OPTIMALLY CLEAN RINGS

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ABSTRACT

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In 1999 Nicholson introduced the definition that an element of a ring is called strongly clean if it can be written as the sum of a unit and an idempotent that commute. Similarly, a ring is called strongly clean if each of its elements is strongly clean. While many popular collections of rings have been shown to possess this characteristic, there are some that do not. Perhaps most surprising is the fact that there are still large collections that have yet to be classified. One such example in this final group is the set of formal power series rings. We know not all these structures are strongly clean, but some are. Which ones?

To answer this question we investigated conditions on a ring that imply the extension to a formal power series ring would still be strongly clean. Using Peirce Decompositions and Corners, we were able to isolate the structure needed. Simply stated as a surjective group homomorphism or the solvability of a ring commutator, it is shown that our definition of optimally clean is sufficient to satisfy the extension in question. Further, we were able to verify the equivalence of strongly and optimally clean within the context of formal power series rings.

Extending on this success, we then investigated similar conditions for an extension to the skew power series ring to be strongly clean. This led to our analogous definition of skew optimally clean and proof of its sufficiency for this result. Additionally, we provide examples verifying our conditions to be distinct from previously established definitions.

Finally, we verify our new conditions to include all other classes that have been shown sufficient to extend to a strongly clean formal power series ring, making this the most general result to date. Unfortunately, there are not yet enough examples to allow us to determine whether or not they are also necessary properties.
To Kelly

for more reasons than I can list
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CHAPTER 1

Preliminaries

Since this is a dissertation about strongly clean rings, we have a fair amount of background and machinery to establish before proceeding. This chapter begins by building up the relevant terms and notation to be used throughout the remainder of the document. Once completed, the subsequent section provides a short explanation of the connections between many of these definitions to establish the landscape our work took place in. All of this background leads to the final piece of the chapter where we motivate and pose the questions to be answered.

1.1 Definitions and Notation

As notational conventions we will let \( \mathbb{R}, \mathbb{Q}, \mathbb{Z} \) and \( \mathbb{N} \) stand for the reals, rationals, integers and natural numbers, respectively.

The first three definitions below are included for completeness and establish the basis for our mathematics.

**Definition 1.1.1.** A *binary operation* \( * \) on a set \( S \) is a function \( * : S \times S \to S \). For any \( a, b \in S \) we will write \( a * b \) for \( *(a, b) \).

**Definition 1.1.2.** A *group* \( G \) is an ordered pair \((G, * )\) where \( G \) is a set and \( * \) is a binary
operation on $G$ satisfying the following conditions.

1. $\star$ is associative: $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in G$.

2. There exists an element $e \in G$, called an identity of $G$, such that for all $a \in G, a \star e = a = e \star a$.

3. For each $a \in G$ there is an element $a^{-1} \in G$, called an inverse of $a$, such that $a \star a^{-1} = e = a^{-1} \star a$.

The group $(G, \star)$ is called abelian if $a \star b = b \star a$ for all $a, b \in G$.

**Definition 1.1.3.** A ring $R$ is a nonempty set together with two binary operations $+$ and $\cdot$ (called addition and multiplication) such that

1. $(R,+)$ is an abelian group,

2. $\cdot$ is associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in R$,

3. the distributive laws hold: $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$ and $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in R$.

The ring $R$ is commutative if $a \cdot b = b \cdot a$ for all $a, b \in R$.

The ring $R$ is said to have an identity if there exists a $1 \in R$ such that $1 \cdot a = a = a \cdot 1$ for all $a \in R$.

**Remark.** We shall use the convention of $ab$ rather than $a \cdot b$ for $a, b \in R$. The additive identity will be denoted $0$ while the additive inverse of $a$ will be $-a$. Unless otherwise stated, all rings are assumed to contain an identity, but they are not necessarily assumed to be commutative. One method of “measuring” the noncommutativity of a ring is given in the following definition.

**Definition 1.1.4.** For any $a, b \in R$ the ring commutator of $a$ and $b$ is given by $[a, b] = ab - ba$. 

Let $R$ be a ring. The following groups of definitions establish elements within a ring and classes of rings that were also necessary to our study.

**Definition 1.1.5.** An element $a \in R$ is in the center if it commutes with every other element of $R$. A nonzero element $a \in R$ is called a zero divisor if there is a nonzero element $b \in R$ such that either $ab = 0$ or $ba = 0$. An element $a \in R$ is called a unit if there is some $a^{-1} \in R$ such that $aa^{-1} = 1 = a^{-1}a$. An element $e \in R$ such that $e^2 = e$ is called an idempotent. An element $x \in R$ such that $x^n = 0$ for some $n \in \mathbb{N}$ is called a nilpotent.

*Remark.* Denote the sets of units, idempotents, nilpotents, and center of a ring $R$ by $U(R), Id(R), Nil(R)$ and $Z(R)$ respectively.

**Definition 1.1.6.** An idempotent $e \in R$ is called central if $e \in Z(R)$. A ring is called abelian if $Id(R) \subseteq Z(R)$.

**Definition 1.1.7.** If every nonzero element of $R$ is a unit, then $R$ is a division ring. A commutative division ring is a field. If $R$ is commutative and has no zero divisors, it called an integral domain. A subring $S$ of $R$ is a subset where $1_S = 1_R \in S$ and the operations of addition and multiplication in $R$ give the structure of a ring on $S$. We say that $R = A \oplus B$ is the internal direct sum of $A$ and $B$ if every element of $R$ can be written uniquely as the sum of an element in $A \subseteq R$ with an element of $B \subseteq R$. If $I$ is some index set and $R_i$ is a ring for every $i \in I$, then the direct product, denoted $\Pi R_i$, is a ring under coordinatewise addition and multiplication.

*Remark.* For $n \in \mathbb{N}$ we will write $\mathbb{M}_n(R)$ and $\mathbb{T}_n(R)$ for the rings of all (respectively, all upper triangular) $n \times n$ matrices over the ring $R$.

Finally consider four more basic definitions before moving on to the specifics of our study.

**Definition 1.1.8.** A subgroup $I$ of $R$ that is closed under multiplication is a left ideal of $R$ if it is also closed under left multiplication by elements from $R$. 
Definition 1.1.9. The left annihilator of $r \in R$, denoted $Ann_l(r)$, is the set of all $a \in R$ such that $ar = 0$.

Definition 1.1.10. A left $R$-module is a set $M$ together with

1. a binary operation $+$ so that $(M, +)$ is an abelian group and

2. a map $R \times M \to M$ denoted by $rm$ satisfying

   - $(r + s)m = rm + sm$,
   - $(rs)m = r(sm)$,
   - $r(m + n) = rm + rn$, and
   - $1m = m$ for all $r, s \in R$ and $m, n \in M$.

Right ideals, annihilators and $R$-modules are defined analogously.

Definition 1.1.11. A (group, ring, $R$-module) homomorphism is a map between (group, ring, $R$-module) structures that respects the (group, ring, $R$-module) operations. An isomorphism is a bijective homomorphism. An endomorphism is a homomorphism from an $R$-module back onto itself.

Remark. Denote the image and kernel of a homomorphism $\phi$ by $Im(\phi)$ and $Ker(\phi)$ respectively. Denote the ring of all endomorphisms on an $R$-module $M$ by $End(M)$.

It can be shown that any ring $R$ is isomorphic to the endomorphism ring of itself (as a module over itself) $End(R)$. Thus units are important because they can be thought of as providing an isomorphism from $R$ back to itself by left (or right) multiplication. Since idempotents can similarly be thought of as projections of a ring onto its direct summands $R = eR \oplus (1-e)R$ (or $Re \oplus R(1-e)$) by left (respectively, right) multiplication, they are also desirable elements. Of course, we cannot expect every element to be a unit or an idempotent. Thus we begin a slow generalization of similar components within a ring.
Definition 1.1.12. An element $r \in R$ is strongly regular if there exist $a, b \in R$ such that $r^2a = r = br^2$.

Equivalently, $r \in R$ is strongly regular if and only if $R = rR \oplus \text{Ann}_r(r)$. That is, there exists a right $R$-module decomposition $R = A \oplus B$ such that $r : A \to A$ and $r : B \to B$ given by left multiplication are an isomorphism and the zero map respectively (see [10] or [3, Proposition 2.5]). In short, there exists a decomposition $R = \text{Im}(r) \oplus \text{Ker}(r)$.

Definition 1.1.13. An element $r \in R$ is strongly $\pi$-regular if there exist $a, b \in R$ such that $r^{n+1}a = r^n = br^{n+1}$.

Equivalently, $r \in R$ is strongly $\pi$-regular if and only if there exists an $n \in \mathbb{N}$ such that $R = r^nR \oplus \text{Ann}_r(r^n)$. That is, there exists a right $R$-module decomposition $R = A \oplus B$ where $r(A) \subseteq A, (1 - r)(B) \subseteq B$ such that $r : A \to A$ and $r : B \to B$ given by left multiplication are an isomorphism and a nilpotent map respectively (see [10] or [3, Proposition 2.4]). In short, there exists a decomposition $R = \text{Im}(r^n) \oplus \text{Ker}(r^n)$.

Definition 1.1.14. An element $r \in R$ is said to be strongly clean if there exists $e^2 = e \in R$ such that $r - e \in U(R)$ and $er = re$.

Equivalently, $r \in R$ is strongly clean if and only if $R = (1 - r)eR \oplus r(1 - e)R$ where $(1 - r)eR \cong eR$ and $r(1 - e)R \cong (1 - e)R$. That is, there exists a right $R$-module decomposition $R = A \oplus B$ such that $r(A) \subseteq A, (1 - r)(B) \subseteq B$, and both $r : A \to A, (1 - r) : B \to B$ given by left multiplication are isomorphisms (see [10] or [3, Proposition 2.3]).

Definition 1.1.15. An element $r \in R$ is said to be clean if there exists $e^2 = e \in R$ such that $r - e \in U(R)$.

Equivalently, $r \in R$ is clean if and only if $R = (1 - r)eR \oplus r(1 - e)R$. That is, there exists a right $R$-module decomposition $R = A \oplus B = C \oplus D$ such that $r(A) \subseteq C, (1 - r)(B) \subseteq D$, and both $r : A \to C, (1 - r) : B \to D$ given by left multiplication are isomorphisms (see [3, Proposition 2.2]).
Definition 1.1.16. A ring $R$ is called strongly regular (respectively strongly $\pi$-regular, strongly clean, or clean) if each of its elements is strongly regular (respectively strongly $\pi$-regular, strongly clean, or clean).

While these four classes of rings served as the basis for most of our work, we were not restricted to them entirely. The next definitions are also required.

Definition 1.1.17. An element $r \in R$ is said to be uniquely (strongly) clean if it is (strongly) clean and the representation is unique [11, 4]. A ring is called uniquely (strongly) clean if all of its elements are.

Remark. By [11, Lemma 4], uniquely clean elements are uniquely strongly clean. Further note that just because an element has a unique strongly clean decomposition does not mean it cannot have other clean (but not strongly clean) decompositions (see [4] for examples).

Finally, our work made extensive use of the following concept of ring theory. See [9, Chapter 21] for more.

Definition 1.1.18. For any idempotent $e \in R$, the Peirce Decomposition of $R$ with respect to $e$ is given by $R = eRe \oplus eR(1 - e) \oplus (1 - e)Re \oplus (1 - e)R(1 - e)$ where the first and last of these four summands are called the Peirce Corners of $R$.

Remark. Due to the structure of idempotents, we can think of multiplication under the Peirce Decomposition as taking place in matrix form. That is, for $w_1, w_2 \in eRe, x_1, x_2 \in eR(1 - e), y_1, y_2 \in (1 - e)Re,$ and $z_1, z_2 \in (1 - e)R(1 - e)$,

$$(w_1 + x_1 + y_1 + z_1)(w_2 + x_2 + y_2 + z_2)$$

can be viewed as

$$\begin{pmatrix} w_1 & x_1 \\ y_1 & z_1 \end{pmatrix} \begin{pmatrix} w_2 & x_2 \\ y_2 & z_2 \end{pmatrix}.$$
1.2 Connections Between Classes of Rings

Given the definitions of the previous section, the following chains of containment can be established for rings:

\[
\text{Strongly Regular} \subseteq \text{Strongly } \pi\text{-Regular} \subseteq \text{Strongly Clean} \subseteq \text{Clean}
\]
\[
\text{Uniquely Clean} \subseteq \text{Uniquely Strongly Clean} \subseteq \text{Strongly Clean} \subseteq \text{Clean}
\]

where none of the containments can be reversed.

This dissertation is centered on strongly clean rings. Thus, we can think of our study as an additive generalization of the strongly regular rings first introduced over 75 years ago.

1.3 Background and Questions to be Answered

Introduced by Nicholson [10], the strongly clean property has garnered a lot of popularity in the past decade for its simple (yet effective) definition and plethora of interesting properties. For example, products and homomorphic images of strongly clean rings are strongly clean. It has been shown that a large class of rings, including local rings, strongly \( \pi \)-regular rings [10], abelian regular rings [8], and right (or left) perfect rings are strongly clean. In fact, all right (or left) artinian rings are strongly clean. Thus we are armed with an abundant class that can be expressed as an “additive analogue” of strongly regular rings.

Historically, one popular question often asked of new classifications of rings is whether or not the defining structure remains when extended to the polynomial ring.

Definition 1.3.1. For a ring \( R \) define the polynomial ring of \( R \), denoted \( R[x] \), to be the set of \( \sum_{i=0}^{n} r_i x^i \) such that \( n \in \mathbb{N}, r_i \in R \) for \( 0 \leq i \leq n \) under the usual polynomial addition and multiplication.

Unfortunately, there is no hope for such a ring to be clean, let alone strongly clean (see [11, Proposition 13]):
Proof. Consider $x \in R[x]$. Assume there exists an idempotent $e = \sum_{i=0}^{n} e_i x^i$ such that $x - e \in U(R[x])$. Then $e_0^2 = e_0$ and $-e_0 \in U(R) \Rightarrow e_0 = 1$. An induction argument on $i$ shows we must have $e_i = 0$ for all $1 \leq i \leq n$. So $e = 1$. But $x-1$ is not a unit of $R[x]$. \qed

Having accepted this result, we then widen our gaze and ask the same question about the more general formal power series ring.

**Definition 1.3.2.** For a ring $R$ define the formal power series ring of $R$, denoted $R[[x]]$, to be the set of $\sum_{i=0}^{\infty} r_i x^i$ such that $r_i \in R$ under the usual polynomial addition and multiplication.

It is fairly straightforward to see that the formal power series ring over a clean ring is clean. The answer to the following question, however, is not as obvious.

**Question.** Given a ring $R$, when is $R[[x]]$ strongly clean?

Here the hopelessness that plagued the polynomial ring has been alleviated to a degree. Some of the problems (in particular - the lack of units) can be avoided in the formal power series ring. In fact, Chen and Zhou [7] proved that a formal power series ring over a strongly $\pi$-regular ring is strongly clean. But there are other examples as well. The question then arises: What characteristic is needed to extend to a strongly clean power series ring?

If this answer is met with some success, it is natural to generalize again and ask when the skew power series ring is strongly clean.

**Definition 1.3.3.** Let $R$ be a ring and $\sigma$ be a ring endomorphism of $R$. Define the formal (left) skew power series ring of $R$ with respect to $\sigma$, denoted $R[[x; \sigma]]$, to be the set of $\sum_{i=0}^{\infty} r_i x^i$ such that $r_i \in R$ under the usual polynomial addition and multiplication subject to the relation $xr = \sigma(r)x$ for all $r \in R$.

