ON FLIPS OF UNITARY BUILDINGS

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ABSTRACT

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In the Gorenstein-Lyons-Solomon revision of the proof of the classification theorem for finite simple groups, one of the techniques in the identification step involves identifying a particular finite simple group (the minimal counterexample to the theorem) with a known finite simple group. Phan’s theorems and the Curtis-Tits theorem provide some techniques for accomplishing this identification. Both these theorems can be phrased in terms of groups acting flag transitively on a simply connected geometry. One method of producing geometries suitable for proving Phan-type theorems is to study geometries induced by flips on twin buildings.

The purpose of this work is to classify flips of the building associated to the geometry of totally isotropic subspaces of a non-degenerate unitary spaces over a finite field of odd characteristic. A secondary goal is to study some properties of geometries related to these flips. We prove that there are up to similarity only four flips, and that for sufficiently large unitary spaces the resulting geometries are simply connected. We then appeal to Tits’ Lemma to prove Phan-type theorems for certain flag-transitive automorphism groups of these geometries.
For Kathleen and Kepler.
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History

The historical genesis of this dissertation lies in the classification of finite simple groups, which states the following:

**Theorem.** Let $G$ be a finite simple group. Then $G$ is isomorphic to one on the following list:

(i) a cyclic group of prime order;

(ii) an alternating group $\text{Alt}(n)$, $n \geq 5$;

(iii) a simple group of Lie type;

(iv) one of 26 sporadic simple groups.

The importance of the classification theorem for finite group theory is hard to overstate. Since the classification was announced, nearly all the major problems of finite group theory that were open before the classification have been resolved (6). Unfortunately the original “proof” of the classification theorem was really a collection of hundreds of papers which, when combined, claimed to prove the theorem. The length and level of complexity of the original proof led Gorenstein, Lyons, and Solomon to launch a revision project, to produce a coherent, readable proof of the classification theorem.

Roughly speaking, the strategy of the revision is to assume we have a minimal counterexample. That is, a group $G$ that is the smallest simple group not isomorphic to one of the known finite simple groups. One then studies the structure of $G$ and tries to argue that in fact $G$ is isomorphic to one of the known finite simple groups. Two important theorems to provide such identification are Phan’s Theorem and the Curtis-Tits Theorem. The statements of both these theorems are taken from 7.
**Theorem** (Phan’s Theorem). Let \( n > 2 \) and let \( q > 2 \) be a prime power. Suppose that \( G \) is a finite group generated by subgroups \( U_1, \ldots, U_n \) where \( U_i \cong SU(2, q^2) \) and for each \( i \), \( D_i \) is a maximal torus in \( U_i \) of size \( q + 1 \) satisfying the following:

(P1) if \( |i - j| > 1 \) then \([x, y] = 1\) for all \( x \in U_i, y \in U_j\); 

(P2) if \( |i - j| = 1 \) then \( U_{ij} = \langle U_i, U_j \rangle \cong SU(3, q^2) \); and 

(P3) \([x, y] = 1\) for all \( x \in D_i, y \in D_j\).

Then \( G \) is isomorphic to a quotient of \( SU(n + 1, q^2) \).

**Theorem** (Curtis-Tits Theorem). Let \( G \) be the universal version of a finite Chevalley group of (twisted) rank at least 3 with root system \( \Sigma \), fundamental system \( \Pi \), and root groups \( X_\alpha \), \( \alpha \in \Sigma \). For each \( J \subseteq \Pi \) let \( G_J \) be the subgroup of \( G \) generated by all root subgroups \( X_\alpha \), \( \pm \alpha \in J \). Let \( D \) be the set of all subsets of \( \Pi \) with at most 2 elements. Then \( G \) is the universal completion of the amalgam \( \bigcup_{J \in D} G_J \).

In [7] it is noted that both Phan’s Theorem and the Curtis-Tits Theorem are equivalent to the statement that certain geometries are simply connected. The important connection is provided by Tits’ Lemma (Theorem 1.3.2 in this dissertation.) This viewpoint led to a series of papers ([9], [8], [12], [25], [26]) and dissertations ([21], [47]) in which further Phan-type theorems are proved.

One technique for producing geometries suitable for proving Phan-type theorems is to study certain automorphisms of twin-buildings called flips. In this dissertation we classify flips of the twin-building associated to a non-degenerate unitary space of rank \( n \) over a finite field \( \mathbb{F}_{q^2} \) where \( q \) is an odd prime power. We then prove that certain geometries related to the flips are flag transitive and simply connected. By applying Tits’ Lemma we are then able to prove new Phan-type theorems.

In [30] geometries related to flips of symplectic and orthogonal buildings are studied, and it is shown that if the field is sufficiently large the resulting geometries are simply connected. The authors of that paper do not consider the case of a unitary building.
The Results of this Dissertation

Let \( q \) be an odd prime power, let \( \mathbb{F} = \mathbb{F}_q^2 \), let \( \sigma \) be the unique involution in \( \text{Aut}(\mathbb{F}) \), let \( V \) be a \( 2n \)-dimensional \( \mathbb{F} \) vector space and let \( \beta \) be a non-degenerate \( \sigma \)-hermitian form on \( V \). Let \( \Delta \) denote the building of totally isotropic subspaces of \((V, \beta)\). These definitions are made precise in Chapters 1 and 2. The definition of a flip appears in Chapter 2.

**Main Theorem 1.** Let \( \varphi \) be a flip of \( \Delta \). Then \( \varphi \) is induced by a semilinear transformation \( f \) of the underlying unitary space \( V \) such that one of the following holds:

(i) \( f \) is a linear isometry of \((V, \beta)\), \( f^2 = \text{id} \) on \( V \), and there is a hyperbolic basis \( \{e_i, f_i\}_{i=1}^n \) for \( V \) such that \( f(e_i) = f_i \) and \( f(f_i) = e_i \) for \( i = 1, \ldots, n \);

(ii) \( f \) is a linear anti-isometry of \((V, \beta)\), \( f^2 = \text{id} \) on \( V \), and there is a hyperbolic basis \( \{e_i, f_i\}_{i=1}^n \) for \( V \) such that \( f(e_i) = \alpha f_i \) and \( f(f_i) = \alpha^{-1}e_i \) for \( i = 1, \ldots, n \), where \( \text{Tr}_\sigma(\alpha) = 0 \);

(iii) \( f \) is a \( \sigma \)-semilinear isometry of \((V, \beta)\), \( f^2 = \text{id} \) on \( V \), and there is a hyperbolic basis \( \{e_i, f_i\}_{i=1}^n \) for \( V \) such that \( f(e_i) = f_i \) and \( f(f_i) = e_i \) for \( i = 1, \ldots, n \);

(iv) \( f \) is a \( \sigma \)-semilinear isometry of \((V, \beta)\), \( f^2 = \text{id} \) on \( V \), and there is a hyperbolic basis \( \{e_i, f_i\}_{i=1}^n \) for \( V \) such that for \( i = 1, \ldots, n-1 \), \( f(e_i) = f_i \), \( f(f_i) = e_i \) and there is a non-square \( \lambda \in \mathbb{F} \) with \( f(e_n) = \lambda f_n \) and \( f(f_n) = \sigma(\lambda^{-1})e_n \).

Conversely any semilinear transformation of \( V \) satisfying one of (i)-(iv) induces a flip of \( \Delta \).

So there are up to similarity only four flips of this building. Each flip gives rise to non-isomorphic geometries which can be used to prove Phan-type theorems about flag-transitive automorphism groups of the geometries when the geometries are simply connected by appealing to Tits’ Lemma.

In the body of the dissertation, Main Theorem 1 is split into four pieces. First, we prove in Lemma 3.1.4 that every flip of \( \Delta \) is induced by some linear isometry, linear anti-isometry, or \( \sigma \)-semilinear isometry of \((V, \beta)\). Then, in Lemma 3.2.2 we prove that a semilinear
transformation of \( V \) satisfying any of (i)-(iv) induces a flip of \( \Delta \). We then prove in Theorem 4.1.1 that if the transformation is linear, then (i) or (ii) holds. Finally in Theorem 5.4.1 we prove that if the transformation is \( \sigma \)-semilinear then either (iii) or (iv) holds.

A flip is called \textbf{linear} if it is induced by a linear transformation of \( V \), and \textbf{semilinear} if it is induced by a \( \sigma \)-semilinear transformation of \( V \).

We also prove the following:

**Main Theorem 2.** Let \( \varphi \) be a linear flip of \( \Delta \).

(i) If \( \varphi \) is induced by an isometry of \((V,\beta)\), then the group \( U_n(q^2) \times U_n(q^2) \) acts flag transitively on the geometry \( \Gamma(n,q) \).

(ii) If \( \varphi \) is induced by an anti-isometry of \((V,\beta)\), then the group \( GL_n(q^2) \) acts flag transitively on the geometry \( \Gamma(n,q) \).

(iii) If \( n \geq 14 \), then the geometry \( \Gamma(n,q) \) is simply connected.

The definition of \( \Gamma(n,q) \) occurs in Section 3.3. Main Theorem 2(i) is proved by combining Theorem 4.2.4(i) with Theorem 4.2.3. Main Theorem 2(ii) is proved by combining Theorem 4.2.4(ii) with Theorem 4.2.3. Main Theorem 2(iii) is proved as Theorem 4.6.7.

**Main Theorem 3.** Let \( \varphi \) be a \( \sigma \)-semilinear flip of \( \Delta \).

(i) If \( \varphi \) satisfies (iii) of Theorem 1 then \( G \) acts flag-transitively on \( \Gamma_1(n,q) \), where:

\( a \) if \( n \) is even or \(-1\) is not a square in \( \mathbb{F}_q \), then \( G \cong O_{2n}^+(q) \);

\( b \) if \( n \) is odd and \(-1\) is a square in \( \mathbb{F}_q \), then \( G \cong O_{2n}^-(q) \).

(ii) If \( \varphi \) satisfies (iv) of Theorem 2 then \( G \) acts flag-transitively on \( \Gamma_1(n,q) \), where:

\( c \) if \( n \) is even or \(-1\) is not a square in \( \mathbb{F}_q \), then \( G \cong O_{2n}^-(q) \);

\( d \) if \( n \) is odd and \(-1\) is a square in \( \mathbb{F}_q \), then \( G \cong O_{2n}^+(q) \).

(iii) If \( n \geq 8 \) or \( n = 7 \) and \( q \geq 5 \) then \( \Gamma_1(n,q) \) is simply connected.
The definition of $\Gamma_1(n, q)$ occurs in Section 5.2. Main Theorem 3(i) is proved by combining Theorem 5.4.1(i) with Theorem 5.4.3. Main Theorem 3(ii) is proved by combining Theorem 5.4.1(ii) with Theorem 5.4.3. Main Theorem 3(iii) is proved as Theorems 6.6.6 and 6.6.11.

We combine Main Theorem 2 and Main Theorem 3 with Tits’ Lemma to prove the following Phan-type theorems.

**Phan-type Theorem 1.** Suppose $n \geq 14$, let $\varphi$ be a linear flip and let $A$ denote the amalgam of maximal parabolic subgroups of $U_{2n}(q^2)^\varphi$ with respect to a maximal flag $F$ of $\Gamma(n, q)$. Then $U_{2n}(q^2)^\varphi$ is the universal completion of $A$.

**Phan-type Theorem 2.** Suppose $n \geq 8$, let $\varphi$ be a $\sigma$-semilinear flip and let $A$ denote the amalgam of maximal parabolic subgroups of $U_{2n}(q^2)^\varphi$ with respect to a maximal flag $F$ of $\Gamma_1(n, q)$. Then $U_{2n}(q^2)^\varphi$ is the universal completion of $A$.

We combine these with Theorem 4.7.1 to prove these even better results.

**Phan-type Corollary 1.** Let $\varphi$ be a linear flip and let $A_{(k)}$ denote the amalgam of rank $k$ parabolic subgroups of $U_{2n}(q^2)^\varphi$ with respect to a maximal flag $F$ of $\Gamma(n, q)$. If $n - k \geq 14$, then $U_{2n}(q^2)^\varphi = \mathcal{U}(A_{(k)})$.

**Phan-type Corollary 2.** Let $\varphi$ be a $\sigma$-semilinear flip and let $A_{(k)}$ denote the amalgam of rank $k$ parabolic subgroups of $U_{2n}(q^2)^\varphi$ with respect to a maximal flag $F$ of $\Gamma_1(n, q)$. If $n - k \geq 8$ or $n - k = 7$ and $q \geq 5$, then $U_{2n}(q^2)^\varphi = \mathcal{U}(A_{(k)})$. 
Notation and Conventions

We have collected here some notation and conventions of which the reader should be aware. Most are reasonably standard, but in cases where another option is possible we have made clear which choice we have made.

Functions

- Functions act on the left, so \( f(x) \) denotes the image of \( x \) under \( f \), rather than \( (x)f \). In particular, groups acting on various objects always act on the left. This means that, in the case when a group \( G \) acts on a set \( X \) with \( g, h \in G \) and \( x \in X \), we have

\[
(gh)(x) = g(h(x)).
\]

The only exception to this rule is that, for a field \( k \), when \( \text{Aut}(k) \) acts on a matrix ring over \( k \), if \( A = (a_{ij}) \) and \( \sigma \in \text{Aut}(k) \) we write

\[
A^\sigma = (a_{ij})^\sigma = (\sigma(a_{ij})).
\]

- Related to the choice of functions acting on the left, functions also compose on the left. So \( fg(x) = f(g(x)) \). This seems to us the more natural choice, but some authors (especially those who have groups act on the right) reverse the order.

- If \( f : A \rightarrow B \) is injective, we may write \( f : A \hookrightarrow B \) or \( A \hookrightarrow B \).
- If $f : A \to B$ is surjective, we may write $f : A \twoheadrightarrow B$ or $A \xrightarrow{f} B$.

**Other Notation**

- The symbol “⊂” always denotes a proper subset. If it is possible for the two sets to be equal we write “⊆” to indicate this possibility.

- The term “field” is reserved for a commutative field. This is reasonably standard, but in some works (for example [15]) a field is not required to be commutative. In this work we will have no need for division rings that are not fields.

- The symbol $\mathbb{F}_n$ denotes the finite field of order $n$. Recall that a finite field has order a prime power, and for every prime power there is a unique finite field of that order.

- If $k$ is a field and $\sigma$ is an automorphism of $k$ then $k^\sigma$ denotes the fixed field of $\sigma$.

- Proofs are ended with the symbol “□”.

- Remarks are ended with the symbol “♦”.

**Numbering**

- Theorems, lemmas, corollaries and propositions are all numbered together, of the form chapter.section.number. So Lemma 3.4.2 is the second result of Chapter 3, Section 4.

- Definitions are also numbered chapter.section.number, but their numbering is separate from the numbering of results.

- Notes are numbered using the same chapter.section.number system, but separately from both results and definitions.
Remarks and Notes

- Notes are important. The information in notes consists of things the reader needs to know to understand the rest of the work. For example, conventions that are followed for the remainder of the text are specified in notes.

- Remarks are less important. They mainly contain motivation and other things that help explain the material and make it more understandable, but are not mathematically necessary.
CHAPTER 1

Groups and Incidence Geometries

Introduction

The primary topic of this dissertation is groups. The geometric notions introduced are used to study groups. The theory of buildings was introduced initially in order to provide a uniform geometric framework to understand certain algebraic groups, and the notion of a flip of a building was developed in order to prove identification results for groups in the spirit of Phan’s theorem. In this chapter we have collected the basic definitions of group theory and (incidence) geometry that we will require. The theory of buildings will be introduced in Chapter 2.

1.1 Group Theory

Definition 1.1.1. A group \( G \) is a non-empty set together with an associative binary operation (denoted by juxtaposition) satisfying the following two axioms:

(G1) there is an element \( e \in G \) such that for all \( g \in G \), \( eg = ge = g \);

(G2) for each \( g \in G \) there is an \( h \in G \) such that \( gh = hg = e \).

If in addition the group satisfies
(Ab) for all $g, h \in G$, $gh = hg$

then the group is called **abelian**.

**Remark.** It can be shown that the element $e$ in axiom (G1) is unique. It is called the **identity** of the group. Similarly for each $g \in G$ it can be shown there is a unique $h \in G$ so that $gh = e$. This element $h$ is called the **inverse** of $g$, and is denoted $g^{-1}$.

**Note 1.1.1.** The group with one element is called the **trivial group**. Throughout it is denoted $\{e\}$.

**Definition 1.1.2.** Let $G$ be a group, and $H$ a subset of $G$. For $g \in G$, we define the set $gH$ by

$$gH = \{gh | h \in H\}$$

This is called a **left coset** of $H$ in $G$. **Right cosets** are defined similarly.

**Definition 1.1.3.** Let $G$ be a group. A **subgroup** of $G$ is a non-empty subset $H \subseteq G$ satisfying the following:

(S1) the identity $e$ of $G$ lies in $H$;

(S2) for all $a, b \in H$, $ab \in H$;

(S3) for all $a \in H$, $a^{-1} \in H$.

We denote this by $H \leq G$. A subgroup $H$ is **proper** if $H \neq G$. A **normal subgroup** of $G$ is a subgroup $H$ that satisfies $gH = Hg$ for all $g \in G$. If $H$ is a normal subgroup of $G$, we write $H \trianglelefteq G$.

**Definition 1.1.4.** Let $G$ be a group and let $g \in G$. The **centralizer** of $g \in G$ is denoted $C_G(g)$ and is the collection of all elements of $G$ that commute with $g$.

**Definition 1.1.5.** A group $G$ is **simple** if its only normal subgroups are $\{e\}$ and $G$ itself. So a group is simple if it has no non-trivial proper normal subgroups.
Definition 1.1.6. Let $G$ be a group, and $H \trianglelefteq G$. Denote by $G/H$ the set of all cosets of $H$ in $G$. We turn this set into a group, called the **quotient of $G$ by $H$** by defining $(g_1H)(g_2H) = (g_1g_2)H$.

*Remark.* In order for this definition to make sense it is necessary that $H$ be a normal subgroup, so that its left and right cosets are equal.  

Definition 1.1.7. Let $X$ be a subset of a group $G$. We define the **subgroup generated by $X$** to be

$$\langle X \rangle = \bigcap_{H \leq G, X \subseteq H} H$$

*Remark.* The subgroup generated by a subset is in fact the minimal subgroup containing the subset. It can also be realized as the set of all finite products of elements of $X$ and their inverses.  

Definition 1.1.8. The **order** of an element $g \in G$ is the cardinality of $\langle g \rangle$. The element $g$ is an **involution** if it has order 2.

Definition 1.1.9. Let $G$ and $H$ be groups. A **group homomorphism** (abbreviated homomorphism when the context permits) from $G$ to $H$ is a set map $f : G \to H$ with the property that for all $g_1, g_2 \in G$, $f(g_1g_2) = f(g_1)f(g_2)$. A bijective homomorphism is called an **isomorphism**. An isomorphism $f : G \to G$ is called a **automorphism**.

Definition 1.1.10. Let $G$ and $H$ be groups with $e_H$ the identity of $H$, and $f : G \to H$ a homomorphism. The **kernel** of $f$ is

$$\ker(f) = \{ g \in G | f(g) = e_H \}.$$ 

*Remark.* It can be shown that the normal subgroups of a group correspond exactly to the kernels of homomorphisms. We therefore see that a group $G$ is simple if and only if it has no non-trivial quotients, that is, every quotient of $G$ is either trivial, or isomorphic to $G$.  

1.2 Amalgams of Groups

Definition 1.2.1. Let $I$ be a non-empty set. An **amalgam of groups** over $I$ is a set $A = \bigcup_{i \in I} G_i$ endowed with a partial multiplication (denoted by juxtaposition) satisfying:

(A1) each $G_i$ is a group with respect to the restriction of the partial multiplication;

(A2) for $a, b \in A$, $ab$ is defined if and only if both lie in some $G_i$;

(A3) for each $i, j \in I$, $G_i \cap G_j$ is a subgroup of both $G_i$ and $G_j$.

The cardinality of $I$ is the **rank** of the amalgam.

*Remark.* This is not the only way to define an amalgam of groups, and in some contexts this is not the best definition to use. A more general definition is given in [50].

Definition 1.2.2. A **completion** of an amalgam $A = \bigcup_{i \in I} G_i$ is a group $G$ together with a map $\phi : A \to G$ such that $\phi|G_i$ is a group homomorphism for each $i$, and $\phi(A)$ generates $G$.

*Remark.* Every amalgam has at least one completion, the trivial group $\{e\}$. An amalgam with a non-trivial completion is called **non-collapsing**.

Definition 1.2.3. Let $(G_1, \varphi_1), (G_2, \varphi_2)$ be completions of an amalgam $A$. A group homomorphism $\phi : G_1 \to G_2$ is called a **homomorphism of completions** if

$$\varphi_2 = \phi \circ \varphi_1.$$

In this case, $(G_2, \varphi_2)$ is called the **quotient** of $(G_1, \varphi_1)$ over $\ker \phi$. Notice that a consequence of the definition is that $\phi$ is surjective.

A completion $(G, \varphi)$ of an amalgam $A$ is **universal** if given any completion $(G_1, \varphi_1)$ there is a unique homomorphism of completions $\phi : G \to G_1$. We denote the universal completion of $A$ by $\mathcal{U}(A)$. 
It is not \textit{a priori} obvious that the universal completion of a particular amalgam even exists. In fact universal completions always exist, and given a fixed amalgam $A$, its universal completion is the (unique up to unique isomorphism) universal object in the category whose objects are completions of $A$, and whose morphisms are homomorphisms of completions. The universal completion of an amalgam can be defined by generators and relations:

**Lemma 1.2.1** ([RS], Lemma 1.3.2). Let $A = \bigcup_{i \in I} G_i$ be an amalgam of groups. Let $X$ be a set in bijective correspondence with the elements of $A$ and let $\rho : A \to X$ be a bijection of sets. Let $U(A)$ be the quotient of the free group on $X$ by the minimal normal subgroup containing

$$\{\rho(x)\rho(y)\rho(z)^{-1} \text{ such that } x, y, z \in G_i \text{ for some } i \text{ and } xy = z\}.$$

Then $(U(A), \varphi)$ is the universal completion of the amalgam $A$ where $\varphi : A \to U(A)$ is defined by $\varphi(x) = \rho(x)$ for all $x \in A$.

### 1.3 Incidence Geometry

**Definition 1.3.1.** Let $I$ be a set. A \textbf{pregeometry} over $I$ is a non-empty set $\Gamma$ together with a \textbf{type function} $t : \Gamma \to I$ and a symmetric, reflexive \textbf{incidence relation} $\sim$ on $\Gamma$ with the property that for $x, y \in \Gamma$, $x \sim y$ and $t(x) = t(y)$ implies $x = y$. The set $I$ is called the \textbf{type set} of the pregeometry. The cardinality of $I$ is called the \textbf{rank} of the pregeometry. The elements of $\Gamma$ are called the \textbf{objects} of the pregeometry.

A pregeometry is often denoted by an ordered quadruple $(\Gamma, I, t, \sim)$. If the context is unambiguous the pregeometry may be denoted $\Gamma$.

**Note 1.3.1.** By convention we assume that the type function is surjective and the type set $I$ has finite cardinality.

**Definition 1.3.2.** Let $\Gamma$ be a pregeometry. A \textbf{flag} is a set of pairwise incident elements.
The type of a flag $\mathcal{F} = \{F_{i_1}, \ldots, F_{i_k}\}$ is $t(\mathcal{F}) = \{t(F_{i_j}) | j = 1, \ldots, k\}$. The cotype of $\mathcal{F}$ is $I \setminus t(\mathcal{F})$. A flag of type $I$ is called a chamber.

A flag $F$ is maximal if it is not properly contained in any larger flag.

We now come to the definition of a geometry.

**Definition 1.3.3.** Let $(\Gamma, I, t, \sim)$ be a pregeometry. $\Gamma$ is transversal if every maximal flag is a chamber. A transversal pregeometry is called a geometry.

**Construction 1.** Let $k$ be a field and $V$ an $n$-dimensional $k$ vector space. The projective geometry $P(V)$ associated to $V$ is defined as follows. The objects are proper, non-zero subspaces of $V$. Let $O$ denote the set of objects. The type set of $P(V)$ is $I = \{0, \ldots, n-2\}$. The type function $t : O \rightarrow I$ is defined by $t(U) = \dim_k(U) - 1$, and incidence is symmetrized inclusion.

**Remark.** Using one less than the vector space dimension for the type is customary, but not necessary. The main advantage is that the minimal objects, which are 1-dimensional subspaces, can be called points and have type 0, just as a point in Euclidean geometry has dimension 0. Similarly a 2-dimensional subspace can be called a line and have the same type (1) as a line in Euclidean geometry has dimensions.

**Definition 1.3.4.** For $i = 1, 2$ let $\Lambda_i = (\Gamma_i, I_i, t_i, \sim_i)$ be a geometry. We define the direct product of $\Lambda_1$ and $\Lambda_2$, denoted by $\Lambda_1 \otimes \Lambda_2$, as follows:

(i) the object set of $\Lambda_1 \otimes \Lambda_2$ is the disjoint union of $\Gamma_1$ and $\Gamma_2$;

(ii) the type set of $\Lambda_1 \otimes \Lambda_2$ is the disjoint union of $I_1$ and $I_2$;

(iii) the type of an object $x$ is $t_1(x)$ if $x \in \Lambda_1$ and $t_2(x)$ if $x \in \Lambda_2$;

(iv) two objects $x, y$ of $\Lambda_1 \otimes \Lambda_2$ are incident if either:

(a) $x$ and $y$ are contained in one $\Lambda_i$, and they are adjacent in $\Lambda_i$;
(b) $x$ and $y$ are contained in different $\Lambda_i$.

This definition extends in the obvious way to any finite family of geometries $\Lambda_1, \ldots, \Lambda_k$. In this case we may use the notation $\prod_{i=1}^k \Lambda_i$ to denote the direct product of the geometries.

It is easy to verify that the direct product of a finite family of geometries is again a geometry.

**Definition 1.3.5.** Let $F$ be a flag of a geometry $\Gamma$. The *residue* of $F$ in $\Gamma$, denoted $\text{res}_F(F)$, is the set of all elements of $\Gamma \setminus F$ that are incident to all elements of $F$. The residue of a flag is a geometry with type set $I \setminus t(F)$. The rank of a residue is called the *corank* of the flag.

**Definition 1.3.6.** A geometry $(\Gamma, I, t, \sim)$ is **connected** if the graph whose vertex set is $\Gamma$ and whose edge set is $E = \{\{x, y\} \mid x, y \in \Gamma \text{ with } x \sim y\}$ is connected. This graph is called the *incidence graph* of the geometry and is denoted $I(\Gamma)$.

**Definition 1.3.7.** A geometry is **residually connected** if the residue of any flag of corank at least 2 is connected.

**Definition 1.3.8.** A path in a geometry $\Gamma$ is a finite sequence of objects $x_1, \ldots, x_n$ with the property that $x_i \sim x_{i+1}$ for $i = 1, \ldots, n - 1$. A path $x_1, \ldots, x_n$ is **closed** if $x_1 = x_n$. In this case, the object $x_1$ is called the *base point*.

One of the properties of geometries in which we are interested is the notion of simple **connectedness**. This concept is best expressed in terms of simplicial complexes.

**Definition 1.3.9.** A *simplicial complex* is a non-empty set $K$ together with a set $S$ of finite non-empty subsets of $K$ with the following properties:

(i) if $v \in K$ then $\{v\} \in S$;

(ii) if $\emptyset \neq U \subseteq V$ and $V \in S$ then $U \in S$. 


The elements of $K$ are called **vertices** and the elements of $S$ are called **simplices**.

A **subcomplex** is a non-empty subset of $K_1$ of $K$ together with a non-empty subset $S_1$ of $S$ so that $(K_1, S_1)$ is a simplicial complex.

**Construction 2.** If $\Gamma$ is a geometry, we obtain a simplicial complex $\Sigma(\Gamma)$ whose vertices are the objects of $\Gamma$, and whose simplices are the non-empty flags of $\Gamma$. This is the **flag complex** of the geometry.

Every simplicial complex is associated to a topological space, and so there is a topological notion of homotopy. In the particular case of a simplicial complex there is a purely combinatorial description, with no reference to continuity. The reader interested in topological spaces and homotopy in that context may consult either [44] or [54].

**Definition 1.3.10.** A **path** in a simplicial complex is a sequence of vertices $x_1, \ldots, x_k$ such that $x_i$ and $x_{i+1}$ are contained in a common simplex of size 2 for all $i = 1, \ldots, k - 1$.

**Definition 1.3.11.** A simplicial complex $K$ is **connected** if any two vertices can be joined by a path.

**Theorem 1.3.1.** A geometry $\Gamma$ is connected if and only if its flag complex $\Sigma(\Gamma)$ is connected.

The proof of Theorem 1.3.1 is straightforward.

**Definition 1.3.12.** Two paths $\gamma_1$ and $\gamma_2$ in a simplicial complex $K$ are **elementary homotopic** if one can be obtained from the other by either:

(i) replacing a subsequence $x, x$ by $x$ (a repetition);

(ii) replacing a subsequence $x, y, x$ by $x$ (a return);

(iii) replacing a subsequence $x, y, z, x$ by $x$ (a triangle) provided $x, y, z$ lie in a common simplex.
Two paths are **homotopic** if one can be obtained from the other by a finite sequence of elementary homotopies.

A **loop** is a path \( x_1 \ldots x_n \) with \( x_1 = x_n \). The vertex \( x_1 \) is called the **base point**. A loop is **null-homotopic** if it is homotopic to a constant loop.

**Definition 1.3.13.** Let \( x \) be a vertex of a simplicial complex \( K \). Let \( \mathcal{L}_x \) denote the set of loops in \( K \) with base point \( x \). Define an equivalence relation \( \simeq \) on \( \mathcal{L}_x \) by \( f \simeq g \) if \( f \) and \( g \) are homotopic paths. The **homotopy class** of a loop \( f \in \mathcal{L}_x \) is denoted \([f]\) and is the part of the partition of \( \mathcal{L}_x \) induced by \( \simeq \) that contains \( f \).

**Definition 1.3.14.** Let \( \gamma_1 = x_1, \ldots, x_n \) and \( \gamma_2 = y_1, \ldots, y_m \) be paths in a simplicial complex \( K \) with \( x_n = y_1 \). The **product path** \( \gamma_1 \gamma_2 \) is the path \( x_1, \ldots, x_n, y_2, \ldots, y_m \), obtained by concatenating \( \gamma_1 \) and \( \gamma_2 \).

**Note 1.3.2.** One can show (see Lemma of [45]) that if \( f \) and \( g \) are paths such that \( fg \) is defined, then \([f][g] = [fg]\) is a well-defined multiplication of homotopy classes of paths.

**Definition 1.3.15.** Let \( x \) be a vertex of a simplicial complex \( K \). The **fundamental group** of \( K \) with base point \( x \) is the set of homotopy classes of loops with base point \( x \) endowed with the multiplication of paths. This group is denoted \( \pi_1(K,x) \).

**Remark.** What we have defined here as the fundamental group of a complex \( K \) with base point \( x \) is in fact the edge-path group. The isomorphism between the fundamental group of a simplicial complex and its edge-edge path group is explored in Section 6 of [54].

**Note 1.3.3.** If a geometry is connected, then its fundamental groups with different base points are all isomorphic. This is a consequence of the fact that a connected simplicial complex is path connected, and so the fundamental groups with different base points are all isomorphic. This is Corollary 52.2 of [45].

**Definition 1.3.16.** A connected simplicial complex \( K \) is **simply connected** if for some (and hence all) vertices \( x \), \( \pi_1(K,x) = \{e\} \). A geometry \( \Gamma \) is **simply connected** if its flag complex \( \Sigma(\Gamma) \) is simply connected.
In this dissertation, the notion of simple connectedness for a geometry is important because it is key to establishing Phan-type theorems for groups acting flag-transitively on the geometry. The connection is made precise by Tits’ Lemma, which is Theorem 1.3.2 in this dissertation.

**Definition 1.3.17.** An automorphism of a geometry $\Gamma$ is a permutation of the objects that preserves incidence and type. We denote the group of all automorphisms of $\Gamma$ by $\text{Aut}(\Gamma)$.

**Definition 1.3.18.** Let $\Gamma$ be a geometry and let $G \leq \text{Aut}(\Gamma)$. A parabolic subgroup of $G$ is the stabilizer in $G$ of a non-empty flag of $\Gamma$. A maximal parabolic subgroup is the stabilizer in $G$ of a flag of rank 1. A minimal parabolic subgroup is the stabilizer in $G$ of a flag of corank 1.

**Definition 1.3.19.** Let $\Gamma$ be a geometry and let $G \leq \text{Aut}(\Gamma)$. If $F = \{F_i\}_{i \in I}$ is a flag in $G$, the amalgam of maximal parabolics with respect to the flag $F$ is the amalgam defined as follows. For each $i \in I$, let $G_i$ denote the stabilizer in $G$ of $F_i$. Let $A = \bigcup_{i \in I} G_i$. Define a partial multiplication on $A$ by defining $g_1g_2$ if and only if $g_1, g_2 \in G_i$ for some $i$, and in this case $g_1g_2$ is the product in $G_i$.

**Definition 1.3.20.** Let $\Gamma$ be a geometry and let $G \leq \text{Aut}(\Gamma)$. We say that $G$ acts flag transitively on $\Gamma$ if, given two flags $C, D$ of $\Gamma$ of the same type, there is an element $g \in G$ so that $g(C) = D$. If $\text{Aut}(\Gamma)$ acts flag transitively on $\Gamma$ then $\Gamma$ is called a flag transitive geometry.

**Note 1.3.4.** In what follows, if $\Gamma$ is a flag transitive geometry and $G \leq \text{Aut}(\Gamma)$ acts flag transitively, then we will omit explicit reference to the flag transitively of the geometry. So the statement “$G \leq \text{Aut}(\Gamma)$ acts flag transitively” implicitly includes the assumption that $\Gamma$ is a flag transitive geometry.

The connection between simple connectedness and amalgams is given by the following theorem, known as Tits’ Lemma.
Theorem 1.3.2 ([59], Corollaire 1). Let $\Gamma$ be a geometry and let $G \leq \text{Aut}(\Gamma)$ act flag transitively. Let $\mathcal{F} = \{F_i\}_{i \in I}$ be a chamber of $\Gamma$. Then $G$ is the universal completion of its amalgam of maximal parabolics with respect to the flag $\mathcal{F}$ if and only if $\Gamma$ is simply connected.

Remark. In Corollaire 1 of [59], Tits actually proves something slightly different, but it is easy to see that the result there implies what we have called Tits’ Lemma.

Note 1.3.5. For the reader who would prefer an English-language proof, Tits’ Lemma is proved as Corollary 3.4.2 of [24]. It is also proved as Corollary 1.4.6 of [38] under the additional hypothesis that the geometry is residually connected. We have recently learned that Tits’ Lemma was also proved in [53], which is likely the earliest proof.

