MODELING STRUCTURAL POLYMERIC FOAMS UNDER COMBINED CYCLIC COMPRESSION-SHEAR LOADING

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Dissertation

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ABSTRACT

The objective of this research was to investigate the mechanical behavior of Divinycell PVC H100 foam under combined cyclic compression-shear loading, and to develop material constitutive models to predict response of the foam under these conditions. Structural polymeric foams are used for the core of sandwich structures in aerospace, marine, transportation, and other industries. They are valued for enabling high specific stiffness and strength as well as energy absorption and impact resistance of sandwich structures. This research addresses energy absorption of the foam due to plastic collapse, damage and hysteresis.

Experiments were done to obtain out-of-plane mechanical properties of Divinycell PVC H100 foam under cyclic compression-shear loading. Stress-strain curves for the Divinycell PVC H100 foam under various combinations of compression-shear deformation and deformation rates were obtained. Rate-dependent behavior was observed before and after foam yielding. Yielding and damage in the foam occurred simultaneously. Foam yielding was associated with permanent change in cell micro-structure either by buckling cell walls when the foam is under compression or by bending and stretching cell walls when they were under shear. The Tsai-Wu failure criterion was shown to be a good predictor of yielding and damage initiation. The foam produced
hysteresis either due to viscoelasticity and/or viscoplasticity if it was allowed to undergo reverse yielding during unloading and reloading.

A phenomenological model was developed to describe the behavior of PVC H100 foam. This model consisted of a standard linear material model for viscoelastic response before yielding/damage initiation. After yielding/damage initiation, combined plastic flow and damage was modeled by modifying the viscoelastic properties of the standard linear model with damage properties and adding a viscoplastic element in series with it in order to control the plastic flow stress. Tsai-Wu plasticity and a specialized hardening function to account for different hardening rates in compression and shear were used to capture plastic flow behavior of the foam under combined compression and shear. The constitutive model was programmed in an ABAQUS user-material subroutine and finite element analysis was used to simulate the tests. Good agreement was found between the predicted and test results, except for specimens which appeared to fail during the tests due to stress concentration effects.
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CHAPTER I

INTRODUCTION

A composite sandwich structure is a special class of laminated composite materials with advantages such as low weight, high stiffness, high strength, high impact resistance, and high corrosion resistance. The sandwich structure is manufactured with lower and upper thin, strong and stiff skins or facesheets, which are bonded to lightweight core materials in between. The facesheets are oftentimes made of steel, aluminum, or fiber-reinforced polymer, while the core material is made of materials such as metallic foam, balsa wood, honeycomb and many kinds of polymeric foams. This research addresses the cyclic behavior of Divinycell PVC H100 foam as it pertains to a composite sandwich panel subjected to impact and blast loading. Recent analysis on the underwater blast response of PVC foam composite sandwich panel shows that PVC foams have blast mitigation effects through energy absorption from plastic core crushing [1,2]. Research is undertaken to examine foam energy absorption beyond initial yield and plastic flow. This research extends the study on cyclic crushing of foams performed by Chen [3] by considering the behavior of the foam under combined compression and shear.
1.1 Cellular Foams

The core materials in lightweight composite structures mostly consist of cellular foams. There are several varieties of foamed plastics and metals with different mechanical properties and densities. Aluminum is often used to create metallic foams. In addition, polystyrene, polyurethane (PUR), and polyvinyl chloride (PVC) are utilized to make plastic foams. Metallic foams are very expensive, while plastic foams are inexpensive. Production of polymeric foam materials is done by making air gas bubbles inside a liquid monomer or hot polymer [4]. The bubbles grow upwards until stabilizing. In general, the cells of cellular plastic foams are elongated in the out-of-plane (through-the-thickness) direction, while in the in-plane direction, while the cells are of equal dimensions [5]. On the other hand, metallic foams are manufactured by mixing organic beads such as carbon inside a melt of metal under regular atmospheric conditions. The carbon must then burn when the metal is cooled and solidified. This makes the end product process [6]. More details about analysis, properties and structure of cellular foams can be found in the book by Gibson and Ashby [4].

Cellular foams are used as core materials in sandwich structures utilized extensively in aerospace, marine, transportation, and other industries were the sandwich configuration offers higher stiffness and strength per unit weight compared to monolithic structures. In addition to this, the foam core allows energy absorption and higher impact resistance. The main purpose of the foam is to support thin facesheets when the sandwich construction is subjected to the load. As a result, facesheets should not deform into the foam but keep the same relative position from each other as the sandwich deforms. Also, the characteristics
of the foam are important. It must be stiff enough to offer a constant distance between the facesheets, and it must be rigid in shear, so that facesheets do not slide over each other. When the sandwich structure is subject to bending loads, the foam must be able to transfer shear between the facesheets.

There are two kinds of the cellular foams materials: closed-cell and open-cell configurations. The density and cell structure of the foam affect the mechanical behavior for the foam. Because of the entrapped gas inside the cell, open-cell foams can absorb less energy compared with closed-cell foams, which absorb a lot of energy [7]. Therefore, the structure of the cell and the density of the foam are important energy absorption.

A closed-cell PVC foam manufactured by the company Diab has excellent mechanical properties. The unique integration of polyurea and PVC achieve awesome mechanical performance with low weight. The Divinycell H series has been widely utilized in virtually every application where sandwich structures are used, such as wind turbine blades, civil infrastructure, land transportation, marine (military, leisure and commercial) industry and other industrial applications. For instance, Figure 1.1 shows a new generation high-speed train in China. The interior components of it are made of Divinycell foam-cored composite sandwich panels, including floors, partition walls, doors and ceilings [8]. A Swedish Visby-class corvette, a stealth patrol boat is displayed in Figure 1.2. The hull is made of a sandwich composite consisting of a PVC foam core with a carbon fiber and vinyl laminate [9]. The polymeric foam is a perfect material for ships when they are subjected to slamming, fatigue or impact loads. Several of Divinycell H products are available for use
in these applications. In addition, Divinycell H has many characteristics including high temperature resistance, low water absorption, superior damage tolerance, good chemical resistance, excellent fatigue properties, and it is fast and easy to process with a wide range of properties [10].

Figure 1.1 Chinese Railway High-speed train [8].
1.2 Notable Foam Material Models

Many researchers have studied the mechanical properties of polymeric foam (PVC) in different Divinycell H grades. For instance, Deshpande and Fleck [11] investigated the yield behavior of PVC H100 and H200 foams, respectively. They have tested closed-cell PVC H100 and H200 foams for a domain of axisymmetric tensile and compressive stresses. They found the yield surface is illustrated by the inner envelope of a maximum compressive principal stress criterion and an effective stress and a mean stress quadratic function. Foams under tension loading are controlled by bending of the cell wall, so the uniaxial and hydrostatic stress magnitudes are comparable. On the other hand, the PVC foams under compressive loadings deform under elastic buckling of the cell walls. Therefore the hydrostatic compressive and uniaxial strengths are almost the same.
Other experiments on the mechanical properties behavior of polymeric foams (PVC) subjected to uniaxial tensile, compressive, and shear states of stress can be found in the open literature. However, studies on the mechanical behaviors of foams under multiaxial loading are limited in the literature. Gdoutos et al. [5] investigated the mechanical behavior for two types of PVC closed cell polymeric foams (Divinycell PVC H100 and H250) under multiaxial loading. The experimental investigations included uniaxial compressive and tension, shear, biaxial tension/compression on strips, combined compression-torsion on ring specimens, combined tension-torsion on thin tubes, and combined tension-torsion-internal pressure on thin tubes. Uniaxial tensile, compressive and shear stress-strain curves for both H100 and H250 foams along the through-thickness and in-plane directions were obtained. They found the H100 foam is an almost isotropic material, and the PVC H250 material displays orthotropic behavior. Likewise, the stiffness is higher along the through-thickness than in the in-plane direction. The values of stress in uniaxial tension, compressive and shear at failure were utilized to construct failure envelopes by the Tsai-Wu failure criterion.

Gibson and coworkers have published on cellular materials under multiaxial loads [12, 13]. They tested flexible and rigid foams materials. The flexible foams were open-cell polyurethane (PUF) and a closed-cell polyethylene (PEF), and the rigid foams were closed-cell polyurethane (PUR) and open-cell aluminum (AL). They developed equations for the combination of stresses to describe various failure modes for both honeycombs and foams.
1.3 Planned Research

This research is an extension of the work performed by Chen and Hoo Fatt [3, 14 and 15]. They performed cyclic material tests on Divinycell PVC H100 foam in uniaxial compression and simple shear both in the out-of-plane and in-plane directions. They obtained stress-strain behavior of the PVC H100 foam after yielding in compression and shear stresses including hysteresis. They found that the stress-strain behaviors were very similar in compression and shear, but the behaviors were not the same after yielding. A plateau (flow) stress occurred after yielding in compression, but it did not occur after yielding in shear. The ratio of stiffness and yield strength in the out-of-plane and in-plane were almost 3/2 in both the compressive and shear modes. The PVC H100 foam underwent permanent damage and showed hysteresis after viscoplastic yielding. The foam followed the pattern of Mullins damage, i.e., damage increased with increasing strain amplitude.

