FILTERS ON POSITIVE CONES OF LATTICE-ORDERED GROUPS

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ABSTRACT

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Abstract: In a recent article “Bazzoni’s Conjecture” the authors used lattice-theoretic techniques to positively answer a conjecture of Bazzoni’s regarding Prüfer domains. Suppose $G$ is an $\ell$-group and $F$ is a filter on $G^+$. Recall that $F$ is a principal filter if it is of the form $\{g \in G^+ : a \leq g\}$ for some $a \in G^+$. We say that $F$ is a cold filter if for all $P \in \text{Min}(G)$, the filter (i.e. the interval) on $(G/P)^+$ defined by $F_P = \{g + P : g \in F\}$ has a minimum. If every filter on $G^+$ is principal (resp. cold) then we say that the group is principally-filtered (resp. cold-filtered) $\ell$-group. In this dissertation we expand on the ideas of cold filters and characterize cold-filtered $\ell$-groups.
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CHAPTER 1

Preliminaries

1.1 Lattices

The setting of the dissertation is the theory of lattice-ordered groups. In this chapter we
shall introduce a few important concepts and definitions that will lay the foundation on these
lattice-ordered groups on which the dissertation is based.

Definition 1.1.1. Let $L$ be a nonempty set. A relation $R$ on $L$ is a subset of $L \times L$. If
$(g_1, g_2) \in R$, we write $g_1 R g_2$. A relation $\leq$ on $L$ is a partial order if $\leq$ satisfies the following
conditions:
Reflexive: For any $g \in L$, $g \leq g$
Antisymmetric: If $g \leq h$ and $h \leq g$, then $g = h$.
Transitive: If $g \leq h$ and $h \leq k$, then $g \leq k$.

Definition 1.1.2. Let $\leq$ be a partial order on $L$ and let $S \subseteq L$. If $z \in L$ satisfies for
all $x \in S$, $x \leq z$ (respectively, $z \leq x$) then $z$ is called an upper bound (respectively, lower
bound) of $S$. If the set of upper bounds of $S$ has a lower bound belonging to the set, it is
straightforward to check that such an element is unique. We call this unique element the
least upper bound of $S$ and denote it by $\vee S$. In case $S = \{x, y\}$, we write $x \vee y$. If every pair
of elements in $L$ has a least upper bound (respectively, a greatest lower bound), we say $L$ is
a join semilattice (respectively, meet semilattice). If \((L, \leq)\), is a join semilattice as well as a meet semilattice, then \(L\) is called a lattice.

**Example 1.1.3.**  
(i) Let \(X\) be a set and let \(\mathcal{P}(X)\) denote its powerset. Then \((\mathcal{P}(X), \subseteq)\) is a lattice.

(ii) Consider \(\mathbb{N}\) and define \(\leq\) on \(\mathbb{N}\) as for all \(a, b \in \mathbb{N}, a \leq b\), if and only if \(a|b\). Then, \((\mathbb{N}, \leq)\) is a lattice. For any \(n, m \in \mathbb{N}, n \wedge m = \gcd\{n, m\}\) and \(n \vee m = \text{lcm}\{n, m\}\).

(iii) Let \(\{L_i : i \in I\}\) be a set of lattices and set \(L = \prod_{i \in I} L_i\). Define \(x \leq y\) if and only if \(x(i) \leq y(i)\forall i \in I\). Then, \(L\) is a lattice in which

\[(x \vee y)(i) = x(i) \vee y(i) \text{ and } (x \wedge y)(i) = x(i) \wedge y(i)\]

\(L\) is defined to be the direct product of the lattices \(\{L_i\}\).

(iv) A partially-ordered set \((L, \leq)\) is called totally-ordered if for all \(a, b \in L\) we have either \(a \leq b\) or \(b \leq a\). A totally-ordered set is a lattice.

**Definition 1.1.4.** A lattice \((L, \leq)\) is said to be complete if every subset of \(L\) has a supremum and an infimum.

Notice that a complete lattice \(L\) always necessarily has a greatest element denoted by \(1_L\) and a least element denoted by \(0_L\). A not necessarily complete lattice that has a top and a bottom element is called a bounded lattice. For example any finite lattice is a bounded lattice. A lattice is said to be distributive if it satisfies the following distributive law (and dually):

\[a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \forall a, b, c \in G\]

It is easy to check that the distributive law implies its dual. A lattice \(L\) is distributive if and only if for \(a, x, y \in L\) we have \(a \vee x = a \vee y\) and \(a \wedge x = a \wedge y\) implies that \(x = y\).
Definition 1.1.5. A boolean algebra $L$ is a bounded, distributive lattice such that for all $a \in L$, there exists $b \in L$ such that $a \land b = 0$ and $a \lor b = 1$. Here $b$ is called the complement of $a$. We shall denote the complement of $a$ as $a^\perp$. The distributivity of the lattice structure forces complements to be unique.

Definition 1.1.6. Given two lattices $L, M$ a function $f : L \rightarrow M$ is called a lattice homomorphism if for any $x_1, x_2 \in L$, the following properties are satisfied:

i) $f(x_1 \lor x_2) = f(x_1) \lor f(x_2)$ and

ii) $f(x_1 \land x_2) = f(x_1) \land f(x_2)$. If in addition $f$ is also bijective, then $f$ is called a lattice isomorphism.

1.2 Lattice-ordered groups

In this section we shall formally define a lattice-ordered group and shall study a few important properties of these groups.

Suppose that $(G, \cdot, 1)$ is a group and is equipped with a lattice order $\leq$. Then $(G, \cdot, 1)$ is called a lattice-ordered group (also denoted as $\ell$-group) if and only if whenever $x, y, g, h \in G$ and $g \leq h$, then $xgy \leq xhy$. If the order on a given $\ell$-group $G$ is actually a total order, then we often say $G$ is an o-group. Finally, it is straightforward to show that the lattice structure on an $l$-group obeys the following distributive law:

$$a \cdot (b \lor c) = (a \cdot b) \lor (a \cdot c).$$

We shall write that $(G, 1, \leq)$ is an $\ell$-group. It is known that a non-trivial $\ell$-group is not a bounded group and therefore there should be no confusion using 1 to denote the multiplicative identity. When it is known that $G$ is abelian we shall use additive notation instead of multiplicative notation, with the same issue using 0 for the additive identity.

Example 1.2.1. (i) $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, all under the usual addition and order, are lattice-ordered
groups. In addition they are abelian and totally ordered groups and are also denoted as \(\mathcal{O}\)-groups.

(ii) Consider \(C(X)\), the additive group of continuous functions from a topological space \(X\) into \(\mathbb{R}\) under the pointwise orderings: for any \(f, g \in C(X)\) we shall mean \(f \leq g\) if and only if \(f(x) \leq g(x)\) for all \(x \in X\). Then, \(C(X)\) is an abelian \(\ell\)-group.

(iii) Let \(I\) be a nonempty set and let \(\{(G_i, 1_i, \leq) : i \in I\}\) be a collection of \(\ell\)-groups. Then the direct product \(\prod_{i \in I} G_i\), as has already been pointed out, is a group and a lattice, it is straightforward to check that the direct product of \(\ell\)-groups is again an \(\ell\)-group.

If \(G = \prod_{i \in I} G_i\), and \(g \in G\), then the component of \(g\) in \(G_i\) will be denoted by \(g(i)\). If \(x \in G_j\) for \(j \in I\), then \(x\) will be identified with the element \(g \in G\) such that

\[
g(i) = \begin{cases} 
x & \text{if } i = j \\
e & \text{if } i \neq j
\end{cases}
\]

We shall identify each \(G_i\) with the set \(\{g \in \prod_{i \in I} G_i : g(j) = e, \text{ for all } j \neq i\}\). Each \(G_i\) is an \(\ell\)-subgroup of \(G\) and is called a cardinal summand of \(G\). The cardinal sum of the \(G_i\) is defined as

\[
\sum_{i \in I} G_i = \{g \in G : g(i) = 1, \text{ for all but finitely many } i\}
\]

The cardinal sum of \(\ell\)-groups is again an \(\ell\)-group.

(iv) Let \(G\) be an \(\ell\)-group and \(A\) be a \(\mathcal{O}\)-group. The lexicographic product of \(A\) and \(G\), denoted by \(\rightarrow \oplus G\), is the direct product of the groups \(A\) and \(G\) ordered by : \((a_1, g_1) \leq (a_2, g_2)\) if \(a_1 < a_2\) or, if \(a_1 = a_2\) and \(g_1 \leq g_2\). Then \(\rightarrow \oplus G\) is an \(\ell\)-group.

Throughout this dissertation we shall use 1 to denote the identity element of an \(\ell\)-group \(G\)(or 0 in the case of abelian \(\ell\)-groups). The collection of positive elements of the group denoted by \(G^+ = \{g \in G : g \geq 1\}\) is called the positive cone of \(G\). For \(g \in G\) we define the
positive part of \( g \) is written as \( g_+ = g \lor 1 \) and the negative part of \( g \) is written as \( g_- = g^{-1} \lor 1 \). Notice that \( g_+ \land g_- = 1 \). Moreover if \( g = a \cdot b^{-1} \) and \( a \land b = 1 \), then \( a = g_+ \) and \( b = g_- \). The absolute value of \( g \) is written as \( |g| = g_+ \lor g_- \) and \( g = g_+(g_-)^{-1} \). Thus \( G \) is generated by its positive cone. Hence, in order to study the properties of an \( \ell \)-group it is sufficient to study the properties of the positive cone.

**Lemma 1.2.2.** If \( (G, 1, \leq) \) is an \( \ell \)-group, then for any positive integer \( n \) and \( g, h, k \in G \), we have the following:

(a) \( g(h \lor k) = gh \lor gk \) and \( g(h \land k) = gh \land gk \)

(b) \( (h \lor k)^{-1} = h^{-1} \land k^{-1} \) and \( (h \land k)^{-1} = h^{-1} \lor k^{-1} \)

(c) \( (g \land 1)^n = g^n \land g^{n-1} \land \ldots \land g \land e \)

(d) \( (g \lor 1)^n = g^n \lor g^{n-1} \lor \ldots \lor g \lor 1 \)

(e) If \( g \land h = 1 \), then \( gh = g \lor h = hg \)

(f) \( |g \lor h| \leq |g| \lor |h| \leq |g||h| \).

(g) \( |gh| \leq |g||h||g| \) (Triangle Inequality).

(h) If \( gh = hg \), then \( |gh| \leq |g||h| \).

**Proof.** (a) Clearly, for \( g, h, k \in G \) we have \( gh, gk \leq gh \lor gk \). If \( c \in G \) be such that \( gh \leq c \), \( gk \leq c \) then \( gh \lor gk \leq c \). Hence \( g(h \lor k) = gh \lor gk \). The dual follows similarly.

(b) For \( h, k \in G \) we have \( h, k \leq h \lor k \) i.e., \( (h \lor k)^{-1} \leq h^{-1} \lor k^{-1} \). If \( c \in G \) be such that \( c \leq h^{-1}, k^{-1} \) then \( c^{-1} \geq h, k \) i.e., \( c^{-1} \geq h \lor k \) i.e., \( c \leq (h \lor k)^{-1} \) proving that \( (h \lor k)^{-1} = h^{-1} \land k^{-1} \). The dual follows similarly.

(c) Clearly the results hold true for \( n = 1 \). Assume that the given proposition holds true for
\[ (g \wedge 1)^n = (g \wedge 1)^{n-1}(g \wedge 1) \]
\[ = (g \wedge e)^{n-1}g \wedge (g \wedge 1)^{n-1} \]
\[ = (g^{n-1} \wedge g^{n-2} \wedge \ldots \wedge g \wedge 1)g \wedge (g^{n-1} \wedge g^{n-2} \wedge \ldots \wedge g \wedge 1) \]
\[ = g^{n-1}g \wedge g^{n-2}g \wedge \ldots \wedge gg \wedge 1g \wedge (g^{n-1} \wedge g^{n-2} \wedge \ldots \wedge g \wedge 1) \]
\[ = g^n \wedge g^{n-1} \wedge g^{n-2} \wedge \ldots \wedge g^2 \wedge g \wedge 1 \]

(d) Similarly we can prove that \((g \vee 1)^n = g^n \vee g^{n-1} \vee \ldots \vee g \vee 1\).