Remark. While the amalgam of maximal parabolics seems to depend on the flag $\mathcal{F}$, the amalgams obtained by two different maximal flags are isomorphic (as amalgams) and so it makes sense to speak of the amalgam of maximal parabolics of $G$ with respect to its action on $\Gamma$, with no explicit reference to the particular maximal flag being considered.

To be more precise, let $C = (C_i)$ and $D = (D_i)$ be chambers. Let $g \in G$ with $g(C) = D$. Let $\mathcal{A}_C$, $\mathcal{A}_D$ be the amalgams of maximal parabolics relative to the flags $C$ and $D$ respectively.

Notice that if $H_i$ is the stabilizer in $G$ of $D_i$ then $g^{-1}H_ig$ is the stabilizer in $G$ of $C_i$. Thus conjugation by $g$ provides an isomorphism of amalgams $\mathcal{A}_D \to \mathcal{A}_C$.

There is another graph besides the incidence graph often associated with geometries. This collinearity graph will be useful in our study of geometries induced by flips of the unitary building. Before defining the collinearity graph we have to make precise what we mean by collinearity.

Definition 1.3.21. Let $\Gamma$ be a geometry with type set $I$ and let $i, j \in I$ be distinct types. Two objects $x, y$ of type $i$ are $j$-collinear if there is an object $z$ of type $j$ with $x \sim z$ and $y \sim z$.

Definition 1.3.22. Let $\Gamma$ be a geometry with type set $I$. Choose distinct types $i, j \in I$. The $(i, j)$-collinearity graph of $\Gamma$ is the graph whose vertices are the objects of type $i$, with
two vertices joined by an edge if they are \( j \)-collinear in \( \Gamma \). The collinearity graph is denoted \( G_{i,j}(\Gamma) \) or if the context permits \( G(\Gamma) \).

**Note 1.3.6.** In the cases seen in this dissertation, the objects will be subspaces of a vector space, and we will choose 1 and 2 dimensional subspaces respectively, so that two 1 dimensional subspaces are collinear if and only if they span a 2 dimensional element of the geometry. In this context we will use the notation \( G(\Gamma) \), since we will only be interested in the collinearity graph coming from the 1 and 2 dimensional objects of the geometry.

We now recall some basic definitions from graph theory. These definitions mirror various definitions for geometries and simplicial complexes.

**Definition 1.3.23.** A (simple) graph is a pair \( (V,E) \) where \( V \) is a non-empty set, and \( E \) is a set of 2-element subsets of \( V \). The elements of \( V \) are called vertices, and the elements of \( E \) are called edges. If \( u, v \) are vertices we say they are joined by an edge if \( \{u,v\} \in E \).

**Definition 1.3.24.** Let \( K = (V,E) \) be a graph. A path in \( K \) is a sequence of vertices \( x_1, x_2, \ldots, x_n \) such that \( \{x_i, x_{i+1}\} \) is an edge for all \( i = 1, \ldots, n - 1 \). The path \( x_1, \ldots, x_n \) is a cycle if \( x_1 = x_n \). The graph is connected if any two vertices can be joined by a path. The length of the path \( x_1, x_2, \ldots, x_n \) is \( n - 1 \). The distance between two vertices \( v_1, v_2 \) is the length of the shortest path between them, and is denoted \( d_K(u,v) \) or \( d(u,v) \) if the context permits. The diameter of \( K \) is \( \max\{d_K(u,v) \mid u,v \in V\} \) provided this maximum exists, and is \( \infty \) otherwise.

**Theorem 1.3.3.** Let \( \Gamma \) be a geometry with type set \( I \). If there exist distinct types \( i, j \in I \) such that \( G_{i,j}(\Gamma) \) is connected then \( \Gamma \) is connected.

**Proof.** Let \( x, y \) be objects of \( \Gamma \). Since \( \Gamma \) is a geometry there are objects of type \( i, x_i \) and \( y_i \) with \( x_i \sim x \) and \( y_i \sim y \). Since \( G_{i,j}(\Gamma) \) is connected there is a path \( \gamma = x_1n_1 \ldots n_ky_i \) of objects of type \( i \) and \( j \) connecting \( x_i \) to \( y_i \), and so \( x\gamma y \) is a path in \( \Gamma \) from \( x \) to \( y \). Hence \( \Gamma \) is connected. \( \Box \)
Remark. If $\Gamma$ is residually connected then the converse of Theorem 1.3.3 also holds. In the geometries studied in this dissertation a partial converse of Theorem 1.3.3 will hold, in particular that if $\Gamma$ is connected then $G_{0,1}(\Gamma)$ is connected. We will not have occasion to use this converse, but it is interesting that this partial converse holds. 

When we study connectedness of geometries related to flips, we will actually study connectedness of the collinearity graph. This is a useful technique, since when we study the homotopy properties of these geometries it will suffice to consider cycles in the collinearity graph.

1.4 Polar Geometries

The unitary building is the building associated to a polar geometry, so in this section we define the required notions related to polar spaces and polar geometries.

Note 1.4.1. Throughout this section, $k$ denotes a finite field, $\sigma$ denotes an automorphism of $k$ and $V$ denotes a finite dimensional $k$-vector space.

Definition 1.4.1. A $\sigma$-sesquilinear form on $V$ is a function $\beta : V \times V \to k$ that is linear in the first argument and $\sigma$-semilinear in the second argument. If $\sigma$ is the identity, we call $\beta$ bilinear. If $\mathcal{U} = \{v_1, \ldots, v_n\}$ is a basis for $V$, then the Gram matrix of $\beta$ with respect to the basis $\mathcal{U}$, denoted $\beta(\mathcal{U})$, is the matrix $(b_{ij})$ where $b_{ij} = \beta(v_i, v_j)$.

Definition 1.4.2. Let $\beta$ be a $\sigma$-sesquilinear form on $\beta$.

(i) The form is called reflexive if $\beta(u, v) = 0$ implies $\beta(v, u) = 0$.

(ii) Two vectors $u, v \in V$ are called orthogonal if $\beta(u, v) = 0$. In this case, we may write $u \perp v$. Notice that $\beta$ is reflexive if and only if $\perp$ defines a symmetric relation on $V$.

(iii) If $X$ is a subset of $V$, we write

$$X^\perp = \{v \in V | \beta(v, x) = 0 \text{ for all } x \in X\}$$
and call $X^\perp$ the **orthogonal complement** of $X$. This is a subspace of $V$.

(iv) The form $\beta$ is called **non-degenerate** if $V^\perp = \{0\}$.

(v) The **radical** of a subspace $U$ of $V$ is $\text{Rad}(U) = U \cap U^\perp$. $U$ is called **non-degenerate** if $\text{Rad}(U) = \{0\}$.

**Theorem 1.4.1** ([56], Theorem 7.1). Let $\beta$ be a reflexive $\sigma$-sesquilinear form. Then $\beta$ is (proportional to) a form of one of the following types:

(i) **Alternating** if $\sigma = 1$ and $\beta(v, v) = 0$ for all $v \in V$;

(ii) **Symmetric** if $\sigma = 1$ and $\beta(u, v) = \beta(v, u)$ for all $u, v \in V$;

(iii) **Hermitian** if $\sigma^2 = 1$, $\sigma \neq 1$ and $\beta(u, v) = \sigma(\beta(v, u))$ for all $u, v \in V$.

**Remark.** If $\beta$ is non-degenerate and $X$ is a subspace of $V$, then $\dim X + \dim X^\perp = \dim V$. It follows that if $X$ is a non-degenerate subspace then $V = X \oplus X^\perp$. ♦

**Definition 1.4.3.** If $\beta$ is a symmetric bilinear form on $V$ over a field of odd characteristic, its **associated quadratic form** is defined by $Q(u) = \frac{1}{2} \beta(u, u)$.

**Remark.** For our purposes, the introduction of the quadratic form will be a notational convenience. In characteristic 2 we can begin with a quadratic form $Q$ and then produce its associated symmetric bilinear form $\beta$ by $\beta(u, v) = Q(u + v) - Q(u) - Q(v)$. In characteristic 2, the quadratic form is often more important than the symmetric bilinear form. In odd characteristic they are equivalent. ♦

**Definition 1.4.4.** Let $\beta$ be a non-degenerate form of one of the types in Theorem 1.4.1

(i) If $u \neq 0$ and $\beta(u, u) = 0$ then $u$ is called **isotropic**.

(ii) If $W$ is a subspace of $V$ and satisfies $W \subseteq W^\perp$ then $W$ is called **totally isotropic**.
(iii) If $\beta$ is a symmetric bilinear form and $Q$ is its associated quadratic form, a vector $u \neq 0$ is called **singular** if $Q(u) = 0$, and a subspace $W$ is called **totally singular** if $Q(u) = 0$ for all $u \in W$.

(iv) If $u$ and $v$ are isotropic vectors and $\beta(u, v) \neq 0$ then we call the pair $(u, v)$ a **pre-hyperbolic pair** and the space $\langle u, v \rangle$ is called a **hyperbolic line**. If $\beta(u, v) = 1$ we will call the pair a **hyperbolic pair**.

(v) If $V = U \oplus W$ and $\beta(u, w) = 0$ for all $u \in U$ and $w \in W$ we say that $V$ is the **orthogonal direct sum** of $U$ and $W$, and we write $V = U \perp W$.

**Note 1.4.2.** Our notion of pre-hyperbolic pair is not standard. Some authors use the term hyperbolic pair to refer to what we have called a pre-hyperbolic pair. The distinction will prove important in this dissertation and so we have elected to introduce the new term and restrict the term hyperbolic pair to the case where $\beta(u, v) = 1$.

**Definition 1.4.5.** Let $\beta$ be a non-degenerate reflexive $\sigma$-sesquilinear form on $V$ of one of the types in Theorem 1.4.1. The **polar geometry** of $(V, \beta)$ is the geometry whose objects are the non-zero $\beta$ totally isotropic proper subspaces of $V$ with incidence given by symmetrized inclusion. The type of an object $U$ is $(\dim_k U - 1)$.

The polar geometry $(V, \beta)$ is called **symplectic** if the form is alternating and bilinear, **orthogonal** if the form is symmetric and bilinear, and **unitary** if the form is $\sigma$-hermitian where $\sigma$ is a non-trivial automorphism of $k$.

**Definition 1.4.6.** Suppose $\beta_1$ and $\beta_2$ are reflexive $\sigma_1$ and $\sigma_2$ sesquilinear forms on vector spaces $V_1$ and $V_2$ over fields $K_1$ and $K_2$. A $\sigma$-semilinear map $f : V_1 \to V_2$ is called an **isometry** if it is injective, $\sigma_2 \sigma = \sigma \sigma_1$ and for all $u, v \in V_1$, $\beta_2(f(u), f(v)) = \sigma(\beta_1(u, v))$.

In the next section we will say more about the isometry groups of polar spaces.

**Lemma 1.4.2** ([56], Lemma 7.3). Let $L$ be a non-degenerate 2-dimensional subspace of $V$ containing an isotropic vector $u$. Then $L = \langle u, v \rangle$ where $(u, v)$ is a hyperbolic pair.
The next theorem is known as Witt’s Lemma.

**Theorem 1.4.3** ([56], Lemma 7.4). Let $U$ be a subspace of $V$, and let $f : U \to V$ be a linear isometry. There is a linear isometry $g : V \to V$ such that $g(u) = f(u)$ for all $u \in U$ if and only if

$$f(U \cap \text{Rad}(V)) = f(U) \cap \text{Rad}(V).$$

**Note 1.4.3.** It follows from Theorem 1.4.3 that any two maximal totally isotropic subspace of $V$ have the same dimension, called the Witt index of the form $\beta$.

There are many interesting things to be said about projective and polar geometries, but we have chosen to omit them because they are not crucial to the results of this dissertation. Some important references are [15], [56] and [58].

## 1.5 Isometry and Similitude Groups of Polar Geometries

**Definition 1.5.1.** Let $V$ be an $n$-dimensional $k$ vector space endowed with a non-degenerate $\sigma$-sesquilinear form $\beta$. In general, the similitude group of the polar space $(V, \beta)$ is the group of all $\tau$-semilinear transformations ($\tau \in \text{Aut}(k)$) $f$ of $V$ with the property that there exists $a \in k$ with $\sigma(a) = a$ so that for all $u, v \in V$

$$\beta(f(u), f(v)) = a\tau(\beta(u, v))$$

(i) If $\beta$ is alternating and bilinear, this is denoted $\Gamma\text{Sp}(V)$.

(ii) If $\beta$ is symmetric and bilinear, this is denoted $\Gamma\text{O}(V)$.

(iii) If $\beta$ is $\sigma$-hermitian, this is denoted $\Gamma\text{U}(V)$.

We will also have occasion to consider groups of linear similitudes, that is similitudes where $\tau = \text{id}$. These are denoted as follows.
(i) If $\beta$ is alternating and bilinear, this is denoted $\text{GSp}(V)$.

(ii) If $\beta$ is symmetric and bilinear, this is denoted $\text{GO}(V)$.

(iii) If $\beta$ is $\sigma$-hermitian, this is denoted $\text{GU}(V)$.

A $\tau$-semilinear similitude $f$ is an **anti-isometry** if for all $u, v \in V$

$$\beta(f(u), f(v)) = -\tau(\beta(u, v)).$$

A $\tau$-semilinear similitude $f$ is an **isometry** if for all $u, v \in V$

$$\beta(f(u), f(v)) = \tau(\beta(u, v)).$$

We will be interested in the group of linear isometries of $(V, \beta)$. The notation for these groups follows.

(i) If $\beta$ is alternating and bilinear, this group is denoted $\text{Sp}(V)$.

(ii) If $\beta$ is symmetric and bilinear, this group is denoted:

(a) $\text{O}(V)$ if $n$ is odd; 
(b) $\text{O}^+(V)$ if $n = 2m$ and the Witt index of $(V, \beta)$ is $m$; 
(c) $\text{O}^-(V)$ if $n = 2m + 2$ and the Witt index of $(V, \beta)$ is $m$.

(iii) If $\beta$ is $\sigma$-hermitian, the group is denoted $\text{U}(V)$.

In all cases, we may include the name of the form in the name of the group. For example, we may write $\text{U}(V, \beta)$ to denote the group of linear isometries of the unitary space $(V, \beta)$. We will only do this if it is necessary to distinguish between different forms in a single result.

Each of these groups has a subgroup consisting of those transformations that have determinant 1 with respect to some (and hence all) bases. These are denoted by placing an “S” in front of the name for the group.
Each of these various groups also have projective cousins, denoted by adding a “P” to the front of the names, and obtained by taking the quotient of each group by its center.

There is one additional notation of which the reader should be aware. If \((V, \beta)\) is a polar space and we write \(O(V, \beta)\) we mean the isometry group of \((V, \beta)\). This notation is convenient if we do not wish to specify the type of form.

Remark. In principle our notation is ambiguous, since there could be different isometry groups on the same space \(V\), arising from different \(\sigma\)-sesquilinear forms. However, when the field is finite there is, up to isomorphism, a unique unitary group, a unique symplectic group, a unique orthogonal group if \(\dim V\) is odd, and only the two listed orthogonal groups if \(\dim V\) is even.

The groups that are mainly of interest to us in this dissertation are the groups \(U(V)\) and the groups \(O'(V)\).

Lemma 1.5.1 ([56] p.61). Let \((V, \beta)\) be a reflexive non-degenerate polar space. Let

\[ \mathcal{V} = \{v_1, \ldots, v_n\} \]

be a basis for \(V\), and let \(M = \beta(\mathcal{V})\). Let \(f \in \text{GL}(V)\) and let \(A\) be the matrix of \(f\) with respect to the basis \(\mathcal{V}\).

(i) If \(\beta\) is alternating and bilinear then \(f \in \text{Sp}(V)\) if and only if

\[ tAMA = M. \]

(ii) If \(\beta\) is symmetric and bilinear then \(f \in \text{O}(V)\) if and only if

\[ tAMA = M. \]
(iii) If $\beta$ is $\sigma$-hermitian, then $f \in U(V)$ if and only if

$$^tAMA^\sigma = M.$$ 

Remark. This result is proved in [56] on page 61, but the author does not present the result as a lemma (or theorem, or anything), it is just a discussion in the text which results in the three conclusions above when the particular type of form is specified.

\[
\diamond
\]

Concluding Remarks

We’ve now met most of the major topics required in this dissertation. The connections between groups and geometries provide most of the techniques employed, however they do not provide all of the necessary motivation. Some excellent references for the geometric concepts are [15] and [56].

In the next chapter we introduce the theory of buildings, which provide an overarching structure to much of the geometric theory, as well as much of the motivation for studying flips.
CHAPTER 2

Buildings

Introduction

In this chapter we introduce the theory of buildings in two different ways. There are at least three different ways to define a building, however only two of them will be useful for us. The chapter begins with a discussion of Coxeter groups and Coxeter complexes, then proceeds to introduce the definition of a building. The third section discusses twin buildings, and flips, which arise as automorphisms of twin buildings. We then discuss the particular case of spherical buildings, and flips of spherical buildings. The chapter finishes with a discussion of the building associated to an incidence geometry, and some properties of these buildings.

2.1 Coxeter Groups and Coxeter Complexes

All the various definitions of buildings are built in some way upon Coxeter groups and Coxeter complexes. The material in this section follows [34], with the exception of the construction of the Coxeter complex, which follows [56].

Definition 2.1.1. A Coxeter matrix of size $n$ is an $n \times n$ matrix $M = (m_{ij})$ with entries in $\mathbb{N} \cup \{\infty\}$ and the following properties:
(CM1) \( m_{ij} = 1 \) if and only if \( i = j \);

(AM2) \( M \) is symmetric, that is \( m_{ij} = m_{ji} \) for all \( i, j \).

**Note 2.1.1.** From now on, we assume that the entries of a Coxeter matrix all lie in \( \mathbb{N} \). In general this is not required, but for the Coxeter matrices of use in this dissertation, all the entries lie in \( \mathbb{N} \). A Coxeter matrix whose entries all lie in \( \mathbb{N} \) is called **2-spherical**.

**Definition 2.1.2.** Let \( M \) be a Coxeter matrix of size \( n \). The **Coxeter diagram** of \( M \) is the graph with vertices indexed with the integers \( 1, \ldots, n \) with edges joining two nodes \( i \) and \( j \) if \( m_{ij} \geq 3 \). If \( m_{ij} = 3 \) there is 1 edge, if \( m_{ij} = 4 \) there are 2 edges, if \( m_{ij} = 6 \) there are 3 edges and if \( m_{ij} = 8 \) there are 4 edges. Other values of \( m_{ij} \) are indicated by writing \( m_{ij} \) over the edge.

Remark. The special cases \( m_{ij} = 2, 3, 4, 6, 8 \) are important because they are the only values that occur in Coxeter diagrams of spherical Moufang buildings. See for example [58].

**Definition 2.1.3.** Let \( M \) be a Coxeter matrix of size \( n \). A **Coxeter system** of type \( M \) is a group \( W \) together with a subset \( S = \{ s_1, \ldots, s_n \} \) of \( W \) with

\[
W = \langle s_1, \ldots, s_n | (s_i s_j)^{m_{ij}} = 1 \text{ for all } i = 1, \ldots, n \rangle.
\]

A **Coxeter group** of type \( M \) is a group \( W \) that contains a subset \( S = \{ s_1, \ldots, s_n \} \) such that \( (W, S) \) is a Coxeter system of type \( M \). The integer \( n \) is the **rank** of the matrix/system/group/diagram.

A Coxeter matrix/system/group/diagram is called **spherical** if its associated group is finite.

**Definition 2.1.4.** A Coxeter system/group/diagram \( (W, S) \) with matrix \( M \) is called **irreducible** if \( S \) cannot be expressed as the disjoint union of two subsets \( J, K \) with \( m_{jk} = 2 \) for all \( j \in J, k \in K \).
Figure 2.1: Irreducible spherical Coxeter diagrams

\[ A_n(n \geq 1) : \quad \begin{array}{c}
\bullet - \bullet - \bullet - \ldots - \bullet - \bullet - \bullet \\
\end{array} \]

\[ B_n/C_n(n \geq 2) : \quad \begin{array}{c}
\bullet - \bullet - \bullet - \ldots - \bullet - \bullet - \bullet \\
\end{array} \]

\[ D_n(n \geq 4) : \quad \begin{array}{c}
\bullet - \bullet - \bullet - \ldots - \bullet - \bullet - \bullet \\
\end{array} \]

\[ E_6 : \quad \begin{array}{c}
\bullet - \bullet - \bullet - \bullet - \bullet - \bullet \\
\end{array} \]

\[ E_7 : \quad \begin{array}{c}
\bullet - \bullet - \bullet - \bullet - \bullet - \bullet - \bullet \\
\end{array} \]

\[ E_8 : \quad \begin{array}{c}
\bullet - \bullet - \bullet - \bullet - \bullet - \bullet - \bullet - \bullet \\
\end{array} \]

\[ F_4 : \quad \begin{array}{c}
\bullet - \bullet - \bullet - \bullet - \bullet \\
\end{array} \]

\[ H_3 : \quad \begin{array}{c}
5 \quad \bullet - \bullet - \bullet \\
\end{array} \]

\[ H_4 : \quad \begin{array}{c}
5 \quad \bullet - \bullet - \bullet - \bullet \\
\end{array} \]

\[ I_n(\text{ or } G_2 \text{ if } n = 6) : \quad \begin{array}{c}
\bullet - \bullet \quad n \quad \bullet \\
\end{array} \]
The irreducible spherical Coxeter diagrams have been classified. We have included this classification because we will need to use the fact that the building of a unitary geometry is of type $C_n$.

**Theorem 2.1.1.** Every irreducible spherical Coxeter diagram is on the list in Figure 2.1.

The reader interested in a proof of this may consult Chapter 2 of [31].

**Definition 2.1.5.** Let $(W, S)$ be a Coxeter system. Define the length of an element $w \in W$ to be the minimal number of elements of $S$ required to express $w$. This is denoted $l(w)$.

**Note 2.1.2.** If $(W, S)$ is a spherical Coxeter system then there is a unique word of maximal length, called the longest word, and denoted $w_0$. It is always an involution. The existence of $w_0$ is important in the study of spherical buildings.

**Construction 3.** We now construct the Coxeter complex associated to the Coxeter system $(W, S)$. It is a simplicial complex crucial to the study (and even definition) of buildings. The Coxeter complex is denoted $\Sigma(W, S)$.

Let $(W, S)$ be a Coxeter system with $S = \{s_i| i \in I\}$. For all $J \subseteq I$, we define

$$W_J = \langle s_j| j \in J\rangle$$

For each $i \in I$, let $W^i = W_{I \setminus \{i\}}$. The vertices of $\Sigma(W, S)$ are the cosets $wW^i$ for $w \in W$, $i \in I$. The simplices of $\Sigma(W, S)$ are sets of the form $\{wW^{i_1}, \ldots, wW^{i_k}\}$ for some $w \in W$, $i_1, \ldots, i_k \in I$.

We can recognize the Coxeter complex as the flag complex of a geometry. The objects of this geometry are the cosets of the subgroups $W^i$. The type of a coset $wW^i$ is $i$, and two objects are incident if they have non-empty intersection. The flag complex of this geometry is the Coxeter complex.
2.2 Two Definitions of Buildings

In this section we will define buildings in two ways, both of which will prove useful. The first definition is older, and makes clear that what we later call the building associated to a polar geometry is in fact a building, while the second definition is the one in which the notion of a flip is more easily expressed. The reader interested in the connection should consult [2] where the equivalence is explored.

**Definition 2.2.1.** A building is a simplicial complex $K$ that is a union of special subcomplexes, called apartments satisfying:

(i) Each apartment $\Sigma$ is a Coxeter complex;

(ii) If $A$, $B$ are simplices of $K$ there is an apartment $\Sigma$ containing both $A$ and $B$.

(iii) If $\Sigma$ and $\Sigma'$ are two apartments both containing simplices $A$ and $B$, then there is an isomorphism $\Sigma \to \Sigma'$ that fixes both $A$ and $B$ point-wise.

**Remark.** A consequence of the axioms for a building is that all the apartments are Coxeter complexes of the same type. To verify this, let $A_1$ and $A_2$ be apartments, and choose simplices $A_1 \in A_1$ and $A_2 \in A_2$. Then by axiom (ii) there is an apartment $A$ containing both $A_1$ and $A_2$, and so by axiom (iii), $A_1 \cong A \cong A_2$.

**Note 2.2.1.** The maximal simplices are called chambers. There is a definition of buildings in terms of chamber systems. We have omitted this definition because we have no need to use any of the properties best expressed in terms of chamber systems, however the notion of a chamber will reappear when we discuss the chamber system induced by a flip. The chamber system induced by a flip really is a chamber system, but we will not need to use this fact.

**Definition 2.2.2.** Let $(W, S)$ be a Coxeter system. A building of type $(W, S)$ is a non-empty set $C$ together with a map $\delta : C \times C \to W$ such that for all $C, D \in C$ we have:
(i) \(\delta(C, D) = 1\) if and only if \(C = D\);

(ii) If \(\delta(C, D) = w\) and \(C' \in \mathcal{C}\) with \(\delta(C', C) = s \in S\) then \(\delta(C', D) = sw\) or \(w\). Moreover if \(l(sw) = l(w) + 1\) then \(\delta(C', D) = sw\).

(iii) If \(\delta(C, D) = w\) then for any \(s \in S\) there is an element \(C' \in \mathcal{C}\) with \(\delta(C', C) = s\) and \(\delta(C', D) = sw\).

The elements of \(\mathcal{C}\) are called chambers.

This definition is sometimes called the \(W\)-metric space definition, since it allows us to view buildings as metric spaces where the metric takes its values in a Coxeter group \(W\).

We will define an automorphism of a building in terms of the \(W\)-metric definition. There is a corresponding definition for the simplicial complex definition, but we will not need that definition.

**Definition 2.2.3.** Let \((\Delta, \delta)\) be a building of type \((W, S)\). An automorphism of \((\Delta, \delta)\) is a bijection \(f : \Delta \to \Delta\) that preserves \(\delta\). The collection of automorphisms of \((\Delta, \delta)\) forms a group denoted \(\text{Aut}((\Delta, \delta))\) or if the context permits, \(\text{Aut}(\Delta)\). This group is the automorphism group of \((\Delta, \delta)\).

**Remark.** What we have called automorphisms are sometimes called isometries of the building, with the term automorphism reserved for a larger class of maps. For the building studied in this work the two terms are equivalent.

**Definition 2.2.4.** A building of type \((W, S)\) is called spherical if \((W, S)\) is a spherical Coxeter system.

**Definition 2.2.5.** Let \(\mathcal{C}\) be a spherical building of type \((W, S)\). Two chambers \(C\) and \(D\) are opposite if \(\delta(C, D) = w_0\), where \(w_0\) is the longest word of \((W, S)\).

**Remark.** The notion of opposition is crucial to the study of spherical buildings, and will prove important in the study of flips.
2.3 Twin Buildings and Flips

Twin buildings were first introduced in an attempt to extend the notion of opposition from spherical buildings to a wider class of buildings. Flips arise as automorphisms of twin buildings. The material on twin buildings in this section is derived from [2].

**Definition 2.3.1.** Let \((W, S)\) be a Coxeter system. A twin building of type \((W, S)\) is a triple \(((C_+, \delta_+), (C_-, \delta_-), \delta^*)\) where \((C_+, \delta_+)\) and \((C_-, \delta_-)\) are buildings of type \((W, S)\) and \(\delta^* : (C_+ \times C_-) \cup (C_- \times C_+) \to W\) is a function that satisfies the following for \(\epsilon \in \{\pm\}, C \in C_\epsilon, D \in C_{-\epsilon}\) with \(w = \delta^*(C, D)\):

(i) \(\delta^*(C, D) = \delta^*(D, C)^{-1}\);

(ii) if \(C' \in C_\epsilon\) with \(\delta_\epsilon(C', C) = s \in S\) and \(l(sw) < l(w)\) then \(\delta^*(C', D) = sw\);

(iii) for any \(s \in S\) there is a chamber \(C' \in C_\epsilon\) with \(\delta_\epsilon(C', C) = s\) and \(\delta^*(C', D) = sw\).

The map \(\delta^*\) is called a codistance or twinning.

**Definition 2.3.2.** Let \(B = ((C_+, \delta_+), (C_-, \delta_-), \delta^*)\) be a twin building of type \((W, S)\). An automorphism of \(B\) is a map \(f : (C_+ \cup C_-) \to (C_+ \cup C_-)\) such that one of the following lists of conditions holds.

(A1) for \(\epsilon \in \{\pm\}, f(C_\epsilon) = C_\epsilon\) and, for all \(u_1, u_2 \in C_\epsilon\) and \(v \in C_{-\epsilon}\):

(i) \(\delta_\epsilon(u_1, u_2) = \delta_\epsilon(f(u_1), f(u_2))\);

(ii) \(\delta^*(u_1, v) = \delta^*(f(u_1), f(v))\).

(A2) for \(\epsilon \in \{\pm\}, f(C_\epsilon) = C_{-\epsilon}\) and, for all \(u_1, u_2 \in C_\epsilon\) and \(v \in C_{-\epsilon}\):

(i) \(\delta_{-\epsilon}(u_1, u_2) = \delta_{-\epsilon}(f(u_1), f(u_2))\);

(ii) \(\delta^*(u_1, v) = \delta^*(f(u_1), f(v))\).

The group of all automorphisms of \(B\) is denoted \(\text{Aut}(B)\).
Note 2.3.1. Notice in particular that an automorphism of a twin building either maps $C_{\epsilon}$ to itself for $\epsilon \in \{\pm\}$ or it maps $C_{\epsilon}$ to $C_{-\epsilon}$.

Remark. Just as in the case of an automorphism of a building, what we have called an automorphism is sometimes called an isometry, with the term automorphism applied to a larger class of maps. Since we will not need any of these other maps we have chosen to restrict the term automorphism to the maps in which we are interested.

Definition 2.3.3. Let $((C_+, \delta_+), (C_-, \delta_-), \delta^*)$ be a twin building of type $(W, S)$. Let $C \in C_{\epsilon}$ and $D \in C_{-\epsilon}$ be chambers. We say that $C$ and $D$ are opposite if $\delta^*(C, D) = e \in W$.

Definition 2.3.4. Let $B = ((C_+, \delta_+), (C_-, \delta_-), \delta^*)$ be a twin building of type $(W, S)$. A flip is an involution in $\text{Aut}(B)$ satisfying:

(i) $f(C_+) = C_-;$

(ii) $\delta_\epsilon(C, D) = \delta_{-\epsilon}(f(C), f(D))$ for all $C, D \in C_{\epsilon}$ and $\epsilon \in \{\pm\}$;

(iii) $\delta^*(C, D) = \delta^*(f(C), f(D))$ for all $C \in C_{\epsilon}$ and $D \in C_{-\epsilon}$;

(iv) There is a chamber $C \in C_{\epsilon}$ with $\delta^*(C, f(C)) = e \in W$.

Remark. The definition of a flip is taken from [7].

Flips are studied in greater detail in [27] and [32], although the authors do not make a systemic study of the flips of any particular building.

We now turn to the particular case where $(W, S)$ is spherical. In this case there is up to isometry a unique twin building, and the axioms for a flip have a slightly different flavor.

Construction 4. Let $(W, S)$ be a spherical Coxeter system with longest word $w_0$. Recall that $w_0 = w_0^{-1}$. Let $(C, \delta)$ be a building of type $(W, S)$. Let $C_+$ and $C_-$ be disjoint copies of $C$. Define $\delta_\epsilon : C_{\epsilon} \to W$ by

$$\delta_+(C, D) = \delta(C, D)$$
$$\delta_-(C, D) = w_0\delta(C, D)w_0$$
and define
\[
\delta^*(C_+, D_-) = \delta(C, D)w_0 \\
\delta^*(C_-, D_+) = w_0\delta(C, D)
\]

Then \(((C_+, \delta_+), (C_-, \delta_-), \delta^*)\) forms a twin building of type \((W, S)\).

**Theorem 2.3.1** ([60], Proposition 2). Let \(\Lambda = ((C_+, \delta_+), (C_-, \delta_-), \delta^*)\) be a twin building of type \((W, S)\) where \((W, S)\) is a spherical Coxeter system. Then \(\Lambda\) is isometric to a twin building of the form produced in Construction 4.

It follows from Theorem 2.3.1 that each spherical building has up to isometry a unique twin, obtained by following Construction 4. It therefore makes sense to talk about the twinning of a spherical building, and so when studying flips of twinned spherical buildings it suffices to assume that the building is produced as in Construction 4. With this in mind we can state a different (but easily seen to be equivalent) definition for a flip of a spherical building.

**Definition 2.3.5.** Let \((C, \delta)\) be a spherical building of type \((W, S)\) and let \(w_0\) be the longest word of \((W, S)\). A flip is a map \(f : C \to C\) such that for all \(C, D \in C\):

(i) \(f^2(C) = C\);

(ii) \(\delta(C, D) = w_0\delta(f(C), f(D))w_0\);

(iii) There exists \(C \in C\) such that \(\delta(C, f(C)) = w_0\).

**Note 2.3.2.** It follows from (ii) that a flip is an isometry of the building if and only if \(w_0\) is central in \(W\). In particular this holds if \((W, S)\) is of type \(B_n, C_n, E_7, E_8, F_4, G_2, D_{2n}\). It is interesting to note that a flip need not be an isometry of a spherical building, however if \(f\) is a flip and we follow Construction 4 and we view \(f\) as mapping \(C_+\) to \(C_-\) then we obtain an isometry of the twin building.
2.4 The Building of a Polar Geometry

We now consider the case of a polar geometry.

**Definition 2.4.1.** Let \((V, \beta)\) be a polar geometry. The **building of** \((V, \beta)\) is the flag complex \(\Sigma((V, \beta))\).

It is shown in [58] that the flag complex of a polar geometry is a building of type \(C_n\) in the sense of Definition 2.2.1. We now describe the apartments of the building of a polar geometry.

**Construction 5.** Let \(k\) be a finite field, let \(V\) be an \(n\)-dimensional \(k\)-vector space and let \(\sigma \in \text{Aut}(k)\) with \(\sigma^2 = e\). Let \(\beta\) be a non-degenerate reflexive \(\sigma\)-sesquilinear form on \(V\) of Witt index \(m > 0\). Let \(\Gamma\) denote the polar geometry of \((V, \beta)\). Recall that \(\Sigma(\Gamma)\) denotes the flag complex of \(\Gamma\).