The objective of this research is to investigate the mechanical behavior of Divinycell PVC H100 foam under combined cyclic compression-shear loading, and to develop material constitutive models to predict response of the foam under these conditions. Chapter II discusses previous work on cellular foams, as well as constitutive material modeling. Then, Chapter III shows the experimental tests including apparatus design, test setup and procedure, and results. Chapter IV explains constitutive modeling and results of experiments are used to validate the constitutive model. Concluding remarks are finally made in Chapter V.
CHAPTER II

LITERATURE REVIEW

Many researchers have studied the mechanical properties of cellular materials. The responses of cellular foams are usually investigated when the core materials are subjected to uniaxial compression, tension, and shear. However, experimental data of polymeric foams under multi-axial loading are limited in the literature even though foams may very well experience multi-axial stress states when used in sandwich configurations. In designing in sandwich structures which consists of outer facesheets and an inner core, the core becomes an important part to increase stiffness and control the mechanisms of failure of the structure [16]. The primary function of the core is to transmit shear loads across a sandwich panel and increase its bending resistance by increasing its thickness or second moment area. It is usually made of lighter and weaker material than the facesheets. On the other hand, choosing too weak of a core can lead to core shear failure and debonding in a sandwich panel.

Several articles are reviewed for cellular foams in this chapter. Stress-strain curves of polymeric foam behaviors under various loading state are described. As mentioned earlier very few papers are written concerning foams under multiaxial loading
[5, 11, 17 and 18]. In most experimental studies the foam is subjected to monotonic loading. The cyclic response in pure compression and shear were found for Divinycell PVC H100 foam in out-of-plane and in-plane directions by Chen and Hoo Fatt [14]. They observed a so-called Mullins effect in the PVC H100 foam once it yielded. This dissertation extends the work of Hoo Fatt and Chen by considering the cyclic response of PVC H100 foam under combined transverse compression and shear.

2.1 General Behavior of Polymeric Foam

The structural response of polymer foam depends on many factors including cell microstructure such as cell size, shape, whether it is closed or open cells and foam densities [4]. However, the shape of the stress-strain curve for the foam subjected to uniaxial compression or tension is approximately the same for all the crushable foams [19]. For uniaxial compression, as shown in Figure 2.1, the curve response has three different regimes:

1) OA or OA is a linear elastic regime.

2) A plateau or plasticity stress is AB, with increasing strains, sometimes stress softening or stress drop is found at A. This may be related to buckling of cell walls of the foam.

3) A regime BC is a densification where stress increase sharply. This occurs when cell walls of foam start touch each other or compact.

On the other hand, the tensile behavior ODE contains of a linear elastic region followed by continued cell wall elongation that causes hardening and then fracture, as
shown in Figure 2.1. Clearly, the strain at fracture in tensile is lower than the densification strain in compression. No fracture occurs in compression.

Many polymeric types of foam are strain-rate dependent. The polymeric foams exhibit strain rate effect as exemplified by an increase of elastic modulus, increase of plateau stress and a decrease of the densification strain, as shown in Figure 2.2. Ouellet and coworkers have studied the mechanical behavior of expanded polystyrene (EPS), high-density polyethylene (HDPE), and polyurethane (PU) foams in compression with strain rates ranging from 0.0087 to 2500/s [48]. The material compression responses were done from quasi-static to medium and high strain rates. For low strain rate, they used a standard compression test machine which is the Instron® Model 4206. They performed the test at quasi-static rates for all materials at a strain rate 0.0087/s. A drop tower apparatus was used for intermediate strain rates, which were between 10 and 100/s. The polymeric split

Figure 2.1 Typical uniaxial stress-strain curve of foams [19].
Hopkinson pressure bar (PSHPB) apparatus was used for high strain rate, ranging from 500 to 2500/s.

Figure 2.2 General compressive stress-strain curve of polymeric foam at varying strain rates [20].
2.2 Behavior under Monotonic Loading

Several tests have been completed to determine the behavior of PVC H100 foam under monotonic loading.

2.2.1 Transverse isotropy

Chen and Hoo Fatt [14] obtained the stress-strain response of PVC H100 foam in different directions. Figure 2.3 explains these directions with respect to the specimen orientation and shows the loading on the specimens in compression and simple shear. Divinycell PVC H100 foam behavior under monotonic compression and simple shear is shown in Figures 2.4 (a) and (b). The monotonic tests were obtained in both out-of-plane and in-plane directions.

![Figure 2.3 Orientation of specimens for PVC H100 foam in compression and simple shear [14].](image)

For monotonic compression, the stress-strain responses of PVC H100 foam at strain rate 0.5 s⁻¹ and strain amplitude 0.1 in the out-of-plane and in-plane directions are shown in Figure 2.4 (a). This graph shows that there are differences between the modulus and
yield strength in the two directions because the foam is a transversely isotropic material. In the in-plane direction, the compressive modulus and the yield strength are lower than values in out-of-plane by around 39 %, or the ratio of out-of-plane to in-plane values are almost 3/2. This is explained by the microstructure of cell. Cells are more elongated in the through-the-thickness direction than they are in the in-plane directions.

For monotonic shear, the shear stress-strain curves in the out-of-plane and in-plane directions at strain rate of 0.001 s\(^{-1}\) until the foam breaks are shown in Figure 2.4 (b). The material behavior under shear exhibited again transversely isotropic material properties. The authors found that the shear modulus and yield strength gave the same ratio of out-of-plane to in-plane as in monotonic compression, i.e. 3/2 modulus and strength in monotonic shear. The load drops are the failure of foams in monotonic shear.
Figure 2.4 Comparison of out-of-plane and in-plane stress-strain curves of Divinycell PVC H100 foam: (a) Compression at strain rate 0.5 s$^{-1}$ and (b) Shear at strain rate 0.001 s$^{-1}$ [14].
2.2.2 Multiaxial loading

Multi-axial behavior of foam can be obtained from several test fixtures. One of the most commonly used test fixture is that of the Arcan test fixture. The standard Arcan fixture has been used to find the mechanical properties of polymer foams and other cellular materials by applying load to a butterfly specimen (BS). The standard Arcan fixture has circular arrangement with holes which used for the loading as shown in Figure 2.5 (a). This fixture is limited to tensile and shear loading. The development of a modified Arcan fixture (MAF) allows the application of any tension or compression and shear loadings with a quasi-spiral distribution of loading holes as shown in Figure 2.5 (b) [21]. Divinycell PVC H100 foam with different shapes specimens were used in these tests. For tensile loading they used a short dogbone (SD) shape geometry, but for shear loading they used a butterfly shape (BS) geometry.

The performance of the MAF was investigated and validated the mechanical properties of PVC H100 foam in tension and shear. By using Digital Image Correlation (DIC), the MAF specimen gauge section strain field was found. A two camera DIC system with cameras on opposite sides of the test specimens was used to ensure that strain fields were free from bending or twist effects. Thermoelastic stress analysis (TSA) was used to validate the uniformity and symmetry of the stress fields obtained in tensile and shear specimen. The mechanical properties obtained from the test were compared with results from other test methods. Authors claimed that the elastic constants, strengths and strains
results were in excellent agreement with data from tensile and shear test fixtures (ASTM) and sheet data provided by DIAB.

Figure 2.5 Bidirectional test fixtures: (a) Standard Arcan fixture and (b) Modified Arcan fixture (MAF) [21].