(e) Since \(g \wedge h = 1\), so it follows that \(gh = g(g \wedge h)^{-1}h = g(g^{-1} \vee h^{-1})h = g \vee h = h \vee g = hg\) by using (a).

(f) \(g, g^{-1} \leq |g| \text{ and } h, h^{-1} \leq |h|\), so \(g \vee h, g^{-1} \wedge h^{-1} \leq |g| \vee |h|\). Thus \(|g \vee h| = (g \vee h) \vee (g \vee h)^{-1} \leq |g| \vee |h| \leq |g||h| \vee |h||h| = |g||h|\) since \(|x| \geq e\) for all \(x\).

(g) \(gh \leq |g||h| \leq |g||h||g|\) and \(h^{-1}g^{-1} \leq |h||g| \leq |g||h||g|\). Therefore \(|gh| \leq |g||h||g|\).

\[ \Box \]

**Proposition 1.2.3.** An \(\ell\)-subgroup is torsion-free.

**Proof.** Suppose that \(g^n = 1\). Then it follows from the above Lemma,

\[ (g \wedge 1)^n = g^n \wedge g^{n-1} \wedge \ldots \wedge g \wedge 1 \]
\[ = g^{n-1} \wedge \ldots \wedge g \wedge 1 \]
\[ = (g \wedge e)^{n-1} \]

and so \(g \wedge 1 = 1\). Thus \(g \geq 1\). Similarly, \(g \leq 1\) and so \(g = 1\).

\[ \Box \]

**Proposition 1.2.4.** The underlying structure of an \(\ell\)-group \(G\) is distributive.

**Proof.** We assume that \(a, b, c \in G\), \(a \wedge b = a \wedge c\) and \(a \vee b = a \vee c\). We want to show that
b = c thereby showing that \( G \) is distributive. To this end we can write,

\[
\begin{align*}
b &= (a \land b)aa^{-1}(a \land b)^{-1}b \\
&= (a \land b)aa^{-1}(a^{-1} \lor b^{-1})b \\
&= (a \land b)a^{-1}(b \lor a) \\
&= (a \land c)a^{-1}(c \lor a) \\
&= (a \land c)a^{-1}(a^{-1} \lor c^{-1})c \\
&= c.
\end{align*}
\]

Thus, \((G, \lor, \land)\) is a distributive lattice.

**Definition 1.2.5.** Let \( L \) be a lattice and \( A \subseteq L \). \( A \) is said to be a sublattice of \( L \) if for any \( a, b \in A \) we have \( a \lor b \in A \) and \( a \land b \in A \).

For a given \( \ell \)-group \((G, 1, \leq)\) a subgroup \( H \) is called an \( \ell \)-subgroup if \( H \) is also a sublattice of \( G \).

**Definition 1.2.6.** Given any two \( \ell \)-groups \( G, H \) a group homomorphism \( \tau : G \to H \) is said to be an \( \ell \)-homomorphism if it preserves both the group and the lattice structures. An \( \ell \)-homomorphism which is a bijection is called an \( \ell \)-isomorphism. Obviously the inverse of an \( \ell \)-isomorphism is also an \( \ell \)-isomorphism. Two \( \ell \)-groups are said to be \( \ell \)-isomorphic to one another if there is an \( \ell \)-isomorphism between them.

### 1.3 Convex \( \ell \)-subgroups

In this section we shall introduce the very important notion of convex \( \ell \)-subgroups of an \( \ell \)-group \( G \). The convex \( \ell \)-subgroups play a very important role in the theory of lattice-ordered groups. In particular, the kernels of \( \ell \)-homomorphisms are convex \( \ell \)-subgroups.

**Definition 1.3.1.** For an \( \ell \)-subgroup \( H \) of \( G \), we say \( H \) is convex if whenever \( h_1, h_2 \in H \)
and \( h_1 \leq g \leq h_2 \), then \( g \in H \). It is known that \( H \) is a convex \( \ell \)-subgroup of \( G \) if and only if whenever \( h \in H \) and \( 1 \leq g \leq h \), then \( g \in G \).

**Example 1.3.2.** (i) \( G \) itself is a convex \( \ell \)-subgroup of \( G \). Also, \( \{1\} \) is a convex \( \ell \)-subgroup of \( G \). These are called **trivial** convex \( \ell \)-subgroups.

(ii) Let \( G = \mathbb{Z} \oplus \mathbb{Z} \) as defined in (iv) of Example 1.2.1. Then \( \{(0, m) : m \in \mathbb{Z}\} \) is the only non-trivial convex \( \ell \)-subgroup of \( G \).

Let \( \mathcal{C}(G) \) denote the set of all convex \( \ell \)-subgroups of \( G \). When partially ordered by inclusion \( \mathcal{C}(G) \) becomes a complete distributive lattice. It is a complete sublattice of the lattice of all subgroups of \( G \). Moreover for all \( A, C_i \in \mathcal{C}(G), \ (i \in I), \) we have \( A \cap \bigvee \{C_i : i \in I\} = \bigvee \{A \cap C_i : i \in I\} \). We elaborate on these points.

Recall that a set of subsets that forms a totally-ordered set under inclusion is called a **chain of subsets**. It is straightforward to check the following. If \( \{C_i : i \in I\} \) is a chain of convex \( \ell \)-subgroups of \( G \), then \( \bigcup \{C_i : i \in I\} \) is again a convex \( \ell \)-subgroup of \( G \). Further, any arbitrary intersection of convex \( \ell \)-subgroups is again a convex \( \ell \)-subgroup. Thus, we may speak of the **convex \( \ell \)-subgroup generated by a subset of \( G \)**. For a subset \( A \subseteq G \), we use \( G(A) \) to denote the convex \( \ell \)-subgroup generated by \( A \). When \( A = \{a\} \), we instead write \( G(a) \) and call this the **principal convex \( \ell \)-subgroup** generated by \( a \). So, if \( C_a \) is the set of all convex \( \ell \)-subgroups of \( G \) that contain \( a \), then \( G(a) = \cap \{C : C \in C_a\} \). The join of two convex \( \ell \)-subgroups is precisely the subgroup generated by the two subgroups. The convex \( \ell \)-subgroup generated by an element \( g \in G \) is denoted by \( G(g) \) and it is known that

\[
G(g) = \{h \in G : |h| \leq |g|^n \text{ for some } n \in \mathbb{N}\}.
\]

Moreover, \( G(g) = G(|g|) \), and for \( a, b \in G^+ \), \( G(a) \cap G(b) = G(a \wedge b) \) and \( G(a) \vee G(b) = G(a \vee b) = G(ab) \).

**Definition 1.3.3.** Suppose \( G \) is an \( \ell \)-group. If \( g \in G \) satisfies \( G(g) = G \), then \( g \) is called a **strong order unit** of \( G \).
Another class of convex \(\ell\)-subgroups play a very important role in the later sections. Let \(G\) be a lattice ordered group and \(C\) be a subgroup of \(G\). We say that \(C\) is closed if for any subset \(\{c_i : i \in I\}\) of \(C\), \(\bigvee_{i \in I} c_i \in C\) whenever it exists in \(G\). If \(\{c_i : i \in I\} \subseteq C\) and \(\bigwedge_{i \in I} c_i\) exists in \(G\), then this infimum belongs to \(C\) if \(C\) is closed. So, all closed subgroups are \(\ell\)-subgroups of \(G\).

**Definition 1.3.4.** For a given \(1 \neq g \in G\) we define a component of \(g\) to be the element \(h\) that is disjoint from \(gh^{-1}\).

In this case disjoint elements commute i.e. \(g\) commutes with any of its components. The collection of components of a given positive element for a boolean algebra under \(\lor, \land\).

**Proposition 1.3.5.** Let \(G\) be an \(\ell\)-group and \(S \in \mathcal{C}(G)\). Let \(\mathcal{R}(S) = \{Sx : x \in G\}\) denote the set of all right cosets of \(S\). On \(\mathcal{R}(S)\), define \(Sx \geq Sy\) if there exists \(s \in S\) such that \(sx \geq y\). Then \(\geq\) is a partial order of \(\mathcal{R}(S)\) and is called the coset ordering of \(\mathcal{R}(S)\).

**Definition 1.3.6.** An \(\ell\)-ideal of \(G\) is a normal convex \(\ell\)-subgroup. We denote the set of all \(\ell\)-ideals of \(G\) by \(\mathcal{L}(G)\). Then \(\mathcal{L}(G)\) forms a complete lattice when partially-ordered by inclusion.

**Proposition 1.3.7.** A subgroup \(L\) is an \(\ell\)-ideal of \(G\) if and only if it is the kernel of an \(\ell\)-homomorphism of \(G\).

Given an \(\ell\)-ideal \(L\) of \(G\), \(G/L\) forms an \(\ell\)-group with \(Lx \lor Ly = L(x \lor y)\) and dually.

Most of the standard theorems about groups and their homomorphisms translate into one about \(\ell\)-groups and \(\ell\)-homomorphisms. For the record we have the following isomorphism theorems in the theory of \(\ell\)-groups.

(i) (First Isomorphism Theorem) Suppose that \(\phi : G \to H\) is a surjective homomorphism between two \(\ell\)-groups. Then \(\ker(\phi)\) is a convex \(\ell\)-subgroup and \(G/\ker(\phi) \cong H\).

(ii) (Second Isomorphism Theorem) Let \(A\) be an \(\ell\)-subgroup of \(G\) and let \(B \in \mathcal{L}(G)\). Then \(AB\) is an \(\ell\)-subgroup of \(G\) and \(A \cap B \in \mathcal{C}(G)\) and we have \(AB/B \cong A/A \cap B\).
(iii) *(Third Isomorphism theorem)* Suppose that $A \subseteq B \subseteq G$ be $\ell$-ideals of $G$. Then, $B/A$ is an $\ell$-ideal of $G/A$ and $G/B \cong (G/A)/(B/A)$.

(iv) *(Lattice Isomorphism theorem)* For any $\ell$-ideal $A$ of $G$, the convex $\ell$-subgroups of $G/A$ are in one-to-one correspondence with the convex $\ell$-subgroups of $G$ that contain $A$.

## 1.4 Prime subgroups

In this section we shall study prime subgroups. These are indeed the analogs of prime ideals in the theory of commutative algebra.

**Definition 1.4.1.** If $P \in \mathcal{C}(G)$ and $g \land h \in P$ implies that either $g \in P$ or $h \in P$, then we call $P$ a prime subgroup.

The following proposition collects a variety of ways to regard prime subgroups.

**Proposition 1.4.2.** Let $P$ be any convex $\ell$-subgroup of $G$. Then the following conditions are equivalent:

a) $P$ is prime.

b) If $g, h \in G$ and $g \land h = 1$, then either $g \in P$ or $h \in P$.

c) For any $A, B \in \mathcal{C}(G)$ if $A \cap B \subseteq P$ then it implies either $A \subseteq P$ or $B \subseteq P$.

d) If $A, B \in \mathcal{C}(G)$ such that $P \subseteq A$ and $P \subseteq B$, then either $A \subseteq B$ or $B \subseteq A$.

We denote the collection of prime subgroups of $G$ by $\text{Spec}(G)$. The intersection of a chain of prime subgroups is a prime subgroup. Zorn’s Lemma argument guarantees the existence of minimal prime subgroups. Thus given any $P \in \text{Spec}(G)$, there exists a minimal prime subgroup $P' \subseteq P$. Unlike $\mathcal{C}(G)$ or $\mathcal{L}(G)$, the set of all prime subgroups of $G$ does not form a lattice since the intersection of two prime subgroups need not be prime. However $\text{Spec}(G)$ forms a very nice structure called a root system.
Definition 1.4.3. A partially-ordered set is a root system if no two incomparable elements have a lower bound.

Since every prime subgroup contains a minimal prime subgroup, hence if two prime subgroups have a lower bound that is also prime then by property (d) of Proposition 1.4.2 it follows that the original subgroups must be comparable. Thus the set of all prime subgroups of a given lattice-ordered group form a root system.