**Question.** Given a ring endomorphism $\sigma$ of $R$, when is $R[[x; \sigma]]$ strongly clean?

Though it may not appear so at first, this problem holds the potential to be a great deal more difficult than the first. Much of the structure of a formal skew power series depends on the endomorphism chosen. Since this choice can vary from ring to ring, we can ask what
is required so that all formal skew power series extensions of a particular ring are strongly clean.

*Question.* Given a ring $R$, when is $R[x;\sigma]$ strongly clean for all ring endomorphisms $\sigma$?

These are the major questions we set out to answer in this dissertation.
CHAPTER 2

Exploration with the Peirce Decomposition

Throughout this chapter let R be a strongly clean unital ring. Since we can think of \( R[[x]] \) as the “end” of the chain \( R \to R[x]/(x^2) \to R[x]/(x^3) \to \cdots \to R[x]/(x^n) \to \cdots \to R[[x]] \), it is natural to explore what it means for elements in \( R[x]/(x^n) \), \( n \in \mathbb{N} \) to be strongly clean. We begin with the trivial extension \( (n = 2) \) in the first section of this chapter, move to the subsequent extension \( (n = 3) \) in the next section, then generalize to the \( n^{th} \) extension in the third section. All this leads to the final section where we summarize our findings to familiarize the reader with the motivation behind our techniques used in later chapters.

As alluded to earlier, our methods make heavy use of ring commutators and Peirce Decompositions. To help alleviate some confusion, we adopt the convention for this chapter that all capital letters will represent elements of the ring while lower case letters represent elements within a piece of the Peirce Decomposition. Finally, as the goal of this chapter is to stimulate (and not prove) our ideas, we will temporarily abandon the rigorous formality of the remainder of the document for a more casual tone.
2.1 The Trivial Extension

We begin with the trivial extension: $R_1 = R[X]/(X^2)$. Let $A = A_0 + A_1X \in R_1$. Then for this element to be strongly clean we need to find $E_0, E_1 \in R$ where:

I): $(E_0 + E_1X)^2 = E_0 + E_1X,$

U): $(A_0 - E_0) + (A_1 - E_1)X \in U(R_1),$

C): $(A_0 + A_1X)(E_0 + E_1X) = (E_0 + E_1X)(A_0 + A_1X).

Considering the constant terms of these three equations we see that $A_0 = E_0 + (A_0 - E_0)$ must be a strongly clean decomposition in R (hence the requirement that R be strongly clean to begin with). Assuming this is satisfied, it is beneficial to note the following simplifications:

Equation (U) is always true as $(A_0 - E_0) \in U(R)$.

Equation (I) is true if and only if $E_0E_1 + E_1E_0 = E_1$. (1)

Equation (C) is true if and only if $A_0E_1 - E_1A_0 = E_0A_1 - A_1E_0$ if and only if $[A_0, E_1] = [E_0, A_1]$. (2)

Thus $A_0 + A_1X$ is strongly clean if (and only if) we can find an $E_1$ that satisfies (1) and (2) for some strongly clean decomposition $A_0 = E_0 + (A_0 - E_0) \in R$.

To this end, consider the matrix form of the Peirce Decomposition of $R$ with respect to the idempotent

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}: R = \begin{pmatrix} E_0RE_0 & E_0R(1 - E_0) \\ (1 - E_0)RE_0 & (1 - E_0)R(1 - E_0) \end{pmatrix}.$$ 

Write

$$A_0 = \begin{pmatrix} w_0 & x_0 \\ y_0 & z_0 \end{pmatrix}$$
where \( w_0 \in E_0 R E_0, x_0 \in E_0 R(1 - E_0), y_0 \in (1 - E_0) R E_0, \) and \( z_0 \in (1 - E_0) R(1 - E_0). \) But as \( E_0 A_0 = A_0 E_0 \) we get that

\[
\begin{pmatrix}
w_0 & x_0 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
w_0 & 0 \\
y_0 & 0
\end{pmatrix}
\]

which implies \( x_0 = 0 \) and \( y_0 = 0. \) Thus

\[
A_0 = \begin{pmatrix}
w_0 & 0 \\
0 & z_0
\end{pmatrix}.
\]

Now we work towards finding an

\[
E_1 = \begin{pmatrix}
q_1 & r_1 \\
s_1 & t_1
\end{pmatrix} \in R
\]

that will satisfy our conditions (1) and (2). According to (1) we must have

\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
q_1 & r_1 \\
s_1 & t_1
\end{pmatrix} + \begin{pmatrix}
q_1 & r_1 \\
s_1 & t_1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
q_1 & r_1 \\
s_1 & t_1
\end{pmatrix},
\]

which simplifies to

\[
\begin{pmatrix}
q_1 + q_1 & r_1 \\
s_1 & 0
\end{pmatrix} = \begin{pmatrix}
q_1 & r_1 \\
s_1 & t_1
\end{pmatrix}.
\]

This gives \( q_1 = 0 \) and \( t_1 = 0. \) So our goal reduces to finding an

\[
E_1 = \begin{pmatrix}
0 & r_1 \\
s_1 & 0
\end{pmatrix},
\]

where \( r_1 \in E_0 R(1 - E_0) \) and \( s_1 \in (1 - E_0) R E_0, \) that satisfies (2).
If \[ A_1 = \begin{pmatrix} w_1 & x_1 \\ y_1 & z_1 \end{pmatrix} \in R, \]
this means we must be able to solve
\[
\begin{pmatrix} w_0 & 0 \\ 0 & z_0 \end{pmatrix} \begin{pmatrix} 0 & r_1 \\ s_1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & r_1 \\ s_1 & 0 \end{pmatrix} \begin{pmatrix} w_0 & 0 \\ 0 & z_0 \end{pmatrix} = \begin{pmatrix} w_1 & x_1 \\ y_1 & z_1 \end{pmatrix} - \begin{pmatrix} w_1 & x_1 \\ y_1 & z_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]
which simplifies to
\[
\begin{pmatrix} w_0r_1 - r_1z_0 \\ z_0s_1 - s_1w_0 \end{pmatrix} = \begin{pmatrix} 0 & x_1 \\ -y_1 & 0 \end{pmatrix}.
\]

Thus we need to find \( r_1 \in E_0R(1 - E_0) \) and \( s_1 \in (1 - E_0)RE_0 \) such that
\[
a_1) : w_0r_1 - r_1z_0 = x_1,
\]
\[
b_1) : z_0s_1 - s_1w_0 = -y_1.
\]

That is, solving for \( E_1 \) under the Peirce Decomposition of \( R \) with respect to \( E_0 \) reveals that we must have some control of the elements of the pieces \( E_0R(1 - E_0) \) and \( (1 - E_0)RE_0 \) while setting the values in the corners \( E_0RE_0 \) and \( (1 - E_0)R(1 - E_0) \) equal to \( 0 \). If we can do this, then we can construct a strongly clean decomposition of an element in \( R_1 \).

### 2.2 The Next Extension

Now we consider the next extension in the chain: \( R_2 = R[X]/(X^3) \). Let \( A_0 + A_1X + A_2X^2 \in R_2 \). Then we need to find \( E_0, E_1, E_2 \in R \) where:
I): \((E_0 + E_1X + E_2X^2)^2 = E_0 + E_1X + E_2X^2\),

U): \((A_0 - E_0) + (A_1 - E_1)X + (A_2 - E_2)X^2 \in U(R_2)\),

C): \((A_0 + A_1X + A_2X^2)(E_0 + E_1X + E_2X^2) = (E_0 + E_1X + E_2X^2)(A_0 + A_1X + A_2X^2)\).

Considering the constant and linear terms of these three equations we see that \(A_0 + A_1X = (E_0 + E_1X) + (A_0 - E_0 + (A_1 - E_1)X)\) must be a strongly clean decomposition in \(R_1\). Assuming this, all the equations from the previous section (trivial extension) are satisfied and the equations above simplify as follows:

Equation (U) is always true as \((A_0 - E_0) \in U(R)\).

Equation (I) is true if and only if \(E_0 E_2 + E_1 E_1 + E_2 E_0 = E_2\). (3)

Equation (C) is true if and only if \(A_0 E_2 - E_2 A_0 = E_0 A_2 - A_2 E_0 + E_1 A_1 - A_1 E_1\) if and only if \([A_0, E_2] = [E_0, A_2] + [E_1, A_1]\). (4)

**Thus** \(A_0 + A_1X + A_2X^2\) **is strongly clean if (and only if) we can find an** \(E_2\) **that satisfies (3) and (4) for some strongly clean decomposition** \(A_0 + A_1X = (E_0 + E_1X) + (A_0 - E_0 + (A_1 - E_1)X) \in R_1\).

We proceed as in the previous section - by considering the Peirce Decomposition with respect to the idempotent \(E_0\). Given the assumption mentioned above and the work in the previous section we can write the following:

\[
A_0 = \begin{pmatrix} w_0 & 0 \\ 0 & z_0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} w_1 & x_1 \\ y_1 & z_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} w_2 & x_2 \\ y_2 & z_2 \end{pmatrix}
\]

\[
E_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & r_1 \\ s_1 & 0 \end{pmatrix}
\]

Now we work towards finding an

\[
E_2 = \begin{pmatrix} q_2 & r_2 \\ s_2 & t_2 \end{pmatrix} \in R
\]
that will satisfy our conditions (3) and (4). According to (3) we must have

\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
q_2 & r_2 \\
s_2 & t_2
\end{pmatrix}
+ \begin{pmatrix}
0 & r_1 \\
s_1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & r_1 \\
s_1 & 0
\end{pmatrix}
+ \begin{pmatrix}
q_2 & r_2 \\
s_2 & t_2
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
= \begin{pmatrix}
q_2 & r_2 \\
s_2 & t_2
\end{pmatrix},
\]

which simplifies to

\[
\begin{pmatrix}
q_2 + r_1 s_1 + q_2 & r_2 \\
s_2 & s_1 r_1
\end{pmatrix}
= \begin{pmatrix}
q_2 & r_2 \\
s_2 & t_2
\end{pmatrix}.
\]

This gives \( q_2 = -r_1 s_1 \) and \( t_2 = s_1 r_1 \) with no restrictions (so far) on \( r_2 \) or \( s_2 \). So our goal reduces to finding an

\[
E_2 = \begin{pmatrix}
-r_1 s_1 & r_2 \\
s_2 & s_1 r_1
\end{pmatrix}
\]

where \( r_2 \in E_0 R (1 - E_0) \) and \( s_2 \in (1 - E_0) R E_0 \).

But to satisfy (4) we must have

\[
\begin{pmatrix}
w_0 & 0 \\
0 & z_0
\end{pmatrix}
\begin{pmatrix}
-w_1 s_1 & r_2 \\
s_2 & s_1 r_1
\end{pmatrix}
- \begin{pmatrix}
w_0 & 0 \\
0 & z_0
\end{pmatrix}
\begin{pmatrix}
-w_1 s_1 & r_2 \\
s_2 & s_1 r_1
\end{pmatrix}
= \begin{pmatrix}
w_2 & x_2 \\
y_2 & z_2
\end{pmatrix}
+ \begin{pmatrix}
w_1 & x_1 \\
y_1 & z_1
\end{pmatrix}
- \begin{pmatrix}
w_2 & x_2 \\
y_2 & z_2
\end{pmatrix}
+ \begin{pmatrix}
w_1 & x_1 \\
y_1 & z_1
\end{pmatrix}
\]

which simplifies to

\[
\begin{pmatrix}
r_1 s_1 w_0 - w_0 r_1 s_1 & w_0 r_2 - r_2 z_0 \\
z_0 s_2 - s_2 w_0 & z_0 s_1 r_1 - s_1 r_1 z_0
\end{pmatrix}
= \begin{pmatrix}
r_1 y_1 - x_1 s_1 & x_2 + r_1 z_1 - w_1 r_1 \\
-y_2 + s_1 w_1 - z_1 s_1 & s_1 x_1 - y_1 r_1
\end{pmatrix}.
\]
Using our assumptions $a_1$ and $b_1$ from the previous section, we can compute that the equations in the main diagonal hold true. Thus we need to find $r_2 \in E_0R(1 - E_0)$ and $s_2 \in (1 - E_0)RE_0$ such that

\[
\begin{align*}
&\quad a_2) : w_0r_2 - r_2z_0 = x_2 + r_1z_1 - w_1r_1, \\
&\quad b_2) : z_0s_2 - s_2w_0 = -y_2 + s_1w_1 - z_1s_1. \end{align*}
\]

Note that these equations are very similar to the ones we computed at the end of the previous section. In fact it seems as if the solutions at the second extension must take into account and “cancel” our choice of solutions $r_1$ and $s_1$ of the first extension.

That is, in the construction of $E_2$ (as in $E_1$), the elements of the main diagonal of the Peirce Decomposition, those in $E_0RE_0$ and $(1 - E_0)R(1 - E_0)$, are completely determined when we are forced to satisfy equation (I), while the elements in the off-diagonal, those in $E_0R(1 - E_0)$ and $(1 - E_0)RE_0$, remain without restriction. This can be more easily seen by rearranging equation (3) to look like $(1 - E_0)E_2 - E_2E_0 = E_1E_1$, which occurs entirely in $E_0RE_0 \oplus (1 - E_0)R(1 - E_0)$. It is only after we attempt to satisfy equation (C) that we run into constraints on this off-diagonal, while the elements we were forced to use on the main diagonal seem to work regardless.

### 2.3 Further Extensions

Finally we consider an arbitrary extension: $R_n = R[X]/(X^{n+1})$. Let $A_0 + A_1X + A_2X^2 + \ldots + A_nX^n = \sum_{i=0}^{n} A_iX^i \in R_n$. Then we need to find $E_0, E_1, E_2, \ldots, E_n \in R$ where:

\[
\begin{align*}
&\quad I): \left( \sum_{i=0}^{n} E_iX^i \right)^2 = \sum_{i=0}^{n} E_iX^i, \\
&\quad U): \sum_{i=0}^{n} (A_i - E_i)X^i \in U(R_n), \\
&\quad C): \left( \sum_{i=0}^{n} A_iX^i \right) \left( \sum_{i=0}^{n} E_iX^i \right) = \left( \sum_{i=0}^{n} E_iX^i \right) \left( \sum_{i=0}^{n} A_iX^i \right). \end{align*}
\]
Considering the first \( n-1 \) terms of these three equations we see that \( \sum_{i=0}^{n-1} A_i X^i = (\sum_{i=0}^{n-1} E_i X^i) + (\sum_{i=0}^{n-1} A_i X^i - \sum_{i=0}^{n-1} E_i X^i) \) must be a strongly clean decomposition in \( R_{n-1} \). Assuming this, the equations above simplify as follows:

Equation (U) is always true as \( (A_0 - E_0) \in U(R) \).

Equation (I) is true if and only if \( \sum_{i=0}^{n} E_i E_{n-i} = E_n \). (5)

Equation (C) is true if and only if \( A_0 E_n - E_n A_0 = \sum_{i=0}^{n-1} (E_i A_{n-i} - A_{n-i} E_i) \) if and only if \( [A_0, E_n] = \sum_{i=0}^{n-1} [E_i, A_{n-i}] \). (6)

Thus \( \sum_{i=0}^{n-1} A_i X^i \) is strongly clean if (and only if) we can find an \( E_n \) that satisfies (5) and (6) for some strongly clean decomposition \( \sum_{i=0}^{n-1} A_i X^i = (\sum_{i=0}^{n-1} E_i X^i) + (\sum_{i=0}^{n-1} A_i X^i - \sum_{i=0}^{n-1} E_i X^i) \in R_{n-1} \).

