat{T})^{n_k}x_k - y\| \leq \|(\hat{T}_{i_0} + \hat{T})^{n_k}x_k - d_{j_0}\| + \|d_{j_0} - y\| < \epsilon
\]

we get that \(x\) is a common hypercyclic vector.

It remains to be shown that for each \(\hat{T}_j\) there exists a dense subset in \(\hat{X}\) of periodic
vectors. For this we fix \( \tilde{T}_u \) and we construct for each vector \( d \) in \( D \) a sequence of periodic vectors \( x_n \) that converges to \( d \). Let \( d = (\hat{d}(0), \hat{d}(1), \ldots, \hat{d}(l), 0, 0, \ldots) \) be a vector in \( D \) and for each \( n > l \) define \( x_n = (\hat{x}_n(0), \hat{x}_n(1), \ldots) \) by:

\[
\hat{x}_n(0) = \hat{d}(0)
\]

\[
\hat{x}_n(j) = \frac{\hat{d}(j-kn)}{a_{kn}}, \quad \text{whenever } kn + 1 \leq j \leq kn + l \text{ and } k \geq 0
\]

\[
\hat{x}_n(j) = \frac{\hat{d}(0) - \sum_{i=0}^{l} a_{i}T_{u}^{n-i}\hat{d}(i)}{a_{kn}}, \quad \text{whenever } j = kn \text{ and } k \geq 1
\]

\[
\hat{x}_n(j) = 0, \quad \text{otherwise}
\]

Notice that for any positive integer \( m \) and for any vector \( y = (\hat{y}(0), \hat{y}(1), \hat{y}(2), \ldots) \),

\[
(\tilde{T} + \tilde{T}_u)^m y = (T_u^m(\hat{y}(0)) + \cdots + a^{k-1}T_u\hat{y}(k-1) + a^k\hat{y}(k), a^k\hat{y}(k+1), \ldots)
\]

and thus by taking \( m = n \) and \( y = x_n \), and directly from the definition of \( x_n \) we get,

\[
(\tilde{T} + \tilde{T}_u)^n x_n
\]

\[
= (T_u^n\hat{x}_n(0) + \cdots + a^{n-1}T_u\hat{x}_n(n-1) + a^n\hat{x}_n(n) + a^n\hat{x}_n(n+1), \ldots)
\]

\[
= (T_u^n\hat{x}_n(0) + \cdots + a^{n-1}T_u\hat{x}_n(n-1) + a^n\hat{d}(0) - \sum_{i=0}^{l} a^i T_u^{n-i}\hat{d}(i), a^n\hat{d}(n+1-n) + \cdots)
\]

\[
= (\hat{d}(0), a^n, \ldots)
\]

\[
= x_n
\]

for each \( n > l \) and we constructed a sequence \( (x_n) \) of periodic vectors. We show now that the sequence \( (x_n) \) converges to \( d \):
\[ \|x_n - d\| \leq \left\| \sum_{j=1}^{\infty} \frac{\hat{d}(0) - \sum_{i=0}^{t} a^i T_u^{n-i} \hat{d}(i)}{a^{jn}} + \sum_{i=1}^{t} \frac{\hat{d}(i)}{a^{jn}} \right\| \]
\[ \leq \|\hat{d}(0)\| + \sum_{i=0}^{t} a^i \|T_u\|^{n-i} \|\hat{d}(i)\| + \sum_{i=1}^{t} \|\hat{d}(i)\| \]
\[ \to 0 \quad \text{as} \quad n \to \infty. \]

Let \( D = \{d_k : k \geq 1\} \) be an enumeration of the dense subset \( D \) and for each \( k \) let \( (x_n^{(k)})_n \) be a sequence of periodic points converging to \( d_k \), then \( \{x_n^{(k)} : k \geq 1, n > l_k\} \) is a dense subset of periodic points and thus \( (\tilde{T} + \tilde{T}_u) \) is a chaotic operator.

Statement (2) in the conclusion of Theorem 20 follows from statement (4). The reason is that the set of hypercyclic vectors \( HC(A) \) of any hypercyclic operator \( A : Y \to Y \) on a Banach space \( Y \) is always a dense \( G_\delta \) subset of \( Y \); see [17], and hence if \( \tilde{T} + \tilde{T}_j \) is hypercyclic, then certainly the set of common hypercyclic vectors \( \bigcap HC(\tilde{T} + \tilde{T}_j) \) is nonempty by the Baire Category Theorem. However, the constructive proof that we presented for Theorem 20 tells us exactly how such a vector may look like, which potentially leads to further study the properties of those hypercyclic vectors. Next we observe that when we construct \( \tilde{T} \) in the above proof of Theorem 20 its definition \( \tilde{T} \) on \( \tilde{X}/X = \bigoplus_{i=1}^{\infty} X_i \) is a shifting operator which does not depend on the actions of any individual operator \( T_j \) on \( X \). The only part of \( \tilde{T} \) that depends on \( T_j \) is the shifting constant \( a \), which is \( 3(\sup(\|T_j\|) + 1) \). Thus, we reach the following conclusion.

**Corollary 21.** Let \( X \) be an infinite dimensional separable Banach space. For any given \( R > 0 \), there exists an infinite dimensional separable Banach space \( \tilde{X} \) and a bounded linear operator \( \tilde{T} \) on \( \tilde{X} \) such that:

1. \( X \) is a complemented subspace of \( \tilde{X} \)

2. whenever \( A : X \to X \) is a bounded linear operator with \( \|A\| < R \), the operator \( (\tilde{T} + \tilde{A}) \)
is a chaotic operator on $\tilde{X}$

3. $(\tilde{T} + \tilde{A})|X = \tilde{A}|X,$

where $\tilde{A} : \tilde{X} \to \tilde{X}$ is the trivial extension of $A : X \to X.$

Another review of the proof of Theorem 20 also provides the following statement in addition to the three statements in the above Corollary.