1.5 Values and Regular subgroups

In this section we shall focus on certain kinds of ℓ-subgroups called regular subgroups. It is a theorem that each convex ℓ-subgroup is the intersection of regular subgroups containing it.

Given any ℓ-group $G$, and $g \in G^+$, let $\nu = \{V \in \mathcal{C}(G) : g \notin V\}$. By the previous discussion, the union of any chain of elements in $\nu$ is back in $\nu$. Since $\{1\} \in \nu$ we have $\nu \neq \emptyset$. Hence by Zorn’s Lemma we have maximal elements in $\nu$ say $M$ that is maximal with respect to not containing $g$. Such a convex ℓ-subgroup is said to be a value of $g$ in $G$ and $M$ is called a regular subgroup. Let, $M^* = \cap\{C \in \mathcal{C}(G) : M \subseteq C\}$. Then $M$ is regular if and only if $M \subset M^*$ and $M^*$ is called the cover of $M$.

Proposition 1.5.1. Given any $g \in G^+$, we have $M \in \Gamma(g)$ if and only if $g \in M^* \setminus M$.

If $M$ is a value of $g \in G$, then by our previous proposition $g \notin M$ yet $g \in C$ for all convex ℓ-subgroups $C$ of $G$ that properly contain $M$. Since Zorn’s Lemma argument guarantees that each $g \in G^+$, has a value it follows that $\cap\{V : V$ is a value of some $g \in G^+\} = \{1\}$. An element $g \in G$ is said to be special if it has a unique value. A regular subgroup is prime and this fact gives our following propositions.

Proposition 1.5.2. Every convex ℓ-subgroup is the intersection of the regular subgroups that contain it and so every convex ℓ-subgroup is the intersection of prime subgroups.
Corollary 1.5.3. A convex \(\ell\)-subgroup \(C \in G\) is prime if and only if it is the intersection of a chain of regular subgroups.

The following are some examples of regular subgroups.

Example 1.5.4. Let \(G = \mathbb{Z}, \mathbb{Q}, \text{ or } \mathbb{R}\). Since \(G\) is totally ordered it follows that there are only two prime subgroups \(G\) and \(\{0\}\) and \(\{0\}\) is the only regular subgroup. Hence \(\Gamma(G) = \{0\}\).

Example 1.5.5. Let \(G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}\) with componentwise addition and define,

\[
(k, m, n) \vee (0, 0, 0) = \begin{cases} (k, m, n), & \text{if } n > 0 \\ (k \lor 0, m \lor 0, 0), & \text{if } n = 0 \\ (0, 0, 0), & \text{if } n < 0 \end{cases}
\]

Then the value of \((0,0,1)\) is \(\{(k, m, 0) : k, m \in \mathbb{Z}\}\), the value of \((0,1,0)\) is \(\{(k, 0, 0) : k \in \mathbb{Z}\}\), and the value of \((0,1,0)\) is \(\{(0, m, 0) : m \in \mathbb{Z}\}\). These are all the proper prime subgroups of \(G\) and the only regular subgroups.

1.6 Polars

In section 1.4 we saw that minimal prime subgroups always exist in any lattice-ordered group. In this section we shall try to characterize minimal prime subgroups in terms of what are known as polars.

Given a lattice ordered group \(G\) and any element \(g \in G\), let \(g^\perp = \{f \in G : \lvert f \rvert \land \lvert g \rvert = 1\}\). Then \(g^\perp\) is called the polar of \(G\). If \(f \in g^\perp\), then \(g, f\) are said to be disjoint.

By property (e) and (f) of Lemma 1.2.2 it follows that \(g^\perp \in \mathcal{C}(G)\). For a subset \(X \subseteq G\), \(X^\perp = \{g \in G : \lvert x \rvert \land \lvert g \rvert = 1\text{ for all } x \in X\}\). For any subset \(X \subseteq G\), \(X^\perp \in \mathcal{C}(G)\) and is called the polar of \(X\). The intersection of polars is again a polar. We denote by \(g^\perp\) the set \(\{g\}^\perp\). Moreover since \(G(g)^\perp = g^\perp\) it follows that \(X^\perp = C^\perp\) where \(C = G(X)\). As we have already seen that Zorn’s Lemma guarantees that every prime subgroup contains a minimal prime subgroup. For each \(g \in G^+, \) let \(P_g\) be the minimal prime subgroup of \(G\) contained in
some value of $g$. Since $g \notin P_g$, we have $\cap\{P_g : g \in G^+\} = \{1\}$. Hence the intersection of all minimal prime subgroups of $G$ is $\{1\}$. The following theorem gives us a characterization of minimal prime subgroups in terms of polars.

**Theorem 1.6.1.** Let $G$ be a lattice ordered group and $P \in \mathcal{C}(G)$. The following are equivalent:

(i) $P$ is a minimal prime subgroup of $G$.

(ii) $P = \bigcup\{g^\perp : g \notin P\}$.

(iii) For all $y \in P$, $y^\perp \notin P$.

We denote the set of all polars of $G$ by $\mathcal{P}(G)$. For any lattice ordered group $G$, it is known that $\mathcal{P}(G)$ is a complete Boolean algebra under inclusion, where

$$\bigwedge \{P_i : i \in I\} = \cap\{P_i\} \text{ and } \bigvee \{P_i : i \in I\} = (\cap\{P_i^\perp : i \in I\})^\perp = (\cup\{P_i : i \in I\})^\perp$$

We now give a few examples of polars in $\ell$-groups.

**Example 1.6.2.** Let $G = \mathbb{Z}, \mathbb{Q}, \text{or } \mathbb{R}$. Then the only polars are $\{0\}$ and $G$ itself. Thus $\mathcal{P}(G) = \{\{0\}, G\}$ and this is the polar structure of any totally ordered group.

**Example 1.6.3.** Let $G = \mathbb{Z} \times \mathbb{Z}$ with componentwise order and addition. Here the polars are $G, \mathbb{Z} \times \{0\}, \{0\} \times \mathbb{Z}, \text{and } \{(0,0)\}$.

**Example 1.6.4.** Let us consider Example 1.5.4.
The polars of $G$, then are given by, $\{(k,0,0) : k \in \mathbb{Z}\}, \text{and } \{(0,m,0) : m \in \mathbb{Z}\} \text{ and } \{(0,0,0)\}$.

**Definition 1.6.5.** An element $g \in G$ for which $g^\perp = \{1\}$ is called a weak order unit.

It is straightforward to check that a strong order unit is in fact a weak order unit but not conversely. When $G$ possess a (positive)strong order unit, say $u \in G^+$, we say that $G$ is a unital $\ell$-group and denote it by $(G,u)$. If $G$ is unital then maximal convex $\ell$-subgroups exist, and a theory of the structure theory of the space of maximal convex $\ell$-subgroups exists.
CHAPTER 2

Classes of ℓ-Groups

2.1 Normal-valued lattice ordered groups

In this section we discuss one of the most important classes of lattice-ordered groups, the normal-valued ℓ-groups.

Recall, if $V$ denotes the value of any $g \in G^+$, then $V^*$ is defined to be the cover of $G$. Also we found that $V^*$ was equal to the intersection of all the convex $l$-subgroups of $G$ containing $V$ and since intersection of convex $l$-subgroups is again convex it follows that $V^* \in \mathcal{C}(G)$. But it is not in general true that $V \in \mathcal{L}(G)$. Let us first provide an equivalent condition for a value to be normal in its cover.

**Proposition 2.1.1.** Let $G$ be a lattice-ordered group and $V$ be a value in $G$. Then $V \vartriangleleft V^*$ if and only if for all $g \in V^*$ and $f \in G^+ \setminus V$, there is a positive integer $n$ such that $Vg \leq Vf^n$.

**Definition 2.1.2.** If every value of $G$ is normal in its cover, then $G$ is called normal-valued and the value is a normal value.

If $V$ is a value in a normal-valued lattice-ordered group and $V^*$ is its cover then, $V \vartriangleleft V^*$ and $V^*/V$ is a lattice-ordered group whose only convex $l$-subgroups are the trivial ones. It follows by Hölder’s Theorem (see Theorem [2.2.2]) that $V^*/V$ is order isomorphic to a subgroup of $\mathbb{R}$. 
Example 2.1.3. Every abelian lattice-ordered group is normal-valued and every ordered group is normal-valued lattice-ordered group. Every special value is a normal value.

When a class of \(\ell\)-groups are closed under taking \(\ell\)-homomorphic images, convex \(\ell\)-subgroups and direct products, it forms a \(\ell\)-variety and by a result by Birkhoff it follows that this class can be represented by a set of equations and are called an *equational class*.

Example 2.1.4. It is known that \(G\) is normal valued if and only if for all \(x, y \in G^+\), \(xy \leq y^2x^2\). It follows that the class of normal-valued \(\ell\)-groups is an equational class.

### 2.2 Archimedean \(\ell\)-groups

Archimedean \(\ell\)-groups have attracted much attention from topologists who study rings of continuous functions. In this section we shall try to characterize archimedean lattice ordered groups.

For any \(\ell\)-group if \(g, h \in G\) be such that \(ng \leq h\) for all \(n \in \mathbb{N}\) we write \(g \ll h\).

**Definition 2.2.1.** An \(\ell\)-group \(G\) is said to be archimedean if \(g, h \in G^+\) and \(ng \leq h\) for all \(n \in \mathbb{N}\) implies that \(g = 1\) i.e. there doesnot exist any \(g, h \in G^+\) such that \(g \ll h\).

We shall denote the class of archimedean \(\ell\)-groups by \(\text{Arch}\).

The following theorem is due to Hölder and it is characterization of the archimedean \(o\)-groups.

**Theorem 2.2.2** (Hölder, 1901). Let \(H\) be an \(o\)-group. The following are equivalent:

(a) \(H\) is a subgroup of the real numbers \(\mathbb{R}\);

(b) \(H\) has no nontrivial convex \(\ell\)-subgroups.

(c) \(H\) is archimedean.
Example 2.2.3. Recall from Example 1.2.1 that $C(X)$ is the set of all real-valued continuous functions on a space $X$. $C(X)$ is an archimedean $\ell$-group. To see this let us suppose that there exists $f_1, f_2 \in C(X)$ such that $0 \leq f_1, f_2$ and $f_1 \ll f_2$ where for all $n \in \mathbb{N}$. If $f_1 \neq 0$, then there exists $x \in X$ such that $f_1(x) > 0$. Now $nf_1(x) = n(f_1(x)) \ll f_2(x)$ for all $n \in \mathbb{N}$ and $f_1(x) \neq 0$ which contradicts that $\mathbb{R}$ is archimedean. Hence $C(X)$ is archimedean.

Other examples of archimedean $\ell$-groups are of course the subgroups of $\mathbb{R}$ and their cardinal products.

One of the main theorems in the theory of $\ell$-groups is the following.

**Theorem 2.2.4.** Every archimedean $\ell$-group is abelian.

The following theorem gives us the conditions under which a quotient of an archimedean $\ell$-group is again archimedean.

**Theorem 2.2.5.** For a convex $\ell$-subgroup $M$ of an archimedean $\ell$-group $G$, the following are equivalent:

(i) $M$ is a special value

(ii) $M$ is a closed prime subgroup.

(iii) $M$ is a maximal polar.

(iv) $M$ is a maximal and a cardinal summand and $G/M$ is archimedean.

It is well known that every archimedean $\ell$-group is abelian. Archimedean $\ell$-groups are closed under direct products and $\ell$-subgroups but not necessarily under $\ell$-homomorphisms and hence these class of $\ell$-groups do not form a $\ell$-variety.