Using the same techniques as in previous sections, namely considering the Peirce Decomposition of \( R \) with respect to the idempotent \( E_0 \) from the constant term, we arrive at some very familiar equations which need to be solved.

In short a pattern emerges. As in earlier sections, equation (I) completely determines what to place in the main diagonal of the Peirce Decomposition (with respect to \( E_0 \)) of \( E_n \). Then, assuming that \( E_1, E_2, \ldots, E_{n-1} \) have been constructed accordingly, we simply need some control on the off-diagonal, that is an ability to solve what would be equations \( a_n \) and \( b_n \) within \( E_0 R(1 - E_0) \) and \( (1 - E_0) R E_0 \), of \( E_n \) in order to obtain a strongly clean decomposition.

### 2.4 Summary and Outline of Plan

All the work from this chapter leads to three questions:

1. What exactly have we discovered?
2. Can we be sure these discoveries extend to $R[x]$?

3. How do these discoveries help answer our questions from Chapter 1?

The answer to question 1 begins with the definition of strongly clean. While it seems like we need to satisfy three separate equations, conveniently labeled (I) for idempotent, (U) for unit, and (C) for commute, the properties of our rings $R_n$ ensure that we really need only satisfy two. Condition (U) becomes trivial if we assume that our base ring $R$ is strongly clean. Since $R$ is a quotient of $R_n$, this must be true by our comments at the end of Chapter 1. Then we are left with equations (I) and (C) to be solved. Here is where the Peirce Decomposition plays an important role. Since $R$ is strongly clean, there is an idempotent in $R$ with respect to which the constant term of our power series is strongly clean. Using this, we begin to build an idempotent power series with respect to which our power series is strongly clean by decomposing elements of $R$ (the coefficients of our power series) into four mutually exclusive pieces. While investigating condition (I) under this format, we discovered a solution that depends only on the two Peirce Corners. This is important as when we combined the properties of strongly clean in each $R_i, i < n$ with our solution to (I), we discovered that the solution to (C) depends only on being able to solve equations in the other pieces of the Peirce Decomposition.

We answer question 2 in two parts. First, the fact that (U) is easily solvable holds true in $R[x]$. This is the structure we alluded to in Chapter 1 - that while extending to the formal power series ring adds many more elements to check for strongly cleanness, it also adds a proportional number of units to increase the likelihood of finding an answer (while the polynomial ring fails in this regard). The answer to the second part is a bit trickier. While it seems like the solvability of (I) and dependence of (C) can be attributed to induction, there is no reason a priori that this pattern will extend to $R[x]$. This is because the chain mentioned at the introduction of the chapter is based more on motivating an idea than maintaining structure. In fact, the operation of multiplication is slightly different at each step of the chain. It must be shown that this pattern of solving (I) and simplifying (C) extends all the
way to \( R[x] \) (which luckily it does).

This leads to question 3. The crux of showing an element to be strongly clean lies in finding the right idempotent. Within a formal power series ring, idempotents (let alone the right ones) are extremely complicated. The results of this chapter allow us to bypass the process of searching through \( R[x] \) for an idempotent and construct one instead. Of course, this construction all depends on us being able to find an idempotent at the constant term of our series that allows us to maintain some degree of control as the induction process unfolds. In what follows, we define this idea of an idempotent with “control” in the off-diagonal pieces of the Peirce Decomposition and show it is sufficient for strongly cleanness not only in \( R_n \), but in \( R[x] \) as well.
CHAPTER 3

Strongly Clean Power Series

Having established all the prerequisites necessary to proceed with the dissertation, we are finally ready to present our first results. This chapter begins as we define the “structure” alluded to earlier as a surjectivity condition on particular pieces of the Peirce Decomposition of a strongly clean ring. We then proceed to show that this condition is sufficient for the power series extension of that ring to be strongly clean. Finally, the chapter is concluded with an interesting result about our definition.

3.1 Optimally Clean

Let us begin by recalling a few fundamental properties of strongly clean elements.

Let $R$ be a ring. If $r = e + (r - e) \in R$ where $e^2 = e$, $(r - e) \in U(R)$ and $er = re$, then:

1. $r \in eRe \oplus (1 - e)R(1 - e),$

2. left multiplication by $r$ and $1 - r$ are right $R$-module automorphisms on $(1 - e)R$ and $eR$ respectively,

3. right multiplication by $r$ and $1 - r$ are left $R$-module automorphisms on $R(1 - e)$ and $Re$ respectively.
These lead to our first new definition.

**Definition 3.1.1.** Let $R$ be a ring. For $r \in R$ with strongly clean decomposition $r = e + (r - e)$ where $e^2 = e, (r - e) \in U(R)$ and $er = re$, let

$$r\Psi_r : eR(1-e) \oplus (1-e)Re \to eR(1-e) \oplus (1-e)Re$$

be the group homomorphism defined as follows:

$$r\Psi_r(x) = rx - xr.$$

First note that by property 1 of strongly clean elements above, $eR(1-e)$ and $(1-e)Re$ are both $r\Psi_r$ invariant. Second, properties 2 and 3 do not necessarily imply that $r\Psi_r$ is even surjective, let alone an automorphism. Ultimately, this is exactly the structure we need to prove our main theorem (notice how our map takes place in the two pieces of the Peirce Decomposition we explored in Chapter 2). We now name this condition and state it in a cleaner fashion.

**Definition 3.1.2.** Let $R$ be a ring. An element $r \in R$ is called **optimally clean** if there exists an idempotent $e^2 = e \in R$, where $(r - e) \in U(R)$ and $er = re$, such that for each $a \in R$ there exists an $x \in R$ satisfying $[r, x] = [e, a]$. The ring $R$ is **optimally clean** if all of its elements are optimally clean.

**Remark.** If $x \in R$ satisfies $[r, x] = [e, a]$ as listed above, we may assume $x \in eR(1-e) \oplus (1-e)Re$.

**Proof.** Suppose that $[r, x] = [e, a]$. Note then that $[r, ex(1-e) + (1-e)xe]$

$$= [r, ex(1-e)] + [r, (1-e)xe]$$
$$= e[r, x](1-e) + (1-e)[r, x]e$$
$$= e[e, a](1-e) + (1-e)[e, a]e$$
$$= ea(1-e) - (1-e)ae$$
$$= [e, a],$$

and $ex(1-e) + (1-e)xe \in eR(1-e) \oplus (1-e)Re$. 

\[\square\]
Lemma 3.1.3. A strongly clean element $r \in R$, where $e^2 = e \in R, (r - e) \in U(R)$ and $er = re$, is optimally clean with respect to $e$ if and only if, $\Psi_r$ is surjective on $eR(1-e) \oplus (1-e)Re$.

Proof. Let $y \in eR(1 - e) \oplus (1 - e)Re$. Set $y^* = ey(1 - e) - (1 - e)ye$. Since $r$ is optimally clean, there exists an $x \in eR(1 - e) \oplus (1 - e)Re$ such that $[r, x] = [e, y^*] = y$. That is, $r\Psi_r$ is surjective. The reverse is proved analogously.

Here are a few basic properties of optimally clean rings.

Proposition 3.1.4. Let $R$ and $R_i$ denote rings for $i \in I$, an indexed set.

1. If $R$ is optimally clean, every homomorphic image of $R$ is optimally clean.

2. The direct product $\prod R_i$ is optimally clean if and only if each $R_i$ is optimally clean.

Proof. (1) Let $\sigma : R \to A$ be a surjective ring homomorphism and let $s \in A$. Then $s = \sigma(r)$ for some $r \in R$. Since $R$ is optimally clean, write $r = e + (r - e)$ where $e^2 = e \in R, (r - e) \in U(R)$ and $er = re$ as an optimally clean expression. Then $s = \sigma(r) = \sigma(e) + \sigma(r - e)$ is a strongly clean expression in $A$. Now let $b \in A$. Then $b = \sigma(a)$ for some $a \in R$. Since $R$ is optimally clean, there is an $x \in R$ such that $[r, x] = [e, a]$. Then $[s, \sigma(x)] = [\sigma(r), \sigma(x)] = \sigma([x, r]) = \sigma([e, a]) = [\sigma(e), \sigma(a)] = [\sigma(e), b]$. Thus $\sigma(r)$ is optimally clean.

(2) This is an obvious coordinate-wise proof.

Our final proposition of the section illustrates how the optimally clean condition maintains one very important property of the strongly clean condition - that it passes to the Peirce Corners.

Proposition 3.1.5. If $R$ is optimally clean then $eRe$ is optimally clean for any $e^2 = e \in R$.

Proof. Let $e^2 = e \in R$ and $ere, eae \in eRe$. By [5, Corollary 1.4], $eRe$ is strongly clean. Since $R$ is optimally clean, there exists an $f^2 = f \in R$ such that $ere - f \in U(R)$, $(ere)f = f(ere)$, and for $a \in R$, there exists an $x \in R$ such that $[ere, x] = [f, a]$. It can be shown that $f = ef = fe = efe$, so $(efe)^2 = efe$ and $ere - efe \in U(eRe)$. Then $ere = efe + (ere - efe)$
is a strongly clean decomposition in $eRe$. More $[ere, exe] = eerexe - exeere = e(ere - xere)e = e(fa - af)e = efae - eafe = efeeae - eaeefe = [efe, eae]$. Thus $eRe$ is optimally clean.

### 3.2 The Extension Theorem

We now show that the optimally clean assumption allows us enough freedom in the necessary pieces of the Pierce Decomposition of a ring to construct a strongly clean expression in the power series extension. We begin with a lemma due to [7, Claim 1].

**Lemma 3.2.1.** Let $R$ be a ring and $n \in \mathbb{N}$. If $e_0^2 = e_0 \in R$ and $e_k \in R, 0 < k < n$, have been defined so that $e_k = \sum_{i=0}^{k} e_i e_{k-i}$, then $e_0 \left( \sum_{i=1}^{n-1} e_i e_{n-i} \right) = \left( \sum_{i=1}^{n-1} e_i e_{n-i} \right) e_0$.

This brings us to the main result of the section.

**Theorem 3.2.2.** Let $R$ be a ring and $r = \sum_{i \geq 0} r_i x^i \in R[x]$. If $r_0$ or $1 - r_0$ is optimally clean in $R$, then $r$ is strongly clean in $R[x]$.

**Proof.** Since $r$ is strongly clean in $R[x]$ if and only if $1 - r$ is strongly clean, we need only to prove the case for $r_0$ being optimally clean. By assumption let $r_0 = e_0 + (r_0 - e_0)$ where $e_0^2 = e_0 \in R, r_0 e_0 = e_0 r_0$, and $r_0 - e_0 \in U(R)$ be an optimally clean decomposition of $r_0$ in $R$. Obviously this expression is strongly clean. For each $k \geq 1$, we must find $e_k \in R$ satisfying

$$
\alpha_k : e_k = \sum_{i=0}^{k} e_i e_{k-i}
$$

$$
\beta_k : [r_0, e_k] = \sum_{i=0}^{k-1} [e_i, r_{k-i}].
$$

Then $\sum_{i \geq 0} e_i x^i = \left( \sum_{i \geq 0} e_i x^i \right)^2$ by $\alpha_k, k \geq 1$, $\sum_{i \geq 0} r_i x^i - \sum_{i \geq 0} e_i x^i \in U(R[x])$ as $r_0 - e_0 \in U(R)$, and $(\sum_{i \geq 0} r_i x^i)(\sum_{i \geq 0} e_i x^i) = (\sum_{i \geq 0} e_i x^i)(\sum_{i \geq 0} r_i x^i)$ by $\beta_k, k \geq 1$. Setting $e = \sum_{i \geq 0} e_i x^i$ gives a strongly clean decomposition $r = e + (r - e)$ in $R[x]$. 

\[ \square \]
We proceed by induction. Let \( n \in \mathbb{N} \) and assume there exists an \( e_k \in R \) satisfying \( \alpha_k, \beta_k \) for \( 1 \leq k \leq n - 1 \). Then we must find an \( e_n \in R \) such that

\[
\text{I): } e_n = \sum_{i=0}^{n} e_i e_{n-i}
\]

\[
\text{C): } [r_0, e_n] = \sum_{i=0}^{n-1} [e_i, r_{n-i}].
\]

To this end let

\[
e_{n*} = (1 - e_0)\left(\sum_{i=1}^{n-1} e_i e_{n-i}\right) - \left(\sum_{i=1}^{n-1} e_i e_{n-i}\right)e_0.
\]

Note that \( e_{n*} \in e_0Re_0 \oplus (1 - e_0)R(1 - e_0) \). Next define

\[
r_n^* = r_n + \sum_{i=1}^{n-1} (e_0[e_i, r_{n-i}])(1 - e_0) - (1 - e_0)[e_i, r_{n-i}]e_0).
\]

Since \( r_0 \) is optimally clean with respect to \( e_0 \), let \( e_n^* \in e_0R(1 - e_0) \oplus (1 - e_0)Re_0 \) such that

\[
[r_0, e_n^*] = [e_0, r_n^*].
\]

Now define

\[
e_n = e_{n*} + e_n^*.
\]

Then by Lemma 3.2.1,

\[
\sum_{i=1}^{n-1} e_i e_{n-i} = (1 - e_0)\left(\sum_{i=1}^{n-1} e_i e_{n-i}\right) + \left(\sum_{i=1}^{n-1} e_i e_{n-i}\right)e_0
\]

\[
= (1 - e_0)e_{n*} - e_{n*}e_0
\]

\[
= (1 - e_0)e_n - e_ne_0.
\]

So \( \sum_{i=0}^{n} e_i e_{n-i} = e_n \) and \( e_n \) satisfies equation (I).

Finally, we must show that \( e_n \) satisfies equation (C). First notice that \( [r_0, e_n^*] \)

\[
= [e_0, r_n^*]
\]

\[
= [e_0, r_n + \sum_{i=1}^{n-1} (e_0[e_i, r_{n-i}])(1 - e_0) - (1 - e_0)[e_i, r_{n-i}]e_0]
\]

\[
= [e_0, r_n] + e_0 \sum_{i=1}^{n-1} (e_0[e_i, r_{n-i}])(1 - e_0) - (1 - e_0)[e_i, r_{n-i}]e_0)
\]

\[
= [e_0, r_n] + e_0 \sum_{i=1}^{n-1} (e_0[e_i, r_{n-i}])(1 - e_0) - (1 - e_0)[e_i, r_{n-i}]e_0)
\]
More, \[ [r_0, e_{n*}] \]
\[ = [r_0, (1 - e_0) \sum_{i=1}^{n-1} e_i e_{n-i}] - [r_0, \sum_{i=1}^{n-1} e_i e_{n-i}] \]
\[ = (1 - e_0)[r_0, \sum_{i=1}^{n-1} e_i e_{n-i}] - [r_0, \sum_{i=1}^{n-1} e_i e_{n-i}] e_0 \]
\[ = (1 - e_0)[r_0, \sum_{i=1}^{n-1} e_i e_{n-i}] (1 - e_0) - e_0[r_0, \sum_{i=1}^{n-1} e_i e_{n-i}] e_0 \]
\[ = (1 - e_0)(1 - e_0) \sum_{i=1}^{n-1} [e_i, r_{n-i}] - \sum_{i=1}^{n-1} [e_i, r_{n-i}] e_0 (1 - e_0) - e_0 \sum_{i=1}^{n-1} [e_i, r_{n-i}] \]
\[ - \sum_{i=1}^{n-1} [e_i, r_{n-i}] e_0 \text{ by our induction assumptions on } e_k, 1 \leq k < n \text{ and } [7, \text{Claim 2}] \]
\[ = (1 - e_0) \sum_{i=1}^{n-1} [e_i, r_{n-i}] (1 - e_0) + e_0 \sum_{i=1}^{n-1} [e_i, r_{n-i}] e_0. \] (2)

Then \[ [r_0, e_n] \]
\[ = [r_0, e_{n*}] + [r_0, e_n^*] \]
\[ = [e_0, r_n] + \sum_{i=1}^{n-1} (e_0[e_i, r_{n-i}] (1 - e_0) + (1 - e_0)[e_i, r_{n-i}] e_0) \]
\[ + (1 - e_0) \sum_{i=1}^{n-1} [e_i, r_{n-i}] (1 - e_0) + e_0 \sum_{i=1}^{n-1} [e_i, r_{n-i}] e_0 \text{ by (1), (2)} \]
\[ = \sum_{i=0}^{n-1} [e_i, r_{n-i}]. \]

Thus \( e_n \) also satisfies equation (C). By induction, \( r \) is strongly clean in \( R[x] \).

