HYPERCYCLIC OPERATORS AND THEIR ORBITAL LIMIT POINTS

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ABSTRACT

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Hypercyclicity is the study of linear and continuous operators that possess a dense orbit. Given a separable, infinite dimensional topological vector space $X$, we say a continuous linear operator $T : X \to X$ is hypercyclic if there exists a vector $x$ in $X$ such that its orbit $\text{Orb}(T, x) = \{x, Tx, T^2x, \ldots\}$ is dense in $X$.

Many interesting phenomena appear when analyzing the behavior of iterates of linear and continuous operators, in particular we emphasize the existence of several zero-one laws. We first note that, if an operator $T$ has a hypercyclic vector, it has a dense $G_δ$ set of such vectors, and hence the set of hypercyclic vectors for an operator is either empty or very large in a topological sense. Next, by proving that a somewhere dense orbit is everywhere dense, P. S. Bourdon and N. S. Feldman showed a second zero-one law which states that either an orbit $\text{Orb}(T, x)$ is nowhere dense or it is dense in the whole space.

In my dissertation we uncovered the existence of another such zero-one law for certain classes of operators. We showed that for a weighted backward shift on $\ell^p$ to be hypercyclic it suffices to require the operator to have an orbit $\text{Orb}(T, x)$ with a single non-zero limit point, thus relaxing Bourdon and Feldman’s condition of having a dense orbit in some open subset of $X$. However, our condition does not guarantee that the original orbit $\text{Orb}(T, x)$ is dense in $X$, nonetheless we can demonstrate how to construct a hypercyclic vector for $T$ by using the non-zero limit point of the orbit. Even more interestingly, the condition above can be relaxed to simply requiring that the orbit has infinitely many members in a ball whose closure avoids the zero vector.

To summarize this behavior of weighted backward shifts, we emphasize that a shift $T$ is not hypercyclic if and only if every set of the form $\text{Orb}(T, x) \cup \{0\}$ is closed in $\ell^p$. Thus we showed the existence of a zero-one law for the hypercyclicity of these shifts, which states that either no orbit has a non-zero limit point in $\ell^p$ or some orbit has every vector in $\ell^p$ as a limit point.
Furthermore we showed that this zero-one law for the hypercyclic behavior of shifts is also shared by other classes of operators, in particular the adjoints of the multiplication operators on the Bergman space $A^2(\Omega)$ for an arbitrary region $\Omega \subset \mathbb{C}$. To achieve this we cannot borrow techniques used for the shift operators, but instead we have to take a function theoretical approach.

However, we also showed that this behavior does not generalize to all classes of operators, namely we provided an example of a linear fractional composition operator on the Hardy space $H^2(\mathbb{D})$ that is not hypercyclic, and yet it has an orbit with a non-constant limit point.

To summarize the importance of our results, we would like to point out that in our endeavor to study the phenomena of hypercyclicity it is important to understand how an operator fails to be hypercyclic. Having proved that for certain classes of operators, a non-hypercyclic operator can at most have the zero vector as an orbital limit point, we have shown that these operators fail at having a dense orbit in quite a dramatic way. Thus we described the hypercyclic behavior of certain operators as a zero-one law of orbital limit points, and so we have uncovered another facet of hypercyclicity associated with dichotomous behavior.
To my mother
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# Table of Contents

## CHAPTER 1: A BRIEF HISTORY OF HYPERCYCLICITY

1.1 Hypercyclicity in the context of analysis ........................................ 1
1.2 How to prove an operator is hypercyclic ........................................ 4
1.3 How many vectors are hypercyclic? ................................................. 6
1.4 How many operators are hypercyclic? .............................................. 7
1.5 How big can a non-dense orbit be? .................................................. 8

## CHAPTER 2: HYPERCYCLICITY OF SHIFTS AS A ZERO-ONE LAW OF ORBITAL LIMIT POINTS

2.1 Elementary properties and hypercyclicity of shifts ......................... 10
2.2 A Zero-One Law for the hypercyclicity of shifts ............................. 16
   2.2.1 Introductory remarks ......................................................... 16
   2.2.2 The unilateral weighted backward shift ............................... 17
   2.2.3 The bilateral weighted backward shift ................................ 28

## CHAPTER 3: ORBITAL LIMIT POINTS AND HYPERCYCLICITY OF OPERATORS ON ANALYTIC FUNCTION SPACES

3.1 Introductory remarks ........................................................................ 41
3.2 The adjoints of multiplication operators ....................................... 42
   3.2.1 Elementary properties and hypercyclicity of the adjoints of multiplication operators ................................. 42
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.2.2 A Zero-One Law for the hypercyclicity of the adjoints of multiplication</td>
<td>48</td>
</tr>
<tr>
<td>3.3 The linear fractional composition operators</td>
<td>52</td>
</tr>
<tr>
<td>3.3.1 Elementary properties and hypercyclicity of linear fractional composition operators</td>
<td>52</td>
</tr>
<tr>
<td>3.3.2 Orbital limit points and hypercyclicity of linear fractional composition operators</td>
<td>57</td>
</tr>
<tr>
<td>3.4 Further remarks</td>
<td>59</td>
</tr>
</tbody>
</table>
List of Tables

Table 1. Entries in the first block for the hypercyclic vector construction 25
Table 2. Entries in the second block for the hypercyclic vector construction 26
Table 3. General term entries for the hypercyclic vector construction 27
Table 4. An example of a bilateral weighted shift 40
Table 5. Summary of the hypercyclic behavior of composition operators 56
CHAPTER 1

A BRIEF HISTORY OF HYPERCYCLICITY

1.1 Hypercyclicity in the context of analysis

One of the primary pursuits in the field of analysis is that of approximating objects to any desired degree of accuracy. The notion of universality in analysis refers to the existence of a single object which, via a usually countable process, allows us to approximate a maximal class of objects. In the course of time many such objects that exhibit this extraordinary behavior have been discovered. It seems the first example is due to Fekete [28] in 1914, who showed the existence of a universal power series \( \sum_{j=1}^{\infty} a_j x^j \) on \([-1, 1]\) that not only diverges at every non-zero \(x\), but it does so in the worst possible way. Namely, for every continuous function \(g\) on \([-1, 1]\) with \(g(0) = 0\) there is a sequence \(n_k \to \infty\) so that \(\sum_{j=1}^{n_k} a_j x^j \to g(x)\) uniformly as \(k \to \infty\), i.e. every continuous function vanishing at zero can be uniformly approximated by this universal power series.

Another interesting example of a universal object was provided in 1935 by Marcinkiewicz [26], who showed the existence of a universal primitive. That is, given a sequence \(\{h_n\} \subset \mathbb{R}\) with \(h_n \to 0\), there is a continuous function \(f\) on \([0, 1]\) so that for every measurable function \(g\), the sequence \(\frac{f(x + h_{n_k}) - f(x)}{h_{n_k}} \to g(x)\) almost everywhere on \([0, 1]\), for some sequence
\{n_k\} that depends on \(g\).

However, one of the most remarkable universalities is due to Menshov \cite{27}, who in 1945 proved the existence of a universal trigonometric series whose coefficients converge to zero. Stated in greater generality, Menshov’s result says that given a complete orthonormal system \(\{\psi_n\}\) in \(L^2[0,1]\), there exists a series \(\sum_{j=1}^{\infty} a_j \psi_j\) with \(a_n \in \mathbb{R}\) and \(a_n \to 0\) having the property that for every measurable function \(f\) on \([0,1]\) there exists a sequence \(n_k \nearrow \infty\) so that \(\sum_{j=1}^{n_k} a_j \psi_j(t) \to f(t)\) almost everywhere in \([0,1]\).

Although one would expect universality to be a rare phenomenon in analysis, the above examples show that quite the opposite is the case. In fact, any process in analysis that diverges or behaves irregularly is likely to produce a universal element. Furthermore, when a process exhibits universality, then in most cases almost every element (in a Baire category sense) is universal. Thus universality is a generic phenomenon in analysis.

One particular type of universality studied in the setting of topological vector spaces is the phenomenon of hypercyclicity. In this particular setting, the countable process by which the universal vector \(x\) allows us to approximate all other vectors in the space is the iterated application of an operator \(T\) to the vector \(x\).

**Definition 1.** Given a topological vector space \(X\) and a continuous linear operator \(T : X \to X\), we say the vector \(x \in X\) is a hypercyclic vector for \(T\) if the orbit \(\text{Orb}(T,x) = \{x, Tx, T^2x, \ldots\}\) is dense in \(X\). We call \(T\) a hypercyclic operator and denote the set of hypercyclic vectors for \(T\) by \(\text{HC}(T)\).

Thus hypercyclicity is the study of linear and continuous operators that possess a dense orbit. The name of hypercyclicity was suggested in 1986 by Beauzamy because of its connection to the much older concept in operator theory of a cyclic operator. As the name suggests, hypercyclicity is a much stronger notion of cyclicity, which only requires the operator \(T\) to have an orbit with a dense linear span.

A first example of a hypercyclic operator was provided in 1929 by Birkhoff \cite{6}, who showed the existence of an entire function \(f\) whose successive translates by a non-zero constant \(a\) are
arbitrarily close to any function in the space of entire functions $H(\mathbb{C})$. That is, there exists an entire function $f$ and a non-zero constant $a \in \mathbb{C}$, so that for every $g \in H(\mathbb{C})$, the sequence $f(z + n_k a) \to g(z)$ locally uniformly on $\mathbb{C}$, for some sequence $\{n_k\}$ that depends on the function $g$. Thus, the translation operator $T_a$ has a dense orbit $\{f(z), f(z+a), f(z+2a), \ldots\}$ in $H(\mathbb{C})$.

A second example of a hypercyclic operator was provided in 1952 by MacLane [25], who showed that there exists a universal entire function $f$ whose successive derivatives are dense in $H(\mathbb{C})$. Namely, there is an entire function $f$ having the property that for every $g \in H(\mathbb{C})$ there exists a sequence $n_k \nearrow \infty$ so that $f^{(n_k)}(z) \to g(z)$ locally uniformly in $\mathbb{C}$, and hence the differentiation operator $D$ on $H(\mathbb{C})$ is hypercyclic.

As it turns out, hypercyclicity is not just a mere curiosity but a wide-spread phenomenon in analysis as there are many natural operators that are hypercyclic, and in fact we will see that every separable, infinite dimensional Fréchet space supports a hypercyclic operator.

Hypercyclicity is also closely related to topological dynamics, which studies among other things continuous maps with dense orbits on compact topological spaces. However, hypercyclicity carries an additional linear structure which allows us to bring into play many tools from the field of functional analysis. As a result, the underlying space in hypercyclicity is never compact. Nevertheless, the diversity of ideas and techniques involved in the study of the linear dynamics of an operator have led to a rich body of interesting results.

The importance of the field of hypercyclicity is made evident by its connection to the famous Invariant Subspace Problem in operator theory, which asks whether every continuous linear operator $T$ on $X$ has a non-trivial closed subspace $Y \subset X$ which is $T$-invariant. Although the problem is still open for Hilbert space operators, Enflo [16] offered a negative solution for a specific Banach space, which was later transferred by Read [31] to an operator on $\ell^1(\mathbb{N})$ that has every non-zero vector hypercyclic. Given that a closed $T$-invariant subspace containing $x$ must also contain the closure of the Orb($T,x$), the hypercyclicity of every non-zero vector for the Read operator yields that $T$ has only the trivial closed $T$-invariant
subspaces \{0\} and \ell^1(\mathbb{N})
, and hence the Invariant Subspace Problem has a negative answer for the space \ell^1(\mathbb{N}).

The field of hypercyclicity was born in 1982 with the Ph.D. dissertation of Kitai [24] and later with the paper of Gethner and Shapiro [19] and in more than two decades since, it has become the focus of active research by numerous authors.

1.2 How to prove an operator is hypercyclic

Our first characterization of hypercyclicity is due to Birkhoff [6], and it relates the concept of hypercyclicity to the well known notion of topological transitivity.

**Theorem 2** (Birkhoff Transitivity Theorem). Let X be a separable F-space and T : X → X a continuous linear operator. The following are equivalent:

(i) T is hypercyclic;

(ii) T is topologically transitive: that is, for each pair of non-empty open sets U and V in X there exists n ∈ \mathbb{N} such that T^n(U) ∩ V ≠ ∅.

We remark that the implication (i) ⇒ (ii) holds for an arbitrary topological vector space X, however the converse follows from a direct application of the Baire category theorem, and therefore requires that the space X be a second countable Baire space.

In her dissertation, Kitai [24] further remarked that an invertible operator is topologically transitive if and only if its inverse T^{-1} is topologically transitive. Thus, a continuous linear operator T on an F-space is hypercyclic if and only if T^{-1} is hypercyclic, however T and T^{-1} may not share the same hypercyclic vectors.

While the above result of Birkhoff characterizes hypercyclicity, the following sufficient condition has proved extremely useful in applications. Although many variants of the criterion have been obtained, the Hypercyclicity Criterion is due, independently, to Kitai [24] and Gethner and Shapiro [19]. We note that the version we adopt here is due to Bés and Peris [5].
Theorem 3 (Hypercyclicity Criterion). Let $X$ be a separable $F$-space and $T : X \to X$ a continuous linear operator. If there exists a sequence $n_k \nearrow \infty$, two dense sets $D_1$ and $D_2$ and a sequence of maps $S_{n_k} : D_2 \to X$ such that:

(i) $T^{n_k} x \to 0$ for all $x \in D_1$;

(ii) $S_{n_k} y \to 0$ for all $y \in D_2$;

(iii) $T^{n_k} S_{n_k} y \to y$ for all $y \in D_2$,

then $T$ is hypercyclic and we say $T$ satisfies the Hypercyclicity Criterion.

For a long time, the question whether the above sufficient condition is in fact necessary for the hypercyclicity of an operator, evaded all attempts at being resolved. In a relatively recent paper, Read [15] answered the question in the negative by constructing a remarkable operator $T$ that is hypercyclic but does not satisfy the Hypercyclicity Criterion. Nevertheless, one should not underestimate the importance of the above theorem, as nearly all known hypercyclic operators were shown to have this property by using this criterion.

Finally, we would like to point out a very useful result of Shapiro [34], which shows that in the search for hypercyclic operators on a space $X$ it can be useful to find hypercyclic operators on smaller spaces.

Theorem 4 (Hypercyclicity Comparison Principle). Suppose $X$ and $Y$ are two normed spaces, $Y$ is continuously and densely embedded in $X$, and $T$ is a linear transformation on $X$ that maps the smaller space $Y$ to itself and is continuous in the topology of each space, then $T$ is hypercyclic on the larger space $X$ whenever $T$ is hypercyclic on $Y$. In particular, if $x$ is a hypercyclic vector for $T|_Y$, then $x$ is a hypercyclic vector for $T$.

$$
\begin{array}{c}
Y \\ \downarrow \\
X
\end{array}
\xrightarrow{T|_Y}

\begin{array}{c}
Y \\ \downarrow \\
X
\end{array}
$$
1.3 How many vectors are hypercyclic?

The answer to this question is given by the following remarkable zero-one law for the set of hypercyclic vectors for an operator \( T \), which states that either the set \( HC(T) \) is empty or it is a dense \( G_\delta \) set. In other words, as soon as an operator is hypercyclic, its set of hypercyclic vectors is very large in a topological sense.

**Theorem 5.** [[24]] Suppose \( X \) is a separable F-space and \( T : X \to X \) a continuous linear operator. If \( T \) is hypercyclic, then its set of hypercyclic vectors \( HC(T) \) is a dense \( G_\delta \) set.

The above standard result follows from an application of the Baire category theorem and describing the set of hypercyclic vectors by \( HC(T) = \bigcap_{j=1}^{\infty} \bigcup_{n=0}^{\infty} T^{-n}(U_j) \), where \( \{U_j\}_{j \in \mathbb{N}} \) forms a countable basis for \( X \).

Maybe the most remarkable result in the field of hypercyclicity is that of Read [[15]], who showed that this zero-one law for the set of hypercyclic vectors can be taken to the extreme. Read was able to construct an operator for which every non-zero vector is hypercyclic, and thus for this operator the set of hypercyclic vectors is not just large in a topological sense but it is maximal.

Furthermore, the preceding result about the topological largeness of \( HC(T) \) also implies that the set of hypercyclic vectors is quite big in an algebraic sense. Since both sets \( HC(T) \) and \( x - HC(T) \) are dense \( G_\delta \) in \( X \), the Baire category theorem yields that their intersection is non-empty. Thus every vector in \( X \) can be written as the sum of two hypercyclic vectors.

**Theorem 6.** [[22]] If \( X \) is a separable F-space and \( T : X \to X \) is a hypercyclic operator, then \( X = HC(T) + HC(T) \).

The above result shows that the set of hypercyclic vectors is not closed under addition, nonetheless it always contains a linear subspace consisting entirely, except the zero vector, of hypercyclic vectors. We call such a subspace a hypercyclic manifold for \( T \).

**Theorem 7.** [[7]] Let \( X \) be a topological vector space and \( T \) a hypercyclic operator. If \( x \in HC(T) \), then \( \{p(T)x : p \text{ polynomial}\} \) is a hypercyclic manifold for \( T \). In particular, \( T \) admits a dense hypercyclic manifold.
1.4 How many operators are hypercyclic?

Perhaps the reason one does not expect hypercyclicity to be a wide-spread phenomenon lies in the simple realization that finite-dimensional spaces do not support hypercyclic operators.

**Theorem 8.** [27] There are no hypercyclic operators on finite dimensional F-spaces.

Nonetheless, Ansari [1] was able to make use of the aforementioned Hypercyclicity Comparison Principle to show the remarkable fact that an infinite dimensional space with nice enough structure always carries a hypercyclic operator.

**Theorem 9.** [1] Every separable, infinite dimensional Fréchet space carries a hypercyclic operator.

Suppose now that a space admits hypercyclic operators. Motivated by the fact that the existence of a single hypercyclic vector implies the existence of a dense set of such vectors, one might ask if an analogous statement is true for operators. However, the set of all hypercyclic operators is not dense in the Banach algebra of linear and continuous operators $\mathcal{B}(X)$ endowed with the norm topology, as every hypercyclic operator has norm greater than one, and in fact $(\mathcal{B}(X), \|\cdot\|)$ is not even separable. The following theorem shows that in the case of Hilbert spaces an even stronger statement holds.

**Theorem 10.** [32] Let $H$ be an infinite dimensional Hilbert space. Then the set of all cyclic operators is nowhere dense in $\mathcal{B}(H)$ with respect to the norm topology.

However, if we endow $\mathcal{B}(X)$ with the strong operator topology, Chan [10] showed that the set of hypercyclic operators on $X$ is dense in $\mathcal{B}(X)$. Recall that the strong operator topology (SOT) is the weakest topology on $\mathcal{B}(X)$ for which the evaluation maps $T \mapsto T(x)$, $x \in X$ are continuous.

**Theorem 11.** [4] Given a separable Fréchet space $X$, if $T : X \to X$ is hypercyclic, then there exists an SOT-dense linear subspace of $\mathcal{B}(X)$ which consists entirely, except the zero operator, of hypercyclic operators.
Finally, continuing the analogy to hypercyclic vectors (see Theorem \[6\]), Grivaux \[21\] showed the following surprising result.

**Theorem 12.** (i) Every operator on a separable, infinite dimensional Hilbert space is the sum of two hypercyclic operators.

(ii) There exists a separable, infinite dimensional Banach space on which not every operator is the sum of two hypercyclic operators.

### 1.5 How big can a non-dense orbit be?

To better understand the phenomenon of hypercyclicity, we also need to address the question about how big the orbits of an operator $T$ can be without ever becoming dense. We remark that the results we obtain in Chapters 2 and 3 provide some surprising insight into this question. Furthermore, our work is a continuation of a result of Bourdon and Feldman \[8\], who showed the following remarkable zero-one law for a topological vector space: either an orbit of an operator $T$ is nowhere dense, or it is everywhere dense, in which case $T$ is hypercyclic.

**Theorem 13.** \[8\] Suppose $T$ is a bounded linear operator on a topological vector space $X$. If $T$ has a somewhere dense orbit $\text{Orb}(T, x)$, then this orbit is dense in $X$, and $T$ is hypercyclic.

In fact we were able to show that for certain classes of operators the statement in Theorem \[13\] can be further relaxed. We show that if an operator $T$ is not hypercyclic, then no orbit of $T$ can have a non-zero limit point; that is if an orbit of $T$ has a limit point, then it must be the zero vector. Thus we uncover the existence of another zero-one law for the hypercyclicity of certain classes of operators, which states that either no orbit has a non-zero limit point or some orbit has every vector in the space as a limit point.

Finally, the question above has also been studied in a different direction by Chan and Sanders \[11\], who showed the existence of a weakly dense orbit that is not norm dense. Here
the weak sense of density is with respect to the weak topology on the topological vector space $X$.

**Theorem 14.** [11] There exists a bounded linear operator $T$ on a Banach space $X$ that is weakly hypercyclic but not norm hypercyclic.
CHAPTER 2

HYPERCYCLICITY OF SHIFTS AS A ZERO-ONE LAW OF ORBITAL LIMIT POINTS

2.1 Elementary properties and hypercyclicity of shifts

The class of weighted shift operators constitutes a favorite testing ground in the literature of operator theory. It is therefore natural to begin our analysis of the phenomenon of hypercyclicity as a zero-one law of orbital limit points by showing that this law does indeed hold for weighted shifts. In this section we define the class of weighted shifts, establish some of the basic properties of the shift operators, and offer a brief summary of the previously established results about the hypercyclicity of shifts. For a classical survey on weighted shifts see Shields [35].