2.2.3 Yield criterion

Various methods use to describe yielding of foam are defined Appendix A. The multi-axial yield behavior under axisymmetric compressive stress states of Alporas (closed-cell) and Duocel (open-cell) metallic foams was investigated by Deshpande and Fleck [18]. The magnitude of the hydrostatic yield strength is similar to the uniaxial yield
strength. Harte et al. [18] also claimed that the yield strengths in uniaxial compression and tension are approximately equal for Alporas and Duocel foams. Also, they found the yield surfaces in the stress space of mean stress versus effective stress are of quadratic form, with the hydrostatic yield strength comparable to the uniaxial yield strength as shown in Figure 2.6. The authors proposed the following yield surface criterion:

\[ \Phi \equiv \hat{\sigma} - Y \leq 0 \]  

(2.1)

where \( \Phi \) is a yield function, \( Y \) is the yield strength, and \( \hat{\sigma} \) is equivalent stress. The equivalent stress is defined as

\[ \hat{\sigma}^2 = \frac{1}{[1 + (\frac{\alpha}{3})^2]}[\sigma_e^2 + \alpha^2 \sigma_m^2] \]  

(2.2)

where \( \sigma_m \) is the mean stress, \( \sigma_e \) is the von Mises or effective stress, and \( \alpha \) is a parameter the defining of the yield surface as shown in Figure 2.6. The mean and von Mises stresses are given by

\[ \sigma_m \equiv \frac{\sigma_{kk}}{3} \]  

(2.3)

and

\[ \sigma_e \equiv \sqrt{\frac{3}{2} \sigma_{ij}' \sigma_{ij}'} \]  

(2.4)

where \( \sigma_{ij}' \sigma_{ij}' \) are the second invariant of the deviatoric stress tensor. The parameter \( \alpha \) is given by

\[ \alpha = 3 \left( \frac{\frac{1}{2} - \nu_p}{1 + \nu_p} \right)^2 \]  

(2.5)

where \( \nu_p \) is the plastic Poisson’s ratio. For most cellular foams \( \nu_p = 0 \).
Deshpande and Fleck [11] also studied the multi-axial yield behavior of Divinycell PVC H100 and H200 foams. The uniaxial tension and compression diagrams of the PVC H100 and H200 were found by using a standard screw driven test machine. Figure 2.7 shows uniaxial tension and compression stress-strain diagrams of PVC H100 foam while Figure 2.8 exhibits uniaxial tension and compression stress-strain curves of PVC H200 foam. The compression stress-strain curves have initial elastic and a plateau stress regime while the tension stress-strain curves the stress increases followed by strain hardening until fracture. Fracture occurred at axial tensile strain of 8 % and 15 % for PVC H100 and H200 foams, respectively. They found that the compressive and tensile strengths of the foams in through-the-thickness direction higher by around 20 % than in the transverse direction. The authors claimed that the PVC H100 and H200 foams are transversely isotropic material.
The shear test results were obtained by Arcan and double-lap shear experiments. Figure 2.9 shows the shear stress-strain responses for PVC H100 and H200 foams. The PVC H100 foam was subjected to both Arcan and double-lap tests, but the PVC H200 foam was subjected to only Arcan test because of failure between foam and the loading platen for double-shear test. The shear stress-strain curves were found for both PVC H100 and H200 foams. The shear strengths of PVC H100 foam from both tests are approximately in agreement with each other. For the shear modulus, the Arcan test gives a slightly higher value than double-lap test.
Figure 2.7 Uniaxial tensile and compressive curves of PVC H100 foam [11].

Figure 2.8 Uniaxial tensile and compressive curves of PVC H200 foam [11].
In hydrostatic compression and tension tests, stress-strain diagrams were obtained for both PVC H100 and H200 foams as shown in Figure 2.10. In hydrostatic compression, the foams behavior displays an initial linear elastic region then a stress plateau. In hydrostatic tension, the foams response shows linear followed by strain hardening with hydrostatic fracture strain at around 20% for both PVC H100 and H200 foams. They mentioned that the strengths of foams are approximately equal in the hydrostatic and uniaxial tensile.
Figure 2.10 Hydrostatic compression and tension stress-strain curves for PVC H100 and H200 foams [11].

The yield data for PVC H100 and H200 foams were plotted by using the axes of axial and radial stress as shown in Figure 2.11, and the yield stresses were fitted by the Deshpande and Fleck [18] yield criterion with $\alpha \approx 1$ (solid line). Dotted lines are buckling line which is a maximum compressive principal stress surface.
Figure 2.11 Yield surfaces for PVC H100 and H200 foams in axial-radial stress space [11].

The mechanical properties of Divinycell PVC H100 and H250 foams under uniaxial and multiaxial states of stress were obtained by Gdoutos et al. [5]. The microstructures of cells are elongated in the thickness direction. Thus, they found that the elastic modulus and strength along out-of-plane (3-direction) were higher than along the in-plane ((1 ≡ 2)-direction). The modulus of elasticity in tension and compression for PVC H100 foam were almost the same, and Divinycell PVC H250 exhibited the same situation. Figure 2.12 stress-strain diagram displayed the through-the-thickness longitudinal and transverse behavior of PVC H100 foam in compression test. Also, the behavior of PVC H100 foam in-plane direction longitudinal and transverse was almost similar as the curve in the through-the-thickness. As a result, the PVC H100 foam was shown nearly isotropic behavior in compression test. For PVC H100 and H250 foams in uniaxial compression, the
stress-strain curve Figure 2.12 and Figure 2.13 response consisted of an initial linear portion followed by a nonlinear part and a yield region where the stress is constant. Densification occurs when the stress increases sharply. In uniaxial tension for both PVC H100 and H250 foams, the stress-strain curve behavior is nonlinear elastic as shown in figures 2.14 and 2.15, respectively.

Figure 2.12 Stress-strain curves of PVC H100 foam under uniaxial compression in the 3-direction [5].
Figure 2.13 Stress-strain curves of PVC H250 foam under uniaxial compression in the 3- and 1-directions [5].

Figure 2.14 Stress-strain curves of PVC H100 foam under uniaxial tension in the 3- and 1-directions [5].
Shear only response was determined by using Arcan test. Shear stress-strain curve of PVC H250 foam is shown in Figure 2.16. The diagram displayed an initial linear region followed by a nonlinear elastic portion up to a plateau section. The flow (plateau) stress was constant then suddenly falls lower at a strain of 12%. The shear strength was determined at maximum stress with the first plateau region which is 5 MPa.
Figure 2.16 Shear stress-strain curve of PVC H250 foam on 1-3 plane the through-the-thickness [5].

From biaxial tension and compression tests, they found the stress-strain response of the material shows the main characteristic features as under uniaxial tension or compression. The stress-strain graphs in compression for a strip specimen of PVC H250 foam were shown in Figure 2.17 along the through-the-thickness and in-plane directions. The failure strengths were determined by maximum stress at the upper yield point. Figure 2.18 exhibited stress-strain response for a strip specimen of PVC 250 foam subjected to tension along the through-the-thickness and in-plane directions. Failure was obtained at maximum stress. The under conditions the specimens with longitudinal axis parallel to the in-plane (1-direction) they obtained $\sigma_1 = \nu_{13}\sigma_3$, and with longitudinal axis parallel to the through-the-thickness (3-direction) they got $\sigma_3 = \nu_{31}\sigma_1$. 
Figure 2.17 Stress-strain responses of PVC H250 strip specimen foam under compression loading in the 3- and 1- directions [5].

Figure 2.18 Stress-strain responses of PVC H250 strip specimen foam under tension loading in the 3- and 1- directions [5].
For ring specimen subjected to compression-torsion loading, compressive stress-strain behavior of PVC H250 foam was shown in Figure 2.19 along the through-the-thickness direction. In the same figure the shear stress-strain diagram is shown. Both curves had an initial linear portion followed by a nonlinear part up to a plateau region. The stress within plateau region was almost constant with increase strain. The failure stresses for both diagrams were obtained as maximum stress at plateau region.

The tube specimen of PVC H250 foam along the through-the-thickness was subjected to combined tension-torsion loading. The tension stress-strain response is shown in Figure 2.20, and the torque versus angle of twist curve on the 3-1 plane is also exhibited on the same plot. The response of PVC H250 for both curves displayed an initial linear region followed by nonlinear behavior until fracture. This behavior was similar to the
uniaxial tension response. The maximum stresses were obtained as failure stresses for both cases.

Figure 2.20 Tension stress-strain curve in 3-direction and torque-angle of twist curve on 3-1 plane for PVC H250 foam under combined tension and torsion loading [5].