Let \(U_1 = \langle e_1, \ldots, e_m \rangle\) be a maximal totally isotropic subspace of \(V\). It can be shown (see Lemma 7.5 of [56]) that there is a totally isotropic subspace \(U_2 = \langle f_1, \ldots, f_m \rangle\) of \(V\) such that \((e_1, f_1), \ldots, (e_m, f_m)\) are pairwise orthogonal hyperbolic pairs. The **polar frame** associated to \(\{U_1, U_2\}\) is

\[
\mathcal{F} = \{\langle e_i \rangle, \langle f_i \rangle | 1 \leq i \leq m\}.
\]

Notice that it is the subspaces \(\langle e_i \rangle, \langle f_i \rangle\) that define the polar frame, not the particular vectors \(e_i, f_i\). Hence different hyperbolic bases for \(V\) may give rise to the same polar frame.

The apartment of \(\mathcal{F}\) in \(\Sigma(\Gamma)\) consists of all flags \(F\) that are spanned by some subset of \(\{e_1, f_1, \ldots, e_m, f_m\}\). This apartment is denoted \(\Sigma(\mathcal{F})\).

Every apartment of \(\Sigma(\Gamma)\) is of the form \(\Sigma(\mathcal{F})\) for some polar frame \(\mathcal{F}\).

The reader interested in more detail should consult Chapter 7 of [56]. The notation here differs slightly from the notation in [56].

**Note 2.4.1.** In what follows, if \(\mathcal{F} = \{\langle e_i \rangle, \langle f_i \rangle | i = 1, \ldots, m\}\) is a polar frame we denote the
apartment $\Sigma(\mathcal{F})$ by

$$\Sigma(\mathcal{F}) = \Sigma(\langle e_i \rangle, \langle f_i \rangle | i = 1, \ldots, m)$$

or if the context permits,

$$\Sigma(\mathcal{F}) = \Sigma(e_i, f_i).$$

This last notation is somewhat of an abuse, since the collection of pairwise orthogonal hyperbolic pairs is not uniquely determined by the polar frame, but if we start with this collection we know the frame, and hence the apartment.

Now that we know what the apartments of $\Sigma(\Gamma)$ look like, we can describe when two chambers are opposite.

**Theorem 2.4.1.** Two chambers $C = (C_i)_{i=1}^n$ and $D = (D_i)_{i=1}^n$ in the building of a non-degenerate polar geometry $(V, \beta)$ of rank $n > 0$ are opposite if and only if for all $i$,

$$C_i^\perp \cap D_i = \{0\}.$$

**Remark.** This result is well known, and in fact is an exercise in [56]. Unfortunately we have been unable to find a reference for a proof, so we have decided to include our own.

**Proof.** Let $(V, \beta)$ be a non-degenerate polar space of Witt index $n > 0$. It can be shown that in a spherical building, two chambers are opposite if and only if they are contained in a unique apartment. Let $C = (C_i)_{i=1}^n$ and $D = (D_i)_{i=1}^n$ be chambers of the building associated to the polar geometry.

Suppose first that $C_i^\perp \cap D_i = \{0\}$ for all $i$. Let $\Sigma = \Sigma(e_i, f_i | i = 1, \ldots, n)$ be an apartment containing both $C$ and $D$. Without loss of generality, we may assume that for all $i$,

$$C_i = \langle e_1, \ldots, e_i \rangle.$$

Since $C_i^\perp \cap D_i = \{0\}$ for all $i$, it follows that $D_i = \langle f_1, \ldots, f_i \rangle$ for all $i$. We now show this
condition forces $\Sigma$ to be the only apartment containing both $C$ and $D$.

Let $\Sigma' = \Sigma'(g_i, h_i)$ be another apartment containing both $C$ and $D$. Without loss of
generality we may assume that for all $i$, $C_i = \langle g_1, \ldots, g_i \rangle$. It follows that $\langle g_1 \rangle = \langle e_1 \rangle$. Since
$C_1^+ \cap D_1 = \{0\}$ it follows that $D_1 = \langle h_1 \rangle$ and so $\langle h_1 \rangle = \langle f_1 \rangle$.

Since $C_2^+ \cap D_2 = \{0\}$ it follows that $D_2 = \langle h_1, h_2 \rangle$. We already know that $D_2 = \langle f_1, f_2 \rangle$
and so $h_2 = \alpha_1 f_1 + \alpha_2 f_2$. If $\alpha_1 \neq 0$ then

$$\beta(g_1, h_2) = \beta(g_1, \alpha_1 f_1 + \alpha_2 f_2) = \beta(g_1, \alpha_1 f_1) \neq 0$$

since $g_1 = \lambda e_1$ and $\beta(e_1, f_1) = 1$, contradicting the assumption that $\{(g_i, h_i)\}_{i=1}^n$ is a collection
of pairwise orthogonal hyperbolic pairs. Thus $h_2 = \alpha_2 f_2$ and so $\langle h_2 \rangle = \langle f_2 \rangle$. Similar
arguments show that for all $i = 1, \ldots, n$

$$\langle e_i \rangle = \langle g_i \rangle \text{ and } \langle f_i \rangle = \langle h_i \rangle.$$ 

Thus $\Sigma = \Sigma'$ and so $C$ and $D$ are contained in a unique apartment, and hence they are
opposite chambers.

To prove the converse, notice that if $C_j^+ \cap D_j \neq \{0\}$ for some $j$, then there is more than
one apartment containing both $C$ and $D$, and so they are not opposite. More precisely,
choose an apartment $\Sigma(e_i, f_i)$ containing both $C$ and $D$. Without loss of generality we may
assume that $C_i = \langle e_1, \ldots, e_i \rangle$ for all $i$. Choose $j$ minimal so that $C_j^+ \cap D_j \neq \{0\}$. This forces
$e_j \in D_j$, and for $i < j$, $D_i = \langle f_1, \ldots, f_i \rangle$ and $D_j = \langle f_1, \ldots, f_{j-1}, e_j \rangle$.

Define a new collection of pairwise orthogonal $\beta$ hyperbolic pairs $\{(e'_i, f'_i)\}_{i=1}^n$ by

$$e'_i = \begin{cases} e_i, & i \neq j + 1; \\ e_{j+1} - e_j, & i = j. \end{cases} \text{ and } f'_i = \begin{cases} f_i, & i \neq j; \\ f_{j+1} + f_j, & i = j + 1. \end{cases}$$

It is easy to calculate that this defines a basis of pairwise orthogonal $\beta$ hyperbolic pairs. It
is also clear that $\Sigma(e_i, f_i) \neq \Sigma(e'_i, f'_i)$, and that $C$ and $D$ lie in both apartments. Thus $C$ and $D$ do not lie in a unique apartment, and so they are not opposite.

**Concluding Remarks**

The theory of buildings extends far beyond what we have hinted at here and there are many excellent books for the interested reader. Some important references are [2], [22], [48], [58], [61] and [62].
CHAPTER 3

Flips of the Unitary Building

Introduction

In this chapter we discuss flips of the unitary building, and prove a preliminary result providing a hint of the eventual classification of these flips. We also begin studying some properties that will prove important in the study of geometries related to flips.

Throughout the remainder of this dissertation, let \( q \) be an odd prime power, let \( \mathbb{F} = \mathbb{F}_{q^2} \), and let \( \sigma \) be the unique involution in \( \text{Aut}(\mathbb{F}) \). Recall that for all \( x \in \mathbb{F} \), \( \sigma(x) = x^q \) and that \( \mathbb{F}^\sigma = \mathbb{F}_q \). Let \( V \) be a \( 2n \)-dimensional (left) \( \mathbb{F} \)-vector space \( (n \geq 2) \), and let \( \beta \) be a non-degenerate \( \sigma \)-hermitian form on \( V \). \( \Delta \) denotes the building associated to the polar geometry of \((V, \beta)\). It is shown in Chapter 7 of [58] that \( \Delta \) is a building of type \( C_n \).

Since we have assumed \( V \) is a left vector space, \( \beta \) satisfies

\[
\beta(\lambda u, \mu v) = \lambda \beta(u, v) \sigma(\mu)
\]

for all \( u, v \in V \) and \( \lambda, \mu \in \mathbb{F} \).

Recall also that the \( \sigma \)-norm of an element \( a \in \mathbb{F} \) is defined by

\[
N_{\sigma}(a) = a \sigma(a)
\]
and the \( \sigma \)-trace of an element \( a \in \mathbb{F} \) is defined by

\[
\text{Tr}_\sigma(a) = a + \sigma(a).
\]

3.1 The Unitary Building and its Flips

We begin with a crucial theorem that relates automorphisms of \( \Delta \) with a group of semilinear transformations of \( V \).

**Theorem 3.1.1.** \( \text{Aut}(\Delta) \cong \text{P\Gamma U}(V) \).

*Sketch of Proof.* Since the polar geometry \((V, \beta)\) is embeddable in a projective geometry and \( \dim V \geq 4 \), the Fundamental Theorem of Projective Geometry (Theorem 3.1 of [56]) applies to ensure that every automorphism of the polar space is induced by a semilinear transformation of \( V \). It follows that the automorphism group of the polar geometry is isomorphic to \( \text{P\Gamma U}(V) \). Finally that every automorphism of the building arises from an automorphism of the geometry is shown on Page 264 of [46].

**Note 3.1.1.** The main interest of Theorem 3.1.1 is that there is a surjective homomorphism

\[
\text{GU}(V) \twoheadrightarrow \text{Aut}(\Delta)
\]

so that in the proof of Lemma 3.1.4 we can argue that any flip of \( \Delta \) is induced by some transformation in \( \text{GU}(V) \).

**Lemma 3.1.2.** Let \( \varphi \) be a flip of \( \Delta \). Then \( \varphi \) is induced by a similitude \( f \) of \( V \) which satisfies \( f^2 = \lambda \text{id} \) on \( V \) for some scalar \( \lambda \).

*Proof.* Recall first that the longest word \( w_0 \) of the Weyl group of type \( C_n \) is central. Thus a flip \( \varphi \) in fact satisfies \( \delta(u, v) = \delta(\varphi(u), \varphi(v)) \) and so is an automorphism of \( \Delta \). It follows from Theorem 3.1.1 that \( \varphi \) is induced by some semilinear map \( f \in \text{GU}(V) \).
Since $\varphi^2 = \text{id}$ on $\Delta$ we see that $f^2$ is in the kernel of the action of $\Gamma U(V)$ on $\Delta$, which is $Z(V) \cap \Gamma U(V)$, the group of scalar transformations that also lie in $\Gamma U(V)$. Thus $f^2 = \lambda \text{id}$ on $V$ for some $\lambda \in \mathbb{F}$.

**Lemma 3.1.3.** Let $\varphi$ be induced by a similitude $f$ of $V$. Then either $f$ is linear or $f$ is $\sigma$-semilinear. Moreover if $\beta(f(u), f(v)) = a \tau(\beta(u, v))$ and $f^2 = \lambda \text{id}$ then $N_\sigma(\lambda) = a^2$.

**Proof.** Suppose $f$ is $\tau$-semilinear for $\tau \in \text{Aut} (\mathbb{F})$. Let $\eta \in \mathbb{F}$ and let $u \in V$ with $u \neq 0$. Since $f^2 = \lambda \text{id}$ on $V$ it follows that $f^2(\eta u) = \lambda \eta u$. But we can calculate directly that

$$f^2(\eta u) = f(\tau(\eta) f(u)) = \tau^2(\eta) f^2(u) = \tau^2(\eta) \lambda u.$$

Thus $\tau^2(\eta) = \eta$ and so $\tau^2$ is the identity of $\text{Aut} (\mathbb{F})$. Since $\text{Aut} (\mathbb{F})$ contains a unique involution it follows that either $\tau = \text{id}$ and $f$ is linear, or $\tau = \sigma$ and $f$ is $\sigma$-semilinear.

In order to prove the second part of the theorem, notice that

$$N_\sigma(\lambda) \beta(u, v) = \beta(f^2(u), f^2(v)) = a \tau(\beta(f(u), f(v))).$$

Since $\beta(f(u), f(v)) = a \tau \beta(u, v)$ it follows that

$$a \tau(\beta(f(u), f(v))) = a \tau(a) \tau^2(\beta(u, v)) = a^2 \beta(u, v).$$

This string of equalities relies on the fact that either $\tau = \text{id}$ or $\tau = \sigma$, and in either case $\tau(a) = a$.

Putting these two strings of equalities together we see that for all $u, v \in V$,

$$N_\sigma(\lambda) \beta(u, v) = a^2 \beta(u, v)$$

and so since there exist $u, v \in V$ with $\beta(u, v) \neq 0$, it follows that $N_\sigma(\lambda) = a^2$.

**Lemma 3.1.4.** Let $\varphi$ be a flip of $\Delta$. Then one of the following holds:
(i) $\varphi$ is induced by a linear isometry $f \in U(V)$ satisfying $f^2 = \text{id}$ on $V$; or

(ii) $\varphi$ is induced by a linear anti-isometry $f$ of $V$ satisfying $f^2 = \text{id}$ on $V$; or

(iii) $\varphi$ is induced by a $\sigma$-semilinear isometry $f \in \Gamma U(V)$ so that $f^2 = \text{id}$ on $V$.

Proof. By Lemma 3.1.2 $\varphi$ is induced by a similitude $f$ of $V$ with $f^2 = \lambda \text{id}$ on $V$ for some scalar $\lambda \in \mathbb{F}$.

Since $\varphi$ maps some chamber of $\Delta$ to an opposite, there is an apartment

$$\Sigma = \Sigma(e_i, f_i | i = 1, \ldots, n)$$

in which $\varphi$ sends the chamber $C = (C_i)_{i=1}^n$ defined by $C_i = \langle e_1, \ldots, e_i \rangle$ to its opposite in $\Sigma$, the chamber $D = (D_i)_{i=1}^n$ defined by $D_i = \langle f_1, \ldots, f_i \rangle$. Since $C$ and $D$ are opposite they are contained in a unique apartment of $\Delta$. It follows that $\varphi(\Sigma) = \Sigma$.

Since $\varphi$ preserves the apartment $\Sigma$, we see that for each $i = 1, \ldots, n$ there exist scalars $\lambda_i, \mu_i \in \mathbb{F}$ so that

$$f(e_i) = \lambda_i f_i$$
$$f(f_i) = \mu_i e_i.$$

(a) Suppose $f$ is linear and for all $u, v \in V$, $a \beta(u, v) = \beta(f(u), f(v))$. Since $a \in \mathbb{F}_q$ there exists $\mu \in \mathbb{F}$ such that $N_\sigma(\mu) = a^{-1}$. Replace $f$ by $\mu f$ and we see that for all $u, v \in V$,

$$\beta((\mu f)(u), (\mu f)(v)) = N_\sigma(\mu) \beta(f(u), f(v)) = a^{-1} a \beta(u, v) = \beta(u, v).$$

Thus $\mu f$ is an isometry which also induces $\varphi$.

Suppose now that we have chosen an isometry $f$ which induces $\varphi$, and $f^2 = \lambda \text{id}$. If
\( \lambda = 1 \) we are in case (i). So assume \( \lambda \neq 1 \). Notice that we have the following equalities:

\[
\beta(u, v) = \beta(f(u), f(v)) = \beta(f^2(u), f^2(v)) = N_\sigma(\lambda)\beta(u, v) \quad (3.1)
\]

\[
\sigma(\lambda_1) = \beta(e_1, f(e_1)) = \beta(f(e_1), f^2(e_1)) = \beta(\lambda_1f_1, \lambda^2e_1) = \lambda_1\sigma(\lambda). \quad (3.2)
\]

It follows from (1) that \( N_\sigma(\lambda) = 1 \), and from (2) that \( \lambda \) is a square in \( \mathbb{F} \). Choose \( \eta \in \mathbb{F} \) so that \( \eta^2 = \lambda^{-1} \). Since the norm map is multiplicative we see that \( N_\sigma(\eta^2) = N_\sigma(\lambda) = 1 \) and so \( N_\sigma(\eta) \in \{ \pm 1 \} \).

Let \( g = \eta f \). Then \( g^2 = \text{id} \) on \( V \), but we have paid a price. We now have that

\[
\beta(g(u), g(v)) = \beta(\eta f(u), \eta f(v)) = \eta\sigma(\eta)\beta(u, v) = N_\sigma(\eta)\beta(u, v).
\]

Thus either \( g \) is an isometry of \( (V, \beta) \) or \( g \) is an anti-isometry of \( (V, \beta) \). If \( g \) is an isometry we are in case (i) again, and if \( g \) is an anti-isometry we are in case (ii).

(b) Suppose now that \( f \) is semilinear but not linear. Then by Lemma 3.1.3 \( f \) is \( \sigma \)-semilinear. We now show that we can replace \( f \) by a scalar multiple \( \mu f \) which still induces \( \varphi \) so that \( (\mu f)^2 = \text{id} \) on \( V \). Namely we have

\[
\lambda e_i = f^2(e_i) = \mu_i\sigma(\lambda_i)e_i
\]

\[
\lambda f_i = f^2(f_i) = \lambda_i\sigma(\mu_i)f_i
\]

and so \( \lambda = \mu_i\sigma(\lambda_i) = \lambda_i\sigma(\mu_i) \). Hence \( \lambda \) lies in \( \mathbb{F}^\sigma = \mathbb{F}_q \), the fixed field of \( \sigma \). Since \( \mathbb{F} \) is finite the norm map \( N_\sigma : \mathbb{F} \rightarrow \mathbb{F}_q \) is surjective. Thus there exists \( \mu \in \mathbb{F} \) so that \( N_\sigma(\mu) = \lambda^{-1} \). Replacing \( f \) by \( \mu f \) does not affect \( \varphi \), and so we may do this and assume \( \lambda = 1 \).

In order to check that \( f \) can be taken to be an isometry, by Lemma 3.1.3 since \( \lambda = 1 \) also \( a^2 = 1 \). Hence \( a \in \{ \pm 1 \} \). The following calculation shows that \( a = 1 \) and so we
are in the situation of (iii):

\[ 1 = \beta(e_1, f_1) = a\sigma(\beta(f(e_1), f(f_1))) = a\sigma(\beta(\lambda_1 f_1, \mu_1 e_1)) = a\lambda = a. \]

**Note 3.1.2.** It follows immediately from Lemma 3.1.3 that if \( f \) induces a flip \( \varphi \) and \( f^2 = \text{id} \) on \( V \) then either \( f \) is an isometry or \( f \) is an anti-isometry. What is interesting about Lemma 3.1.4 is that if \( f \) is linear we have to consider both the isometry and anti-isometry possibilities, whereas if \( f \) is \( \sigma \)-semilinear we can assume it is an isometry.

**Definition 3.1.1.** Let \( \varphi \) be a flip of the unitary building \( \Delta \). We say \( \varphi \) is **linear** if it is induced by a linear transformation of \( V \). We say \( \varphi \) is **\( \sigma \)-semilinear** if it is induced by a \( \sigma \)-semilinear transformation of \( V \).

**Note 3.1.3.** From now on we identify \( \varphi \) with a transformation of \( V \) that induces \( \varphi \) and satisfies the appropriate conclusion of Lemma 3.1.4.

We now define a new form on \( V \). The definition is for any \( f \in \Gamma(U(V)) \), however we will be interested in applying the construction to transformations that induce flips.

**Definition 3.1.2.** Given \( f \in \Gamma(U(V)) \), define \( \beta_f(u, v) = \beta(u, f(v)) \).

**Lemma 3.1.5.** Let \( f \in \Gamma(U(V)) \) be \( \tau \)-semilinear, and assume \( f^2 = \text{id} \). Then \( \beta_f \) is a non-degenerate, reflexive, \( \sigma\tau \)-sesquilinear form. In particular,

(i) if \( \sigma = \tau \), then \( \beta_f \) is a non-degenerate, reflexive, bilinear form, and

(ii) if \( f \) is linear, then \( \beta_f \) is a non-degenerate, reflexive \( \sigma \)-sesquilinear form.

**Proof.** That \( \beta_f \) is non-degenerate follows since \( f \) is bijective. Left homogeneity follows since \( \beta \) is left homogeneous and \( f \) acts in the second argument.
In order to check that $\beta_f$ is reflexive, suppose $\beta_f(u,v) = 0$. Then

$$
\beta_f(v,u) = \beta(v,f(u)) \\
= \tau^{-1}(a^{-1})\tau^{-1}(\beta(f(v),u)) \\
= \tau^{-1}(a^{-1})\tau^{-1}(\sigma(\beta(u,f(v)))) \\
= \tau^{-1}(a^{-1})\tau^{-1}(\sigma(\beta_f(u,v))) \\
= 0
$$

where $f \in \Gamma U(V)$ satisfies $\beta(f(u),f(v)) = a\sigma(\beta(u,v))$ for all $u, v \in V$.

Both (i) and (ii) now follow from the following calculation:

$$
\beta_f(u,v + \lambda v') = \beta(u,f(v + \lambda v')) \\
= \beta(u,f(v) + \tau(\lambda)f(v')) \\
= \beta(u,f(v)) + \sigma(\tau(\lambda))\beta(u,f(v')) \\
= \beta_f(u,v) + \sigma(\tau(\lambda))\beta_f(u,v')
$$

where $\sigma \tau = \sigma$ if $\tau = 1$ and $1$ if $\tau \neq 1$, since by Lemma 3.1.3, $f^2 = \text{id}$ on $V$ implies that $\tau = 1$ or $\tau = \sigma$. Thus $\beta_f$ is a $\sigma \tau$-sesquilinear form on $V$. \hfill $\square$

**Lemma 3.1.6.** Let $\varphi$ be a $\sigma$-semilinear flip of $\Delta$. Then $\beta_\varphi$ is a non-degenerate, reflexive, symmetric, bilinear form.

**Proof.** All except the fact that $\beta_f$ is symmetric follows from Lemma 3.1.5. The following
calculation shows that $\beta_f$ is symmetric. Let $u, v \in V$. Then:

\[
\beta_f(u, v) = \beta(u, f(v)) = \sigma(\beta(f(u), f^2(v))) = \sigma(\beta(f(u), v)) = \beta(v, f(u)) = \beta_f(v, u). \quad \Box
\]

Lemma 3.1.7. Let $\varphi$ be a linear flip of $\Delta$.

(i) If $\varphi$ is a linear isometry then $\beta_\varphi$ is a non-degenerate, reflexive, $\sigma$-hermitian form.

(ii) If $\varphi$ is a linear anti-isometry then $\beta_\varphi$ is a non-degenerate, reflexive, $\sigma$-antihermitian form.

Proof. All that remains is to show that in (i) the form is hermitian and in (ii) the form is antihermitian.

(i) Suppose $\varphi$ is an isometry and let $u, v \in V$. Then

\[
\beta_\varphi(u, v) = \beta(u, \varphi(v)) = \beta(\varphi(u), \varphi^2(v)) = \beta(\varphi(u), v) = \sigma(\beta(v, \varphi(u))) = \sigma(\beta_\varphi(v, u)).
\]

Thus $\beta_\varphi$ is hermitian.
(ii) Suppose \( \varphi \) is an anti-isometry and let \( u, v \in V \). Then

\[
\beta_{\varphi}(u, v) = \beta(u, \varphi(v)) \\
= -\beta(\varphi(u), \varphi^2(v)) \\
= -\beta(\varphi(u), v) \\
= -\sigma(\beta(v, \varphi(u))) \\
= -\sigma(\beta_{\varphi}(v, u)).
\]

Thus \( \beta_{\varphi} \) is anti-hermitian.

\[
\square
\]

**Definition 3.1.3.** Given a flip \( \varphi \), set \( Q_{\varphi}(v) = \frac{1}{2} \beta_{\varphi}(v, v) \).

Notice that \( Q_{\varphi} \) is the pseudo-quadratic form that polarizes to \( \beta_{\varphi} \).

### 3.2 The Chamber System Induced by a Flip

We now define a chamber system left invariant by a flip. We shall use this chamber system to classify flips, and we shall also be interested in automorphism groups of these chamber systems. Recall \( \varphi \) denotes both a flip of \( \Delta \) and a semilinear transformation of \( V \) that induces \( \varphi \) and satisfies the appropriate conclusion of Lemma 3.1.4.

**Definition 3.2.1.** Let \( (\Lambda, \lambda) \) be a spherical building with Weyl group \( W \). Let \( w_0 \) be the longest word of \( W \), and let \( \rho \) be a flip of \( \Lambda \). Define

\[
\Lambda^\rho = \{ u \in \Lambda | \lambda(u, \rho(u)) = w_0 \}
\]

This is the **chamber system induced by** \( \rho \).

In the context of twinned spherical buildings, this is the definition of the chamber system induced by a flip. In the case of the unitary building, we can be more specific.
Definition 3.2.2. Recall that a pair of vectors $u, v \in V$ are $\beta$-orthogonal if $\beta(u, v) = 0$, this is denoted $u \perp v$. The vectors are $\beta_\varphi$-orthogonal if $\beta_\varphi(u, v) = 0$, this is denoted $u \perp_\varphi v$. The vectors are biorthogonal if $\beta(u, v) = \beta_\varphi(u, v) = 0$, this is denoted $u \perp\perp v$. If $U$ is a subspace of $V$ we use $U^\perp, U^\perp_\varphi,$ and $U^\perp\perp$ to refer to the $\beta$-orthogonal complement, $\beta_\varphi$-orthogonal complement, and biorthogonal complement respectively.

Recall that a pair of $\beta$ isotropic vectors $u, v$ is called a hyperbolic pair if $\beta(u, v) = 1$. We define a pre-hyperbolic pair to be a pair of $\beta$ isotropic vectors $u, v$ with $\beta(u, v) \neq 0$. This is not standard, but there are instances where the distinction will be important.

Theorem 3.2.1. Let $\varphi$ be a flip of the unitary building $\Delta$.

(1) A chamber $C = (C_i)_{i=1}^n$ of $\Delta$ lies in $\Delta^\varphi$ if and only if $C_i$ is non-degenerate with respect to $\beta_\varphi$ for all $i = 1, \ldots, n$.

(2) If $\{e_i, f_i\}_{i=1}^n$ is a $\beta$ pre-hyperbolic basis for $V$ with $\varphi(e_i) = f_i$, then the chambers $(C_i)_{i=1}^n$ and $(D_i)_{i=1}^n$ defined by $C_i = \langle e_1, \ldots, e_i \rangle$ and $D_i = \langle f_1, \ldots, f_i \rangle$ are opposite in $\Delta$ and so lie in $\Delta^\varphi$. Conversely if $C = (C_i)_{i=1}^n$ is a chamber of $\Delta^\varphi$, then there is a $\beta$ pre-hyperbolic basis $\{e_i, f_i\}$ for $V$ so that $C_i = \langle e_1, \ldots, e_i \rangle$, $\varphi(e_i) = f_i$ for all $i = 1, \ldots, n$, and the chamber $D = (D_i)_{i=1}^n$ defined by $D_i = \langle f_1, \ldots, f_i \rangle$ lies in $\Delta^\varphi$ and is opposite to $C$.

Proof. (1) By assumption we may view $\varphi$ as acting on the vector space $V$, and have $\varphi^2 = \text{id}$ on $V$. Suppose $C = (C_i)_{i=1}^n$ is a chamber of $\Delta^\varphi$. Then since $C$ is also a chamber of $\Delta$, each $C_i$ is $\beta$ isotropic.

Recall from Theorem 2.4.1 that $C$ is opposite to $\varphi(C)$ in $\Delta$ if and only if for each $i,$

$$\varphi(C_i) \cap C_i^\perp = \{0\}.$$  

Notice that

$$\varphi(\text{Rad}_{\beta_\varphi}(C_i)) = \varphi(C_i \cap C_i^\perp_\varphi) = \varphi(C_i) \cap C_i^\perp$$
where the last equality is justified since \( \varphi \) is a bijective transformation of \( V \). Thus the \( \beta_\varphi \) radical of \( C_i \) is \( \{0\} \) if and only if \( \varphi(C_i) \cap C_i^\perp = \{0\} \).

(2) The first part follows from (1) by noting that if \( \{e_i, f_i\} \) is a \( \beta \) pre-hyperbolic basis for \( V \) with \( f_i = \varphi(e_i) \) for all \( i \), then for each \( i \), \( \{e_1, \ldots, e_i\} \) and \( \{f_1, \ldots, f_i\} \) form \( \beta_\varphi \) orthogonal bases for \( C_i \) and \( D_i \) respectively, and satisfy the hypotheses of (1).

Conversely suppose \( C = (C_i)_{i=1}^n \) is a chamber of \( \Delta^c \). Choose \( e_1, \ldots, e_n \) as follows. Pick \( e_1 \in C_1 \setminus \{0\} \). Then for \( i = 2, \ldots, n \) pick \( e_i \in C_i \cap \varphi(C_{i-1})^\perp \). The vectors \( e_1, \ldots, e_n \) are then biorthogonal. Moreover none can be \( \beta_\varphi \) isotropic as this would contradict the \( \beta_\varphi \) non-degeneracy of \( C_i \). Finally, define \( f_i = \varphi(e_i) \) for \( i = 1, \ldots, n \). Then \( \{e_i, f_i\}_{i=1}^n \) gives the desired basis.

Lemma 3.2.2. If \( f \in \Gamma(U(V) \) satisfies any of (i)-(iv) in the statement of Theorem 1 then \( f \) induces a flip of \( \Delta \).

Proof. Since \( f \in \Gamma(U(V) \) f induces an automorphism of \( \Delta \). It therefore suffices to show that \( f \) maps some chamber to \( \Delta \) to an opposite chamber. Let \( \{e_i, f_i\}_{i=1}^n \) be a hyperbolic basis for \( V \) as in the hypotheses of Theorem 1. Let \( C_i = \langle e_1, \ldots, e_i \rangle \) for \( i = 1, \ldots, n \). Then \( \varphi(C_i) = \langle f_1, \ldots, f_i \rangle \), and clearly \( C_i^\perp \cap \varphi(C_i) = \{0\} \) for all \( i \). Hence the chamber \( C = (C_i)_{i=1}^n \) is sent to an opposite chamber by \( f \), and so \( f \) induces a flip of \( \Delta \).

\[ \square \]

3.3 The Geometry Induced by a Flip

Definition 3.3.1. Let \( \Gamma(n, q) \) denote the set of all \( \beta \)-isotropic and \( \beta_\varphi \) non-degenerate subspaces \( U \) of \( V \). Let \( I = \{1, \ldots, n\} \) and define \( \tau : \Gamma(n, q) \to I \) by \( \tau(U) = \dim(U) \). Finally, define an incidence relation \( \sim \) on \( \Gamma(n, q) \) by \( U \sim W \) if either \( U \subseteq W \) or \( W \subseteq U \).

Note 3.3.1. It is clear that \( (\Gamma(n, q), I, \tau, \sim) \) is a pregeometry, since \( \Gamma \) a subset of the set of objects of the full projective geometry, \( \mathcal{P}(V) \), of \( V \) and we have inherited the type and incidence structure. We will prove in Theorem 3.5.8 that \( \Gamma(n, q) \) is in fact a geometry. In
order to achieve this goal we will have to study the properties of the vector space endowed with both forms $\beta$ and $\beta_\varphi$ in more detail.

**Lemma 3.3.1.** Let $U$ be a $\beta$-isotropic subspace of $V$. Then $U \in \Gamma(n, q)$ if and only if $U^\perp \cap \varphi(U) = \{0\}$.

*Proof. *Since $U$ is $\beta$-isotropic it suffices to check when it is $\beta_\varphi$ non-degenerate. This occurs if and only if $U^\perp \cap \varphi(U) = \{0\}$. \hfill $\square$

**Definition 3.3.2.** A point of $\Gamma(n, q)$ is an object of type 1 of $\Gamma(n, q)$. A line of $\Gamma(n, q)$ is an object of type 2 of $\Gamma(n, q)$.

**Note 3.3.2.** From now on, we identify a point of the geometry, which is really a 1-dimensional subspace of $V$, with any non-zero vector in that subspace.

**Lemma 3.3.2.** Let $U$ be a subspace of $V$. Then

$$U^\perp = \varphi(U)^{\perp_{\varphi}}$$

and

$$\langle U, \varphi(U) \rangle^\perp = \langle U, \varphi(U) \rangle^{\perp_{\varphi}} = U^\perp.$$ 

**Lemma 3.3.3.** Let $U, U' \in \Gamma(n, q)$ with $U \subset U'$. Then

$$\langle U, \varphi(U) \rangle^\perp \cap U' = \varphi(U)^\perp \cap U' = U^{\perp_{\varphi}} \cap U' = U^\perp \cap U'.$$

The proofs of Lemmas 3.3.2 and 3.3.3 are straightforward.

**Lemma 3.3.4.** Let $U, U' \in \Gamma(n, q)$ with $U \subset U'$. Then

$$\langle U, \varphi(U) \rangle^\perp \cap U' \in \Gamma(n, q).$$
Proof. Let $W = \langle U, \varphi(U) \rangle^\perp \cap U'$ where $U \subset U' \in \Gamma(n, q)$. Since $W \subseteq U'$ and $U'$ is $\beta$ isotropic, it follows that $W$ is $\beta$ isotropic. It therefore suffices to show that $W$ is $\beta_\varphi$ non-degenerate. We accomplish this by showing that $W$ is a $\beta_\varphi$ orthogonal complement to $U$ in $U'$. This is immediate since a vector $w \in U'$ is $\beta_\varphi$ orthogonal to $U$ if and only if $w \in W$, and $U$ itself is $\beta_\varphi$ non-degenerate. \qed

Remark. In essence Lemma 3.3.4 says that if $U \subseteq U'$ are elements of $\Gamma(n, q)$ then $U$ has a vector space complement $W$ in $U'$ that is also in $\Gamma(n, q)$. \hfill \diamondsuit

Lemma 3.3.5. Let $U \in \Gamma(n, q)$. Then $U$ contains a point of $\Gamma(n, q)$.

Proof. If $\dim U = 1$ then $U$ is a point. Suppose $\dim U > 1$. Since $U \in \Gamma(n, q)$ it is $\beta_\varphi$ non-degenerate and so there exists $u, v \in U$ so that $\beta_\varphi(u, v) \neq 0$. If either $Q_\varphi(u) \neq 0$ or $Q_\varphi(v) \neq 0$ then $u$ or $v$ is a point respectively. Otherwise it is straightforward to check that there exists $\lambda \in \mathbb{F}$ so that $u + \lambda v$ is a point of $\Gamma(n, q)$. \qed

3.4 Further Properties of $\beta$ and $\beta_\varphi$

In this section we have collected some results regarding the relationship between $\beta$ and $\beta_\varphi$. Unless otherwise stated, these results hold for both linear and $\sigma$-semilinear flips.

Note 3.4.1. Recall that $\varphi$ denotes both a flip and a semilinear transformation of $V$ that induces the flip and satisfies the appropriate conclusion of Lemma 3.1.4.