We define the space $\ell^p(I)$ for $1 \leq p < \infty$ to be the space of all sequences $\{\hat{x}(j)\}_{j \in I}$ with $\|x\| = \left(\sum_{j \in I} |\hat{x}(j)|^p\right)^{\frac{1}{p}} < \infty$ for $I = \mathbb{N} \cup \{0\}$ or $I = \mathbb{Z}$. For notational simplicity we will use $\mathbb{Z}_+$ to denote $\mathbb{N} \cup \{0\}$. We note that $\ell^p(I)$ is a separable, infinite dimensional Banach space that has the canonical basis $\{e_n\}_{n \in I}$, where each $e_n = \{\hat{e}_n(j)\}_{n \in I}$ with $\hat{e}_n(j) = 0$ if $j \neq n$ and $\hat{e}_n(j) = 1$ if $j = n$. 
A linear operator $T : \ell^p(\mathbb{Z}_+) \to \ell^p(\mathbb{Z}_+)$ is said to be a unilateral weighted backward shift if there is a sequence of complex weights $\{w_n\}_{n \geq 1}$ such that $Te_n = w_ne_{n-1}$, if $n \geq 1$ and $Te_0 = 0$. Similarly, for the canonical basis $\{e_n : n \in \mathbb{Z}\}$, a linear operator $T : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ is said to be a bilateral weighted backward shift if there is a sequence of complex weights $\{w_n\}_{n \in \mathbb{Z}}$ such that $Te_n = w_ne_{n-1}$ for all $n \in \mathbb{Z}$.

Weighted forward shifts are also defined on the space $\ell^p(I)$ in the obvious way. In what follows we use $B$ to denote the unweighted backward shift and $S$ to denote the unweighted forward shift. The following proposition [35] relates the forward to the backward shift for the Hilbert space $\ell^2(I)$.

Proposition 15. If $T$ is a bilateral forward weighted shift with weight sequence $\{w_n\}_{n \in \mathbb{Z}}$, then its adjoint $T^*$ is given by $T^*e_n = \overline{w_{n-1}}e_{n-1}$, for all $n \in \mathbb{Z}$. If $T$ is a unilateral forward weighted shift with weight sequence $\{w_n\}_{n \in \mathbb{Z}_+}$, then its adjoint $T^*$ is given by $T^*e_n = \overline{w_{n-1}}e_{n-1}$ for all $n \geq 1$, and $T^*e_0 = 0$.

Proof. Note that in the bilateral case, $\langle Te_j, e_i \rangle = \langle w_je_{j+1}, e_i \rangle = w_j \langle e_{j+1}, e_i \rangle$ and $\langle e_j, \overline{w_{i-1}}e_{i-1} \rangle = w_{i-1} \langle e_j, e_{i-1} \rangle$. Both expressions are non-zero if $j + 1 = i$, in which case they equal $w_j$. For $x, y \in \ell^p(\mathbb{Z})$ with $x = \sum_{j \in \mathbb{Z}} \hat{x}(j)e_j$ and $y = \sum_{j \in \mathbb{Z}} \hat{y}(j)e_j$, we have $\langle Tx, y \rangle = \sum_{i,j \in \mathbb{Z}} \hat{x}(j)\overline{\hat{y}(i)}\langle w_je_{j+1}, e_i \rangle = \sum_{j \in \mathbb{Z}} \hat{x}(j)\overline{\hat{y}(j+1)}w_j$. On the other hand, $\sum_{i,j \in \mathbb{Z}} \hat{x}(j)\overline{\hat{y}(i)}\langle e_j, \overline{w_{i-1}}e_{i-1} \rangle = \sum_{j \in \mathbb{Z}} \hat{x}(j)\overline{\hat{y}(j+1)}w_j$. So for $T^*$ defined as in the proposition, $\langle Tx, y \rangle = \langle x, T^*y \rangle$, for all $x, y \in \ell^p(\mathbb{Z})$.

A similar argument can be made in the unilateral case.

We remark that given the weighted forward shift $T$ with weights $\{w_n\}_{n \in I}$, the adjoint operator $T^*$ is not the weighted backward shift with weight sequence $\{w_n\}_{n \in I}$. Nevertheless, the forward shift with weight sequence $\{\overline{w_{n+1}}\}_{n \in I}$ has as its adjoint the weighted backward shift with the desired weights $\{w_n\}_{n \in I}$.

In the following proposition [35] we establish that a weighted backward shift $T$ is bounded if and only if its sequence of weights $\{w_n\}_{n \in I}$ is bounded, in particular $\|T\| = \sup\{|w_n| : n \in I\}$.
Proposition 16. $\|T^n\| = \sup_{k \in I} |w_{k-n+1} \cdot \cdot \cdot w_k|$ for all $n \geq 1$.

Proof. Let $n \geq 1$. Since $T^n e_k = (w_{k-n+1} \cdot \cdot \cdot w_k) e_{k-n}$, we have that $|w_{k-n+1} \cdot \cdot \cdot w_k| = \|T^n e_k\| \leq \|T^n\| \|e_k\| = \|T^n\|$. Thus taking supremum over all $k \in I$, we obtain that the sup

$$\sup_{k \in I} |w_{k-n+1} \cdot \cdot \cdot w_k| \leq \|T^n\|.$$

On the other hand,

$$\|T^n\| = \inf\{c > 0 : \|T^nx\| \leq c \|x\|, x \in \ell^p(I)\}$$

$$\leq \inf\{c > 0 : \|T^ne_k\| \leq c \|e_k\|, k \in I\}$$

$$= \inf\{c > 0 : |w_{k-n+1} \cdot \cdot \cdot w_k| \leq c, k \in I\}$$

$$= \sup_{k \in I} |w_{k-n+1} \cdot \cdot \cdot w_k|.$$

As a consequence of the above result, we will assume that the weight sequence $\{w_n\}_{n \in I}$ for the weighted shift $T$ is always bounded above. In the following we will see that a lower bound on the weights yields another important property for the shift operator $T$.

We first observe that a unilateral weighted backward shift is never injective, as its kernel is the linear span of the vector $e_0$. On the other hand, a bilateral weighted shift $T$ is injective if and only if none of its weights is zero. We remark that if one of its weights is zero, say $w_0 = 0$, we can write $T = T_1 \oplus T_2$, where $T_1$ is a unilateral weighted shift on the closed subspace spanned by $\{e_k : k \geq 0\}$ and $T_2$ is a weighted shift on the orthogonal complement. We will assume from now on that none of the weights is zero.

Moreover, the vector $y \in \ell^p(I)$ is in the image of $T$ if and only if $\sum_{j \in I} \left| \frac{\hat{y}(j)}{w_{j+1}} \right|^p < \infty$, in which case the vector $x$ given by $\hat{x}(j) = \frac{\hat{y}(j)}{w_j}$ for all $j \in I$ is the unique vector $T$ sends to $y$. Thus a weighted shift $T$ is surjective if and only if its weight sequence is bounded away from zero.

Therefore, a bilateral weighted backward shift $T$ is invertible if and only if there exists $m > 0$ so that $|w_j| \geq m$ for all $j \in I$. In fact we will show in the following that in the context of hypercyclicity we can assume without loss of generality that the weight sequence
is always positive, i.e. \( w_j > 0 \) for all \( j \in I \), and thus the absolute value in the statement above can be dropped.

We say two bounded linear operators \( A \) and \( B \) on a Banach space \( X \) are isometrically isomorphic, if there is a linear homeomorphism \( U : X \to X \) with \( \|U\| = 1 \) such that \( UAU^{-1} = B \).

**Proposition 17.** \([35]\) If \( \{\lambda_n\} \) is a sequence of complex numbers with \( |\lambda_n| = 1 \), then a weighted backward shift \( T \) with weight sequence \( \{w_n\} \) is isometrically isomorphic to the backward weighted shift operator with weight sequence \( \{\lambda_{n-1}\lambda_nw_n\} \).

**Proof.** Without loss of generality assume that \( T \) is a bilateral weighted backward shift, and let \( U : \ell^p(Z) \to \ell^p(Z) \) be defined by setting \( Ue_n = \lambda_ne_n \) for all \( n \in Z \), and extending by linearity and continuity. Since \( |\lambda_n| = 1 \) for all \( n \in Z \), given \( g \in \ell^p(Z) \) with \( g = \sum_{n \in I} \hat{g}(n)e_n \) we can define a vector \( h \) in \( \ell^p(Z) \) by setting \( h = \sum_{n \in I} \frac{\hat{g}(n)}{\lambda_n}e_n \), and so \( Uh = g \). Thus \( U \) is surjective.

Also, for any \( h \in \ell^p(Z) \) we have \( \|Uh\| = \left\| \sum_{n \in I} \lambda_n\hat{h}(n)e_n \right\| = \left( \sum_{n \in I} |\hat{h}(n)|^p \right)^{\frac{1}{p}} = \|h\| \), so \( U \) is injective and has norm \( \|U\| = 1 \). Thus \( U \) is an isometric isomorphism.

Finally, define \( \tilde{T} \) on \( \ell^p(Z) \) by setting \( \tilde{T} = UTU^{-1} \). Then \( \tilde{T} \) is isometrically isomorphic to \( T \). Furthermore, for all \( n \in Z \), we have \( U^{-1}e_n = \frac{1}{\lambda_n}e_n = \overline{\lambda}_ne_n \). So \( \tilde{T}e_n = U(\lambda_{n-1}\overline{\lambda}_nw_n)e_{n-1} = (\lambda_{n-1}\overline{\lambda}_nw_n)e_{n-1} \) for all \( n \in Z \). Note that \( \tilde{T} \) is linear and continuous. Thus \( \tilde{T} \) is a weighted backward shift with weights \( \{\lambda_{n-1}\overline{\lambda}_nw_n\} \).

From the above proposition we can conclude that a weighted backward shift \( T \) is isometrically isomorphic to a weighted backward shift operator with positive weights.

**Corollary 18.** \([35]\) A weighted backward shift operator \( T \) with weight sequence \( \{w_n\} \) is isometrically isomorphic to the weighted backward shift operator with weight sequence \( \{|w_n|\} \).
Proof. Let $\lambda_0 = 1$. Set $\lambda_1 = \frac{|w_1|}{w_1}$ and $\lambda_{-1} = \frac{|w_{-1}|}{w_{-1}}$ so that $|w_1| = \lambda_0 \lambda_1 w_1$ and $|w_0| = \lambda_{-1} \lambda_0 w_0$. Next, we set $\lambda_2 = \frac{1}{\lambda_1} \frac{|w_2|}{w_2}$ and $\lambda_{-2} = \frac{1}{\lambda_{-1}} \frac{|w_{-2}|}{w_{-2}}$, etc. Clearly $|\lambda_n| = 1$ and $\lambda_n \lambda_{n-1} |w_n| = |w_n|$ for all $n \in I$, and hence $T$ is isometrically isomorphic to the weighted backward shift $\tilde{T}$ with weight sequence $\{|w_n|\}$.

Finally, the following standard proposition shows that in the context of hypercyclicity we can always assume that the weight sequence for the shift operator $T$ is positive.

**Proposition 19.** Hypercyclicity is preserved under isomorphism; that is, if a hypercyclic operator $T$ is isomorphic to $\tilde{T}$, then $\tilde{T}$ is also hypercyclic.

**Proof.** Suppose there exists an isomorphism $U : X \to X$ so that $U^{-1}TU = \tilde{T}$, and suppose further that there exists a vector $x \in X$ with $\text{Orb}(T, x) = X$. Since $U$ is surjective, let $\tilde{x} \in X$ so that $U \tilde{x} = x$, and let $y$ in $X$. We want to show that there exists a sequence $m_k \not\to \infty$ so that $\tilde{T}^{m_k} \tilde{x} \to y$ in $X$. Note that since $Uy \in X$ and $T$ is hypercyclic, there exists a sequence $n_k \not\to \infty$ so that $T^{n_k} x \to Uy$ in $X$. Since $x = U \tilde{x}$, we get that $T^{n_k} (U \tilde{x}) \to Uy$ in $X$. Finally, the continuity of $U^{-1}$ yields that $\tilde{T}^{n_k} \tilde{x} = U^{-1} T^{n_k} U \tilde{x} \to y$ in $X$, so $\text{Orb}(\tilde{T}, \tilde{x}) = X$.

Given our previous considerations about the weight sequence of a shift, we are now ready to give the definition of unilateral and bilateral shifts that we will use in the sequel.

**Definition 20.** Given the canonical basis $\{e_n : n \in \mathbb{Z}\}$ for $\ell^p(\mathbb{Z}_+)$ for $1 \leq p < \infty$, a bounded linear operator $T : \ell^p(\mathbb{Z}_+) \to \ell^p(\mathbb{Z}_+)$ is said to be a unilateral weighted backward shift if there is a bounded sequence of positive weights $\{w_n\}_{n \geq 1}$ such that $Te_n = w_n e_{n-1},$ if $n \geq 1$ and $Te_0 = 0$.

Similarly, a bounded and linear operator $T : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ is said to be a bilateral weighted backward shift if there is a bounded sequence of positive weights $\{w_n : n \in \mathbb{Z}\}$ such that $Te_n = w_n e_{n-1}$ for all $n \in \mathbb{Z}$.
Having defined the weighted shift operators and established some of their elementary properties, we now turn our attention to the hypercyclic behavior of these shifts. We choose the backward direction for a shift, since the unilateral forward shift is never hypercyclic.

We first note that relatively early on, Salas [33] characterized the hypercyclicity of weighted shifts by providing a necessary and sufficient condition in terms of the weights.

**Theorem 21.** A bilateral weighted backward shift $T$ is hypercyclic if and only if for every $\epsilon > 0$ and every $q \in \mathbb{N}$ there exists $n$ arbitrarily large such that for every $j \in \mathbb{Z}$ with $|j| \leq q$ we have $\prod_{s=0}^{n-1} w_{j+s} > \frac{1}{\epsilon}$ and $\prod_{s=1}^{n} w_{j-s} < \epsilon$.

As an easy consequence, one deduces the following analogous statement for the unilateral weighted backward shift.

**Corollary 22.** [33] A unilateral weighted backward shift $T$ is hypercyclic if and only if there exists a sequence $n_k \nearrow \infty$ so that $\prod_{j=1}^{n_k} w_j \to \infty$.

We note that the above condition for unilateral shifts can be used to characterize the hypercyclic behavior of a shift with greater ease than its bilateral counterpart. Nonetheless, Feldman [17] showed that an analogous condition can be obtained for a bilateral weighted shift $T$, provided we assume the operator $T$ is invertible.

**Proposition 23.** A bilateral weighted backward shift $T : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ with weight sequence $\{w_n\}_{n \in \mathbb{Z}}$ where $w_n \geq m$ for some $m > 0$ is hypercyclic if and only if there exists a sequence $n_k \nearrow \infty$ so that $\prod_{j=1}^{n_k} w_j \to \infty$ and $\prod_{j=1}^{n_k} w_{-j} \to 0$.

We note that in fact Feldman [17] showed that the above proposition still holds if we only assume that the negative indexed weights are bounded below, i.e. there exists $m > 0$ so that $w_n \geq m$ for all $n < 0$ (or alternatively all $n > 0$).

Finally, Chan and Sanders [11] showed that the condition in Corollary 22 is also equivalent to another weaker notion of hypercyclicity, which is appropriately called weak hypercyclicity. We say an operator $T$ on a separable, infinite dimensional Banach space $X$ is *weakly*...
**h\textit{ypercyclic}** if there is a vector $x$ in $X$ whose orbit $Orb(T, x)$ is dense in $X$ with respect to the weak topology. Since the norm topology is strictly stronger than the weak topology, it follows that every (norm-)hypercyclic operator is also weakly hypercyclic. In [11] Chan and Sanders showed that for a unilateral weighted backward shift the converse also holds true.

\textbf{Theorem 24.} Let $T : \ell^p(\mathbb{Z}^+) \rightarrow \ell^p(\mathbb{Z}^+)$ be a unilateral weighted backward shift. Then $T$ is (norm-)hypercyclic if and only if $T$ is weakly hypercyclic.

However, Chan and Sanders [11] also showed that there exists a bilateral weighted backward shift $T$ on $\ell^p(\mathbb{Z})$ that is weakly hypercyclic but not (norm-)hypercyclic.

### 2.2 A Zero-One Law for the hypercyclicity of shifts

#### 2.2.1 Introductory remarks

As noted in the previous section, the class of weighted backward shifts has been well studied in the field of hypercyclicity, in fact relatively early on Salas [33] characterized the hypercyclic shifts by offering a necessary and sufficient condition in terms of their weight sequences. In the following we relate the concept of hypercyclicity to the geometry of an orbit, thus obtaining a new equivalent condition for the hypercyclicity of these shifts.

We would like to point out that our results are also a continuation of the work of Bourdon and Feldman [8]. The authors proved that if an operator $T$ has a somewhere dense orbit $Orb(T, x)$, then the same orbit will be everywhere dense in $X$, and thus the operator $T$ is hypercyclic. As it turns out, in the case of weighted backward shifts their remarkable insight can be carried even further. Indeed, in the next sections we show that for a shift to be hypercyclic it suffices to require the operator to have an orbit $Orb(T, x)$ with a single non-zero limit point, thus relaxing Bourdon and Feldman’s condition of having a dense orbit in some open subset of $X$. However, our condition does not guarantee that the original orbit $Orb(T, x)$ is dense in $X$, nonetheless we can demonstrate how to construct a hypercyclic
vector for $T$ using the non-zero limit point of the orbit. Even more interestingly, the condition above can be relaxed to simply requiring that the orbit has infinitely many members in a ball whose closure avoids the zero vector.

To summarize this behavior of weighted backward shifts we emphasize that a shift $T$ is not hypercyclic if and only if every set of the form $\text{Orb}(T, x) \cup \{0\}$ is closed in $X$. Thus we uncover the existence of a zero-one law for the hypercyclicity of these shifts, which states that either no orbit has a non-zero limit point in $X$ or some orbit has every vector in $X$ as a limit point.

In Section 2.2.2, we provide several conditions that characterize the hypercyclicity of a unilateral weighted backward shift. We then offer a technique for constructing a hypercyclic vector for a unilateral weighted backward shift $T$ having $e_0$ as a limit point of one of its orbits $\text{Orb}(T, x)$. In Section 2.2.3, we give a proof for the bilateral analogue of the above results which calls for different techniques.

### 2.2.2 The unilateral weighted backward shift

Let $\{e_n : n \geq 0\}$ be the canonical base for $\ell^p(\mathbb{Z}^+)$ for $p \geq 1$, denoted by $\ell^p$ in the following. A vector $x$ in $\ell^p$ is denoted by $x = (\hat{x}(0), \hat{x}(1), \ldots) = \sum_{i=0}^{\infty} \hat{x}(i)e_i$, where $\sum_{i=0}^{\infty} |\hat{x}(i)|^p < \infty$. A bounded and linear operator $T : \ell^p \to \ell^p$ is said to be a unilateral weighted backward shift if there is a sequence of bounded positive weights $\{w_n\}_{n \geq 1}$ such that $Te_n = w_ne_{n-1}$, if $n \geq 1$ and $Te_0 = 0$.

We recall that for a unilateral weighted backward shift $T$ to be hypercyclic, Salas [33] provided a necessary and sufficient condition on the weights that $\sup_{n \geq 1} \prod_{j=1}^{n} w_j = \infty$. Another characterization was obtained by Chan and Sanders [11], who showed that $T$ is hypercyclic if and only if $T$ is weakly hypercyclic, which means that $T$ has an orbit $\text{Orb}(T, x)$ that is dense in the weak topology of $\ell^p$. In the following, we show that the above equivalent conditions can be carried forward to other conditions in terms of the geometry of an orbit.
Theorem 25. Let $T : \ell^p \to \ell^p$ be a unilateral weighted backward shift, where $1 \leq p < \infty$.

The following are equivalent:

(i) $T$ is hypercyclic.

(ii) For any vector $f$ in $\ell^p$, there exists a vector $x = x(f)$ in $\ell^p$ whose orbit under $T$ 

$\text{Orb}(T, x) = \{x, Tx, T^2x, \ldots\}$ has $f$ as a limit point.

(iii) The vector $e_0$ is a limit point of a certain orbit $\text{Orb}(T, x)$ with $x$ in $\ell^p$.

(iv) There exists an $x$ in $\ell^p$ whose orbit $\text{Orb}(T, x)$ has a non-zero limit point.

(v) There exists an $x$ in $\ell^p$ whose orbit $\text{Orb}(T, x)$ has a non-zero weak limit point.

(vi) There exists a vector $x$ in $\ell^p$ whose orbit $\text{Orb}(T, x)$ has infinitely many members in an open ball whose closure avoids the origin; that is, there are a non-zero vector $f$ in $\ell^p$ and a positive $r$ with $r < \|f\|$ such that $\text{Orb}(T, x) \cap B(f, r)$ is infinite.

Before we provide a proof for the theorem, we have a few remarks to illustrate the result.

1. For statement (iv), the limit point $f$ cannot be chosen to be the zero vector. Take $w_j = \frac{1}{2}$ for all $j \geq 1$. Let $x = (x_0, x_1, x_2, \ldots)$ be a vector in $\ell^p$ with infinitely many non-zero entries $x_i$. Then $T^nx \to 0$ as $n \to \infty$, so the zero vector is a limit point of $\text{Orb}(T, x)$, but $T$ is clearly not hypercyclic.