**Corollary 22.** Each operator $\tilde{T} + \tilde{A}$ in Corollary 21 satisfies Kitai's Hypercyclicity Criterion; that is, Theorem 14.

**Proof.** We identify $\tilde{X}$ as $\bigoplus_{i=0}^{\infty} X_i = \{(x_0, x_1, \ldots) : x \in X$ and $\sum_{i=0}^{\infty} \|x_i\| < \infty\}$ and the original space $X$ as $X_0$ and so $\tilde{A} : \tilde{X} \to \tilde{X}$ is defined by $\tilde{A}(x_0, x_1, \ldots) = (Ax_0, 0, 0, \ldots).$ Let $S : \tilde{X} \to \tilde{X}$ be defined by

$$S(h_0, h_1, h_2, \ldots) = \frac{1}{a}(0, h_0, h_1, h_2, \ldots).$$

Since $a > 1,$ $\lim_{n \to \infty} S^n(h_0, h_1, h_2, \ldots) = 0.$ Note that

$$\begin{align*}
(\tilde{T} + \tilde{A})S(h_0, h_1, h_2, \ldots) &= (\tilde{T} + \tilde{A})(0, \frac{1}{a} h_0, \frac{1}{a} h_1, \frac{1}{a} h_2, \ldots) \\
&= (h_0, h_1, h_2, \ldots).
\end{align*}$$

Thus $(\tilde{T} + \tilde{A})S$ is the identity on $\tilde{X}.$ It suffices to find a dense subset $D$ of $\tilde{X}$ so that

$$\lim (\tilde{T} + \tilde{A})^n d = \lim S^n d = 0$$

for each $d$ in $D.$ For that we let $\Delta$ be a countable dense subset of $X = X_0 = X_1 = X_2 = \ldots.$ Note that $C = \{(d_0, d_1, d_2, \ldots, d_k, 0, 0, 0, \ldots) \in \tilde{X} : d_i \in \Delta, k \in \mathbb{N}\}$ is a countable dense subset.
of $X$, also that if $n \geq k$ and $x = (d_0, d_1, d_2, \ldots, d_k, 0, 0, 0, \ldots) \in C$ then

$$
(\tilde{T} + \tilde{A})^n(d_0, d_1, d_2, \ldots, d_k, 0, 0, 0, \ldots)
= (A^{n-k}(A^kd_0 + aA^{k-1}d_1 + \cdots + a^kd_k), 0, 0, 0, \ldots).
$$

(3.8)

We now observe that if $n \geq k$ then

$$
S^n(\tilde{T} + \tilde{A})^nx
= S^n(\tilde{T} + \tilde{A})^n(d_0, d_1, d_2, \ldots, d_k, 0, 0, 0, \ldots)
= (0, 0, \ldots, 0, \frac{1}{a^n}A^{n-k}(A^kd_0 + aA^{k-1}d_1 + \cdots + a^kd_k), 0, 0, 0, \ldots)
$$

(3.9)

in which the nonzero expansion appears as a member of $X_{n-1}$. Hence,

$$
(\tilde{T} + \tilde{A})^n(x - S^n(\tilde{T} + \tilde{A})^nx)
= (\tilde{T} + \tilde{A})^nx - (\tilde{T} + \tilde{A})^nS^n(\tilde{T} + \tilde{A})^nx
= (\tilde{T} + \tilde{A})^nx - (\tilde{T} + \tilde{A})^nx
= 0,
$$

(3.10)

and thus for all $m \geq n$, $(\tilde{T} + \tilde{A})^m(x - S^n(\tilde{T} + \tilde{A})^nx) = 0$, and trivially

$$
\lim_{m \to \infty} (\tilde{T} + \tilde{A})^m(x - S^n(\tilde{T} + qa)^nx) = 0.
$$

Note also that for any $x \in C$

$$
\lim_{n \to \infty} \|S^n(\tilde{T} + \tilde{A})^nx\|
= \lim_{n \to \infty} \frac{1}{a^n}\|A^{n-k}(A^kd_0 + aA^{k-1}d_1 + \cdots + a^kd_k)\|
= 0.
$$

(3.11)

Since $C$ is dense in $\tilde{X}$, the set $D = \{x - S^n(\tilde{T} + \tilde{A})^nx : x \in C\}$ is also dense in $\tilde{X}$. Since
we have shown that if \( y \in D \), then \( \lim S^n y = \lim (\tilde{T} + \tilde{A})^n y = 0 \), the hypothesis of Kitai’s Hypercyclicity Criterion is satisfied.

Although we have shown in the proof of Theorem 20 that the operator \( \tilde{T} + \tilde{A} \) has a dense set of periodic points, it is worth a few lines here to reprove it using the ideas and notation in the proof of Corollary 22. Using the same set \( D \) as in the proof of Corollary 22, we let \( x = (d_0, d_1, d_2, \ldots, d_k, 0, 0, 0, \ldots) \) be a vector in \( D \). We are to show that for any \( \epsilon > 0 \), there is a periodic point \( y \) with \( \|x - y\| < \epsilon \). We have shown in the proof of Corollary 22 that

\[
\|S^n(\tilde{T} + \tilde{A})^n(x - S^n(\tilde{T} + \tilde{A})^nx)\| \leq \frac{1}{a^n}\|A\|^{n-k}\|A^kd_0 + \cdots + a^kd_k\| \quad (3.12)
\]

and that

\[
(\tilde{T} + \tilde{A})^n(x - S^n(\tilde{T} + \tilde{A})^nx) = 0. \quad (3.13)
\]

Since \( x \) is fixed and \( a > \|A\| \), there is an integer \( N_1 \), such that if \( n > N_1 \) then

\[
\|S^n(\tilde{T} + \tilde{A})^nx\| < \frac{1}{a^{2n}}\|A\|^{n-k}\|A^kd_0 + \cdots + a^kd_k\| \quad (3.14)
\]

By (3.12) we can further assume that \( N_1 \) is chosen large enough so that \( \|S^n(\tilde{T} + \tilde{A})^nx\| < 1 \).