**Example 2.2.6.** The following is an example of a archimedean $\ell$-group with a non-archimedean homomorphic image. Consider $G = \mathbb{R}^\mathbb{N} = \prod_{n \in \mathbb{N}} \mathbb{R}$ and let $L$ be the $\ell$-ideal of all eventually zero sequences in $G$; so $G/L$ is an $\ell$-homomorphic image of $G$. Let $f, g \in G$ be defined by $f(n) = 1$ and $g(n) = n$ for all $n \in \mathbb{N}$. Then $G$ is an archimedean. But in $G/L$, we shall have $f + L \ll g + L$ and $f + L \neq L$. Thus $G/L$ is nonarchimedean.
2.3 Hyperarchimedean $\ell$-groups

As we have seen in the previous example that archimedean $\ell$-groups are not necessarily closed under $\ell$-homomorphic images. Hence the study of the $\ell$-groups for which every $\ell$-homomorphic image is archimedean have been a point of interest. These are called hyperarchimedean $\ell$-groups. For any infinite set $\lambda$, $\sum_{\lambda} \mathbb{Z}$ is hyperarchimedean but $\prod_{\lambda} \mathbb{Z}$ is not.

Definition 2.3.1. If $G$ is an $\ell$-group for which every $\ell$-homomorphic image is archimedean, then $G$ is said to be hyperarchimedean. These groups were originally called epiarchimedean by Conrad (see [13]). Observe that a hyper-archimedean $\ell$-group is archimedean.

We shall denote the class of all hyperarchimedean $\ell$-groups by $\mathcal{H}_{\text{Arch}}$.

The following result gives some other ways to think about these hyperarchimedean $\ell$-groups.

Theorem 2.3.2 (Conrad, see [13]). For an $\ell$-group $G$, the following are equivalent:

(a) $G$ is hyperarchimedean.

(b) Every prime subgroup is maximal and hence minimal.

(c) $G = G(g) \boxplus g^+$ for all $g \in G$.

(d) For all $f, g \in G^+$, there exists $N \in \mathbb{Z}$ such that for all $m, n \geq N, mf \wedge g = nf \wedge g$.

(e) For all $f, g \in G^+$, there exists $n \in \mathbb{N}$ such that $nf \wedge g = (n + 1)f \wedge g$.

(f) $G$ is $\ell$-isomorphic to a subdirect of the reals such that for all $0 \leq f, g \in G$, there exists an integer $n$ such that for all $x \in \text{supp}(f), g(x) \leq nf(x)$.

By applying property (g) we get the following corollary:

Corollary 2.3.3. Every $\ell$-subgroup of a hyperarchimedean lattice-ordered group is hyperarchimedean.
Notice that in Theorem 2.3.2 no assumptions regarding the commutativity of the group is made. A subclass of the class of hyperarchimedean ℓ-groups consists of those ℓ-groups for which every totally-ordered homomorphic image of $G$ is cyclic. Such ℓ-groups are called *hyper-Z* and were first introduced and studied by Martinez in [12] *Doubling chains, Singular elements and Hyper-Z ℓ-groups*. Of course not every hyperarchimedean ℓ-group is hyper-Z, e.g. $\mathbb{R}$.

**Remark.** The class of hyperarchimedean ℓ-groups is closed with respect to ℓ-subgroups and ℓ-homomorphic images. But the class is not closed under direct products. Consider $G = \prod \mathbb{R}$, a direct product of the hyper-archimedean ℓ-group $\mathbb{R}$. But $G \notin \mathcal{HA}rch$. Hence $\mathcal{HA}rch$ does not form a ℓ-variety. However, they do form a torsion class; consider the following definition.

**Definition 2.3.4.** A nonempty class $\mathcal{R}$ of ℓ-groups is a *radical class* if it is closed under convex ℓ-subgroups, closed under ℓ-isomorphic images, and joins of convex ℓ-subgroups. This latter property means that whenever $\{A_i \in C(G) : i \in I\}$ and each $A_i \in \mathcal{R}$, then $\bigvee_{i \in I} A_i$ also belongs to $\mathcal{R}$. A radical class $\mathcal{R}$ is called a *torsion class* if it is also closed under ℓ-homomorphic images.

**Example 2.3.5.** $Arch$ is a radical class but not a torsion class; it is not closed under ℓ-homomorphic images.

**Remark.** Suppose $\mathcal{R}$ is a radical class. Then for any ℓ-group $G$ define $\mathcal{R}(G) = \{A \in C(G) : A \in \mathcal{R}\}$. Clearly $\mathcal{R}(G) \neq \emptyset$ since the trivial subgroup always belongs to $\mathcal{R}$. Define $\overline{\mathcal{R}} = \bigvee \mathcal{R}(G)$, and observe that $\overline{\mathcal{R}} \in \mathcal{R}$ by our definition of radical class. In other words $\overline{\mathcal{R}}$ denotes the largest convex ℓ-subgroup of $G$ which lies in $\mathcal{R}$. this ℓ-subgroup is called the *radical kernel with respect to $\mathcal{R}$.*

It can be verified that the join of hyperarchimedean convex ℓ-groups is also hyperarchimedean. The following theorem gives us an example of a torsion class.

**Theorem 2.3.6 (Darnel, see [6]).** The class of hyperarchimedean ℓ-groups forms a torsion class.
Remark. In his proof Darnel constructs the hyperarchimedean kernel. We mention this because we will be interested in studying a different torsion class of $\ell$-groups, where we shall construct the kernel.

**Definition 2.3.7.** An element $1 \leq s$ in an $\ell$-group $G$ is said to be *singular* if for any $1 \leq t \leq s$, then $t \wedge st^{-1} = 1$. A $\ell$-group that is generated as a group by all of its singular elements is called a *Specker group*.

Specker $\ell$-groups have been fully investigated and one of the main characterizations is that every Specker group can be embedded as an $\ell$-subring of bounded integer-valued functions generated by characteristic functions or alternatively as an $\ell$-subgroup of bounded integer-valued functions generated by characteristic functions and hence is hyper-$\mathbb{Z}$. It was observed that Specker groups form a torsion-class of $\ell$-groups. Examples of hyper-$\mathbb{Z}$ $\ell$-groups that are not Specker will be constructed later.
CHAPTER 3

Filters on positive cones

3.1 Filters and principal filters

In this section we shall talk about filters which are natural objects to study on a lattice. The main aim of this chapter is to discuss these different types of filters and study the characteristics of the classes \( \ell \)-groups generated by the different kinds of filters.

**Definition 3.1.1.** A subset \( F \) of an \( \ell \)-group \( G \) is said to be a filter on \( G \) if

1. \( \emptyset \neq F \),
2. whenever \( f, g \in F \), then \( f \wedge g \in F \), and
3. if \( g, h \in G \) with \( g \leq h \) and \( g \in F \), then \( h \in F \).

**Example 3.1.2.** (i) Let \( S \) be a nonempty set and \( C \subseteq S \). Then the collection of all subsets containing \( C \) is a filter of the power set of \( S \).

(ii) Consider \( C(\mathbb{R}) \). Let \( M_0 = \{ f \in C(\mathbb{R}) : f(0) = 0 \} \). Then \( M_0 \) is a maximal ideal in \( C(\mathbb{R}) \) and the set of all zero set neighborhoods of 0 forms a filter on \( C(\mathbb{R}) \).

(iii) Consider \( \mathbb{N} \), the set of natural numbers and consider the set \( A = \{ (n, \infty) : n \in \mathbb{N} \} \). The collection of all subsets containing \( A \) is a filter on \( \mathbb{N} \).
(iv) Let $X$ be a topological space and $x \in X$. Let $N_x$ be set of all neighbourhoods of $x \in X$.

Then, $\{N_x : x \in X\}$ is called the \textit{filter of neighbourhoods of $x$}.

As we have already seen in Chapter 1, the positive cone of an $\ell$- group generates the group which explain our interest in investigating filters on the positive cone $G^+$ of an $\ell$-group $G$.

**Definition 3.1.3.** Given any lattice $(L, \wedge, \vee)$, a filter is called a \textit{principal filter} if it is of the form $\{x \in L : x \geq a\}$ for some $a \in L$. For any $g \in G^+$ we denote this filter by $\mathcal{F}_g = \{h \in G^+ : h \geq g\}$ and it is also referred as the filter generated by $g$. An $\ell$-group $G$ in which every filter on $G^+$ is a principal filter is called \textit{principally filtered $\ell$-group}.

**Example 3.1.4.**

(i) Example 2.1.2(i) is an example of a principal filter.

(ii) By the well-ordering principle of the natural, $\mathbb{Z}$ is a principally-filtered $\ell$-group. Also, as we shall shortly demonstrate any direct sum of copies of $\mathbb{Z}$ is also a principally-filtered $\ell$-group.

### 3.2 Principally-filtered groups

In this section we develop a theory of principally-filtered $\ell$-groups. We begin by showing that a principally-filtered $\ell$-groups is archimedean.

**Proposition 3.2.1.** Suppose $G$ is principally-filtered, then $G$ is archimedean.

**Proof.** By means of contradiction suppose that $G$ is not archimedean. So, there exists $g \geq 0$ such that $ng \leq h$, for all $n$. Consider $\mathcal{F} = \{x \in G^+ : x \geq h - ng\}$ for some $n \in \mathbb{N}$.

Claim. $\mathcal{F}$ is a filter on $G^+$.

Since $G$ is principally-filtered, there exists $x \in G^+$ such that $\mathcal{F} = \mathcal{F}_x$. Consider the chain,

$$h \geq h - g \geq h - 2g \geq \ldots \geq h - ng \geq \ldots$$
Notice for all \( n \in \mathbb{N} \) we have \( h - ng \in \mathcal{F} = \mathcal{F}_x \) which implies that \( h - ng \geq x. \) Also \( h - ng \geq x, \) for all \( n \in \mathbb{N}. \) So, in particular for \( n = n_1, h - n_1 g \geq x. \) Now, \( x \in \mathcal{F}_x = \mathcal{F} \) so, \( x = h - n_1 g \geq h - (n + 1)g \geq x. \) Thus, \( h - n_1 g = h - (n_1 + 1)g. \) It follows that, \( n_1 g = (n_1 + 1)g = n_1 g + g \) which implies that \( g = 0 \) which is a contradiction. \( \square \)

**Proposition 3.2.2.** Suppose \( G \) is a principally filtered \( \ell \)-group. Then every homomorphic image of \( G \) is principally filtered and hence every principally-filtered \( \ell \)-group is hyperarchimedean.

**Proof.** Suppose \( \tau : G \to E \) is an \( \ell \)-homomorphism. We shall show that \( H \) is principally filtered. To this end, consider, a filter \( \mathcal{F} \) on \( H^+. \) Then \( \tau^{-1}(\mathcal{F}) = \{ g \in G^+ : \tau(g) \in \mathcal{F} \} \) is a filter on \( G^+. \) Notice \( g_1, g_2 \in \tau^{-1}(\mathcal{F}). \) i.e. \( \tau(g_1), \tau(g_2) \in \mathcal{F}. \) i.e. \( \tau(g_1 \wedge g_2) = \tau(g_1) \wedge \tau(g_2) \in \mathcal{F}. \) Thus \( g_1 \wedge g_2 \in \tau^{-1}(\mathcal{F}). \) Also, \( g \leq h, g \in \tau^{-1}(\mathcal{F}). \) Therefore, \( \tau(g) \leq \tau(h). \) Hence, \( \tau(h) \in \mathcal{F} \) since \( \mathcal{F} \) is a filter. Therefore, \( h \in \tau^{-1}(F). \) Thus \( \tau^{-1}(\mathcal{F}) \) is a filter on \( G^+. \) Since, \( G \) is principally filtered, there exists \( g \in G \) such that \( \tau^{-1}(\mathcal{F}) = \mathcal{F}_g. \)

**Claim.** \( \mathcal{F} = \mathcal{F}_{\tau(g)} \).

Let \( h \in H^+ \) and \( h \in \mathcal{F}. \) Since \( \tau \) is onto, so there exists some \( j \in G^+ \) such that \( \tau(j) = h \) implies \( j \in \tau^{-1}(\mathcal{F}) = \mathcal{F}_g. \) Thus, \( j \geq g. \) Therefore, \( h = \tau(j) \geq \tau(g). \) Hence, \( h \in \mathcal{F}_{\tau(g)}. \) Thus \( \mathcal{F} \subseteq \mathcal{F}_{\tau(g)}. \) Also, \( \mathcal{F}_{\tau(g)} = \{ h \in H^+ : h \geq \tau(g) \text{ where } \tau(g) \in H^+ \} \subseteq \mathcal{F}. \) This implies that, \( h \in \mathcal{F} \). Thus, \( \mathcal{F} = \mathcal{F}_{\tau(g)}. \) Hence \( H \) is principally-filtered. \( \square \)

**Proposition 3.2.3.** If \( \{ G_i \}_{i \in I} \) is principally-filtered, then \( \oplus G_i \) is also principally-filtered.