2. Regarding the equivalence of statements (i) and (iv), we remark that if an orbit $\text{Orb}(T, x)$ has a non-zero limit point, the vector $x$ that generates the orbit is not necessarily a hypercyclic vector. Take, for instance, the unilateral weighted backward shift $T : \ell^1 \to \ell^1$ whose weight sequence is given by $w_j = 2$ for all $j \geq 1$, and the vector $x = (x_0, x_1, x_2, \ldots)$ given by $x_j = 2^{-j}$ if $j = 2^k$ and $x_j = 0$ otherwise. Clearly $x \in \ell^1$ and $\|T^{2^k}x - e_0\| = \sum_{j=k+1}^{\infty} 2^{2^k} 2^{-2j}$, which goes to 0 as a limit, as $k \to \infty$. However, $\hat{T}^nx(0) = 0$ if $n \neq 2^k$, and $\hat{T}^nx(0) = 1$ if $n = 2^k$. Hence $x$ is not a hypercyclic vector. In fact, for a similar reason $x$ is not even a supercyclic vector because the scalar multiples of $\text{Orb}(T, x)$ cannot approximate $e_0 + e_1$. For a definition, we say an operator $T : X \to X$ is supercyclic if there is a vector $x \in X$ such that the set $\{\lambda T^nx : n \geq 0, \lambda \in \mathbb{C}\}$ is dense in $X$. 

3. Statement (vi) cannot be relaxed to consider weakly open sets. That is, a unilateral weighted backward shift $T$ on $\ell^p$ may not be hypercyclic even if $T$ has an orbit $\text{Orb}(T, x)$ with infinitely many members inside a weakly open set whose weak closure does not contain zero.

Let $g = \sum_{j=1}^{\infty} \frac{1}{2^j} e_{3 \cdot 4^{j+1}}$. Clearly $\|g\|^2 = \sum_{j=1}^{\infty} \left( \frac{1}{2^j} \right)^2 = \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}$. Consider the weakly open set $U = \{ f \in \ell^2 : |\langle f - g, g \rangle| < \frac{1}{10} \}$. Then if $f \in U$, we have that $|\langle f, g \rangle| < \frac{1}{10}$, and thus $\frac{7}{30} < |\langle f, g \rangle| < \frac{13}{30}$. Let $V = \{ f \in \ell^2 : |\langle f, g \rangle| < \frac{1}{30} \}$. Then if $f \in V$, $f \notin U$, and hence $U \cap V = \emptyset$. But $0 \in V$, so $U$ is a weakly open set whose weak closure avoids the origin.

We now proceed to define a bounded positive weight sequence for $T$ as follows. For any positive integer $j$ in an interval of the form $[1 + 2 \cdot 4^k, 4^{k+1}]$, where $k \geq 1$, we define

$$w_{1+2 \cdot 4^k} = \ldots = w_{3 \cdot 4^k} = \left[ \frac{1}{k \cdot 2^{k-1}} \right] \frac{1}{4^k},$$

$$w_{1+3 \cdot 4^k} = \ldots = w_{4k+1} = \left[ k \cdot 2^{k-1} \right] \frac{1}{4^k}.$$  

For those positive integers $j$ outside the intervals of the form $[1 + 2 \cdot 4^k, 4^{k+1}]$, for $k \geq 1$, we simply take $w_j = 1$.

Note that $\sup \left\{ x^{\frac{1}{x^2}} : x \geq 1 \right\} < \infty$, and so $\{w_j\}_{j \geq 1}$ is a bounded sequence. Hence the unilateral weighted backward shift $T$ with the weight sequence $\{w_j\}_{j \geq 1}$ is a bounded linear operator. Furthermore, from Salas’ criterion for hypercyclicity for unilateral backward shifts (see [33]) we immediately see that $T$ is not hypercyclic since $w_1 \ldots w_j < 1$ for all integers $j \geq 1$.

Now, let $x = \sum_{j=1}^{\infty} \frac{1}{3j} e_{4^{j+1}}$. Clearly, $\|x\|^2 = \sum_{j=1}^{\infty} \left( \frac{1}{3j} \right)^2 = \frac{1}{9} \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$.

Furthermore, for any integer $k \geq 2$,

$$T^{4k} x = \frac{1}{3(k-1)} e_0 + \frac{2k-1}{3} e_{3 \cdot 4^k} + \sum_{j=k+1}^{\infty} \frac{1}{3j} w_{4^j+1-4^{j+1}} \ldots w_{4^{j+1}} e_{4^{j+1}-4^k}.$$  

Since $4^{j+1}-4^k > 3 \cdot 4^{j+1}$ whenever $j \geq k+1$, the above summation is obviously orthogonal to $g$. Hence, $|\langle T^{4k} x, g \rangle - \|g\|^2| = \left| \left\langle \frac{2k-1}{3} e_{3 \cdot 4^k}, g \right\rangle - \frac{1}{3} \right| = \left| \frac{1}{3} - \frac{1}{3} \right| = 0$ for all $k \geq 2$. So, $T^{4k} x \in U$ for all $k \geq 2$, however $T$ is not hypercyclic.
We are now ready to prove Theorem 25.

**Proof.** It is clear that (i) implies (vi), (ii) implies (iii), (iii) implies (iv), which in turn implies (v).

To show (i) implies (ii), suppose that $T$ is hypercyclic. Then by definition, there exists $x \in \ell^p$ such that $\text{Orb}(T, x)$ is dense in $\ell^p$. Let $f \in \ell^p$. If $f \notin \text{Orb}(T, x)$, then clearly $f$ is a limit point of $\text{Orb}(T, x)$. On the other hand, if $f \in \text{Orb}(T, x)$ and $f$ is not a limit point of the orbit, then there is a neighborhood $U$ of $f$ that contains no point of $\text{Orb}(T, x)$ other than $f$. But then the points of $U \setminus \{f\}$ are not in the closure of $\text{Orb}(T, x)$, which gives a contradiction.

To show (iv) implies (i), we suppose that there exist a vector $x$ and a non-zero vector $f = (f_0, f_1, f_2, f_3, \ldots)$ in $\ell^p$ such that $f$ is a limit point of the orbit $\text{Orb}(T, x)$. Since $f_j \neq 0$ for some $j \geq 0$, we assume without loss of generality that $f_0 \neq 0$. Hence there exists an increasing sequence $\{n_k\}_{k \geq 1} \subset \mathbb{N}$ and an $N > 0$ such that

$$\|T^{n_k}x - f\| < \frac{1}{2^k} < \frac{|f_0|}{2}, \text{ for all } k \geq N.$$  

Let $x = (x_0, x_1, x_2, \ldots) \in \ell^p$. Then

$$T^{n_k}x = T^{n_k}(x_0, x_1, x_2, \ldots) = (w_1 \cdot \ldots \cdot w_{n_k}x_{n_k}, \ldots).$$

Hence $\|T^{n_k}x - f\| \geq |w_1 \cdot \ldots \cdot w_{n_k}x_{n_k} - f_0|$. So there exists a sequence $\{n_k\}_{k \geq 1}$ such that $|w_1 \cdot \ldots \cdot w_{n_k}x_{n_k} - f_0| < \frac{|f_0|}{2}$, for all $k \geq N$.

Thus $\frac{|f_0|}{2} < |w_1 \cdot \ldots \cdot w_{n_k}x_{n_k}|$ and so $\frac{|f_0|}{2(w_1 \cdot \ldots \cdot w_{n_k})} < |x_{n_k}|$ for all $k \geq N$. Hence we get that

$$\frac{|f_0|^p}{2^p(w_1 \cdot \ldots \cdot w_{n_k})^p} < |x_{n_k}|^p, \text{ for all } k \geq N.$$

Now, since $x \in \ell^p$ we have

$$\frac{|f_0|^p}{2^p} \sum_{k=N}^{\infty} \frac{1}{(w_1 \cdot \ldots \cdot w_{n_k})^p} \leq \sum_{k=N}^{\infty} |x_{n_k}|^p \leq \|x\|^p < \infty.$$
It follows that $\frac{1}{(w_1 \ldots w_n)^p} \to 0$, i.e. there exists an increasing sequence $\{n_k\}$ such that $w_1 \cdot \ldots \cdot w_{n_k} \to \infty$ as $k \to \infty$.

Thus by Salas’ criterion for hypercyclicity of unilateral backward shifts that $\sup_{n \geq 1} \prod_{j=1}^{n} w_j = \infty$, we have that $T$ is hypercyclic.

To show (v) implies (iv), we suppose there exists a vector $x$ in $\ell^p$ such that the $\text{Orb}(T, x)$ has $f \in \ell^p$ as a non-zero weak limit point. Since $f \neq 0$, let $k \geq 0$ such that $f_k \neq 0$.

Considering the weakly open sets that contain $f$, we get that for all $j \geq 1$ there exists an $n_j \geq 1$ such that for $k$ as above, $|\langle T^{n_j} x - f, e_k \rangle| < \frac{1}{j}$.

That is $|w_{k+1} \cdot \ldots \cdot w_{k+n_j} x_{k+n_j} - f_k| < \frac{1}{j}$, for all $j \geq 1$.

Next, we inductively pick a subsequence $\{n_{j_k}\}$ of $\{n_j\}$ as follows:

1. Let $j_1 = 1$.

2. Once we have chosen $j_m$ we pick $j_{m+1} > j_m$ such that $k + n_{j_m} < n_{j_m+1}$ and $\sum_{i=j_m+1}^{\infty} |x_{k+n_i}|^p \leq \frac{1}{j_m \cdot \|T\|^p n_{j_m}}$. This can be done since $x \in \ell^p$, so $\sum_{i=1}^{\infty} |x_{k+n_i}|^p \leq \|x\|^p < \infty$ and $\frac{1}{j_m \cdot \|T\|^p n_{j_m}}$ has been fixed in the previous $m$-th step.

Now, without loss of generality we can assume, by taking a subsequence if necessary, that $\{n_j\}$ satisfies $k + n_j < n_{j+1}$ and $\sum_{i=j+1}^{\infty} |x_{k+n_i}|^p \leq \frac{1}{j \cdot \|T\|^p n_j}$.

Let $y = \sum_{i=1}^{\infty} x_{k+n_i} \cdot e_{k+n_i}$. Clearly since $x$ is in $\ell^p$, so is $y$, as $\|y\| \leq \|x\| < \infty$.

Then $T^{n_m} y = \sum_{i=1}^{\infty} x_{k+n_i} \cdot T^{n_m} e_{k+n_i}$. But $k + n_i < n_{i+1}$ for all $i \geq 1$, and so $k + n_i < n_m$ for all $i < m$. Thus since $T$ is a unilateral backward shift we conclude that $T^{n_m} y = \sum_{i=m}^{\infty} x_{k+n_i} \cdot T^{n_m} e_{k+n_i}$.

Furthermore, since the vectors $T^{n_m} e_{k+n_i}$ and $T^{n_m} e_{k+n_j}$ have disjoint support for $i \neq j$, that is $\hat{T^{n_m} e_{k+n_i}}(s) = 0$ whenever $\hat{T^{n_m} e_{k+n_j}}(s) \neq 0$, we have that
Thus \( T^{nm} y \to f_k e_k \) in norm as \( m \to \infty \), where \( f_k e_k \neq 0 \) in \( \ell^p \). So \( \text{Orb}(T, y) \) has a non-zero limit point.

To show (vi) implies (i), we suppose there exist non-zero vectors \( x \) and \( f \) in \( \ell^p \), and a positive number \( r \) with \( 0 < r < \| f \| \) such that \( \text{Orb}(T, x) \cap B(f, r) \) is infinite.

For \( p > 1 \), we have that \( \ell^p \) is reflexive, so the convex unit ball \( \overline{\text{Ball}(\ell^p)} := \text{Ball}(\ell^p)^w = \text{Ball}(\ell^p)^w \) is weakly compact.

Now, since \( \ell^p \) is a Banach space, by the Eberlein-Smulian Theorem, \( \overline{\text{Ball}(\ell^p)} \) is weak limit point compact, so every infinite set has a weak limit point.

Since \( \text{Orb}(T, x) \cap B(f, r) \) is infinite and included in \( \overline{B(f, r)} \) which is weak limit point compact, we conclude that the \( \text{Orb}(T, x) \) has a non-zero weak limit point in \( \ell^p \) as \( 0 \notin \overline{B(f, r)} \). Thus since (v) implies (i) we have that \( T \) is hypercyclic.

For the remaining case that \( p = 1 \), we will show that there exists a vector \( y \in \ell^1 \) such that \( \text{Orb}(T, y) \) has a non-zero limit point. Thus, since (iv) implies (i), it follows that \( T \) is hypercyclic.

**Claim 1:** Without loss of generality we can assume that \( f \) has at most finitely many non-zero entries.

**Proof of Claim 1.** Suppose \( f \) has infinitely many non-zero entries.

Since the set \( \{ h \in \ell^1 : h \text{ has finitely many non-zero entries} \} \) is dense in \( \ell^1 \), we have that there exists \( h \in \ell^1 \) with finitely many non-zero entries such that \( \| f - h \| < \frac{\| f \| - r}{2} \).
Thus for \( g \in \ell^1 \) with \( \|g - f\| < r \), we have \( \|g - h\| \leq \|g - f\| + \|f - h\| < r + \frac{\|f\| - r}{2} = \frac{\|f\| + r}{2} < \|h\| \) (since \( \|f\| + \frac{\|f\| - r}{2} < \|h\| \)).

Therefore \( B(f, r) \subset B(h, \frac{\|f\| + r}{2}) \), and hence for \( r' = \frac{\|f\| + r}{2} \) we have that the intersection \( \text{Orb}(T, x) \cap B(h, r') \) is infinite with \( 0 < r' < \|h\| \).

Now, by our Claim 1 assume that there exists an \( N > 0 \) such that \( f_k = 0 \) for all \( k \geq N \). By assumption there exist a vector \( x \in \ell^1 \) and a strictly increasing sequence \( \{n_j\} \subset \mathbb{N} \) such that \( T^{n_j}x \in B(f, r) \) for all \( j \), where \( 0 < r < \|f\| \).

Let \( E = \{i \geq 0 : 0 \leq i \leq N, f_i \neq 0\} \).

**Claim 2:** For all \( j \geq 0 \) there exists \( i \in E \) such that \( \left| \left( \widehat{T^{n_j}x} - f \right)(i) \right| < \frac{r \cdot |f_i|}{\|f\|} \).

**Proof of Claim 2.** Suppose that there exists \( j_0 \geq 0 \) such that for all \( i \in E \), \( \left| \left( \widehat{T^{n_{j_0}}x} - f \right)(i) \right| \geq \frac{r \cdot |f_i|}{\|f\|} \). Then \( \|T^{n_{j_0}}x - f\| \geq \sum_{i \in E} \left| \left( \widehat{T^{n_{j_0}}x} - f \right)(i) \right| \geq \sum_{i \in E} \frac{r \cdot |f_i|}{\|f\|} = \frac{r \cdot \sum_{i \in E} |f_i|}{\|f\|} = r, \) which gives a contradiction with \( T^{n_{j_0}}x \in B(f, r) \).

Now, since \( E \subset \{1, 2, \ldots, N\} \) is a finite set we get by our Claim 2 that there exists an \( i \in E \) such that for infinitely many \( j \) we have \( \left| \left( \widehat{T^{n_j}x} - f \right)(i) \right| < \frac{r \cdot |f_i|}{\|f\|} \).

Without loss of generality we can assume, by taking a subsequence of \( \{n_j\} \) if necessary, that there exists \( i \in E \) such that for all \( j \geq 0 \), \( \left| \left( \widehat{T^{n_j}x} - f \right)(i) \right| < \frac{r \cdot |f_i|}{\|f\|} \).

For notational simplicity assume further that \( i = 0 \).

By the reverse triangle inequality, \( |f_0| - \left| \widehat{T^{n_j}x}(0) \right| \leq \frac{r \cdot |f_i|}{\|f\|} \), and hence for \( \alpha := \frac{r \cdot |f_0|}{\|f\|} > 0 \) we get that \( \left| \widehat{T^{n_j}x}(0) \right| > |f_0| - \frac{r \cdot |f_0|}{\|f\|} = |f_0| \cdot \frac{\|f\| - r}{\|f\|} = \alpha > 0 \). That is, \( \left| \widehat{T^{n_j}x}(0) \right| > \alpha > 0 \) for all \( j \geq 0 \).

(A)

Next, we pick a subsequence \( \{n_{j_k}\} \) of \( \{n_j\} \) as follows:

1. Set \( j_1 = 1 \).

2. Inductively choose \( j_{k+1} > j_k \) and \( \sum_{i=j_{k+1}}^{\infty} |x_{n_i}| \leq \frac{1}{\|T\|^{n_{j_k}} \cdot (j_k + 1)} \). Without loss of generality, we assume that \( \sum_{i=k+1}^{\infty} |x_{n_i}| \leq \frac{1}{\|T\|^{n_k} \cdot (k + 1)} \).

(B)
Let $y = \sum_{j=1}^{\infty} \alpha \left| x_{n_j} \right| e_{n_j}$. Clearly, by (A) we have that $\|y\| \leq \|x\| < \infty$, and thus $y \in \ell^1$.

Now since $T$ is a backward shift and $n_j < n_{j+1}$ for all $j \geq 1$ we have that $T^{n_m} y - \alpha e_0 = \left( \frac{\alpha}{\left| T^{n_m} x(0) \right|} \right) x_{n_m} \cdot T^{n_m} e_{n_m} - \alpha e_0 + \sum_{j=m+1}^{\infty} \alpha \left| \left| x_{n_j} \right| \cdot T^{n_m} e_{n_m} = |x_{n_m}| \cdot w_1 \cdot w_2 \cdot \ldots \cdot w_{n_m} \cdot e_0 = \left| T^{n_m} x(0) \right| e_0$.

So $\|T^{n_m} y - \alpha e_0\| \leq 0 + \sum_{j=m+1}^{\infty} \alpha \left| x_{n_j} \right| \|T^{n_m} e_{n_j}\|$.

Hence (A), (B) and continuity of $T$ give that $\|T^{n_m} y - \alpha e_0\| \leq \sum_{j=m+1}^{\infty} \left| x_{n_j} \right| \|T\|^{n_m} \leq \|T\|^{n_m} \cdot \frac{1}{\|T\|^{n_m (m+1)}} = \frac{1}{m+1}$ for all $m \geq 1$.

Thus $\|T^{n_m} y - \alpha e_0\| \to 0$ as $m \to \infty$, and hence $\text{Orb}(T, y)$ has the non-zero limit point $\alpha e_0$. So $T$ is hypercyclic.

As an easy consequence of the equivalence of statements (i) and (iv) in the above theorem, we have the following result.

**Corollary 26.** A unilateral weighted backward shift $T : \ell^p \to \ell^p$ is not hypercyclic if and only if for every $x$ in $\ell^p$ the set $\text{Orb}(T, x) \cup \{0\}$ is closed.

As we have pointed out in Remark 2, an orbit $\text{Orb}(T, x)$ may have a non-zero limit point without having the vector $x$ that generates the orbit be hypercyclic for $T$. Nonetheless, we can demonstrate how to construct a hypercyclic vector for $T$ using the non-zero limit point of the orbit, which is assumed to be $e_0$ in the following.

Let $1 \leq p < \infty$. Let $T : \ell^p \to \ell^p$ be a unilateral weighted backward shift with weight sequence $\{w_j\}_{j \geq 1}$ and let $c = \|T\| < \infty$.

Suppose that there exists a vector $x$ in $\ell^p$ such that the orbit $\text{Orb}(T, x)$ has $e_0$ as a limit point. We want to construct a hypercyclic vector $y$ for $T$. 

Let \( D = \{(a_0, a_1, a_2, \ldots) \in \ell^p : a_i \in \mathbb{Q} \text{ and } a_i = 0 \text{ for all but finitely many } i \in \mathbb{Z}_+\}. \)

Clearly \( D \) is a dense and countable set, so we can enumerate \( D = \{d_1, d_2, d_3, \ldots\} \).

Since \( \text{Orb}(T,x) \) has \( e_0 \) as a limit point, there exists a sequence of positive integers \( n_k \to \infty \) such that \( \|T^{n_k}x - e_0\| < \frac{1}{2^k} < \frac{1}{2} \) for all \( k \geq 1 \). By our proof in Theorem 25 we get that \( \frac{1}{w_1 \cdots w_{n_k}} \to 0 \) as \( k \to \infty \).

In the next steps we will construct a sequence of positive integers \( \{k_j\}_{j \geq 1} \) whose terms we will then use to define the desired hypercyclic vector \( y \) for \( T \).

For this purpose we will first construct the sequence \( \{k_j\}_{j \geq 1} \) subject to the restriction that \( y \in \ell^p \), that is \( \sum_{j=0}^{\infty} |\hat{y}(j)|^p < \infty \). Furthermore, we require that there exists a sequence \( \{m_l\}_{l \geq 1} \) such that for each \( l \geq 1 \) and each \( \epsilon > 0 \) there exists an \( l_0 \geq 1 \) having \( \|T^{m_l}y - d_l\| < \epsilon \). Thus each \( d_l \) in \( D \) can be approximated arbitrarily close by an element in the \( \text{Orb}(T,y) \).

We note that for the second condition it suffices to require that there exists a sequence \( \{m_l\}_{l \geq 1} \) such that \( \|T^{m_l}y - d_l\| < \frac{1}{2^l} \) for all \( l \geq 1 \). For if \( l \geq 1 \) and \( \epsilon > 0 \), there exists \( l_0 \) large enough such that \( \frac{1}{2^{l_0}} < \frac{\epsilon}{2} \) and \( \|d_l - d_{l_0}\| < \frac{\epsilon}{2} \), by the density of the set \( D \). Then \( \|T^{m_{l_0}}y - d_l\| < \|T^{m_{l_0}}y - d_{l_0}\| + \|d_l - d_{l_0}\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon \).

**Step 1.** In this step we will choose \( k_1 \in \mathbb{N} \) and define the first block of entries of \( y \). For this we require the following two conditions:

(a) Since \( d_1 \in D \) we can write \( d_1 = \sum_{j=0}^{N_1} \alpha_j(1)e_j \) for some \( N_1 \geq 0 \).