In short, we have taken a big step towards answering one of our initial questions.

**Corollary 3.2.3.** If \( R \) is optimally clean, then \( R[x] \) is strongly clean.

Since the strongly clean condition is stable under quotients, we get the following from Theorem 3.2.2.

**Theorem 3.2.4.** Let \( n \in \mathbb{N} \) and \( r = \sum_{i=0}^{n-1} r_i x^i \in R[x]/(x^n) \). If \( r_0 \) or \( 1 - r_0 \) is optimally clean in \( R \), then \( r \) is strongly clean in \( R[x]/(x^n) \).


3.3 The Invariant Theorem

Unfortunately, we have not been able to prove the reverse implication for Theorem 3.2.2.

Our next lemma builds to a Theorem which implies this may be more difficult than initially anticipated.

Lemma 3.3.1. Let \( R \) be a ring and \( n \in \mathbb{N} \). If \( e_0^2 = e_0 \in R \) and for \( 0 \leq k < n, e_k \in R \) and \( y_k \in R \) have been defined so that \( e_k = \sum_{i=0}^{k} e_i e_{k-i} \) and \( y_k = \sum_{i=0}^{k} (y_i e_{k-i} + e_{k-i} y_i) \), then

\[
e_0 \sum_{i=0}^{n-1} (y_i e_{n-i} + e_{n-i} y_i) = \sum_{i=0}^{n-1} (y_i e_{n-i} + e_{n-i} y_i) e_0.
\]

Proof. Note that

\[
e_0 \sum_{i=0}^{n-1} y_i e_{n-i} = \sum_{i=0}^{n-1} e_0 y_i e_{n-i} = \sum_{i=0}^{n-1} y_i e_{n-i} - \sum_{j=0}^{i-1} e_j e_{n-i-j} + \sum_{j=1}^{n-i-1} e_j e_{n-i-j}
\]

by our assumptions on \( y_k, 0 \leq k < n \)

\[
= \sum_{i=0}^{n-1} y_i e_{n-i} - \sum_{j=0}^{n-i-1} e_j e_{n-i-j} - \sum_{j=1}^{n-i-1} e_j e_{n-i-j} = \sum_{i=0}^{n-1} y_i e_{n-i} e_0 - \sum_{j=1}^{n-i-1} e_j y_i e_{n-i-j}
\]

by our assumptions on \( e_k, 0 \leq k < n \).

Similarly

\[
\sum_{i=0}^{n-1} (e_{n-i} y_i) e_0 = \sum_{i=0}^{n-1} (e_0 e_{n-i} y_i - \sum_{j=1}^{n-i-1} e_j y_i e_{n-i-j}).
\]

Then

\[
e_0 \sum_{i=0}^{n-1} y_i e_{n-i} - \sum_{i=0}^{n-1} e_{n-i} y_i e_0 = \sum_{i=0}^{n-1} (y_i e_{n-i} e_0 - \sum_{j=1}^{n-i-1} e_j y_i e_{n-i-j}) - \sum_{i=0}^{n-1} (e_0 e_{n-i} y_i - \sum_{j=1}^{n-i-1} e_j y_i e_{n-i-j})
\]

\[=
\sum_{i=0}^{n-1} y_i e_{n-i} e_0 - \sum_{i=0}^{n-1} e_0 e_{n-i} y_i \text{ and the result holds.}
\]

And now we present the result of the section.
Remark. For ease of presentation in the following proof, we adopt the slight abuse of notation that \( \sum_{i=0}^{-1} r_i = 0 \) and \( \sum_{i=1}^{0} r_i = 0 \) for \( r_i \in R \) a ring.

**THEOREM 3.3.2.** Let \( R \) be a ring. Then \( r = \sum_{i \geq 0} r_i x^i \in R[x] \) is optimally clean if and only if \( r_0 \in R \) is optimally clean.

**Proof.** By Proposition 3.1.4, the forward implication is clear. For the reverse, let \( e^2 = e = \sum_{i \geq 0} e_i x^i \in R[x] \) be as constructed in the proof of Theorem 3.2.2 so that \( re = er \) and \( r - e \in U(R[x]) \). Let \( a = \sum_{i \geq 0} a_i x^i \in R[x] \) and \( y_0 \in e_0 R(1 - e_0) \oplus (1 - e_0)Re_0 \) such that \( [r_0, y_0] = [e_0, a_0] \) by our assumption on \( R \). Then for all \( k \geq 1 \), it will suffice to find \( y_k \in R \) such that

\[
\gamma_k : y_k = \sum_{i=0}^{k} (y_i e_{k-i} + e_{k-i} y_i)
\]

\[
\delta_k : [r_0, y_k] = \sum_{i=0}^{k} [e_i, a_{k-i}] + \sum_{i=1}^{k} [y_{k-i}, r_i]
\]

Then \( y = \sum_{i \geq 0} y_i x^i \) satisfies \( y = ye + ey \) by \( \gamma_k \Rightarrow y \in eR[x](1 - e) \oplus (1 - e)R[x]e \) and

\[
\sum_{i=0}^{k} [r_i, y_{k-i}] x^k = \sum_{i=0}^{k} [e_i, a_{k-i}] x^k \quad \text{for all} \quad k \in \mathbb{N} \quad \text{by} \quad \delta_k \Rightarrow [r, y] = [e, a].
\]

We proceed by induction. Obviously \( y_0 \) satisfies the conditions. Let \( n \in \mathbb{N} \) and assume there exists \( y_k \in R \) satisfying \( \gamma_k, \delta_k \) for \( 1 \leq k \leq n - 1 \). Then we must find an \( y_n \in R \) such that

\[
A) : y_n = \sum_{i=0}^{n} (y_i e_{n-i} + e_{n-i} y_i)
\]

\[
B) : [r_0, y_n] = \sum_{i=0}^{n} [e_i, a_{n-i}] + \sum_{i=1}^{n} [y_{n-i}, r_i]
\]

To this end set \( S = \sum_{i=0}^{n} [e_i, a_{n-i}] + \sum_{i=1}^{n} [y_{n-i}, r_i] \) and

\[
y_n^* = e_0 S(1 - e_0) - (1 - e_0)Se_0.
\]
Since \( r_0 \) is optimally clean with respect to \( e_0 \), let \( y_{n*} \in e_0 R(1 - e_0) \oplus (1 - e_0)Re_0 \) such that 
\([r_0, y_{n*}] = [e_0, y_n']\). Then define 
\[ y_n = y_{n*} + (1 - e_0) \sum_{i=0}^{n-1} (y_ie_{n-i} + e_{n-i}y_i)(1 - e_0) - e_0 \sum_{i=0}^{n-1} (y_ie_{n-i} + e_{n-i}y_i)e_0. \]

First note that \( y_n \)
\[ = e_0y_n(1 - e_0) + (1 - e_0)y_ne_0 + (1 - e_0) \sum_{i=0}^{n-1} (y_ie_{n-i} + e_{n-i}y_i)(1 - e_0) \]
\[ - e_0 \sum_{i=0}^{n-1} (y_ie_{n-i} + e_{n-i}y_i)e_0 \]
\[ = e_0y_n + y_ne_0 - 2e_0y_ne_0 + 2e_0 \sum_{i=0}^{n-1} (y_ie_{n-i} + e_{n-i}y_i)e_0 - 2e_0 \sum_{i=0}^{n-1} (y_ie_{n-i} + e_{n-i}y_i)e_0 \]
by Lemma 3.3.1
\[ = \sum_{i=0}^{n} (y_ie_{n-i} + e_{n-i}y_i) + 2e_0 \sum_{i=0}^{n-1} (y_ie_{n-i} + e_{n-i}y_i)e_0 - 2e_0 \sum_{i=0}^{n-1} (y_ie_{n-i} + e_{n-i}y_i)e_0 \]
\[ = \sum_{i=0}^{n} (y_ie_{n-i} + e_{n-i}y_i). \] Thus \( y_n \) satisfies (A).

Further, \([r_0, \sum_{i=0}^{n-1} (y_ie_{n-i} + e_{n-i}y_i)]\)
\[ = \sum_{i=0}^{n-1} ([r_0, y_ie_{n-i}] + [r_0, e_{n-i}y_i]) \]
\[ = \sum_{i=0}^{n-1} (y_i[r_0, e_{n-i}] + [r_0, y_i]e_{n-i} + e_{n-i}[r_0, y_i] + [r_0, e_{n-i}]y_i) \]
\[ = \sum_{i=0}^{n-1} (y_i \sum_{j=0}^{n-i-1} [e_j, r_{n-i-j}] + \sum_{j=0}^{i-1} [y_j, r_{i-j}] + \sum_{j=0}^{i} [e_j, a_{i-j}] + \sum_{j=0}^{i-1} [y_j, r_{j-i}])e_{n-i} + e_{n-i}(\sum_{j=0}^{i-1} [y_j, r_{j-i}]) \]
\[ + \sum_{j=0}^{i} [e_j, a_{i-j}] + \sum_{j=0}^{i} [e_j, r_{n-i-j}y_i] \] by our assumptions on \( e_k, y_k, 1 \leq k \leq n - 1 \). (*)

Where \( \sum_{i=0}^{n-1} (\sum_{j=0}^{i} [e_j, a_{i-j}]e_{n-i} + e_{n-i}(\sum_{j=0}^{i} [e_j, a_{i-j}])) \)
\[ = \sum_{i=0}^{n-1} (\sum_{j=0}^{i} [e_j, a_i]e_{n-i-j} + e_{n-i-j}(\sum_{j=0}^{i} [e_j, a_i])) \] after reindexing
\[ = \sum_{i=0}^{n-1} (\sum_{j=0}^{n-i-1} ([e_j e_{n-i-j}, a_i] - e_j[e_{n-i-j}, a_i] + e_{n-i-j}[e_j, a_i])) \]
\[ = \sum_{i=0}^{n-1} (\sum_{j=0}^{n-i-1} ([e_j e_{n-i-j}, a_i] - e_0[e_{n-i}, a_i] + e_{n-i}[e_0, a_i])) \]
\[\begin{align*}
\sum_{i=0}^{n-1} (\sum_{j=0}^{n-i-1} (e_j e_{n-i-j}, a_i) - e_0[e_{n-i}, a_i] + [e_{n-i} e_0, a_i] - [e_{n-i}, a_i] e_0) \\
= \sum_{i=0}^{n-1} \left( \sum_{j=0}^{n-i-1} (e_j e_{n-i-j}, a_i) - e_0[e_{n-i}, a_i] - [e_{n-i}, a_i] e_0 \right) \\
= \sum_{i=0}^{n-1} \left( [e_{n-i}, a_i] - e_0[e_{n-i}, a_i] - [e_{n-i}, a_i] e_0 \right) \text{ by our assumptions on } e_k, 1 \leq k \leq n-1 \\
= (1 - e_0) \sum_{i=0}^{n-1} [e_{n-i}, a_i] - \sum_{i=0}^{n-1} [e_{n-i}, a_i] e_0. 
\end{align*}\]

And 
\[\begin{align*}
\sum_{i=0}^{n-1} (y_i \sum_{j=0}^{n-i-1} [e_j, r_{n-i-j}] + \sum_{j=0}^{n-i-1} [y_j, r_{i-j}] e_{n-i} + e_{n-i} \sum_{j=0}^{i-1} [y_j, r_{i-j}] + \sum_{j=0}^{n-i-1} [e_j, r_{n-i-j}] y_i) \\
= \sum_{i=0}^{n-1} (y_i e_0, r_{n-i}) + [e_0, r_{n-i}] y_i + \sum_{j=1}^{n-i-1} (y_i [e_j, r_{n-i-j}] + [y_i, r_j] e_{n-i-j} \\
+ e_{n-i-j} [y_i, r_j] + [e_j, r_{n-i-j}] y_i)) \text{ after reindexing} \\
= \sum_{i=0}^{n-1} \left( [y_i e_j, r_{n-i-j}] + [e_j y_i, r_{n-i-j}] \right) - \sum_{i=0}^{n-1} \left( [y_i, r_{n-i}] e_0 + e_0 [y_i, r_{n-i}] \right) \\
= \sum_{i=0}^{n-1} \left( \sum_{j=0}^{n-i} ([y_j e_{i-j} + e_{i-j} y_j, r_{n-i}] - \sum_{i=0}^{n-1} ([y_i, r_{n-i}] e_0 + e_0 [y_i, r_{n-i}]) \right) \text{ after returning to our original index} \\
= \sum_{i=0}^{n-1} ([y_i, r_{n-i}]) - \sum_{i=0}^{n-1} ([y_i, r_{n-i}] e_0 + e_0 [y_i, r_{n-i}]) \text{ by our assumptions } \gamma_k, 1 \leq k \leq n-1 \\
= (1 - e_0) \sum_{i=0}^{n-1} [y_i, r_{n-i}] - \sum_{i=0}^{n-1} [y_i, r_{n-i}] e_0. 
\end{align*}\]

So in short \([r_0, \sum_{i=0}^{n} (y_i e_{n-i} + e_{n-i} y_i)] = (\ast) = (1) + (2) = (1 - e_0)S - Se_0.\]

Finally, 
\[\begin{align*}
[r_0, y_n] \\
= [r_0, y_n] + (1 - e_0) \sum_{i=0}^{n-1} (y_i e_{n-i} + e_{n-i} y_i) (1 - e_0) - e_0 \sum_{i=0}^{n-1} (y_i e_{n-i} + e_{n-i} y_i) e_0 \\
= [e_0, y_n^*] + (1 - e_0) [r_0, \sum_{i=0}^{n-1} (y_i e_{n-i} + e_{n-i} y_i)] (1 - e_0) - e_0 [r_0, \sum_{i=0}^{n-1} (y_i e_{n-i} + e_{n-i} y_i)] e_0 \\
= e_0 S(1 - e_0) + (1 - e_0) S e_0 + (1 - e_0) S (1 - e_0) + e_0 S e_0 \text{ by (3)}
\end{align*}\]
Thus $y_n$ satisfies (B) and $r$ is optimally clean.

So the optimally clean condition not only guarantees passage to a strongly clean formal power series, but passage to an optimally clean formal power series. In fact, by Proposition 3.1.4 and Theorem 3.3.2 we get the following.

**Corollary 3.3.3.** Let $R$ be a ring and let $n \in \mathbb{N}$. The following are equivalent:

1.) $R$ is optimally clean

2.) $R[x]/(x^n)$ is optimally clean

3.) $R[x]$ is optimally clean.
CHAPTER 4

Classes of Optimally Clean Rings

Having firmly established the concept of optimally clean in the previous chapter, we now classify some classes of rings which satisfy the condition. In the first section, many popular strongly clean examples from the literature are shown to be optimally clean. While we are still unable to find an example of a strongly clean ring which is not optimally clean, we culminate the chapter with a strongly clean element that fails to be.