Lemma 3.4.1. Let $U$ be a $\varphi$-invariant subspace of $V$. Then

$$\text{Rad}_\beta(U) = \text{Rad}_{\beta_\varphi}(U) = \varphi(\text{Rad}_\beta(U)).$$

Proof. A vector $u$ lies in the $\beta$ radical of $U$ if and only if $\beta(u, v) = 0$ for all $v \in U$. Since $U$ is $\varphi$-invariant, $U = \varphi(U)$ and so this is also equivalent to requiring that $\beta_\varphi(u, v) = 0$ for all $v \in V$. Thus a vector lies in the $\beta$ radical of $U$ if and only if it lies in the $\beta_\varphi$ radical of $U$. \qed
Note 3.4.2. From now on, when referring to the radical of a $\varphi$-invariant subspace we need not always identify the form.

Lemma 3.4.2. Let $U$ be a $\varphi$-invariant subspace of $V$, and let $R$ be a $\varphi$-invariant subspace of $U$. Then $R$ has a $\varphi$-invariant complement in $U$.

Proof. This is a special case of Maschke’s Theorem, see for example Theorem 1.9 of [37]. □

Corollary 3.4.3. Let $U$ be a $\varphi$-invariant subspace of $V$ and let $R$ be its radical. Then $R$ has a $\varphi$-invariant complement in $U$.

Proof. By Lemma 3.4.1 $R$ is $\varphi$-invariant and so by Lemma 3.4.2 it has a $\varphi$ invariant complement in $U$. □

Lemma 3.4.4. Let $W \in \Gamma(n, q)$ with $\dim W = k$. Then $W \cap \varphi(W) = \{0\}$. Hence $W' = \langle W, \varphi(W) \rangle$ is $2k$-dimensional, $\varphi$-invariant, and non-degenerate.

Proof. Since $W$ is $\beta$ isotropic, $W \subseteq W^\perp$. Since $W$ is $\beta_\varphi$ non-degenerate it follows that

$$W^\perp \cap \varphi(W) = \{0\}$$

and so also $W \cap \varphi(W) = \{0\}$. This shows that $\dim(W, \varphi(W)) = 2k$.

That $W'$ is $\varphi$-invariant is clear. To show that $W'$ is non-degenerate, notice first that by Lemma 3.4.1 $\operatorname{Rad}_{\beta}(W') = \varphi(\operatorname{Rad}_{\beta}(W'))$. Furthermore, $\operatorname{Rad}_{\beta}(W') \subseteq W^\perp \cap W' = W$. But also $\varphi(\operatorname{Rad}_{\beta}(W')) \subseteq \varphi(W^\perp \cap W') = \varphi(W \cap W') = \varphi(W) \cap \varphi(W') = \varphi(W)$. It follows that $\operatorname{Rad}_{\beta}(W') \subseteq W \cap \varphi(W) = \{0\}$ and so $W'$ is non-degenerate. □

Corollary 3.4.5. If $W \in \Gamma(n, q)$ with $\dim W = k$ then there is a basis $\{w_i\}_{i=1}^k$ for $W$ of biorthogonal points.

Proof. We induct on $k$. If $k = 1$ the result is trivial. If $k > 1$ then by Lemma 3.3.5 $W$ contains a point $w_1$ of $\Gamma(n, q)$. Let $W' = (w_1, \varphi(w_1))^\perp \cap W$. Then by Lemma 3.3.4 $W' \in \Gamma(n, q)$ and so by the inductive hypothesis there exists a collection of biorthogonal points $\{w_2, \ldots, w_k\}$
that is a basis for \( W' \). Since \( W' \subseteq \langle w_1, \varphi(w_1) \rangle^\perp \) it follows that \( \{w_1, \ldots, w_k\} \) forms a basis of biorthogonal points for \( W \).

\[ \square \]

### 3.5 \( \Gamma(n, q) \) is a Geometry

In this section we prove that \( \Gamma(n, q) \) is a geometry.

**Lemma 3.5.1.** Let \( U \) be a subspace of \( V \) with \( \dim U > n \). Then \( \varphi \) does not act as a scalar on \( U \).

**Proof.** Let \( M \) be an \( n \)-dimensional \( \beta \) isotropic \( \beta \varphi \) non-degenerate subspace of \( V \). Then by Lemma 3.4.4, \( M \cap \varphi(M) = \{0\} \). If \( U \) is a subspace of dimension greater than \( n \) and \( \varphi \) acts as a scalar on \( U \), then \( \varphi \) acts as a scalar on \( M \cap U \neq \{0\} \). Thus there is a non-zero vector \( v \in M \cap U \) with \( \varphi(v) = \mu v \) for some non-zero \( \mu \in \mathbb{F} \). But then \( v \in M \cap \varphi(M) = \{0\} \), a contradiction.

\[ \square \]

**Lemma 3.5.2.** Suppose \( \varphi \) is linear. Let \( X \) be a \( 2k \)-dimensional, \( \varphi \) invariant, non-degenerate subspace of \( V \). Then one of the following three holds:

(i) \( X \) contains a point of \( \Gamma(n, q) \);

(ii) \( \varphi(x) = x \) for all \( x \in X \);

(iii) \( \varphi(x) = -x \) for all \( x \in X \).

**Proof.** Suppose that \( X \) does not contain any points of \( \Gamma(n, q) \). We will show that either (ii) or (iii) holds. Since \( X \) is \( \beta \) non-degenerate and even dimensional we can write it as an orthogonal direct sum of \( \beta \) hyperbolic lines,

\[ X = \perp_{i=1}^m \langle a_i, b_i \rangle \]

where each \( \langle a_i, b_i \rangle \) is a hyperbolic pair.

We proceed now in a series of steps to show that \( \varphi \) acts on \( X \) as either \( \text{id}_X \) or \( -\text{id}_X \).
Step 1: If \( u, v \in X \) are \( \beta \)-isotropic then \( \beta(u, v) = 0 \) if and only if \( \beta_\varphi(u, v) = 0 \).

Proof. Notice first that since \( X \) contains no points of \( \Gamma(n, q) \), \( Q_\varphi(u) = Q_\varphi(v) = 0 \).

Suppose \( \beta(u, v) = 0 \) but \( \beta_\varphi(u, v) \neq 0 \). If \( \varphi \) is an isometry and \( \lambda \) is chosen so that \( \text{Tr}_\sigma(\sigma(\lambda)\beta_\varphi(u, v)) \neq 0 \) then \( u + \lambda v \) is a point of \( \Gamma(n, q) \). If \( \varphi \) is an anti-isometry and \( \lambda \) is chosen so that \( \sigma(\lambda)\beta_\varphi(u, v) - \lambda\sigma(\beta_\varphi(u, v)) \neq 0 \) then \( u + \lambda v \) is a point of \( \Gamma(n, q) \). In either case, such \( \lambda \) exist and so since by hypothesis \( X \) contains no points of \( \Gamma(n, q) \) we conclude that if \( \beta(u, v) = 0 \) then \( \beta_\varphi(u, v) = 0 \).

Conversely if \( \beta_\varphi(u, v) = 0 \) but \( \beta(u, v) \neq 0 \) then \( \beta(u, \varphi(v)) = 0 \) while \( \beta_\varphi(u, \varphi(v)) \neq 0 \), which we have already shown cannot happen.

Thus \( \beta(u, v) = 0 \) if and only if \( \beta_\varphi(u, v) = 0 \) for all \( \beta \)-isotropic \( u, v \in X \).

Step 2: For all \( i = 1, \ldots, m \), \( \varphi(a_i) \in \langle a_i \rangle \) and \( \varphi(b_i) \in \langle b_i \rangle \).

Proof. We perform the calculation only for \( a_1 \), the others are similar. Suppose

\[
\varphi(a_1) = \sum_{i=1}^{m} (x_i a_i + y_i b_i)
\]

for some scalars \( x_i, y_i \in F, i = 1, \ldots, m \).

Since \( \beta(b_i, a_1) = 0 \) for all \( i \neq 1 \), also \( \beta_\varphi(b_i, a_1) = 0 \) for all \( i \neq 1 \). But we can calculate that \( \beta_\varphi(b_i, a_1) = \sigma(x_i) \), and so \( x_i = 0 \) if \( i \neq 1 \).

Similarly for all \( i \), \( \beta(a_i, a_1) = 0 \) and so also \( \beta_\varphi(a_i, a_1) = 0 \), but \( \beta_\varphi(a_i, a_1) = \sigma(y_i) \) and so \( y_i = 0 \).

Hence \( \varphi(a_1) = x_1 a_1 \).

Step 3: For all \( i \), \( \varphi(a_i) = a_i \) or \( \varphi(a_i) = -a_i \). Similarly \( \varphi(b_i) = b_i \) or \( \varphi(b_i) = -b_i \).

Proof. We prove the result for \( a_i \), the result for \( b_i \) is proved similarly. Since \( \varphi^2 = \text{id} \) on \( V \), \( \varphi^2(a_i) = x_i^2 a_i = a_i \) and so \( x_i^2 = 1 \). Hence \( x_i \in \{ \pm 1 \} \).
Step 4: \( \varphi(a_i) = -a_i \) if and only if \( \varphi(b_i) = -b_i \).

Proof. Assume first that \( \varphi \) is an isometry of \((V, \beta)\) and that \( \varphi(a_i) = -a_i \). Then

\[
1 = \beta(a_i, b_i) = \beta(\varphi(a_i), \varphi(b_i)) = -\beta(a_i, \varphi(b_i))
\]

which forces \( \varphi(b_i) = -b_i \). Similarly if \( \varphi(b_i) = -b_i \) then \( \varphi(a_i) = -a_i \).

Assume next that \( \varphi \) is an anti-isometry of \((V, \beta)\) and that \( \varphi(a_i) = -a_i \) while \( \varphi(b_i) = b_i \). Consider the vector \( x = a_i + \lambda b_i \) where \( \lambda \) is any non-zero element of trace 0 in \( F \). An easy calculation shows that \( \beta(x, x) = 0 \) while \( \beta_{\varphi}(x, x) = -2\lambda \neq 0 \). Thus \( x \) is a point of \( \Gamma(n, q) \) which lies in \( X \), contradicting the assumption that \( X \) contains no points of \( \Gamma(n, q) \). \( \square \)

Step 5: If \( \varphi(a_1) = a_1 \) then \( \varphi(a_i) = a_i \) for all \( i \) and if \( \varphi(a_1) = -a_1 \) then \( \varphi(a_i) = -a_i \) for all \( i \).

Proof. Suppose that \( \varphi(a_1) = a_1 \) but \( \varphi(a_i) = -a_i \) for some \( i \). Then also \( \varphi(b_1) = b_1 \) and \( \varphi(b_i) = -b_i \). Let \( x = a_1 + b_1 + a_i - b_i \). Then two easy calculations show that \( x \) is a point of \( \Gamma(n, q) \). Since by assumption \( X \) contains no points of \( \Gamma(n, q) \) we conclude that if \( \varphi(a_1) = a_1 \) then \( \varphi(a_i) = a_i \) for all \( i \). Similarly if \( \varphi(a_1) = -a_1 \) then \( \varphi(a_i) = -a_i \) for all \( i \). \( \square \)

Thus for all \( x \in X \), either \( \varphi(x) = x \) or \( \varphi(x) = -x \). \( \square \)

In the \( \sigma \)-semilinear case, the situation is even better.

Lemma 3.5.3. Suppose \( \varphi \) is \( \sigma \)-semilinear. Let \( U \) be a 2k-dimensional \((k \geq 1)\) subspace of \( V \) that is \( \beta \) non-degenerate. Then either \( U \) is \( \beta_{\varphi} \) totally singular or \( U \) contains a point of \( \Gamma(n, q) \).
Proof. Assume $U$ is not $\beta_\varphi$ totally singular. Since $U$ is $\beta$ non-degenerate we can write $U = \sum_{i=1}^{k} \langle a_i, b_i \rangle$ where each $(a_i, b_i)$ is a hyperbolic pair. If any $a_i$ or $b_j$ is a point of $\Gamma(n, q)$ then it is the desired point, so we may assume that for all $i$, $Q_\varphi(a_i) = Q_\varphi(b_i) = 0$.

Since $U$ is not $\beta_\varphi$ totally singular we must have one of the following:

(i) there is some $i$ so that $\beta_\varphi(a_i, b_i) \neq 0$ and so for any non-zero $\lambda$ of trace 0, $a_i + \lambda b_i$ is a point of $\Gamma(n, q)$;

(ii) there are $i, j$ so that $\beta_\varphi(a_i, a_j) \neq 0$ and so $a_i + a_j$ is a point of $\Gamma(n, q)$;

(iii) there are $i, j$ so that $\beta_\varphi(a_i, b_j) \neq 0$ and so $a_i + b_j$ is a point of $\Gamma(n, q)$;

(iv) there are $i, j$ so that $\beta_\varphi(b_i, b_j) \neq 0$ and so $b_i + b_j$ is a point of $\Gamma(n, q)$.

Theorem 3.5.4 is crucial to the proof that $\Gamma(n, q)$ is a geometry.

**Theorem 3.5.4.** Let $U \in \Gamma(n, q)$. If $\dim U < n$ then the space $X = \langle U, \varphi(U) \rangle^\perp$ contains a point of $\Gamma(n, q)$.

Proof. Notice first that since $X^\perp = \langle U, \varphi(U) \rangle$ is non-degenerate by Lemma 3.4.4 and $V$ is non-degenerate by hypothesis, also $X$ is non-degenerate.

If $\varphi$ is $\sigma$-semilinear the result now follows immediately from Lemma 3.5.3.

Now suppose that $\varphi$ is linear. We proceed by contradiction. Suppose $X$ does not contain a point of $\Gamma(n, q)$. Then by Lemma 3.5.2 $\varphi$ acts either as $\text{id}_X$ or $-\text{id}_X$ on $X$. Let $k = \dim U$.

Choose a basis $\{a_1, \ldots, a_{2(n-k)}\}$ for $X$.

Let $\{u_1, \ldots, u_k\}$ be a basis of biorthogonal points for $U$. Recall that such a basis exists by Corollary 3.4.5. Then

$$\{u_1, \ldots, u_k, \varphi(u_1), \ldots, \varphi(u_k)\}$$

forms a basis for $\langle U, \varphi(U) \rangle$. We define a new basis for $\langle U, \varphi(U) \rangle$ by:

$$\{u_i + \varphi(u_i), u_i - \varphi(u_i) | i = 1, \ldots, k\}.$$
If $\varphi$ acts on $X$ as $\text{id}_X$, define a subspace $A$ of $V$ by

$$A = \langle u_1 + \varphi(u_1), \ldots, u_k + \varphi(u_k), a_1, \ldots, a_{2(n-k)} \rangle.$$ 

Then $A$ is a $2n - k > n$ dimensional subspace of $V$ on which $\varphi$ acts as multiplication by 1, contradicting Lemma 3.5.1.

Thus by Lemma 3.5.2, $\varphi$ must act on $X$ as $-\text{id}_X$. In this case we define a subspace $B$ of $V$ by

$$B = \langle u_1 - \varphi(u_1), \ldots, u_k - \varphi(u_k), a_1, \ldots, a_{2(n-k)} \rangle.$$ 

Then $B$ is a $2n - k > n$ dimensional subspace of $V$ on which $\varphi$ acts as multiplication by $-1$, contradicting Lemma 3.5.1. Hence $X$ must contain a point of $\Gamma(n,q)$.

**Corollary 3.5.5.** If $U$ is a maximal object of $\Gamma(n,q)$ then $\dim U = n$.

**Proof.** We proceed by contraposition. If $U \in \Gamma(n,q)$ with $\dim U < n$ then by Theorem 3.5.4 there is a point of $\Gamma(n,q)$, $u \in \langle U, \varphi(U) \rangle^\perp$. It is easy to see that $\langle U, u \rangle \in \Gamma(n,q)$ and so $U$ is not maximal. \hfill \Box

**Definition 3.5.1.** Given an object $U \in \Gamma(n,q)$ and a subspace $X$ of $V$, define $r_U(X) = X \cap U^\perp$.

**Lemma 3.5.6.** Let $U$ be an $m$-object of $\Gamma(n,q)$ with $m < n$ and let $W = U^\perp$. Then $\varphi|_W$ is a flip of the building of totally isotropic subspaces of $(W, \beta|_W)$.

**Proof.** Let $M$ be a maximal object of $\Gamma(n,q)$ containing $U$. Let $M' = W \cap M$. By Corollary 3.5.10, $M'$ has a basis $\{m_1, \ldots, m_{n-m}\}$ of biorthogonal points of $\Gamma(n,q)$. After possibly scaling we see that $\{m_i, \varphi(m_i)\}_{i=1}^m$ is a basis for $W$ and that with respect to this basis $\varphi|W$ satisfies one of the hypotheses of Lemma 3.2.2. Thus $\varphi|W$ is a flip of $(W, \beta|W)$. \hfill \Box

We have ignored a subtle point: since there is more than one type of flip, which sort of flip is $\varphi|W$? Once we finish proving Main Theorem 1 it will be easy to see that if $\varphi$ satisfies
(i) or (ii) of Main Theorem then so does $\phi|_W$. If $\phi$ satisfies (iii) (resp. (iv)) of Main Theorem and the determinant of the $\beta_\phi$ Gram matrix of $U$ is a square in $F$ then $\phi|_W$ also satisfies (iii) (resp. (iv)). If $\phi$ satisfies (iii) (resp. (iv)) and the determinant of the $\beta_\phi$ Gram matrix of $U$ is a non-square in $F$ then $\phi|_W$ satisfies (iv) (resp. (iii)).

**Corollary 3.5.7.** If $u$ is a point of $\Gamma(n,q)$ then $r_u$ induces an isomorphism of geometries $\text{res}_{\Gamma(n,q)}(u) \rightarrow \Gamma(n-1,q)$.

**Proof.** Notice that the objects in the residue of $u$ correspond to $\beta$ isotropic $\beta_\phi$ non-degenerate subspaces of $W = \langle u, \phi(u) \rangle^\perp$, and this correspondence preserves incidence. By Lemma 3.5.6 $\phi|_W$ is a flip of $(W, \beta|_W)$ and it is clear from the construction that the geometry induced on $W$ by $\phi|_W$ and the geometry on $W$ induced by $\phi$ agree. Hence $\text{res}_{\Gamma(n,q)}(u) \cong \Gamma(n-1,q)$ and the isomorphism is induced by $r_u$. 

**Theorem 3.5.8.** $\Gamma(n,q)$ is a geometry with type and incidence as defined in Definition 3.3.1.

**Proof.** We induct on $n$. If $n = 1$ then the result is trivial. Suppose $n > 1$ and let $F$ be a flag of $\Gamma(n,q)$. By Lemma 3.3.5 we can assume that $F$ contains a point $u$ of $\Gamma(n,q)$. By Corollary 3.5.7 the residue of $u$ is isomorphic to $\Gamma(n-1,q)$, which by the inductive hypothesis is a geometry. Thus $r_u(F)$ is a chamber in $\Gamma(n-1,q)$. It follows easily that $F$ is a chamber of $\Gamma(n,q)$. 

With this in hand, we can also prove the following:

**Lemma 3.5.9.** Let $W$ be an object of $\Gamma(n,q)$ and let $M$ be an $n$-dimensional object of $\Gamma(n,q)$ that contains $W$. Then any $\beta_\phi$ orthogonal basis for $W$ extends to a $\beta_\phi$ orthogonal basis for $M$. Furthermore if $\{w_i\}_{i=1}^n$ is any $\beta_\phi$ orthogonal basis for $M$ then $\{w_i, \phi(w_i)\}_{i=1}^n$ forms a $\beta_\phi$ orthogonal basis for $V$.

**Proof.** Notice first that since $\Gamma(n,q)$ is a geometry, $W$ is contained in an $n$-dimensional object $M$ of $\Gamma(n,q)$. Let $d = \dim(W)$ and let $\{w_1, \ldots, w_m\}$ be a $\beta_\phi$ orthogonal basis for $W$. Then $\{w_1, \ldots, w_d\}$ is a basis of biorthogonal points for $W$. Let $\{w_{d+1}, \ldots, w_n\}$ be a basis of
biorthogonal points for $\langle W, \varphi(W) \rangle^\perp \cap M$. Then $\{w_1, \ldots, w_n\}$ forms a basis of biorthogonal points for $M$, which is in particular a $\beta_\varphi$ orthogonal basis for $M$.

That $\{w_i, \varphi(w_i)\}_{i=1}^n$ forms a $\beta_\varphi$ orthogonal basis for $V$ follows since $\dim M = n$ and $M \in \Gamma(n, q)$.

**Corollary 3.5.10.** If $W$ is a $k$-object of $\Gamma(n, q)$ with $\beta_\varphi$ orthogonal basis $\{w_i\}_{i=1}^k$ then there is a basis for $V$ of $\beta$ pre-hyperbolic pairs $\{e_i, \varphi(e_i)\}_{i=1}^n$ with $e_i = w_i$ for $i = 1, \ldots, k$.

**Proof.** This follows immediately from Theorem 3.2.1 and Lemma 3.5.9.

**Concluding Remarks**

Lemma 3.1.4 gives us some insight into the structure of a flip on the unitary building. In Chapter 4 we will fully classify linear flips and explore the large rank connectedness and simple connectedness of $\Gamma(n, q)$ in this case. In Chapter 5, we will fully classify $\sigma$-semilinear flips, and in Chapter 6, we explore the large rank connectedness and simple connectedness of a certain subgeometry of $\Gamma(n, q)$ in the case of a $\sigma$-semilinear flip.
CHAPTER 4

Linear Flips and the Geometry $\Gamma(n, q)$

Introduction

In this chapter we will classify linear flips of the unitary building in Theorem 4.1.1. The classification of linear flips is achieved more easily than the classification of $\sigma$-semilinear flips. We prove in Section 4.2 that if $\varphi$ is induced by an isometry of $(V, \beta)$ then the group $U_n(q^2) \times U_n(q^2)$ acts flag transitively on $\Gamma(n, q)$ and if $\varphi$ is induced by an anti-isometry of $(V, \beta)$ then the group $GL_n(q^2)$ acts flag transitively on $\Gamma(n, q)$. We finish by showing that if $n \geq 8$ then $\Gamma(n, q)$ is connected, and if $n \geq 14$ then $\Gamma(n, q)$ is simply connected. This allows us to prove Phan-type theorems for the groups $U_n(q^2) \times U_n(q^2)$ and $GL_n(q^2)$.

Throughout this chapter, $\varphi$ denotes a linear flip. Recall that we have identified a linear flip $\varphi$ with a linear transformation of $V$ that satisfies the appropriate conclusion of Lemma 3.1.4.

4.1 Classification of Linear Flips

Theorem 4.1.1 (Classification of Linear Flips). Let $\varphi$ be a linear flip of $\Delta$

(i) If $\varphi$ is induced by an isometry of $(V, \beta)$ then there is a basis for $V$, $\{e_i, f_i\}_{i=1}^n$ of $\beta$
hyperbolic pairs so that \( \varphi(e_i) = f_i \) and \( \varphi(f_i) = e_i \) for all \( i = 1, \ldots, n \).

(ii) If \( \varphi \) is induced by an anti-isometry of \((V, \beta)\) then there is a basis for \( V \), \( \{e_i, f_i\}_{i=1}^n \) of \( \beta \) hyperbolic pairs so that \( \varphi(e_i) = \alpha f_i \) and \( \varphi(f_i) = \alpha^{-1} e_i \) for all \( i = 1, \ldots, n \) where \( \text{Tr}_\sigma(\alpha) = 0 \).

Conversely any linear transformation of \( V \) which satisfies (i) or (ii) induces a flip of \( \Delta \).

Proof. It follows easily from the proof of Lemma 3.1.4 there is a basis of orthogonal \( \beta \)-hyperbolic pairs \( \{h_i, g_i\}_{i=1}^n \) so that

\[
\varphi(h_i) = \lambda_i g_i, \quad \text{and} \quad \varphi(g_i) = \lambda_i^{-1} h_i
\]

for some \( \lambda_i \in \mathbb{F} \).

(i) Suppose that \( \varphi \) is induced by an isometry of \((V, \beta)\). Since \( \beta_\varphi \) is \( \sigma \)-hermitian it follows that for all \( i = 1, \ldots, n \), \( \sigma(\lambda_i) = \beta_\varphi(h_i, h_i) \in \mathbb{F}_q \) and so in fact \( \sigma(\lambda_i) = \lambda_i \).

For \( i = 1, \ldots, n \), let \( g'_i = \lambda_i g_i \). Choose \( \gamma_i \in \mathbb{F} \) so that \( N_\sigma(\gamma_i) = \lambda_i^{-1} \). Define

\[
e_i = \gamma_i h_i, \quad \text{and} \quad f_i = \gamma_i g'_i.
\]

We now calculate to show that \( \{e_i, f_i\}_{i=1}^n \) is a basis of \( \beta \) hyperbolic pairs with \( \varphi(e_i) = f_i \) and \( \varphi(f_i) = e_i \).

\[
\beta(e_i, e_j) = \beta(\gamma_i h_i, \gamma_j h_j) = 0 \\
\beta(f_i, f_j) = \beta(\gamma_i g'_i, \gamma_j g'_j) = \beta(\gamma_i \lambda_i g_i, \gamma_j \lambda_j g_j) = 0 \\
\beta(e_i, f_i) = \beta(\gamma_i h_i, \gamma_i \lambda_i g_i) = N_\sigma(\gamma_i) \lambda_i = 1 \\
\beta(e_i, f_j) = \beta(\gamma_i h_i, \gamma_j \lambda_j g_j) = 0 \text{ if } i \neq j.
\]
Thus \( \{e_i, f_i\}_{i=1}^n \) forms a \( \beta \) hyperbolic basis for \( V \), and finally

\[
\varphi(e_i) = \varphi(\gamma_i h_i) = \gamma_i \varphi(h_i) = \gamma_i \lambda_i g_i = \gamma_i g_i' = f_i
\]

\[
\varphi(f_i) = \varphi(\gamma_i g_i') = \gamma_i \varphi(g_i') = \gamma_i \lambda_i \varphi(g_i) = \gamma_i h_i = e_i.
\]

(ii) Suppose now that \( \varphi \) is induced by an anti-isometry of \( (V, \beta) \). Since \( \beta \varphi \) is \( \sigma \)-antihermitean it follows that for all \( i \), \( \sigma(\lambda_i) = \beta \varphi(h_i, h_i) = -\lambda_i \), and so \( \lambda_i \) is of trace 0. Let \( \alpha \) be any non-zero element of trace 0 in \( \mathbb{F} \). For each \( i = 1, \ldots, n \) choose \( a_i \in \mathbb{F}_q \) so that \( a_i \lambda_i = \alpha \). Let \( \gamma_i \in \mathbb{F} \) be chosen so that \( N_\sigma(\gamma_i) = a_i \). Set \( e_i = \gamma_i h_i \) and \( f_i = \alpha^{-1} \gamma_i \lambda_i g_i \).

Direct calculation shows that \( \{e_i, f_i\}_{i=1}^n \) is a \( \beta \) hyperbolic basis with the property that \( \varphi(e_i) = \alpha f_i \) and \( \varphi(f_i) = \alpha^{-1} e_i \).

The converse follows from Lemma 3.2.2.

\[
\text{Note 4.1.1.}\] It is now clear that the geometry \( \Gamma(n, q) \) depends on whether \( \varphi \) is an isometry or an anti-isometry. With the basis found in Theorem 4.1.1 we can see that when \( n = 1 \) the number of points in the geometry depends on whether the flip is an isometry or an anti-isometry, implying that in larger rank the geometries are also not isomorphic.

### 4.2 The Group Preserving \( \beta \) and \( \beta \varphi \)

We are interested in finding a group that acts in a natural way on \( \Gamma(n, q) \). The obvious choice for this group is the group of linear transformations of \( V \) that preserve both the forms \( \beta \) and \( \beta \varphi \). In this section we study some properties of this common group of linear isometries.

**Definition 4.2.1.** Let \( U_{2n}(q^2)\varphi = \{ f \in U_{2n}(q^2) | \beta \varphi(u, v) = \beta \varphi(f(u), f(v)) \text{ for all } u, v \in V \} \).

Notice that \( U_{2n}(q^2)\varphi \) is the group of linear transformations of \( V \) that preserve both \( \beta \) and \( \beta \varphi \).

In this section we will prove three results regarding \( U_{2n}(q^2)\varphi \). First we will prove that it is precisely the centralizer in \( U_{2n}(q^2) \) of \( \varphi \). Then we prove that \( U_{2n}(q^2)\varphi \) acts flag transitively on
Finally, we prove that if \( \varphi \) is an isometry of \((V, \beta)\) then \( U_{2n}(q^2)^\varphi \cong U_n(q^2) \times U_n(q^2) \), and that if \( \varphi \) is an anti-isometry of \((V, \beta)\) then \( U_{2n}(q^2)^\varphi \cong GL_n(q^2) \).

**Lemma 4.2.1.** \( U_{2n}(q^2)^\varphi \cong C_{\Gamma U_{2n}(q^2)}(\varphi) \cap U_{2n}(q^2) \).

**Proof.** Let \( f \in U_{2n}(q^2)^\varphi \) and \( v \in V \). We show \( \varphi f = f \varphi \). To achieve this, we will show that for all \( w \in V \), \( \beta(w, f(\varphi(v))) = \beta(w, \varphi(f(v))) \). Since \( \beta \) is non-degenerate this will force \( f(\varphi(v)) = \varphi(f(v)) \).

Let \( w \in V \). Choose \( x \in V \) so that \( f(x) = w \). Then

\[
\beta_\varphi(x, v) = \beta(x, \varphi(v)) = \beta(f(x), f(\varphi(v))) = \beta(w, f(\varphi(v)))
\]

and

\[
\beta_\varphi(x, v) = \beta_\varphi(f(x), f(v)) = \beta(f(x), \varphi(f(v))) = \beta(w, \varphi(f(v))).
\]

Conversely if \( f \in U_{2n}(q^2) \) commutes with \( \varphi \) then for all \( u, v \in V \) we have

\[
\beta_\varphi(f(u), f(v)) = \beta(f(u), \varphi(f(v))) = \beta(f(u), f(\varphi(v))) = \beta(u, \varphi(v)) = \beta_\varphi(u, v). \quad \Box
\]

We now turn to the problem of showing that \( U_{2n}(q^2)^\varphi \) acts flag transitively on \( \Gamma(n, q) \). Before we can prove that \( U_{2n}(q^2)^\varphi \) acts flag transitively on \( \Gamma(n, q) \) we require one more lemma.

**Lemma 4.2.2.** Let \( C = \{C_i\}_{i=1}^n \) be a chamber of \( \Gamma(n, q) \).

(a) If \( \varphi \) is induced by an isometry of \((V, \beta)\) as in Lemma 3.1.4 then there is a basis \( \{e_i, f_i\}_{i=1}^n \) for \( V \) with the following properties:

(i) \( \{e_i, f_i\}_{i=1}^n \) is hyperbolic with respect to \( \beta \);

(ii) for all \( i = 1, \ldots, n \), \( \varphi(e_i) = f_i \) and \( \varphi(f_i) = e_i \); and

(iii) for all \( i = 1, \ldots, n \), \( C_i = \langle e_1, \ldots, e_i \rangle \).
(b) If \( \varphi \) is induced by an anti-isometry of \((V, \beta)\) as in Lemma 3.1.4 then there is a basis \( \{e_i, f_i\}_{i=1}^n \) for \( V \) with properties (i) and (iii), and (ii') for all \( i = 1, \ldots, n, \varphi(e_i) = \alpha f_i \) and \( \varphi(f_i) = \alpha^{-1} e_i \) where \( \alpha \) has trace 0.

Proof. (a) Let \( e_1 \) be a non-zero vector in \( C_1 \). Then after scaling as in the proof of Theorem 4.1.1(i) we may assume that \((e_1, \varphi(e_1))\) is a hyperbolic pair. Since \( C_1^\perp \cap C_2 \) is an element of \( \Gamma(n, q) \) by Lemma 3.3.4 we can choose \( e_2 \in C_1^\perp \cap C_2 \) so that after scaling, \((e_2, \varphi(e_2))\) is a hyperbolic pair. Repeating this procedure we produce the desired basis.

(b) This is proved in the same fashion as (a), with the scaling as in the proof of Theorem 4.1.1(ii) replacing the scaling in Main Theorem 1A(i).

Theorem 4.2.3. If \( \varphi \) is a linear flip then \( U_{2n}(q^2)^\varphi \) acts flag transitively on \( \Gamma(n, q) \).

Proof. Since \( \Gamma(n, q) \) is a geometry, it suffices to show that \( U_{2n}(q^2)^\varphi \) acts chamber transitively. Let \( C = (C_i)_i^\varphi \) and \( D = (D_i)_i^\varphi \) be two chambers of \( \Gamma(n, q) \). By Lemma 4.2.2 there is a basis \( \{e_i, \varphi(e_i)\}_{i=1}^n \) of \( \beta \) pre-hyperbolic pairs such that \( C_i = \langle e_1, \ldots, e_i \rangle \) for all \( i \). Similarly there is a basis \( \{f_i, \varphi(f_i)\}_{i=1}^n \) of \( \beta \) pre-hyperbolic pairs such that \( D_i = \langle g_1, \ldots, g_i \rangle \).

Define \( T : V \rightarrow V \) by \( T(e_i) = g_i \) and \( T(\varphi(e_i)) = \varphi(g_i) \) and extend linearly. It is easy to see that \( T \) preserves \( \beta \) and commutes with \( \varphi \), and hence also preserves \( \beta_\varphi \). Thus \( T \) is an element of \( U_{2n}(q^2)^\varphi \) with \( T(C) = D \).

Theorem 4.2.4. Let \( \varphi \) be a linear flip of \( \Delta \).

(i) If \( \varphi \) is induced by an isometry of \((V, \beta)\) as in Lemma 3.1.4 then \( U_{2n}(q^2)^\varphi \cong U_n(q^2) \times U_n(q^2) \).

(ii) If \( \varphi \) is induced by an anti-isometry of \((V, \beta)\) as in Lemma 3.1.4 then \( U_{2n}(q^2)^\varphi \cong \text{GL}_n(q^2) \).