We choose \( k_1 \) by requiring first that \( n_{k_1} > N_1 \), and thus having the entire first block defined below fit inside \( y \in \ell^p \). The first block of \( y \) will have \( (N_1 + 1) \) entries and will end at position \( n_{k_1} \in \{n_k\}_{k \geq 1} \). We further note that in between the blocks of \( y \) defined in each step, we set the vector \( y \) to have only zero entries.

(b) Dealing with the requirement that \( y \in \ell^p \), we need

<table>
<thead>
<tr>
<th>Position</th>
<th>( n_{k_1} - N_1 )</th>
<th>( n_{k_1} - (N_1 - 1) )</th>
<th>\ldots</th>
<th>( n_{k_1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Entry</td>
<td>( \alpha_{N_1}(1) )</td>
<td>( \alpha_1(1) )</td>
<td>\ldots</td>
<td>( \alpha_{N_1}(1) )</td>
</tr>
<tr>
<td></td>
<td>( w_1^{-1} w_2^{-1} \cdots w_{n_k}^{-1} )</td>
<td>( w_2^{-1} w_3^{-1} \cdots w_{n_k}^{-1} )</td>
<td>\ldots</td>
<td>( w_{N_1}^{-1} w_{N_1+1}^{-1} \cdots w_{n_k}^{-1} )</td>
</tr>
</tbody>
</table>
\[\frac{|\alpha_0(1)|}{w_1 \cdot w_2 \cdots w_{n_k_1} - N_1} < \frac{1}{2} \quad (1)\]
\[\frac{|\alpha_1(1)|}{w_2 \cdot w_3 \cdots w_{n_k_1} - (N_1 - 1)} < \frac{1}{2^2} \quad (2)\]
\[\vdots\]
\[\frac{|\alpha_{N_1}(1)|}{w_{N_1 + 1} \cdot w_{N_1 + 2} \cdots w_{n_k_1}} < \frac{1}{2^{N_1 + 1}} \quad (N_1 + 1)\]

Similar conditions for \(k_2, k_3, \ldots\) will give us that \(\|y\|^p = \sum_{j=1}^\infty \left(\frac{1}{2^j}\right)^p \leq \sum_{j=1}^\infty \frac{1}{2^j} = 1\), so \(y \in \ell^p\).

We achieve the above finite number of inequalities by using the fact that the product \(\frac{1}{w_1 \cdots w_{n_k}} \to 0\) as \(k \to \infty\). Namely if \(k_1\) is large enough, \(\frac{1}{w_1 \cdots w_{n_k_1}}\) can be made as small as needed.

Now we note that
\[\frac{|\alpha_0(1)|}{w_1 \cdot w_2 \cdots w_{n_k_1} - N_1} = \frac{|\alpha_0(1)|w_{n_k_1} - (N_1 - 1) \cdots w_{n_k_1}}{w_1 \cdot w_2 \cdots w_{n_k_1} - N_1 - (N_1 - 1) \cdots w_{n_k_1}} < \frac{|\alpha_0(1)| \cdot w_{n_k_1}^{N_1}}{w_1 \cdot w_2 \cdots w_{n_k_1}}.\]

So condition (1) can be satisfied by choosing the term \(k_1\) large enough such that \(\frac{1}{w_1 \cdot w_2 \cdots w_{n_k_1}} < \frac{1}{2|\alpha_0(1)| \cdot w_{n_k_1}^{N_1}}\). Similarly, all conditions (1) through \((N_1 + 1)\) can be satisfied by choosing \(k_1\) large enough such that \(\frac{1}{w_1 \cdot w_2 \cdots w_{n_k_1}} < P_1\), where \(P_1 = \min \left\{ \frac{1}{2^j + 1 \cdot |\alpha_j(1)| \cdot w_{n_k_1}^{N_1}} : j = 0, \ldots, N_1 \right\}\).

We note that \(P_1 = \frac{1}{2^{N_1 + 1} \cdot M_1 \cdot w_{n_k_1}^{N_1}}\), where \(M_1 = \max \left\{ |\alpha_j(1)| : j = 0, 1, \ldots, N_1 \right\}\).

We have now chosen \(k_1 \in \mathbb{N}\).

**Step 2.** This step is for choosing \(k_2 \in \mathbb{N}\) and defining the second block of entries of \(y\).

Write \(d_2 = \sum_{j=0}^{N_2} \alpha_j(2)e_j\) for some \(N_2 \geq 0\). We require the following of \(k_2\):

(a) To avoid the overlapping of entries in the second block of \(y \in \ell^p\) with entries in the first block, we need that \(n_{k_2} - n_{k_1} > N_2\). Also we would need that \(n_{k_2} > N_2\), but this follows from the previous condition. Clearly we require that \(k_2 > k_1\). The second block will have \((N_2 + 1)\) entries, ending at position \(n_{k_2} \in \{n_k\}_{k \geq 1}\).

<table>
<thead>
<tr>
<th>Position</th>
<th>(n_{k_2} - N_2)</th>
<th>(n_{k_2} - (N_2 - 1))</th>
<th>(\ldots)</th>
<th>(n_{k_2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Entry</td>
<td>(\frac{\alpha_0(2)}{w_1 \cdot w_2 \cdots w_{n_k_2} - N_2})</td>
<td>(\frac{\alpha_1(2)}{w_2 \cdot w_3 \cdots w_{n_k_2} - (N_2 - 1)})</td>
<td>(\ldots)</td>
<td>(\frac{\alpha_{N_2}(2)}{w_{N_2 + 1} \cdot w_{N_2 + 2} \cdots w_{n_k_2}})</td>
</tr>
</tbody>
</table>
(b) For the requirement that \( y \in \ell^p \) we need to choose \( k_2 \) large enough such that 
\[
\frac{1}{w_1\cdot w_2\cdots w_{n+k_2}} < P_2, \quad \text{where} \quad P_2 = \min \left\{ \frac{1}{2^{j+1+(N_1+1)|\alpha_j(2)|}c^{N_2^2}} : j = 0, 1, \ldots, N_2 \right\}.
\]
We note that \( P_2 = \frac{1}{2^n+1+n_2+n_2^2+M_2c^{N_2^2}} \), where \( M_2 = \max\{|\alpha_j(2)| : j = 0, \ldots, N_2\} \).

(c) For the choice of \( k_2 \), we need an extra condition since we now want to also verify the 
condition that \( ||T^{m_1}y - d_1|| < \frac{1}{2} \).

Let \( m_1 = n_{k_1} - N_1 \). The next conditions will clearly give us this last requirement.

Shifting the vector \( y \) by \( m_1 = n_{k_1} - N_1 \) will produce the following changes to the coefficients of the second block:

<table>
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<tr>
<th>Position</th>
<th>Entry</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>( \alpha_0(2)\cdot w_{n+k_2} - N_2 - (m_1 - 1)\cdots w_{n+k_2} - N_2 )</td>
<td>( \alpha_1(2)\cdot w_{n+k_2} - (m_1 - 1)\cdots w_{n+k_2} )</td>
<td>( \alpha_{N_2}(2)\cdot w_{n+k_2} - (m_1 - 1)\cdots w_{n+k_2} )</td>
</tr>
<tr>
<td></td>
<td>( w_1\cdot w_2\cdots w_{n+k_2} - N_2 )</td>
<td>( w_1\cdot w_2\cdots w_{n+k_2} - M_2c^{N_2^2} )</td>
<td>( w_1\cdot w_2\cdots w_{n+k_2} - M_2c^{N_2^2} )</td>
</tr>
</tbody>
</table>

But each entry above is in absolute value bounded above by \( \frac{|\alpha_j(2)|c^{N_2^2}M_1}{w_1\cdots w_{n+k_2}} \), which can be made as small as needed.

So we now choose \( k_2 \) large enough such that 
\[
\frac{1}{w_1\cdot w_2\cdots w_{n+k_2}} < Q_2, \quad \text{where} \quad Q_2 = \min \left\{ \frac{1}{2^{j+1+(N_1+1)|\alpha_j(2)|}c^{N_2^2}} : j = 0, 1, \ldots, N_2 \right\} = \frac{1}{2^{j+1+(N_1+1)|\alpha_j(2)|}c^{N_2^2}}.
\]
We have now chosen \( k_2 \in \mathbb{N} \).

**Step 3.** We will choose the general term \( k_j \in \mathbb{N} \) for \( j \geq 3 \) satisfying:

(a) \( k_j > k_{j-1} \) and \( n_{k_j} - n_{k_{j-1}} > N_j \);

(b) \( \frac{1}{w_1\cdot w_2\cdots w_{n+k_j}} < \frac{1}{M_j\cdot c^{N_j^2+\sum_{l=1}^{N_j} n_l}} \);

(c) \( \frac{1}{w_1\cdot w_2\cdots w_{n+k_j}} < \frac{1}{M_j\cdot c^{N_j^2+m_1} \cdot (N_{j+1}+1)2^{j-1}} \), for all \( l = 1, 2, \ldots, j - 1 \), where \( m_l = n_{k_l} - N_l \)
and \( m_1 < m_2 < m_3 < \ldots \).

Note that the entries of \( y \) in the \( j \)-th block are:

\[
\frac{\alpha_0(j)}{w_1\cdot w_2\cdots w_{n+k_j} - N_j} \cdot \frac{\alpha_1(j)}{w_2\cdot w_3\cdots w_{n+k_j} - (N_{j+1})} \cdots \frac{\alpha_{N_j}(j)}{w_{N_{j+1}}\cdot w_{N_{j+2}}\cdots w_{n+k_j}}.
\]

Thus in steps 1–3 we have chosen the sequence \( \{k_l\}_{l \geq 1} \) such that \( y \in \ell^p \) and \( ||T^{m_l}y - d_l|| < \frac{1}{2} \) for all \( l \geq 1 \) where \( d_l \in D \). Thus for all \( d_l \) in the dense set \( D \), we have that \( d_l \in \text{Orb}(T, y) \), so \( y \) is a hypercyclic vector for \( T \).
2.2.3 The bilateral weighted backward shift

After examining how hypercyclicity relates to having an orbit with a non-zero limit point for the unilateral weighted backward shifts in Section 2.2.2, we turn to the study of bilateral weighted shifts.

Let \( \{e_n : n \in \mathbb{Z}\} \) be the canonical basis for \( \ell^p(\mathbb{Z}) \) for \( p \geq 1 \). Then, a bounded and linear operator \( T: \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z}) \) is said to be a bilateral weighted backward shift if there is a sequence of bounded positive weights \( \{w_n : n \in \mathbb{Z}\} \) such that \( Te_n = w_ne_{n-1} \) for all \( n \in \mathbb{Z} \).

Analogous to the unilateral weighted shift we have the following result.

**Theorem 27.** Suppose that \( T: \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z}) \) for \( 1 \leq p < \infty \) is a bilateral weighted backward shift. The following are equivalent:

(i) \( T \) is hypercyclic.

(ii) There exists an \( x \) in \( \ell^p(\mathbb{Z}) \) whose orbit \( \text{Orb}(T,x) \) has a non-zero limit point.

(iii) There exists a vector \( x \) in \( \ell^p(\mathbb{Z}) \) whose orbit \( \text{Orb}(T,x) \) has infinitely many members in an open ball \( B(h,r) \), where \( 0 < r < \|h\| \).

Before we give a proof of the theorem, we remark that statement (ii) cannot be relaxed to having an orbit with a non-zero weak limit point in \( \ell^p(\mathbb{Z}) \), as is the case in Theorem 25. Chan and Sanders [11] showed the existence of a bilateral weighted backward shift that is weakly hypercyclic, and thus has a non-zero weak limit point, but the shift is not norm hypercyclic.

Recall that for a bilateral weighted shift to be hypercyclic, Salas [33] provided a necessary and sufficient condition in terms of the weights: for every \( \epsilon > 0 \) and every \( q \in \mathbb{N} \) there exists \( n \) arbitrarily large such that for every \( j \in \mathbb{Z} \) with \( |j| \leq q \) we have \( \prod_{s=0}^{n-1} w_{s+j} > \frac{1}{\epsilon} \) and \( \prod_{s=1}^{n} w_{j-s} < \epsilon \). However, as it turns out, this condition is not as helpful in proving the above theorem as its counterpart for the unilateral case that we have studied in Section 2.2.2. For that reason, we now offer a constructive argument to prove Theorem 27.
Proof. It is clear that (i) implies (iii).

To show that statement (ii) implies (i), we suppose without loss of generality that \( x = (\ldots, \hat{x}(-1), \hat{x}(0), \hat{x}(1), \ldots) \) is a vector in \( \ell^p(\mathbb{Z}) \) such that \( e_0 \) is the non-zero limit point of the orbit \( \text{Orb}(T,x) \). We set \( p = 2 \) for notational simplicity.

Hence there is an increasing sequence of positive integers \( \{n_i\} \) such that

\[
\|T^{n_i}x - e_0\| < \frac{1}{2^i} \text{ for all } i \geq 1.
\]

Thus \( \left| w_1 \cdot w_2 \cdot \ldots \cdot w_{n_i} \hat{x}(n_i) - 1 \right| < \frac{1}{2^i} < \frac{1}{2} \) for all \( i \geq 1 \). This then implies that if \( i \geq 1, \)

\[
\frac{1}{2w_1 \cdot w_2 \cdot \ldots \cdot w_{n_i}} < \left| \hat{x}(n_i) \right|. \quad (A)
\]

Since \( \sum_{i=1}^{\infty} \left| \hat{x}(n_i) \right|^2 \leq \|x\|^2 < \infty \), the above inequality gives that

\[
\sum_{i=1}^{\infty} \left[ \frac{1}{w_1 \cdot w_2 \cdot \ldots \cdot w_{n_i}} \right]^2 < \infty,
\]

and thus \( \frac{1}{w_1 \cdot w_2 \cdot \ldots \cdot w_{n_i}} \to 0 \) as \( i \to \infty \). \( \quad (B) \)

By considering the terms of the vector \( T^{n_j}x \) with negative indices we have that

\[
\sum_{i=1}^{j-1} \left| w_{-(n_j-n_i-1)} \cdot \ldots \cdot w_0 \cdot \ldots \cdot w_{n_i} \cdot \hat{x}(n_i) \right|^2 < \frac{1}{2^{2j}}.
\]

Now, focusing on the \( i \)-th term of the above summation and using \( (A) \), we see that if \( 1 \leq i \leq j - 1, \) then \( w_{-(n_j-n_i-1)} \cdot \ldots \cdot w_0 < \frac{1}{2^{2j}}. \) \( \quad (C) \)

Claim: Let \( y = (\ldots, 0, \hat{y}(0), \ldots, \hat{y}(k), 0, \ldots) \) and \( a = (\ldots, 0, \hat{a}(-l), \ldots, \hat{a}(0), \ldots, \hat{a}(l), 0, \ldots) \) in \( \ell^2(\mathbb{Z}) \) with \( k \) and \( l \geq 1 \). For all positive \( \epsilon \), there exist an integer \( m > k + l \) and a vector \( z \) of the form \( z = (\ldots, 0, \hat{z}(m-l), \ldots, \hat{z}(m), \ldots, \hat{z}(m+l), 0, \ldots) \) such that:

(1) \( T^mz = a, \)

(2) \( \|z\| < \epsilon, \)

(3) \( \|T^sz\| < \epsilon, \text{ if } 1 \leq s \leq k, \)

(4) \( \|T^my\| < \epsilon. \)
Proof of Claim. Note that \( \|T\| = \sup \{|w_i| : i \in \mathbb{Z}\} > 1 \).

Choose an integer \( i \) such that \( n_i > k + l \), where \( k \) and \( l \) are positive integers given in the vectors \( y \) and \( a \) in the statement of the Claim. For this fixed \( n_i \) and for any \( j > i \) we denote \( m = n_j - n_i + k \).

We first observe by \((C)\) that

\[
\|T^m y\|^2 = \sum_{r=0}^{k} \left| w_{-(m-r-1)} \cdot \ldots \cdot w_{-(n_j-n_i-1)} \cdot \ldots \cdot w_r \cdot \hat{y}(r) \right|^2
\]

\[
\leq \|T\|^{2k} \cdot \|y\|^2 \cdot \sum_{r=0}^{k} \left| w_{-(m-r-1)} \cdot \ldots \cdot w_{-(n_j-n_i-1)} \cdot \ldots \cdot w_0 \right|^2
\]

\[
\leq \|T\|^{2k} \cdot \|y\|^2 \cdot \|T\|^{2k} \sum_{r=0}^{k} \left[ \sum_{j > i} \left| \frac{1}{w_{r+1} \cdot \ldots \cdot w_{m+r}} \hat{a}(r) \right| \right]^2 \leq \|a\|^2 \sum_{r=0}^{k} \left[ \sum_{j > i} \left| \frac{1}{w_{r+1} \cdot \ldots \cdot w_{m+r}} \right| \right]^2
\]

Note that if \(-l \leq r \leq l\), then \( m + r \leq m + l = n_j - n_i + k + l = n_j - (n_i - k - l) < n_j \) since \( n_i > k + l \).

Thus, \( \|z\|^2 \leq \|a\|^2 \sum_{r=-l}^{l} \left| \frac{w_{m+r+1} \cdot \ldots \cdot w_{n_j}}{w_{r+1} \cdot \ldots \cdot w_{n_j}} \right|^2 \).

We now let \( c_\ell = \max \{1, \frac{1}{w_0}, \frac{1}{w_{-1}w_0}, \ldots, \frac{1}{w_{-l+1} \cdot \ldots \cdot w_{-1}w_0} \} \) and observe that the numerator \( w_{m+r+1} \cdot \ldots \cdot w_{n_j} \) is a product of \( n_j - (m + r) = n_j - [n_j - n_i + k + r] = n_i - k - r \) factors.

Hence, \( \|z\|^2 \) is bounded above by

\[
\|a\|^2 \cdot \|T\|^{2(n_i-k+l)} \cdot \sum_{r=-l}^{0} \left[ \frac{1}{w_{r+1} \cdot \ldots \cdot w_{n_j}} \right]^2 + \|a\|^2 \cdot \|T\|^{2(n_i-k-1)} \cdot \sum_{r=1}^{l} \left[ \frac{1}{w_{r+1} \cdot \ldots \cdot w_{n_j}} \right]^2
\]

\[
\leq \|a\|^2 \cdot \|T\|^{2(n_i-k+l)} \cdot c^2 \sum_{r=-l}^{0} \left[ \frac{1}{w_{r+1} \cdot \ldots \cdot w_{n_j}} \right]^2 + \|a\|^2 \cdot \|T\|^{2(n_i-k-1)} \cdot \sum_{r=1}^{l} \left[ \frac{1}{w_{r+1} \cdot \ldots \cdot w_{n_j}} \right]^2
\]

\[
\leq \left[ \frac{1}{w_1 \cdot \ldots \cdot w_{n_j}} \right]^2 \left\{ \|a\|^2 \cdot \|T\|^{2(n_i-k+l)} \cdot c^2 \cdot (l + 1) + \|a\|^2 \cdot \|T\|^{2(n_i-k-1)} \cdot \|T\|^{2l} \right\}.
\]
Note that if $1 \leq s \leq k$, then $\|T^s z\| \leq \|T\|^k \|z\|$.

It is now evident that for any given $\epsilon > 0$, we can use $(B)$ to choose an integer $j > i$ so that $(2)$, $(3)$ and $(4)$ in the Claim are satisfied.

\[\square\]

To finish the proof, it remains to show that our Claim implies that $T$ is hypercyclic. For this, let $D = \{d_1, d_2, \ldots\}$ be a dense subset of $\ell^2(\mathbb{Z})$, where each $d_i$ is of the form $d_i = (\ldots, 0, \hat{d}(-l_i), \ldots, \hat{d}(0), \ldots, \hat{d}(l_i), 0, \ldots)$.

First, take $y_1 = 0$ with $k_1 = 1$, $a_1 = d_1$ and $\epsilon = \frac{1}{4}$ as in the statement of the Claim. Then our Claim gives us that there exists an $m_1 \in \mathbb{N}$ and a vector $z_1$ with $\hat{z}_1(i) = 0$ whenever $i \geq m_1 + l_1$ such that

(1) $T^{m_1} z_1 = d_1$,

(2) $\|z_1\| < \frac{1}{4}$,

(3) $\|T z_1\| < \frac{1}{4}$,

(4) $\|T^{m_1} y_1\| = 0$.

Inductively, we take $k_j = m_{j-1} + l_{j-1}$ and $y_j = z_1 + z_2 + \ldots + z_{j-1}$. Clearly, $\{k_j\}_{j \geq 1}$ is an increasing sequence, as $k_{j+1} = m_j + l_j > k_j + 2l_j$ for all $j \geq 1$.

Thus we note that $\hat{y}_j(i) = 0$ whenever $i \geq k_j$.

Let $\epsilon = \frac{1}{4}$ in the statement of the Claim. Then our Claim implies that there exist $m_j \in \mathbb{N}$ and a vector $z_j$ with $\hat{z}_j(i) = 0$ whenever $i \geq m_j + l_j$ so that

(1) $T^{m_j} z_j = d_j$,

(2) $\|z_j\| < \frac{1}{4}$,

(3) $\|T^s z_j\| < \frac{1}{4}$, if $1 \leq s \leq k_j$,

(4) $\|T^{m_j} y_j\| < \frac{1}{4}$.
Define the vector \( b \) by setting \( b = \sum_{i=1}^{\infty} z_i \). Since the sum is absolutely convergent, we have that \( \|b\| = \left\| \sum_{i=1}^{\infty} z_i \right\| < \sum_{i=1}^{\infty} \frac{1}{4^i} < \infty \), so \( b \in \ell^2(\mathbb{Z}) \).