Combined axial tension, internal pressure and torsion loading were applied on a tube specimen of PVC H250 foam. Stress-strain curve along the 3-direction for combined axial tension, internal pressure and torsion loading are shown in Figure 2.21. Also, in the same graph the torque versus angle of twist diagram is displayed. The diagrams consisted of an initial linear region followed by nonlinear response until fracture. Again, this response shape was as in the uniaxial tension case. The failure stress was determined at maximum stress from the diagram.
Composite materials are anisotropic and also have different strength in compression and tension. Therefore, the Tsai – Wu criterion is widely utilized for this kind of materials. Gdoutos et al. [5] investigated mechanical properties for closed-cell PVC cellular foam under multiaxial loadings to determine the failure behavior of foam. For a general two-dimensional state of stress in the 1-3 plane, the Tsai – Wu failure criterion was proposed as

\[ f_1 \sigma_1 + f_3 \sigma_3 + f_{11} \sigma_1^2 + f_{33} \sigma_3^2 + 2f_{13} \sigma_1 \sigma_3 = 1 - k^2 \]  

(2.6)

where

\[ f_1 = \frac{1}{F_{1t}} - \frac{1}{F_{1c}}, \quad f_3 = \frac{1}{F_{3t}} - \frac{1}{F_{3c}}, \quad f_{11} = \frac{1}{F_{1t} F_{1c}} \]
\[ f_{33} = \frac{1}{F_{3t}^*F_{3c}^*} \quad f_{13} = -\frac{1}{2} \sqrt{f_{11}f_{33}} \quad k = \frac{\tau_{13}}{F_{13}} \] \hspace{1cm} (2.7)

\( F_{1t} \) and \( F_{1c} \) are the tensile and compressive strengths along the in-plane direction, \( F_{3t} \) and \( F_{3c} \) are the tensile and compressive strengths along the through-the-thickness direction, and \( F_{13} \) is the shear strength on the 1-3 plane. The values of the strength parameters were obtained from separate uniaxial tension, compression and shear tests.

Divinycell PVC H250 failure surfaces under combined normal and shear stresses along the out-of-plane and in-plane directions were found from biaxial tests. The failure envelopes were built with several values of \( k = \tau_{13}/F_{s} = 0, 0.8 \) and 1. The points obtained experimentally from biaxial tests and plotted in Figure 2.2. The results were predicted well by the Tsai – Wu failure criterion as shown in Figure 2.2 at the \( \sigma_1 - \sigma_3 \) plane.
In addition, Gdoutos et al. [17] did a similar study on Divinycell PVC H250 foam as discussed earlier. The stress-strain curves of PVC H250 foam in out-of-plane subjected to tension, compression and shear were obtained. The shear stress-strain curve presents an initial elastic region followed by a nonlinear elastic region up to a plateau region. Also, tensions in out-of-plane and in-plane directions were compared. The authors claimed that in the out-of-plane direction the elastic modulus and strength were higher by 65 and 40 percent compared with the values in-plane direction. The biaxial test results were predicted by the Tsai – Wu yield criterion.
2.3 Behavior under Cyclic Loading

Cyclic compression and shear experiments were done on Divinycell PVC H100 foam as shown in Figure 2.23 by Chen and Hoo Fatt [14]. The material tests were in both out-of-plane and in-plane directions. The compressive strain amplitudes were determined to a maximum limit of 0.1, and the engineering shear strain amplitudes were found to maximum value of 0.2. The strain rates were ranged from 0.0005 to 5 s\(^{-1}\). The strain amplitudes of compression tests were varied from 0.02 to 0.1 with 0.02 increments. The strain amplitudes of simple shear tests were varied from 1.5 to 0.2 with 1.5 increments.

2.3.1 Cyclic compression

The stress-strain curves with five different strain rates at strain amplitude 0.1 in out-of-plane and in-plane directions are shown in Figures 2.24 (a) and (b), respectively. Ten loading-unloading cycles were applied for different specimens. The first loading-unloading cycle and the second and subsequent cycles are dissimilarity to compare to each other. The first cycle of the PVC H100 foam displays viscoelasticity and then viscoplasticity with a permanent strain when unloaded to zero strain. Viscoelasticity is shown by a rise of the modulus with increasing strain rate. Also, viscoplasticity is shown by increasing yield strength and flow (plateau) stress of the foam response with increasing strain rates. In the second cycle, the foam softens where the stresses to the maximum strain are lower compare with the first cycle. This is because of the deformation of cell microstructures.
Figure 2.23 Apparatus design tests for PVC H100 foam: (a) cyclic compression test and (b) cyclic simple shear test [14].
Figure 2.24 Cyclic compression stress-strain curves at strain amplitude 0.1 with different strain rates: (a) out-of-plane and (b) in-plane [14].
Always hysteresis follows from the second cycle and subsequent cycles. Figure 2.25 shows the characteristic shape of the hysteresis after the foam crushes in out-of-plane or in-plane direction. The modulus is parallel to the initial elastic slope when each cycle is reloading from Point A to B. After an elastic limit, the foam exhibits secondary yielding following by strain hardening in compression. The same manner occurs for unloading from B to A. After an initial elastic level, secondary yielding and strain hardening take place again. The cell geometry is shown at the bottom of Figure 2.25 in compression where the cell is buckling. This figure explains how cell geometries change during hysteresis. For instance, at loading the cells are closing and yielding, and at unloading the cells are opening and yielding. When the PVC foam is subjected to bending, yielding, reverse bending and yielding in each cycle, energy is dissipated. The hysteresis shows a significant advantage of the PVC foam. For example, after initial loading from a dynamic load such as shock and impact most structure will vibrate. Therefore, the sandwich structure with PVC foam core material will dampen these vibrations because of the hysteresis. Recent crushable foam models do not use this hysteresis and are limited to monotonic loading cases.
Cyclic compression stress-strain diagrams in out-of-plane and in-plane directions at fixed strain rate with various strain amplitudes are displayed in Figures 2.26 (a) and (b), respectively. Ten loading-unloading for each one specimen were tested at specific strain amplitude with strain rate of 0.5 s\(^{-1}\). The strain amplitudes were applied in different specimen where strain amplitudes were starting from 0.02, 1.5, 0.06, 0.08, and to 0.1. These values of strain amplitudes were repeated for different strain rates which are 0.0005, 0.005, 0.05, 0.5, and 5 s\(^{-1}\). The damage increases with increasing strain amplitude. Also, hysteresis is illustrated the energy dissipation that the foam is increasing damage with increasing strain amplitude.
Figure 2.26 Cyclic compression stress-strain curves at strain rate $0.5 \text{ s}^{-1}$ with different strain amplitudes: (a) out-of-plane and (b) in-plane [14].
2.3.2 Cyclic simple shear

Cyclic shear stress-strain curves with different strain rates at strain amplitude of 0.2 in out-of-plane and in-plane directions are shown in Figures 2.27 (a) and (b), respectively. Ten loading-unloading cycles were done in different specimens. The behavior of the foam starts linear followed by strain hardening after yielding, and the hysteresis is similar to those of the cyclic compression tests. Again, shear modulus increases with increasing strain rate before the foam yields. This is typical of a viscoelasticity material. Because yield shear and flow stress increase with increasing strain rate, the foam is also a viscoplastic material. Also, the shear stresses in the foam soften within second cycle because of damage in the cells. Permanent shear strains are still in the foam after first cycle.

For strain amplitude, the shear stress-strain diagram in out-of-plane and in-plane directions at specific strain rate of 0.5 s\(^{-1}\) and various strain amplitudes are shown in Figures 2.28 (a) and (b). Each specimen was subjected to ten loading-unloading cycles shear loading. The different shear strain amplitudes at different specimens are 1.5, 0.08, 0.12, 0.16, and 0.2. More damage occurred with increasing shear strain amplitude. The shear modulus in the second cycle decreases compared with initial cycle because of damage in the foam. Hysteresis is also obvious as the damage increases with increasing shear strain amplitude. These are the same behavior at fixed strain rate as those in the compression tests.
Figure 2.27 Cyclic shear stress-strain curves at strain amplitude 0.2 and different strain rates: (a) out-of-plane and (b) in-plane [14].
Figure 2.28 Cyclic shear stress-strain curves at a strain rate of 0.5 s\(^{-1}\) and different strain amplitudes: (a) out-of-plane and (b) in-plane [14].
The hysteresis developed in shear is due to different microstructural mechanisms. Figure 2.29 explains how cells bend and stretch in shear. Such bending and stretching of cells during cyclic shear loading cause yielding and reverse yielding in addition to viscoelastic dissipation. This is contrast to the hysteresis exhibited in uniaxial compression, which was primary due to cell buckling.

![Figure 2.29 Hysteresis behavior of Divinecell PVC H100 foam with diagram of cell geometry under shear loading.](image)

Figure 2.29 Hysteresis behavior of Divinecell PVC H100 foam with diagram of cell geometry under shear loading.
2.4 Mullins Effect

2.4.1 Mullins effect for polymeric foam

In addition to above, the Divinycell PVC H100 foam under cyclic compression loading was investigated by Chen and Hoo Fatt [14]. They found the stress-strain behavior of PVC H100 foam displays similar Mullins behavior to rubber under cyclic loading. The Mullins effect originated in 1948 for rubber materials to describe their mechanical behavior under cyclic loading [73]. The Mullins effect is used to describe damage in rubber, whereby the stress-strain behavior depends on the maximum loading previously encountered. Figure 2.30 shows the typical Mullins effect stress-strain behavior for rubber [23]. The material is subjected to loading-unloading at a given strain amplitude $\varepsilon_1$. When the material is reloaded, it follows the path of the previous unloading curve until $\varepsilon_1$. After which, it follows a new loading path until $\varepsilon_2$. New damage is seen to occur when the material unloads at $\varepsilon_2$.