4.1 Examples

Our first proposition illustrates how important it is to allow some level of non-commutativity within our strongly clean rings when trying to construct a counterexample to the optimally clean condition.

Proposition 4.1.1. Let $R$ be a ring and $r \in R$. If there exists an $e^2 = e \in Z(R)$ such that $r - e \in U(R)$, then $r$ is optimally clean.

Proof. For any $a \in R$, letting $x = 0$ solves the equation $[r, x] = [e, a] = ea - ae = ea - ea = 0$.

Thus, all commutative (strongly) clean rings are optimally clean. But we can actually be a little more general in this statement.
Corollary 4.1.2. If $R$ is an abelian (strongly) clean ring, then $R$ is optimally clean.


Perhaps a more interesting class follows next.

Proposition 4.1.3. Let $R$ be a ring. If $r \in R$ is strongly $\pi$-regular, then $r$ is optimally clean.

Proof. Let $e^2 = e \in R$ be the idempotent used in the strongly clean decomposition constructed in [10, Theorem 1]. Let $f = 1 - e$. Then $ere \in \text{Nil}(eRe)$ and $frf \in U(fRf)$. Let $n \in N$ be the smallest such that $(ere)^n = 0$ and let $(frf)^{-1}$ denote the inverse of $frf$ in $fRf$. Then for any $y \in eRf \oplus fRe$,

$$x = \sum_{i=1}^{n}((frf)^{-i}y(ere)^{i-1} - (ere)^{i-1}y(frfr)^{-i}) \in eRf \oplus fRe$$

is the unique solution to $r\Psi_r(x) = y$ (as any other solution would contradict the minimality of $n$). Thus $r\Psi_r$ is surjective and, by Lemma 3.1.3, $r$ is optimally clean. \hfill $\square$

This leads to the following corollaries, which were first proved in [7].

Corollary 4.1.4. Let $R$ be a ring and $r = \sum_{i \geq 0} r_i x^i \in R[x]$. If either $r_0$ or $1 - r_0$ is a strongly $\pi$-regular element of $R$, then $r$ is a strongly clean element of $R[x]$.

Corollary 4.1.5. If $R$ is strongly $\pi$-regular, then $R[x]$ is strongly clean.

It is useful at this point to note that our optimally clean definition is a strict generalization of both the strongly $\pi$-regular and abelian conditions, as illustrated by the following example.

Example. Consider the strongly clean ring $R = T_2(\mathbb{Z}(2))$ which is not abelian and has elements which are not strongly $\pi$-regular. However, it can be shown (to come in Proposition 4.1.8) that $R$ is optimally clean. To support this claim, in [10, Example 2] it was shown that $R[x] = T_2(\mathbb{Z}(2))[x] \cong T_2(\mathbb{Z}(2)[x])$ is strongly clean.

We now prove a converse to Theorem 3.2.2 under one added assumption. Recall that an element is uniquely strongly clean if it has a unique strongly clean decomposition [4].
Proposition 4.1.6. Let \( R \) be uniquely strongly clean. TFAE:

1. \( R \) is optimally clean.

2. \( R[x] \) is strongly clean.

3. \( R[x]/(x^n) \) is strongly clean \( \forall n \in \mathbb{N} \).

4. \( R[x]/(x^2) \) is strongly clean.

Proof. 1) \( \Rightarrow \) 2) By Theorem 3.2.2 and 2) \( \Rightarrow \) 3) \( \Rightarrow \) 4) since the strongly clean condition is stable under quotients. Thus we need only show 4) \( \Rightarrow \) 1). To this end let \( r \in R \) and \( e^2 = e \in R \) where \( er = re, r - e \in U(R) \) be the idempotent providing its uniquely strongly clean decomposition. Let \( a \in R \) and consider \( r + ax \in R[x]/(x^2) \). By assumption, there exists an \( e_0 + e_1 x \in R[x]/(x^2) \) such that

I) \((e_0 + e_1 x)^2 = e_0 + e_1 x,

C) \((e_0 + e_1 x)(r + ax) = (r + ax)(e_0 + e_1 x),

U) \( r - e_0 \in U(R) \).

But the conditions \( e_0^2 = e_0, e_0 r = re_0, \) and \( r - e_0 \in U(R) \) derived from the constant terms of I), C), and U) show that \( e_0 = e \) as \( R \) is uniquely strongly clean. Substituting this fact back into equation C) yields \([r, e_1] = [e, a] \). Thus \( r \) is optimally clean.

We now conclude this section with some examples using both full matrix and upper triangular matrix rings. A lemma starts us off.

Lemma 4.1.7. Let \( R \) be an abelian ring. For each \( i \), let

\[
E_i = \begin{pmatrix} e_i & g_i \\ 0 & f_i \end{pmatrix} \in \mathbb{T}_2(R).
\]

Then

\[
\sum_{i \geq 0} E_i X^i = \sum_{i \geq 0} \begin{pmatrix} e_i & g_i \\ 0 & f_i \end{pmatrix} X^i
\]
is an idempotent in \( T_2(R)[X] \) if and only if \( E_0^2 = E_0 \), \( e_i = f_i = 0 \), and \((e_0 + f_0)g_i = g_i \) for all \( i > 0 \).

Proof. From considering \( \sum_{i \geq 0} E_i X^i = (\sum_{i \geq 0} E_i X^i)^2 \) we get \( e_0 e_1 + e_1 e_0 = e_1 \Rightarrow e_0 e_1 + e_0 e_1 = e_1 \Rightarrow e_0 e_1 = e_0 e_1 = 0 \) since \( R \) is abelian. Substituting back into the first equation gives \( e_1 = 0 \). Now let \( n \in \mathbb{N} \) and assume \( e_i = 0 \) for all \( 0 < i < n \). By our assumptions we get \( e_0 e_n + e_n e_0 = e_n \), which, by earlier arguments, yields \( e_n = 0 \). By induction, \( e_i = 0 \) for all \( i > 0 \). A similar proof holds for \( f_i \). Given this, the rest of the proof is obvious.

Proposition 4.1.8. Let \( S \) be a local ring and let \( R = T_2(S) \). Then \( R \) is strongly clean if and only if \( R \) is optimally clean.

Proof. We need only prove the forward implication. Let

\[
A_0 = \begin{pmatrix} a_0 & c_0 \\ 0 & b_0 \end{pmatrix} \in R.
\]

By assumption there exists an \((E_0)^2 = E_0 \in R\) such that \( E_0 A_0 = A_0 E_0 \) and \( A_0 - E_0 \in U(R) \).

If \( a_0 \) and \( b_0 \) are both units or if \( a_0 - 1 \) and \( b_0 - 1 \) are both units so that \( E_0 \) can be

\[
\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

then \( A_0 \) is optimally clean by Proposition 4.1.1. This leaves us only two cases: when \( a_0, b_0 - 1 \in J(S) \) or \( a_0 - 1, b_0 \in J(S) \) (see \([1]\)). Then \( E_0 \) must be

\[
\begin{pmatrix} 1 & y_0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & y_0 \\ 0 & 1 \end{pmatrix}
\]

respectively for some \( y_0 \in S \). Without loss of generality, assume that \( a_0, b_0 - 1 \in J(S) \) so
$E_0$ is of the first form. Let

$$A_1 = \begin{pmatrix} a_1 & c_1 \\ 0 & b_1 \end{pmatrix} \in R.$$ 

Then we need to find an $X \in R$ such that $[A_0, X] = [E_0, A_1]$.

To this end, set $w = c_1 + y_0b_1 - a_1y_0$ and consider

$$A = \begin{pmatrix} a_0 & w \\ 0 & b_0 \end{pmatrix} \in R.$$ 

Then there exists an $E^2 = E \in R$ such that $A - E \in U(R)$ and $EA = AE$. By our earlier restrictions on $a_0$ and $b_0$, we must have

$$A = E + (A - E) = \begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a_0 - 1 & w - z \\ 0 & b_0 \end{pmatrix}$$

for some $z \in S$ satisfying $a_0z - zb_0 = w$ as $EA = AE$. Let

$$X = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}.$$ 

Then

$$[A_0, X] = \begin{pmatrix} 0 & a_0z - zb_0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & c_1 + y_0b_1 - a_1y_0 \\ 0 & 0 \end{pmatrix} = [E_0, A_1]$$

so that $A_0$ is optimally clean. \qed

**Proposition 4.1.9.** Let $S$ be a commutative uniquely clean ring. Then $R = T_2(S)$ is optimally clean.

**Proof.** By [4, Theorem 10], $R$ is uniquely strongly clean. Since $S$ is (commutative) uniquely clean, we get by [11, Corollary 10] that $S[x]$ is as well. But then $S[x]/x^2$ is also (commutative)
uniquely clean as a quotient of $S[x]$. Thus, by [4] again, $T_2(S[x]/x^2)$ is uniquely strongly clean. But then $T_2(S[x]/x^2) \cong T_2(S)[x]/x^2 = R[x]/x^2$ is strongly clean, so by Proposition 4.1.6, $R$ is optimally clean.

Lemma 4.1.10. Let $R$ be a ring, $r \in R$ and $u \in U(R)$. Then $r$ is optimally clean if and only if $uru^{-1}$ is optimally clean.

Proof. For the forward implication, let $r = e + (r - e)$ where $e^2 = e \in R, re = er$, and $(r - e) \in U(R)$ be an optimally clean decomposition of $r$. Note then that $(ueu^{-1})^2 = ueu^{-1}, uru^{-1}ueu^{-1} = ueu^{-1}uru^{-1}$, and $(uru^{-1} - ueu^{-1}) \in U(R)$. Thus $uru^{-1} = ueu^{-1} + (uru^{-1} - ueu^{-1})$ is a strongly clean decomposition of $uru^{-1}$. Now let $a \in R$. Since $r$ is optimally clean, there exists an $x \in R$ such that $[r, x] = [e, u^{-1}au]$. But then $[uru^{-1}, xu^{-1}]$

$$= uru^{-1}uxu^{-1} - uxu^{-1}uru^{-1}$$

$$= u[r, x]u^{-1}$$

$$= u[e, u^{-1}au]u^{-1}$$

$$= ueu^{-1}a - au^{-1}$$

$$= [ueu^{-1}, a]$$ and $uru^{-1}$ is optimally clean. The reverse implication is proved analogously.

Proposition 4.1.11. Let $D$ be a commutative domain. Then $R = M_2(D)$ is strongly clean if and only if $R$ is optimally clean.

Proof. One direction is clear. For the other let $C \in R$. By Proposition 4.1.1 we may assume neither $C$ nor $1 - C$ is invertible. Since $C$ is strongly clean (as an endomorphism of the $R$-module $D^2$), [10, Theorem 3] states that we have an $R$-module decomposition $D^2 = A \oplus B$, where $A$ and $B$ are $C$-invariant, such that $C$ is invertible on $A$ and $1 - C$ is invertible on $B$. Because we are assuming that neither $C$ nor $1 - C$ is invertible, neither of the submodules $A$ nor $B$ can be all of $D^2$. But $D$, as a corner ring of $R$, is strongly clean and, as a domain, has only trivial idempotents. So $D$ is local [11, Lemma 14]. Hence $A \cong D \cong B$ and there exists a $U \in U(R)$ such that $UCU^{-1}$ is diagonal. But by our assumptions on $C$ again, we
have that neither $UCU^{-1}$ nor $1 - UCU^{-1}$ is invertible. As a diagonal matrix over a local ring, this means one of the diagonal entries of $URU^{-1}$ must be from $J(D)$ while the other from $1 + J(D)$. That is, $UCU^{-1}$ has the form

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in R$$

where either $a, b - 1 \in J(D)$ or $a - 1, b \in J(D)$. Without loss of generality we assume the first case. Notice then that $(a - b) - 1 = a + ((-b) - 1) \in J(D)$, so $(a - b) \in U(D)$. Now let

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R.$$ 

Obviously $UCU^{-1} = E + (UCU^{-1} - E)$ is a strongly clean decomposition in $R$. Finally let

$$M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \in R.$$ 

Then

$$X = \begin{pmatrix} 0 & m_2(a - b)^{-1} \\ -m_3(b - a)^{-1} & 0 \end{pmatrix} \in R$$

is a solution to the equation $[UCU^{-1}, X] = [E, M]$ and $C$ is optimally clean by Lemma 4.1.10. \qed

### 4.2 A Non-Example

We now conclude the chapter by presenting an example of a strongly clean element that is not optimally clean. The following relies heavily on [1, Example 45]

**Example.** Let $k$ be a field. Let $R = k[t_1, t_2, \ldots]_{(t_1)}$ be the ring of polynomials in countably
many indeterminates over $k$, localized at the prime ideal $(t_1)$. Define $\sigma$ and $\tau$ to be endomorphisms of $R$ which fix $k$ and such that $\sigma(t_i) = t_{i+1}$ and $\tau(t_i) = t_{i+1} + 1$ extended to the localization. Construct the left-right skew power series ring $A = \sigma A_r = \sum_{i,j \geq 0} y^i a_{ij} x^j$ with coefficients $a_{ij} \in R$ subject to $xy = yx, ry = y\tau(r)$ and $xs = \sigma(s)x$ for all $r, s \in R$. Note that $A$ (viewed as a right skew power series ring over a left skew power series ring of a local ring with commuting endomorphisms) is also a local ring. Thus the only idempotents are 0 and 1. Now consider the ring $S = T_2(A)$. Let

$$M_0 = \begin{pmatrix} t_1 & 0 \\ 0 & 1 + t_1 \end{pmatrix} \in S.$$ 

Claim. $M_0$ is strongly clean.

Proof. Since $t_1 \in J(R)$ and $1 + t_1 \in 1 - J(R)$, we have the unique strongly clean decompositions $t_1 = 1 + (t_1 - 1)$ and $1 + t_1 = 0 + (1 + t_1)$ respectively in $A$ where $t_1 - 1, 1 + t_1 \in U(R) \subseteq U(A)$. Let

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in S.$$ 

Then it can then be easily checked that $E_0^2 = E_0, E_0 M_0 = M_0 E_0$, and $M_0 - E_0 \in U(S)$ so that $M_0$ is strongly clean in $S$. \qed

Note that this decomposition is not unique, but any strongly clean decomposition of $M_0$ must have an idempotent of the form

$$E_0 = \begin{pmatrix} 1 & c \\ 0 & 0 \end{pmatrix}$$

for some $c \in A$ that satisfies $c = t_1 c - c t_1$. Given any idempotent of this form, we can compute that
Claim. $M_0$ is not optimally clean.

Proof. Let

$$M_1 = \begin{pmatrix} 0 & xy \\ 0 & 0 \end{pmatrix} \in S.$$ 

Then we need to show that for any possible $(E_0)^2 = E_0$ listed above, there does not exist a $Z = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \in E_0S(1-E_0) \oplus (1-E_0)SE_0$ such that $[M_0, Z] = [E_0, M_1]$. If such a $Z$ exists, then we get the identity $xy = t_1z - z(1+t_1)$.