Also note that \( b = y_j + z_j + \sum_{i=j+1}^{\infty} z_i \), so

\[
\|T^{m_j} b - d_j\| \leq \|T^{m_j} y_j\| + \|T^{m_j} z_j - d_j\| + \left\| T^{m_j} \left( \sum_{i=j+1}^{\infty} z_i \right) \right\|
\]

\[
< \frac{1}{4^j} + 0 + \sum_{i=j+1}^{\infty} \|T^{m_j} z_i\| < \frac{1}{4^j} + \sum_{i=j+1}^{\infty} \frac{1}{4^i} = \sum_{i=j}^{\infty} \frac{1}{4^i} \rightarrow 0 \quad \text{as} \quad j \rightarrow \infty.
\]

Thus \( \|T^{m_j} b - d_j\| \rightarrow 0 \) as \( j \rightarrow \infty \), so by density of the set \( D = \{d_1, d_2, \ldots \} \) we have that \( b \) is a hypercyclic vector for \( T \).

To show that (iii) implies (ii), we suppose that there exists a vector \( x \) in \( \ell^p(\mathbb{Z}) \) whose orbit \( \text{Orb}(T, x) \) has infinitely many members in an open ball \( B(h, r) \), where \( 0 < r < \|h\| \). For notational simplicity we set \( p = 2 \).

Let \( s > 0 \) such that \( \frac{1-s}{1+s} > \frac{r}{\|h\|} \). Since \( h \in \ell^2(\mathbb{Z}) \), there exists \( N \geq 1 \) such that \( \frac{\|h\|^2}{(1+s)^2} < \sum_{i=-N}^{N} \left| \hat{h}(i) \right|^2 \). Let \( \{n_j\}_{j \geq 1} \) be an increasing sequence of integers such that \( T^{m_j} x \in B(h, r) \) and set \( f_j = T^{m_j} x \).

We now show that for each \( f_j \) there exists an integer \( i \) with \( i \leq N \) such that \( 0 < s \cdot \left| \hat{h}(i) \right| < \left| \hat{f_j}(i) \right| \) whenever \( |i| \leq N \).

To do that, we suppose on the contrary that there exists an \( f_j \) so that \( \left| \hat{f_j}(i) \right| \leq s \cdot \left| \hat{h}(i) \right| \) whenever \( |i| \leq N \). This would imply that, by our choice of \( N \),

\[
(1-s)^2 \frac{\|h\|^2}{(1+s)^2} < (1-s)^2 \sum_{i=-N}^{N} \left| \hat{h}(i) \right|^2
\]

\[
\leq \sum_{i=-N}^{N} \left| \hat{h}(i) - \hat{f_j}(i) \right|^2
\]

\[
\leq \|h - f_j\|^2 < r^2,
\]

\[
< r^2,
\]
which would contradict our choice of $s$.

Since there are finitely many integers $i$ with $|i| \leq N$, but infinitely many $j \geq 1$, there exists an integer $i_0$ with $|i_0| \leq N$ such that $0 < s \cdot |\hat{h}(i_0)| < |\hat{f}_j(i_0)|$, for infinitely many $j$.

We now assume $i_0 = 0$ for notational simplicity. In addition, by taking a subsequence of $\{n_j\}$ if necessary, we further assume that

$$0 < s \cdot |\hat{h}(0)| < |\hat{f}_j(0)|, \text{ whenever } j \geq 1.$$ 

Recall that $f_j = T^{n_j}x$, and thus

$$\hat{f}_j(0) = w_1 \cdots w_{n_j} \hat{x}(n_j),$$

from which it follows that

$$\infty > \|x\|^2 \geq \sum_{j=-\infty}^{\infty} |\hat{x}(n_j)|^2 = \sum_{j=-\infty}^{\infty} \frac{|\hat{f}_j(0)|^2}{(w_1 \cdots w_{n_j})^2} \geq s^2 \|\hat{h}(0)\|^2 \sum_{j=-\infty}^{\infty} \left(\frac{1}{w_1 \cdots w_{n_j}}\right)^2.$$ 

Hence $\frac{1}{w_1 \cdots w_{n_j}} \to 0$ as $j \to \infty$. \hfill (D)

This means that $\|T\| = \sup \{w_j : j \in \mathbb{Z}\} > 1$.

We now switch our attention to the weights with negative indices. First, we note that $T^{n_j}x \in B(h, r)$ for all $j$, and hence, there exists $M > 0$ so that $\|T^{n_j}x\| \leq M$ for all $j$.

Second we note that if $1 \leq i < j$, then

$$T^{n_j}(\hat{x}(n_i)e_{n_i}) = w_{-(n_j-n_i-1)} \cdots w_0 \cdots w_{n_i} \hat{x}(n_i)e_{n_i-n_j}.$$ 

Thus,
\[ M^2 \geq \|T^m x\|^2 \]
\[ \geq \left\| \sum_{i=1}^{j-1} T^m_j (\hat{x}(n_i) e_{n_i}) \right\|^2 \]
\[ = \sum_{i=1}^{j-1} (w_{-(n_j-n_i-1)} \cdots w_0 \cdots w_{n_i})^2 |\hat{x}(n_i)|^2 \]
\[ = \sum_{i=1}^{j-1} (w_{-(n_j-n_i-1)} \cdots w_0)^2 |\hat{f}_i(0)|^2, \text{ by (B)} \]
\[ > s^2 |\hat{h}(0)|^2 \sum_{i=1}^{j-1} (w_{-(n_j-n_i-1)} \cdots w_0)^2, \text{ by (A)}. \]

In other words, if \( 1 \leq i < j \), then
\[ \sum_{i=1}^{j-1} (w_{-(n_j-n_i-1)} \cdots w_0)^2 < \frac{M^2}{s^2 |\hat{h}(0)|^2}. \] (E)

**Claim:** For every \( \epsilon > 0 \) and every integer \( k \geq 1 \), there exist positive integers \( j \) and \( m \) with \( m < j \) so that:

1. \( k < m, \)
2. \( 2n_m < n_j, \)
3. \( w_{-(n_j-n_m-1)} \cdots w_0 < \frac{\epsilon}{\sqrt{n_k \|T\|^{n_k}}}, \)
4. \( \frac{1}{w_1 \cdots w_{n_j}} < \frac{\epsilon}{\|T\|^{2n_m}}. \)

**Proof of Claim.** Let \( \epsilon > 0 \) and \( k \geq 1 \) be given. Determine a positive integer \( t > k + 1 \) so that
\[ \frac{M^2}{s^2 |\hat{h}(0)|^2 (t-k)} < \left[ \frac{\epsilon}{\sqrt{n_k \|T\|^{n_k}}} \right]^2. \]

For that choice of \( t \), we can determine \( j > t \) such that \( n_j > 2n_t \) and \( \frac{1}{w_1 \cdots w_{n_j}} < \frac{\epsilon}{\|T\|^{2n_m}} \), by (D).

Since \( k + 1 < t < j \), we have, by (E),
\[ \sum_{i=k+1}^{t} (w_{-(n_j-n_i-1)} \cdots w_0)^2 < \frac{M^2}{s^2 |\hat{h}(0)|^2}. \]
The summation on the left-hand side has \( t - k \) positive terms, so there must exist an integer \( m \) with \( k + 1 \leq m \leq t \) such that
\[
\left( w_{-(n_j-n_m-1)} \cdot \ldots \cdot w_0 \right)^2 < \frac{M^2}{s^2 |\hat{h}(0)|^2 (t-k)} < \left[ \frac{\epsilon}{\sqrt{n_k T^{2k}}} \right]^2,
\]
and hence (3) is satisfied.

Since \( k + 1 \leq m \leq t < j \), we see that (1), (2) and (4) are satisfied too.

To finish the proof, we use our Claim to construct a vector \( z \) whose orbit \( \text{Orb}(T, z) \) has \( e_0 \) as a limit point.

Take \( k = 1 \) and \( \epsilon = \frac{1}{2} \) in our Claim. We then have positive integers \( j \) and \( m \) with \( j > m \) so that (1) through (4) are satisfied. For notational simplicity, we can assume that \( m = 2 \) and \( j = 3 \), because we can certainly achieve that by taking a subsequence of \( \{n_i\} \) having the first three terms \( n_k, n_m \) and \( n_j \). Hence, \( 2n_2 < n_3, w_{-(n_3-n_2-1)} \cdot \ldots \cdot w_0 < \frac{1}{2\sqrt{\|T\|} \sqrt{n_2}}, \) and
\[
\frac{1}{w_1 \cdots w_{n_3}} < \frac{1}{2\|T\|^{2n_2}}.
\]

Inductively, for every odd integer \( k \geq 1 \), we take \( \epsilon = 2^{-\frac{k+1}{2}} \) in the Claim, and by choosing a subsequence of \( \{n_i\} \), we get three consecutive integers \( k < m < j \) such that (1) through (4) are satisfied. In this way we can assume that the original sequence \( \{n_i\} \) satisfies that for any \( q \geq 1 \):

1. \( n_{2q-1} < n_{2q} \),
2. \( 2n_{2q} < n_{2q+1} \),
3. \( w_{-(n_{2q+1}-n_{2q-1})} \cdot \ldots \cdot w_0 < \frac{1}{2^q \sqrt{n_{2q-1}}\sqrt{n_{2q+1}}} \),
4. \( \frac{1}{w_1 \cdots w_{n_{2q+1}}} < \frac{1}{2^q \|T\|^{2n_{2q}}} \).

Let \( m_i = n_{2i+1} - n_{2i} + n_{2i-1} \) for \( i \geq 1 \) and set \( z = \sum_{i=1}^{\infty} \frac{1}{w_1 \cdots w_{m_i}} e_{m_i} \). Then,
$$\|z\|^2 = \sum_{i=1}^{\infty} \left[ \frac{1}{w_1 \cdots w_{m_i}} \right]^2 = \sum_{i=1}^{\infty} \left[ \frac{w_{m_i+1} \cdots w_{n_{2i+1}}}{w_1 \cdots w_{n_{2i+1}}} \right]^2$$

$$\leq \sum_{i=1}^{\infty} \left[ \frac{\|T\|_{n_{2i+1}}}{w_1 \cdots w_{n_{2i+1}}} \right]^2 \leq \sum_{i=1}^{\infty} \left[ \frac{\|T\|_{n_{2i}}}{2^i \|T\|_{2n_{2i}}} \right]^2 \quad \text{(by (4))}$$

$$< \sum_{i=1}^{\infty} \frac{1}{4^i \|T\|_{2n_{2i}}} < \sum_{i=1}^{\infty} \frac{1}{4^i} < \infty,$$

so clearly $z$ is in $\ell^2(\mathbb{Z})$. Now,

$$T^{m_k}z = \sum_{i=1}^{k-1} w_{-(m_k-m_i-1)} \cdots w_0 e_{m_i-m_k} + e_0 + \sum_{i=k+1}^{\infty} \frac{1}{w_1 \cdots w_{m_i-m_k}} e_{m_i-m_k},$$

and so

$$\|T^{m_k}z - e_0\|^2 = \sum_{i=1}^{k-1} (w_{-(m_k-m_i-1)} \cdots w_0)^2 + \sum_{i=k+1}^{\infty} \left( \frac{1}{w_1 \cdots w_{m_i-m_k}} \right)^2.$$

To estimate the first summation, we first note that its subindex $m_k - m_i - 1 = n_{2k+1} - n_{2k} + n_{2k-1} - m_i - 1 = (n_{2k+1} - n_{2k}) - (n_{2k-1} - m_i)$.

Since the summation index runs between 1 and $k-1$, we have $m_i \leq m_{k-1} = n_{2k-1} - n_{2k-2} + n_{2k-3}$, and so $n_{2k-1} - m_i \geq 0$. Hence, using (3) and the fact that $\{n_i\}$ is increasing and so $n_{2k-1} \geq 2k - 1 > k - 1$, we have that

$$\sum_{i=1}^{k-1} (w_{-(m_k-m_i-1)} \cdots w_0)^2 \leq \sum_{i=1}^{k-1} \|T\|^{2(n_{2k-1}-m_i)} (w_{-(n_{2k+1}-n_{2k})} \cdots w_0)^2$$

$$< \sum_{i=1}^{k-1} \frac{1}{4^i n_{2k-1}} < \frac{1}{4^i}.$$
Note that the number of weights in the numerator of the previous expression is given by 
\[ n_{2i+1} - m_i + m_k = n_{2i+1} - (n_{2i+1} - n_{2i} + n_{2i-1}) + m_k < n_{2i} + m_k < 2n_{2i}, \]
because \( i \geq k + 1 \) and \( m_k < n_{2k+1} \) by its definition.

Hence,
\[
\sum_{i=k+1}^{\infty} \left( \frac{1}{w_1 \cdot \ldots \cdot w_{m_i} - m_k} \right)^2 < \sum_{i=k+1}^{\infty} \left( \frac{\|T\|^{2n_{2i}}}{w_1 \cdot \ldots \cdot w_{n_{2i+1}}} \right)^2 < \sum_{i=k+1}^{\infty} \left( \frac{\|T\|^{2n_{2i}}}{2^i \|T\|^{2n_{2i}}} \right)^2 \text{ by (4)} \]
\[
= \sum_{i=k+1}^{\infty} \frac{1}{4^i} < \frac{1}{3 \cdot 4^k}.
\]

By combining both estimates, we see that \( \|T^{m_k} z - e_0\| < \frac{1}{4^k} + \frac{1}{3 \cdot 4^k} \), which goes to 0 as \( k \to \infty \). Hence, \( e_0 \) is a limit point of the orbit Orb\((T, z)\).

\[\square\]

A much simpler argument for showing (ii) implies (i) can be made if we require the bilateral weighted shift having \( e_0 \) as a limit point of Orb\((T, x)\) to be invertible.

Since \( e_0 \) is a limit point, there exists a sequence of integers \( n_k \not\to \infty \) such that \( \|T^{n_k} x - e_0\| < \frac{1}{2^k} < \frac{1}{2} \) for all \( k \geq 1 \). Thus we have shown previously that \( |w_1 \cdot w_2 \cdot \ldots \cdot w_{n_k} x(n_k) - 1| < \frac{1}{2} \) for all \( k \geq 1 \), and consequently that \( \prod_{j=1}^{n_k} w_j \to \infty \).

It also follows that \( |w_{-(n_k-n_1-1)} \cdot \ldots \cdot w_{-1} \cdot w_0 \cdot w_1 \cdot \ldots \cdot w_{n_1} x(n_1)| < \frac{1}{2^k} \) for all \( k \geq 2 \), and thus since \( \frac{1}{w_1 \cdot \ldots \cdot w_{n_1} |x(n_1)|} < 2 \) we observe that \( w_{-(n_k-n_1-1)} \cdot \ldots \cdot w_{-1} \cdot w_0 < \frac{1}{2^k} \cdot 2 = \frac{1}{2^{k-1}} \) for all \( k \geq 2 \).

Furthermore, \( w_{-n_k} \cdot \ldots \cdot w_{-1} \cdot w_0 < \frac{w_{-n_k} \cdot \ldots \cdot w_{-n_k-n_{k+1}}} {2^{k-1}} \) for all \( k \geq 2 \).

Hence \( 0 < w_{-n_k} \cdot \ldots \cdot w_{-1} \cdot w_0 < \frac{\|T^{n_{k+1}}\|} {2^{k-1}} \) for all \( k \geq 2 \). Letting \( k \to \infty \) we get that \( w_{-n_k} \cdot \ldots \cdot w_{-1} \cdot w_0 \to 0 \), and therefore \( \prod_{j=1}^{n_k} w_{-j} \to 0 \).

Thus there is a sequence \( n_k \not\to \infty \) so that \( \prod_{j=1}^{n_k} w_j \to \infty \) and \( \prod_{j=1}^{n_k} w_{-j} \to 0 \), so by Feldman's
criterion for invertible bilateral shifts \[17\] we have that \(T\) is hypercyclic.

From this we can now deduce the following statement, using the fact that if \(T\) is hypercyclic then so is \(T^{-1}\) (see Kitai \[24\]).

**Corollary 28.** If \(T : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})\) is an invertible bilateral weighted shift having an orbit with a non-zero limit point, then \(T^{-1}\) also has an orbit with a non-zero limit point.

As we have pointed out before, Salas \[33\] proved that a bilateral weighted shift \(T\) is hypercyclic if and only if for every \(\epsilon > 0\) and every \(q \in \mathbb{N}\) there exists \(n\) arbitrarily large such that for every \(j \in \mathbb{Z}\) with \(|j| \leq q\) we have \(\prod_{s=0}^{n-1} w_{s+j} > \frac{1}{\epsilon}\) and \(\prod_{s=1}^{n} w_{-s} < \epsilon\). We would like to point out that Theorem 27 offers a new equivalent condition to check whether or not an operator \(T\) is hypercyclic. To illustrate the applicability of our condition we offer an example of an operator \(T\) for which it is not easy to check hypercyclically using Salas’ condition, but quite easy using ours.

Let \(T\) be a bilateral weighted backward shift on \(\ell^2(\mathbb{Z})\) with weight sequence \(\{w_j\}_\infty^{-\infty}\) defined as follows. Let \(k_1 = 1\) and \(n_1 = 1\). Recursively we take \(k_j = n_{j-1} + j\) and \(n_j = n_{j-1} + k_j + 1\) for all \(j \geq 2\). Let \(w_0 = 1\). We define the weights with positive indices in consecutive blocks, setting the \(j\)-th block to have \(k_j\) entries of 2’s followed by the entry \(\frac{1}{2^j}\).

Similarly, we define the weights with negative indices in consecutive blocks having the following entries:

- **block 1:** one weight with value \(\frac{1}{2^{1+l_1}}\), followed by \(l_1\) entries of 2’s;
- **block 2:** \(\frac{1}{2^{1+l_2}}\), then \(l_3\) entries of 2’s; \(\frac{1}{2^{1+l_3}}\), then \(l_2\) entries of 2’s;
- **block 3:** \(\frac{1}{2^{1+l_4}}\), then \(l_6\) entries of 2’s; \(\frac{1}{2^{1+l_5}}\), then \(l_5\) entries of 2’s; \(\frac{1}{2^{1+l_4}}\), then \(l_4\) entries of 2’s, where

\[
\begin{align*}
l_1 &= n_2 - n_1 - 2, \\
l_2 &= n_3 - n_2 - l_1 - 3, \\
l_3 &= n_3 - n_1 - (l_1 + l_2) - 4, \\
l_4 &= n_4 - n_3 - (l_1 + l_2 + l_3) - 5, \\
l_5 &= n_4 - n_2 - (l_1 + l_2 + l_3 + l_4) - 6, \\
l_6 &= n_4 - n_1 - (l_1 + l_2 + l_3 + l_4 + l_5) - 7, 
\end{align*}
\]
Finally for all \( j \geq 1 \) we let \( p_j = -(l_1 + \ldots + l_j + 1 + \ldots + j) \).

We list below the values of the terms of the sequences \( n_j, k_j \) and \( l_j \), that we have used to construct the next two tables, where the \( n_j \) positions have been marked by *:

\[
\begin{align*}
  k_1 &= 1, \quad n_1 = 1, \\
  k_2 &= 3, \quad n_2 = 5, \quad l_1 = 2, \\
  k_3 &= 8, \quad n_3 = 14, \quad l_2 = 4, \quad l_3 = 3, \\
  k_4 &= 18, \quad n_4 = 33, \quad l_4 = 5, \quad l_5 = 8, \quad l_6 = 3.
\end{align*}
\]

We define the vector \( x \in \ell^2(\mathbb{Z}) \) by setting \( \hat{x}(n_i) = \frac{1}{2^{k_i}} \) for \( i \geq 1 \) and 0 otherwise.

Looking at the effect of applying \( T^{n_j} \) to the coordinates of the vector \( x \), we first note that the product of the positive weights in each block is 1. Thus, since \( \hat{x}(n_j) = \frac{1}{2^{k_j}} \) we have that \( T^{n_j}x(0) = 1 \), for all \( j \geq 1 \). Furthermore the non-zero coordinate of the vector \( x \) in the \( (j + 1) \)-block is shifted by \( T^{n_j} \) to the value \( \frac{1}{2^{k_j+1}} \cdot 2^{n_j} = \frac{1}{2^{k_j+1+j}} \cdot 2^{n_j} = \frac{1}{2^{k_j+1}} \). The other non-zero entries with positive indices of \( T^{n_j}x \) are given by the value \( \frac{1}{k_{j+s} - n_j} \), where each \( k_{j+s} - n_j > 2^{j+1} \) and the sequence \( \{k_{j+s} - n_j\}_{s>1} \) is strictly increasing to infinity as \( s \to \infty \).

Considering the entries with negative indices of \( T^{n_j}x \), we find that shifting by \( T^{n_j} \) moves the coordinate \( \hat{x}(n_j) \) in the 0-th position, while \( \hat{x}(n_s) \) for \( 1 \leq s < j \) is moved in the \( p - 1 \) position for some \( p \).

Furthermore the non-zero terms with negative indices of the vector \( T^{n_j}x \) are:

\[
2^{-[1+2+\ldots+(j+S_j)]}, 2^{-[1+2+\ldots+(j-1)+S_j)]}, \ldots, 2^{-[1+2+\ldots+(1+S_j)]},
\]

where \( S_j = \sum_{i=1}^{j-1} i \). Thus as \( j \to \infty \) we have that the entries above go to zero.

Finally to avoid any overlapping of the entries with negative indices while shifting by \( T^{n_j} \) (that is we want the non-zero entries of \( T^{n_j} \) to be to the left of the non-zero entries of \( T^{n_j-1} \)), we verify that \( k_j > n_{j-1} + 2 \).