![Figure 2.30 Typical stress-strain response for rubber showing Mullins effect [23].](image)
More recently the Mullins effect with hysteresis on the tensile stress-stretch behavior of an ethylene-propylene-diene terpolymer (EPDM) rubber was studied based on the experimental results [24]. Quasi-static uniaxial tensile experiments were done at room temperature using MTS 810 servo-hydraulic machine by displacement control. The EPDM specimens were subjected to cycles of loading at various constant strain rates 0.15, 1.5, 15/s as shown in Figure 2.31. More details are in Reference [24].

![Figure 2.31 Stress-strain behavior of EPDM rubber under cyclic loading rate 0.15/s](image)

Figure 2.31 Stress-strain behavior of EPDM rubber under cyclic loading rate 0.15/s [24].

In the out-of-plane direction of PVC H100 foam at strain rate of 0.5 s⁻¹, the stress-strain curves from monotonic compression test was superimposed with cyclic compression test results with increasing strain amplitude as shown in Figure 2.32. The dash line is a monotonic compression response which is the primary curve. Loading-unloading stress-strain response from increasing strain amplitude ranging from 0 to 1.5, shows that the foam softens. The stress at strain amplitude of 0.06 is lower than initial strain amplitude. The
stress-strain curve from 0.06 to 0 to 0.06 shows unique softening curve because the monotonic and cyclic curves are identical at this range. For 0.08 and 0.1 amplitudes, the stress-strain curve follows the primary curve. Therefore, the damage of the foam is irreversible. This phenomenon explains Mullins effect on Divinycell PVC H100 foam.

Figure 2.32 Mullins effect compression stress-strain curve of Divinycell PVC H100 foam (monotonic stress-strain is the primary curve) [14].
2.5 Material Models

2.5.1 Viscoelasticity

Viscoelastic materials show that the relationship between stress and strain depends on rate of time. Viscoelastic behavior is explained in several publications [25-28]. The materials that display viscoelastic are synthetic polymers, wood, human tissue, and metals at high temperature. A viscoelastic material shows hysteresis is exhibited in the stress-strain curve when cyclic loading is applied, leading to dissipation energy. Stress relaxation occurs at constant strain, resulting in decreasing stress, over time, and creep occurs at constant stress, resulting in increasing strain over time.

Viscoelastic materials can be modeled to predict their stress and strain responses. Figures 2.33 (a), (b) and (c) show three kinds of models: Maxwell, Voigt and Stander linear solid, respectively. The elastic component can be modeled as spring stiffness (E), and the viscous component can be modeled as dashpot viscosity (η). The Maxwell model has a viscous damper and elastic spring in series connection that appears in Figure 2.33 (a). For this model, when material is under a constant strain, the stress relaxes. For a constant stress, the strain consists of two components. First, an elastic component happens immediately corresponding to the spring, and relaxes onto the stress release. The second, a viscous component increases with time as long as the stress is used. This model is accurate for most polymers because the stress decays exponentially with time. For creep, this model is limitation for predict accurately. The Voigt model or the Kelvin model consists of a spring and dashpot connected in parallel as shown in Figure 2.33 (b). It is used for the creep behavior of polymer. The Standard Linear Solid Model has combinations of the Maxwell
Model and a spring in parallel which is displayed in Figure 2.33 (c). For predicting material response, the Stander Linear Solid Model is in more accurate than Maxwell and Voigt models [53]. It is used for both creep and relaxation.

![Diagram of viscoelastic models](image)

Figure 2.33 Types of viscoelastic models. E is spring stiffness, and η is dashpot viscosity: (a) Maxwell, (b) Voigt and (c) Standard linear solid model [28].
2.5.2 Plasticity

For uniaxial one dimensional state, a specimen deforms after the Yield Point (A) with hardening as shown in Figure 2.34. In the case of perfectly-plastic material, the stress remains approximately constant and equal to the yield strength value at Point A. Elastic unloading occurs when the stress decreases. For hardening materials, the stresses after Point (A) continue increasing with the present of plastic deformation. If elastic unloading occurs at Point B, elastic strains are recovered. This is shown in Figure 2.34 as $\varepsilon_e$. Permanent plastic strains $\varepsilon_p$ remain after unloading.

Figure 2.34 Uniaxial stress-strain diagram for metal.
2.5.2.1 Isotropic hardening

Describing the evolution of the yielding surface could be done by defining several new parameters that characterize the effect of hardening. The easy approach, presented by Odqvist, uses a one-parameter family of yield surfaces. They are all similar and relative with respect to the origin [30]. Therefore, the yield surface is still the same shape but extends with increasing stress as shown in Figure 2.35. This is called isotropic hardening.

![Figure 2.35 Isotropic hardening: (a) uniaxial stress-strain curve and (b) yield surface in the biaxial plane stress.](image)

The basic form of the yield function can be used for the yield surfaces, in which different yield stresses are utilized. The initial yield criterion is

$$ F(\sigma) - \sigma_0 = 0 \quad (2.6) $$

where \( \sigma_0 \) is the initial yield stress. This equation is re-writing as

$$ F(\sigma) - \sigma_y = 0 \quad (2.7) $$

where \( \sigma_y \) is a function that increases with increasing plastic strain.
The plastic flow occurs after yield stress should be described by the hardening law equation. Under monotonic loading, hardening laws can be assumed as an explicit dependence of the yield stress on the plastic strain:

\[ \sigma_y = h(\varepsilon_p) \]  \hspace{1cm} (2.8)

This is called the strain hardening. However, \( \varepsilon_p \) is the plastic region of the normal strain \( \varepsilon \) in the direction of applied stress \( \sigma \). The function \( h \) is taken from the monotonic uniaxial stress-strain diagram. The plastic modulus \( H \) is given by

\[ H(\varepsilon_p) = \frac{dh}{d\varepsilon_p} \]  \hspace{1cm} (2.9)

For special case when the hardening law is linear,

\[ \sigma_y = \sigma_0 + H\varepsilon_p \]  \hspace{1cm} (2.10)

where the plastic modulus is constant \( H(\varepsilon_p) = H_0 \).

A positive plastic modulus \( H > 0 \) gives hardening. When \( H = 0 \), it is perfect plastic. When \( H < 0 \), softening occurs, thus, a negative plastic modulus is called the softening modulus.

2.5.2.2 Kinematic hardening

Isotropic hardening was characterized by a single parameter. However, more complicated hardening rules are needed, especially for unloading and cyclic loading cases [64]. For alternative hardening rule, the current loading yield surface is supposed not to extend but also to move as a rigid body during the stress space as display in Figure 2.36. This is known as kinematic hardening.
The utilization of kinematic hardening is needed to model the Bauschinger effect. This effect is oftentimes shown in metals subjected to cyclic loading. Kinematic hardening results in a translation of the yield surface. This means a shift of the origin of the initial yield surface. The initial yield surface is described by

$$ f(\sigma) = F(\sigma) - \sigma_0 \quad (2.11) $$

and the shift surface is described by

$$ f(\sigma, \sigma_b) = F(\sigma - \sigma_b) - \sigma_0 \quad (2.12) $$

where $\sigma_b$ is the back stress which appears the center of the shift elastic range. Many materials may exhibit combination of isotropic and kinematic hardening.
2.5.3 Viscoplasticity

Viscoplasticity is description the rate-dependent behavior of materials. Rate-dependence is the deformation of material depends on that loads are applied. Most researchers used viscoplasticity in solid mechanics. The elastic behavior of viscoplastic materials is presented by spring components in one dimension while the rate-dependence is presented by nonlinear dashpot components which are the same manner to viscoelasticity. Theories of viscoplasticity are useful to calculate permanent deformation of materials also to predict the plastic collapse of structures and other applications [31].
CHAPTER III
EXPERIMENTS

Testing of material behavior is important to determine the mechanical properties of materials. Data from material tests are used to develop constitutive models that can be used to predict the response of structures. In this study, experiments are designed to find the mechanical properties of Divinycell PVC H100 foam under cyclic compression-shear loading in the out-of-plane or through-thickness direction. This chapter describes materials and specimen, apparatus design and test setup, the test plan/procedure and experimental results.