Proof. Let \( \{e_i, f_i\}_{i=1}^n \) be a basis for \( V \) as in Theorem 4.1.1.
(i) Define a new basis for $V$ by $g_i = e_i + f_i$ for $i = 1, \ldots, n$ and $h_i = e_i - f_i$ for $i = 1, \ldots, n$. Order this basis as \{g_1, \ldots, g_n, h_1, \ldots, h_n\}. Direct calculation shows that with respect to this (ordered) basis, $\beta$ and $\beta_\varphi$ have Gram matrices

$$M_1 = \begin{pmatrix} 2I_n & 0 \\ 0 & -2I_n \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 2I_n & 0 \\ 0 & 2I_n \end{pmatrix}$$

respectively. Given a linear transformation $T$ of $V$, we can express $T$ as a block matrix

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$ 

Since both $\beta$ and $\beta_\varphi$ are hermitian, it follows that $T$ preserves both forms if and only if for $i = 1, 2$, $^tT^\sigma M_i T = M_i$. These two requirements are by direct calculation equivalent to the following four equalities:

\begin{align*}
^tA^\sigma A - ^tC^\sigma C &= ^tD^\sigma D - ^tB^\sigma B = I_n \quad (4.1) \\
^tA^\sigma A + ^tC^\sigma C &= ^tB^\sigma B + ^tD^\sigma D = I_n \quad (4.2) \\
^tA^\sigma B - ^tC^\sigma D &= ^tB^\sigma A - ^tD^\sigma C = 0 \quad (4.3) \\
^tA^\sigma B + ^tC^\sigma D &= ^tB^\sigma A + ^tD^\sigma C = 0. \quad (4.4)
\end{align*}

Adding (4.1) to (4.2) shows that $^tA^\sigma A = ^tD^\sigma D = I_n$, and so $A$ and $D$ are unitary matrices. Adding (4.3) to (4.4) and using the fact that both $A$ is invertible shows that $B = 0$. Similarly subtracting (4.4) from (4.3) and using the fact that $D$ is invertible shows that $C = 0$. Thus in fact

$$T = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}.$$
where $A$ and $D$ are unitary matrices. Conversely it is easy to check that if $A$ and $D$ are unitary matrices then

$$
\begin{pmatrix}
A & 0 \\
0 & D
\end{pmatrix}
$$

preserves both $\beta$ and $\beta_\varphi$, and so lies in $U_{2n}(q^2)^\varphi$.

(ii) The technique here is the same as in (i), but the details are different. We only outline this part of the proof. Define a new basis for $V$ by setting $h_i = e_i + \alpha f_i$ for $i = 1, \ldots, n$ and $h_i = e_i - \alpha f_i$ for $i = n + 1, \ldots, 2n$. Instead of constructing the Gram matrix for $\beta_\varphi$ in this case, it simplifies the calculation to consider the hermitian form $\alpha^{-1}\beta_\varphi$. Considering a linear transformation of $V$ as a block matrix $T$ as above, direct calculation shows that $T$ preserves both $\beta$ and $\beta_\varphi$ if and only if $B = C = 0$ and $^tA^\sigma D = I_n$. Conversely for any $A \in \text{GL}_n(q^2)$ the matrix

$$
\begin{pmatrix}
A & 0 \\
0 & (^tA^\sigma)^{-1}
\end{pmatrix}
$$

preserves both $\beta$ and $\alpha\beta_\varphi$ and so lies in $U_{2n}(q^2)^\varphi$.

It is worth noting at this point that if $\varphi$ is $\sigma$-semilinear then the group preserving both $\beta$ and $\beta_\varphi$ will not act flag transitively on $\Gamma(n, q)$. In that case we will look at the orbit of a given chamber.

### 4.3 Parabolic Subgroups

We now determine the parabolic subgroups of $U_{2n}(q^2)^\varphi$ with respect to its action on $\Gamma(n, q)$. This is important for understanding the amalgam of maximal parabolic subgroups with respect to the groups’ action on $\Gamma(n, q)$.
Theorem 4.3.1. Let $\mathcal{F} = (C_i)_{i=1}^k$ be a flag of $\Gamma(n,q)$. For each $i = 2, \ldots, k$ let $C_i' = C_i \cap \varphi(C_{i-1})^\perp$ and let $C_1' = C_1$. For $i = 1, \ldots, k$ let $m_i = \dim C_i'$ and let $d_k = \dim C_k$. Then the stabilizer of $\mathcal{F}$ in $U_{2n}(q^2)^\varphi$ is isomorphic to
\[
H = \left( \bigoplus_{i=1}^k U_{m_i}(q^2) \right) \oplus U_{2(n-d_k)}(q^2)^\phi
\]
where $\phi$ is a linear flip of the building of totally isotropic subspaces of a unitary space of dimension $2(n-d_k)$ over $\mathbb{F}$.

Proof. First, notice that we can write $V$ as a $\beta$ and $\beta_\varphi$ orthogonal direct sum of subspaces
\[
V = [(C_1' \oplus \varphi(C_1')) \perp \ldots \perp (C_k' \oplus \varphi(C_k'))] \perp \langle C_k, \varphi(C_k) \rangle^\perp.
\]
Notice also that each $(C_i', \beta, \beta_\varphi|C_i')$ is a non-degenerate unitary space over $\mathbb{F}$ of dimension $m_k$.

We obtain an embedding of $H$ into the stabilizer of $\mathcal{F}$ in $U_{2n}(q^2)^\varphi$ as follows. For $i = 1, \ldots, k$ let $U_{m_i}(q^2)$ act on $(C_i' \oplus \varphi(C_i'))$ as the group of matrices of the form
\[
\begin{pmatrix}
A & 0 \\
0 & A
\end{pmatrix}
\]
where $A \in U_{m_i}(q^2)$ and let $U_{m_i}(q^2)$ act trivially on all the other pieces of the decomposition. Finally, let $U_{2(n-d_k)}(q^2)^\phi$ act in the obvious way on $\langle C_k, \varphi(C_k) \rangle^\perp$ and trivially on the other pieces of the decomposition. It is easy to see that this action embeds $H$ into $\text{Stab}_{U_{2n}(q^2)^\varphi}(\mathcal{F})$.

Conversely, if $f \in U_{2n}(q^2)^\varphi$ stabilizes $\mathcal{F}$ then $f$ must stabilize the decomposition given above. Furthermore, for each $i = 1, \ldots, k$ the transformation $f$ must act on $(C_i' \oplus \varphi(C_i'))$ as a group of matrices of the form
\[
\begin{pmatrix}
A & 0 \\
0 & A
\end{pmatrix}
\]
where $A \in U(C_i', \beta_\varphi|C_i') \cong U_{m_k}(q^2)$. Finally, $f$ restricted to $\langle C_k, \varphi(C_k) \rangle^\perp$ must preserve
β and commute with the flip \( \Phi = \varphi|\langle C_k, \varphi(C_k) \rangle \perp \) and so the action of \( f \) on this space is induced by some element of \( U_{2(n-d_k)}(q^2) \) where \( \phi \) is a linear flip of the building of totally isotropic subspaces of a unitary space of dimension \( 2(n-d_k) \) over \( \mathbb{F} \).

Hence the embedding of \( H \) into \( \text{Stab}_{U_{2n}(q^2)}(\mathcal{F}) \) described above is surjective, and hence the stabilizer is isomorphic to \( H \).

\[ \square \]

**Note 4.3.1.** The notation in Theorem 4.3.1 is slightly imprecise, as it is possible that for \( i \neq j \), \( m_i = m_j \). However the two resulting copies of the group \( U_{m_i}(q^2) \) are distinct groups, one corresponding to the space \( (C_i' \oplus \varphi(C_i')) \) and the other corresponding to the space \( (C_j' \oplus \varphi(C_j')) \).

### 4.4 Residues

We now determine the residues of \( \Gamma(n,q) \). In many circumstances the notion of residual connectedness contributes to proofs that a geometry is simply connected. As we will see, we do now know if \( \Gamma(n,q) \) is in general residually connected, but Lemma 4.6.2 achieves the goal usually achieved by appealing to residual connectedness of the geometry.

**Theorem 4.4.1.** Let \( \mathcal{F} = (C_i)_{i=1}^k \) be a flag of \( \Gamma(n,q) \). Let \( C_0 = \{0\} \). For \( i = 1, \ldots, k \) let \( D_i = C_i \cap \varphi(C_{i-1}) \perp \) and let \( D_{k+1} = \langle C_k, \varphi(C_k) \rangle \perp \). For \( i = 1, \ldots, k \) let \( \Lambda_i \) denote the geometry of non-degenerate subspaces of the unitary space \( (D_i, \beta|D_i) \), and let \( \Lambda_{k+1} \) denote the geometry of \( \beta \) isotropic \( \beta \varphi \) non-degenerate subspaces of \( D_{k+1} \). Then the residue of \( \mathcal{F} \) in \( \Gamma(n,q) \) is isomorphic to \( \prod_{i=1}^{k+1} \Lambda_i \).

**Proof.** We will construct a map from \( \prod_{i=1}^{k+1} \Lambda_i \) to \( \text{res}_{\Gamma(n,q)}(\mathcal{F}) \) which is an isomorphism of geometries.

First, notice that for \( U \in \text{res}_{\Gamma(n,q)}(\mathcal{F}) \), since \( U \) is incident to each object of \( \mathcal{F} \) one of the following must hold:

(i) \( U \subset C_1 \);
(ii) \( C_k \subset U \);

(iii) there is an index \( i \) with \( 1 \leq i \leq k \) such that \( C_i \subset U \subset C_{i+1} \).

Define a map \( \alpha : \prod_{i=1}^{k+1} \Lambda_i \rightarrow \text{res}_{\Gamma(n,q)}(\mathcal{F}) \) by setting, for \( U \in \prod_{i=1}^{k+1} \Lambda_i \),

\[
\alpha(U) = (C_i, U) \text{ where } U \in \Lambda_i.
\]

It is straightforward to check that the image of \( \alpha \) in \( \Gamma(n,q) \) lies in the residue of \( \mathcal{F} \), and when the codomain of \( \alpha \) is restricted to \( \text{res}_{\Gamma(n,q)}(\mathcal{F}) \) it is bijective and preserves incidence, and so is an isomorphism of geometries. Hence \( \text{res}_{\Gamma(n,q)}(\mathcal{F}) \cong \prod_{i=1}^{k+1} \Lambda_i \). \hfill \Box

4.5 Connectedness of \( \Gamma(n,q) \)

Recall that \( \mathcal{G}(\Gamma(n,q)) \) denotes the \((0,1)\)-collinearity graph of \( \Gamma(n,q) \).

In this section we show that if \( n \geq 8 \) then \( \mathcal{G}(\Gamma(n,q)) \) is connected of diameter 2. The key result is Corollary 4.5.1, which is a variation of Lemma 3.5.2 for odd-dimensional subspaces.

Note 4.5.1. Throughout the remainder of this chapter, if \( u \) is a point of \( \Gamma(n,q) \) then we assume \( \beta_\varphi(u,u) = 1 \). This assumption is easily justified if \( \varphi \) is a linear isometry by appropriate scaling. If \( \varphi \) is a linear anti-isometry then we must instead consider the form \( \lambda \beta_\varphi \) where \( \lambda \) is a non-zero trace 0 element of \( \mathbb{F} \). Since \( \beta_\varphi \) is anti-hermitian, \( \lambda \beta_\varphi \) is hermitian and moreover a subspace \( U \) of \( V \) is \( \beta_\varphi \) non-degenerate if and only if \( U \) is \( \lambda \beta_\varphi \) non-degenerate. Thus we may as well consider the hermitian form \( \lambda \beta_\varphi \). To avoid over complicating the notation, we will suppress the \( \lambda \), although the reader should be aware that if \( \varphi \) is an anti-isometry we are considering this multiple of \( \beta_\varphi \), rather than \( \beta_\varphi \) itself.

Corollary 4.5.1. Let \( X \) be a \( \varphi \)-invariant non-degenerate subspace of \( V \) with \( \dim X \geq n+2 \). Then \( X \) contains a point of \( \Gamma(n,q) \).
Proof. If dim $X = 2k$ then this follows by combining Lemma \ref{lemma:3.5.1} with Lemma \ref{lemma:3.5.2} as in the proof of Theorem \ref{thm:3.5.4}. So we may assume dim $X = 2k + 1$. We first show there is a vector $c \in X$ such that $\beta(c, c) \neq 0$ and $\langle c \rangle = \langle \varphi(c) \rangle$.

Let $c_1 \in X$ with $\beta(c_1, c_1) = 1$. If $\beta_\varphi(c_1, c_1) \neq -1$ then $c = c_1 + \varphi(c_1)$ works. If $\beta_\varphi(c_1, c_1) = -1$ then $c = c_1 - \varphi(c_1)$ works. In either case, we can find such a vector $c$.

Next, let $W = \langle c \rangle^\perp \cap X$. Since $X$ and $\langle c \rangle$ are $\varphi$-invariant also $W$ is $\varphi$-invariant. Furthermore $W$ is non-degenerate and $2k$-dimensional. Since $2k \geq n + 1$ it follows that $W$ contains a point of $\Gamma(n, q)$ and so also $X$ contains a point of $\Gamma(n, q)$. □

Lemma 4.5.2. Let $\{u_1, \ldots, u_k\}$ be a collection of points of $\Gamma(n, q)$. If $n \geq 4k$ then there is a point $z$ of $\Gamma(n, q)$ biorthogonal to each of the $u_i$.

Proof. Let $X = \langle u_1, \varphi(u_1), \ldots, u_k, \varphi(u_k) \rangle$. Then $X$ is $2k$ dimensional and $\varphi$-invariant with at most a $2k - 2$ dimensional radical. Hence $X^\perp$ is at least $2n - 2k$ dimensional with at most a $2k - 2$ dimensional radical. Let $W$ be a $\varphi$-invariant complement to the radical of $X^\perp$ in $X^\perp$. If $n \geq 4k$ then dim $W \geq n + 2$ and so by Corollary \ref{cor:4.5.1} $W$ contains a point $z$ of $\Gamma(n, q)$. By construction $z$ is biorthogonal to each of the $u_i$. □

Lemma 4.5.3. Let $u$, $v$ be biorthogonal points of $\Gamma(n, q)$. Then $\langle u, v \rangle$ is a line of $\Gamma(n, q)$.

Proof. That $\langle u, v \rangle$ is $\beta$ totally isotropic follows since both $u$ and $v$ are isotropic, and they are $\beta$ orthogonal. That $\langle u, v \rangle$ is $\beta_\varphi$ non-degenerate follows since the $\beta_\varphi$ Gram matrix is

$$
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
$$

which has determinant 1. □

Theorem 4.5.4. Let $n \geq 8$. Then $G(\Gamma(n, q))$ is connected of diameter 2.

Proof. Let $u$ and $v$ be points of $\Gamma(n, q)$. Then by Lemma \ref{lemma:4.5.2} there is a point $z$ biorthogonal to both $u$ and $v$, and so $z$ is a common neighbor of $u$ and $v$ in $G(\Gamma(n, q))$. Thus $G(\Gamma(n, q))$ is connected of diameter 2. □
We can now see why we are unable to prove that $\Gamma(n, q)$ is residually connected in general. The residues of rank at least 2 in $\Gamma(n, q)$ are of one of three types:

(i) $\Gamma(s, q)$ for some $2 \leq s < n$;

(ii) The geometry of non-degenerate subspaces of a unitary space of rank at least 2;

(iii) A direct product of geometries of types (ii) with a single geometry of type (i).

We prove in Lemma 4.6.1 that geometries of type (ii) are connected, and it is easy to see that residues of type (iii) are connected. So the difficulty arises because we do not know if all the geometries $\Gamma(s, q)$ for $s \geq 2$ are connected. If they are, then $\Gamma(n, q)$ is residually connected, and otherwise $\Gamma(n, q)$ is not residually connected. Fortunately we will not need $\Gamma(n, q)$ to be residually connected in order to study its homotopy properties.

4.6 Simple Connectedness of $\Gamma(n, q)$

In this section we prove that $\Gamma(n, q)$ is simply connected if $n \geq 14$. We begin by showing that every path in $\Gamma(n, q)$ that begins and ends with a point is homotopic to a path consisting only of points and lines. Then we will study cycles in $\Gamma(n, q)$ by studying cycles in the collinearity graph $G(\Gamma(n, q))$ and argue that if $n \geq 14$ all such cycles correspond to null-homotopic cycles in the flag complex.

Lemma 4.6.1. Let $\mathcal{U}(n, q)$ denote the geometry of non-degenerate subspaces of an $(n + 1)$ dimensional unitary space $(V, \rho)$ over the field $\mathbb{F}_{p^2}$, where $p$ is an odd prime power and $\rho$ is a $\tau$-hermitian form. Then $G_{0,1}(\mathcal{U}(n, q))$ is connected of diameter 2 for all $n \geq 2$.

Proof. Let $u$ and $v$ be distinct points of $\mathcal{U}(n, q)$. If $\langle u, v \rangle$ is $\rho$ non-degenerate then they are joined by an edge in $G_{0,1}(\mathcal{U}(n, q))$.

Suppose next that $\langle u, v \rangle$ is $\rho$ degenerate. We consider three cases depending on $n$. 
**Case 1:** Suppose $n = 2$. Let $e$ be any non-zero vector in the radical of $\langle u, v \rangle$. Then $e \in \langle u \rangle^\perp$, and so since $\langle u \rangle^\perp$ is $\rho$ non-degenerate and 2 dimensional there is an isotropic vector $f \in \langle u \rangle^\perp$ such that $\rho(e, f) \neq 0$. Notice that $\rho(f, v) \neq 0$, since if $\rho(f, v) = 0$ then $\langle v \rangle^\perp = \langle e, f \rangle$ and so $u$ and $v$ represent the same point of $U(n, q)$.

After possibly scaling $f$ we may assume that $\rho(f, v) = 1$. Similarly after possibly scaling $e$ we may assume that $\rho(e, f) = 1$. Choose $\alpha \in \mathbb{F}_{q^2}$ so that $\text{Tr}_{\tau}(\alpha) \notin \{0, 1\}$. Notice that this is always possible if $q$ is an odd prime power. Let $w = f + \alpha e$. Then we have the following:

\[
\rho(w, w) = \text{Tr}_{\tau}(\alpha) \neq 0
\]
\[
\rho(w, u) = 0
\]
\[
\rho(w, v) = 1.
\]

An easy calculation now shows that both $\langle u, w \rangle$ and $\langle v, w \rangle$ are lines of $U(n, q)$, and so $G_{0,1}(U(n, q))$ is connected of diameter 2.

**Case 2:** Suppose $n = 3$. Let $W = \langle u, v \rangle^\perp$. If $W$ is $\rho$ totally isotropic then $W = W^\perp$, and so in particular both $u, v \in W$. But neither $u$ nor $v$ is $\rho$ isotropic, contradicting the assumption that $W$ is $\rho$ totally isotropic. Hence $W$ is not $\rho$ totally isotropic, and so there exists a vector $w \in W$ such that $\rho(w, w) \neq 0$. Thus $u$ and $v$ are both adjacent to $w$ in $G_{0,1}(U(n, q))$ and so $G_{0,1}(U(n, q))$ is connected of diameter 2.

**Case 3:** Suppose $n \geq 4$. Let $W = \langle u, v \rangle^\perp$. Then $W$ is not $\rho$ totally isotropic, since $\dim W = n - 1$ which is strictly greater than the Witt index of $(\mathcal{V}, \rho)$. Hence $W$ contains a vector $w$ with $\rho(w, w) \neq 0$, and so $u$ and $v$ are both adjacent to $w$ in $G_{0,1}(U(n, q))$. It follows that $G_{0,1}(U(n, q))$ is connected of diameter 2. □

**Lemma 4.6.2.** Every path in $\Gamma(n, q)$ that begins and ends at a point is homotopic to a path only passing through points and lines.
Proof. Notice that if \( n = 1 \) or \( n = 2 \) the result is trivial, so we can assume that \( n \geq 3 \). Let \( \gamma = x_1 \ldots x_n \) be a path in \( \Gamma(n, q) \) with no repetition and \( x_1 \) and \( x_n \) points. Let \( m(\gamma) \) denote the number of elements of the path that are neither points nor lines. We induct on \( m(\gamma) \). If \( m(\gamma) = 0 \) then \( \gamma \) is already a path consisting of only points and lines.

Suppose \( m(\gamma) > 0 \) and choose \( i \) so that \( x_i \) is the first object in \( \gamma \) that is neither a point nor a line. If \( x_{i+1} \) has dimension greater than \( x_i \) then \( x_{i-1} \) is contained in \( x_{i+1} \), and so \( \gamma \) is homotopic to the shorter path \( \gamma' = x_1 \ldots x_{i-1} \hat{x}_i \ldots x_n \) which satisfies \( m(\gamma') = m(\gamma) - 1 \) and so is homotopic to a point line path by the inductive hypothesis.

If \( x_{i+1} \) is neither a point nor a line and has dimension less than the dimension of \( x_i \), then choose \( j \in \mathbb{N} \) maximal such that for all \( s = 1, \ldots, j \), \( x_{i+s} \subset x_{i+s-1} \). Then we have a string of subspaces

\[
    x_i \supset x_{i+1} \supset \ldots \supset x_{i+j}
\]

but \( x_{i+j} \subset x_{i+j+1} \). Let \( v \) be any point of \( \Gamma(n, q) \) contained in \( x_{i+j} \). Then we can replace \( x_{i+1} \ldots x_{i+j} \) by \( v \) to obtain a path \( \gamma' \) homotopic to \( \gamma \) with \( m(\gamma') < m(\gamma) \). By the inductive hypothesis \( \gamma' \) is homotopic to a point-line path, and so \( \gamma \) is homotopic to a point-line path.

Finally, if \( x_{i+1} \) is a point or a line, then both \( x_{i-1} \) and \( x_{i+1} \) are contained in \( x_i \). It follows from Lemma 4.6.1 that there is a path from \( x_{i-1} \) to \( x_{i+1} \) contained entirely in \( x_i \). This new path may introduce new objects that are neither points nor lines, but by induction on the dimension of the first object that is neither a point nor a line we can produce a path \( \gamma' \) homotopic to \( \gamma \) but with no elements that are neither points nor lines until after \( x_{i+1} \). By the inductive hypothesis \( \gamma' \) is homotopic to a point line path, and hence \( \gamma \) is homotopic to a point line path.

\[\square\]

Definition 4.6.1. A path in a geometry is geometric if there is an object of the geometry incident to every object on the path.

Lemma 4.6.3 ([25], Proposition 5.2). A geometric cycle in a geometry is null-homotopic.

Remark. Since we are interested in the homotopy type of \( \Gamma(n, q) \), Lemma 4.6.2 will greatly
simplify our later work. First, since $\Gamma(n, q)$ is connected for $n \geq 8$, its flag complex is path-connected as a topological space, and so the isomorphism type of its fundamental groups is independent of base point. It therefore suffices to fix a point $x$ of $\Gamma(n, q)$, and then compute the fundamental group with base point $x$. Since every path is homotopic to a point line path, we only have to consider point-line paths with base $x$. These cycles correspond to cycles in $G(\Gamma(n, q))$, which is the object we will actually study when proving the geometry is simply connected.

To be more precise, we will study cycles in $G(\Gamma(n, q))$ and show that the cycles to which they correspond in the flag complex are null-homotopic.

For the remainder of this section, lengths of cycles refer to the collinearity graph of $\Gamma(n, q)$. In somewhat of an abuse of notation, we will refer to cycles in the collinearity graph as being null-homotopic, when really we mean the cycles to which they correspond in the flag complex are null-homotopic.

**Lemma 4.6.4.** If $n \geq 10$ every 3-cycle is a product of geometric 3-cycles and hence null-homotopic.

**Proof.** Let $abca$ be a 3-cycle. Let $W = \langle a, b, c, \varphi(a), \varphi(b), \varphi(c) \rangle^\perp$. Then $W$ is at least $2n - 6$ dimensional with at most a 2 dimensional radical, since it contains the non-degenerate subspace $\langle a, b, \varphi(a), \varphi(b) \rangle^\perp$. Let $W'$ be a $\varphi$-invariant non-degenerate complement to this radical. Then $W'$ is at least $2n - 8$ dimensional, $\varphi$ invariant and non-degenerate. If $n \geq 10$ then $\dim W' \geq 2n - 8 \geq n + 2$ and so by Corollary [4.5.1] there is a point $w \in W'$. Then $abca$ is the product of $abwa$, $bwcw$ and $awca$ each of which is geometric (being incident to the 3-space spanned by its vertices) and hence null-homotopic. \[
\]

**Lemma 4.6.5.** If $n \geq 10$ every 5-cycle is a product of 3-cycles and 4-cycles.

**Proof.** Let $abcdea$ be a 5-cycle. If $n \geq 10$ then by the same arguments as in the proof of Lemma [4.6.4] there is a point $w$ biorthogonal to each of $a$, $b$ and $d$. We can therefore decompose $abcdea$ as a product of $abwa$, $bcdwb$ and $awca$.
Lemma 4.6.6. If $n \geq 14$ every 4-cycle is a product of 3-cycles.

Proof. Let $abcd$ be a 4-cycle. Let $W = \langle a, b, c, d, \varphi(a), \varphi(b), \varphi(c), \varphi(d) \rangle^\perp$. Then $W$ has codimension at most 8 with at most a 4 dimensional radical, since it contains the non-degenerate subspace $\langle a, b, \varphi(a), \varphi(b) \rangle^\perp$.

Let $W'$ be a $\varphi$-invariant non-degenerate complement to the radical of $W$ in $W$. If $n \geq 14$ then $\dim W' \geq 2n - 12 \geq n + 2$. Hence $W'$ contains a point $w$ of $\Gamma(n, q)$ biorthogonal to each of $a$, $b$, $c$ and $d$. Thus we can decompose $abcd$ as a product of the 3-cycles $abwa$, $bcwb$, $cdwc$ and $dawd$. \hfill $\Box$
Theorem 4.6.7. If \( n \geq 14 \) then \( \Gamma(n, q) \) is simply connected.

Proof. Since every path in \( \Gamma(n, q) \) is homotopic to a point-line path it suffices to consider cycles in the collinearity graph. Since \( \mathcal{G}(\Gamma(n, q)) \) is connected of diameter 2 for \( n \geq 8 \), in particular it is connected of diameter 2 for \( n \geq 14 \), and so every cycle in \( \mathcal{G}(\Gamma(n, q)) \) can be expressed as a product of cycles of length 3, 4 and 5. These are all null-homotopic by Lemmas 4.6.4, 4.6.6 and 4.6.5 respectively. \( \square \)

4.7 Resulting Phan-Type Theorems

By applying Tits’ Lemma together with Theorem 4.6.7 and Theorem 4.2.3 we immediately obtain the following:

Phan-type Theorem 1. Suppose \( n \geq 14 \), let \( \varphi \) be a linear flip and let \( A \) denote the amalgam of maximal parabolic subgroups of \( U_{2n}(q^2)^\varphi \) with respect to a maximal flag \( F \) of \( \Gamma(n, q) \). Then \( U_{2n}(q^2)^\varphi \) is the universal completion of \( A \).

In fact we can do a little better than this, by applying the following theorems of [25] and [26]:

Theorem 4.7.1 ([25], Theorem 8.2). Let \( \Gamma \) be a geometry over some finite set \( I \) with a flag-transitive group of automorphisms \( G \), let \( k \leq |I| \), let \( A \) and \( A_{(k-1)} \) be the amalgam of parabolics resp. rank-\( k \)-parabolics with respect to some maximal flag \( F \), and assume that all residues of rank greater than or equal to \( k \) with respect to subsets of \( F \) are simply connected. Then \( G = \mathcal{U}(A) = \mathcal{U}(A_{(k-1)}) \).

Lemma 4.7.2 ([26], Lemma 7.2). Assume that \( \Sigma = \Sigma_1 \otimes \Sigma_2 \) with \( \Sigma_1 \) connected of rank at least 2. Then \( \Sigma \) is simply connected.

With these we have the following:
Phan-type Corollary 1. Let $\varphi$ be a linear flip and let $A(k)$ denote the amalgam of rank $k$ parabolic subgroups of $U_{2n}(q^2)^{\varphi}$ with respect to a maximal flag $\mathcal{F}$ of $\Gamma(n, q)$. If $n - k \geq 14$ then $U_{2n}(q^2)^{\varphi} = U(A(k))$.

Proof. All we need to prove is that the residues of rank at least 14 are simply connected. This is easy, since the residues are of one of three types:

(i) Direct products of unitary geometries with $\Gamma(s, q)$ for some integer $s$;

(ii) The geometry of non-degenerate subspaces of a unitary space of rank at least 14;

(iii) $\Gamma(s, q)$ where $s \geq 14$.

That residues of type (i) are simply connected follows from Lemma 4.7.2. That residues of type (ii) are simply connected is equivalent to Phan’s Theorem on the special unitary group in rank at least 27 (see for example Section 3.2 of [7]). Finally that residues of type (iii) are simply connected follows from Theorem 4.6.7. \hfill $\square$

Concluding Remarks

The obvious remaining question is whether $\Gamma(n, q)$ is connected or simply connected for smaller values of $n$. This is a question to which we do not yet have an answer.

We now move to the case of a $\sigma$-semilinear flip. The classification of $\sigma$-semilinear flips is more difficult than the linear flips, however we know more about the resulting geometries.
CHAPTER 5

σ-Semilinear Flips

Introduction

We now turn our attention to the study of σ-semilinear flips of the unitary building. In particular we prove in Theorem 5.4.1 that there are only two similarity classes of σ-semilinear flips of the unitary building.

Unlike the case of a linear flip, if \( \phi \) is a σ-semilinear flip of the unitary building then the group \( U_{2n}(q^2)^{\phi} \) does not act flag transitively on the geometry \( \Gamma(n,q) \). Since Tits’ Lemma requires flag transitivity, we introduce a geometry \( \Gamma_1(n,q) \) that is a subgeometry of \( \Gamma(n,q) \) on which \( U_{2n}(q^2)^{\phi} \) acts flag transitively. In Chapter 6 we will explore the properties of \( \Gamma_1(n,q) \) in more detail.

Throughout this chapter, \( \varphi \) denotes a σ-semilinear flip. Recall that we have identified the flip with a σ-semilinear transformation of \( V \) that satisfies the conclusion of Lemma 3.1.4.

5.1 Further Properties of \( \beta \) and \( \beta_\varphi \)

We now explore some properties of \( (V,\beta,\beta_\varphi) \) in the particular case where \( \varphi \) is a σ-semilinear flip.
Lemma 5.1.1. Let $U$ be a $2k + 1$-dimensional ($k \geq 1$) subspace of $V$ that is $\beta$ and $\beta_\varphi$ non-degenerate. Then $U$ contains a point of $\Gamma(n, q)$.

Proof. Since $U$ is $\beta$ non-degenerate we can write

$$U = \left( \bigwedge_{i=1}^{k} \langle a_i, b_i \rangle \right) \perp \langle c \rangle$$

where each $(a_i, b_i)$ is a hyperbolic pair and $c$ is not $\beta$ isotropic. Let $L = \bigwedge_{i=1}^{k} \langle a_i, b_i \rangle$. We show that $L$ is not $\beta_\varphi$ totally singular and then appeal to Lemma 3.5.3 to conclude that $L$ contains a point of $\Gamma(n, q)$ and so also $U$ contains a point of $\Gamma(n, q)$.

Notice first that $\langle \varphi(c) \rangle \perp L \neq \{0\}$ since $\langle \varphi(c) \rangle$ has codimension 1 and $L$ has dimension at least 2. If $L$ is $\beta_\varphi$ totally singular then any non-zero element of $\langle \varphi(c) \rangle \perp L$ lies in the $\beta_\varphi$ radical of $U$, contradicting the assumption that $U$ is $\beta_\varphi$ non-degenerate.

Thus $L$ is not $\beta_\varphi$ totally singular and so by Lemma 3.5.3 $L$ contains a point of $\Gamma(n, q)$, and since $L \subset U$ it follows that $U$ also contains a point of $\Gamma(n, q)$.

Lemma 5.1.2. Let $W$ be $2k$-dimensional, $\varphi$-invariant and non-degenerate. Then

$$W = \langle U, \varphi(U) \rangle$$

where $U$ is a $k$-object of $\Gamma(n, q)$.

Proof. We induct on $k$. If $k = 1$ this follows easily from Lemma 3.5.3. In general by Lemma 3.5.3 $W$ contains a point $u$ of $\Gamma(n, q)$. Since $W$ is $\varphi$-invariant $W$ also contains $\varphi(u)$ and so it contains the subspace $U_1 = \langle u, \varphi(u) \rangle$.

Let $U_2 = U_1^\perp \cap W$. Then $U_2$ is $(2k - 1)$ dimensional, $\varphi$ invariant and non-degenerate and so by the inductive hypothesis

$$U_2 = \langle U', \varphi(U') \rangle$$
where $U'$ is a $k - 1$ dimensional object of $\Gamma(n, q)$. Finally, set

$$U = \langle u \rangle \oplus U'.$$

\[\square\]

**Lemma 5.1.3.** Let $W$ be $2k + 1$-dimensional, $\varphi$-invariant and non-degenerate. Then

$$W = \langle U, \varphi(U) \rangle \perp \langle c \rangle$$

where $U$ is a $k$-object of $\Gamma(n, q)$, $\langle c \rangle = \langle \varphi(c) \rangle$ and $\beta(c, c) \neq 0$.

**Proof.** We induct on $k$. If $k = 1$ then by Lemma 5.1.1 $W$ contains a point $u$ of $\Gamma(n, q)$. Since $W$ is $\varphi$-invariant, $W$ also contains $\varphi(u)$ and so $W$ contains the subspace $U_1 = \langle u, \varphi(u) \rangle$. Let $W' = \langle u, \varphi(u) \rangle^\perp$. Then $W'$ is a $\varphi$-invariant $\beta$ orthogonal complement to $U_1$ in $W$. Let $c \in W$ be any non-zero vector. Then $W' = \langle c \rangle$ and so $\langle c \rangle = \langle \varphi(c) \rangle$. Moreover since $W$ is non-degenerate, $U_1$ is non-degenerate and $W'$ is a $\beta$ orthogonal complement to $U_1$ in $U$, it follows that $W'$ is $\beta$ non-degenerate, and hence $\beta(c, c) \neq 0$. Finally if we set $U = \langle u \rangle$ then $W = \langle U, \varphi(U) \rangle \perp \langle c \rangle$ as desired.

In general, by Lemma 5.1.1 $W$ contains a point $u$ of $\Gamma(n, q)$. Applying the inductive hypothesis to $W_1 = W \cap \langle u, \varphi(u) \rangle^\perp$ which is also $\varphi$-invariant and non-degenerate, we obtain an element $U_1$ of $\Gamma(n, q)$ and a vector $c$ satisfying $\langle c \rangle = \langle \varphi(c) \rangle$ and $\beta(c, c) \neq 0$ so that $W_1 = \langle U_1, \varphi(U_1) \rangle \perp \langle c \rangle$, and finally by setting $U = U_1 \oplus \langle u \rangle$ we see that $W = \langle U, \varphi(U) \rangle \perp \langle c \rangle$. $\square$

**Corollary 5.1.4.** Let $W$ be a $\varphi$-invariant non-degenerate subspace of dimension at least 2. Then $W$ contains a point of $\Gamma(n, q)$.

**Proof.** This follows from Lemmas 5.1.2 and 5.1.3. In particular, if $W$ is even dimensional then $W = \langle U, \varphi(U) \rangle$ where $U \in \Gamma(n, q)$ and then $U$ contains a point of $\Gamma(n, q)$. Hence also $W$ contains a point of $\Gamma(n, q)$. If $W$ is odd dimensional then $W = \langle U, \varphi(U) \rangle \perp \langle c \rangle$ where $U$ is an object of $\Gamma(n, q)$ and so contains a point of $\Gamma(n, q)$, which then is contained in $W$. $\square$
Notice that Corollary 5.1.4 does not follow immediately from Lemma 3.5.3 since Lemma 3.5.3 requires $W$ be even-dimensional.