We can now chose an integer \( N_2 > N_1 \) so that

\[
\sum_{m=1}^{\infty} \frac{1}{a^{mn}} = \frac{1}{a^n - 1} < \frac{\epsilon}{2(\|x\| + 1)} \quad (3.15)
\]

To construct a periodic point \( y \) in the ball \( \text{Ball}(x, \epsilon) \), we let

\[
y = x - S^n(\tilde{T} + \tilde{A})^nx + \sum_{m=1}^{\infty} S^{nm}(x - S^n(\tilde{T} + \tilde{A})^nx),
\]
where $n$ is a fixed integer with $n > N_2$.

It follows that if $n > N_2$, then

$$\|y - x\|$$

$$\leq \|S^n(\tilde{T} + \tilde{A})^nx\| + \sum_{m=1}^{\infty} \frac{1}{a^{mn}}\|x - S^n(\tilde{T} + \tilde{A})^nx\|$$

$$< \frac{\epsilon}{2} + \sum_{m=1}^{\infty} \frac{1}{a^{mn}}(\|x\| + 1)$$

$$< \frac{\epsilon}{2} + (\|x\| + 1)\frac{\epsilon}{2(\|x\| + 1)}$$

$$= \epsilon$$

(3.16)

Thus $y \in \text{Ball}(x, \epsilon)$. To show that $y$ is a periodic point, we observe that

$$(\tilde{T} + \tilde{A})^ny$$

$$= (\tilde{T} + \tilde{A})^nx - (\tilde{T} + \tilde{A})^nS^n(\tilde{T} + \tilde{A})^nx + \sum_{m=1}^{\infty} S^{n(m-1)}(x - S^n(\tilde{T} + \tilde{A})^nx)$$

$$= \sum_{m=0}^{\infty} S^{mn}(x - S^n(\tilde{T} + \tilde{A})^nx)$$

$$= y.$$  

(3.17)

The simultaneous chaotic extension is stated for a countable set of prescribed operators on a Banach space $X$. But if the countable set has precisely one operator, then we can have a very precise statement.

**Corollary 23. Chaotic extension for Banach spaces** Let $X$ be a separable Banach space, and $T$ be a bounded linear operator on $X$. Then there is an an operator $\bar{T}$ on a Banach space $\tilde{X}$ such that the following conditions are satisfied:

(i) $T : \tilde{X} \to \tilde{X}$ is chaotic

(ii) $X$ is a closed subspace of $\tilde{X}$

(iii) $T|X = T$
Proof. We apply the previous theorem to the operator $T$ to find $\tilde{T} : \bigoplus_{i=0}^{\infty} X_i \to \bigoplus_{i=0}^{\infty} X_i$, where each $X_i = X$, then we define the operator $\tilde{T}$ by $\tilde{T} = \tilde{T} + T$, where $Tx = x$ for $x \in X = X_0$ and $Tx = 0$ for $x \in \bigoplus_{i=1}^{\infty} X_i$. Note that the original $X$ in the statement of the corollary is identified with $X_0$. It is now easy to see that the second condition is satisfied by the way we constructed the Banach space $\tilde{X}$. First and third conditions follow directly from third and fourth conditions in the theorem:

$$\tilde{T}|X = \tilde{T}|X_0 = (\tilde{T} + T)|X_0 = T|X_0 = T|X = T$$

and

$$T = \tilde{T} + T \text{ is a chaotic operator.}$$

As a consequence of the above result, if $T$ is a hypercyclic operator on a Banach space $X$ that does not satisfy the Hypercyclicity Criterion, then it has an extension that satisfies the Hypercyclicity Criterion. This is because every chaotic operator satisfies the Hypercyclicity Criterion, as proved by Bes and Peris [6]. For Hilbert spaces we can prove a more general result. We extend an operator defined on $M$, a subspace with infinite codimension of a Hilbert space $H$.

**Corollary 24. Chaotic extension on a Hilbert space** Let $H$ be an infinite dimensional separable Hilbert space and let $M$ be a closed subspace with $\dim H/M = \infty$.

If $T : M \to M$ is a bounded linear operator, then $T$ has a chaotic extension $\tilde{T} : H \to H$.

Proof. Since all separable infinite dimensional Hilbert spaces are isomorphic we may write $H = \bigoplus_{i=0}^{\infty} M_i$, where $M_i = M$ and for each $i$, $M_0$ is identified with the original $M$ given in the statement of the corollary. Using the previous corollary together with the proof of the chaotic extension theorem with $M$ instead of $X$ and $H$ instead of $\tilde{X}$ we get the desired extension $\tilde{T} = \tilde{T} + T$. 

□
Remark 25. We observe that the condition that the codimension of the subspace $M$ is infinite is required. Indeed, suppose that $\dim H/M < \infty$ and $\tilde{T}$ is a hypercyclic extension. Then we consider the linear map $S : H/M \to H/M$ defined by $S[x] = [\tilde{T}x]$. If $h$ is a hypercyclic vector for $\tilde{T}$, then $\{h, \tilde{T}h, \tilde{T}^2h, \ldots\}$ is dense in $H$. If in addition, $\pi : H \to H/M$ is the quotient map defined by $\pi h = [h]$ then $\pi$ is a continuous linear map. Thus $\{\pi h, \pi(\tilde{T}h), \pi(\tilde{T}^2h), \ldots\}$ is dense in $H/M$. That is, $\{[h], S[h], S^2[h], S^3[h], \ldots\}$ is dense in $H/M$, which means the linear map $S : H/M \to H/M$ is hypercyclic. But it is impossible because $\dim H/M < \infty$.