Therefore by the observations listed above, we have that for the sequence \( \{n_j\}_{j \geq 1} \), the vector \( x \) in \( \ell^2(\mathbb{Z}) \) and the bilateral weighted shift \( T \) with weights \( \{w_j\}_{j \in \mathbb{Z}} \), \( T^{n_j}x \to e_0 \) as \( j \to \infty \). Hence by our theorem, \( T \) is hypercyclic.
Example of a bilateral weighted backward shift having an orbit Orb(T, x) with e_0 as a limit point

(i) The positive indices

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(ii) The negative indices

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CHAPTER 3

ORBITAL LIMIT POINTS AND HYPERCYCLICITY OF OPERATORS ON ANALYTIC FUNCTION SPACES

3.1 Introductory remarks

As we saw in Chapter 2, the result of Bourdon and Feldman that an operator $T$ with a somewhere dense orbit is everywhere dense, can be relaxed to a great extent in the case of weighted backward shifts on the sequence spaces $\ell^p$, for $1 \leq p < \infty$. Indeed we showed if a unilateral or bilateral weighted backward shift $T$ has an orbit $\text{Orb}(T,x)$ with a single non-zero limit point, then $T$ is hypercyclic. In fact, we were able to relax the above condition for the hypercyclicity of shifts even further to simply requiring that the orbit has infinitely many members contained in a ball whose closure avoids the zero vector. A natural question that arises is whether the above results can be generalized to other classes of operators.

In the field of hypercyclicity, the adjoints of multiplication operators and the composition operators are two well-studied classes of operators on analytic function spaces. In this chapter we show the above results for the shifts hold true also for the adjoints of multiplication
operators on the Bergman spaces. To achieve this we cannot borrow techniques used for the shift operators in Chapter 2, but instead we have to take a function theoretical approach. On the other hand, the situation for the composition operators on the Hardy space is totally different, as we show that a composition operator may have an orbit with a non-zero limit point and yet not be hypercyclic, and in fact not even cyclic. Recall that, an operator $T : X \to X$ is said to be cyclic if there is a vector $x \in X$ such that the linear span of $\text{Orb}(T, x)$ is dense in $X$. Thus in general, the existence of an orbit with a non-zero limit point does not determine the cyclic behavior of the operator.

In Section 3.2.2, we prove that hypercyclicity for the adjoint of a multiplication operator on the Bergman spaces is equivalent to having an orbit with infinitely many members contained in a ball whose closure avoids the zero vector. We then offer some considerations regarding the hypercyclicity of this class of operators on the Dirichlet space. In Section 3.3.2 we construct an example of a linear fractional composition operator on the Hardy space that has an orbit with a non-zero limit point, but the operator is not hypercyclic. Finally, motivated by the work in the previous sections we study in Section 3.4 the set of all vectors $x$ in $X$ whose orbit has a non-zero limit point.

### 3.2 The adjoints of multiplication operators

#### 3.2.1 Elementary properties and hypercyclicity of the adjoints of multiplication operators

In this section we define multiplication operators on analytic function spaces, examine some of their basic properties, and briefly characterize the hypercyclic behavior of the adjoints of multiplication operators.

**Definition 29.** Let $\Omega$ be a region in the complex plane $\mathbb{C}$. We define the Bergman space $A^2(\Omega)$ of $\Omega$ to be the space of all analytic functions $f : \Omega \to \mathbb{C}$ that are square integrable with respect to the area measure, i.e. $\|f\|^2 = \int |f|^2 dA < \infty$. 
We note that $A^2(\Omega)$ is a separable, infinite dimensional Hilbert space with the inner product $\langle f, g \rangle = \int f \overline{g} dA$ for all $f, g \in A^2(\Omega)$. Furthermore, it is sometimes useful to express the norm of the Bergman spaces in terms of power series coefficients. The well known proposition below establishes that, given an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ of the unit disk $\mathbb{D}$, the norm of the Bergman space $A^2(\mathbb{D})$ can also be expressed as $\|f\|^2 = \pi \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}$.

**Proposition 30.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function on the unit disk $\mathbb{D}$. Then $f \in A^2(\mathbb{D})$ if and only if $\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} < \infty$.

**Proof.** For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic on $\mathbb{D}$ we have

\[
\int_{\mathbb{D}} |f(z)|^2 dA(z) = \int_{\mathbb{D}} f(z) \overline{f(z)} dA(z) = \int_{\mathbb{D}} \left( \sum_{n=0}^{\infty} a_n z^n \right) \left( \sum_{n=0}^{\infty} a_n^{-1} \bar{z}^n \right) dA(z) = \int_0^{2\pi} \int_0^1 \left( \sum_{n=0}^{\infty} a_n r^n e^{in\theta} \right) \left( \sum_{m=0}^{\infty} a_m r^{m} e^{-im\theta} \right) r dr d\theta = \int_0^{2\pi} \int_0^1 \sum_{n,m=0}^{\infty} a_n a_m r^{n+m+1} e^{i(n-m)\theta} dr d\theta = \int_0^{2\pi} \int_0^1 \sum_{n,m=0}^{\infty} a_n a_m r^{n+m+1} e^{i(n-m)\theta} d\theta dr = \int_0^{\infty} \sum_{n,m=0}^{\infty} a_n a_m r^{n+m+1} \left( \int_0^{2\pi} e^{i(n-m)\theta} d\theta \right) dr = \int_0^{\infty} \sum_{n=0}^{\infty} a_n a_m r^{2n+1} 2\pi dr = 2\pi \sum_{n=0}^{\infty} |a_n|^2 \int_0^1 r^{2n+1} dr = \pi \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}.
\]
We now proceed in defining the multiplication operators on $A^2(\Omega)$. In the following we let $H$ denote a Hilbert space of analytic functions on $\Omega$ with bounded point evaluations.

**Definition 31.** For the Hilbert space $H$, we say an analytic function $\varphi : \Omega \to \mathbb{C}$ is a multiplier of $H$, if $\varphi f \in H$ whenever $f \in H$.

Each multiplier $\varphi$ of the Hilbert space $H$ induces a linear multiplication operator $M_{\varphi}$ on $H$ by the formula $M_{\varphi}(f) = \varphi \cdot f$ for all $f \in H$.

As it turns out, in the case of the Bergman spaces $A^2(\Omega)$, if $\varphi$ is a function in $H_\infty(\Omega) = \{ h : \Omega \to \mathbb{C} \text{ analytic}, \ |h|_\infty < \infty \}$, then for every $f \in A^2(\Omega)$

$$\int |\varphi f|^2 dA \leq |\varphi|_\infty^2 \int |f|^2 dA < \infty,$$

and thus $\varphi f \in A^2(\Omega)$. Thus every bounded analytic function on $\Omega$ is a multiplier of $A^2(\Omega)$ and it induces a linear multiplication operator $M_{\varphi}$. Moreover, the reverse inclusion also holds, which was established by Godefroy and Shapiro [20] for the more general setting of the Hilbert space $H$.

**Theorem 32.** Every multiplier of the Hilbert space $H$ is a bounded analytic function on $\Omega$.

**Proof.** Suppose $\varphi$ is a multiplier of $H$, and let $f$ be an element of $H$ that does not vanish identically on $\Omega$. If $z$ is not a zero of $f$, then

$$|\varphi(z)|^n \cdot |f(z)| = |M^nf(z)| = |\langle M^nf, k_z \rangle| \leq \|M\varphi\| \cdot \|f\| \cdot \|k_z\|,$$

where $k_z$ is the unique reproducing kernel for $z$ given by the Riesz Representation Theorem from the boundedness of evaluation maps, i.e. $f(z) = \langle f, k_z \rangle$ for all $f \in H$.

Taking $n$-th roots and letting $n \to \infty$, we obtain $|\varphi(z)| \leq \|M\varphi\|$ for all $z$ in $\Omega' = \Omega \setminus \{\text{zeros of } f\}$.

Now, since $\varphi$ is a multiplier of $H$, $\varphi \cdot f = g$ for some $g \in H$, and so $\varphi = \frac{g}{f}$ is analytic on $\Omega'$ and bounded by $\|M\varphi\|$.

Consequently, $\varphi$ has an analytic extension on $\Omega$, which is also bounded by $\|M\varphi\|$.

$\square$
The above theorem together with our previous considerations allow us to conclude that in the case of the Bergman spaces $A^2(\Omega)$, the set of multipliers is in fact equal to $H^\infty(\Omega)$. Furthermore, if $\varphi$ is a multiplier of $A^2(\Omega)$ then the induced multiplication operator has norm $\|M_\varphi\| = \|\varphi\|_\infty$. Clearly, since for every $f \in A^2(\Omega)$, $\|M_\varphi f\|^2 = \int |\varphi f|^2 dA \leq \|\varphi\|^2_\infty \int |f|^2 dA$, we have that $\|M_\varphi\| \leq \|\varphi\|_\infty$.

To show the reverse inequality, let $k_z$ be the reproducing kernel of $z$. Then for each $f \in A^2(\Omega)$ and $z \in \Omega$ we have

$$|\varphi(z)| \cdot |\langle f, k_z \rangle| = |\varphi(z)| \cdot |f(z)| = |\langle \varphi f, k_z \rangle| = |\langle M_\varphi f, k_z \rangle| \leq \|M_\varphi\| \cdot \|f\| \cdot \|k_z\|.$$ 

Recall that for any $g$ in a Hilbert space, $\|g\| = \sup_{\|f\|=1} |\langle g, f \rangle|$. Then, by taking supremum in the above inequality over all $f \in A^2(\Omega)$ with $\|f\| = 1$, we obtain $|\varphi(z)| \leq \|M_\varphi\|$ for all $z \in \Omega$. Thus $\|\varphi\|_\infty \leq \|M_\varphi\|$, which completes the proof.

While the statement in Theorem 32, as well as the inequality $\|M_\varphi\| \geq \|\varphi\|_\infty$, hold in the general case of the Hilbert space $H$, we observe that the proof of the converse and the reverse inequality were intrinsically linked to the norm of $A^2(\Omega)$. Thus the question arises as to whether the above results can be generalized to other analytic function spaces.

We remark that the same conclusion about the norm of $M_\varphi$, that $\|M_\varphi\| = \|\varphi\|_\infty$, holds also for the Hardy space $H^2(\mathbb{D})$ of the open unit disk $\mathbb{D}$, which consists of all analytic functions $f$ on $\mathbb{D}$ with $\lim_{r \to 1} \left( \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\theta})|^2 d\theta \right)^{1/2} < \infty$. Furthermore, the set of multipliers for the Hardy space $H^2(\mathbb{D})$ consists, analogous to the Bergman spaces, of all bounded analytic functions on $\mathbb{D}$.

However, this is not the case for all analytic function spaces. We note that in the case of the Dirichlet space $\text{Dir}(\Omega)$, consisting of all analytic functions $f : \Omega \to \mathbb{C}$ on the region $\Omega$ with $\|f\|^2 = |f(0)|^2 + \frac{1}{\pi} \int_{\Omega} |f'|^2 dA < \infty$, not every function in $H^\infty(\Omega)$ is a multiplier of $\text{Dir}(\Omega)$. As Theorem 32 clearly states, we certainly have that the set of multipliers is contained in $H^\infty(\Omega)$, but the reverse inclusion does not hold. Furthermore, in section 3.2.2 we show that the multiplication operator induced by $\varphi(z) = z$ on $\text{Dir}(\mathbb{D})$ has norm...
\[ \|M_z\| = \sqrt{\frac{3}{2}} > 1 = \|z\|_\infty, \] and so the Dirichlet space fails to share another property with the Bergman and Hardy spaces.

We now move on to summarizing the hypercyclicity results for the adjoints of the multiplication operators on the Bergman, Hardy and Dirichlet spaces.

As is made evident by the next theorem, the relation of the set of multipliers to the space of bounded analytic functions plays a crucial role in the hypercyclicity phenomena on the aforementioned analytic function spaces. However, it is not the multiplication operator \( M_{\varphi} \) that exhibits hypercyclic behavior on these three spaces, but rather its adjoint \( M_{\varphi}^* \). In fact, no multiplication operator \( M_{\varphi} \) can be hypercyclic since with a little work one can show that for any multiplier \( \varphi \) having \( |\varphi(\alpha)| \neq 1 \) for some \( \alpha \), a function \( g \) in the closure of an orbit \( \text{Orb}(M_{\varphi}, f) \) must vanish at \( \alpha \). We note that convergence in norm implies pointwise convergence for all of the above three spaces of analytic functions.

In [20], Godefroy and Shapiro offer the following characterization for the hypercyclicity of the adjoints of multiplication operators.

**Theorem 33.** Suppose every bounded analytic function \( \varphi \) on \( \Omega \) is a multiplier of the Hilbert space \( H \) and \( \|M_{\varphi}\| = \|\varphi\|_\infty \). Then for each such non-constant \( \varphi \), the operator \( M_{\varphi}^* \) is hypercyclic if and only if \( \varphi(\Omega) \) intersects the unit circle.

We note that the above theorem asserts that in the case of the Bergman and Hardy spaces, the adjoint of the multiplication operator \( M_{\varphi}^* \) is hypercyclic if and only if \( \varphi(\Omega) \cap \partial D \neq \emptyset \). On the other hand, in the case of the Dirichlet spaces we cannot once again draw the same conclusion. Whilst [20] Theorem 4.5] states that for a non-constant multiplier \( \varphi \), the adjoint operator \( M_{\varphi}^* \) is hypercyclic whenever \( \varphi(\Omega) \) intersects the unit circle, the converse does not hold. We demonstrate this in the next section by providing an example of an operator \( M_{\varphi}^* \) on \( \text{Dir}(D) \) that is hypercyclic, but nevertheless has \( \varphi(D) \cap \partial D = \emptyset \).

Finally, we would like to remark that even though in the case of the Dirichlet spaces, \( \varphi(\Omega) \) need not intersect the unit disk if \( M_{\varphi}^* \) is hypercyclic, we can however infer that the
closure of $\varphi(\Omega)$ must intersect the unit circle. Kitai [24] showed that if $T$ is a hypercyclic operator on a Hilbert space, then the spectrum of $T$ intersects the unit circle. Thus, since one can show that in the case of the multiplication operator $M_\varphi$ on the Dirichlet space $\text{Dir}(\Omega)$ the spectrum $\sigma(M_\varphi^*) = \sigma(M_\varphi) = \overline{\varphi(\Omega)}$, we can deduce that if $M_\varphi^*$ is hypercyclic then $\overline{\varphi(\Omega)} \cap \partial \mathbb{D} \neq \emptyset$.

**Proposition 34.** The spectrum of the multiplication operator $M_\varphi$ on $A^2(\Omega)$, $H^2(\mathbb{D})$ or $\text{Dir}(\Omega)$ is $\sigma(M_\varphi) = \overline{\varphi(\Omega)}$.

**Proof.** In the following we present the proof for the Dirichlet space $\text{Dir}(\Omega)$, and note that the proof for the other two spaces can be derived in a similar fashion.

Suppose $\varphi : \Omega \rightarrow \mathbb{C}$ is a multiplier on $\text{Dir}(\Omega)$, and consider the induced multiplication operator $M_\varphi$.

Let $\lambda \in \varphi(\Omega)$, i.e. there exists $z_0 \in \Omega$ so that $\varphi(z_0) = \lambda$. Since $(M_\varphi - \lambda I)(f) = M_\varphi f - \lambda f = \varphi f - \lambda f = (\varphi - \lambda) f$, we have that $(M_\varphi - \lambda I)(f)(z_0) = (\varphi(z_0) - \lambda) \cdot f(z_0) = 0$ for all $f \in \text{Dir}(\Omega)$. Since the constant function $f = 1$ in $\text{Dir}(\Omega)$ does not vanish at $z_0$, we get that $M_\varphi - \lambda I$ is not surjective on $\text{Dir}(\Omega)$, i.e. $\lambda \in \sigma(M_\varphi)$. Since the spectrum $\sigma(M_\varphi)$ is compact, we can further conclude that $\overline{\varphi(\Omega)} \subset \sigma(M_\varphi)$.

To show the reverse inclusion, suppose $\lambda \notin \overline{\varphi(\Omega)}$. Then $\left\| \frac{1}{\varphi - \lambda} \right\|_\infty < \infty$, as otherwise there exists a sequence $\{z_n\}$ in $\Omega$ so that $|\varphi(z_n) - \lambda| \rightarrow 0$, which contradicts $\lambda \notin \overline{\varphi(\Omega)}$. Thus, $\frac{1}{\varphi - \lambda} \in H^\infty(\Omega)$. We will now show that $M_\varphi - \lambda I$ is invertible and hence $\lambda \notin \sigma(M_\varphi)$.

We first note that if $f$ is in the kernel of the operator $M_\varphi - \lambda I$, then $(\varphi - \lambda) f = 0$. But $\varphi(z) - \lambda \neq 0$ for every $z \in \Omega$, and thus $f = 0$ on $\Omega$. So $M_\varphi - \lambda I$ is injective.

On the other hand, if we are able to show that $\frac{1}{\varphi - \lambda}$ is a multiplier of $\text{Dir}(\Omega)$, then for every $g \in \text{Dir}(\Omega)$ the function $\frac{1}{\varphi - \lambda} g \in \text{Dir}(\Omega)$, and hence $(M_\varphi - \lambda I) \left( \frac{1}{\varphi - \lambda} g \right) = (\varphi - \lambda) \frac{1}{\varphi - \lambda} g = g$. Thus $M_\varphi - \lambda I$ is surjective, and so $\lambda \notin \sigma(M_\varphi)$, which gives the reverse inclusion.

To complete the proof, we note it suffices to show that for $g \in \text{Dir}(\Omega)$, the func-
\[ \frac{1}{(\varphi - \lambda^2)^2} \varphi' \frac{g'}{\varphi - \lambda} + \frac{1}{\varphi - \lambda}g' \] is square integrable with respect to the area measure. Since \( \varphi \) is a multiplier of \( \text{Dir}(\Omega) \) and \( g \in \text{Dir}(\Omega) \), we can deduce that \( (\varphi g)' \in L^2(dA) \). Furthermore, Theorem 32 also gives that \( \varphi \in H^\infty(\Omega) \), and thus we have that \( \varphi g' \in L^2(dA) \). Therefore, \( \varphi'g = (\varphi g)' - \varphi g' \in L^2(dA) \). Consequently, since we have shown that \( \frac{1}{\varphi - \lambda} \in H^\infty(\Omega) \), it follows that the above derivative of \( \frac{1}{\varphi - \lambda}g \) is also square integrable. Thus \( \frac{1}{\varphi - \lambda}g \in \text{Dir}(\Omega) \) for all \( g \in \text{Dir}(\Omega) \), which completes the proof.

\[ \square \]

### 3.2.2 A Zero-One Law for the hypercyclicity of the adjoints of multiplication operators

Let \( \Omega \) be a region in the complex plane \( \mathbb{C} \). Recall that the Bergman space \( A^2(\Omega) \) of \( \Omega \) is the Hilbert space of all analytic functions \( f \) on \( \Omega \) such that \( |f|^2 \) is integrable with respect to the area measure \( dA \) on \( \Omega \). The norm of this space is given by \( \|f\|^2 = \int |f|^2 \, dA \). We say an analytic function \( \varphi \) on \( \Omega \) is a multiplier, if \( \varphi \cdot f \) is in \( A^2(\Omega) \) whenever \( f \) is in \( A^2(\Omega) \). It is easy to verify that if \( \varphi \) is a function in \( H^\infty(\Omega) = \{ h : \Omega \to \mathbb{C} \text{ analytic} \mid \|h\|_\infty < \infty \} \), then \( \varphi \) is a multiplier and it induces a bounded linear multiplication operator \( M_\varphi : A^2(\Omega) \to A^2(\Omega) \) given by \( M_\varphi(f) = \varphi \cdot f \). Indeed, one can show that the set of all multipliers for the Bergman space is equal to \( H^\infty(\Omega) \).

A result of Godefroy and Shapiro states that for a non-constant multiplier \( \varphi \), the adjoint \( M_\varphi^* \) of the multiplication operator is hypercyclic if and only if \( \varphi(\Omega) \cap \partial \mathbb{D} \neq \emptyset \). Using the above result we will show that if \( M_\varphi^* \) has an orbit with a non-zero limit point, then it is hypercyclic. In fact, a more general result holds:

**Theorem 35.** Let \( \Omega \) be a region in \( \mathbb{C} \) and \( \varphi \in H^\infty(\Omega) \) be a non-constant function. Suppose \( M_\varphi^* : A^2(\Omega) \to A^2(\Omega) \) has an orbit \( \text{Orb}(M_\varphi^*, f) \) which has infinitely many members contained in an open ball \( B(h, r) \) where \( 0 < r < \|h\| \). Then \( M_\varphi^* \) is hypercyclic.

**Proof.** By our hypothesis, there exists an increasing sequence \( \{n_k\}_{k \geq 1} \) of positive integers
such that \( \| (M_\varphi^*)^{nk} f - h \| < r \) for all \( k \geq 1 \). In light of the above result of Godefroy and Shapiro, to prove that \( M_\varphi^* \) is hypercyclic, it suffices to show that \( \varphi(\Omega) \cap \partial \mathbb{D} \neq \emptyset \). By way of contradiction, we suppose \( \varphi(\Omega) \cap \partial \mathbb{D} = \emptyset \). Since \( \varphi(\Omega) \) is an open connected subset of \( \mathbb{C} \), we have only two cases to consider:

**Case 1:** Suppose \( |\varphi(z)| < 1 \) for all \( z \) in \( \Omega \).

We note that for any \( g \in A^2(\Omega) \) with \( \|g\| \leq 1 \) we have that \( \| (M_\varphi^*)^{nk} f - h \| < r \) for all \( k \geq 1 \).