But notice that for any such $z = \sum_{i,j \geq 0} y^i z_{ij} x^j \in A$, $t_1z - z(1+t_1)$

$$= t_1 \sum_{i,j \geq 0} y^i z_{ij} x^j - \sum_{i,j \geq 0} y^i z_{ij} x^j (1+t_1)$$

where the “$xy$” term cancels to 0 once we apply $\tau$ and $\sigma$. Thus $xy \neq t_1z - z(1+t_1)$ for any $z \in A$ and consequently there is no $Z \in E_0S(1-E_0) \oplus (1-E_0)SE_0$ such that $[M_0, Z] = [E_0, M_1]$. Therefore $M_0$ is not optimally clean. \qed

Note that this does not necessarily imply that $M_0 + B_1X + B_2X + ... \in S[X]$ is not strongly clean for any set of $B_i \in S$, but we can compute at least one which is not.

Claim. $M_0 + M_1X \in S[X]$ is not strongly clean.

Proof. By Lemma 4.1.7, any strongly clean decomposition of $M_0 + M_1X$ must have the form

$$\begin{pmatrix} 1 & c \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & z_1 \\ 0 & 0 \end{pmatrix} X + \begin{pmatrix} 0 & z_2 \\ 0 & 0 \end{pmatrix} X^2 + ... \]

$$+ \begin{pmatrix} t_1 - 1 & -c \\ 0 & 1 + t_1 \end{pmatrix} + \begin{pmatrix} 0 & xy - z_1 \\ 0 & 0 \end{pmatrix} X + \begin{pmatrix} 0 & -z_2 \\ 0 & 0 \end{pmatrix} X^2 + ... \].$$
But such elements commute only if there exist \( z_1 \in A \) s.t. \( xy = t_1 z_1 - z_1(1+t_1) \). By the same arguments as in the previous claim, this equation cannot be solved. Therefore, \( M_0 + M_1 X \) is not strongly clean.

It is interesting to note that while our elements \( M_0 \) and \( M_1 \) are both strongly clean, the ring \( S \) is not (consider the element \( M_0 + M_1 \), as shown in [1]). Hence there is no contradiction with Proposition 4.1.8. So, while we know that optimally clean elements are not the same as strongly clean elements, we have yet to draw the same conclusions about rings.
CHAPTER 5

Strongly Clean Skew Power Series

In previous chapters we established a condition that allows for extension to strongly clean formal power series rings. It is natural then to explore what analogous condition is required to extend to the more general strongly clean formal (left) skew power series rings. We begin this chapter by investigating the analogous generalization of the optimally clean definition. In the second and final section we verify the merits of this broader view by proving the corresponding result to Theorem 3.2.2.

5.1 Skew Optimally Clean

Let $R$ be a ring. Since we make extensive use of endomorphisms in this chapter, a few remarks are in order.

First note if $r \in R$ is an idempotent (respectively unit), then $\sigma(r)$ is an idempotent (respectively unit) for any ring endomorphism $\sigma$ of $R$. As a result, if $r = e + (r - e)$ where $e^2 = e \in R, (r - e) \in U(R)$, and $er = re$ is a strongly clean decomposition, then $\sigma(r) = \sigma(e) + (\sigma(r) - \sigma(e))$ where $\sigma(e)^2 = \sigma(e^2) = \sigma(e) \in R, \sigma(r) - \sigma(e) = \sigma(r - e) \in U(R)$, and $\sigma(r)\sigma(e) = \sigma(re) = \sigma(er) = \sigma(e)\sigma(r)$ is a strongly clean decomposition as well.

More, since $R = eR \oplus (1 - e)R$, we can write $R = eR\sigma(e) \oplus eR\sigma(1-e) \oplus (1-e)R\sigma(e) \oplus (1-e)R\sigma(1-e)$. 
Finally, the fact that we are working with a formal left skew power series ring guarantees that coefficients of power series being multiplied on the right are subject to the endomorphism. This leads to the following definition.

**Definition 5.1.1.** For ring endomorphisms $\sigma, \tau$ of $R$ and $a, b \in R$ let $[a, b]_\sigma^\tau = a\tau(b) - b\sigma(a)$ be the $\sigma - \tau$ commutator.

A few properties of this new commutator:

1. Let $1$ be the identity endomorphism of $R$, then $[a, b]_1^1 = [a, b]$.

2. $[a + b, c]_\sigma^\tau = [a, c]_\sigma^\tau + [b, c]_\sigma^\tau$ and $[a, b + c]_\sigma^\tau = [a, b]_\sigma^\tau + [a, c]_\sigma^\tau$.

We are now prepared to generalize our conditions from Chapter 3.

**Definition 5.1.2.** Let $R$ be a ring, $\sigma$ a ring endomorphism of $R$, and $n \in \mathbb{N}$. For $r \in R$ with strongly clean decomposition $r = e + (r - e)$ where $e^2 = e, (r - e) \in U(R)$ and $er = re$, let

$$r \Psi_{\sigma^n(r)} : eR\sigma^n(1 - e) \oplus (1 - e)R\sigma^n(e) \to eR\sigma^n(1 - e) \oplus (1 - e)R\sigma^n(e)$$

be the group homomorphism defined as follows:

$$r \Psi_{\sigma^n(r)}(x) = rx - x\sigma^n(r).$$

As before, the structure sufficient to extend to a strongly clean formal skew power series ring is surjectivity of this homomorphism. Let us state an equivalent form for this.

**Definition 5.1.3.** Let $R$ be a ring and $\sigma$ a ring endomorphism of $R$. An element $r \in R$ is called $\sigma$-optimally clean if there exists an idempotent $e^2 = e \in R$, where $er = re$ and $r - e \in U(R)$, such that for each $a \in R$ and $n \in \mathbb{N}$, there exists an $x_n \in R$ satisfying $[r, x_n]_{\sigma^n} = [e, a]_{\sigma^n}$. The ring $R$ is $\sigma$-optimally clean if every element is $\sigma$-optimally clean.

**Remark.** If $x_n \in R$ satisfies $[r, x_n]_{\sigma^n} = [e, a]_{\sigma^n}$ as listed above, we may assume $x_n \in eR\sigma^n(1 - e) \oplus (1 - e)R\sigma^n(e)$. 
Proof. Suppose that \([r, x_n]_{\sigma^n} = [e, a]_{\sigma^n}\). Note then that
\[
[r, e x_n \sigma^n (1 - e) + (1 - e) x_n \sigma^n (e)]_{\sigma^n}
= [r, e x_n \sigma^n (1 - e)]_{\sigma^n} + [r, (1 - e) x_n \sigma^n (e)]_{\sigma^n}
= e [r, x_n]_{\sigma^n} \sigma^n (1 - e) + (1 - e) [r, x_n]_{\sigma^n} \sigma^n (e)
= e [e, a]_{\sigma^n} \sigma^n (1 - e) + (1 - e) [e, a]_{\sigma^n} \sigma^n (e)
= e a \sigma^n (1 - e) - (1 - e) a \sigma^n (e)
= [e, a]_{\sigma^n},
\]
and \(e x_n \sigma^n (1 - e) + (1 - e) x_n \sigma^n (e) \in e R \sigma^n (1 - e) \oplus (1 - e) R \sigma^n (e)\).

Lemma 5.1.4. A strongly clean element \(r \in R\) where \(e^2 = e \in R, (r - e) \in U(R)\) and \(er = re\) is \(\sigma\)-optimally clean with respect to \(e\) if and only if \(\Psi_{\sigma^n(r)}\) is surjective on \(e R \sigma^n (1 - e) \oplus (1 - e) R \sigma^n (e)\) for all \(n \in \mathbb{N}\).

Proof. Let \(n \in \mathbb{N}\) and \(y \in e R \sigma^n (1 - e) \oplus (1 - e) R \sigma^n (e)\). Set \(y^* = e y \sigma^n (1 - e) - (1 - e) y \sigma^n (e)\).
Since \(r\) is \(\sigma\)-optimally clean, there exists an \(x_n \in e R \sigma^n (1 - e) \oplus (1 - e) R \sigma^n (e)\) such that \([r, x_n]_{\sigma^n} = [e, y^*]_{\sigma^n} = y\). That is, \(\Psi_{\sigma^n(r)}\) is surjective. The reverse is proved analogously.

Obviously, the definition of \(\sigma\)-optimally clean depends heavily on the endomorphism \(\sigma\). It is natural therefore to ask when (or even if) elements satisfy this condition for every ring endomorphism \(\sigma\) of \(R\). While this may seem like a fairly hefty restriction, there are at least two large collections of strongly clean rings to which it applies (as we will see in the next chapter). For now, let us make rigorous our ideas.

Definition 5.1.5. An element \(r \in R\) that is \(\sigma\)-optimally clean for every ring endomorphism \(\sigma\) of \(R\) is called skew optimally clean. A ring is called skew optimally clean if every element is skew optimally clean.

While the skew optimally clean condition will be sufficient to satisfy our goal in the next section, it does not appear to maintain many of the nice structure properties of strongly or optimally clean. This can be attributed to the variation provided by the added endomorphism.
5.2 The Skew Extension Theorem

Our goal now is to prove that a skew optimally clean ring extends to a strongly clean formal skew power series ring. While the proof is very similar to that of Theorem 3.2.2, we provide it here for completeness. Let us first restate the lemma due to [7, Claim 1] under our new condition.

**Lemma 5.2.1.** Let $R$ be a ring, $\sigma$ be a ring endomorphism of $R$ and $n \in \mathbb{N}$. If $e_0^2 = e_0 \in R$ and for $0 < k < n$, $e_k$ have been defined so that $\sum_{i=0}^{k} e_i \sigma^i(e_k-i)$, then $e_0(\sum_{i=1}^{n-1} e_i \sigma^i(e_{n-i})) = \sum_{i=1}^{n-1} e_i \sigma^i(e_{n-i})) \sigma^n(e_0)$.

This brings us to the main proof of the section.

**Theorem 5.2.2.** Let $R$ be a ring, $\sigma$ a ring endomorphism of $R$, and $r = \sum_{i \geq 0} r_i x^i \in R[[x; \sigma]]$. If either $r_0$ or $1 - r_0$ is $\sigma$-optimally clean in $R$, then $r$ is strongly clean in $R[[x; \sigma]]$.

**Proof.** Since $r$ is strongly clean in $R[[x; \sigma]]$ if and only if $1 - r$ is strongly clean, we need only to prove the case for $r_0$ being $\sigma$-optimally clean. By assumption let $r_0 = e_0 + (r_0 - e_0)$ be a $\sigma$-optimally clean decomposition of $r_0$. Obviously this expression is strongly clean. For each $k \geq 1$ we must find $e_k \in R$ satisfying

$$\alpha_k : e_k = \sum_{i=0}^{k} e_i \sigma^i(e_{k-i})$$

$$\beta_k : [r_0, e_k]_{\sigma^k} = \sum_{i=0}^{k-1} [e_i, r_{k-i}]_{\sigma^{k-i}}. $$

Then $\sum_{i \geq 0} e_i x^i$ by $\alpha_k, k \geq 1$, $\sum_{i \geq 0} r_i x^i - \sum_{i \geq 0} e_i x^i \in U(R[[x, \sigma]])$ as $r_0 - e_0 \in U(R)$, and $\sum_{i \geq 0} r_i x^i) \sum_{i \geq 0} e_i x^i) = (\sum_{i \geq 0} e_i x^i) (\sum_{i \geq 0} r_i x^i)$ by $\beta_k, k \geq 1$. Setting $e = \sum_{i \geq 0} e_i x^i$ gives a strongly clean decomposition $r = e + (r - e)$ in $R[[x; \sigma]]$.

We proceed by induction. Let $n \in \mathbb{N}$ and assume there exists an $e_k \in R$ satisfying $\alpha_k, \beta_k$ for $1 \leq k \leq n - 1$. Then we must find an $e_n \in R$ such that
I) \[ e_n = \sum_{i=0}^{n} e_i \sigma^i(e_{n-i}) \]

C) \[ [r_0, e_n]_{\sigma^n} = \sum_{i=0}^{n-1} [e_i, r_{n-i}]_{\sigma^n_{n-i}}. \]

To this end let

\[ e_{n*} = (1 - e_0) \sum_{i=1}^{n-1} e_i \sigma^i(e_{n-i}) - \sum_{i=1}^{n-1} e_i \sigma^i(e_{n-i}) \sigma^n(e_0). \]

Note that \( e_{n*} \in e_0 R \sigma^n(e_0) \oplus (1 - e_0) R \sigma^n(1 - e_0). \) Next define

\[ r_n^* = r_n + \sum_{i=1}^{n-1} (e_0([e_i, r_{n-i}]_{\sigma^n_{n-i}}) \sigma^n(1 - e_0) - (1 - e_0)([e_i, r_{n-i}]_{\sigma^n_{n-i}}) \sigma^n(e_0)). \]

Since \( r_0 \) is \( \sigma \)-optimally clean with respect to \( e_0 \), let \( e_n^* \in e_0 R \sigma^n(1 - e_0) \oplus (1 - e_0) R \sigma^n(e_0) \) such that \([r_0, e_n^*]_{\sigma^n} = [e_0, r_n^*]_{\sigma^n} \). Now define

\[ e_n = e_{n*} + e_n^*. \]

Then by Lemma 5.2.1, \( \sum_{i=1}^{n-1} e_i \sigma^i(e_{n-i}) \)

\[ = (1 - e_0) \left( \sum_{i=1}^{n-1} e_i \sigma^i(e_{n-i}) \right) + \sum_{i=1}^{n-1} e_i \sigma^i(e_{n-i}) \sigma^n(e_0) \]

\[ = (1 - e_0) e_{n*} - e_{n*} \sigma^n(e_0) \]

\[ = (1 - e_0) e_n - e_n \sigma^n(e_0). \]

So \( \sum_{i=0}^{n} e_i \sigma^i(e_{n-i}) = e_n \) and \( e_n \) satisfies equation (I).