**Proof.** Let \( G = \oplus G_i^+ \) and \( \mathfrak{F} \) be a filter on \( G. \) We want to show that \( \mathfrak{F} \) is principal. To this end, consider

\[
\Pi_i(\mathfrak{F}) = \{ x \in G_i^+ : \exists g \in \mathfrak{F} \text{ such that } g(i) = x \text{ for all } i \}
\]

. We claim that \( \Pi_i(\mathfrak{F}) \) is a filter on \( G_i^+. \) To this end, let \( x_1, x_2 \in \Pi_i(\mathfrak{F}) \), then there exists \( g_1, g_2 \in (\mathfrak{F}) \) such that \( g_1(i) = x_1, g_2(i) = x_2. \) Thus, \( x_1 \wedge x_2 = g_1(i) \wedge g_2(i) = (g_1 \wedge g_2)(i) \)
which implies that \( x_1 \land x_2 \in \Pi_i(\mathfrak{F}) \) since \( g_1 \land g_2 \in \mathfrak{F} \). Again let, \( x, y \in G_i^+ \) and \( x \leq y \) and \( x \in \Pi_i(\mathfrak{F}) \). Now, \( x \in \Pi_i(\mathfrak{F}) \) implies that there exists \( g \in \mathfrak{F} \) such that \( g(i) = x \). Consider,

\[
\hat{g}(j) = \begin{cases} 
  y & \text{if } i = j \\
  g(j) & \text{if } i \neq j 
\end{cases}
\]

Then, \( \hat{g} = (x, x, \ldots, y, x, \ldots) \). Thus \( g(j) = \hat{g}(j) \) for all \( i \neq j \) and \( \hat{g}(i) = y \). So \( \hat{g} \geq g \) and since \( g \in \mathfrak{F} \) and \( \mathfrak{F} \) is a filter on \( G^+ \) it implies that \( \hat{g} \in \mathfrak{F} \). Then \( \Pi_i(\mathfrak{F}) \) is a filter on \( G_i^+ \). Since \( G_i^+ \) is principally filtered, so there exists \( x_i \in G_i^+ \) such that \( \Pi_i(\mathfrak{F}) = \mathfrak{F}_{x_i} \). Define \( f \) such that \( f(i) = x_i \).

**Claim.**

i) \( f \in \bigoplus_{i \in I} G_i \)

ii) \( \mathfrak{F} = \mathfrak{F}_f \).

Clearly, \( \mathfrak{F}_f \) is a filter. Consider \( g \in \mathfrak{F} \). Since \( \mathfrak{F} \subseteq G \), hence \( g(i) = 1 \) for all but finitely many \( i \)'s. Let \( \{i_1, i_2, \ldots, i_n\} \) be the set such that \( g(i) \neq 1 \) for all \( i = i_1, i_2, \ldots, i_n \). For each \( i_j \) choose \( g_{i_j} \in \mathfrak{F} \) such that \( g_{i_j}(i_j) = x_{i_j} \). Consider \( h = g \land g_{i_1} \land g_{i_2} \land \cdots \land g_{i_n} \). Notice \( g \in \mathfrak{F} \), \( g_{i_k} \in \mathfrak{F} \) for all \( k = 1, 2, \ldots, n \) and since \( \mathfrak{F} \) is a filter so \( h \in \mathfrak{F} \). Now the following cases may arise:

**Case I:** If \( k \neq i_j \) for all \( j = 1, 2, \ldots, n \) then \( g(k) = 1 \) and \( h(k) = (g \land g_{i_1} \land g_{i_2} \land \cdots \land g_{i_n})(k) = g(k) \land g_{i_1}(k) \land g_{i_2}(k) \land \cdots \land g_{i_n}(k) \leq g(k) = 1 \) which implies that \( h(k) = 1 \) and since \( g \in \mathfrak{F} \), hence \( g(k) = 1 \geq x_k = f(k) \) for all \( k \) implies that \( f(k) = 1 \). Hence for all \( k \neq i_j \) we have \( f(k) = g(k) \). Thus for all \( k \neq i_j \), we have \( f = g \) and since \( g \in \mathfrak{F} \) it implies that \( f \in \mathfrak{F} \).

**Case II:** If \( k = i_j \) then \( g_k(k) = g_{i_j}(i_j) = x_{i_j} \). Now \( h(i_j) \in \mathfrak{F}_{i_j} = \Pi_{i_j}(\mathfrak{F}) = \mathfrak{F}_{x_{i_j}} \). Thus, \( h(i_j) = (g \land g_{i_1} \land g_{i_2} \land \cdots \land g_{i_n})(i_j) = g(i_j) \land g_{i_1}(i_j) \land g_{i_2}(i_j) \land \cdots \land g_{i_n}(i_j) \geq x_{i_j} = g_{i_j}(i_j) = x_{i_j} \) for all \( j = 1, 2, \ldots, n \) implies that \( h(i_j) = (g \land g_{i_1} \land g_{i_2} \land \cdots \land g_{i_n})(i_j) = g(i_j) \land g_{i_1}(i_j) \land g_{i_2}(i_j) \land \cdots \land g_{i_n}(i_j) = g(i_j) = x_{i_j} = f_{i_j} \) for all \( j = 1, 2, \ldots, n \) and hence \( f(i) = h(i) \) for all \( i = i_j \) and \( j = 1, 2, \ldots, n \) which implies that \( f \in \mathfrak{F} \). By the above cases \( f \in \mathfrak{F} \). This establishes our
claim(i). Hence $f \in \mathcal{F}_f$ implies that $f \in \mathcal{F}$. We have $\mathcal{F}_f \subseteq \mathcal{F}$. Conversely for the reverse containment notice that for any $\hat{h} \in \mathcal{F}$ we have $\hat{h}(i) \in \Pi_i(\mathcal{F}) = \mathcal{F}_x$, i.e. $\hat{h}(i) \geq x = f(i)$ for all $i$ i.e. $\hat{h} \geq f$ i.e. $\hat{h} \in \mathcal{F}_f$. Hence $\mathcal{F} \subseteq \mathcal{F}_f$. This establishes our claim(ii).

Thus $\oplus_{i \in I} G_i$ is principally-filtered.

\[ \square \]

**Proposition 3.2.4.** If $G$ is principally-filtered and $H \leq G$, then $H$ is principally-filtered.

**Proof.** Let $G$ be a principally-filtered group and $H \leq G$. Let $\mathcal{F}$ be a filter on $H^+$. We want to show that $\mathcal{F}$ is a principal filter. Consider $\mathcal{F}' = \{ g \in G^+ : g \geq h \text{ for some } h \in \mathcal{F} \}$. It is easy to verify that $\mathcal{F}'$ is a filter on $G^+$. To this end, let $g_1, g_2 \in \mathcal{F}'$. Then $g_1 \geq h_2, g_2 \geq h_2$ for some $h_1, h_2 \in H^+$. Thus $g_1 \land g_2 \geq h_1 \land h_2$. Clearly, $h_1 \land h_2 \in H^+$ since $\mathcal{F}'$ is a filter.

Let, $g_1 \in \mathcal{F}'$ and $g_1 \leq g_2$. Now, $g_1 \geq h_1$ for some $h_1 \in H^+$ i.e. $h_1 \leq g_1 \leq g_2$. This implies that $g_2 \in \mathcal{F}'$. Thus, $\mathcal{F}'$ is a filter on $G^+$ and $G$ is principally-filtered. This implies that there exists some $\hat{g} \in G^+$ such that $\mathcal{F}' = \mathcal{F}_{\hat{g}}'$.

**Claim.** $\mathcal{F} = \mathcal{F}' = \mathcal{F}_{\hat{g}}'$.

Clearly, $\mathcal{F} \subseteq \mathcal{F}'$ since, for all $h \in \mathcal{F}, h \geq h$ which implies that $h \in \mathcal{F}' = \mathcal{F}_{\hat{g}}'$.

Conversely, we want to show that $\mathcal{F}_{\hat{g}} \subseteq \mathcal{F}$. Notice, $\hat{g} \in \mathcal{F}_{\hat{g}} = \mathcal{F}'$ which implies that $g \geq \hat{h}$ where $\hat{h} \in \mathcal{F}$. Since, $\mathcal{F}$ is a filter, hence, $\hat{g} \in \mathcal{F}$. Thus $\mathcal{F}_{\hat{g}} \subseteq \mathcal{F}$ i.e. $\mathcal{F} = \mathcal{F}_{\hat{g}}$. Hence $H$ is principally filtered. \[ \square \]

Next, we completely characterize principally-filtered $\ell$-groups. To obtain our result we need to remind the reader of Conrad’s Property (F) (see [3]).

**Definition 3.2.5.** The $\ell$-group $G$ is said to satisfy property (F) for each $g \in G^+$, $g$ is greater than at most a finite number of disjoint elements.

**Proposition 3.2.6.** Suppose that $G$ is a principally-filtered $\ell$-group, then $G$ satisfies property (F). In particular, each $g \in G^+$ belongs to all but at most a finite number of minimal prime subgroups of $G$. 

Proof. The proof of this follows readily from the works in the paper (see [3]) which states that every cold-filter on $G$ is principally-filtered, then $G$ satisfies property (F).

**Theorem 3.2.7** (Conrad, see [5]). $G$ is a direct sum of archimedean o-groups if and only if $G$ is archimedean and satisfies (F).

**Theorem 3.2.8.** Suppose $G$ is an $\ell$-group. $G$ is principally-filtered if and only if there exists an index set $I$ such that $G \cong \oplus_{i \in I} \mathbb{Z}_i$, that is $G$ is a direct sum of copies of $\mathbb{Z}$.

**Proof.** The sufficiency of the fact that a direct sum of copies of $\mathbb{Z}$ is a principally-filtered $\ell$-group follows from Proposition 3.2.3 together with the fact that $\mathbb{Z}$ is a principally-filtered $\ell$-group. As to the converse suppose that $G$ is principally-filtered. Then in particular $G$ is hyperarchimedean. Consider the embedding (which holds for all abelian $\ell$-groups)

$$\phi : G \to \prod_{P \in \text{Min}(G)} G/P$$

Since $G$ is principally-filtered so each $G/P$ and hence $G/P$ is $\ell$-isomorphic to $\mathbb{Z}$. Furthermore this induces an embedding

$$G \to \prod_{P \in \text{Min}(G)} \mathbb{Z}$$

The fact that $G$ satisfies property (F) means that $G$ possess a basis (see [3]) which contains the full cardinal sum of copies of $\mathbb{Z}$. By Proposition 3.2.6 $G$ satisfies the property that every $g \in G^+$ belongs to all but a finite number of minimal prime subgroups of $G$, which means that the above embedding is actually an embedding

$$G \to \oplus_{P \in \text{Min}(G)} \mathbb{Z}$$

This means that the embedding is an $\ell$-isomorphism. Consequently, $G$ is $\ell$-isomorphic to a cardinal sum of copies of $\mathbb{Z}$.

\[\square\]
CHAPTER 4

Cold-filtered $\ell$-groups

4.1 Definition

Definition 4.1.1. Let $G$ be an $\ell$-group and let $\mathcal{F}$ be any filter on $G^+$. Then $\mathcal{F}$ is said to be a cold filter if for all $P \in \text{Min}(G)$, the filter $\mathcal{F}_P = \{g + P : g \in G\}$ has a minimum element on $(G/P)^+$. (By a minimum, we mean there is some $f \in \mathcal{F}$ such that for every $g \in \mathcal{F}$, $f + P \leq g + P$). This definition is due to the works of the authors in [3]. If every filter on $G$ is a cold filter, then $G$ is cold-filtered.