Furthermore, \( \langle (M_\varphi^*)^{nk} f, g \rangle = \langle f, M_\varphi^{nk} g \rangle = \langle f, \varphi^{nk} g \rangle = \int f : \varphi^{nk} \cdot \overline{g} \, dA \), where \( |f : \varphi^{nk} \cdot \overline{g}| \leq |f \cdot \varphi^{nk}| \cdot |\overline{g}| < |f \cdot g| \), which is a function in \( L^1(dA) \) by the Cauchy-Schwarz inequality. Hence by the Dominated Convergence Theorem, we have \( \lim_{k \to \infty} \langle (M_\varphi^*)^{nk} f, g \rangle = \int \lim_{k \to \infty} f : \varphi^{nk} \cdot \overline{g} \, dA = 0 \) for all \( g \in A^2(\Omega) \) with \( \|g\| \leq 1 \). This implies along with \( (i) \) that \( |\langle h, g \rangle| \leq r \) for all \( g \in A^2(\Omega) \) with \( \|g\| \leq 1 \). But if \( g = \frac{h}{\|h\|} \), then \( |\langle h, g \rangle| = \|h\| \) which is a contradiction.

**Case 2:** Suppose \( |\varphi(z)| > 1 \) for all \( z \) in \( \Omega \).

As in Case 1 given any \( g \in A^2(\Omega) \) with \( \|g\| \leq 1 \) we have \( |\langle (M_\varphi^*)^{nk} f - h, g \rangle| < r \) for all \( k \geq 1 \). Now, since \( f \in A^2(\Omega) \) and \( \varphi \in H^\infty(\Omega) \) with \( |\varphi(z)| > 1 \) on \( \Omega \) we have that
\[
\int \left| \frac{1}{\varphi^{nk}} f \right|^2 \, dA \\
\leq \int |f|^2 \, dA.
\]
Thus \( \frac{f}{\varphi^{nk}} \in A^2(\Omega) \), and consequently \( \lim_{k \to \infty} \left\| \frac{f}{\varphi^{nk}} \right\| = \left( \int \lim_{k \to \infty} \left| \frac{f}{\varphi^{nk}} \right|^2 \, dA \right)^{1/2} = 0 \). We now want to show that \( f = 0 \). If this is not the case, we let \( g_k = \frac{f}{\|f/\varphi^{nk}\|} \) be a unit vector in \( A^2(\Omega) \), and consequently
\[
\langle (M_\varphi^*)^{nk} f, g_k \rangle = \int f \varphi^{nk} \overline{g_k} \, dA = \int f \varphi^{nk} \frac{\overline{f/\varphi^{nk}}}{\|f/\varphi^{nk}\|} \, dA = \left\| \frac{f}{\varphi^{nk}} \right\|^{-1} \int |f|^2 \, dA.
\]
Hence we have that \( \langle (M_\varphi^*)^{nk} f, g_k \rangle \to \infty \) as \( k \to \infty \). But we observe that \( \left| \langle (M_\varphi^*)^{nk} f, g_k \rangle \right| \leq \left| \langle (M_\varphi^*)^{nk} f, g_k \rangle - \langle h, g_k \rangle \right| + |\langle h, g_k \rangle| \), and since we assumed \( \left| \langle (M_\varphi^*)^{nk} f - h, g_k \rangle \right| < r \) for all \( k \geq 1 \) we have that \( \lim_{k \to \infty} |\langle h, g_k \rangle| = \infty \), which contradicts that \( \|g_k\| = 1 \). Thus \( f = 0 \) and so since \( (M_\varphi^*)^{nk} f = 0 \) for all \( k \geq 1 \), we get that \( 0 \in B(h, r) \), which is a contradiction.

Therefore by Cases 1 and 2, we get that \( \varphi(\Omega) \cap \partial \mathbb{D} \neq \emptyset \) and thus \( M_\varphi^* \) is hypercyclic.

\[\square\]
From the last theorem, one can deduce the following corollary.

**Corollary 36.** Let $\Omega$ be a region in $\mathbb{C}$ and $\varphi \in H^\infty(\Omega)$ be a non-constant function. If $M_\varphi^* : A^2(\Omega) \to A^2(\Omega)$ has an orbit with a non-zero limit point, then $M_\varphi^*$ is hypercyclic.

While the converse of Corollary 36 is absolutely trivial, if we only require $M_\varphi^*$ to be cyclic and not hypercyclic we cannot conclude that $M_\varphi^*$ has an orbit with a non-zero limit point.

**Example 1.** A cyclic $M_\varphi^*$ on $A^2(\mathbb{D})$ having no non-zero orbital limit point.

Consider the adjoint of the multiplication operator $M_\varphi^*$ on the Bergman space $A^2(\mathbb{D})$ with $\varphi(z) = z$. Since $\varphi$ is a non-constant multiplier we have that $M_\varphi^*$ is supercyclic (see Godefroy and Shapiro [20, page 248]) and thus cyclic. But $|\varphi(z)| < 1$ on $\mathbb{D}$, so by Theorem 35 we can infer that $M_\varphi^*$ does not have an orbit with a non-zero limit point. □

We now turn our attention to the **Hardy space** $H^2(\mathbb{D})$ for the open unit disk $\mathbb{D}$, which consists of all analytic functions $f$ on $\mathbb{D}$ with $\|f\|_2 = \left(\frac{1}{2\pi} \int |f(re^{i\theta})|^2 \, d\theta\right)^{1/2} < \infty$. This limit is defined to be the norm of $f$. Clearly, every function $\varphi \in H^\infty(\mathbb{D})$ is a multiplier for $H^2(\mathbb{D})$; that is if $\varphi \in H^\infty(\mathbb{D})$ then $\varphi \cdot H^2(\mathbb{D}) \subseteq H^2(\mathbb{D})$, and hence $M_\varphi : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ given by $M_\varphi(f) = \varphi \cdot f$ is a bounded linear operator. As in the case of the Bergman spaces given $\varphi$ to be a non-constant multiplier, the same result of Godefroy and Shapiro [20, Theorem 4.9] gives us that the adjoint $M_\varphi^*$ is hypercyclic if and only if $\varphi(\Omega) \cap \partial \mathbb{D} \neq \emptyset$. However, the arguments in Cases 1 and 2 of the proof of Theorem 35 do not apply directly for the Hardy space.

Another interesting space is the **Dirichlet space** $\text{Dir}(\Omega)$ consisting of all analytic functions $f : \Omega \to \mathbb{C}$ on the region $\Omega$ with $\|f\|^2 = |f(0)|^2 + \frac{1}{\pi} \int |f'|^2 \, dA < \infty$. As different from the Bergman and the Hardy spaces, not every function in $H^\infty(\Omega)$ is a multiplier of $\text{Dir}(\Omega)$.

Suppose $\varphi \in H^\infty(\Omega)$ and $\varphi$ is a non-constant multiplier of $\text{Dir}(\Omega)$. If $\varphi(\Omega) \cap \partial \mathbb{D} \neq \emptyset$, then a result of Godefroy and Shapiro [20, Theorem 4.5] gives us that the adjoint $M_\varphi^*$ of the multiplication operator $M_\varphi : \text{Dir}(\Omega) \to \text{Dir}(\Omega)$ is hypercyclic, however the converse does not hold and as a result, the argument in the proof of Theorem 35 does not apply. We offer
below an example of an operator $M^*_\varphi$ on $\text{Dir}(\Omega)$ that is hypercyclic but $\varphi(\Omega) \cap \partial\mathbb{D} = \emptyset$.

**Example 2.** A hypercyclic $M^*_\varphi$ on $\text{Dir}(\mathbb{D})$ having $\varphi(\mathbb{D}) \cap \partial\mathbb{D} = \emptyset$.

We will show that given the identity function $\varphi(z) = z$ on the open unit disk $\mathbb{D}$ we have that $M^*_\varphi : \text{Dir}(\mathbb{D}) \rightarrow \text{Dir}(\mathbb{D})$ is hypercyclic, but obviously $|\varphi(z)| < 1$ for all $z$ in $\mathbb{D}$. Consider $f_n : \mathbb{D} \rightarrow \mathbb{C}$, with $f_n(z) = z^n$ for $n \geq 1$. Then each $f_n$ is analytic and we have $\|f_n\|^2 = \|z^n\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |n \cdot z^{n-1}|^2 \, d\theta = \frac{n^2}{2\pi} \int_0^{2\pi} |z|^{n-1} \cdot e^{i(n-1)\theta}|^2 r \, d\theta = n$. Thus $\|z^n\| = \sqrt{n}$ for $n \geq 1$. Define further $e_n(z) = \frac{z^n}{\sqrt{n}}$, if $n \geq 1$ and $e_0(z) = 1$ on $\mathbb{D}$. Clearly $\|e_j\| = 1$ for all $j \geq 0$. Also, if $n \neq m$ we have that $\langle e_n, e_m \rangle = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{n} \cdot \sqrt{m} \cdot z^{n-1} \cdot \overline{z}^{m-1} \, dA = 0$. Thus \{e_n\}_{n \geq 1} is an orthonormal basis for $\text{Dir}(\mathbb{D})$.

Let $M_z : \text{Dir}(\mathbb{D}) \rightarrow \text{Dir}(\mathbb{D})$ be defined by $M_z f(z) = z \cdot f(z)$. Then $M_z e_n = \frac{z^n + 1}{\sqrt{2}} \frac{e^{n+1}}{\|z^n\|} = \frac{\|z^{n+1}\|}{\|z^n\|} = \sqrt{\frac{n+1}{n}} e_{n+1}$ for $n \geq 1$. Letting $w_n = \sqrt{\frac{n+1}{n}}$ if $n \geq 2$ and $w_1 = 1$, we have $M_z e_n = w_{n+1} \cdot e_{n+1}$ for all $n \geq 0$.

We can now identify $e_j$ in $\text{Dir}(\mathbb{D})$ with $e_j = (0, 0, \ldots, 0, 1, 0, \ldots)$ in $\ell^2$, and consequently the analytic function $f = \sum_{n=0}^{\infty} a_n e_n$ in $\text{Dir}(\mathbb{D})$ is identified with the vector $(a_0, a_1, a_2, \ldots)$ in $\ell^2$. Finally we can then identify the operator $M_z : \text{Dir}(\mathbb{D}) \rightarrow \text{Dir}(\mathbb{D})$ defined above with $M_z : \ell^2 \rightarrow \ell^2$, where $M_z(a_0, a_1, \ldots) = (0, w_1 a_0, w_2 a_1, \ldots)$ is the unilateral forward shift with weight sequence \{w_n\}_{n \geq 1}.

Note that for $(a_0, a_1, a_2, \ldots)$ and $(b_0, b_1, b_2, \ldots)$ in $\ell^2$, we have

$$\langle M^*_z(b_0, b_1, \ldots), (a_0, a_1, \ldots) \rangle = \langle (b_0, b_1, \ldots), M_z(a_0, a_1, \ldots) \rangle = \langle (b_0, b_1, \ldots), (0, w_1 a_0, w_2 a_1, \ldots) \rangle = \sum_{i=0}^{\infty} w_{i+1} \cdot a_i \cdot b_{i+1} = \langle (w_1 b_1, w_2 b_2, \ldots), (a_0, a_1, \ldots) \rangle.$$  

So since the above identity holds for any $(a_0, a_1, a_2, \ldots)$ in $\ell^2$, we conclude that $M^*_z(b_0, b_1, b_2, \ldots) = (w_1 b_1, w_2 b_2, w_3 b_3, \ldots)$.

Thus $M^*_z : \text{Dir}(\mathbb{D}) \rightarrow \text{Dir}(\mathbb{D})$ is isometrically isomorphic to the unilateral backward weighted shift on $\ell^2$ with weights $w_n = \sqrt{\frac{n+1}{n}}$ if $n \geq 2$ and $w_1 = 1$. Note that $\|M^*_z\| = \sup_{n \geq 1} \{w_n\} = \sqrt{\frac{3}{2}}$. So since $\lim_{n \rightarrow \infty} (w_1 \cdots \cdot w_n) = \lim_{n \rightarrow \infty} \sqrt{\frac{n+1}{2}} = \infty$, Salas’ criterion for hypercyclicity of unilateral backward weighted shifts gives us that $M^*_z$ is hypercyclic. □
We would like to point out that the above example of the operator $M^*_z$ on $\text{Dir}(\mathbb{D})$ also shows that the well known Hypercyclicity Comparison Principle (see [34]) cannot be applied to conclude the hypercyclicity of adjoint operators. The Hypercyclicity Comparison Principle states that if $X$ and $Y$ are two normed spaces, $Y$ is continuously and densely embedded in $X$, and $T$ is a linear transformation on $X$ that maps the smaller space $Y$ to itself and is continuous in the topology of each space, then $T$ is hypercyclic on the larger space $X$ whenever $T$ is hypercyclic on $Y$.

Clearly in this setting $\text{Dir}(\mathbb{D}) \hookrightarrow H^2(\mathbb{D}) \hookrightarrow A^2(\mathbb{D})$ continuously and densely, and $M_z$ is a continuous linear operator on each of the these spaces, but it would be erroneous to conclude that $M^*_z$ is hypercyclic on the Hardy or Bergman space, given that $M^*_z$ is indeed hypercyclic on the Dirichlet space. In fact using Godefroy and Shapiro’s result [20] since the analytic function $\varphi(z) = z$ satisfies $\varphi(\mathbb{D}) \cap \partial \mathbb{D} = \emptyset$, we have that $M^*_z$ is not hypercyclic on $H^2(\mathbb{D})$, respectively $A^2(\mathbb{D})$. Indeed since the operation of taking the adjoint of an operator on a given space is intrinsically linked to the inner product of that space, different norms will induce different adjoints. Thus restricting the adjoint $M^*_z$ on $H^2(\mathbb{D})$ or $A^2(\mathbb{D})$ to the Dirichlet space will not produce the desired hypercyclic operator $M^*_z : \text{Dir}(\mathbb{D}) \to \text{Dir}(\mathbb{D})$ and hence one cannot apply the Hypercyclicity Comparison Principle.

### 3.3 The linear fractional composition operators

#### 3.3.1 Elementary properties and hypercyclicity of linear fractional composition operators

Having examined the relationship between hypercyclicity and possessing an orbit with a non-zero limit point for the adjoints of multiplication operators on the Bergman spaces, we now turn our attention to the class of composition operators on the Hardy space.

In the theory of composition operators it is useful to express the norm of the Hardy
space both in terms of the integral of its functional values and in terms of its power series coefficients. Having defined the Hardy space from the first viewpoint in section 3.2, we now focus on the latter perspective. We say that an analytic function $f$ on the unit disk $\mathbb{D}$ having the power series representation $\sum_{n=0}^{\infty} \hat{f}(n) z^n$ belongs to the Hardy space $H^2(\mathbb{D})$ if the sequence of power series coefficients is square summable, namely $\|f\|^2 = \sum_{n=0}^{\infty} \left| \hat{f}(n) \right|^2 < \infty$.

We define a linear fractional transformation on $\mathbb{D}$ to be an analytic map $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ with $\varphi(z) = \frac{az + b}{cz + d}$ subject to the condition $ad - bc \neq 0$, which is necessary and sufficient for $\varphi$ to be non-constant. Clearly, a linear fractional transformation $\varphi$ taking $\mathbb{D}$ into $\mathbb{D}$ is a restriction to the unit disk of an automorphism of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We denote by $\text{LFT}(\mathbb{D})$ the set of all linear fractional transformations of $\mathbb{D}$ and by $\text{Aut}(\mathbb{D})$ the set of automorphisms of $\mathbb{D}$. We note that the class of linear fractional transformations $\text{LFT}(\mathbb{D})$ is diverse enough to reflect the character of the more general situation of arbitrary analytic self-maps of the unit disk, and hence in the following we will focus our attention on composition operators induced by these rather well-behaved maps.

Given a linear fractional transformation $\varphi$, we define the linear fractional composition operator $C_\varphi$ on the Hardy space $H^2(\mathbb{D})$ by setting $C_\varphi(f) = f \circ \varphi$ for any $f$ in $H^2(\mathbb{D})$. As a consequence of the Littlewood Subordination Principle (see Shapiro [34, page 13]) we have that whenever $f$ belongs to the Hardy space so does $f \circ \varphi$. Thus the linear fractional composition operator $C_\varphi$ on $H^2(\mathbb{D})$ takes the Hardy space to itself. The principle also supplies a uniform estimate from which the boundedness of the linear operator $C_\varphi$ on $H^2(\mathbb{D})$ can be determined.

**Theorem 37** (Littlewood’s Theorem). Suppose $\varphi$ is an analytic self-map of the unit disk $\mathbb{D}$. Then $C_\varphi$ is a bounded linear operator on the Hardy space $H^2(\mathbb{D})$, and

$$\|C_\varphi\| \leq \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}}$$

In [34] Shapiro showed that the hypercyclic behavior of these composition operators is strongly influenced by the dynamical properties of the inducing linear fractional map $\varphi$. In
particular, we note that the location of the fixed point(s) of the map $\varphi$ plays an essential role for the hypercyclicity of the operator $C_\varphi$, so it is important to classify the linear fractional transformations with respect to their fixed point location.

We note that if $\varphi$ is a linear fractional transformation on $\hat{\mathbb{C}}$, then $\varphi$ has at least one and at most two fixed points in $\hat{\mathbb{C}}$. A map $\varphi$ is called parabolic if it has a single fixed point in $\hat{\mathbb{C}}$. Suppose $\varphi$ is parabolic and has its fixed point $\xi \in \mathbb{C}$. If $\psi$ is a linear fractional transformation that takes $\xi$ to $\infty$, then $\rho = \psi \circ \varphi \circ \psi^{-1}$ is also a linear fractional transformation on $\hat{\mathbb{C}}$ that fixes only the point $\infty$. Therefore $\psi(z) = z + \tau$ for some non-zero complex number $\tau$. Thus, every parabolic linear fractional map is conjugate to a translation.

If the map $\varphi$ is not parabolic, then it has two fixed points $\xi_1$ and $\xi_2$ in $\hat{\mathbb{C}}$. Let $\psi$ be a linear fractional map that takes $\xi_1$ to 0 and $\xi_2$ to $\infty$. Then the map $\rho = \psi \circ \varphi \circ \psi^{-1}$ is also a linear fractional transformation on $\hat{\mathbb{C}}$ and it fixes both 0 and $\infty$, so it must have the form $\rho(z) = \lambda z$ for some complex number $\lambda$. Thus, $\varphi(z) = \psi^{-1}(\lambda \psi(z))$ for all $z \in \mathbb{C}$, where $\lambda$ is called the multiplier for $\varphi$. We note that multipliers give the following classification for non-parabolic maps. If $|\lambda| \neq 1$, we say the linear fractional transformation $\varphi \in LFT(\mathbb{D})$ is elliptic if $|\lambda| = 1$, hyperbolic if $\lambda > 0$, and loxodromic if $\varphi$ is neither elliptic nor hyperbolic.

Thus, we identify four categories of linear fractional transformations on the unit disk $\mathbb{D}$: (1) parabolic: $\varphi$ is conjugate to a complex translation and has one attractive fixed point on $\partial \mathbb{D}$; (2) hyperbolic: $\varphi$ is conjugate to a positive dilation and has one attractive fixed point in $cl(\mathbb{D})$ and one repellent fixed point on $\partial \mathbb{D}$ or outside $cl(\mathbb{D})$; (3) elliptic: $\varphi$ is conjugate to a rotation and has one fixed point in $\mathbb{D}$ and the other fixed point outside $cl(\mathbb{D})$; (4) loxodromic: $\varphi$ is conjugate to a complex dilation and has one attractive fixed point in $\mathbb{D}$ and one repellent fixed point outside $cl(\mathbb{D})$.

As noted above, the dynamics of the linear fractional transformation $\varphi$ on $\mathbb{D}$ determine the hypercyclic behavior of the induced composition operator $C_\varphi$. The next result of Shapiro [34] establishes that composition operators induced by loxodromic and elliptic maps, as well as hyperbolic non-automorphisms with a fixed point in $\mathbb{D}$, are never hypercyclic.
Proposition 38. If φ is an analytic self-map of $\mathbb{D}$ that fixes a point $\xi$ in $\mathbb{D}$, then $C_{\varphi}$ is not hypercyclic on $H^2(\mathbb{D})$. Moreover, if φ is not an elliptic automorphism, then for each $f \in H^2(\mathbb{D})$, only the constant function $g = f(\xi)$ on $\mathbb{D}$ is in the closure of the orbit Orb($C_{\varphi}, f$).

Proof. Suppose the map φ fixes the point $\xi$ in $\mathbb{D}$ and $f \in H^2(\mathbb{D})$ is a hypercyclic vector for $C_{\varphi}$. Assume now that the function $g \in H^2(\mathbb{D})$ is in the closure of the orbit Orb($C_{\varphi}, f$), that is there exists a sequence $n_k \to \infty$ with $f \circ \varphi^{n_k} \to g$ in $H^2(\mathbb{D})$. Since convergence in $H^2(\mathbb{D})$ implies pointwise convergence, we have $f \circ \varphi^{n_k} \to g$ pointwise on $\mathbb{D}$, and in particular $f \circ \varphi^{n_k}(\xi) \to g(\xi)$. But $\xi$ is the fixed point of φ, and so $f \circ \varphi^{n_k}(\xi) = f(\xi)$, which implies $g(\xi) = f(\xi)$. Therefore, any function $g$ in the closure of Orb($C_{\varphi}, f$) must agree with $f$ at the fixed point $\xi$, and thus $\text{Orb}(C_{\varphi}, f) \neq H^2(\mathbb{D})$, a contradiction.

Moreover, if φ is not an elliptic automorphism, then $\xi \in \mathbb{D}$ is an attractive fixed point for φ, and so $\varphi^n(z) \to \xi$ for all $z \in \mathbb{D}$. Thus by continuity of $f$, we get that $f \circ \varphi^n(z) \to f(\xi)$ for all $z \in \mathbb{D}$. Thus, any $g$ in the closure of the orbit Orb($C_{\varphi}, f$) has $g(z) = f(\xi)$ for all $z \in \mathbb{D}$.

Furthermore, an application of the Hypercyclicity Criterion gives that parabolic and hyperbolic automorphisms of $\mathbb{D}$ always induce a hypercyclic composition operator on $H^2(\mathbb{D})$.