5.2 Geometries Induced by a $\sigma$-Semilinear Flip

We’ve already seen that the geometry induced by a flip $\varphi$ is related to a form $\beta_\varphi$ defined on $V$. In the case of a linear flip we saw this form is $\sigma$-hermitian. As we saw in Lemma 3.1.6, when $\varphi$ is a $\sigma$-semilinear flip of the unitary building, the induced form $\beta_\varphi$ is symmetric. It is known that in the geometry of non-degenerate subspaces of an orthogonal space, there are two distinct classes of objects and those classes cannot be intermingled by any group that preserves the form. Since the objects of $\Gamma(n,q)$ are subspaces that are totally isotropic for $\beta$ and non-degenerate for $\beta_\varphi$, we will need to understand these two classes.

**Definition 5.2.1.** Let $U$ be a subspace of $V$. The **discriminant** of $U$ is defined as:

$$\text{disc}(U) = \begin{cases} 
1, & \text{if } \det(\beta_\varphi(U)) \text{ is a square in } \mathbb{F}; \\
-1, & \text{if } \det(\beta_\varphi(U)) \text{ is a non-square in } \mathbb{F}; \\
0, & \text{if } \det(\beta_\varphi(U)) = 0. 
\end{cases}$$

**Definition 5.2.2.** A **square type** (resp. **non-square type**) $i$-space is an $i$-dimensional subspace $U$ of $V$ with $\text{disc}(U) = 1$ (resp. $\text{disc}(U) = -1$).

**Lemma 5.2.1.** Let $U, U' \in \Gamma(n,q)$ with $U \subseteq U'$. Let $W = \langle U, \varphi(U) \rangle^\perp \cap U'$. Then

$$\text{disc}(U') = \text{disc}(U)\text{disc}(W).$$

**Proof.** Since $W$ is a $\beta_\varphi$ orthogonal complement to $U$ in $U'$, we can choose a basis relative to
this decomposition and so represent $\beta_\varphi(U)$ as a matrix with the form
\[
\beta_\varphi(U') = \begin{pmatrix}
\beta_\varphi(W) & 0 \\
0 & \beta_\varphi(U)
\end{pmatrix}
\]
which has determinant $(\det \beta_\varphi(W))(\det \beta_\varphi(U))$. 

We now define two pregeometries contained in $\Gamma(n,q)$. We will prove shortly that these are in fact geometries. We will also later define in more generality a class of geometries of which the following two are examples. The group $U_{2n}(q^2)$ acts flag transitively on all these geometries.

**Definition 5.2.3.** Define the following pregeometries in $\Gamma(n,q)$:

<table>
<thead>
<tr>
<th>Pregeometry</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_1(n,q)$</td>
<td>${U \in \Gamma(n,q)</td>
</tr>
<tr>
<td>$\Gamma_{-1}(n,q)$</td>
<td>${U \in \Gamma(n,q)</td>
</tr>
</tbody>
</table>

**Note 5.2.1.** Just as in the study of $\Gamma(n,q)$, a **point** of $\Gamma_1(n,q)$ (resp. $\Gamma_{-1}(n,q)$) is a 1-dimensional object of $\Gamma_1(n,q)$ (resp. $\Gamma_{-1}(n,q)$) and a **line** of $\Gamma_1(n,q)$ (resp. $\Gamma_{-1}(n,q)$) is a 2-dimensional object of $\Gamma_1(n,q)$ (resp. $\Gamma_{-1}(n,q)$).

**Remark.** It may not be immediately clear why we take all $n$-dimensional elements of $\Gamma(n,q)$ in both $\Gamma_1(n,q)$ and $\Gamma_{-1}(n,q)$. We will see shortly (Theorem 5.3.3) that the $n$-dimensional objects of $\Gamma(n,q)$ all have the same $\beta_\varphi$ type. In order that $\Gamma_1(n,q)$ and $\Gamma_{-1}(n,q)$ are both geometries of rank $n$, we must include these $n$-dimensional objects.

We now explore the properties of the geometries $\Gamma(n,q)$, $\Gamma_1(n,q)$ and $\Gamma_{-1}(n,q)$ in some more detail. Looking at these geometries will give insight into the structure of the flip.

**Lemma 5.2.2.** If $L$ is a line of $\Gamma(n,q)$ then there are biorthogonal points $u$, $v$ of $\Gamma(n,q)$ so that $L = \langle u, v \rangle$. If $L$ is a line of $\Gamma_1(n,q)$ we can assume both $u$ and $v$ are points of $\Gamma_1(n,q)$.
If $L$ is a line of $\Gamma_{-1}(n,q)$ then one of $u$ or $v$ is of square type and the other is of non-square type.

**Proof.** Since $L$ is already $\beta$ totally isotropic it suffices to only consider the form $\beta_{\varphi}$. Since $L$ is a line of $\Gamma(n,q)$ and $\Gamma(n,q)$ is a geometry it follows that $L$ contains a point $u$ of $\Gamma_1(n,q)$. Let $L' = \langle u \rangle^{\perp} \cap L$. Since $L$ is $\beta_{\varphi}$ non-degenerate and also $\langle u \rangle$ is $\beta_{\varphi}$ non-degenerate it follows that $L'$ is $\beta_{\varphi}$ non-degenerate. Since $L' \subset L$ and $L$ is $\beta$ isotropic also $L'$ is $\beta$ isotropic. Hence $L'$ is a point of $\Gamma(n,q)$. Choose any non-zero vector $v \in L'$. By Lemma 5.2.1 it follows that $u$ is of square type if and only if $L$ is of square type. Thus $u$ and $v$ are the desired points. 

**Corollary 5.2.3.** Let $L$ be a line of $\Gamma(n,q)$. Then $L$ contains points of both $\Gamma_1(n,q)$ and $\Gamma_{-1}(n,q)$.

**Proof.** If $L$ is of non-square type then this follows immediately from Lemma 5.2.2. If $L$ is of square type then choose $u$ and $v$ biorthogonal square type points so that $L = \langle u, v \rangle$. Choose $\lambda \in \mathbb{F}$ so that $1 + \lambda^2$ is a non-square in $\mathbb{F}$. Then $x + \lambda y$ is a non-square type point.

**Lemma 5.2.4.** If $\dim U \geq 2$ and $U \in \Gamma(n,q)$, then $U$ contains points of both $\Gamma_1(n,q)$ and $\Gamma_{-1}(n,q)$.

**Proof.** We induct on $\dim U$. Assume $\dim U = 2$. Then $U$ is a line of $\Gamma(n,q)$ and so by Corollary 5.2.3 $U$ contains points of both types.

In general, by Lemma 3.3.5 $U$ contains a point $x$ of $\Gamma(n,q)$. Set $W = \langle x, \varphi(x) \rangle^{\perp} \cap U$. Then $W$ is an object of $\Gamma(n,q)$ that is at least 2 dimensional and has dimension strictly less than $\dim(U)$. Hence by the inductive hypothesis $W$ contains points of both types, and so also $U$ contains points of both types.

**Corollary 5.2.5.** If $U \in \Gamma_{\epsilon}(n,q)$ then $U$ contains a point of $\Gamma_{\epsilon}(n,q)$.

**Proof.** If $\dim(U) = 1$ then $U$ is a point of $\Gamma_{\epsilon}(n,q)$. If $\dim(U) > 1$ then we can apply Lemma 5.2.4 to conclude that $U$ contains a point of $\Gamma_{\epsilon}(n,q)$.
Corollary 5.2.6. A $k$-element $U$ of $\Gamma_1(n,q)$ with $k < n$ has a basis $\{u_1, \ldots, u_k\}$ of square type points that are pairwise biorthogonal. Consequently any $k$-element $U$ of $\Gamma_1(n,q)$ contains a $\{1, \ldots, k\}$-flag of $\Gamma_1(n,q)$.

Proof. We induct on $k$, the case $k = 1$ being obvious. If $U$ is $k$-dimensional, $k > 1$ then choose a point of $\Gamma_1(n,q)$, $u_1 \in U$. Let $W = \langle u_1, \varphi(u_1) \rangle^\perp \cap U$. Then $W$ is a $(k - 1)$-dimensional element of $\Gamma_1(n,q)$ and so we can apply the inductive hypothesis to produce the remaining points. Finally, the flag is $(c_i)_{i=1}^k$ where $c_i = \langle u_1, \ldots, u_i \rangle$.

Corollary 5.2.7. Let $M$ be an $n$-object of $\Gamma(n,q)$. If $M$ is of square type then $M$ has a basis $\{u_1, \ldots, u_n\}$ of pairwise biorthogonal square type points. If $M$ is of non-square type then $M$ has a basis $\{v_1, \ldots, v_n\}$ of pairwise biorthogonal points, where for $i = 1, \ldots, n - 1$, $v_i$ is of square type and $v_n$ is of non-square type.

Proof. If $M$ is of square type it contains a square type point $u_1$. By applying Corollary 5.2.6 to $M' = M \cap \langle u_1 \rangle^\perp$ we produce the remaining points. If $M$ is of non-square type it contains a non-square type point $v_n$ and applying Corollary 5.2.6 to $M' = \langle v \rangle^\perp$ produces the remaining points.

Remark. We will prove in Theorem 5.3.3 that for a fixed flip $\varphi$, every maximal object has the same $\beta_\varphi$ type.

Lemma 5.2.8. Let $u$ be a point of $\Gamma_\epsilon(n,q)$ for $\epsilon \in \{1, -1\}$. Then $\text{res}_{\Gamma_\epsilon(n,q)}(u) \cong \Gamma_1(n - 1, q)$.

Proof. This follows immediately from the proof of Corollary 3.5.1 once we note that if $u \in \Gamma_1(n,q)$ then $r_u$ sends square type subspaces to square type subspaces, and non-square type subspaces to non-square type subspaces. Similarly if $u \in \Gamma_{-1}(n,q)$ then $r_u$ sends square type subspaces to non-square type subspaces, and non-square type subspaces to square type subspaces.

Theorem 5.2.9. $\Gamma_1(n,q)$ and $\Gamma_{-1}(n,q)$ are geometries with type and incidence inherited from $\Gamma(n,q)$. 
Proof. The same arguments as in the proof of Theorem 3.5.8 work in this case, with Lemma 5.2.8 replacing Corollary 3.5.7.

5.3 The Group Preserving $\beta$ and $\beta_\varphi$

We now study the group of linear transformations of $V$ that preserve both $\beta$ and $\beta_\varphi$. This group acts as an automorphism group of the geometry $\Gamma(n,q)$ although it does not act flag transitively on that geometry. We prove in this section that this group acts flag transitively on $\Gamma_1(n,q)$.

Definition 5.3.1. Let $U_2n(q^2)_\varphi = \{ f \in U_2n(q^2) | \beta_\varphi(u,v) = \beta_\varphi(f(u),f(v)) \text{ for all } u,v \in V \}$. Notice that this is the same notation we used in the case of a linear flip.

Lemma 5.3.1. $U_2n(q^2)_\varphi = C_{\Gamma(U)}(\varphi) \cap U_2n(q^2)$.

Proof. The same proof as in Lemma 4.2.1 holds in this case.

Lemma 5.3.2. If $U$ is a $\varphi$-invariant non-degenerate subspace of $V$ of dimension 2, then $U = \langle u, \varphi(u) \rangle$ for some $u \in \Gamma(n,q)$. Moreover every point of $U$ has the same $\beta_\varphi$ type as $u$.

Proof. That $U = \langle u, \varphi(u) \rangle$ for some point $u$ of $\Gamma(n,q)$ follows from Lemma 5.1.2.

In order to prove the second part of the theorem, we show that if $\lambda$ has trace 0 in $\mathbb{F}$ then $1 + \lambda^2$ lies in $\mathbb{F}_q$ and so in particular is a square in $\mathbb{F}$. Since $\text{Tr}_\sigma(\lambda) = 0$ it follows that $\sigma(\lambda) = -\lambda$ and so $\sigma(\lambda^2) = \sigma(\lambda)^2 = \lambda^2$. Hence $\lambda^2 \in \mathbb{F}_q$ and thus also $1 + \lambda^2 \in \mathbb{F}$.

If $u$ is a point in $U$, then all the other points of $\Gamma(n,q)$ that lie in $U$ (except for $\varphi(u)$) are of the form $u + \lambda \varphi(u)$ for some non-zero $\lambda$ of trace 0, and since these points have $Q_\varphi$ value $(1 + \lambda^2)Q_\varphi(u)$, we conclude that all these points have the same type as $u$. Since also $\varphi(u)$ has the same $\beta_\varphi$ type as $u$ we conclude that all the points of $U$ have the same $\beta_\varphi$ type.

Theorem 5.3.3. Let $M$ and $M'$ be $n$-dimensional objects of $\Gamma(n,q)$. Then $M$ and $M'$ have the same $\beta_\varphi$ type.
Proof. If $n = 1$ then this follows because $V = \langle u, \varphi(u) \rangle$ for some point $u$ of $\Gamma(1, q)$. Since every point on $\langle u, \varphi(u) \rangle$ has the same $\beta_\varphi$ type as $u$ (by Lemma 5.3.2), when $n = 1$ the maximal objects of $\Gamma(1, q)$ all have the same $\beta_\varphi$ type.

Assume $n > 1$, suppose $M$ is of square type and suppose $M'$ is of non-square type. Then by Corollary 5.2.7 $M$ has a basis $\{e_1, \ldots, e_n\}$ of biorthogonal square type points. We may then scale each $e_i$ so that $\beta_\varphi(e_i, e_i) = 1$ for all $i$. Setting $f_i = \varphi(e_i)$ for $i = 1, \ldots, n$ we obtain a $\beta$ hyperbolic basis $B_1 = \{e_i, \varphi(e_i)\}_{i=1}^n$ for $V$.

Similarly by Corollary 5.2.7 $M'$ has a basis $\{g_1, \ldots, g_n\}$ of biorthogonal points where $g_1, \ldots, g_{n-1}$ are of square type and $g_n$ is of non-square type. After scaling we can assume that $\beta_\varphi(g_i, g_i) = 1$ for $i = 1, \ldots, n$, and $\beta_\varphi(g_n, g_n) = \alpha$ where $\alpha$ is a non-square in $\mathbb{F}$. Let $h_n = \alpha^1 g_n$. Notice that $\varphi(h_n) = \sigma(\alpha^{-1})\varphi(g_n)$ and so $\varphi(g_n) = \sigma(\alpha)\varphi(h_n)$. It is easy to see that $(h_n, \varphi(g_n))$ forms a $\beta$ hyperbolic pair, and that $\beta_\varphi(h_n, h_n) = \alpha^{-1}$, and $\beta_\varphi(\varphi(g_n), \varphi(g_n)) = \sigma(\alpha)$. This gives another $\beta$ hyperbolic basis, $B_2 = \{g_1, \varphi(g_1), \ldots, g_{n-1}, \varphi(g_{n-1}), h_n, \sigma(\alpha)\varphi(h_n)\}$.

Let $T$ be the transition matrix from $B_1$ to $B_2$. For $i = 1, 2$ let $B_i$ denote the Gram matrix of $\beta$ with respect to $B_i$, and let $C_i$ denote the Gram matrix of $\beta_\varphi$ with respect to $B_i$. Since $T$ is the transition matrix from $B_1$ to $B_2$ it follows that

$$
TB_1^\sigma T^\sigma = B_2 
$$

(5.1)

$$
TC_1^\sigma T = C_2.
$$

(5.2)

Since $C_1 = I_{2n}$ it follows from (5.2) that $T^\sigma T = C_2$. It is easy to check that $\det(C_2) = \alpha^{q-1}$ and so also $\det(T)^2 = \alpha^{q-1}$. Hence $\det(T) = \pm\alpha^{(q-1)/2}$ and it is easy to check that this implies $N_{\sigma}(\det(T)) = -1$.

On the other hand, since both $B_1$ and $B_2$ are $\beta$ hyperbolic bases, we see that $B_1 = B_2$, which combined with (5.1) forces $N_{\sigma}(\det(T)) = 1$, a contradiction. Hence $M$ and $M'$ have the same $\beta_\varphi$ type.  \qed
Remark. Theorem 5.3.3 is important for two very different reasons. First, it will be important in proving that $U_{2n}(q^2)^\sigma$ acts flag transitively on $\Gamma_1(n, q)$. Second, it shows that there are exactly two distinct classes of $\sigma$-semilinear flips of the unitary building. One of these classes has square-type maximal objects, and the other has non-square type maximal objects. In fact these are the only two possibilities, that is the $\beta_\varphi$ type of a maximal object determines $\varphi$ up to a unitary change of basis. We state this precisely as Theorem 5.4.1.

Definition 5.3.2. A $\sigma$-semilinear flip $\varphi$ is of square type if the maximal objects of $\Gamma(n, q)$ are of square $\beta_\varphi$ type. A $\sigma$-semilinear flip $\varphi$ is of non-square type if the maximal objects of $\Gamma(n, q)$ are of non-square $\beta_\varphi$ type.

5.4 Classification of $\sigma$-Semilinear Flips of the Unitary Building

We are now in a position to fully classify the flips of the unitary building that are induced by $\sigma$-semilinear transformations of the underlying vector space.

Theorem 5.4.1 (Classification of $\sigma$-Semilinear Flips). Let $\varphi$ be a $\sigma$-semilinear flip on the unitary building. Then there is a basis for $V$, $\{e_i, f_i\}_{i=1}^n$ of $\beta$ hyperbolic pairs so that for $i = 1, \ldots, n-1$, we have $\varphi(e_i) = f_i$, $\varphi(f_i) = e_i$ and either

(i) $\varphi(e_n) = f_n$, $\varphi(f_n) = e_n$ or

(ii) $\varphi(e_n) = \lambda f_n$, $\varphi(f_n) = \sigma(\lambda^{-1})e_n$ where $\lambda$ is a non-square in $\mathbb{F}$.

Case (i) occurs if $\varphi$ is of square type, and Case (ii) occurs if $\varphi$ is of non-square type. Conversely any $\sigma$-semilinear transformation of $V$ that satisfies either (i) or (ii) induces a flip of $\Delta$.

Proof. The forward implication follows immediately from the proof of Theorem 5.3.3. The converse follows from Lemma 3.2.2.
What is mainly interesting here is that these two types of flips are in fact distinct, and the resulting geometries are not isomorphic. In order to verify that the geometries are not isomorphic, one can count the number of \( n - 1 \) dimensional objects contained in a fixed \( n \)-object. This is done in Theorem 6.1.5.

**Lemma 5.4.2.** Let \( C = (C_i)_{i=1}^{n} \) be a chamber of \( \Gamma_1(n, q) \).

(a) If \( \varphi \) is of square type then there is a basis \( \{e_i, f_i\}_{i=1}^{n} \) for \( V \) with the following properties:

(i) \( \{e_i, f_i\} \) is hyperbolic with respect to \( \beta \);

(ii) for all \( i = 1, \ldots, n \), \( \varphi(e_i) = f_i \) and \( \varphi(f_i) = e_i \); and

(iii) for all \( i = 1, \ldots, n \), \( C_i = \langle e_1, \ldots, e_i \rangle \).

(b) If \( \varphi \) is of non-square type then there is a basis \( \{e_i, f_i\}_{i=1}^{n} \) for \( V \) with properties (i) and (iii) and

(ii') for all \( i = 1, \ldots, n - 1 \), \( \varphi(e_i) = f_i \) and \( \varphi(f_i) = e_i \) and \( \varphi(e_n) = \lambda f_n, \ \varphi(f_n) = \sigma(\lambda^{-1})e_n \), where \( \lambda \) is a non-square in \( \mathbb{F} \).

**Proof.** The same arguments as in the proof of Lemma 4.2.2 work with Corollary 5.2.5 replacing Lemma 3.3.5 and the scaling arguments from the proof of Theorem 5.3.3 replacing the scaling arguments from the proof of Theorem 4.1.1.

**Note 5.4.1.** The proof of Lemma 5.4.2 shows that we can choose any non-square \( \lambda \) to appear in (ii').

**Note 5.4.2.** Lemma 5.4.2 is strictly stronger than Theorem 5.4.1. In Theorem 5.4.1 we only prove that, given a maximal object we can find a basis satisfying (i) and (ii) (resp. (ii'))), what is interesting about Lemma 5.4.2 is that given any chamber of \( \Delta \varphi \) we can additionally require that (iii) be satisfied.

**Theorem 5.4.3.** \( U_{2n}(q^2)^\varphi \) acts flag transitively on \( \Gamma_1(n, q) \).
Proof. The same proof as in Main Theorem 2 applies here, with Lemma 5.4.2 replacing Lemma 4.2.2.

The proof that \(U_{2n}(q^2)^\varphi\) acts flag transitively on \(\Gamma_{-1}(n, q)\) is similar.

Remark. Now we are in a position to see why our interest in flips will prove useful. Since \(U_{2n}(q^2)^\varphi\) acts flag transitively on \(\Gamma_1(n, q)\), whenever \(\Gamma_1(n, q)\) is simply connected the group \(U_{2n}(q^2)^\varphi\) will be the universal completion of its amalgam of maximal parabolics. We will explore this further in the next chapter. 

\[\diamondsuit\]

5.5 The Isomorphism Type of \(U_{2n}(q^2)^\varphi\)

We now determine the isomorphism type of \(U_{2n}(q^2)^\varphi\) when \(\varphi\) is a \(\sigma\)-semilinear flip. We begin with a general lemma regarding 2-dimensional non-degenerate orthogonal spaces that contain singular vectors. In particular this result applies to the space \(\langle u, \varphi(u) \rangle\) when \(u\) is a point of \(\Gamma(n, q)\).

**Lemma 5.5.1.** Let \((W, \rho)\) be a non-degenerate orthogonal space over a field \(k\). Let \(U\) be a 2-dimensional non-degenerate subspace of \(W\).

1. If \(-1\) is a square in \(k\), then \(U\) is of + type if and only if \(U\) is of square type.
2. If \(-1\) is not a square in \(k\), then \(U\) is of + type if and only if \(U\) is of non-square type.

**Proof.** Suppose first that \(-1\) is a square in \(k\). If \(U\) is of + type then \(U = \langle u, v \rangle\) where \((u, v)\) is a pre-hyperbolic pair. By taking an appropriate linear combination we may choose a vector \(u_1 \in U\) so that \(\rho(u_1, u_1) = 1\). Choose any non-zero vector \(u_2 \in U \cap u_1^\perp\). Then we have a basis \(U = \{u_1, u_2\}\) for \(U\) so that the Gram matrix of \(\rho\) with respect to this basis is

\[
\begin{pmatrix}
1 & 0 \\
0 & \text{disc}(U)
\end{pmatrix}.
\]
If \( x = x_1u_1 + x_2u_2 \) then \( \rho(x, x) = x_1^2 + \text{disc}(U)x_2^2 \). Solving for \( \text{disc}(U) \) we find that

\[
\text{disc}(U) = -x_1^2x_2^{-2}
\]

and so \( \text{disc}(U) \) is a square.

Conversely suppose \( \text{disc}(U) \) is a square. Let \( u_1 \) be any non-singular vector in \( U \). Let \( u_2 \) be any non-zero vector in \( U \cap u_1^\perp \). Then with respect to this basis, \( U \) has \( \rho \) Gram matrix

\[
\begin{pmatrix}
\rho(u_1, u_1) & 0 \\
0 & \rho(u_2, u_2)
\end{pmatrix}
\]

which has determinant \( \rho(u_1, u_1)\rho(u_2, u_2) \). Since \( U \) is of square type, either both \( \rho(u_1, u_1) \) and \( \rho(u_2, u_2) \) are squares or both are non-squares. In either case there is a linear combination \( x = x_1u_1 + x_2u_2 \) such that \( \rho(x, x) = 0 \). Thus \( U \) contains singular vectors and so \( U \) is of + type.

The proof of (ii) is similar.

The next theorem is crucial to determining the isomorphism type of \( U_{2n}(q^2)^{\varphi} \) when \( \varphi \) is a \( \sigma \)-semilinear flip.

**Theorem 5.5.2.** There is a basis for \( V \) relative to which the Gram-matrices of \( \beta \) and \( \beta_{\varphi} \) coincide.

**Proof.** Choose a basis for \( V \) as provided by Main Theorem 1. Then we have a basis \( \{e_i, f_i\}_{i=1}^n \) so that each \( (e_i, f_i) \) is a \( \beta \) hyperbolic pair, and for \( i = 1, \ldots, n - 1 \) we have

\[
\begin{align*}
\varphi(e_i) &= f_i \\
\varphi(f_i) &= e_i
\end{align*}
\]
If \( \varphi \) is a square type flip then we may assume that \( \lambda = 1 \). If \( \varphi \) is a non-square type flip then \( \lambda \) is a non-square in \( \mathbb{F} \).

Choose \( a \in \mathbb{F}_q \) that is not a square (in \( \mathbb{F}_q \)) and choose \( \alpha \in \mathbb{F} \) so that \( \alpha^2 = a \). Notice that \( \sigma(\alpha) = -\alpha \). Define a new basis as follows. For \( i = 1, \ldots, n \) set

\[
\begin{align*}
g_i &= e_i + f_i \\
g_{i+n} &= \alpha(e_i - f_i) \\
g_n &= e_n + \lambda f_n \\
g_{2n} &= \alpha(e_n - \lambda f_n)
\end{align*}
\]

Then each vector \( g_i \) is fixed by \( \varphi \), and so since \( \varphi \) acts trivially on the basis \( \{ g_i | i = 1, \ldots, 2n \} \) we conclude the Gram matrices of \( \beta \) and \( \beta_\varphi \) agree with respect to this basis.

**Construction 6.** For convenience, we reorder the basis \( \{ g_i | i = 1, \ldots, 2n \} \) from the proof of Theorem 5.5.2 as follows: for \( i = 1, \ldots, n \) set

\[
\begin{align*}
h_{2i-1} &= g_i \\
h_{2i} &= g_{i+n}
\end{align*}
\]

With respect to the basis \( \{ h_i \}_{i=1}^{2n} \), the common Gram matrix of \( \beta \) and \( \beta_\varphi \) is a block diagonal matrix with \( 2 \times 2 \) blocks \( \{ M_i \}_{i=1}^n \). If \( \varphi \) is of square type then for \( i = 1, \ldots, n \) we have
have
\[ M_i = \begin{pmatrix} 2 & 0 \\ 0 & 2a \end{pmatrix}. \]

If \( \varphi \) is of non-square type then for \( i = 1, \ldots, n-1 \) we have the same matrix \( M_i \) as above and
\[
M_n = \begin{pmatrix} \text{Tr}_\sigma(\lambda) & \alpha(\sigma(\lambda) - \lambda) \\ \alpha(\sigma(\lambda) - \lambda) & a\text{Tr}_\sigma(\lambda) \end{pmatrix}.
\]

We now prove a final lemma that will be important in determining the isomorphism type of \( U_{2n}(q^2)^\sigma \) for a non-square type flip.

**Lemma 5.5.3.** Suppose \( \gamma \) is a non-square in \( \mathbb{F} \). Then \( N_\sigma(\gamma) \) is a non-square in \( \mathbb{F}_q \).

**Proof.** Recall first that
\[
N_\sigma(\gamma) = \gamma \sigma(\gamma) = \gamma \gamma^q = \gamma^{q+1}.
\]

Let \( \alpha \) generate the multiplicative group of units of \( \mathbb{F} \). Since \( N_\sigma(\alpha) \in \mathbb{F}_q \) it is a square in \( \mathbb{F} \), and we can identify its square roots in \( \mathbb{F} \). They are
\[
\pm \alpha^{\frac{q+1}{2}}.
\]

We show that these do not lie in \( \mathbb{F}_q \), and so \( N_\sigma(\alpha) \) has no square roots in \( \mathbb{F}_q \), which will prove that \( N_\sigma(\alpha) \) is a non-square in \( \mathbb{F}_q \).

In order to verify that \( \alpha^{\frac{q+1}{2}} \notin \mathbb{F}_q \) we will show it is not fixed by \( \sigma \). This follows by the
following calculation:

\[
\sigma\left(\alpha^{\frac{q+1}{2}}\right) = \left(\alpha^{\frac{q+1}{2}}\right)^q \\
= \alpha^{\frac{(q+1)q}{2}} \\
= \alpha^{\frac{q^2+q}{2}} \\
= \alpha^{\frac{q^2-1+q+1}{2}} \\
= \left(\alpha^{\frac{q^2-1}{2}}\right)\left(\alpha^{\frac{q+1}{2}}\right) \\
= -\alpha^{\frac{q+1}{2}}.
\]

The last equality holds because \(\alpha\) generates the multiplicative group of units of \(\mathbb{F}\) and so \(\alpha^{\frac{q^2-1}{2}} = -1\). It follows that \(N_\sigma(\alpha)\) has no square roots in \(\mathbb{F}_q\) and so it is a non-square in \(\mathbb{F}_q\).

If \(\gamma\) is any other non-square in \(\mathbb{F}\), then \(\gamma = \alpha^{2k+1}\) for some \(k \in \mathbb{N}\).

Since \(N_\sigma\) is multiplicative and \(N_\sigma(\alpha)\) is a non-square in \(\mathbb{F}_q\) also \(N_\sigma(\alpha^{2k+1}) = N_\sigma(\alpha)^{2k+1}\) is a non-square in \(\mathbb{F}_q\). Hence \(N_\sigma(\gamma)\) is a non-square in \(\mathbb{F}_q\). \(\square\)

**Theorem 5.5.4.** Let \(\varphi\) be a \(\sigma\)-semilinear flip of \(\Delta\).

1. Suppose \(\varphi\) is a square type flip.
   
   (i) If \(n\) is even or \(-1\) is not a square in \(\mathbb{F}_q\) then \(U_{2n}(q^2)\varphi \cong O_{2n}^+(q)\).
   
   (ii) If \(n\) is odd and \(-1\) is a square in \(\mathbb{F}_q\) then \(U_{2n}(q^2)\varphi \cong O_{2n}(q)\).

2. Suppose \(\varphi\) is a non-square type flip.
   
   (i) If \(n\) is even or \(-1\) is not a square in \(\mathbb{F}_q\) then \(U_{2n}(q^2)\varphi \cong O_{2n}^{-}(q)\).
   
   (ii) If \(n\) is odd and \(-1\) is a square in \(\mathbb{F}_q\) then \(U_{2n}(q^2)\varphi \cong O_{2n}^{+}(q)\).

**Proof.** Let \(M\) denote the common Gram matrix of \(\beta\) and \(\beta_\varphi\) with respect to the basis \(\{h_i|i = 1, \ldots, 2n\}\) produced at the end of Construction 6. Then, with respect to this basis,
a transformation $A \in \text{GL}(V)$ lies in $U_{2n}(q^2)^\varphi$ if and only if both

$$^tAMA^\varphi = M^t$$

and

$$AMA = M.$$  

Since $M$ and $A$ are both invertible, it follows that $A = A^\sigma$ and so $A$ is defined over $\mathbb{F}_q$. Hence $U_{2n}(q^2)^\varphi$ consists of all $q$-rational matrices in $\text{GL}(V)$ that satisfy

$$^tAMA = M.$$  

Since the matrix $M$ is also defined over $\mathbb{F}_q$, we see that the group of matrices satisfying $^tAMA = M$ is the full orthogonal group over $\mathbb{F}_q$ with respect to the symmetric bilinear form whose Gram matrix is $M$. So now we must determine this group. That is, $U_{2n}(q^2)^\varphi$ is isomorphic to either $O_{2n}^+(q)$ or $O_{2n}^-(q)$.

Recall that if $L_1$ and $L_2$ are $\beta$-orthogonal elliptic lines, then $L_1 \perp L_2$ can be written as $H_1 \perp H_2$ where each $H_i$ is a hyperbolic line. It follows that the isomorphism type of $U_{2n}(q^2)^\varphi$ is determined by the last two blocks $M_{n-1}$ and $M_n$ if $n$ is even, and the last block $M_n$ if $n$ is odd.

(1) Suppose now that $\varphi$ is a square type flip. If $n$ is even, then since the last two blocks both represent non-square type spaces, either both are elliptic lines or both are hyperbolic lines. In either case the resulting group is $O_{2n}^+(q)$.

If $n$ is odd, then the group is determined by whether a non-square type line is elliptic or hyperbolic. If $-1$ is not a square in $\mathbb{F}_q$ then the non-square type line corresponding to $M_n$ is hyperbolic, and hence the resulting group is $O_{2n}^+(q)$. If $-1$ is a square in $\mathbb{F}_q$ then the non-square type line corresponding to $M_n$ is elliptic and hence the resulting group is $O_{2n}^-(q)$.
Suppose now that $\varphi$ is a non-square type flip. Then in the last block $\lambda$ is a non-square in $\mathbb{F}$. An easy calculation shows that

$$\det M_n = \det \begin{pmatrix} \text{Tr}_\sigma(\lambda) & \alpha(\lambda - \sigma(\lambda)) \\ \alpha(\lambda - \sigma(\lambda)) & a\text{Tr}_\sigma(\lambda) \end{pmatrix} = 4aN_\sigma(\lambda).$$

This determinant is a square in $\mathbb{F}_q$ since both $a$ and $N_\sigma(\lambda)$ are non-squares in $\mathbb{F}_q$, and so the space corresponding to $M_n$ is a square type space.

If $n$ is even, then since the last two blocks represent spaces of different types, the resulting group is $O_{2n}^-(q)$. If $n$ is odd then the group is determined by whether a square type line is elliptic or hyperbolic. If $-1$ is a square in $\mathbb{F}_q$, then the square type line corresponding to $M_n$ is of $+$ type and so the group is $O_{2n}^+(q)$. If $-1$ is not a square in $\mathbb{F}_q$, then the square type line corresponding to $M_n$ is of $-$ type, and so the group is $O_{2n}^-(q)$.

### 5.6 Parabolic Subgroups

Now that we have identified the group acting flag transitively on our geometries, we need to identify the parabolic subgroups so that we can understand the particular amalgams produced when the geometries are simply connected. We determine the parabolic subgroups for flags of $\Gamma(n,q)$ because this does not increase the work involved.

**Theorem 5.6.1.** Let $C = (C_i)_{i=1}^k$ be a flag of $\Gamma(n,q)$ and let $m_k = \dim C_k$. For each $i = 2, \ldots, k$, let $C'_i = C_i \cap \varphi(C_{i-1})^\perp$ and let $C'_1 = C_1$. Let $\eta_i = \beta_\varphi|C_i$. Then the stabilizer of $C$ in $U_{2n}(q^2)\varphi$ is isomorphic to

$$H = \bigoplus_{i=1}^k O(C'_i, \eta_i) \oplus U_{2(n-m_k)}(q^2)^\varphi,$$

where $\phi$ is a $\sigma$-semilinear flip on the $2(n-m_k)$ dimensional unitary space $\langle C_k, \varphi(C_k) \rangle^\perp$ and
is:

(i) square type if \( \varphi \) and \( C_k \) have the same type; and

(ii) non-square type if \( \varphi \) and \( C_k \) have different types.