As a corollary of Remark 25, the same idea shows that no hypercyclic operator can have an invariant subspace of finite codimension. To see that, we let $T : H \to H$ be a hypercyclic operator and $M$ be an invariant subspace of $T$. If $h \in H$ is a hypercyclic vector for $T$ and $\pi : H \to H/M$ is the quotient map, then $\pi \{h, Th, T^2h, \ldots\} = \{[h], [Th], [T^2h], \ldots\}$ is dense in $M/H$. But then it would follow that the linear map $A : [x] = [Tx]$ is a hypercyclic map on a finite dimensional space, which is impossible; see [17].

For our next result we use the fact that an operator satisfying the Hypercyclicity Criterion and an extra condition has a hypercyclic subspace. We state here this result (theorem 3.6 in [9]).

**Theorem 26.** Suppose $T : H \to H$ is a bounded linear operator satisfying the Hypercyclicity Criterion. If $H$ has a closed infinite dimensional subspace $K$ such that for every vector $f$ in $K$

$$\lim_{n \to \infty} \|T^n f\| = 0,$$

then $H$ has an infinite dimensional subspace consisting entirely, except for zero, of hypercyclic vectors of $T$.

Using Theorem 20 and our previous results, we obtain the following corollary.

**Corollary 27.** Let $H$ be an infinite dimensional separable Hilbert space and let $M$ be a closed subspace of $H$ with $\dim H/M = \infty$.

If $T : M \to M$ is a bounded linear operator, then $T$ has a chaotic extension with a hypercyclic...
proof. Let $N$ be an infinite dimensional subspace in $M^\perp$ such that $\dim H/(N \oplus M) = \infty$. We define the bounded linear operator $T_1$ on $(N \oplus M)$ by

$$T_1 h = \begin{cases} 0, & \text{whenever } h \in N \\ Th, & \text{whenever } h \in M \end{cases}$$

The previous corollary provides $T_1$ with a chaotic extension $\tilde{T} : H \to H$ which satisfies the Hypercyclicity Criterion [6]. Since $T_1 = 0$ on $N$ the extension $\tilde{T} = 0$ on an infinite dimensional subspace $N$. By the previous theorem $\tilde{T}$ has a hypercyclic subspace.

As a consequence of the previous result, we have the following corollary. Here we let $B(H)$ be the algebra of all bounded linear operators $T : H \to H$ and $B_{ch}(H)$ be the subset of chaotic operators having a hypercyclic subspace.

**Corollary 28. SOT-denseness of $B_{ch}(H)$**

The set $B_{ch}(H)$ is dense in $B(H)$ in the strong operator topology.

**Proof.** Let $\epsilon > 0$ and let $A \in B(H)$. Let $h_1, h_2, \ldots, h_k \in H$ and let $U = \{ T \in B(H) : \| (T - A)h_i \| < \epsilon, \text{ for all } i = 1, \ldots, k \}$ be an SOT-basic open set. Let $M = \text{span}\{h_1, h_2, \ldots, h_k\}$ be an finite dimensional subspace spanned by $h_1, h_2, \ldots, h_k$. So, $M$ is closed. By Corollary 27 there exists a chaotic operator $T \in B(H)$ such that $T = A$ on $M$ and hence $T \in U$. }

The above result was first obtained by Chan [9]. Our chaotic extension result of Corollary 27 is a generalization of his result. One may wonder what will happen if the Hilbert space $H$ in Corollary 22 is a Banach space $X$. In that case, we cannot expect to have a chaotic extension because not every Banach space can support a chaotic operator. This result was proved by Bonet, Martinez-Gimenez and Peris [8]. One may then ask whether we can have a hypercyclic extension to $X$ instead, because Ansari [2] and Bernal-González [5] proved that

subspace; i.e. a closed infinite dimensional subspace consisting entirely of hypercyclic vectors, except the zero vector.
every Banach space supports a hypercyclic operator. However our techniques in Corollary 24 cannot be generalized to a Banach space setting. Clearly Theorem 20 shows us how to extend a bounded linear operator $T$ on a larger Banach space containing $X$. If we are given a Banach space $X$ and a bounded linear operator $T$ defined on a closed subspace $M$ of $X$, then it will be much harder to extend $T$ to be a hypercyclic operator on $X$, analogous to the Hilbert space result in Corollary 24. There is a historical reason for that. The first hypercyclic operator on a Hilbert space was obtained by Rolewicz [23], who also asked whether every Banach space supports a hypercyclic operator. This problem was open for about thirty years, until Ansari [2] and Bernal-González [5] gave a positive answer. Since the Banach space extension analogous to Corollary 24 is asking for a lot more, it potentially is a much harder question.
CHAPTER 4

Compressions of hypercyclic operators

We now turn our attention to the compression of a hypercyclic operator. We show in Theorem 30 below that a compression of a hypercyclic operator onto a closed subspace of infinite codimension can coincide with any given operator on the subspace. Let $H$ be an infinite dimensional separable Hilbert space, and $M$ be a closed subspace of $H$ with infinite codimension. For all $i \in \mathbb{Z}$, let $M_i = M$ and we write $H = \bigoplus_{-\infty}^{\infty} M_i$, where $M_i \perp M_j$ whenever $i \neq j$. We identify the original closed subspace $M$ with $M_0$ in $H$. Let $T : M \to M$ be a bounded linear operator and let $a > 0$. Let $\{e_j, e_{j+1}, e_{j+2}, \ldots\}$ be an orthonormal basis of $M_j$. Hence $e_{ji} \perp e_{mn}$ whenever $(j, i) \neq (m, n)$. Let $S : H \to H$ be the unitary operator given by $Se_{j,i} = e_{j+1,i}$ whenever $i, j \in \mathbb{Z}$, then $S|M_j$ is an isomorphism from $M_j$ to $M_{j+1}$.