3.1 Materials and Specimen

The polymer foam material which is utilized in this research is a closed-cell cross-linked Divinycell PVC H100 foam with an average cell size 400μm. It is manufactured by DIAB. The mechanical properties of Divinycell H grade series provided by DIAB are listed in Table 3.1[32]. The data in this table is treated as average values because DIAB has specified that PVC H100 foam can have a ±10% variation in properties. The nominal value is an average of the mechanical property, while the minimum value is a minimum guaranteed mechanical property. It is expected that there will be a small difference in the properties given in Table 3.1 and those measured in the laboratory.
The Divinycell PVC H100 foam materials were purchased from Aircraft Spruce & Specialty Company in 305mm × 254mm × 25.4mm sheets. A Bridgeport mill machine was used to cut the specimens into 25.4mm × 25.4mm × 25.4mm cubes using a slitting saw. The material directions of the PVC H100 foam sheet are shown in Figure 3.1, where the 3-direction is the out-of-plane direction and 1- and 2-directions the in-plane directions. In most industrial applications, loads are applied to the PVC H100 foam in out-of-plane directions. Therefore, loads in this study were subjected to the Divinycell PVC H100 specimens in out-of-plane direction. For the experiments, a typical cubic specimen is shown in right lower corner of the sheet in Figure 3.1. Out-of-plane compression-shear loading is applied to this specimen in the manner shown in this figure.

### Table 3.1 Mechanical properties Divinycell® H Series [32]

<table>
<thead>
<tr>
<th>Property</th>
<th>Test Procedure</th>
<th>Unit</th>
<th>H35</th>
<th>H45</th>
<th>H60</th>
<th>H80</th>
<th>H100</th>
<th>H130</th>
<th>H200</th>
<th>H250</th>
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<tbody>
<tr>
<td>Compressive Stress(^1)</td>
<td>ASTM D 1621</td>
<td>MPa</td>
<td>Nominal 0.5</td>
<td>0.6</td>
<td>0.8</td>
<td>1.4</td>
<td>2.0</td>
<td>3.0</td>
<td>5.4</td>
<td>7.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Minimum 0.3</td>
<td>0.5</td>
<td>0.7</td>
<td>1.15</td>
<td>1.85</td>
<td>2.4</td>
<td>4.5</td>
<td>6.1</td>
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<td>ASTM D1621-B-73</td>
<td>MPa</td>
<td>Nominal 46</td>
<td>50</td>
<td>70</td>
<td>90</td>
<td>130</td>
<td>170</td>
<td>310</td>
<td>400</td>
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<td></td>
<td></td>
<td></td>
<td>Minimum 29</td>
<td>35</td>
<td>45</td>
<td>60</td>
<td>115</td>
<td>145</td>
<td>200</td>
<td>300</td>
</tr>
<tr>
<td>Tensile Strength(^1)</td>
<td>ASTM D 1623</td>
<td>MPa</td>
<td>Nominal 1.0</td>
<td>1.4</td>
<td>1.8</td>
<td>2.2</td>
<td>3.5</td>
<td>4.8</td>
<td>7.1</td>
<td>9.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Minimum 0.8</td>
<td>1.1</td>
<td>1.5</td>
<td>2.2</td>
<td>2.5</td>
<td>3.5</td>
<td>6.3</td>
<td>8.0</td>
</tr>
<tr>
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<td>75</td>
<td>95</td>
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<td>176</td>
<td>250</td>
<td>320</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>Minimum 37</td>
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<td>57</td>
<td>85</td>
<td>105</td>
<td>135</td>
<td>210</td>
<td>280</td>
</tr>
<tr>
<td>Shear Strength</td>
<td>ASTM C 273</td>
<td>MPa</td>
<td>Nominal 0.4</td>
<td>0.55</td>
<td>0.75</td>
<td>1.15</td>
<td>1.6</td>
<td>2.2</td>
<td>3.5</td>
<td>4.5</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Minimum 0.3</td>
<td>0.45</td>
<td>0.63</td>
<td>0.95</td>
<td>1.4</td>
<td>1.9</td>
<td>3.2</td>
<td>3.9</td>
</tr>
<tr>
<td>Shear Modulus</td>
<td>ASTM C 273</td>
<td>MPa</td>
<td>Nominal 12</td>
<td>15</td>
<td>20</td>
<td>27</td>
<td>35</td>
<td>50</td>
<td>73</td>
<td>97</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Minimum 0</td>
<td>12</td>
<td>16</td>
<td>23</td>
<td>28</td>
<td>40</td>
<td>65</td>
<td>81</td>
</tr>
<tr>
<td>Shear Strain</td>
<td>ASTM C 273</td>
<td>%</td>
<td>Nominal 9</td>
<td>12</td>
<td>20</td>
<td>30</td>
<td>40</td>
<td>40</td>
<td>45</td>
<td>45</td>
</tr>
<tr>
<td>Density</td>
<td>ISO 345</td>
<td>kg/m³</td>
<td>Nominal 38</td>
<td>48</td>
<td>60</td>
<td>80</td>
<td>100</td>
<td>130</td>
<td>200</td>
<td>250</td>
</tr>
</tbody>
</table>

\(^1\) Perpendicular to the plane. All values measured at +23°C
3.2 Apparatus Design and Test Setup

Figure 3.2 is a schematic of the compression-shear test fixture for PVC H100 foam, showing the force and displacement acting on the specimen. Each specimen experiences a compression force of $F \cos \theta/2$ and a shear force of $F \sin \theta/2$. By adjusting the angle $\theta$, different combination of compression and shear could be realized. Fixtures were made for cyclic compression-shear loading with angles $\theta = 15^\circ$, $\theta = 30^\circ$, $\theta = 45^\circ$, $\theta = 60^\circ$, and $\theta = 75^\circ$. Figures 3.3 and 3.4 (a) – (d) are photographs for the actual compression-shear fixtures at different angles. The upper and lower holders, platen, and support cylinder were made out of 1018 Steel. The Divinycell H100 foam was glued onto the holders so that it could carry both tension and compression loads during cyclic compression-shear tests. Permanent strains in the foam after it crushes would induce residual tensile forces if the foam returned back to its original position at the start of the experiment. The lower platen was connected to the lower holder by stainless steel socket head cap screws. This setup is useful for easily getting specimens in and out of the MTS 831 servo-hydraulic machine.
The support cylinder below the lower platen prevents bending during the cyclic compression-shear loading. A pair of “C” clamp was used to connect the upper holder of the test piece to the actuator arm of the MTS machine. The “C” clamp was tightened by screws orientated horizontally.

![Diagram of compression-shear test fixture for PVC H100 foam.](image)

Figure 3.2 Schematic of compression-shear test fixture for PVC H100 foam.

As mentioned earlier, the specimen was glued onto surfaces of upper and lower holders during cyclic compression-shear loading. Loctite® Epoxy Heavy Duty, which is a two-part adhesive of an epoxy resin and a hardener [33], was used to glue the Divinycell PVC H100 specimens. First, Scotch Magic Tape was put on the surrounding in-plane sides of the specimen to keep them clear of the overflowing glue. Second, the surface of lower holder was cleaned by sandpaper and then wiped with Acetone. The surface had to be clean, dry and free from oil or grease before bonding. Also, a roughened surface is better for
adhesion. After that, the resin and the hardener were mixed together in a paper cup with a wooden stick in equal volumes a 1:1 ratio for one minute, and the glue was used to connect the specimen to the lower holder by leaving the mixture to cure for about 24 hours at room temperature. The specimen and lower holder were then placed in the MTS machine. Next, the upper holder surface was cleaned as explained earlier and it was glued to the other side of the specimen in the MTS machine to avoid misalignment. The Scotch Magic Tape was used to create a channel between the upper holder and foam specimens so that glue could be placed between the surfaces without sliding down the holder and specimen. Extra glue was used to fill out gaps between the surfaces of the foam and upper holder. The specimens were kept in the MTS machine for at least 25 minutes, which was enough time for the epoxy to harden or set. After the epoxy hardened, the specimen was taken out of the MTS machine, and allowed to cure for another 24 hours.

Each side of the glued specimen with various angles which are $\theta = 15^\circ$, $\theta = 30^\circ$, $\theta = 45^\circ$, $\theta = 60^\circ$, and $\theta = 75^\circ$ were fully cured for 24 hours at room temperature. Preparation of each specimen therefore took roughly two and a half days to be prepared. The Scotch Magic Tape was eventually taken off from the PVC H100 foams before each test. The final specimens ready for cyclic compression-shear testing are shown in Figures 3.3, 3.4 (a) – (d). After testing, the specimens were broken using displacement control of the MTS machine. A razor blade was used to remove the foam specimens from the surfaces for lower and upper holders.
Figure 3.3 Setup of cyclic compression-shear test for PVC H100 foam at $\theta = 15^\circ$. 
Figure 3.4 Setup of cyclic compression-shear test for PVC H100 foam: (a) $\theta = 30^\circ$, (b) $\theta = 45^\circ$, (c) $\theta = 60^\circ$, and (d) $\theta = 75^\circ$. 
3.3 Test Plan / Procedure

Ten displacement-controlled compression-shear cycles were applied to Divinycell PVC H100 foam in the out-of-plane direction under fixed displacement rate and displacement amplitude. The displacement control is shown in Figure 3.5, where $\delta$ is the time-varying displacement. The waveform chosen was a sawtooth shape. In Figure 3.5, the displacement amplitude is $\Delta$ and the period of one cycle is $T$. The displacement rate is $\dot{\Delta} = 2\Delta/T$. For each specimen angle the components of compression and shear amplitude are given by $\delta_c = \delta \cos \theta$ and $\delta_s = \delta \sin \theta$, respectively.