Example 4.1.2. It follows directly that every principal filter is a cold-filter. So, let us try to give an example of a cold-filter that is not a principal filter. Let $X$ be an infinite compact, zero-dimensional space, e.g. the Cantor space. Let $G = C(X, \mathbb{Z})$. The minimal prime subgroups of $G$ are precisely the subgroups of the form $M_x = \{g \in C(X, \mathbb{Z}) : g(x) = 0\}$. Let $p \in X$ be a non-isolated point and consider $\mathcal{F} = \{g \in G^+ : g(p) \geq 1\}$. A direct proof demonstrates that $\mathcal{F}$ is a filter on $G^+$. To this end, let $g_1, g_2 \in \mathcal{F}$. Then, $g_1(p) \geq 1$, $g_2(p) \geq 1$. Thus, $g_1(p) \wedge g_2(p) \geq 1 \wedge 1 = 1$. i.e. $g_1 \wedge g_2 \in \mathcal{F}$. Also, let $f \in \mathcal{F}$ be such that $f \leq g$. We claim that $g \in \mathcal{F}$. Notice, $g(p) \geq f(p) \geq 1$. Then, $g \in \mathcal{F}$. Thus, $\mathcal{F}$ is a filter on $G^+$.

To show that $\mathcal{F}$ is a cold filter let $x \in X$. Then $\mathcal{F}_{M_x}$ has a minimum on $G/M_x$, namely $1 + M_x \in \mathcal{F}_{M_p}$. We show that $\mathcal{F}$ is not principal. Choose $x \in X$ different than $p$. Since, $X$ is
zero dimensional, so it has a base of clopen sets. Thus, there exists a clopen set $C$ such that $p \in C$ and $x \notin C$. The function $\chi_C \in G$ and moreover $\chi_C \in \mathcal{F}$. Thus, any minimum of $\mathcal{F}$ would have to be 0 at every point but $p$. This cannot happen unless $p$ is isolated. It follows that $\mathcal{F}$ is not a principal filter unless $p$ is an isolated point.

### 4.2 Some basic features of Cold-filtered groups

**Proposition 4.2.1.** Let $G$ be an $\ell$-group. Let $\mathcal{F}$ be any filter on $G^+$. The following statements are equivalent:

(i) $\forall P \in \text{Min}(G)$, the filter $\mathcal{F}_P$ has a minimum element on $(G/P)^+$. 

(ii) $\forall P \in \text{Spec}(G)$, the filter $\mathcal{F}_P$ has a minimum element on $(G/P)^+$. 

(iii) $\forall P \in \text{Val}(G)$, the filter $\mathcal{F}_P$ has a minimum element on $(G/P)^+$. 

**Proof.** $(ii) \Rightarrow (i)$: Clear. $(i) \Rightarrow (ii)$: Let $\mathcal{F}$ be a cold filter on $G$ and $P \in \text{Spec}(G)$. Now by Zorn’s Lemma it can be shown that every prime subgroup contains a minimal prime subgroup. Thus there exists a minimal prime subgroup $P' \in \text{Min}(G)$ such that $P' \leq P$. Since (i) holds, i.e. $\mathcal{F}'_P$ has a minimum, hence there exists $g \in \mathcal{F}$ such that $g + P' \leq f + P'$, for all $f \in \mathcal{F}$. We want to show that $g + P \leq f + P$, for all $f \in \mathcal{F}$. Notice, since $g + P' \leq f + P'$, for all $f \in \mathcal{F}$. Thus there exists $p' \in P'$ such that $g + p' \leq f$, for all $f \in \mathcal{F}$. Since $p' \in P' \leq P$ we have $g + P \leq f$, for all $f \in \mathcal{F}$ i.e. $g + P \leq f + P$. Thus for all $P \in \text{Spec}(G)$, $\mathcal{F}_P$ has a minimum. So, (ii) holds.

$(iii) \Rightarrow (ii)$: This follows from the fact that every value is prime. $(ii) \Rightarrow (iii)$: Since $P \in \text{Spec}(G)$, hence $P$ is the intersection of a chain of regular subgroups containing it. □

**Proposition 4.2.2.** Every cold-filtered $\ell$-group is archimedean.

**Proof.** Let $G$ be a cold-filtered $\ell$-group. We want to show that $G$ is archimedean. To this end, suppose by means of contradiction that $G$ is not archimedean. So, there exists some $g \neq 0$
and \( g, h \in G^+ \) such that \( ng \leq h \) for all \( n \in \mathbb{N} \). Since, \( g \neq 0 \) there exists \( P \in \text{Min}(G) \) such that \( g \notin P \). Consider the chain, \( h - g \geq h - 2g \geq h - 3g \geq \ldots \geq h - ng \geq h - (n+1)g \geq \ldots \).

Let \( \mathcal{F} = \{ x \in G^+ : x \geq h - ng \text{ for some } n \in \mathbb{N} \} \). Notice that \( \mathcal{F} \) is a filter on \( G^+ \). Consider the filter, \( \mathcal{F}_P = \{ g + P : g \in \mathcal{F} \} \). Then, \( \mathcal{F}_P \) is a filter on \((G/P)^+\). Since, \( G \) is cold-filtered \( \mathcal{F}_P \) has a minimum say \( k + P \) where \( k \in \mathcal{F} \). Hence, \( k + P \leq (h - ng) + P \), for all \( n \in \mathbb{N} \).

Since, \( k \in \mathcal{F} \), so, \( k \geq h - n_1g \) for some \( n = n_1 \in \mathbb{N} \). Hence, \( k + P \geq (h - n_1g) + P \). So, in particular, for \( n = n_1 \), we have the following:

\[
(h - n_1g) + P \leq k + P \leq (h - n_1g) + P.
\]

Thus, \( k + P = (h - n_1g) + P \).

Therefore, \((h - (n_1 + 1)g) + P \geq k + P = (h - n_1g) + P \geq (h - (n_1 + 1)g) + P \). Hence, \( (h - (n_1 + 1)g) + P = (h - n_1g) + P \) which implies that

\[
h - n_1g - g - (h - n_1g) \in P.
\]

Thus, \(-g \in P\) which is a contradiction. Hence, \( G \) is archimedean. \( \square \)

**Proposition 4.2.3.** Let \( G \) be a cold-filtered \( \ell \)-group and \( S \) be an \( \ell \)-subgroup of \( G \), then \( S \) is cold-filtered.

**Proof.** Let \( \mathcal{F} \) be a filter on \( S^+ \). We want to show that for all \( P \in \text{Spec}(S) \), the filter \( \mathcal{F}_P = \{ s + P : s \in S \} \) has a minimum on \((S/P)^+\), i.e. there exists some \( s' \in \mathcal{F} \) such that for all \( s \in \mathcal{F} \) we have \( s' + P \leq s + P \). Since \( \mathcal{F} \) is a filter on \( S \), so \( \mathcal{F} \) must be a base for some filter on \( G \). Consider, \( \mathcal{F}_S = \{ g \in G : g \geq s \text{ for some } s \in \mathcal{F} \} \). We shall show that \( \mathcal{F}_S \) is a filter on \( G \). Let, \( g_1, g_2 \in \mathcal{F}_S \). Then \( g_1 \geq s_1, g_2 \geq s_2 \), for some \( s_1, s_2 \in \mathcal{F} \). Now, \( g_1 \land g_2 \geq s_1 \land s_2 \) where \( s_1 \land s_2 \in \mathcal{F} \) since \( \mathcal{F} \) is a filter. So, \( g_1 \land g_2 \in \mathcal{F}_S \). Also, let \( x \leq y \) and \( x \in \mathcal{F}_S \). Hence, \( x \geq s \) for some \( s \in \mathcal{F} \). Thus, \( y \geq x \geq s \) implies that \( y \in \mathcal{F}_S \). Hence \( \mathcal{F}_S \) is a filter on \( G \).

Let \( P \in \text{Spec}(S) \), so there exists \( P' \in \text{Spec}(G) \) such that \( P = S \cap P' \) (see [6]). Since \( G \) is a cold-filtered \( \ell \)-group, \((\mathcal{F}_S)_{P'}\) must have a minimum on \((G/P')^+\). Hence, there exists \( g' \in \mathcal{F}_S \) such that for all \( g \in \mathcal{F}_S \), \( g' + P' \leq g + P' \). Thus, there exists \( p' \in P' \) such that \( g' + p' \leq g \). Since \( g' \in \mathcal{F}_S \) there exists \( s' \in \mathcal{F} \) such that \( s' \leq g' \). Thus, \( s' + P' \leq g' + P' \). Now, \( s' \in \mathcal{F} \) implies that \( s' \in \mathcal{F}_S \) since \( \mathcal{F} \subseteq \mathcal{F}_S \). So, without any loss of generality, suppose
that \( g' = s' \). So, \( s' + P' \) is a minimum on \((\mathcal{F}_S)_{P'}\). We claim that \( s' + P \) is a minimum on \( \mathcal{F}_P \). Let, \( s + P \in \mathcal{F}_P \), where \( s \in \mathcal{F} \). Thus \( s \in \mathcal{F} \subseteq \mathcal{F}_S \) which implies that \( s \in \mathcal{F}_S \) which implies that \( s' + P' \leq s + P \). Since \( S/P \) is totally ordered, we have either \( s' + P \leq s + P \) or \( s + P \leq s' + P \). Now, if \( s' + P \leq s + P \), then we are done. If \( s + P \leq s' + P \), then \( s + p \leq s' \), where \( p \in P \subseteq P' \). Thus we have,

\[ s + p' \leq s' + p' \Rightarrow s + P' = s' + P' \Rightarrow s - s' \in P' \cap S = P \Rightarrow s + P = s' + P. \]

Consequently, \( S \) is cold-filtered.

**Proposition 4.2.4.** Let \( G \) be a cold-filtered \( \ell \)-group. Then every \( \ell \)-homomorphic image of \( G \) is also cold-filtered.

**Proof.** Let \( G \) be a cold-filtered \( \ell \)-group and let \( \tau : G \to E \) be a \( \ell \)-homomorphism. We want to show that \( H = \tau(G) \) is cold-filtered. To this end, let \( \mathfrak{F}_P \) be a filter on \( H^+ \) and let \( P \in \text{Spec}(H) \). We want to show that \( \mathfrak{F} = \{ h + P : h \in \mathfrak{F} \} \) has a minimum on \((H/P)^+\).

Consider, \( F' = \tau^{-1}(\mathfrak{F}) = \{ g \in G^+ : \tau(G) \in \mathfrak{F} \} \). It is easy to check that \( \tau^{-1}(\mathfrak{F}) \) is a filter on \( G^+ \). Notice, \( \tau^{-1}(P) \in \text{Spec}(G) \) for all \( P \in \text{Spec}(G) \). Let us denote \( \mathfrak{F}' = \tau^{-1}(\mathfrak{F}) \). Since, \( G \) is cold-filtered, hence \( \mathfrak{F}'_{\tau^{-1}(P)} \) must have a minimum on \((G/\tau^{-1}(P))^+\). So, there exists \( f' \in \mathfrak{F}' \) such that for all \( g \in \mathfrak{F}' \), we have, \( f' + \tau^{-1}(P) \leq g + \tau^{-1}(P) \). We want to show that \( \tau(f') + P \) is the minimum on \((H/P)^+\). Notice, since \( f' \in \tau^{-1}(\mathfrak{F}) \), so it implies that \( \tau(f') \in \mathfrak{F} \). Let, \( h \in \mathfrak{F} \). Also let, \( \tau(x) = h \), where \( x \in \tau^{-1}(h) \subseteq \tau^{-1}(\mathfrak{F}) = \mathfrak{F}' \). We already have \( f' + \tau^{-1}(P) \leq g + \tau^{-1}(P) \), for all \( g \in \mathfrak{F}' \). So, in particular, for \( g = \tau^{-1}(h) \in \mathfrak{F}' \), we have the following:

\[ f' + \tau^{-1}(P) \leq \tau^{-1}(h) + \tau^{-1}(P) \]
\[ \Rightarrow f' + \tau^{-1}(P) \leq x + \tau^{-1}(P) \]
\[ \Rightarrow \text{there exists } p_x \in \tau^{-1}(P) \text{ such that } f' + p_x \leq x \]
\[ \Rightarrow \tau(f' + p_x) \leq \tau(x) = h \]
\[ \Rightarrow \tau(f') + \tau(p_x) \leq h \]
\[\tau(f') + P \leq h + P,\text{ since } p_x \in \tau^{-1}(P). \text{ Thus } \tau(f') \text{ is a minimum on } (H/P)^+.\]

**Proposition 4.2.5.** If \((G_i)_{i \in I}\) are cold-filtered, for each \(i \in I\), then so is \(\bigoplus_{i \in I} G_i\).