Proof. The same proof as in Theorem 4.3.1 does most of the work. All that differs is that in this case, the spaces \((C_i', \eta_i)\) are orthogonal rather than unitary. Finally we have to determine the type of the flip \( \phi \). In Theorem 4.3.1 there was only one type of flip available, in this case we have to determine if \( \phi \) is of square type or non-square type.

The type of the flip \( \phi \) is given by the type of the maximal objects of the geometry induced by the restriction of \( \varphi \) to the unitary space \( \langle C_k, \varphi(C_k) \rangle \). Let \( M \) be an arbitrary maximal object of the geometry induced by this restriction. If \( \varphi \) and \( C_k \) are of the same type, so is \( M \) and hence so is \( \phi \). Similarly if \( \varphi \) and \( C_k \) are of different types then \( M \) is of non-square type and hence so is \( \phi \).

5.7 Other Geometries

The construction of \( \Gamma_1(n, q) \) can be generalized to produce a number of other geometries on which \( U_{2n}(q^2) \) acts flag transitively. It follows that if any of these geometries are simply connected, then they can be used to produce amalgams for \( U_{2n}(q^2) \). We now describe this construction and then prove that \( U_{2n}(q^2) \) acts flag transitively on all these geometries.

Construction 7. For each \( i = 1, \ldots, n - 1 \) let \( \epsilon_i \in \{ \pm 1 \} \), and let \( \epsilon = (\epsilon_i)_{i=1}^{n-1} \). Let \( \Psi_i(n, q) \) denote the set of \( i \)-objects \( U \) of \( \Gamma(n, q) \) with disc\((U) = \epsilon_i \). Let \( \Psi_n(n, q) \) denote the set of \( n \)-objects of \( \Gamma(n, q) \). By Theorem 5.3.3 all the elements of \( \Psi_n(n, q) \) have the same \( \beta_\varphi \) type.

Let \( \Gamma_\epsilon(n, q) = \bigcup_{i=1}^{n} \Psi_i(n, q) \), with type and incidence inherited \( \Gamma(n, q) \).

Theorem 5.7.1. \( \Gamma_\epsilon(n, q) \) is a geometry.

Proof. It suffices to show that every maximal flag of \( \Gamma_\epsilon(n, q) \) is a chamber. Let \( C = (C_i)_{i=1}^{k} \) be a flag that is not a chamber. Let \( d_i = \dim C_i \).
If \(d_1 > 1\) then \(C_1\) contains points of both \(\Gamma_1(n, q)\) and \(\Gamma_{-1}(n, q)\). If \(\epsilon_1 = 1\) then any square-type point contained in \(C_1\) can be added to the flag; if \(\epsilon_1 = -1\) then any non-square type point contained in \(C_1\) can be added to the flag. In either case, the flag is not maximal.

If \(d_k < n\) then \(W = \langle C_k, \varphi(C_k) \rangle^\perp\) is \(2(n-k)\) dimensional, \(\varphi\) invariant and non-degenerate. If \(k = n-1\) then choose any point \(u\) of \(\Gamma(n, q)\) in \(W\). Then \(C_n = \langle C_k, u \rangle\) can be added to the flag, so the flag is not maximal. If \(k < n-1\) then \(W\) contains points of both types. Thus we can choose a point \(u\) of \(\Gamma(n, q)\) in \(W\) so that \(\langle C_k, u \rangle \in \Psi_{k+1}(n, q)\). Thus the flag \(C\) is not maximal.

Finally, if there is an index \(j\) so that \(d_j - d_{j-1} > 1\) then \(X = C_j \cap \varphi(C_{j-1})^\perp\) contains points of both \(\Gamma_1(n, q)\) and \(\Gamma_{-1}(n, q)\). Thus we can find a point \(u\) of \(\Gamma(n, q)\) in \(X\) so that \(\langle C_{j-1}, x \rangle\) is contained in \(C_j\) and \(\Psi_{d_{j-1}+1}\). Hence again we see the flag \(C\) is not maximal.

Hence every maximal flag of \(\Gamma_\epsilon(n, q)\) is a chamber, and so \(\Gamma_\epsilon(n, q)\) is a geometry. 

\[\square\]

**Note 5.7.1.** We could use an argument involving the residue of a point to prove Theorem 5.7.1 just as we did in the proof of Theorem 5.2.9 however in this case that argument is somewhat less pleasant. The difficulty in applying that argument to prove Theorem 5.7.1 is that in the inductive step we must assume that for every \(\epsilon' : \{1, \ldots, n-1\} \rightarrow \{\pm 1\}\) the resulting \(\Psi_\epsilon(n-1, q)\) is a geometry. While this presents no technical difficulty, it seems less natural than the direct argument presented above.

**Theorem 5.7.2.** \(U_{2n}(q^2)^\epsilon\) acts flag transitively on \(\Gamma_\epsilon(n, q)\).

**Proof.** Let \((C_i)\) and \((D_i)\) be flags of \(\Gamma_\epsilon(n, q)\). By Theorem 5.7.1 we may assume that both are chambers.

Choose a vector \(0 \neq a_1 \in C_1\), notice that \(a_1\) is a point of \(\Gamma_\epsilon(n, q)\). Choose a point of \(\Gamma(n, q)\), \(a_2 \in C_2 \cap \varphi(C_1)^\perp\). For \(k \geq 2\), choose points of \(\Gamma(n, q)\), \(a_k \in C_k \cap \varphi(C_{k-1})^\perp\). Then for each \(k\), \(a_k\) is a point of \(\Gamma(n, q)\) and has the same \(\beta_\varphi\) type as \(C_k \cap \varphi(C_{k-1})^\perp\). Furthermore, \(C_k = \langle a_1, \ldots, a_k \rangle\), and each \(\langle a_i, \varphi(a_i) \rangle\) is a hyperbolic line. Notice also that \(a_i\) is biorthogonal to \(a_j\) for \(i \neq j\).
We may similarly construct a sequence of points \((b_i)\) with the same properties relative to the flag \((D_i)\). After scaling we may assume that \(Q_\varphi(a_i) = Q_\varphi(b_i)\) for all \(i\). Define a transformation \(T : V \rightarrow V\) by \(T(a_i) = b_i\) and \(T(\varphi(a_i)) = \varphi(b_i)\). Then \(T \in U_{2n}(q^2)^\varphi\), and \(T(C_i) = d_i\) for all \(i\). Thus \(U_{2n}(q^2)^\varphi\) acts flag transitively on \(\Gamma_\epsilon(n, q)\). □

**Concluding Remarks**

Now that we have classified the \(\sigma\)-semilinear flips on the unitary building and identified a class of geometries induced by these flips on which the unitary group acts flag transitively, we can ask more questions about these geometries. For example, when are they connected? When are they simply connected? What do their residues look like? In the next chapter we have studied these problems for the geometry \(\Gamma_1(n, q)\) induced by a \(\sigma\)-semilinear flip and achieved some partial results. In particular we will prove that for \(n \geq 4\) and all odd prime powers \(q\), the geometry \(\Gamma_1(n, q)\) is connected, and for all \(n \geq 8\) and all odd prime powers \(q\) the geometry \(\Gamma_1(n, q)\) is simply connected. We also prove that if \(q \geq 5\) and \(n = 7\) then \(\Gamma_1(n, q)\) is simply connected, and that if \(-1\) is a square in \(\mathbb{F}_q\) then \(\Gamma_1(2, q)\) is connected.
CHAPTER 6

The Geometry $\Gamma_1(n, q)$

Introduction

In this chapter we explore the properties of $\Gamma_1(n, q)$ in more detail. In particular we prove that if $n \geq 5$ then $G(\Gamma_1(n, q))$ is connected of diameter 2, $G(\Gamma_1(4, q))$ is connected of diameter at most 4, and if $-1$ is a square in $\mathbb{F}_q$ then $G(\Gamma_1(2, q))$ is connected of diameter at most 4.

We also prove that if $n \geq 8$ or if $n = 7$ and $q \geq 5$ then $\Gamma_1(n, q)$ is simply connected. It follows that $U_{2n}(q^2)\varphi$ is the universal completion of its amalgam of maximal parabolics in these cases.

Throughout this chapter if $u$ and $v$ are points of $\Gamma_1(n, q)$, we use $d(u, v)$ to denote their distance in the collinearity graph $G(\Gamma_1(n, q))$.

6.1 Combinatorial Properties of $\Gamma(n, q)$

In this section we include some preliminary combinatorial properties of $\Gamma_1(n, q)$. We also prove that the geometries induced by square type and non-square type flips are not isomorphic if $n \geq 2$. We will then discuss some of the difficulties in dealing with the geometry $\Gamma_1(n, q)$ that are not issues in some other geometries studied in the context of Phan theory.
Lemma 6.1.1 ([17], Lemma 43). On a plus-type line in an orthogonal space over a finite field of order $s$ there are $(s - 1)/2$ square-type points and $(s - 1)/2$ nonsquare-type points.

Lemma 6.1.2 ([17], Lemma 44). On a minus-type line in an orthogonal space over a finite field of order $s$ there are $(s + 1)/2$ square-type points and $(s + 1)/2$ nonsquare-type points.

These have consequences for the geometries considered in this dissertation, which we now record.

Corollary 6.1.3. Let $L$ be a square-type line of $\Gamma(n, q)$. Then $L$ contains $(q^2 - 1)/2$ square-type points and $(q^2 - 1)/2$ nonsquare-type points.

Corollary 6.1.4. Let $L$ be a nonsquare-type line of $\Gamma(n, q)$. Then $L$ contains $(q^2 + 1)/2$ square-type points and $(q^2 + 1)/2$ nonsquare-type points.

Proof. The proof of these Corollaries 6.1.3 and 6.1.4 is essentially the same. Since $-1$ is a square in $\mathbb{F}_{q^2}$ a line $L$ is plus-type if and only if it is of square type. \qed

For clarity purposes in the next theorem, we need to introduce some new notation. Let $\Gamma_1(n, q, +)$ be the geometry $\Gamma_1(n, q)$ induced by a square-type flip, and let $\Gamma_1(n, q, -)$ be the geometry $\Gamma_1(n, q)$ induced by a nonsquare-type flip.

Theorem 6.1.5. For all $n \geq 2$, $\Gamma_1(n, q, +)$ is not isomorphic to $\Gamma_1(n, q, -)$.

Proof. We induct on $n$. When $n = 2$ this follows since Corollaries 6.1.3 and 6.1.4 show that the lines of $\Gamma_1(2, q, +)$ have $(q^2 - 1)/2$ points and the lines of $\Gamma_1(2, q, -)$ have $(q^2 + 1)/2$ points. Since isomorphic geometries must have the same number of points on a line, we conclude that the geometries are not isomorphic.

Now suppose $n > 2$. If $f : \Gamma_1(n, q, +) \to \Gamma_1(n, q, -)$ is an isomorphism of geometries, then choose any point $u \in \Gamma_1(n, q, +)$. Since $f$ is an isomorphism of geometries, $f|\text{res}(u)$ is also an isomorphism of geometries with $f|\text{res}(u) : \text{res}(u) \to \text{res}(f(u))$. But we know that $\text{res}(u) \cong \Gamma_1(n - 1, q, +)$ and $\text{res}(f(u)) \cong \Gamma_1(n - 1, q, -)$, and so $f|\text{res}(u)$ is an isomorphism
from $\Gamma_1(n-1, q, +)$ to $\Gamma_1(n-1, q, -)$ which are not isomorphic by the inductive hypothesis. Thus no such isomorphism $f$ exists.

\[\square\]

Remark. Another proof of Theorem 6.1.5 is possible. When $n = 2$ we use Corollaries 6.1.3 and 6.1.4 again. Suppose now that $n > 2$. Let $M$ be an $n$-object, and let $M_1$ be an $(n-2)$-object in $M$. Then every $(n-1)$-object $M_2$ with $M_1 \subset M_2 \subset M$ corresponds to a point on the line $M \cap \varphi(M_1)^\perp$. Since the number of points on that line depends on the type of flip, the number of such $(n-1)$-objects depends on the type of the flip, and so the geometries cannot be isomorphic.

\[\diamondsuit\]

The issue of the distinction between geometries induced by square-type and non-square type flips is one which we mostly ignored in Chapter 5, especially in Lemma 5.2.8. To be more precise, that result should indicate that the residue of a point $u$ in $\Gamma_1(n, q)$ is isomorphic as a geometry to $\Gamma_1(n-1, q)$ where both geometries are induced by $\sigma$-semilinear flips of the same type (square or non-square.)

Remark. In the study of many other geometries related to Phan-type theorems, there are enough points in the space to ensure that, even for small rank geometries, if the field on which the polar geometry is defined is sufficiently large then the diameter of the geometry is 2. This tends to work because each point is collinear to more than half the points of the geometry, and so any pair of points is either collinear to has a common neighbor. Notice that in the case of a square type flip of the unitary building this technique will not work. There are simply too many points, and the number of points grows faster than the number of points collinear to a given point as the size of the field increases. It is possible to use the same techniques as in the above results to show that the same holds for non-square type flips, that is no matter how large the underlying field, it will (in principle) be possible to have two points with no common neighbor. We prove in the next section that if the rank of the geometry is sufficiently large ($n \geq 5$) then in fact any two points are either collinear or have a common neighbor.
6.2 Residues

In this section we determine the residues of flags of the various types in $\Gamma_1(n, q)$. We have been unable to determine if $\Gamma_1(n, q)$ is in general residually connected, but as explained below this is not a significant problem.

**Theorem 6.2.1.** Let $F = (C_i)_{i=1}^k$ be a flag of $\Gamma_1(n, q)$. Write $C_0 = \{0\}$. For each $i = 1, \ldots, k$, let $D_i = C_i \cap \varphi(C_{i-1})^\perp$. Let $D_{k+1} = \langle C_k, \varphi(C_k) \rangle^\perp$. For $i = 1, \ldots, k$ let $\Lambda_i$ denote the geometry of square type subspaces of the orthogonal space $(D_i, \beta|D_i)$. Let $\Lambda_{k+1}$ denote the geometry of $\beta$-isotropic $\beta$-square type subspaces of $D_k$ together with all maximal $\beta$-isotropic $\beta$-non-degenerate subspaces of $D_k$. Then the residue of $F$ in $\Gamma_1(n, q)$ is isomorphic to $\prod_{i=1}^{k+1} \Lambda_i$.

**Proof.** We will construct a map from $\prod_{i=1}^{k+1} \Lambda_i$ to $\text{res}_{\Gamma_1(n, q)}(F)$ which is an isomorphism of geometries.

First, notice that for $U \in \text{res}_{\Gamma_1(n, q)}(F)$, since $U$ is incident to each object of $F$ one of the following must hold:

(i) $U \subset C_1$;

(ii) $C_k \subset U$;

(iii) there is an index $i$ with $1 \leq i \leq k$ such that $C_i \subset U \subset C_{i+1}$.

Define a map $\alpha : \prod_{i=1}^{k+1} \Lambda_i \to \text{res}_{\Gamma_1(n, q)}(F)$ by, for $U \in \prod_{i=1}^{k+1} \Lambda_i$,

$$\alpha(U) = \langle C_i, U \rangle$$

where $U \in \Lambda_i$.

It is straightforward to check that the image of $\alpha$ in $\Gamma_1(n, q)$ lies in the residue of $F$, and when the codomain is restricted to $\text{res}_{\Gamma_1(n, q)}(F)$, $\alpha$ is bijective and preserves incidence, and so is an isomorphism of geometries. Hence $\text{res}_{\Gamma_1(n, q)}(F) \cong \prod_{i=1}^{k+1} \Lambda_i$. 

\[\square\]
It is possible that residues of cotype \( \{n, n-1\}, \{n, n-1, n-2\} \) are disconnected for some prime powers, or types of flips. The connectedness of those residues relies on connectedness of the geometry in rank 2 and 3. Since we have been unable to prove (or disprove) the connectedness in those cases in general we are unable to determine whether or not for larger primes the geometry is residually connected. As it turns out this is not a significant problem. Residual connectedness is usually employed to prove that every path in a geometry is homotopic to a path passing only through points and lines. We have been able to prove this result for \( \Gamma_1(n, q) \) in the absence of residual connectedness.

We now record, for completeness, a description of the residues of arbitrary flags of \( \Gamma(n, q) \) in \( \Gamma(n, q) \). The reasoning involved is identical to the reasoning in the proof of Theorem 6.2.1, but the details are slightly different.

**Theorem 6.2.2.** Let \( \mathcal{F} = (C_i)_{i=1}^k \) be a flag of \( \Gamma(n, q) \). Write \( C_0 = \emptyset \). For each \( i = 1, \ldots, k \) let \( D_i = C_i \cap \varphi(C_{i-1})^\perp \). Let \( D_{k+1} = \langle C_k, \varphi(C_k) \rangle^\perp \). For \( i = 1, \ldots, k \) let \( \Lambda_i \) denote the geometry of non-degenerate subspaces of the orthogonal space \( (D_i, \beta|_{D_i}) \). Let \( \Lambda_{k+1} \) denote the geometry of \( \beta \)-isotropic \( \beta \)-non-degenerate subspaces of \( D_k \). Then the residue of \( \mathcal{F} \) in \( \Gamma(n, q) \) is isomorphic to \( \prod_{i=1}^{k+1} \Lambda_i \).

We conclude this section by describing the residues of flags of the geometries \( \Psi_\epsilon(n, q) \). Again the reasoning involved is identical to the reasoning in the proof of Theorem 6.2.1, but the details are slightly different.

**Theorem 6.2.3.** Let \( \epsilon \) be a function mapping the set \( \{1, \ldots, n-1\} \) to \( \{\pm 1\} \). Let \( \Psi_\epsilon(n, q) \) denote the corresponding geometry as constructed in Construction 7. Let \( \mathcal{F} = (C_i)_{i=1}^k \) be a flag of \( \Psi_\epsilon(n, q) \). Write \( C_0 = \emptyset \). For each \( i = 1, \ldots, k \), let \( D_i = C_i \cap \varphi(C_{i-1})^\perp \). Let \( D_{k+1} = \langle C_k, \varphi(C_k) \rangle^\perp \). For \( i = 1, \ldots, k \), define \( \Lambda_i \) to be the geometry of:

(i) square type subspaces of the non-degenerate orthogonal space \( (D_i, \beta|_{D_i}) \) if \( \epsilon(i) = \epsilon(i-1) \);
(ii) non-square type subspaces of the non-degenerate orthogonal space $(D_i, \beta_i | D_i)$ if $\epsilon(i) \neq \epsilon(i - 1)$.

Let $\Lambda_{k+1}$ denote the geometry of $\beta$ isotropic $\beta$ non-degenerate subspaces $U$ of $D_{k+1}$ with the property that $\langle C_k, U \rangle \in \Psi(\epsilon(n, q))$. Then the residue of $F$ in $\Psi_{\epsilon}(n, q)$ is isomorphic to $\prod_{i=1}^{k+1} \Lambda_i$.

6.3 Connectedness of $\Gamma_1(n, q)$ for $n \geq 5$

In this section we prove that if $n \geq 5$ then $G(\Gamma_1(n, q))$ is connected of diameter 2 for every odd prime power $q$. This result holds whether $\varphi$ is a square type or a non-square type flip.

**Note 6.3.1.** Throughout the rest of this chapter, a square type point $u$ is assumed to satisfy $Q_\varphi(u) = 1$. This is not necessary, but since every square type point is represented by a vector satisfying $Q_\varphi(u) = 1$ it makes things easier to assume from the start that we have chosen one of these vectors.

**Lemma 6.3.1.** Let $U$ be a $\varphi$-invariant non-degenerate subspace of $V$ of dimension at least 4. Then $U$ contains a point of $\Gamma_1(n, q)$.

**Proof.** Suppose first that $U$ is even dimensional. Then by Lemma 5.1.2 $U = \langle W, \varphi(W) \rangle$ where $W \in \Gamma(n, q)$ and is at least 2 dimensional. It follows from Lemma 5.2.4 that $W$ contains a point of $\Gamma_1(n, q)$ and so also $U$ contains a point of $\Gamma_1(n, q)$.

If $U$ is odd-dimensional then by Lemma 5.1.3 $U = \langle W, \varphi(W) \rangle \perp \langle e \rangle$ where $W \in \Gamma(n, q)$ and is at least 2 dimensional. It follows from Lemma 5.2.4 that $W$ contains a point of $\Gamma_1(n, q)$ and so also $U$ contains a point of $\Gamma_1(n, q)$. \hfill $\square$

**Lemma 6.3.2.** Let $u, v \in \Gamma_1(n, q)$ be points with $u \perp v$. Then $L = \langle u, v \rangle \in \Gamma_1(n, q)$. Conversely if $L$ is a line of $\Gamma_1(n, q)$ then there are biorthogonal points $u, v$ of $\Gamma_1(n, q)$ with $L = \langle u, v \rangle$. 
Proof. Since $u$ and $v$ are biorthogonal, the $\beta_\varphi$ Gram matrix of $L$ with respect to the basis \{u, v\} is
\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]
which has determinant 1, a square in $\mathbb{F}$.

The converse holds by Lemma 5.2.2.

Lemma 6.3.3. Let $\{u_1, \ldots, u_k\}$ be a collection of points of $\Gamma_1(n, q)$. If $n \geq 2k + 1$ then there is a point of $\Gamma_1(n, q)$ biorthogonal to each of the $u_i$.

Proof. Let $W = \langle u_1, \ldots, u_k, \varphi(u_1), \ldots, \varphi(u_k) \rangle^\perp$, let $R = \text{Rad}(W)$ and let $U$ be a $\varphi$ invariant non-degenerate complement to $R$ in $W$, which exists by Lemma 3.4.2. Notice that $W$ has codimension at most $2k$ with at most a $2k - 2$ dimensional radical, since it contains the non-degenerate subspace $\langle u_1, \varphi(u_1) \rangle^\perp$. So anytime $2n - (2k + (2k - 2)) \geq 4$ by Lemma 6.3.1 we can conclude that $U$ contains a point of $\Gamma_1(n, q)$ which will be collinear to each of the $u_i$ by Lemma 6.3.2. An easy calculation shows that $2n - (2k + (2k - 2)) \geq 4$ if and only if $n \geq 2k + 1$.

Note 6.3.2. For the remainder of this dissertation, $\mathcal{G}(\Gamma_1(n, q)) = \mathcal{G}_{0,1}(\Gamma_1(n, q))$, that is we are considering the collinearity graph whose vertices are the points of $\Gamma_1(n, q)$ and whose edges correspond to lines of $\Gamma_1(n, q)$.

Theorem 6.3.4. If $n \geq 5$ then $\mathcal{G}(\Gamma_1(n, q))$ is connected of diameter 2.

Proof. Let $u, v$ be points of $\Gamma_1(n, q)$. Then by Lemma 6.3.3 if $n \geq 2(2) + 1 = 5$ there is a point $z$ collinear to both $u$ and $v$. Thus $uzv$ is a path in $\mathcal{G}(\Gamma_1(n, q))$ from $u$ to $v$ and so $d(u, v) \leq 2$. 

\[\square\]
In this section we prove that $G(\Gamma_1(4, q))$ is connected of diameter at most 4 for every odd prime power $q$.

**Lemma 6.4.1.** Let $u, v$ be points of $\Gamma_1(4, q)$. If $\beta(u, v) \neq 0$ then there is a point $z$ of $\Gamma_1(4, q)$ that is $\beta$-orthogonal to both $u$ and $v$.

**Proof.** Notice first that $U = \langle u, \varphi(u), v, \varphi(v) \rangle$ is $\varphi$-invariant, at least 4 dimensional and has at most a 2 dimensional radical $R$. Let $W$ be a $\varphi$-invariant complement to $R$ in $U$, which exists by Lemma 3.4.2. Then $W$ is at least 2 dimensional, is $\varphi$-invariant and is non-degenerate. It follows from Corollary 5.1.4 that $W$ contains a point $a$ of $\Gamma(n, q)$. If $a$ is a point of $\Gamma_1(n, q)$ it is the desired point. If $a$ is not a point of $\Gamma_1(n, q)$ then consider the space $X = \langle u, v, a, \varphi(a) \rangle$. Notice that $X$ is 4 dimensional and $\beta$ non-degenerate. If $X$ is $\beta_\varphi$ totally singular then choose $\lambda$ so that $\lambda^2 = -1$. Then $Q_\varphi(a + \lambda \varphi(a)) = 0$ and so the space $X \oplus \langle a + \lambda \varphi(a) \rangle$ is 5-dimensional and $\beta_\varphi$ totally singular, a contradiction since the $\beta_\varphi$ Witt index of $V$ is 4. Thus $X$ is not $\beta_\varphi$ totally singular and so by Lemma 3.5.3 $X$ contains a point $b$ of $\Gamma(n, q)$.

Since $b \in X$, $b$ is biorthogonal to $a$ and so $\langle a, b \rangle$ is a line of $\Gamma(n, q)$ which therefore contains a point $z$ of $\Gamma_1(n, q)$ by Lemma 6.3.2. 

We now state a theorem from [47] which we will need to prove that $\Gamma_1(4, q)$ is connected of diameter at most 4.

**Theorem 6.4.2** ([47], Theorem 87). Let $W$ be a non-square type orthogonal space of dimension at least 3 over a finite field of order at least 7. Then the collinearity graph of the geometry of square-type subspaces of $W$ is connected of diameter 2.

**Note 6.4.1.** Since we have assumed $q$ is an odd prime, and $V$ is defined over $\mathbb{F}_{q^2}$, the hypotheses of Theorem 6.4.2 are satisfied if one considers the $\beta_\varphi$ non-degenerate subspaces of any object of $\Gamma(n, q)$ of dimension at least 3. This will prove important in the proof of Lemma 6.4.3.
Lemma 6.4.3. Let $u$, $v$ be points of $\Gamma_1(4, q)$ that are $\beta$-orthogonal. Then $d(u, v) \leq 2$.

Proof. We proceed by a series of cases. Notice that since $u$, $v$ are $\beta$ orthogonal either $\langle u, v \rangle \in \Gamma(4, q)$, or it is $\beta_\varphi$ degenerate. If $\langle u, v \rangle$ is $\beta_\varphi$ degenerate then $\langle u, v, \varphi(u), \varphi(v) \rangle^\perp$ is either 3 or 4 dimensional. So the three necessary cases are:

Case 1: Suppose $\langle u, v \rangle \in \Gamma(4, q)$. Then $\langle u, v, \varphi(u), \varphi(v) \rangle^\perp$ is 4 dimensional, $\varphi$-invariant and non-degenerate, and so contains a point $z \in \Gamma_1(4, q)$ biorthogonal to both $u$ and $v$. Thus $d(u, v) \leq 2$.

Case 2: Suppose $\dim \langle u, v, \varphi(u), \varphi(v) \rangle = 3$. Then $\langle u, v, \varphi(u), \varphi(v) \rangle^\perp$ has at most a 1 dimensional radical, and so $\langle u, v, \varphi(u), \varphi(v) \rangle^\perp$ is at least 5 dimensional, $\varphi$-invariant with at most a 1 dimensional radical. Thus there is a $\varphi$-invariant non-degenerate complement to the radical of dimension at least 4, which contains a point $z \in \Gamma_1(4, q)$ biorthogonal to both $u$ and $v$. Thus $d(u, v) \leq 2$.

Case 3: Suppose $\dim \langle u, v, \varphi(u), \varphi(v) \rangle = 4$ but $\langle u, v \rangle$ is $\beta_\varphi$ degenerate. In this case we will show that if $\varphi$ is a square type flip, there is a point biorthogonal to both $u$ and $v$, and if $\varphi$ is a non-square type flip then there is a maximal object $A$ of $\Gamma_1(4, q)$ containing both $u$ and $v$, and so by Theorem 6.4.2 $d(u, v) \leq 2$ in $A$, and hence in $\Gamma_1(4, q)$.

Since $n = 4$ and $\langle u, v, \varphi(u), \varphi(v) \rangle$ is 4 dimensional with at most a 2 dimensional radical, the space $\langle u, v, \varphi(u), \varphi(v) \rangle^\perp$ contains a point $z$ of $\Gamma(4, q)$ biorthogonal to both $u$ and $v$.

Since $\langle u, v \rangle$ is $\beta_\varphi$ degenerate there is a vector $w \in \langle u, v \rangle$ so that

$$v = u + w \quad \text{and} \quad Q_\varphi(w) = 0.$$
\( \langle u, v \rangle \) is
\[
\begin{pmatrix}
1 & \beta_\varphi(u, v) \\
\beta_\varphi(v, u) & 1
\end{pmatrix}
\]
which has determinant \( 1 - \beta_\varphi(u, v)^2 \) since \( \beta_\varphi \) is symmetric. Hence \( \beta_\varphi(u, v) \in \{ \pm 1 \} \), and so after scaling \( v \) we may assume \( \beta_\varphi(u, v) = 1 \). Next, write \( V = \langle u, \varphi(u) \rangle \oplus W \) where \( W = \langle u, \varphi(u) \rangle ^\perp \) and let \( W = \langle w_1, w_2 \rangle \). Then we can write
\[
v = \alpha_1 u + \alpha_2 \varphi(u) + \alpha_3 w_1 + \alpha_4 w_2.
\]
Since \( \beta(u, v) = 0 \) it follows that \( \alpha_2 = 0 \). Since \( \beta_\varphi(u, v) = 1 \) it follows that \( \alpha_1 = 1 \). Thus \( v = u + (\alpha_3 w_1 + \alpha_4 w_2) = u + w \). In order to check that \( Q_\varphi(w) = 0 \), notice that \( 1 = Q_\varphi(v) = Q_\varphi(u + w) = Q_\varphi(u) + Q_\varphi(w) \), and so \( Q_\varphi(w) = 0 \).

Moreover we may assume that \( \varphi(w) \notin \langle w \rangle \), since otherwise \( \langle u, v, \varphi(u), \varphi(v) \rangle \) is 3-dimensional. The space \( \langle u, w, z \rangle ^\perp \) is 5 dimensional, with radical contained entirely in \( \langle u, w, z \rangle \). Thus we can write
\[
\langle u, w, z \rangle ^\perp = \langle u, w, z \rangle \perp \langle a, b \rangle
\]
where \( (a, b) \) is a \( \beta \)-hyperbolic pair.

Consider the space \( A = \langle u, w, z, a \rangle \). It is \( \beta \)-isotropic with \( \beta_\varphi \) Gram matrix
\[
\begin{pmatrix}
1 & 0 & 0 & \beta_\varphi(u, a) \\
0 & 0 & 0 & \beta_\varphi(w, a) \\
0 & 0 & \beta_\varphi(z, z) & \beta_\varphi(z, a) \\
\beta_\varphi(u, a) & \beta_\varphi(w, a) & \beta_\varphi(z, a) & \beta_\varphi(a, a)
\end{pmatrix}
\]
which has determinant \(-\beta_\varphi(z, z)\beta_\varphi(w, a)^2\). Notice that if \( A \) is \( \beta_\varphi \) non-degenerate then it is a hyperplane of \( \Gamma_1(4, q) \) containing both \( u \) and \( v \). Moreover the \( \beta_\varphi \) type of \( A \) is
the same as that of $z$.

Now suppose $A$ is $\beta_\varphi$ degenerate. Then consider the space $B = \langle u, w, z, b \rangle$. The same argument (with $b$ replacing all appearances of $a$) shows that the $\beta_\varphi$ Gram matrix of $B$ has determinant $-\beta_\varphi(z, z)\beta_\varphi(w, b)^2$. Again if $B$ is $\beta_\varphi$ non-degenerate then it is a hyperplane of $\Gamma_1(4, q)$ containing both $u$ and $v$, with the same $\beta_\varphi$ type as $z$.

The next step is to show that one of $A$ or $B$ is not $\beta_\varphi$ degenerate. Notice that $A$ is $\beta_\varphi$ degenerate if and only if $\beta_\varphi(w, a) = 0$ and $B$ is $\beta_\varphi$ degenerate if and only if $\beta_\varphi(w, b) = 0$.

Suppose $\beta_\varphi(w, a) = 0$. Then since $\varphi(w) \notin \langle w \rangle$ the space $\langle u, w, z, a, \varphi(w) \rangle$ is $\beta$-isotropic, and so since the Witt index of $(V, \beta)$ is 4, $a \in \langle u, w, z, \varphi(w) \rangle$. Since $\langle u, w, z, a, b \rangle$ is 5 dimensional, in this case we conclude that $b \notin \langle u, w, z, \varphi(w) \rangle$ and so $\beta_\varphi(w, b) \neq 0$ and hence $B$ is not $\beta_\varphi$ degenerate.

Without loss of generality we may assume that $A$ is $\beta_\varphi$ non-degenerate. Then if $\varphi$ is a square type flip, $z$ is a square type point and by construction it is biorthogonal to both $u$ and $v$. Thus $d(u, v) \leq 2$. If $\varphi$ is a non-square type flip then by Theorem 6.4.2 $u$ and $v$ have a common neighbor in $A$, and hence in $\Gamma_1(4, q)$.

Thus in any case, $d(u, v) \leq 2$. \hfill \Box

**Theorem 6.4.4.** $\mathcal{G}(\Gamma_1(4, q))$ is connected of diameter at most 4.

**Proof.** Let $u, v$ be points of $\Gamma_1(4, q)$. If $\beta(u, v) = 0$ then by Lemma 6.4.3 either $d(u, v) \leq 2$.

If $\beta(u, v) \neq 0$ then by Lemma 6.4.1 we can choose a point $z \in \Gamma_1(4, q)$ that is $\beta$-orthogonal to both $u$ and $v$. It follows that $d(u, z) \leq 2$ and $d(v, z) \leq 2$. Thus $d(u, v) \leq 4$. \hfill \Box

### 6.5 Connectedness of $\Gamma_1(2, q)$

We now turn our attention to the $n = 2$ case. We will prove that if $-1$ is a square in $\mathbb{F}_q$ and $\varphi$ is a square type flip then $\mathcal{G}(\Gamma_1(2, q))$ is connected of diameter at most 4.
Note 6.5.1. In the proof of Theorem 6.5.1 the terms “point” and “line” are reserved for points and lines of $\Gamma_1(n,q)$.

Theorem 6.5.1. If $-1$ is a square in $\mathbb{F}_q$ and $\varphi$ is a square type flip then $G(\Gamma_1(2,q))$ is connected of diameter at most 4.

Proof. The technique in this proof is to show that when $q \geq 5$, if $u$ is any point of $\Gamma_1(2,q)$ then more than half the lines $L$ of $\Gamma_1(2,q)$ have the property that if $z \in L$ then $d(u,z) \leq 2$.