![Displacement histories at constant displacement rate.](image)

The compressive stress $\sigma_c$ and shear stress $\tau_s$ for various angles $\theta$ were calculated by

$$\sigma_c = \frac{F_c}{2A} = \frac{F \cos \theta}{2A}$$

and

$$\tau_s = \frac{F_s}{2A} = \frac{F \sin \theta}{2A}$$
where $F_c$ is compression force, $F_s$ is shear force, $F$ is the total reaction force and $A$ is cross-sectional area of cubic PVC H100 foam. To calculate compressive strain $\varepsilon_c$ and shear strain $\gamma_s$ at different angles $\theta$ the following equations are used:

$$\varepsilon_c = \frac{\delta_c}{H} = \frac{\delta \cos \theta}{H}$$  \hspace{1cm} (3.3)$$

and

$$\gamma_s = \frac{\delta_s}{H} = \frac{\delta \sin \theta}{H}$$  \hspace{1cm} (3.4)$$

where $\delta$ is the time-varying displacement and $H$ is the height of the specimen.

It should be emphasized that the stresses and strains calculated by Equations (3.1) – (3.4) are average values. The cubic specimen does not have uniform stresses or strains.
3.4 Limitation on Maximum Displacement

The maximum displacement imposed by the MTS actuator depends on the angles because the PVC H100 foam has a propensity to break under transverse shear or higher values of $\theta$. Table 3.2 shows maximum displacement allowable for each angle. Also, Figures 3.6 to 3.10 show fractures at edges in PVC H100 foam specimens after exceed maximum displacement for different angles.

Table 3.2 Maximum compression-shear displacements for different angles.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>15°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>75°</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta_{max}$ (mm)</td>
<td>12.7</td>
<td>10.16</td>
<td>5.08</td>
<td>2.54</td>
<td>1.905</td>
</tr>
</tbody>
</table>

Figure 3.6 Fracture of PVC H100 foam specimens at $\Delta = -2.54$ mm for $\theta = 75^\circ$. 
Figure 3.7 Fracture of PVC H100 foam specimens at Δ = - 3.81 mm for θ = 60°.

Figure 3.8 Fracture of PVC H100 foam specimens at Δ = - 7.62 mm for θ = 45°.
Figure 3.9 Fracture of PVC H100 foam specimens at $\Delta = -12.7$ mm for $\theta = 30^\circ$.

Figure 3.10 Fracture of PVC H100 foam specimens at $\Delta = -15.24$ mm for $\theta = 15^\circ$. 
3.5 Cyclic Compression-Shear Test Results

The following sections summarize test results with varying displacement amplitudes and rates.

3.5.1 Varying displacement amplitudes

Stress-strain curves at constant displacement rates (DR) and varying displacement amplitudes for $\theta = 15^\circ$, $\theta = 30^\circ$, $\theta = 45^\circ$, $\theta = 60^\circ$, $\theta = 75^\circ$ with ten loading-unloading cycles were found.

For $\theta = 15^\circ$, compressive stress - strain and shear stress - strain at displacement rates 0.015 mm/s are shown in Figures 3.11 (a) and (b), respectively. The results corresponding to $\Delta = 2.54, 5.08, 7.62, 10.16, $ to 12.7mm were obtained from different specimens. The cyclic loading shows loading-unloading cycles for both combinations of compression-shear stresses. The values in compression are greater than values in shear because of the small angle.

For $\theta = 30^\circ$ and displacement rates of 0.15 mm/s, several results from different specimens with $\Delta = 1.27, 2.54, 5.08, 7.62, $ and 10.16mm are shown in Figures 3.12(a) and (b), respectively. Figure 3.12 (a) shows compression stress - strain, while Figure 3.12 (b) shows shear stress - strain. At this angle, the stresses in compression are still larger than the values in shear.

For $\theta = 45^\circ$ and displacement rates at 15 mm/s, results from specimens with $\Delta = 2.54, 3.556, 4.064, 4.572, $ and 5.08mm are shown in Figures 3.13 (a) and (b). These
diagrams also exhibited the compression stress - strain and shear stress - strain, but now the stresses in compression and shear are about the same. The behaviors of the PVC H100 foams under cyclic compression and shear loading at the angle of 45 degrees are equal.

For $\theta = 60^\circ$ and an displacement rates of 0.15 mm/s, results from various specimens are shown in Figure 3.14 (a) for compression and Figure 3.14 (b) for shear with $\Delta = 1.016, 1.524, 2.032,$ and 2.54mm. The stresses in compression are now less than values in shear.

For $\theta = 75^\circ$ and an displacement rates of 0.015 mm/s, results from different specimens with $\Delta = 1.143, 1.397, 1.651,$ and 1.905mm are shown in Figure 3.15 (a) for compression and 3.15 (b) for shear. In this case because of the high angle, the shear stresses are greater than the values in compression. Appendix C gives more details about varying displacement amplitudes for the different angles.
Figure 3.11 Stress-strain curves with various displacements control for θ = 15° at DR = 0.015 mm/s: (a) Compression and (b) Shear.
Figure 3.12 Stress-strain curves with various displacements control for \( \theta = 30^\circ \) at DR = 0.15 mm/s: (a) Compression and (b) Shear.
Figure 3.13 Stress-strain curves with various displacements control for $\theta = 45^\circ$ at DR = 15 mm/s: (a) Compression and (b) Shear.
Figure 3.14 Stress-strain curves with various displacements control for $\theta = 60^\circ$ at DR = 0.15 mm/s: (a) Compression and (b) Shear.
Figure 3.15 Stress-strain curves with various displacements control for \( \Theta = 75^\circ \) at DR = 0.015 mm/s: (a) Compression and (b) Shear.
On a whole, the stress-strain curves of the combination compression-shear loadings exhibit elastic, plastic, and hysteretic responses. Note that the plastic flow stress where hardening occurred decreases when the angle is increased. The dissipation energy or hysteresis during cyclic loading-unloading increases when the displacement increases at a constant displacement rate. In addition to this, the first cycle of the total ten cycles is always higher than the following cycles, and this indicates mechanical degradation after the first cycle. For instance, Figure 3.16 (a) shows compression stress-strain curves for the 1st, 5th, and 10th cycles, while Figure 3.16 (b) shows shear stress-strain curves for the same 1st, 5th, and 10th cycles for specimen with $\theta = 15^\circ$ at a displacement rate 0.015 mm/s and maximum displacement of 12.7mm. These graphs show a large difference between the 1st and 5th cycles but little difference between 5th and 10th cycles. The specimens with other angles exhibit similar patterns with respect to the 1st, 5th, and 10th cycles. The material reaches a steady state hysteresis by the 5th cycle.
Figure 3.16 Stress-strain curves for various cycles loading-unloading for $\theta = 15^\circ$ at DR = 0.015 mm/s and displacement = 12.7 mm: (a) Compression and (b) Shear.
3.5.2 Varying displacement rates

The cyclic compression-shear test results of the Divinycell PVC H100 foams were obtained for displacement rates of 0.015, 0.15, 1.5, 15, and 150 mm/s and fixed displacements of $\Delta = 10.16, 7.62, 3.556, 2.54, 1.905\text{mm}$ at angles of $\theta = 15^\circ, \theta = 30^\circ, \theta = 45^\circ, \theta = 60^\circ, \theta = 75^\circ$, respectively. The cyclic compression-shear stress-strain responses for different displacement rates are shown in Figure 3.17 to Figure 3.21 at selected displacements. More details of results showing the effect of displacement rates with other displacements are offered in Appendix D. The behavior of PVC H100 foam displays viscoelasticity then viscoplasticity with flow stress before unloading to a tensile of stress. At zero stress (no loading), PVC has permanent compressive and shear strain. Curiously these permanent strains have very little rate effect. Viscoelasticity is said to occur before foam yielding because the modulus increases with increasing strain rate. Viscoplasticity occurs after yielding because the yield strengths and flow stresses increase with increasing strain rate.
Figure 3.17 Stress-strain curves with various displacement rates for $\theta = 15^\circ$ at compression displacement = 10.16 mm: (a) Compression and (b) Shear.
Figure 3.18 Stress-strain curves with various displacement rates for $\theta = 30^\circ$ at compression displacement = 7.62 mm: (a) Compression and (b) Shear.
Figure 3.19 Stress-strain curves with various displacement rates for $\theta = 45^\circ$ at compression displacement = 3.556 mm: (a) Compression and (b) Shear.
Figure 3.20 Stress-strain curves with various displacement rates for $\theta = 60^\circ$ at compression displacement $= 2.54$ mm: (a) Compression and (b) Shear.
Figure 3.21 Stress-strain curves with various displacement rates for $\theta = 75^\circ$ at compression displacement = 1.905 mm: (a) Compression and (b) Shear.
3.5.3 Mullins effect