We now first need a lemma.

Lemma 29. Let $H$ be an infinite dimensional Hilbert space, and $M$ be a closed subspace of $H$ with infinite codimension. Let $k$ be a positive integer and for each $i$ with $|i| \leq k$ let $w_i$ be a positive weight such that $0 < w_i \leq a$ and let $x_i$ and $y_i$ be vectors in $M_i$. Then for any $\epsilon > 0$ there exists an integer $n \geq 2k + 2$ and for all integers $i$ with $k + 1 \leq |i| \leq n + k$ there exist weights $w_i$ with $0 < w_i \leq a$ and vectors $x_i \in M_i$ (in fact $x_i = 0$ whenever
\(-n - k \leq i \leq -k - 1\) such that if \(D\) is defined by

\[
D_{e_{j,i}} = \begin{cases} 
  w_{j} e_{j+1,i} & \text{whenever } i \in \mathbb{Z} \text{ and } |j| \leq n + k \\
  0 & \text{otherwise}
\end{cases}
\]  

(4.1)

then

1. \(\left\| (D + T)^n \left( \sum_{-k \leq i \leq n-k-1} x_i \right) \right\| < \epsilon\)

2. \(\sum_{k+1 \leq i \leq n+k} \|x_i\|^2 < \epsilon\)

3. \((D + T)^n \sum_{i=n-k}^{n+k} x_i = \sum_{i=-k}^{k} y_i\)

Proof. Suppose \(D\) is given in the form of (4.1) and \(n\) is the integer given in its definition. We first make the following observations. for any integer \(m\) with \(k + 1 \leq m\) and \(m + k + 1 \leq n\) and \(x_m \in M_m\), we have

\[(D + T)^m \left( \sum_{-k \leq i \leq k} x_i + x_m \right) \perp M_j\]

whenever \(j \geq 1\).

In fact,

\[(D + T)^k \left( \sum_{|i| \leq k} x_i \right) \perp M_j\]

whenever \(j \geq 1\).

Hence if we define \(w_i = a\) for all integers \(i\) with \(k + 1 \leq i \leq m\) and if we take \(P_0 : H \to H\) to be the orthogonal projection onto \(M_0\), and if we let

\[
x_m = \frac{-1}{w_1 \cdots w_k w_{k+1} \cdots w_m} S^m P_0(D + T)^m \sum_{|i| \leq k} x_i = \frac{-1}{w_1 \cdots w_k w_{k+1} \cdots w_m} S^m P_0 T^{m-k} (D + T)^k \sum_{|i| \leq k} x_i.
\]  

(4.2)
Then

$$(D + T)^k \left( \sum_{|i| \leq k} x_i \right) \perp M_j \quad \text{whenever } j \geq 0. \quad (4.3)$$

Since $w_{k+1} = \cdots w_m = a > \|T\|$, we can first choose $m$ large enough, before we determine the integer $n$ in the definition of $D$ satisfying $n \geq m + k + 1$ so that the definition of $x_m$ gives

$$\|x_m\| < \frac{\epsilon}{\sqrt{2}} \quad (4.4)$$

Then for large enough integer $n \geq m + k + 1$, if we define $w_{-k-1} = \cdots = w_{n-k} = a^{-1} < 1$, then it will follow from (4.3) that

$$\left\| (D + T)^n \left( \sum_{|i| \leq k} x_i + x_m \right) \right\| < \epsilon.$$

If we define $x_{k+1} = \cdots x_{m-1} = 0$ and $x_m = \cdots = x_{n-k-1} = 0$, then (1) is clearly satisfied.

It remains to show that if $n$ is chosen large enough, then (2) and (3) can be satisfied too.

For that, we first check how (3) can be satisfied whenever $n > 2k$ by choosing appropriate $x_{n-k}, \ldots, x_{n+k}$. Let $w_{m+1} = \cdots = w_{n+k} = a$.

Let

$$x_{n-k} = \frac{1}{w_{-k+1} \cdots w_{n-k}} S^n y_{-k}.$$

Furthermore for every integer $i$ with $-k + 1 \leq i \leq 0$ let

$$x_{n+i} = \frac{1}{w_{i+1} \cdots w_{n+i}} (S^n y_{i-1}).$$

Finally for every integer $i$ with $1 \leq i \leq k$ let

$$x_{n+i} = \frac{1}{w_{i+1} \cdots w_{n+i}} S^{n+i} y_i.$$

To verify that the above definitions of $x_{n-k}, \ldots, x_{n+k}$ satisfy (3), we need to show an inter-
mediate step that whenever \(-k \leq i \leq 0\),

\[
(D + T)^{n+i}x_{n+i} = \frac{1}{w_{i+1} \cdots w_{0}}S^{-i}y_{i} \in M_{0},
\]

which follows easily from the fact that \((D + T)|M_{i} = D|M_{i}\) if \(i \neq 0\).