Then, when $u$ and $v$ are arbitrary points of $\Gamma_1(2,q)$ we will conclude that there is some line $L$ with the property that every point $z$ on that line has distance at most 2 from both $u$ and $v$, in particular if $z$ is on the line it is in the same connected component as both $u$ and $v$, and $d(u,v) \leq d(u,z) + d(v,z) = 4$.

Let $u$ be a fixed point of $\Gamma_1(2,q)$. Let $v$ be a point of $\Gamma_1(2,q)$ that is biorthogonal to $u$, so $\langle u, \varphi(u) \rangle \perp \langle v, \varphi(v) \rangle$ forms a basis that looks like the basis guaranteed to exist by Theorem 5.4.1.

Furthermore if $a$ is on a line of the form $L_1 = \langle u, v + \lambda \varphi(v) \rangle$ and $b$ is on a line of the form $L_2 = \langle u, v + \rho \varphi(v) \rangle$ where these are different lines, then $\beta(a,b) \neq 0$ since if $\beta(a,b) = 0$ then $\langle u, a, b \rangle$ is $\beta$ totally isotropic, and hence $b \in \langle u, a \rangle$ contradicting the assumption that $L_1 \neq L_2$.

It follows that in this case, $a$ and $b$ lie on no common line.

Consider the various lines $\langle u, v + \lambda \varphi(v) \rangle$ for $\text{Tr}_T(\lambda) = 0$, together with $\langle u, \varphi(v) \rangle$. Each of these $q + 1$ lines contains $(q^2 - 3)/2$ points other than $u$ by Lemma 6.1.1 and each of these points lies on $q$ lines other than its line with $u$. Furthermore, any points on any of these lines has distances at most 2 from $u$. Thus there are at least

$$q(q+1)\frac{q^2-3}{2} = \frac{(q^2+q)(q^2-3)}{2} = \frac{q^4 + q^3 - 3q^2 - 3q}{2}$$

lines with the property that any point on one of these lines has distance at most 2 from $u$. Since $\Gamma_1(2,q)$ has $q^4 - q^2$ lines, if $q \geq 5$ more than half the lines are accounted for, and so
the geometry is connected of diameter at most 4. Since all prime powers $q$ with $-1$ a square in $\mathbb{F}_q$ are at least 5, we conclude $\Gamma_1(2, q)$ is connected for all odd prime powers $q$ with $-1$ a square in $\mathbb{F}_q$.

The question of whether $\Gamma_1(2, q)$ is connected when $-1$ is not a square in $\mathbb{F}_q$ is one to which we do not yet have a full answer. A GAP calculation shows that when $q = 3$ the (collinearity graph of the) geometry is disconnected. The technique used to prove Theorem 6.5.1 will not work when $-1$ is not a square in $\mathbb{F}_q$, since in that case the number of lines on any given point is $q - 1$ rather than $q + 1$, and the difference is just enough to make the proof fail to go through. A second GAP calculation shows that $G(\Gamma_1(2, 5))$ is connected of diameter 3 and a third calculation shows that $G(\Gamma_1(2, 7))$ is connected of diameter 3.

**Proposition 6.5.2.** If $\varphi$ is a square type flip then $G(\Gamma_1(2, 3))$ is disconnected, while $G(\Gamma_1(2, 5))$ and $G(\Gamma_1(2, 7))$ are connected of diameter 3.

**Proof.** This is proved by a GAP calculation, the code used is contained in the Appendix. □

### 6.6 Simple Connectedness of $\Gamma_1(n, q)$

We now prove that if $n \geq 8$ or $n = 7$ and $q \geq 5$, the geometry $\Gamma_1(n, q)$ is simply connected. This implies that, in these cases, $U_{2n}(q^2)^\varphi$ is the universal completion of its amalgam of maximal parabolics with respect to its action on $\Gamma_1(n, q)$.

We now state another theorem of [47] which will be used, along with 6.4.2 to show that every path starting and ending in a point in $\Gamma_1(n, q)$ is homotopic to a point-line path.

**Theorem 6.6.1** ([47], Theorem 57). Let $W$ be a square-type orthogonal space of dimension at least 3 over a finite field of order at least 7. Then the geometry of square-type subspaces of $W$ is connected.

**Lemma 6.6.2.** Every path in $\Gamma_1(n, q)$ that begins and ends in a point is homotopic to one passing only through points and lines.
Remark. The proof of this is very similar to the proof of Lemma 4.6.2, but some of the details are slightly different.

Proof. Notice first that if \( n = 1 \) or \( n = 2 \) the result is trivial, so we can assume that \( n \geq 3 \). Let \( \gamma = x_1 \ldots x_n \) be a path with no repetition. Let \( m(\gamma) \) be the number of elements of \( \gamma \) that are neither points nor lines. We induct on \( m(\gamma) \). If \( m(\gamma) = 0 \) then \( \gamma \) passes only through points and lines, so the conclusion of the lemma holds.

Assume \( m(\gamma) > 0 \) and let \( x_i \) be the first element of \( \gamma \) that is neither a point nor a line. If \( x_{i+1} \) has dimension greater than \( x_i \), then \( x_{i-1} \) is contained in \( x_{i+1} \), and so \( \gamma \) is homotopic to the path \( \gamma' = x_1 \ldots \hat{x}_i \ldots x_n \), which has one fewer element that is neither a point nor a line. By the inductive hypothesis, \( \gamma' \) is homotopic to a point line path and so also \( \gamma \) is homotopic to a point line path.

If \( x_{i+1} \) is neither a point nor a line, and has dimension less than the dimension of \( x_i \) then choose \( j \in \mathbb{N} \) maximal so that for all \( s = 1, \ldots, j \) \( x_{i+s} \subset x_{i+s-1} \). Then we have a string of subspaces

\[
x_i \supset x_{i+1} \supset x_{i+2} \supset \ldots \supset x_{i+j}
\]

but \( x_{i+j} \subset x_{i+j+1} \). Let \( v \) be any point of \( \Gamma_1(n, q) \) contained in \( x_{i+j} \). Then we can replace \( x_{i+1}x_{i+2} \ldots x_{i+j} \) by \( v \) in \( \gamma \) to obtain a cycle \( \gamma' \) homotopic to \( \gamma \) with \( m(\gamma') < m(\gamma) \). By the inductive hypothesis \( \gamma' \) is homotopic to a point line path, and so also \( \gamma \) is homotopic to a point line path.

Finally, if \( x_{i+1} \) is a point or a line, then both \( x_{i-1} \) and \( x_{i+1} \) are contained in \( x_i \). If \( n = 3 \) and \( \varphi \) is a non-square type flip then it follows from Theorem 6.4.2 that \( x_{i-1} \) and \( x_{i+1} \) can be connected inside of \( x_i \). This new path is homotopic to \( \gamma \), and omits \( x_i \), and so has fewer objects that are not points or lines. By induction we conclude that \( \gamma \) is homotopic to a point line path.

Finally if \( n = 3 \) and \( \varphi \) is a square type flip or \( n > 3 \) then it follows from Theorem 6.6.1 that there is a path from \( x_{i-1} \) to \( x_{i+1} \) inside of \( x_i \). This path may introduce more objects
that are not points or lines, but by induction on the dimension of $x_i$ we can produce a cycle $\gamma'$ homotopic to $\gamma$ that has no elements that are not points or lines until after $x_{i+1}$, and so by the inductive hypothesis on $m(\gamma')$ is homotopic to a point line path.

We now turn to the task of studying cycles in the collinearity graph, and showing that the cycles to which they correspond in the flag complex are null-homotopic. Since we are going to be studying cycles in the collinearity graph, we have to consider a notion of length different from that in the geometry. A point-line cycle of length 3 in the collinearity graph corresponds to a cycle of length 7 in the incidence graph of the geometry. In what follows, lengths of cycles refer to lengths in the collinearity graph, not in the incidence graph.

The proofs of Lemmas 6.6.3, 6.6.4 and 6.6.5 follow the same basic ideas as the proofs of Lemmas 4.6.4, 4.6.6 and 4.6.5, the main difference being that the bounds are better in this case.

Recall that a path $x_1 \ldots x_k$ is geometric if there is an object $x$ such that $x$ is incident to $x_i$ for all $i = 1, \ldots, k$. Recall also that a geometric cycle is null-homotopic (Lemma 4.6.3).

**Lemma 6.6.3.** Let $n \geq 6$. Then all 3-cycles are homotopic to products of geometric 3-cycles and thus are null-homotopic.

*Proof.* Let $abca$ be a 3-cycle. Consider the space $W = \langle a, b, c, \varphi(a), \varphi(b), \varphi(c) \rangle^\perp$. This space is of codimension at most 6, with a most a 2-dimensional radical (since it contains a 4-dimensional non-degenerate subspace.) If $n \geq 6$ then a $\varphi$-invariant complement to the radical of $W$ is at least 4 dimensional and so contains a point $z$ of $\Gamma_1(n, q)$ by Lemma 6.3.1. Since this point is biorthogonal to each of $a$, $b$ and $c$, we can express $abca$ as the product of $abza$, $acza$, and $bvzb$ each of which is a geometric triangle since the spaces spanned by their points lie in $\Gamma_1(n, q)$. Each of these three is therefore null-homotopic, and since $abca$ is their product we conclude that $abca$ is null-homotopic.

**Lemma 6.6.4.** Let $n \geq 8$. Then all 4-cycles decompose into 3-cycles.
Proof. Let $abcd$ be a 4-cycle. Consider the space $\langle a, b, c, d, \varphi(a), \varphi(b), \varphi(c), \varphi(d) \rangle^\perp$. Since $n \geq 8$, a $\varphi$-invariant complement to its radical (in the subspace) is at least 4 dimensional and so by Lemma 6.3.1 contains a point of $\Gamma_1(n,q)$, say $z$.

Thus the 4-cycle $abcd$ decomposes as a product of the 3-cycles $abza$, $bczb$, $cdzc$ and $azda$.  

\[ \square \]

Figure 6.2: Graphical representation of the proof of Lemma 6.6.4.

\[ \square \]

Lemma 6.6.5. Let $n \geq 6$. Then 5-cycles decompose into 4-cycles and 3-cycles.

Proof. Let $abcdea$ be a 5-cycle. Consider the space $\langle a, c, d, \varphi(a), \varphi(c), \varphi(d) \rangle^\perp$. If $n \geq 6$, a $\varphi$ invariant complement to its radical contains a point of $\Gamma_1(n,q)$ by Lemma 6.3.1. Call this point $z$. Then the 5-cycle decomposes as a product of the cycles $abcza$, $cdzc$ and $deazd$.  

\[ \square \]

Theorem 6.6.6. Let $n \geq 8$. Then $\Gamma_1(n,q)$ is simply connected.

Proof. By Theorem 6.3.4, if $n \geq 5$, then $\mathcal{G}(\Gamma_1(n,q))$ is connected of diameter 2. In particular this holds if $n \geq 8$. It therefore suffices to show that every 3-cycle, 4-cycle, and 5-cycle in
\( \mathcal{G}(\Gamma_1(n,q)) \) corresponds to a null-homotopic cycle in the flag complex. This follows from Lemmas 6.6.3, 6.6.4, and 6.6.5.

A more detailed analysis will allow us to conclude that \( \Gamma_1(n,q) \) is simply connected when \( n = 7 \) and \( q \geq 5 \).

**Lemma 6.6.7.** Assume \( q \geq 5 \). In a rectangle \( abcd \) of dimension 4 with \( \beta(b,d) \neq 0 \) there is a point \( u \) of \( \Gamma_1(7,q) \) biorthogonal to both \( a \) and \( c \), and with the property that \( \langle u, b \rangle \) and \( \langle u, c \rangle \) are lines of \( \Gamma(7,q) \). Moreover if \( a \) is biorthogonal to \( b \), then \( \langle a, u, b \rangle \) is an element of \( \Gamma(7,q) \), if \( a \) is biorthogonal to \( d \) then \( \langle a, u, d \rangle \) is an element of \( \Gamma(7,q) \) and analogous results hold if \( a \) is replaced by \( c \).

**Proof.** Let \( W = \langle a, c, \varphi(a), \varphi(c) \rangle^\perp \). This space is 10 dimensional with at most a 2 dimensional radical. Let \( W' \) be a \( \varphi \)-invariant non-degenerate complement to the radical of \( W \) that contains both \( b \) and \( d \). Then \( W' \) is 8 dimensional, and in \( W' \), \( W'' = \langle b, d \rangle^\perp \) is 6 dimensional and \( \beta \) non-degenerate. We know there is a point \( u \) of \( \Gamma(7,q) \) biorthogonal to both \( b \) and \( d \). Write \( W'' = \langle u, \varphi(u) \rangle^\perp \perp U \). Then \( U \) is \( \beta \) non-degenerate, 4 dimensional, and not \( \beta_\varphi \) totally singular, since if it is \( \beta_\varphi \) totally singular, adjoining a \( \beta_\varphi \) singular vector \( u + \lambda \varphi(u) \) produces a 5-dimensional \( \beta_\varphi \) totally singular subspace in \( W' \), which has 4 as its \( \beta_\varphi \) Witt index.

Thus there is a point \( z \) of \( \Gamma(7,q) \) in \( U \). It follows that \( \langle u, z \rangle \) is a line of \( \Gamma(7,q) \). The points of \( \Gamma(7,q) \) on this line are of the form \( u + \lambda z \) for certain choices of \( \lambda \in \mathbb{F} \), together
with $z$. For fixed $\lambda$, the $\beta_\varphi$ Gram matrix of $\langle b, u + \lambda z \rangle$ is

$$
\begin{pmatrix}
1 & \lambda \beta_\varphi(b, z) \\
\lambda \beta_\varphi(b, z) & \beta_\varphi(u, u) + \lambda^2 \beta_\varphi(z, z)
\end{pmatrix}
$$

which has determinant $\beta_\varphi(u, u) + \lambda^2 [\beta_\varphi(z, z) - \beta_\varphi(b, z)]^2$ which is quadratic in $\lambda$ and so has at most 2 zeroes. Since $q \geq 5$, there are at least 12 square type points on this line, at most 4 of which give degenerate lines when combined with either $b$ or $d$. Thus there are some (in fact at least 4) square type points that give non-degenerate lines when combined with both $b$ and $d$. Choose any of these points.

The second part of the lemma is proved by noting that if $a$ is biorthogonal to $b$ then the $\beta_\varphi$ Gram matrix of $\langle a, b, u \rangle$ has the same determinant as the $\beta_\varphi$ Gram matrix of $\langle b, u \rangle$.  

**Lemma 6.6.8.** If $n \geq 6$ then every 4-cycle $abcd$ is homotopic to a product of 3-cycles and one 4-cycle with the property that adjacent points of the 4-cycle are biorthogonal.

**Proof.** Since $n \geq 6$ there exists a point $u_1$ biorthogonal to each of $a$, $b$, and $c$. Thus $abcd$ can be written as the product of $abu_1a$, $bucb$, and $au_1cd$. Choose $u_2$ biorthogonal to each of $a$, $d$, and $c$. Then $au_1cd$ can be written as the product of $au_1cu_2a$, $au_2da$ and $du_2cd$, and the rectangle $au_1cu_2a$ has its adjacent points biorthogonal, as desired.

---

Figure 6.4: Graphical representation of the proof of Lemma 6.6.8.

**Note 6.6.1.** From now on, all 4-cycles are assumed to satisfy the conclusion of Lemma 6.6.8, that is, adjacent points are biorthogonal.
**Theorem 6.6.9.** If $q \geq 5$ and $n = 7$ any non $\beta$-isotropic 4-cycle is null-homotopic.

*Proof.* Let $abcd$ be such a 4-cycle with $\beta(b, d) \neq 0$. Choose $u$ to be biorthogonal to $a$, $d$, and $c$. Then $abcd$ can be written as a product $abduaucda$, and $abdua$ and $aucda$ are both geometric. This follows since $\langle a, b, d, u, \varphi(a), \varphi(b), \varphi(d), \varphi(u) \rangle^\perp$ has codimension at most 8 with at most a 2 dimensional radical, since $\langle a, b, u \rangle$ is an element of $\Gamma(n, q)$. This leaves a 4 dimensional $\varphi$ invariant non-degenerate complement to the radical, which then contains a point $z$ biorthogonal (and hence collinear) to each of $a, b, d, u$. Similarly $aucda$ is geometric.

**Corollary 6.6.10.** When $n = 7$ a $\beta$ totally isotropic 4-cycle is null-homotopic.

*Proof.* Let $abcd$ be a $\beta$ totally isotropic 4-cycle with adjacent points biorthogonal. If $\langle a, b, c, d \rangle$ is $\beta_\varphi$ non-degenerate then there is a point collinear to all 4 of the corners, and so the 4-cycle is geometric and hence null-homotopic.

So we can assume that $\langle a, b, c, d \rangle$ is $\beta_\varphi$ degenerate. In this case it can have a 1 dimensional or a 2 dimensional radical. If the radical is 1 dimensional then $\langle a, b, c, d, \varphi(a), \varphi(b), \varphi(c), \varphi(d) \rangle$ has a 2 dimensional radical, and so its orthogonal complement has at least a 4 dimensional $\varphi$ invariant non-degenerate subspace which then contains a point of $\Gamma_1(n, q)$, and that point is collinear to all of $a, b, c,$ and $d$, so the 4-cycle is geometric and hence null-homotopic.

So we can assume that $\langle a, b, c, d \rangle$ has a 2 dimensional $\beta_\varphi$ radical. This corresponds (after possibly multiplying some of the vectors by -1) to having $\beta_\varphi(a, c) = \beta_\varphi(b, d) = 1$.

We can then decompose $abcd$ as the product of $abc\varphi(b)a$ and $a\varphi(b)cda$, neither of which is $\beta$ totally isotropic. It follows from Theorem 6.6.9 that each of these is null-homotopic, and so also $abcd$ is null-homotopic. □

We now collect all this into a theorem.

**Theorem 6.6.11.** Let $q \geq 5$. Then $\Gamma_1(7, q)$ is simply connected.

*Proof.* By Theorem 6.3.4 the collinearity graph of the geometry is connected of diameter 2, and so it suffices to consider cycles in the collinearity graph of length at most 5. That
Figure 6.5: Graphical representation of the proof of Corollary 6.6.10

all 3-cycles are null-homotopic follows from Lemma 6.6.3. That all 5-cycles are products of 3-cycles and 4-cycles follows from Lemma 6.6.5. Finally that all 4-cycles are null-homotopic follows from Theorem 6.6.9 and Corollary 6.6.10.

6.7 Resulting Phan-Type Theorems

By applying Tits’ Lemma together with Theorems 6.6.6, 6.6.11 and 5.4.3 we immediately obtain the following:

Phan-type Theorem 2. Suppose $n \geq 8$, let $\varphi$ be a $\sigma$-semilinear flip and let $A$ denote the amalgam of maximal parabolic subgroups of $U_{2n}(q^2)^{\varphi}$ with respect to a maximal flag $F$ of $\Gamma_1(n, q)$. Then $U_{2n}(q^2)^{\varphi}$ is the universal completion of $A$.

Again we can appeal to Theorem 4.7.1 and Lemma 4.7.2 to prove the following:

Phan-type Corollary 2. Let $\varphi$ be a $\sigma$-semilinear flip and let $A_{(k)}$ denote the amalgam of rank $k$ parabolic subgroups of $U_{2n}(q^2)^{\varphi}$ with respect to a maximal flag $F$ of $\Gamma_1(n, q)$. If $n - k \geq 8$ or $n - k = 7$ and $q \geq 5$ then $U_{2n}(q^2)^{\varphi} = U(A_{(k)})$.

Proof. All we need to prove is that the residues of rank at least 8 (or 7 if $q \geq 5$) are simply connected. This is easy, since the residues are of one of three types:

(i) Direct products of orthogonal geometries with $\Gamma(s, q)$ for some integer $s$;

(ii) The geometry of non-degenerate subspaces of an orthogonal space of rank at least 8;
(iii) $\Gamma_1(s,q)$ where $s \geq 8$, or if $q \geq 5$, $\Gamma_1(7,q)$.

That residues of type (i) are simply connected follows from Lemma 1.7.2. That residues of type (ii) are simply connected follows from Theorem 118 of [47]. That residues of type (iii) are simply connected follows from Theorems 6.6.6 and 6.6.11.

Concluding Remarks

In this chapter we have partially explored one particular geometry associated to a $\sigma$-semilinear flip on the unitary building. Recall that by Theorem 6.6.6 over any odd prime power, if $n \geq 8$ then $\Gamma_1(n,q)$ is simply connected. There are some obvious remaining questions to which we have some partial answers.

(1) Are there odd prime powers $q$ so that the geometry $\Gamma_1(n,q)$ is simply connected even for small values of $n$?

When $n = 1$ or $n = 2$ we can conclude that for no odd prime power $q$ is the geometry simply connected. This is because when $n = 1$ the geometry has no lines, and so is not connected. When $n = 2$ the geometry may be connected, but the universal completion of a rank 2 amalgam is always infinite. Since the group $U_d(q^2)^\epsilon$ is always finite it is never the universal completion of its amalgam of maximal parabolics with respect to its action on $\Gamma_1(n,q)$, since that amalgam is of rank 2.

(2) Under what (if any) conditions are the geometries $\Gamma_\epsilon(n,q)$ connected? Simply connected?

For the exact same reasons as above, for any choice $\epsilon$ we have that $\Gamma_\epsilon(1,q)$ is not connected and hence not simply connected, and $\Gamma_\epsilon(2,q)$ may or may not be connected, but cannot be simply connected since again the universal completion of the amalgam of maximal parabolics is infinite, but $U_{2n}(q^2)^\epsilon$ is finite.
If $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$ then the arguments used to show that $\Gamma_1(n, q)$ is simply connected for $n \geq 8$ or $n = 7$ and $q \geq 5$ will show that $\Gamma_\epsilon(n, q)$ is simply connected for $n \geq 8$, so we can already conclude that those geometries are simply connected. This provides a further collection of Phan-type theorems for the corresponding orthogonal groups.
CHAPTER 7

Summary of Main Results and Proofs

Introduction

In this chapter we state and prove our three main results and the corresponding Phan-type theorems. The hard work has already been done, so these proofs mainly consist of putting the pieces together.

7.1 Main Result 1

Main Theorem 1. Let \( \varphi \) be a flip of \( \Delta \). Then \( \varphi \) is induced by a semilinear transformation \( f \) of the underlying unitary space \( V \) such that one of the following holds:

(i) \( f \) is a linear isometry of \( (V, \beta) \), \( f^2 = \text{id} \) on \( V \), and there is a hyperbolic basis \( \{e_i, f_i\}_{i=1}^n \) for \( V \) such that \( f(e_i) = f_i \) and \( f(f_i) = e_i \) for \( i = 1, \ldots, n \);

(ii) \( f \) is a linear anti-isometry of \( (V, \beta) \), \( f^2 = \text{id} \) on \( V \), and there is a hyperbolic basis \( \{e_i, f_i\}_{i=1}^n \) for \( V \) such that \( f(e_i) = \alpha f_i \) and \( f(f_i) = \alpha^{-1} e_i \) for \( i = 1, \ldots, n \), where \( \text{Tr}_\sigma(\alpha) = 0 \).

(iii) \( f \) is a \( \sigma \)-semilinear isometry of \( (V, \beta) \), \( f^2 = \text{id} \) on \( V \), and there is a hyperbolic basis \( \{e_i, f_i\}_{i=1}^n \) for \( V \) such that \( f(e_i) = f_i \) and \( f(f_i) = e_i \) for \( i = 1, \ldots, n \);
(iv) \( f \) is a \( \sigma \)-semilinear isometry of \((V, \beta)\), \( f^2 = \text{id} \) on \( V \), and there is a hyperbolic basis \( \{e_i, f_i\}_{i=1}^n \) for \( V \) such that for \( i = 1, \ldots, n - 1 \), \( f(e_i) = f_i, \ f(f_i) = e_i \) and there is a non-square \( \lambda \in \mathbb{F} \) with \( f(e_n) = \lambda f_n \) and \( f(f_n) = \sigma(\lambda^{-1})e_n \).

Conversely any semilinear transformation of \( V \) satisfying one of (i)-(iv) induces a flip of \( \Delta \).

**Proof.** By Lemma 3.1.3 we know that \( \varphi \) is induced by either a linear transformation of \( V \), or a \( \sigma \)-semilinear transformation of \( V \). By Theorem 4.1.1 if \( \varphi \) is induced by a linear transformation of \( V \) either (i) or (ii) holds. Finally by Theorem 5.4.1 if \( \varphi \) is induced by a \( \sigma \)-semilinear transformation of \( V \) then either (iii) or (iv) holds.

The converse is the content of Lemma 3.2.2.

\[ \square \]

### 7.2 Main Result 2

**Main Theorem 2.** Let \( \varphi \) be a linear flip of \( \Delta \).

(i) If \( \varphi \) is induced by an isometry of \((V, \beta)\) then the group \( U_{2n}(q^2)\varphi = U_n(q^2) \times U_n(q^2) \) acts flag transitively on the geometry \( \Gamma(n,q) \).

(ii) If \( \varphi \) is induced by an anti-isometry of \((V, \beta)\) then the group \( U_{2n}(q^2)\varphi = \text{GL}_n(q^2) \) acts flag transitively on the geometry \( \Gamma(n,q) \).

(iii) If \( n \geq 14 \), then the geometry \( \Gamma(n,q) \) is simply connected.

**Proof.** It follows from Theorem 4.2.3 that, whether \( \varphi \) is induced by an isometry or an anti-isometry, the group \( U_{2n}(q^2)\varphi \) acts flag transitively on \( \Gamma(n,q) \). Combining this with Theorem 4.2.4 shows that both (i) and (ii) hold.

Finally, (iii) holds by Theorem 4.6.7.

\[ \square \]

### 7.3 Main Result 3

**Main Theorem 3.** Let \( \varphi \) be a \( \sigma \)-semilinear flip of \( \Delta \).
(i) If $\varphi$ satisfies (iii) of Main Theorem 1 then $U_{2n}(q^2)^{\varphi}$ acts flag transitively on $\Gamma_1(n, q)$ where:

- (a) if $n$ is even or $-1$ is not a square in $F_q$, then $U_{2n}(q^2)^{\varphi} \cong O_{2n}^+(q)$;
- (b) if $n$ is odd and $-1$ is a square in $F_q$, then $U_{2n}(q^2)^{\varphi} \cong O_{2n}^-(q)$.

(ii) If $\varphi$ satisfies (iv) of Main Theorem 2 then $U_{2n}(q^2)^{\varphi}$ acts flag transitively on $\Gamma_1(n, q)$ where:

- (c) if $n$ is even or $-1$ is not a square in $F_q$, then $U_{2n}(q^2)^{\varphi} \cong O_{2n}^-(q)$;
- (d) if $n$ is odd and $-1$ is a square in $F_q$, then $U_{2n}(q^2)^{\varphi} \cong O_{2n}^+(q)$.

(iii) If $n \geq 8$ or $n = 7$ and $q \geq 5$ then $\Gamma_1(n, q)$ is simply connected.

Proof. It follows from Theorem 5.4.3 that, whether $\varphi$ is square type or non-square type, the group $U_{2n}(q^2)^{\varphi}$ acts flag transitively on $\Gamma_1(n, q)$. Both (i) and (ii) now follow from Theorem 5.5.4.

Finally, (iii) holds by combining Theorem 6.6.6 with Theorem 6.6.11. $\square$

### 7.4 Resulting Phan-Type Theorems

We include the resulting Phan-type theorems in this section so that all our main results are contained in a single chapter. The proofs of Phan-type Theorem 1 and Phan-type Corollary 1 are contained in Section 4.7. The proofs of Phan-type Theorem 2 and Phan-type Corollary 2 are contained in Section 6.7.

**Phan-type Theorem 1.** Suppose $n \geq 14$, let $\varphi$ be a linear flip and let $\mathcal{A}$ denote the amalgam of maximal parabolic subgroups of $U_{2n}(q^2)^{\varphi}$ with respect to a maximal flag $\mathcal{F}$ of $\Gamma(n, q)$. Then $U_{2n}(q^2)^{\varphi}$ is the universal completion of $\mathcal{A}$.
Phan-type Corollary 1. Let $\varphi$ be a linear flip and let $A_{(k)}$ denote the amalgam of rank $k$ parabolic subgroups of $U_{2n}(q^2)^{\varphi}$ with respect to a maximal flag $F$ of $\Gamma(n, q)$. If $n - k \geq 14$ then $U_{2n}(q^2)^{\varphi} = U(A_{(k)})$.

Phan-type Theorem 2. Suppose $n \geq 8$, let $\varphi$ be a $\sigma$-semilinear flip and let $A$ denote the amalgam of maximal parabolic subgroups of $U_{2n}(q^2)^{\varphi}$ with respect to a maximal flag $F$ of $\Gamma_1(n, q)$. Then $U_{2n}(q^2)^{\varphi}$ is the universal completion of $A$.

Phan-type Corollary 2. Let $\varphi$ be a $\sigma$-semilinear flip and let $A_{(k)}$ denote the amalgam of rank $k$ parabolic subgroups of $U_{2n}(q^2)^{\varphi}$ with respect to a maximal flag $F$ of $\Gamma_1(n, q)$. If $n - k \geq 8$ or $n - k = 7$ and $q \geq 5$ then $U_{2n}(q^2)^{\varphi} = U(A_{(k)})$. 
BIBLIOGRAPHY


Appendix: GAP Code

In this appendix we have included the GAP code used to show that for a square type flip, $\mathcal{G}(\Gamma_1(2, 3))$ is not connected and both $\mathcal{G}(\Gamma_1(2, 7))$ and $\mathcal{G}(\Gamma_1(2, 7))$ are connected of diameter 3. This program works in a straightforward way. The key is that there is a basis for $V$ with respect to which the $\beta$ Gram matrix is

$$
M = \begin{pmatrix}
-2 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & -2 & 0 \\
0 & 0 & 0 & 2
\end{pmatrix}
$$

and $\varphi$ acts trivially on half the basis vectors, and acts by multiplication by $-1$ on the other half.

To produce this basis, let $\{u, v\}$ be biorthogonal square type points in $\Gamma_1(2, 3)$. Then we obtain a basis for $V$ by $\{u, \varphi(u), v, \varphi(v)\}$. Let $e_1 = u - \varphi(u)$, $e_2 = u + \varphi(u)$, $e_3 = v - \varphi(v)$ and $e_4 = v + \varphi(v)$. Then with respect to the basis $\{e_1, e_2, e_3, e_4\}$ $\beta$ has Gram matrix $M$.

The program works by first identifying all the points of $\Gamma_1(2, q)$. Then the program chooses a point to start with, and finds all the neighbors of the chosen point. Then the program finds all neighbors of the neighbors of the starting point.

```gap
n:=2;
d:=2*n;
p:=t;
```
q:=p^2;
F:=GF(q);
V:=F^d;
umat:=[[−2,0,0,0],[0,2,0,0],[0,0,−2,0],[0,0,0,2]];
fsigma:=FrobeniusAutomorphism(F);

vsigma:=function(v)
local i,vsigma;
vsigma:=[];
for i in [1..d] do
    vsigma[i]:=v[i]^fsigma;
od;
return(vsigma);
end;;

beta:=function(x,y)
return x*umat*vsigma(y);
end;;

BetaSingularVectors:=function(U)
local u,betasingularvectors;
betasingularvectors:=[[];
for u in U do
    if beta(u,u)=Zero(F) then
        Append(betasingularvectors,[u]);
    fi;
od;
return(betasingularvectors);
end;;

Flip:=function(v)
local i,vphi;
vphi:=[ ];
for i in [1..d] do
vphi[i]:=(((-1^i)*v[i])^fsigma);
od;
return(vphi);
end;;

betahat:=function(x,y)
return x*umat*vsigma(Flip(y));
end;;

BetahatNormalVectors:=function(U)
local u,betahatnormalvectors;
betahatnormalvectors:=[ ];
for u in U do
if betahat(u,u)=Z(p)^0 then
Append(betahatnormalvectors,[u]);
fi;
od;
return(betahatnormalvectors);
end;;
UniqueRepresentation:=function(U)
local u,urep;
urep:=[ ];
for u in U do
if (-u in urep) = false then
Append(urep,[u]);
fi;
od;
return(urep);
end;;

FirstCollinearPoints:=function(U)
local u,v,fcop;
v:=U[1];
fcop:=[ ];
Append(fcop,[v]);
for u in U do
if (beta(u,v)=Zero(F) and (Z(p)^0-betahat(u,v)^2)<Zero(F)) then
Append(fcop,[u]);
fi;
od;
return(fcop);
end;;

NextCollinearPoints:=function(U,W)
local u,w,ncop;
ncop:=[ ];

for w in W do
    Append(ncop,[w]);
for u in U do
    if (beta(u,w)=Zero(F) and (Z(p)^0-betahat(u,w)^2)<Zero(F)) then
        Append(ncop,[u]);
    fi;
od;
od;
return(ncop);
end;;

bsv:=BetaSingularVectors(V);;

bhnv:=BetahatNormalVectors(bsv);;

unrep:=UniqueRepresentation(bhnv);;

When \( t = 3 \) we obtain the following outputs:

Size(unrep);
144
dist1:=FirstCollinearPoints(unrep);;
dist2:=Set(NextCollinearPoints(unrep,dist1));;
Size(dist2);
16
dist3:=Set(NextCollinearPoints(unrep,dist2));;
Size(dist3);
16

Since \( \Gamma_1(2, 3) \) has 144 points for a square type flip, and only 16 in the connected component of our starting point we can conclude that \( \mathcal{G}(\Gamma_1(2, 3)) \) is not connected. In fact it has 9 connected components.
When $t = 5$ we obtain the following outputs:

\begin{verbatim}
Size(unrep);
1200
dist1:=FirstCollinearPoints(unrep);;
dist2:=Set(NextCollinearPoints(unrep,dist1));;
Size(dist2);
1128
dist3:=Set(NextCollinearPoints(unrep,dist2));;
Size(dist3);
1200
\end{verbatim}

It follows that $G(\Gamma_1(2,5))$ is connected of diameter 3. Notice that this is an improvement on Theorem 6.5.1 which concludes that $G(\Gamma_1(2,5))$ is connected of diameter at most 4.

When $t = 7$ we obtain the following outputs:

\begin{verbatim}
Size(unrep);
9408
dist1:=FirstCollinearPoints(unrep);;
dist2:=Set(NextCollinearPoints(unrep,dist1));;
Size(dist2);
7560
dist3:=Set(NextCollinearPoints(unrep,dist2));;
Size(dist3);
9408
\end{verbatim}

It follows from this calculation that $G(\Gamma_1(2,7))$ is connected of diameter 3.

As a result of these calculations, we suspect that for a square type flip, $G(\Gamma_1(2,q))$ is connected of diameter 3 for all $q \geq 5$, but have been unable find a proof.
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