Finally we must show \( e_n \) satisfies (C). First notice that \([r_0, e_n^*]_{\sigma^n}\)

\[ = [e_0, r_n^*]_{\sigma^n} \]

\[ = [e_0, r_n + \sum_{i=1}^{n-1} (e_0([e_i, r_{n-i}]_{\sigma^n_{n-i}}) \sigma^n(1 - e_0) - (1 - e_0)([e_i, r_{n-i}]_{\sigma^n_{n-i}}) \sigma^n(e_0))]_{\sigma^n} \]

\[ = [e_0, r_n]_{\sigma^n} + [e_0, \sum_{i=1}^{n-1} (e_0[e_i, r_{n-i}]_{\sigma^n_{n-i}}) \sigma^n(1 - e_0) - (1 - e_0)[e_i, r_{n-i}]_{\sigma^n_{n-i}}] \sigma^n(e_0))]_{\sigma^n} \]

\[ = e_0[e_0, r_n]_{\sigma^n} \sigma^n(1 - e_0) + (1 - e_0)[e_0, r_n]_{\sigma^n} \sigma^n(e_0) \]

\[ + e_0(\sum_{i=1}^{n-1} [e_i, r_{n-i}]_{\sigma^n_{n-i}}) \sigma^n(1 - e_0) + (1 - e_0)(\sum_{i=1}^{n-1} [e_i, r_{n-i}]_{\sigma^n_{n-i}}) \sigma^n(e_0) \]

\[ = e_0(\sum_{i=0}^{n-1} [e_i, r_{n-i}]_{\sigma^n_{n-i}}) \sigma^n(1 - e_0) + (1 - e_0)(\sum_{i=0}^{n-1} [e_i, r_{n-i}]_{\sigma^n_{n-i}}) \sigma^n(e_0). \] (1)
More, $[r_0, e_{n*}]_{\sigma^n}$\[
= [r_0, (1 - e_0) \sum_{i=1}^{n-1} e_i \sigma^i (e_{n-i})]_{\sigma^n} - [r_0, \sum_{i=1}^{n-1} e_i \sigma^i (e_{n-i})]_{\sigma^n} (e_0) \\
= (1 - e_0)[r_0, \sum_{i=1}^{n-1} e_i \sigma^i (e_{n-i})]_{\sigma^n} - [r_0, \sum_{i=1}^{n-1} e_i \sigma^i (e_{n-i})]_{\sigma^n} \sigma^n (e_0) \\
= (1 - e_0)[r_0, \sum_{i=1}^{n-1} e_i \sigma^i (e_{n-i})]_{\sigma^n} \sigma^n (1 - e_0) - e_0[r_0, \sum_{i=1}^{n-1} e_i \sigma^i (e_{n-i})]_{\sigma^n} \sigma^n (e_0) \\
= (1 - e_0)((1 - e_0) \sum_{i=0}^{n-1} [e_i, r_{n-i}]_{\sigma^n-i}) - (\sum_{i=0}^{n-1} [e_i, r_{n-i}]_{\sigma^n-i}) \sigma^n (e_0)) \sigma^n (1 - e_0) \\
- e_0((1 - e_0) \sum_{i=0}^{n-1} [e_i, r_{n-i}]_{\sigma^n-i}) - (\sum_{i=0}^{n-1} [e_i, r_{n-i}]_{\sigma^n-i}) \sigma^n (e_0)) \sigma^n (e_0) \text{ by our induction assumptions and [7, Claim 2]} \\
= (1 - e_0) \sum_{i=0}^{n-1} [e_i, r_{n-i}]_{\sigma^n-i}) \sigma^n (1 - e_0) + e_0(\sum_{i=0}^{n-1} [e_i, r_{n-i}]_{\sigma^n-i}) \sigma^n (e_0). \tag{2}
\]

Thus $[r_0, e_n]_{\sigma^n}$\[
= [r_0, e_{n*}]_{\sigma^n} + [r_0, e_{n*}]_{\sigma^n} \\
= e_0 \sum_{i=0}^{n-1} [e_i, r_{n-i}]_{\sigma^n-i}) \sigma^n (1 - e_0) + (1 - e_0)(\sum_{i=0}^{n-1} [e_i, r_{n-i}]_{\sigma^n-i}) \sigma^n (e_0) \\
+ (1 - e_0)(\sum_{i=0}^{n-1} [e_i, r_{n-i}]_{\sigma^n-i}) \sigma^n (1 - e_0) + e_0(\sum_{i=0}^{n-1} [e_i, r_{n-i}]_{\sigma^n-i}) \sigma^n (e_0) \text{ by (1), (2)} \\
= (\sum_{i=0}^{n-1} [e_i, r_{n-i}]_{\sigma^n-i}).
\]

Thus $e_n$ also satisfies equation (C). By induction, $r$ is strongly clean in $R[x; \sigma]$.

In short, we have evidence towards answering another of our motivating questions.

**Corollary 5.2.3.** If $R$ is $\sigma$-optimally clean, then $R[x; \sigma]$ is strongly clean.

Since the strongly clean property holds under homomorphic images we get the following from Theorem 5.2.2 as well.

**Theorem 5.2.4.** Let $n \in \mathbb{N}$ and $r = \sum_{i=0}^{n-1} r_i x^i \in R[x; \sigma]/(x^n)$. If $r_0$ or $1-r_0$ is $\sigma$-optimally clean in $R$, then $r$ is strongly clean in $R[x; \sigma]/(x^n)$.

Finally, considering the definition of skew optimally clean and Theorem 5.2.2, we end this section with the true result we were after.
THEOREM 5.2.5. If either $r_0$ or $1 - r_0$ is skew optimally clean in $R$, then for any ring endomorphism $\sigma$ of $R$ and $r = \sum_{i \geq 0} r_i x^i \in R[x; \sigma]$, $r$ is strongly clean in $R[x; \sigma]$.

Corollary 5.2.6. If $R$ is skew optimally clean, then $R[x; \sigma]$ is strongly clean for any ring endomorphism $\sigma$ of $R$. 
CHAPTER 6

Classes of Skew Optimally Clean Rings

As promised, we now devote a chapter to skew optimally clean rings. This new condition is much stricter than simply being optimally clean, so our first section provides only a few examples. However, the chapter does culminate with an example of an optimally clean ring which is not skew optimally clean.

6.1 Examples

Let us begin with a fairly straightforward example.

**Proposition 6.1.1.** Let $R$ be a ring. If $R$ is local then $R$ is skew optimally clean.

**Proof.** Recall that the only idempotents $e^2 = e$ in a local ring are 1 and 0. Then, for any ring endomorphism $\sigma$ of $R$ and $n \in \mathbb{N}$, $[e, a]_{\sigma^n} = ea - a\sigma^n(e) = ea - ae = ae - ae = 0.$ Letting $x = 0$ solves $[r, x]_{\sigma^n} = [e, a]_{\sigma^n} = 0$ for any $r \in R.$

At this point it may seem the skew optimally clean condition is too restrictive to be useful, but our next class of rings serves to dispel this concern.
Proposition 6.1.2. Let $R$ be a ring. If $R$ is strongly $\pi$-regular, then $R$ is skew optimally clean.

Proof. Let $r = e + (r - e) \in R$ be the strongly clean decomposition utilized in [10, Theorem 1] and let $\sigma : R \rightarrow R$ be any ring endomorphism. Let $f = 1 - e$. Then $ere \in \text{Nil}(eRe), \sigma^n(ere) \in \text{Nil}(\sigma^n(e)R\sigma^n(e)), frf \in U(fRf)$, and $\sigma^n(frf) \in U(\sigma^n(f)R\sigma^n(f))$ for all $n \in \mathbb{N}$. Fix $n$. Let $p, q \in \mathbb{N}$ be the smallest such that $(ere)^p = 0 = \sigma^n(ere)^q$. Finally let $(frf)^{-1}$ and $\sigma^n(frf)^{-1}$ denote the inverses of $frf$ and $\sigma^n(frf)$ in $fRf$ and $\sigma^n(f)R\sigma^n(f)$ respectively. Then for any $y \in eR\sigma^n(f) \oplus fR\sigma^n(e)$,

$$x = \sum_{i=1}^{q} (frf)^{-i}y\sigma^n(ere)^{i-1} - \sum_{i=1}^{p} (ere)^{i-1}y\sigma^n(frf)^{-i} \in eR\sigma^n(f) \oplus fR\sigma^n(e)$$

is the unique solution to $r \Psi_{\sigma^n(r)}(x) = y$ (as any other solution would contradict the minimality of $p$ and $q$). Thus $r \Psi_{\sigma^n(r)}$ is surjective for all $n \in \mathbb{N}$ and, by Lemma 5.1.4, $r$ is $\sigma$-optimally clean. Since $r$ and $\sigma$ were arbitrary, $R$ is skew optimally clean.

This leads to the following corollary, which simplifies the results in [7].

Corollary 6.1.3. Let $\sigma : R \rightarrow R$ be a ring endomorphism and $r = \sum_{i \geq 0} r_ix^i \in R[[x; \sigma]]$. If either $r_0$ or $1 - r_0$ is a strongly $\pi$-regular element of $R$, then $r$ is a strongly clean element of $R[[x; \sigma]]$.

Corollary 6.1.4. If $R$ is strongly $\pi$-regular, then $R[[x; \sigma]]$ is strongly clean for any ring endomorphism $\sigma$ of $R$.

6.2 A Non-Example

We now present an example of an optimally clean ring $S$ and ring endomorphism $\sigma$ such that $S[[x, \sigma]]$ is not strongly clean. As before, this example tries to imitate the contradiction from [1, Example 45].
**Example.** Let $k$ be a field. Let $R = k[t_1, t_2, \ldots]_{(t_1)}$ be the ring of polynomials in countably many indeterminates over $k$ localized at the prime ideal $(t_1)$. Define $\tau$ to be the endomorphism of $R$ which fixes $k$ and such that $\tau(t_i) = t_{i+1} + 1$ extended to the localization. Construct the ring of right skew power series $A = \sum_{i \geq 0} y^i a_i$ with coefficients $a_i \in R$ written on the right subject to $ry = y\tau(r)$ for all $r \in R$. Note that $A$ is a local ring. Now consider the ring $S = T_2(A)$. It was shown in [1, Theorem 41] that $S$ is strongly clean so, by Proposition 4.1.8, $S$ is optimally clean. We now construct an endomorphism $\sigma$ such that $S$ is not $\sigma$-optimally clean.

To this end let $\sigma^*: R \to R$ be the identity on $k$ such that $\sigma(t_i) = t_{i+1}$ extended to the localization. Extend to $\sigma': A \to A$ by $\sigma'(\sum_{i \geq 0} y^i a_i) = \sum_{i \geq 0} y^i \sigma(a_i)$.

Finally define

$$
\sigma : S \to S \text{ by } \sigma \left( \begin{array}{cc} a & b \\ 0 & c \end{array} \right) = \left( \begin{array}{cc} \sigma'(a) & \sigma'(b) \\ 0 & \sigma'(c) \end{array} \right).
$$

Note that $\sigma$ is a ring endomorphism of $S$. To simplify, we shall use $\sigma$ in place of $\sigma^*$ and $\sigma'$ where context permits.

Let $A_0 = \left( \begin{array}{cc} t_1 & 0 \\ 0 & 1 + t_1 \end{array} \right) \in S$.

Then any strongly clean decomposition of $A_0$ must have an idempotent of the form

$$
E_0 = \left( \begin{array}{cc} 1 & c \\ 0 & 0 \end{array} \right)
$$

for some $c \in A$ that satisfies $c = t_1 c - c t_1$. Given any idempotent of this form, we can compute that

$$
E_0 S \sigma(1 - E_0) \oplus (1 - E_0) S \sigma(E_0) \subseteq \left( \begin{array}{cc} 0 & A \\ 0 & 0 \end{array} \right).
$$
Claim. $A_0$ is not $\sigma$-optimally clean.

Proof. Let

$$A_1 = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \in S.$$  

Then we need to show that for any possible $E_0$ listed above, there does not exist a

$$Z = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \in E_0 S \sigma(1 - E_0) \oplus (1 - E_0) S \sigma(E_0)$$

such that $[A_0, Z]_\sigma = [E_0, A_1]_\sigma$. If such an $Z$ exists, then we get the equation $y = t_1 z - z \sigma(1 + t_1)$ for some $z \in A$. But notice that for any such $z = \sum_{i \geq 0} y^i z_i$, $t_1 z - z \sigma(1 + t_1)$

$$= t_1 \sum_{i \geq 0} y^i z_i - \sum_{i \geq 0} y^i z_i \sigma(1 + t_1)$$

$$= \sum_{i \geq 0} y^i \tau^i(t_1) z_i - \sum_{i \geq 0} y^i z_i \sigma(1 + t_1)$$

$$= \sum_{i \geq 0} y^i (t_{i+1} + i) z_i - \sum_{i \geq 0} y^i z_i (1 + t_2)$$

where the “$y$” term cancels to 0. Thus $y \neq t_1 z - z \sigma(1 + t_1)$ for any $z \in A$. Therefore, there is no $Z \in E_0 S \sigma(1 - E_0) \oplus (1 - E_0) S \sigma(E_0)$ such that $[A_0, Z]_\sigma = [E_0, A_1]_\sigma$ and $A_0$ is not $\sigma$-optimally clean.

Note that this does not necessarily imply that $A_0 + B_1 X + B_2 X + ... \in S[X; \sigma]$ is not strongly clean for any set of $B_i \in S$, but we can compute at least one which is not.

Claim. $A_0 + A_1 X \in S[X; \sigma]$ is not strongly clean.

Proof. By earlier arguments, any strongly clean decomposition of $A_0 + A_1 X$ must have the form

\[
\begin{bmatrix}
1 & c \\
0 & 0 \\
0 & 0
\end{bmatrix}
+ \begin{bmatrix}
0 & z_1 \\
0 & 0 \\
0 & 0
\end{bmatrix} X
+ \begin{bmatrix}
0 & z_2 \\
0 & 0 \\
0 & 0
\end{bmatrix} X^2 + ...
\]

\[
+ \begin{bmatrix}
t_1 - 1 & -c \\
0 & 1 + t_1
\end{bmatrix}
+ \begin{bmatrix}
0 & y - z_1 \\
0 & 0 \\
0 & 0
\end{bmatrix} X
+ \begin{bmatrix}
0 & -z_2 \\
0 & 0 \\
0 & 0
\end{bmatrix} X^2 + ...
\]

\]
which commute only if there exist $z_1 \in A$ such that $y = t_1z_1 - z_1\sigma(1 + t_1)$. But by the argument above, this equation cannot be solved. Therefore, $A_0 + A_1X$ is not strongly clean.

In short, we have found an optimally clean ring which is not skew optimally clean. Thus the seemingly restrictive requirements of skew optimally clean truly are necessary when working with formal skew power series rings.
CHAPTER 7

Strongly Clean Left-Right Skew Power Series

Having established a condition that allows for extension to strongly clean formal (left) skew power series rings, it is natural to explore what analogous condition is required to extend to the more general strongly clean formal left-right skew power series rings. In this chapter we investigate this generalization with interesting results.

7.1 Left-Right Skew Optimally Clean

Let us begin by defining exactly what we mean by a “formal left-right skew power series ring.”

**Definition 7.1.1.** Let \( \sigma, \tau : R \to R \) be commuting ring endomorphisms. Define the formal left-right skew power series ring of \( R \) with respect to \( \sigma \) and \( \tau \), denoted \( R[y; \tau][x; \sigma] \), to be the set of \( \sum_{i, j=0}^{\infty} y^j r_{j,i} x^i \) such that \( r_{j,i} \in R \) under the usual polynomial addition and multiplication subject to the relations \( yx = xy, ry = y \tau(r) \) and \( xs = \sigma(s)x \) for all \( r, s \in R \).

The ring \( R[y; \tau][x; \sigma] \) may be viewed as a right skew power series ring over a left skew power series ring. For this reason we need the following definition.
Definition 7.1.2. Let $\rho, \sigma, \tau, \upsilon$ be ring endomorphisms of $R$ and let $a, b \in R$. Define

$$\sigma^\rho_{\tau}[a, b]_\upsilon = \sigma(a)\upsilon(b) - \rho(b)\tau(a).$$

A few properties of this new commutator:

1. Let $1$ be the identity endomorphism of $R$. Then $1[a, b]_1 = [a, b]$.

2. $\sigma^\rho_{\tau}[a + b, c]_\upsilon = \sigma^\rho_{\tau}[a, c]_\upsilon + \sigma^\rho_{\tau}[b, c]_\upsilon$ and $\sigma^\rho_{\tau}[a, b + c]_\upsilon = \sigma^\rho_{\tau}[a, b]_\upsilon + \sigma^\rho_{\tau}[a, c]_\upsilon$.

We are now prepared to generalize our conditions from Chapter 5.

Definition 7.1.3. Let $\sigma$ and $\tau$ be ring endomorphisms of $R$ such that $\sigma\tau = \tau\sigma$, and let $n, m \in \mathbb{N}$.

For $r \in R$ with strongly clean decomposition $r = e + (r - e)$ where $e^2 = e, (r - e) \in U(R)$ and $er = re$, let

$$\tau^m(r)\Psi_{\sigma^n(r)} : \tau^m(e)R\sigma^n(1 - e) \oplus \tau^m(1 - e)R\sigma^n(e) \to \tau^m(e)R\sigma^n(1 - e) \oplus \tau^m(1 - e)R\sigma^n(e)$$

be the group homomorphism defined as follows:

$$\tau^m(r)\Psi_{\sigma^n(r)}(x) = \tau^m(r)x - x\sigma^n(r).$$

As before, we believe the structure sufficient to extend to a strongly clean formal left-right skew power series ring is surjectivity of this homomorphism. Let us state an equivalent form for this.