Now one can verify (3) inductively for the increasing order of the integer \(i\). Since

\[
w_{m+1} = \cdots = w_{n+k} = a > 1
\]

, and \(S\) is an isometry, we can assume that \(n\) is chosen large enough so that the definitions for \(x_{n-k}, \ldots, x_{n+k}\) give

\[
\sum_{n-k \leq i \leq n+k} \|x_{1}\|^{2} < \frac{\epsilon}{2}.
\]

This inequality, along with (4.4) and the fact that \(x_{k+1} = \cdots = x_{m} = x_{m+1} = \cdots = x_{n-k} = 0\), yields (2). The whole proof of the lemma is finished by letting \(x_{-k-1} = \cdots = x_{-n-k} = 0\). \(\square\)

Using this technical lemma we are now ready to prove our main result on compressions.

**Theorem 30.** Let \(H\) be an infinite dimensional Hilbert space, and \(M\) be a closed subspace of \(H\) with infinite codimension. Let \(P : H \to H\) be the orthogonal projection onto \(M\). If \(T : M \to M\) is a bounded linear operator, then there is an operator \(A : H \to H\) such that

1. both \(A\) and \(A^{*}\) are hypercyclic
2. \(PAP|M = T\)
3. \(PA^{*}P|M = T^{*}\)

**Proof.** Since \(M\) has infinite codimension in \(H\), we can use an orthonormal basis argument to write \(H\) as an orthogonal sum \(H = \bigoplus_{j=-\infty}^{\infty} M_{j} = \{(\ldots, m_{-2}, m_{-1}, m_{0}, m_{1}, m_{2}, \ldots) : m_{i} \in M_{i}\}\), where \(M_{0} = M\), and each \(M_{j}\) is isomorphic to \(M_{0}\).
In the rest of the proof, we assume \( \dim M = \infty \) and the same argument works if \( \dim M \) is finite. For that, we let \( \{e_{j1}, e_{j2}, e_{j3}, \ldots\} \) be an orthonormal basis of \( M_j \). Since \( T \) takes \( M \) to \( M \), we can view \( T \) as an operator from \( H \) to \( H \) with \( T|M^\perp = 0 \). This view allows us to define an operator \( A : H \to H \) by \( A = B + T \), where \( B : H \to H \) is a linear map defined by \( Be_{j,i} = w_j e_{j-1,i} \) for all integers \( i, j \) and \( \{w_j : j \in \mathbb{Z}\} \) is a bounded two-sided sequence of positive numbers. Thus \( B \) takes each \( M_j \) to \( M_{j-1} \) and \( B|M_j \) is simply \( w_j \) times a Hilbert space isomorphism. In fact, if \( S : H \to H \) is the unitary operator given by \( Se_{j,i} = e_{j+1,i} \) whenever \( i, j \in \mathbb{Z} \), then \( S|M_j \) is an isomorphism from \( M_j \) to \( M_{j+1} \), and \( B|M_j = w_j S^{-1}|M_j \).

One can easily verify that \( B \) is a bounded linear operator because \( \{w_j\} \) is bounded. In fact, \( B \) is a bilateral weighted backward shift with infinite multiplicity, with \( \|B\| = \sup |w_j| \).

In addition, \( A^* = B^* + T^* \), where \( B^* \) is given by \( B^* e_{j,i} = w_{j+1} e_{j+1,i} \). Thus \( B^* \) takes \( M_j \) to \( M_{j+1} \) and \( B^*|M_j \) is simply \( w_{j+1} \) times a Hilbert space isomorphism. Clearly, the above definition of \( A \) gives \( PAP|M = T \) and we need to choose \( w_j \) so that \( A \) and \( A^* \) are hypercyclic, in order to finish the proof.

Let \( a \) be a positive number strictly greater than \( \max\{1, \|T\|\} \), say \( a = 2 + \|T\| \) and let \( v_1, v_2, v_3, \ldots \) be an enumeration of all vectors \( v \) with finite number of nonzero coefficients \( \langle v, e_{j,i} \rangle \) all of which are rational. The set of \( v_i \) is dense in \( H \). Let \( k_1 \geq 1 \) such that \( v_i \in \bigoplus_{|i| \leq k_1} M_i \). With that integer \( k_1 \), we let \( w_i = a \) and \( x_i = 0 \) whenever \( |i| \leq k_1 \) and \( v_i = \sum_{|i| \leq k_1} y_i \), with each \( y_i \in M_i \). With \( \epsilon_1 = (2a^{2k_1})^{-1} \) the lemma provides an integer \( n_1 \) and weights \( w_i \) with \( 0 < w_i \leq a \) and vectors \( x_i \in M_i \), whenever \( k_1 + 1 \leq |i| \leq n_1 + k_1 \) and in fact \( x_i = 0 \) for those \( i \) in the range \(-n_1 - k_1 \leq i \leq -k_1 - 1 \) such that if \( D_1 \) is given by

\[
D_1 e_{j,i} = \begin{cases} 
  w_j e_{j-1,i} & \text{whenever } i \in \mathbb{Z} \text{ and } |j| \leq n_1 + k_1 \\
  0 & \text{otherwise}
\end{cases}
\]

then

\[
\sum_{i=k_1+1}^{n_1+k_1} \|x_i\|^2 < \frac{1}{2a_1^k}
\]
and
\[\left\| (D_1 + T)^{n_1} \sum_{|i| \leq n_1 + k_1} x_i - v_1 \right\| = \left\| (D_1 + T)^{n_1} \sum_{i=-n_1-k_1}^{n_1-k_1-1} x_i \right\| < \frac{1}{2a^{2k_1}}\]

Although the operator \( D \) provided by the lemma is shifting backward, the techniques in the lemma provide an analogous result for an operator that shifts forward. Hence, we can continue our definitions of \( w_i \) and \( x_i \) by settling \( k'_1 = n + k_1 \) and \( \epsilon'_1 = \frac{1}{2a^{2k_1}} \). The lemma provides an integer \( n'_1 \) and weights \( w_i \) with \( 0 < w_i \leq a \) whenever \( k'_1 + 1 \leq |i| \leq n'_1 + k'_1 \), and vectors \( x_i \in M_i \) whenever \( k'_1 + 1 \leq |i| \leq n'_1 + k'_1 \) and in fact \( x_i = 0 \) for those \( i \) in the range \( k'_1 + 1 \leq i \leq n'_1 + k'_1 \) such that
\[\sum_{-n'_1-k'_1 \leq i \leq -k'_1-1} \|x_i\|^2 < \frac{1}{2a^{2k_1}}.\]