**Proof.** Set \(G = \bigoplus_{i \in I} G_i\) and without loss of generality we assume that each \(G_i\) is not trivial. Let \(\mathfrak{F}\) be a filter on \(G^+\). Let \(P \in \text{Min}(G)\). We want to show that \(\mathfrak{F}_P = \{g + P : g \in \mathfrak{F}\}\) has a minimum. Now there is some \(i_P\) such that \(P = \bigoplus_{i = i_P} P_i\) where \(P_i \in \text{Min}(G)\) if \(i = i_P\) and \(P_i = G_i\) otherwise. Furthermore, \(G/P \cong G_i/P_i\), thus if is some \(g \in \mathfrak{F}\) such that \(g(i_P) = 0\) then \(g + P\) will yield a minimum. So suppose that for all \(g \in \mathfrak{F}, g(i_P) \neq 0\). Set

\[\phi(\mathfrak{F}) = \{x \in G_{i_P} : \exists g \in \mathfrak{F} \text{ such that } g(i) = x\}.\]

Then, \(\phi(\mathfrak{F})\) is a filter on \(G_{i_P}^+\) and since \(G_{i_P}\) is cold-filtered, there is some \(x \in \phi(\mathfrak{F})\) such that \(x + P_{i_P}\) is a minimum. Let \(g \in \mathfrak{F}\) satisfy \(g(i_P) = x\). Since \(G/P \cong G_i/P_i\) it follows that \(g + P\) is a minimum for \(\mathfrak{F}\). \(\Box\)

**Proposition 4.2.6.** Cold-filtered \(\ell\)-groups form a torsion class.

**Proof.** Let \(G\) be any \(\ell\)-group. Let \(C = \{g \in G : \text{every value } V \text{ of } g \text{ is a minimal prime subgroup and } G/V \text{ has a least strictly positive element}\}\). Let,

\[D = \{N \in \text{Spec}(G) : N \notin \text{Min}(G) \text{ or } N \in \text{Min}(G) \text{ and } (G/N)^+ \text{ does not have a least element }\}\]

We want to show that \(C = \cap D\).

To this end let \(g \in C\) and \(N \in D\) where \(D\) is the set as in the left hand side. So, \(N \in \text{Spec}(G)\). Now the following cases arise:

Case 1: When \(N \notin \text{Min}(G)\).

Then there exists \(M \in \text{Min}(G)\) such that \(M \subseteq N\). Since \(g \in C\), every value of \(g\) must be a minimal prime. If \(g \notin N\) then \(N\) can be extended to a value \(P\) of \(g\). Since \(P\) has to be a minimal prime so, \(M = N = P\) which contradicts \(N \notin \text{Min}(G)\). Thus \(g \in N\). Therefore \(C \subseteq N\) for all \(N \in D\). So \(C \subseteq \cap N\).
Case 2: If $N \in \text{Min}(G)$.

If $g \notin N$, then $N$ is a value of $G$. Since $N \in C$, then $G/N$ has a least positive element which contradicts that $N \in D$. So, $g \in N$, for all $N \in D$. Hence in both the cases we have, $C \subseteq \bigcap_{N \in D} N$.

Next we want to show that $\bigcap_{N \in D} N \subseteq C$.

To this end let $h \in \bigcap_{N \in D} N$. This implies that $h \in N$, for all $N \in D$. So for any value $P$ of $h$, $h \notin P$. Hence $P \notin D$. So $P \in \text{Min}(G)$ and $(G/P)^+$ has a least element. Thus $P$ is a value of $h$ such that $P \in \text{Min}(G)$ and $(G/P)^+$ has a least element. Therefore, $h \in C$. Hence, $\bigcap_{N \in D} N \subseteq C$. Thus $C = \bigcap_{N \in D} N$.

Claim. $C$ is cold-filtered.

Let $R \in \text{Spec}(C)$. Then $R = C \cap P$ where $P \in \text{Spec}(G)$ and $C$ is not a subset of $P$. This implies $P \in \text{Min}(G)$ and $(G/P)^+$ has a least element. Thus $C \cap P \subseteq P$ and $P \in \text{Min}(G)$. Thus $C \cap P = P$. Hence every prime subgroup of $C$ is minimal.

We want to show that there exists an element $c + R$ which is the minimal element of $(C/R)^+$. By means of contradiction suppose $(C/R)^+$ has no minimum. Thus $R \in \text{Min}(G)$ and $(C/R)^+$ has no minimum. This implies that $R \in C$ which is a contradiction. Thus $C$ is cold-filtered. This establishes our claim.

Claim. For any cold-filtered convex $\ell$-subgroup $A$ of $G$, $A \subseteq C$

Let $A$ be any cold-filtered convex $\ell$-subgroup. We claim that $A \subseteq C$. By means of contradiction suppose it is not true. So, there exists $0 \leq a \in A$ with value $P$ in $G$ such that either $P \notin \text{Min}(G)$ or $(G/P)^+$ has a least positive element and $P \in \text{Min}(G)$.

Case 1: Suppose there exists $0 \leq a \in A$ with value $P$ in $G$ such that $P \notin \text{Min}(G)$. So, there exists $N \in G$ such that $N \subseteq P$. Let $x \in P \setminus N$. Let $N$ be extended to a value $M$ of $x$ such that $M \subseteq P$ (This can be done because the set of prime subgroups form a root system). Notice $M$ is a value of $a \wedge x$. So, assume $x \leq a$ and $A$ is convex. Hence $x \in A$. Thus $N \subseteq M \subseteq P$ which implies that $M \cap A \subseteq P \cap A \subseteq A$. Since $A$ is cold-filtered, so $A$ is hyperarchimedean. Hence $M \cap A = P \cap A$. Notice $x \in P \cap A = M \cap A$ which is a
contradiction since \( x \notin M \cap A \) implies that \( x \notin M \).

Case 2: Suppose there exists \( 0 \leq a \in A \) with value \( P \) in \( G \) such that \( P \in \text{Min}(G) \) and \( (G/P)^+ \) has a least positive element. We shall then contradict that \( A \) is cold-filtered. Consider for any \( 0 \leq b \in A \) the element \( b + (P \cap A) \). Since \( (G/P)^+ \) has no minimum, so there exists \( g \in G^+ \setminus P \) such that \( g + P \leq b + P \). So then, \( g \land b + (P \cap A) \leq b + (P \cap A) \).

If \( g \land b + (P \cap A) = b + (P \cap A) \), then \( b - (g \land b) \in P \cap A \).

Now, \( 0 + P \leq g \land b + P \leq g + P \leq b + P \). This implies \( g \land b + P \neq b + P \). Thus we have \( b - (g \land b) \notin P \) which is a contradiction. Hence \( A \subseteq C \). Thus the union of any cold-filtered, convex \( \ell \)-subgroups is contained in \( C \) and hence is again cold-filtered by Proposition 4.2.3.

Notice, here \( C \) denotes the cold-filtered kernel. Thus the class of cold-filtered \( \ell \)-groups is closed under homomorphic images, convex \( \ell \)-subgroups and under finite products. Also, if \( G \) denotes any cold-filtered \( \ell \)-group, and \( CF(G) \) denotes the cold-filtered kernel then for any \( \{A_\lambda\} \leq CF(G) \) we would have \( \lor\{A_\lambda\} \leq CF(G) \), so \( \lor\{A_\lambda\} \) is cold-filtered. Hence, \( \lor\{A_\lambda\} \) is cold-filtered. Thus cold-filtered \( \ell \)-groups form a torsion class.

We are now in a position to give a characterization of cold-filtered \( o \)-groups. The following proposition gives this characterization.

**Proposition 4.2.7.** Suppose \( G \) is a totally-ordered group. Then the following statements are equivalent.

(i) \( G \) is principally-filtered.

(ii) \( G \) is cold-filtered.

(iii) \( G \) is \( \ell \)-isomorphic to \( \mathbb{Z} \).

**Proof.** A principally-filtered \( \ell \)-group is cold-filtered. The most common example of a principally-filtered \( \ell \)-group is \( \mathbb{Z} \). Thus we are left proving that a cold-filtered totally-ordered group is \( \ell \)-isomorphic to \( \mathbb{Z} \). Since a cold-filtered \( \ell \)-groups is archimedean, we know that such a group is \( \ell \)-isomorphic to a subgroup of \( \mathbb{R} \), by Hölder’s Theorem. Let \( \mathfrak{F} \) be a filter on \( G^+ \), hence
\(F\) is cold filter. Since 0 is the unique minimal prime subgroup of \(G\) it follows that \(F\) has a minimum. Thus every filter on \(G\) has a minimum. Consequently, \(G\) is cyclic.

We now give an explicit characterization of cold-filtered \(\ell\)-groups.

**Theorem 4.2.8.** The \(\ell\)-group \(G\) is cold-filtered if and only if for every minimal prime subgroup \(P \in \text{Min}(G)\), we have \(G/P\) is \(\ell\)-isomorphic to \(\mathbb{Z}\).

**Proof.** Suppose that \(G\) is cold-filtered. Then \(G\) is hyperarchimedean and hence abelian. Now, for each minimal prime subgroup \(P \in \text{Min}(G)\), \(G/P\) is also cold-filtered totally-ordered group. Thus, \(G/P\) is \(\ell\)-isomorphic to \(\mathbb{Z}\), for every \(P \in \text{Min}(G)\).

Conversely, suppose that \(G/P\) is \(\ell\)-isomorphic to \(\mathbb{Z}\) for all \(P \in \text{Min}(G)\). Then in particular, it follows that every minimal prime subgroup is maximal, thus \(G\) is hyperarchimedean. Next, let \(F\) be any filter on \(G^+\) and consider \(F_P\) for \(P \in \text{Min}(G)\). By hypothesis \(G/P\) is \(\ell\)-isomorphic to \(\mathbb{Z}\). Thus \(F_P\) possess a minimum by the Well-Ordering Principle of the Naturals. It follows that \(F\) is a cold-filter. Consequently \(G\) is a cold-filtered \(\ell\)-group.

**Proposition 4.2.9.** Let \(G\) be an \(\ell\)-group. If \(G\) is Specker then \(G\) is cold-filtered but not conversely.

Consider the following example:

**Example 4.2.10.** Take \(G\) as the set of all integer valued sequences that are eventually zero; \(G\) is hyperarchimedean. For each \(n \in \mathbb{N}\), the set

\[M_n = \{g \in G : g(n) = 0\}\]

is a prime subgroup of \(G\) and every prime subgroup of \(G\) is of this form.

Let,

\[U(x) = \begin{cases} 
1 & \text{if } x \text{ is even} \\
2 & \text{if } x \text{ is odd}
\end{cases}\]
Consider $F$ to be the $\ell$-group generated by all eventually zero sequences and $U$. Then $U$ is a strong order unit of $F$.

It is straightforward to show that $G$ is a prime subgroup of $F$. In particular,

$$F/G = \{ nU + G : g \in G, n \in \mathbb{Z} \} \cong \mathbb{Z}.$$ 

Moreover, the prime subgroups of $F$ are given by

$$Spec(F) = \{ M_n : n \in \mathbb{N} \} \cup \{ G \}$$

It follows from Theorem 4.2.8 that $F$ is cold-filtered.