Definition 7.1.4. Let $\sigma$ and $\tau$ be ring endomorphisms of $R$ such that $\sigma\tau = \tau\sigma$. An element $r \in R$ is called $\sigma, \tau$-optimally clean if there exists an idempotent $e^2 = e \in R$ where $er = re$ and $r - e \in U(R)$ such that for each $a \in R$ and $n, m \in \mathbb{N}$, there exists an $x_{m,n} \in R$ satisfying

$$\tau^m[r, x_{m,n}]_{\sigma^n} = \tau^m[e, a]_{\sigma^n}.$$ 

The ring $R$ is $\sigma, \tau$-optimally clean if every element is $\sigma, \tau$-optimally clean.

Remark. If $x_{n,m} \in R$ satisfies $\tau^m[r, x_{m,n}]_{\sigma^n} = \tau^m[e, a]_{\sigma^n}$ as listed above, we may assume $x_{m,n} \in \tau^m(e)R\sigma^n(1 - e) \oplus \tau^m(1 - e)R\sigma^n(e)$.
\textbf{Proof.} Suppose that }\tau^m[r, x_{m,n}]_{\sigma^n} = \tau^m[e, a]_{\sigma^n}. \text{ Note then that } \tau^m[r, \tau^m(e)x_{m,n}\sigma^n(1-e) + \tau^m(1-e)x_{m,n}\sigma^n(e)]_{\sigma^n} \\
= \tau^m[r, \tau^m(e)x_{m,n}\sigma^n(1-e)]_{\sigma^n} + \tau^m[r, \tau^m(1-e)x_{m,n}\sigma^n(e)]_{\sigma^n} \\
= \tau^m(e)\tau^m[r, x_{m,n}]_{\sigma^n}\sigma^n(1-e) + \tau^m(1-e)\tau^m[r, x_{m,n}]_{\sigma^n}\sigma^n(e) \\
= \tau^m(e)\tau^m[e, a]_{\sigma^n}\sigma^n(1-e) + \tau^m(1-e)\tau^m[e, a]_{\sigma^n}\sigma^n(e) \\
= \tau^m(e)a\sigma^n(1-e) - \tau^m(1-e)a\sigma^n(e) \\
= \tau^m[e, a]_{\sigma^n},
\text{ and } \tau^m(e)x_{m,n}\sigma^n(1-e) + \tau^m(1-e)x_{m,n}\sigma^n(e) \in \tau^m(e)R\sigma^n(1-e) \oplus \tau^m(1-e)R\sigma^n(e). \qed

\textbf{Lemma 7.1.5.} A strongly clean element } r \in R \text{ where } e^2 = e \in R, (r-e) \in U(R) \text{ and } er = re \text{ is } \sigma, \tau\text{-optimally clean if and only if } \tau^m(r)\Psi_{\sigma^n(r)} \text{ is surjective on } \tau^m(e)R\sigma^n(1-e) \oplus \tau^m(1-e)R\sigma^n(e) \text{ for all } n, m \in \mathbb{N}.

\textbf{Proof.} Let } m, n \in \mathbb{N} \text{ and } y \in \tau^m(e)R\sigma^n(1-e) \oplus \tau^m(1-e)R\sigma^n(e). \text{ Set } y^* = \tau^m(e)y\sigma^n(1-e) - \tau^m(1-e)y\sigma^n(e). \text{ Since } r \text{ is } \sigma, \tau\text{-optimally clean, there exists an } x_{m,n} \in \tau^m(e)R\sigma^n(1-e) \oplus \tau^m(1-e)R\sigma^n(e) \text{ such that } \tau^m[r, x_{m,n}]_{\sigma^n} = \tau^m[e, y^*]_{\sigma^n} = y. \text{ That is } \tau^m(r)\Psi_{\sigma^n(r)} \text{ is surjective. The reverse is proved analogously.} \qed

\text{Obviously, the definition of } \sigma, \tau\text{-optimally clean depends heavily on the endomorphisms } \sigma \text{ and } \tau. \text{ It is natural therefore to ask when (or even if) elements satisfy this condition for every pair of commuting endomorphisms } \sigma, \tau \text{ of } R. \text{ While this may seem like a fairly hefty restriction, there are at least two large collections of strongly clean rings to which it applies (as we will see in a later section). For now, let us make rigorous our ideas.}

\textbf{Definition 7.1.6.} An element } r \in R \text{ that is } \sigma, \tau\text{-optimally clean for every pair of commuting endomorphisms } \sigma \text{ and } \tau \text{ of } R \text{ is called } \textit{left-right skew optimally clean}. \text{ A ring is called } \textit{left-right skew optimally clean} \text{ if every element is left-right skew optimally clean.}
7.2 The Left-Right Skew Extension Conjecture

Our goal now is to prove that a left-right skew optimally clean ring extends to a strongly clean formal left-right skew power series ring. While we have been unable to complete this task at the current date, there is enough evidence to make this a reasonable conjecture.

**Conjecture 7.2.1.** If $R$ is left-right skew optimally clean, then $R[y; \tau][x; \sigma]$ is strongly clean for any commuting endomorphisms $\sigma$ and $\tau$ of $R$.

Proof of this conjecture would obviously lead to analogous corollaries to those of Chapter 5.

7.3 Classes of Left-Right Skew Optimally Clean Rings

As promised, we now present classes of left-right skew optimally clean rings.

**Proposition 7.3.1.** Let $R$ be a ring. If $R$ is local then $R$ is left-right skew optimally clean.

*Proof.* Recall that the only idempotents $e^2 = e$ in a local ring are 1 and 0. Then, for any commuting ring endomorphisms $\sigma, \tau$ of $R$ and $n, m \in \mathbb{N}$, $\tau^m[e, a]_{\sigma^n} = \tau^m(e)a - a\sigma^n(e) = ea - ae = ae - ae = 0$. Letting $x = 0 \in \tau^m(e)R\sigma^n(1-e) \oplus \tau^m(1-e)R\sigma^n(e)$ solves $\tau^m[r, x]_{\sigma^n} = \tau^m[e, a]_{\sigma^n} = 0$ for any $r \in R$. □

**Proposition 7.3.2.** Let $R$ be a ring. If $R$ is strongly $\pi$-regular, then $R$ is left-right skew optimally clean.

*Proof.* Let $r = e + (r - e) \in R$ be the strongly clean decomposition utilized in [10, Theorem 1] and let $\sigma, \tau$ be commuting ring endomorphisms of $R$. Let $f = 1 - e$. Then $\sigma^n(ere) \in \text{Nil}(\sigma^n(e)R\sigma^n(e)), \tau^m(ere) \in \text{Nil}(\tau^m(e)R\tau^m(e)), \sigma^n(fr) \in U(\sigma^n(f)R\sigma^n(f))$, and $\tau^m(fr) \in U(\tau^m(f)R\tau^m(f))$ for all $n, m \in \mathbb{N}$. Fix $n, m$. Let $p, q \in \mathbb{N}$ such that $\tau^m(ere)^p = 0 = \sigma^n(ere)^q$. Finally let $\tau^m(fr)^{-1}$ and $\sigma^n(fr)^{-1}$ denote the inverses of $\tau^m(fr)$ and $\sigma^n(fr)$ in $\tau^m(f)R\tau^m(f)$ and $\sigma^n(f)R\sigma^n(f)$ respectively. Then for any $y \in \tau^m(e)R\sigma^n(f) \oplus \tau^m(f)R\sigma^n(e)$,
\[ x = \sum_{i=1}^{q} \tau^m(ffe)^{-i}y\sigma^n(ere)^{i-1} - \sum_{i=1}^{p} \tau^m(ere)^{i-1}y\sigma^n(ffe)^{-i} \in \tau^m(e)R\sigma^n(f) \oplus \tau^m(f)R\sigma^n(e) \]
is a solution to \( \tau^m(r)\Psi_{\sigma^n(r)}(x) = y \). Thus \( \tau^m(r)\Psi_{\sigma^n(r)} \) is surjective for all \( n, m \in \mathbb{N} \) and, by Lemma 7.1.5, \( r \) is \( \sigma, \tau \)-optimally clean. Since \( r, \sigma, \) and \( \tau \) were arbitrary, \( R \) is left-right skew optimally clean.

This leads us to make the following conjecture.

**Conjecture 7.3.3.** If \( R \) is strongly \( \pi \)-regular, then \( R[y;\tau][x;\sigma] \) is strongly clean for any commuting ring endomorphisms \( \sigma \) and \( \tau \) of \( R \).
CHAPTER 8

Conclusions

We now conclude the document with a quick recap of our major results and a list of open questions for possible future work.

8.1 Summary

In summary, we have established the following findings within this dissertation.

1. We discovered the structure needed to extend to a strongly clean formal power series ring by using the technique of Peirce Decompositions.

2. We were able to define this structure (optimally clean) by two distinct, yet equivalent, methods.

3. We verified that an optimally clean ring extends to a strongly clean formal power series ring and that this is the most general classification to do so to date.

4. We showed that strongly clean and optimally clean are equivalent characterizations within formal power series rings.

5. We established that numerous classes of strongly clean rings are in fact optimally clean.

6. We produced an example of a strongly clean element that is not optimally clean.
7. We discovered that we could generalize the optimally clean structure to extend to a strongly clean formal skew power series ring.

8. We defined this more general condition (skew optimally clean) by two distinct, yet equivalent, methods.

9. We verified that a skew optimally clean ring extends to a strongly clean formal skew power series ring.

10. We established a few classes of optimally clean rings that are in fact skew optimally clean.

11. We produced an example of an optimally clean ring that is not skew optimally clean.

We have also established the following chain of containment for rings:

\[
\text{strongly } \pi\text{-regular } \subseteq \text{skew optimally clean } \subseteq \text{optimally clean } \subseteq \text{strongly clean}
\]

with all but possibly the final implication non-reversible.

### 8.2 Open Questions

Of course, there are still a few questions left unanswered with this dissertation. We begin with the obvious.

**Question.** Is there an example of a strongly clean ring which is not optimally clean?

By Proposition 4.1.2, any example satisfying this question must be highly noncommutative. A relatively small pool of such rings from the literature has, at present, hindered our progress. Perhaps [2] would be a good place to continue the investigation.

While the optimally clean classification is sufficient to extend to a strongly clean formal power series, we do not yet know if it is necessary.

**Question.** Does the reverse implication to Theorem 3.2.2 hold?
It should be noted that the following weaker condition does hold when assuming the formal power series extension to be strongly clean.

**Definition 8.2.1.** Let $R$ be a ring and $a \in R$. An element $r \in R$ is called *a-optimally clean* if there exists an idempotent $e^2 = e \in R$, where $(r - e) \in U(R)$ and $er = re$, and $x \in R$ such that $[r, x] = [e, a]$. The ring $R$ is *a-optimally clean* if all of its elements are $a$-optimally clean.

**Proposition 8.2.2.** Let $R$ be a ring. If $R[x]$ is strongly clean then $R$ is $a$-optimally clean for each $a \in R$.

*Proof.* Let $a$ and $r \in R$. Since $R[x]$ is strongly clean there exists a decomposition $r + ax = (\sum_{i \geq 0} e_i x^i) + (r + ax - \sum_{i \geq 0} e_i x^i) \in R[x]$ where $\sum_{i \geq 0} e_i x^i = (\sum_{i \geq 0} e_i x^i)^2$, $(r + ax)(\sum_{i \geq 0} e_i x^i) = (\sum_{i \geq 0} e_i x^i)(r + ax)$ and $(r + ax - \sum_{i \geq 0} e_i x^i) \in U(R[x])$. But then $e_0$ satisfies $e_0 = (e_0)^2$, $re_0 = e_0 r$ and $(r - e_0) \in U(R)$, and $e_1$ satisfies the equation $[r, e_1] = [e_0, a]$. Thus $r$ is $a$-optimally clean. \hfill $\Box$

While we were able to show that most popular classifications of strongly clean rings are in fact optimally clean, one such example from the literature still eluded us.

**Question.** Are uniquely strongly clean elements optimally clean?

Going along with Definition 3.1.1, the following question naturally arises.

**Question.** Is there a module decomposition classification of optimally clean along the lines of [3, Propositions 2.2 - 2.5]?

There are many open questions surrounding the more general classification of skew optimally clean. For starters, we do not know if skew optimally clean is necessary to extend to a strongly clean formal skew power series ring.

**Question.** Does the reverse implication to Theorem 5.2.2 hold?

As before, the following weaker condition does hold when assuming the formal skew power series extension to be strongly clean.
Definition 8.2.3. Let $\sigma$ be a ring endomorphism of $R$ and let $a \in R$. An element $r \in R$ is called $\sigma$-a-optimally clean if for each $n \in \mathbb{N}$ there exists an idempotent $e^2 = e \in R$, where $(r - e) \in U(R)$ and $er = re$, and $x_n \in R$ such that $[r, x_n]_{\sigma^n} = [e, a]_{\sigma^n}$. The ring $R$ is $\sigma$-a-optimally clean if all of its elements are $\sigma$-a-optimally clean.

Proposition 8.2.4. Let $\sigma$ be a ring endomorphism of $R$. If $R[x; \sigma]$ is strongly clean then $R$ is $\sigma$-a-optimally clean for each $a \in R$.

Proof. Let $a, r \in R$ and $n \in \mathbb{N}$. Since $R[x; \sigma]$ is strongly clean there exists a decomposition $r + ax^n = \left(\sum_{i \geq 0} e_i x^i\right) + \left(r + ax^n - \sum_{i \geq 0} e_i x^i\right) \in R[x; \sigma]$ where $\sum_{i \geq 0} e_i x^i = (\sum_{i \geq 0} e_i x^i)^2$, $(r + ax^n)(\sum_{i \geq 0} e_i x^i) = (\sum_{i \geq 0} e_i x^i)(r + ax)$ and $(r + ax - \sum_{i \geq 0} e_i x^i) \in U(R[x; \sigma])$. But then $e_0$ satisfies $e_0 = (e_0)^2$, $re_0 = e_0r$ and $(r - e_0) \in U(R)$, and $e_n$ satisfies the equation $[r, e_n]_{\sigma^n} = [e_0, a]_{\sigma^n}$. Thus $r$ is $\sigma$-a-optimally clean.

This leads to the following definition and corollary.

Definition 8.2.5. Let $a \in R$. An element $r \in R$ that is $\sigma$-a-optimally clean for every ring endomorphism $\sigma$ of $R$ is called skew $a$-optimally clean. A ring is called skew $a$-optimally clean if every element is skew $a$-optimally clean.

Corollary 8.2.6. Let $R$ be a ring. If $R[x, \sigma]$ is strongly clean for all ring endomorphisms $\sigma$ of $R$, then $R$ is skew $a$-optimally clean for every $a \in R$.

Perhaps just as interesting as the answer to the last question would be knowing if the analogous result of Theorem 3.3.2 holds in the more general setting of Chapter 5.

Question. Is the skew optimally clean condition invariant under extending to a formal skew power series ring?

It is because of difficulties caused by the ring endomorphisms that we were able to verify only a few classes of optimally clean rings as skew optimally clean. In fact, the outcomes of some of the most obvious possible examples continue to evade us.
Question. Is it true that commutative clean elements are skew optimally clean?

Question. If the previous question holds, is it true that abelian clean elements are skew optimally clean?

Even with this list of questions still remaining, we feel this dissertation has provided a substantial introduction and basis for studying strongly clean formal power series rings.
BIBLIOGRAPHY