In addition, the lemma provides an operator \( D'_1 \) shifting forward by
\[D'_1 e_{j,i} = \begin{cases} w_{j+1} e_{j+1,i} & \text{if } i \in \mathbb{Z} \text{ and } -n'_1 - k'_1 \leq j \leq n'_1 + k'_1 - 1 \\ 0 & \text{otherwise} \end{cases}\]
such that
\[\left\| (D'_1 + T^*)^{n'_1} \sum_{|i| \leq n'_1 + k'_1} x_i - v_1 \right\| = \left\| (D'_1 + T^*)^{n'_1} \sum_{i=-n'_1-k'_1+1}^{k'_1} x_i \right\| < \frac{1}{2a^{2k_1}}\]

In the second step, we let \( k_2 = n'_1 + k'_1 \) and assume that, without loss of generality, \( v_2 \in \bigoplus_{|i| \leq k_2} M_i \). In the case that \( v_2 \) is not in the space, we can choose \( v_j \) with the least integer \( j \) such that \( v_j \) is in that space. We can define operator \( D_2 \) and then \( D'_2 \) as in the previous case for \( v_1 \). Inductively, in the \( m \)-th step, we take \( k_m = n'_m + k'_m \) and \( \epsilon_m = \frac{1}{2a^{2k_m}} \) and we can assume, without loss of generality, that \( v_m \in \bigoplus_{|i| \leq k_m} M_i \). By the lemma, we have an integer \( n_m \) and weights \( w_i \) with \( 0 < w_i \leq a \), where \( |i| \leq n_m + k_m \) and vectors \( x_i \) in \( M_i \), where \( |i| \leq n_m + k_m \) and in fact \( x_i = 0 \) for these \( i \) in the range \(-n_m - k_m \leq i \leq -k_m - 1\).
such that if $D_m$ is given by

$$D_m e_{j,i} = \begin{cases} w_j e_{j-1,i} & \text{whenever } i \in \mathbb{Z} \text{ and } |j| \leq n_m + k_m \\ 0 & \text{otherwise} \end{cases}$$

then

$$\sum_{k_m+1 \leq i \leq n_m + k_m} \| x_i \|^2 \leq \frac{1}{2m a^{2k_m}}$$

and

$$\left\| (D_m + T)^{n_m} \sum_{|i| \leq n_m + k_m} x_i - v_m \right\| = \left\| (D_m + T)^{n_m-k_m-1} \sum_{-n_m-k_m \leq i \leq -k_m-1} x_i \right\| < \frac{1}{2m a^{2k_m}}$$

Next, we let $k'_m = n_m + k_m$ and $\epsilon'_m = \frac{1}{2m a^{2k_m}}$. by the lemma, we have an integer $n'_m$ and weights $w_i$ with $0 < w_i < a$ where $|i| \leq n'_m + k'_m$ and vectors $x_i$ in $M$, whenever $k'_m + 1 \leq |i| \leq n'_m + k'_m$ and in fact $x_i = 0$ for those $i$ in the range $k'_m + 1 \leq i \leq n'_m + k'_m$, such that

$$\sum_{-n'_m-k'_m \leq i \leq -k'_m-1} \| x_i \|^2 < \frac{1}{2m a^{2k'_m}} \quad (\dagger)$$

In addition, the lemma gives an operator $D'_m$ given by

$$D'_m e_{j,i} = \begin{cases} w_{j+1} e_{j+1,i} & \text{if } i \in \mathbb{Z} \text{ and } -n'_m - k'_m \leq j \leq n'_m + k'_m - 1 \\ 0 & \text{otherwise} \end{cases}$$

such that

$$\left\| (D'_m + T^*)^{n'_m} \sum_{|i| \leq n'_m + k'_m} x_i - v_m \right\| = \left\| (D'_m + T^*)^{n'_m-k'_m} \sum_{i=-n'_m-k'_m+1}^{k'_m} x_i \right\| < \frac{1}{2m a^{2k'_m}}$$
with all weights \( w \) with \( 0 < w \leq a \) and all vectors \( x \) given by the inductive process above, we define an operator \( B : H \to H \) by \( B|_{M_j} = w_j S^{-1}|_{M_j} \) and the vector \( x \) by \( x = \sum_{i \in \mathbb{Z}} x_i \).

Hence,

\[
\|(B + T)^n x - v_m\|^2 = \|(B + T)^{n_m} \sum_{i = -\infty}^{n_m - k_m - 1} x_i\|^2 + \|(B + T)^{n_m} \sum_{i = n_m + k_m + 1}^{\infty} x_i\|^2. \tag{††}
\]