Theorem 39. If φ is a non-elliptic automorphism of $\mathbb{D}$, then $C_{\varphi}$ is hypercyclic on $H^2(\mathbb{D})$.

The following result of Shapiro [31] completes the characterization of the hypercyclicity of linear fractional composition operators on $H^2(\mathbb{D})$.

Theorem 40. Suppose φ is linear fractional non-automorphism of the unit disk $\mathbb{D}$ with no fixed point in $\mathbb{D}$.

(a) If φ is hyperbolic, then $C_{\varphi}$ is hypercyclic on $H^2(\mathbb{D})$.

(b) If φ is parabolic, then $C_{\varphi}$ fails to be hypercyclic on $H^2(\mathbb{D})$ in a very strong sense: Only constant functions can be limit points of $C_{\varphi}$ orbits.
As a final remark about the hypercyclic behavior of this class of operators, we note that a common mistake in the literature is to classify all composition operators induced by hyperbolic linear fractional transformations on the Hardy space as hypercyclic. Bourdon and Shapiro [9] showed that the location of the fixed points of the inducing map determines whether or not these composition operators are hypercyclic. For example, the functions \( \varphi(z) = \frac{1}{2}z \) and \( \gamma(z) = \frac{z}{2-z} \) are hyperbolic non-automorphisms with fixed points 0, \( \frac{3}{2} \) and 0, 1, respectively, but they induce composition operators on the Hardy space that are not hypercyclic. On the other hand the function \( \phi(z) = \frac{z + 2}{4 - z} \) is a hyperbolic non-automorphism with fixed points 1, 2 that induces a hypercyclic composition operator on \( H^2(\mathbb{D}) \). To check that the above linear fractional transformations are indeed hyperbolic, one easily verifies that the multiplier \( \lambda \) of each map is positive. We recall that the multiplier \( \lambda \) of a linear fractional transformation \( \rho \) is a complex number such that \( \Psi \circ \rho \circ \Psi^{-1}(z) = \lambda z \), where \( \Psi(z) = \frac{z - \xi_1}{z - \xi_2} \) for \( \xi_1, \xi_2 \) the fixed points of \( \rho \).

We summarize the hypercyclicity behavior of composition operators on the Hardy space as follows:

<table>
<thead>
<tr>
<th>Type of ( \varphi )</th>
<th>Fixed point(s) location</th>
<th>Hypercyclicity of ( C_{\varphi} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>parabolic automorphism</td>
<td>( \xi \in \partial \mathbb{D} )</td>
<td>Yes</td>
</tr>
<tr>
<td>parabolic non-automorphism</td>
<td>( \xi \in \partial \mathbb{D} )</td>
<td>No, in a strong sense*</td>
</tr>
<tr>
<td>hyperbolic automorphism</td>
<td>( \xi_1, \xi_2 \in \partial \mathbb{D} )</td>
<td>Yes</td>
</tr>
<tr>
<td>hyperbolic non-automorphism</td>
<td>( \xi_1 \in \mathbb{D}, \xi_2 \in \partial \mathbb{D} )</td>
<td>No, in a strong sense*</td>
</tr>
<tr>
<td></td>
<td>( \xi_1 \in \mathbb{D}, \xi_2 \notin cl \mathbb{D} )</td>
<td>No, in a strong sense*</td>
</tr>
<tr>
<td></td>
<td>( \xi_1 \in \partial \mathbb{D}, \xi_2 \notin cl \mathbb{D} )</td>
<td>Yes</td>
</tr>
<tr>
<td>loxodromic</td>
<td>( \xi_1 \in \mathbb{D}, \xi_2 \notin cl \mathbb{D} )</td>
<td>No, in a strong sense*</td>
</tr>
<tr>
<td>elliptic</td>
<td>( \xi_1 \in \mathbb{D}, \xi_2 \notin cl \mathbb{D} )</td>
<td>No</td>
</tr>
</tbody>
</table>

*Only constant functions can be limit points of an orbit of \( C_{\varphi} \).
3.3.2 Orbital limit points and hypercyclicity of linear fractional composition operators

In the following we investigate which linear fractional transformations $\varphi$ induce composition operators $C_{\varphi}$ on the Hardy space $H^2(D)$ that have the property patterned in Corollary 36.

As noted in the preceding section, there exists a complete characterization of linear fractional composition operators with respect to their hypercyclic behavior based on the dynamics of the inducing linear fractional transformation $\varphi$.

Bourdon and Shapiro [9] emphasized the existence of a zero-one law for the hypercyclic behavior of these composition operators, wherein a linear fractional composition operator is either hypercyclic or it fails at it in a strong sense. This property makes it easy for us to determine the validity of the implication whether hypercyclicity follows from having an orbit with a non-zero limit point.

Firstly, since all parabolic and hyperbolic automorphisms, as well as all hyperbolic non-automorphisms with no interior fixed point, induce hypercyclic composition operators, our implication holds trivially for these classes.

Secondly, the composition operators induced by parabolic non-automorphisms, loxodromic linear fractional transformations and hyperbolic non-automorphisms with an interior fixed point fail to be hypercyclic in the strong sense that only constant maps can be limit points of any orbit of $C_{\varphi}$. This shows that our implication fails for these classes of composition operators, but it does so in a not so aggravating way.

Finally, the remaining class of composition operators, those induced by elliptic linear fractional transformations, provides us with a non-trivial example of a composition operator that has an orbit with a non-zero limit point but is not hypercyclic. Since no elliptic composition operator is hypercyclic, it suffices to construct an example of an elliptic linear fractional transformation that induces an operator with an orbit having a non-constant limit point.
Example 3. A non-hypercyclic $C_\varphi$ with a non-constant orbital limit point.

We recall from the study of ergodic theory that irrational rotations are ergodic and measure-preserving for the Lebesque measure (see Petersen [29, page 49]). Thus, for a given irrational $\alpha$, the $\text{Orb}(T_\alpha, x)$ is dense for almost all $x$ in $[0,1)$, where $T_\alpha : [0,1) \to [0,1)$ is defined by setting $T_\alpha(x) = x + \alpha \ (\text{mod} \ 1)$.

Let $\alpha > 0$ be an irrational number and let $x^*$ be a point in $(0,1)$ such that $\text{Orb}(T_\alpha, x^*)$ is dense in $[0,1)$. Then there exists a sequence $n_k \nearrow \infty$ such that $T_\alpha^{n_k}x^* \to 0$ as $k \to \infty$, or equivalently $e^{2\pi i(x^* + n_k \alpha)} \to e^0 = 1$.

Now, let $\varphi : \mathbb{D} \to \mathbb{D}$ be defined by $\varphi(z) = e^{2\pi i \alpha} \cdot z$. Clearly, $\varphi$ is an elliptic linear fractional transformation of the unit disk. Further we let $f : \mathbb{D} \to \mathbb{D}$ be defined by $f(z) = e^{2\pi i x^*} \cdot z$, where $f$ is in $H^2(\mathbb{D})$ with $\|f\| = 1$.

We claim that the identity function $\psi : \mathbb{D} \to \mathbb{D}$, $\psi(z) = z$ in $H^2(\mathbb{D})$ is a non-zero limit point of $\text{Orb}(C_\varphi, f)$. To show this let $n_k \nearrow \infty$ as above. Then $\|C_\varphi^n f - \psi\|^2 = \|e^{2\pi i(x^* + n_k \alpha)} z - z\|^2 = |e^{2\pi i(x^* + n_k \alpha)} - 1|^2 \to 0$, as $k \to \infty$.

Hence the identity function is a non-zero limit point of $C_\varphi$, but the operator is not hypercyclic. □

We summarize this behavior of composition operators in the following proposition.

Proposition 41. There exists a linear fractional composition operator $C_\varphi$ on $H^2(\mathbb{D})$ that has an orbit with a non-constant limit point but $C_\varphi$ is not hypercyclic.

In fact, hypercyclicity may fail in a more dramatic way, as the operator need not even be cyclic.

Example 4. A non-cyclic $C_\varphi$ with a non-zero orbital limit point.

Consider the linear fractional composition operator $C_\varphi : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ induced by the hyperbolic non-automorphism $\varphi(z) = \frac{z^2}{2-z}$ with fixed points 0 and 1. A result of Bourdon and Shapiro [1], Theorem 2.2, see also Table I] gives us that $C_\varphi$ is not cyclic. To show that some orbit of $C_\varphi$ has a non-zero limit point, consider the function $f(z) = 1 + z$ in $H^2(\mathbb{D})$. 
Then, since 0 is the attractive fixed point for \( \varphi \) in \( \mathbb{D} \), we have that \( f \circ \varphi^n \rightarrow f(0) \) pointwise on \( \mathbb{D} \). Furthermore, as \( f \) is an analytic function defined in a neighborhood of the closed disk \( cl(\mathbb{D}) \) we have that \( f \circ \varphi^n \rightarrow f(0) \) pointwise on \( \partial \mathbb{D} \setminus \{1\} \). Thus, since the sequence \( \{f \circ \varphi^n\}_{n \geq 1} \) is uniformly bounded on \( \partial \mathbb{D} \) we can conclude by the Dominated Convergence Theorem that \( \|f \circ \varphi^n\|^2 = \frac{1}{2\pi} \int_{\partial \mathbb{D}} |f \circ \varphi^n(e^{i\theta})|^2 \, d\theta \rightarrow |f(0)|^2 = 1 \). Thus \( f \circ \varphi^n \rightarrow 1 \) in the norm of \( H^2(\mathbb{D}) \); see for instance [18, page 187], and hence the function \( g = 1 \) on \( \mathbb{D} \) is a non-zero limit point of \( Orb(C\varphi, f) \) but \( C\varphi \) is not cyclic. □

3.4 Further remarks

Related to what we have studied in the previous sections, another interesting object to explore for a given operator \( T \) is the set \( LP(T) = \{x \in X : Orb(T, x) \) has a non-zero limit point \}, which is a generalization of the set of hypercyclic vectors. We note that different variations of this set have been studied by Costakis and Manoussos [14] and Prăjitura [30]. Clearly the cardinality of the set \( LP(T) \) is either zero or infinite, as once the vector \( x \) belongs to the set \( LP(T) \) we have that \( Tx, T^2x, \ldots \in LP(T) \). In fact, if the set \( LP(T) \) contains the non-zero vector \( x \) it necessarily contains the complex lines \( \mathbb{C} \{T^n x\} \) with the exception of the zero vector.

To illustrate the set \( LP(T) \) with different examples, we first note that the unilateral backward shift \( B \) with constant weights \( w_j = 1 \) on \( \ell^2 \) is an operator having an empty \( LP(T) \) set, as only the zero vector can be a limit point of an orbit. On the other hand, if the operator \( T \) is hypercyclic the set \( LP(T) \) is certainly dense in \( X \) as it contains the dense set of hypercyclic vectors. Furthermore, the work of Beauzamy [3], Enflo [16] and Read [31] provided us with an example of an operator having all non-zero vectors as hypercyclic vectors, so the same containment as above gives us that in this case the set \( LP(T) \) is maximal, meaning that \( LP(T) = X \setminus \{0\} \). However, the set \( LP(T) \) can be dense but not maximal as is demonstrated by the unilateral backward weighted shift \( 2B \) on \( \ell^2 \).
Finally, we would like to point out that an operator $T$ may have a dense $LP(T)$ set even if $T$ is not hypercyclic.

**Example 5.** A non-hypercyclic $C_\varphi$ having a dense $LP(C_\varphi)$ set.

Consider the operator $C_\varphi : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ induced by the loxodromic linear fractional transformation $\varphi(z) = \frac{-z}{2z+4}$. Since 0 is the attractive fixed point for $\varphi$ in $\mathbb{D}$, we have that for any $f$ in $H^2(\mathbb{D})$ $f \circ \varphi^n \to f(0)$ pointwise on $\mathbb{D}$. We now show that for the given $\varphi(z) = \frac{-z}{2z+4}$ and any $f$ in $H^2(\mathbb{D})$ there exists a sequence $n_k \nearrow \infty$ so that $f \circ \varphi^{n_k} \to f(0)$ also in the norm of $H^2(\mathbb{D})$. For this we first note that since $\|\varphi\|_\infty < 1$, the induced composition operator $C_\varphi$ is compact on $H^2(\mathbb{D})$; see Shapiro [34, page 23]. Next, since $\varphi(0) = 0$, the Littlewood Subordination Principle gives us that $\|C_\varphi f\| \leq \|f\|$ and thus the sequence $\{f \circ \varphi^{n-1}\}_{n \geq 1}$ is bounded in $H^2(\mathbb{D})$. Now, $|\varphi(z)| \leq \frac{|z|}{4-2|z|} < \frac{|z|}{2}$ on $\mathbb{D}$ and thus $\varphi^n \to 0$ uniformly on compact subsets of $\mathbb{D}$, so $f \circ \varphi^n \to f(0)$ uniformly on compact subsets of $\mathbb{D}$. Now, for notational simplicity let $g_n = f \circ \varphi^{n-1}$ for all $n \geq 1$. By the above we clearly have the sequence $\{g_n\}_{n \geq 1}$ bounded in $H^2(\mathbb{D})$ and so the compactness of the operator $C_\varphi$ yields that the sequence $\{g_n \circ \varphi\}_{n \geq 1}$ is relatively compact. Thus there exists an increasing sequence $\{n_k\}_{k \geq 1}$ so that $g_{n_k} \circ \varphi \to \tilde{g}$ in $H^2(\mathbb{D})$ for some function $\tilde{g}$ in $H^2(\mathbb{D})$. But convergence in $H^2(\mathbb{D})$ implies uniform convergence on compact subsets of $\mathbb{D}$, so we can conclude that $g_{n_k} \circ \varphi \to \tilde{g}$ uniformly on compact subsets of $\mathbb{D}$. However, $g_{n_k} \circ \varphi = f \circ \varphi^{n_k}$ which as a subsequence of $\{f \circ \varphi^n\}_{n \geq 1}$ converges on compact subsets of $\mathbb{D}$ to the constant function $f(0)$. Thus $\tilde{g} = f(0)$ on $\mathbb{D}$ and hence $f \circ \varphi^{n_k} \to f(0)$ in the norm of $H^2(\mathbb{D})$. Thus we can conclude that the set $LP(C_\varphi)$ consists of all non-constant functions $f$ in $H^2(\mathbb{D})$ with $f(0) \neq 0$ and consequently is dense in $H^2(\mathbb{D})$. □

In closing we note that for the classes of weighted backward shifts and adjoints of multiplication operators on the Bergman spaces, having an orbit with a non-zero limit point implies hypercyclicity. However, the class of linear fractional composition operators on the Hardy space does not enjoy the same property. Thus the question remains for which operators does the geometry of an orbit determine the hypercyclic behavior of the operators.
BIBLIOGRAPHY


APPENDIX

Definition 42. A **Hilbert space** $H$ is a complete inner product space.

Definition 43. A **Banach space** $X$ is a complete normed space.

Definition 44. A **topological vector space** $X$ is a vector space $X$ with a topology $\tau$ so that addition of vectors is a continuous operation from $X \times X$ into $X$, and multiplication by scalars is a continuous operation from $\mathbb{F} \times X$ into $X$.

Definition 45. An **F-space** is a topological vector space $X$ whose topology is induced by a translation invariant metric $d$ so that $(X,d)$ is complete.

Definition 46. A **Fréchet space** is a locally convex F-space, that is it has a basis consisting of convex sets.

Clearly, a Hilbert space is a Banach space, which is a Fréchet space, which in turn is an F-space, and they are all topological vector spaces.

The space $\ell^2(\mathbb{Z}_+)$ defined in Section 2.1 is a Hilbert space, whereas the spaces $\ell^p(\mathbb{Z}_+)$ for $1 \leq p < \infty$ and $p \neq 2$ are Banach spaces that are not Hilbert spaces. On the other hand, the spaces $\ell^p(\mathbb{Z}_+)$ for $0 < p < 1$ are F-spaces that are not Fréchet spaces as they are not locally convex. Finally, $C(-1,1)$ given by the metric $d$ with

$$d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\|f-g\|_n}{1 + \|f-g\|_n},$$

and $\|f-g\|_n = \sup\{|f(x) - g(x)| : x \in [-1 + \frac{1}{n}, 1 + \frac{1}{n}]\}$, is a Fréchet space that is not a Banach space.
Definition 47. If $H$ and $K$ are Hilbert spaces and $T$ is in the Banach algebra of bounded linear operators $\mathcal{B}(H, K)$, then the unique operator $S \in \mathcal{B}(K, H)$ satisfying $\langle Th, k \rangle = \langle h, Sk \rangle$ is called the **adjoint of** $T$ and is denoted by $S = T^*$.

Definition 48. Given $T \in \mathcal{B}(X)$, the **spectrum** $\sigma(T) = \{ \alpha \in \mathbb{F} : T - \alpha I \text{ is not invertible} \}$.

Definition 49. If $X$ is a Banach space, then the **strong operator topology (SOT)** on $\mathcal{B}(X)$ is the topology defined by the family of seminorms $\{ p_f : f \in X \}$, where $p_f(A) = \|Af\|$.

We note that a basic open set for the SOT topology is given by $\{ A : X \to X : \|(A - A_0)f_i\| < \varepsilon, \ i = 1, \ldots, k \}$. Also the norm topology on $\mathcal{B}(X)$ is finer than the SOT topology, so norm convergence implies SOT convergence. Finally, we note that $A_n \to A$ strongly if and only if $A_n f \to Af$ for every $f \in X$.

Definition 50. Given a Banach space $X$, the **weak topology** on $X$ is the topology induces by the seminorms $\{p_{\alpha^*}\}_{\alpha^* \in X^*}$, where each $p_{\alpha^*}$ is given by $p_{\alpha^*}(x) = |\alpha^*(x)|$.

We note that a basic weakly open set is given by $\{ x \in X : |\alpha_i^*(x - x_0)| < \varepsilon, \ i = 1, \ldots, k \}$. Furthermore, we remark that the weak topology is weaker than the norm topology.

**Fact:** $\text{Dir}((\mathbb{D})) \hookrightarrow H^2(\mathbb{D}) \hookrightarrow A^2(\mathbb{D})$

To see that $\text{Dir}((\mathbb{D})) \hookrightarrow H^2(\mathbb{D}) \hookrightarrow A^2(\mathbb{D})$ we note that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then $\|f\|_{\nu}^2 = \sum_{n=0}^{\infty} |a_n|^2(n + 1)^\nu$, where $\nu = \frac{1}{2}$ for the Dirichlet space $\text{Dir}((\mathbb{D}))$, $\nu = 0$ for the Hardy space $H^2(\mathbb{D})$ and $\nu = -\frac{1}{2}$ for the Bergman space $A^2(\mathbb{D})$.

**Fact:** $M_\varphi$ can never be hypercyclic on $\text{Dir}(\Omega)$, $H^2(\mathbb{D})$ or $A^2(\Omega)$

To see that $M_\varphi$ can never be hypercyclic on either of the three spaces $\text{Dir}(\Omega)$, $H^2(\mathbb{D})$ or $A^2(\Omega)$, we let $\varphi$ be a non-constant multiplier and $g$ be a function in the closure of an orbit Orb$(M_\varphi, f)$. Then there exists a $z_0$ in the space $\Omega$ or $\mathbb{D}$ so that $|\varphi(z_0)| < 1$ or $|\varphi(z_0)| > 1$. 
Since convergence in norm implies pointwise convergence for all of the above three spaces of analytic functions, we have that \(|\varphi(z_0)|^{n_k}|f(z_0)| \to |g(z_0)|\) for some sequence \(\{n_k\}\). Thus if \(|\varphi(z_0)| < 1\) we get that \(|g(z_0)| = 0\), and if \(|\varphi(z_0)| > 1\) then \(|f(z_0)| = 0\) which in turn gives that \(|g(z_0)| = 0\). Therefore only maps that vanish at \(z_0\) are in the closure of an orbit \(\text{Orb}(M_{\varphi}, f)\), and hence \(M_{\varphi}\) can never be hypercyclic.

**Fact:** \((B(X), \|\|)\) is not separable

Let \(H\) be a separable infinite dimensional Hilbert space. Then \(H \simeq L^2[0, 1]\). Since we can identify \(L^\infty[0, 1]\) as a subset of \(B(L^2[0, 1])\) (if \(\varphi \in L^\infty[0, 1]\) define \(M_{\varphi} : L^2[0, 1] \to L^2[0, 1]\) by \(M_{\varphi}(f) = \varphi f\) and \(\|M_{\varphi}\| = \|\varphi\|_\infty\)), it suffices to show that \(L^\infty[0, 1]\) is not separable in the \(\|\|_\infty\) norm.

Let \(a \in [0, 1)\) and define \(f_a = \chi_{[0,a]}\). Suppose by means of contradiction that \(D\) is a countable dense subset of \(L^\infty[0, 1]\). Then for \(f_a \in L^\infty[0, 1]\) there exists \(g_a \in D\) so that \(\|f_a - g_a\|_\infty < \frac{1}{4}\). Thus, \(\frac{3}{4} < g_a(z) < \frac{5}{4}\) for all \(z \in [0, a]\) and \(-\frac{1}{2} < g_a(z) < \frac{1}{2}\) for all \(z \in (a, 1]\).

Hence, if the function \(g_a\) is used to approximate \(f_a\), it cannot be used to approximate \(f_b\) for \(b \neq a\) (if \(\|f_a - g_a\|_\infty < \frac{1}{4}\), then for \(b > a\) we have \(|(f_a - g_a)(\frac{a+b}{2})| > \frac{1}{4}\) and so \(\|f_a - g_a\|_\infty \geq \frac{1}{4}\)).

Since there are uncountably many functions \(f_a\), there are uncountably many functions \(g_a\) in \(D\), so \(D\) cannot be countable. Therefore \((B(H), \|\|)\) is not separable.