**Claim.** $F$ is not Specker.

It is straightforward to demonstrate the the singular elements of $F$ are precisely the singular elements of $G$, which are characteristic functions defined on finite subsets of $\mathbb{N}$. Since $G$ is Specker it follows that the Specker subgroup of $F$ is $G$. Hence $F$ is not Specker.
CHAPTER 5

Clean Unital ℓ-Groups

5.1 Clean Unital ℓ-Groups

In this section we shall let $G$ be ℓ-group written additively.

Suppose $(G,u)$ is a unital ℓ-group and let $Max(G)$ denote the collection of values of $u$. Since $G$ is unital, $Max(G)$ is the collection of maximal convex ℓ-subgroups of $G$. $Max(G)$ can be equipped with a topology where the closed sets are sets of the form

$$V(H) = \{ M \in Max(G) : H \subseteq M \}$$

for some $H \in C(G)$. This topology is called the hull-kernel topology on $Max(G)$ and it is a compact Hausdorff topology. For an $H \in C(G)$ we let $U(H) = Max(G) \setminus V(H)$. For $g \in G^+$ we let $U(g) = \{ M \in Max(G) : g \notin M \}$.

Lemma 5.1.1. Suppose $(G,u)$ is a unital ℓ-group. The collection of sets $\{ U(g) : g \in G^+ \}$ is a base for the hull-kernel topology on $Max(G)$.

Definition 5.1.2. Suppose $(G,u)$ is a unital ℓ-group. Recall from Definition 1.3.4 that an element $e \in G^+$ is called a component of $u$ if $0 \leq e \leq u$ and $e \wedge (u - e) = 0$. The collection of components of $u$ is denoted by $B(G,u)$ and it forms a boolean algebra.
Lemma 5.1.3. Suppose \((G,u)\) is a unital \(\ell\)-group. For any \(e \in B(G,u)\), the set \(U(e)\) is a clopen subset of \(\text{Max}(G)\).

Proof. In a unital \(\ell\)-group maximal convex \(\ell\)-subgroups have the property that if \(H_1, H_2 \in \mathcal{C}(G)\) and \(H_1 \cap H_2 \subseteq M\), then either \(H_1 \subseteq M\) or \(H_2 \subseteq M\). Therefore since \(e\) is a component of \(u\) it follows that \(G(e) \cap G(u - e) = \{0\}\) and so either \(e \in M\) or \(u - e \in M\). It follows that \(U(e) \cap U(u - e) = \phi\). That \(U(e) \cup U(u - e) = \text{Max}(G)\) is straightforward. Therefore, \(U(e)\) and \(U(u - e)\) are disjoint open sets whose union is the whole space, whence they are complementary clopen subsets of \(\text{Max}(G)\). \(\square\)

Lemma 5.1.4. Suppose \((G,u)\) is a unital \(\ell\)-group and \(K\) is a clopen subset of \(\text{Max}(G)\). Then there is some component \(e \in B(G,u)\) such that \(K = U(e)\).

Proof. Since \(K\) is closed it follows that there is some ideal \(H_1 \in \mathcal{C}(G)\) such that \(K = V(H_1)\). Similarly, \(\text{Max}(G) \setminus K = V(H_2)\) for some \(H_2 \in \mathcal{C}(G)\). Notice that \(\text{Max}(G) = V(H_1) \cup V(H_2) = V(H_1 \cap H_2)\). \(\square\)

Definition 5.1.5. The unital \(\ell\)-group \((G,u)\) is said to be a clean \(\ell\)-group if for every \(g \in G\) there exists an order unit \(v \in G\) and a component \(e\) of \(u\) such that \(g = ve\).

The notion of a clean object first arose in the theory of rings. A ring is called clean if every element is the sum of an unit and an idempotent. To our knowledge this is the first application of the theorem to the theory of lattice-ordered. It is our aim to characterize a clean unital \(\ell\)-group. For rings the idempotents play an integral part, while for unital \(\ell\)-groups the components of a strong order unit play the central role. In general, we say a positive element \(e \in G^+\) is a component of \(G\) if there is another positive element, say \(f \in G^+\), such that \(e \wedge f = 1\) and \(ef\) is a strong order unit. Notice that since disjoint elements commute it follows that both \(fe\) and \(e \cup f\) are also strong order units.

Proposition 5.1.6. The (unital) homomorphic image of a clean \(\ell\)-group is clean.
Proof. Suppose \((G, u)\) and \((H, v)\) are unital \(\ell\)-groups and \(\phi : G \to H\) is a surjective \(\ell\)-homomorphism with \(\phi(u) = v\). Suppose further that \((G, u)\) is a clean \(\ell\)-group. Let \(h \in H\). There there is some \(g \in G\) for which \(\phi(g) = h\). Write \(g = w + e\) where \(e\) is a component of \(u\) and \(w\) is a strong order unit. Then there is some \(n \in \mathbb{N}\) such that \(u \leq nw\). Since \(\phi\) is a \(\ell\)-homomorphism applying \(\phi\) to both sides we have \(v \leq n\phi(w)\). This implies that \(\phi(w)\) is a strong order unit. We claim that a component of \(u\) maps to a component of \(\phi(u) = v\). To see this observe that

1. \(\phi(e) \land v\phi(e)^{-1} = \phi(e) \land \phi(u)\phi(e)^{-1} = \phi(e \land (u - e)) = \phi(1_G) = 1_H\) and
2. \(\phi(e) \lor \phi(ue^{-1}) = \phi(u) = v\).

Therefore, \(h = \phi(g) = \phi(w)\phi(e)\) is a clean expression of \(h\). Consequently, \((H, v)\) is a clean \(\ell\)-group. \(\square\)

Remark. It can be shown that clean unital \(\ell\)-group is in no way related to the strong order unit \(u\). Hence we have the following proposition.

**Proposition 5.1.7.** Suppose \((G, u)\) is a clean \(\ell\)-group. Then for any other strong order unit \(v \in G^+\), \((G, v)\) is also a clean unital \(\ell\)-group.

**Theorem 5.1.8.** Suppose \((G, u)\) is a unital \(\ell\)-group. The following statements are equivalent.

1. \((G, u)\) is a clean unital \(\ell\)-group.
2. The collection \(\{U(e) : e \in B(G)\}\) forms a base for the hull-kernel topology on \(\text{Max}(G)\).
3. \(\text{Max}(G)\) is a boolean space.
4. \((G, v)\) is a clean unital \(\ell\)-group for any order unit \(v \in G^+\).
5.2 Suitable Clean Unital \( \ell \)-Groups

Nicholson in his paper on “Lifting Idempotents and Exchange Rings” (see [8]) studied the conditions under which given any noncommutative ring with identity, idempotents can be lifted modulo every one-sided (left) ideal of the ring and he defined them as “suitable rings”. These rings turned out to be indeed very “suitable” for lifting idempotents. For rings idempotents play an integral part and for unital \( \ell \)-groups, components of a strong order unit play the same role. In this dissertation we looked at the conditions that help us to construct components of a strong order unit in the context of \( \ell \)-groups.

As we have seen before that \( \mathcal{L}(G) \subseteq \mathcal{C}(G) \). As was mentioned before if \( C \in \mathcal{C}(G) \), then \( G/C \) is a lattice but is not always an \( \ell \)-group unless \( C \in \mathcal{L}(G) \). This paves the way for the next definition in this section.

**Definition 5.2.1.** If every convex \( \ell \)-subgroup of \( G \) is also normal in \( G \) i.e. \( \mathcal{C}(G) = \mathcal{L}(G) \), then \( G \) is called *Hamiltonian*.

Notice for any \( H \in \mathcal{C}(G) \) if we try to say the same that \( He \) is a component of \( Hu \) when \( He \land Hu(He)^{-1} = H \), then this coset multiplication will not make any sense unless \( H \triangleleft G \). So throughout this section we shall assume that \( G \) is Hamiltonian.

**Definition 5.2.2.** A positive element \( He \) in \( (G/H)^+ \) is defined to be a component of \( Hu \) in \( G/H \) if \( H \leq He \leq Hu \) and \( He \land (Hu)(He)^{-1} = H \).

**Lemma 5.2.3.** For any Hamiltonian \( \ell \)-group \( (G,u) \), the components of \( G/H \) are precisely given by \( \{Hg : 1 \leq g \leq u \text{ and } g \land ug^{-1} \in H\} \).

Following the footsteps of Nicholson we propose to have the following definitions.

**Definition 5.2.4.** We say components can be lifted modulo a normal convex \( \ell \)-subgroup \( H \) of \( G \) such that for all \( g \in G \), if \( Hg \) is a component of \( Hu \) in \( G/H \), then there exist a component \( e \) of \( u \) in \( G \) i.e. \( e \in B(G,u) \) such that \( He = Hg \).
Definition 5.2.5. Given any unital $\ell$-group $(G, u)$ we say that $G$ is $u$-suitable if and only if for all $H \in \mathfrak{C}(G)$ every component of $Hu$ can be lifted to a component of $u$ in $G$ i.e. for all $g \in G$ there exists a component $e$ of $u$ in $G$ such that $He = Hg$.

Definition 5.2.6. A unital $\ell$-group $(G, u)$ is said to be $u$-clean if for all $g \in G$, there exists a strong order unit $v \in G$ and a component $e$ of $u$ such that $g = ve$.

Proposition 5.2.7. Let $(G, u)$ be a unital $\ell$-group. Then the following are equivalent:

(i) For all $g \in G$ there exists some $e \in B(G, u)$ such that $e \in G(g)$ and $G = G(e) \lor G(ug^{-1})$.

(ii) $G$ is $u$-clean.

(iii) For all $g \in G$ there exists $e \in B(G, u)$ such that $ue^{-1} \in G(ug^{-1})$.

Proof. (i) $\Rightarrow$ (ii) : Let $(G, u)$ be a unital $\ell$-group satisfying the hypothesis. For any $g \in G$ fix, $\bar{g} := g^{-1}u$. Then $\bar{g} \in G$. Hence by the hypothesis, there exists $e \in G(\bar{g})$ such that $G = G(e) \lor G(u\bar{g}^{-1}) = G(e) \lor G(u(g^{-1}u)^{-1}) = G(e) \lor G(ug^{-1}g) = G(e) \lor G(g)$. Notice that $g = e(e^{-1}g)$ where $e \in B(G, u)$. Thus in order to show that $(G, u)$ is $u$-clean it suffices to show that $e^{-1}g$ is a strong order unit in $G$ i.e. it suffices to show that $G(e^{-1}g) = G$. Now by means of contradiction, suppose that $M \in \text{Max}(G)$ be such that $e^{-1}g \in M$. Also, $G(e^{-1}g) \subseteq M$. Now, $e \lor ue^{-1} = 1_G \in M$ but $e \in M$. Since, $M$ is maximal, hence $M \in \text{Spec}(G)$. Since $e \notin M$, it implies that $ue^{-1} \in M$. Thus $g^{-1}u = (g^{-1}e)(e^{-1}u) = g^{-1}e(ue^{-1}) \in M$ since both $g^{-1}e$ and $ue^{-1}$ are in $M$. Consequently, $e \in G(g^{-1}u) \subseteq M$ which is a contradiction. Thus $G$ is $u$-clean.

(ii) $\Rightarrow$ (iii) : Let $(G, u)$ be $u$-clean. Now,

$$G = G(u) = G(ug^{-1}g) = G(ug^{-1}) \lor G(g)$$
it can be shown that there exists $e \in B(G, u)$ such that $G(e) \leq G(g)$ and $G(ue^{-1}) \leq G(ug^{-1})$. Hence $ue^{-1} \in G(ue^{-1}) \leq G(ug^{-1})$. Thus (iii) holds.

(iii) $\Rightarrow$ (i): Suppose that for all $g \in G$ there exists $e \in B(G, u)$ such that $e \in G(g)$ and $ue^{-1} \in G(ug^{-1})$. Now $e \in G(g)$ implies that $G(e) \leq G(g)$. Again $ue^{-1} \in G(ug^{-1})$ implies that $G(ue^{-1}) \leq G(ug^{-1})$. Thus $G(e) \lor G(ue^{-1}) \leq G(e) \lor G(ug^{-1})$ i.e. $G = G(u) = G(e \lor ue^{-1}) \leq G(e) \lor G(ug^{-1})$. Hence (i) holds. \qed
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