We continue our computations with the two summands separately. For the first summand, we note that \( k'_m = n_m + k_m \) and so it is bounded above by

\[
\|(B + T)^{n_m} \sum_{i = -\infty}^{n_m - k_m - 1} x_i\|^2 + \|(B + T)^{n_m} \sum_{i = -\infty}^{n_m - k_m - 1} x_i\|^2 < \frac{1}{2^m a^{2k_m}} + a^{2n_m} \sum_{m = 1}^{\infty} \sum_{i = -k_{j+1}}^{-n_j - k_j - 1} \|x_i\|^2,
\]

because \( x_i = 0 \) whenever \( -n'_j - k'_j = -k_j \leq i \leq -k_{j+1} - 1 \) for some \( j \in \mathbb{Z} \). Note that

\[
a^{2n_m} \sum_{j = m}^{\infty} \sum_{i = -k_{j+1}}^{-n_j - k_j - 1} \|x_i\|^2 = a^{2n_m} \sum_{j + m = 1}^{\infty} \sum_{i = -n'_j - k'_j}^{-k'_j - 1} \|x_i\|^2 < a^{2n_m} \sum_{j = m}^{\infty} \frac{1}{2^m a^{2k'_m}} \leq \sum_{j = m}^{\infty} \frac{1}{2^m} = \frac{1}{2^{m-1}}.
\]

Hence the first summand is bounded above by

\[
\frac{1}{2^m a^m} + \frac{1}{2^{m-1}} \to 0, \text{ as } m \to \infty.
\]

For the second summand of (††) we note that \( x_i = 0 \) for all index \( i \) in the range \( n_j + k_j + 1 \leq i \leq n'_j + k'_j = k_{j+1} \) and so the second summand is bounded
above by

\[
\|B_{nm} \sum_{j=m}^{\infty} \sum_{i=k_{j+1}+1}^{n_{j+1}+k_{j+1}} x_i \|^2 \\
\leq \|B^{2nm} \| \sum_{j=m}^{\infty} \sum_{i=k_{j+1}+1}^{n_{j+1}+k_{j+1}} \|x_i\|^2 \\
\leq a^2 n_m \sum_{j=m}^{\infty} \left( \frac{1}{2j a^{2k_j}} \right) \\
\leq \sum_{j=m}^{\infty} \left( \frac{1}{2j} \right) \\
= \frac{1}{2^{m-1}} \to 0 \text{ as } m \to \infty.
\]

Hence, \(\|(B + T)^{nm} x - v_m\|^2 \to 0\) as \(m \to \infty\).

The exact argument shows that \(\|(B^* + T^*)^{nm} x - v_m\|^2 \to 0\) as \(m \to \infty\). Hence both 
\(B + T\) and \(B^* + T^*\) are hypercyclic.

\[
\square
\]

The existence of a hypercyclic operator with a hypercyclic adjoint was a surprising fact. Herrero [16] asked whether such an operator can ever exist. Salas [25] provided the first example of such an operator, and also a second example using a general result about bilateral weighted shifts in [26]. Theorem 30 shows that it is not uncommon to have a hypercyclic operator whose adjoint is also hypercyclic, and indeed we can choose one whose compression to a closed subspace with infinite codimension can coincide with a prescribed operator on the subspace.

We remark that Corollary 23 and Theorem 30 call for two different constructions of an operator, but one may wonder if the two constructions may produce the same operator. The answer is never, because if both \(T\) and \(T^*\) are hypercyclic then \(T\) cannot be chaotic. To show that by the way of contradiction, suppose \(T\) has a periodic point \(x\), say \(T^n x = x\), then \((T^*)^n = (T^n)^*\) is not hypercyclic, which contradicts Ansari’s result [1] that the operator \(S^n\)
is hypercyclic whenever \( S \) is.

An operator is said to be a universal model if every operator in \( B(H) \) is similar to a multiple part of it. Rota’s Theorem says that canonical backward unilateral shifts are universal models.

**Theorem 31. Rota** Let \( T \) be an operator on a Hilbert space \( H \). If \( r(T) < 1 \), then \( T \) is similar to a part of the canonical backward unilateral shift on \( \ell^2_+(H) \).

Let \( K = \bigoplus_{i=0}^{\infty} H_i \) where each \( H_i \) is isomorphic to \( H \), and \( H_0 = H \). Suppose \( B : K \to K \) is the canonical backward unilateral shift; that is \( B \) takes \( H_0 \) to 0 and for \( i = 1, 2, 3, \ldots \), \( B|H_i \) is an isomorphism from \( H_i \) onto \( H_{i-1} \).

**Theorem 32.** Let \( a > 1 \) and let \( T : H \to H \) be a bounded linear operator. If its spectral radius \( r(T) \leq 1 \), then \( T \) is similar to a part of a hypercyclic operator \( aB : K \to K \).

**Proof.** The proof is exactly the same as the argument for Rota’s Theorem [24] on the existence of universal models. For all vectors \( x \) in \( H \) we denote

\[
Wx = (x, \frac{1}{a}Tx, \frac{1}{a^2}x, \ldots).
\]

Since \( r(T) = \lim_{n \to \infty} \|T^n\|^{\frac{1}{n}} \leq 1 \) we have that \( \|T^n\| \leq 1 \) for all \( n \). Thus

\[
\sum_{n=0}^{\infty} \frac{1}{a^{2n}} \|T^n x\|^2 < \infty.
\]

Note that

\[
\|x\| \leq \|Wx\| \leq \left( \sum_{n=0}^{\infty} \frac{1}{a^{2n}} \right)^{\frac{1}{2}} \|x\|
\]

and hence \( W : H \to H \) is a bounded linear operator that is bounded below. Thus \( K := \text{ran}W \) is closed in \( K \), and \( W : H \to K \) is one-to-one and onto. Observe that

\[
WTx = (Tx, \frac{1}{a}Tx, \frac{1}{a^2}T^2x, \ldots)
\]
and

\[(aB) W x = aB(x, \frac{1}{a} T x, \frac{1}{a^2} T^2 x, \ldots) = WT x \quad (4.5)\]

Hence ranW is an invariant subspace of aB and

\[T = W^{-1} (aB|H) W.\]

Although the techniques we used here are pretty much standard for the Rota model, it is interesting for us to point out that the operator aB is indeed hypercyclic, which has never been observed before.
BIBLIOGRAPHY