The Divinycell PVC H100 foams were tested in the MTS machine for the Mullins effect. The Mullins effect describes progressive damage that is dependent on the range of deformation or strain. Figure 3.22 shows displacements history at displacement rate 1.5 mm/s for $\theta = 45^\circ$. A variety of maximum displacements are applied to the specimen in order to explain Mullins effect: 2.54, 3.556, 4.064, and 4.572mm. The resulting stress-strain curves are shown in Figures 3.23 (a) and (b). The monotonic stress-strain curves in compression and shear (dashed line) for the same $\theta = 45^\circ$ and 1.5 mm/s displacement rate are superimposed with cyclic stress-strain responses in compression and shear (solid line). The monotonic compression and shear dashed line is called the primary curve. After the first loading/unloading cycle where the displacement is 2.54mm, the foam becomes softer. Then, this phenomenon repeats itself for the all displacements. The compression stress-strain response and shear stress-strain response are followed the primary curve as shown in Figures 3.23 (a) and (b), respectively.
Figure 3.22 Mullins effect for different displacements versus time for $\theta = 45^\circ$ at DR = 1.5 mm/s.
Figure 3.23 Mullins effect stress-strain curves in various displacements with primary curve monotonic test for $\theta = 45^\circ$ at DR = 1.5 mm/s: (a) Compression and (b) Shear.
In addition, a second experiment was done on a new Divinycell PVC H100 foam specimen with a different displacement history and displacement rate of 1.5 mm/s for $\theta = 45^\circ$. The displacement history is shown in Figure 3.24. The compression stress-strain curves (solid line) with superimposed primary curve (dashed line) are displayed in Figure 3.25 (a), while shear stress-strain curves (solid line) with superimposed primary curve (dashed line) are shown in Figure 3.25 (b). The first two cycles with displacements 3.556mm show elastic-plastic and hysteresis behavior of the PVC H100 foam, so that the foam softens. The stress-strain curve response follows the primary curve. The next two cycles with displacements 4.064mm show that the response after 3.556mm follows the primary curve. The 5th and 6th cycles at 3.556mm compare not with the first displacements at 3.556mm but with displacement at 4.064mm because more damage of cells occur when displacement exceeded 3.556mm to the 3rd and 4th cycle. In other words, permanent and specific damage occurred at 4.064mm even though the 5th and 6th cycles ended at 3.556mm, the material response is characterized by damage associated with 4.064mm.

Next displacements at 4.064mm are used as end points of the 7th and 8th cycles. No more damage of cells occurred after 4.064mm. Only the 9th and 10th cycles at 4.572mm show additional damage following the primary curve and then reloading/unloading to produce hysteresis. Subsequent cycles with displacements at 4.064mm give the same results as in cycles 5th and 6th. Finally the last two cycles with displacements at 4.572mm return to the primary curve. The conclusion from this second set of experiment is that damage is unique to particular strain amplitude and permanent. It is this damage that will be used to construct a constitutive model for hysteresis of PVC H100 foam.
Figure 3.24 Mullins effect for different displacements versus time for $\theta = 45^\circ$ at DR = 1.5 mm/s.
Figure 3.25 Mullins effect stress-strain curves in various displacements with primary curve monotonic test for $\theta = 45^\circ$ at DR = 1.5 mm/s: (a) Compression and (b) Shear.
3.6 Experiments for Viscoelastic Damage

Reversed yielding during hysteresis is observed when the unloading displacement amplitude was large, such as in the $\theta = 15^\circ$, $30^\circ$, $45^\circ$ and some of the $\theta = 60^\circ$ tests. A diagram explaining this is shown below in Figures 3.26 (a). Energy dissipation is due to both viscoelasticity and viscoplasticity in this case. When the unloading displacement amplitude is small, such as in the $\theta = 75^\circ$ and some of the $\theta = 60^\circ$ tests, hysteresis is primarily viscoelastic. Figure 3.26 (b) shows this. Note the different shapes of hysteresis produced under large and small unloading amplitudes.

The constitutive model presented herein will be limited to viscoelastic hysteresis after initial yielding and damage. Additional tests were done to capture the viscoelastic hysteresis response under combined compression and shear. These tests were only performed at the lowest displacement rate, 0.015 mm/s, for each angle and results from these tests are given in Figure 3.27 to Figure 3.36.

Figure 3.27 shows the displacement control for the $\theta = 15^\circ$ specimen. The first set of cycles were performed to a maximum displacement of 2.54 mm, which was enough to yield and damage the foam. Unloading displacement amplitudes were then imparted to the specimen in a range as follows:

a. from 2.54 mm to 2.159 mm

b. from 2.54 mm to 1.905 mm
c. from 2.54 mm to 1.397 mm

d. from 2.54 mm to 1.016 mm

The first few amplitudes were small enough to ensure that some hysteresis would be viscoelastic, but eventually reversed yielding could occur and the hysteresis would be a combination of viscoelasticity and viscoplasticity. This process was repeated for a higher maximum displacement and sequence of unloading displacement amplitudes. Each of these new displacement amplitudes exhibited similar hysteresis response curves, i.e., viscoelastic hysteresis at low displacement amplitudes and eventually viscoelastic-viscoplastic hysteresis as the displacement amplitude increased. It was observed that a viscoelastic-viscoplastic hysteresis started to occur approximately when the unloading curve was near zero or when the foam was in an unloaded state with permanent plastic deformation and damage.

In these experiments, we took advantage of the fact that Mullins damage would ensure that new damage would occur beyond the last maximum displacement so that data between the last maximum displacement and the new maximum displacement would be the same had we used new specimens to perform each sequence of loading-unloading cycles. Keep in mind that preparation of each specimen takes a few days so that it would have taken substantially more time to produce the same result.
Figure 3.26 Behavior of stress-strain hysteresis curves of PVC H100 foam to show reverse yielding at Displacement rate = 0.015 mm/s: (a) Theta = 15° and (b) Theta = 75°.
Figure 3.27 Viscoelastic damage for different displacements versus time for $\theta = 15^\circ$
at DR = 0.015 mm/s.
Figure 3.28 Viscoelastic damage stress-strain curves in various displacements for $\theta = 15^\circ$ at DR = 0.015 mm/s: (a) Compression and (b) Shear.
Figure 3.29 Viscoelastic damage for different displacements versus time for $\theta = 30^\circ$ at $DR = 0.015$ mm/s.
Figure 3.30 Viscoelastic damage stress-strain curves in various displacements for $\theta = 30^\circ$ at DR = 0.015 mm/s: (a) Compression and (b) Shear.
Figure 3.31 Viscoelastic damage for different displacements versus time for $\theta = 45^\circ$ at DR = 0.015 mm/s.
Figure 3.32 Viscoelastic damage stress-strain curves in various displacements for $\theta = 45^\circ$ at DR = 0.015 mm/s: (b) Compression and (c) Shear.
Figure 3.33 Viscoelastic damage for different displacements versus time for $\theta = 15^\circ$ at
DR = 0.015 mm/s.
Figure 3.34 Viscoelastic damage stress-strain curves in various displacements for $\theta = 60^\circ$ at $DR = 0.015$ mm/s: (a) Compression and (b) Shear.
Figure 3.35 Viscoelastic damage for different displacements versus time for $\theta = 75^\circ$ at
DR = 0.015 mm/s.
Figure 3.36 Viscoelastic damage stress-strain curves in various displacements for θ = 75° at DR = 0.015 mm/s: (a) Compression and (b) Shear.
CHAPTER IV

CONSTITUTIVE MODELING

This chapter involves the development of a constitutive model that can be used to predict response for Divinycell PVC H100 foam under cyclic compression and shear. The stress-strain curves for Divinycell PVC H100 foam that were found from combined cyclic compression-shear tests as well as those under uniaxial compression and simple shear in Chen [3] are used to develop this constitutive model.

4.1 Phenomenological Model

At a given deformation rate (DR), the stress-strain behavior of PVC H100 foam has a combination of elasticity, plasticity and hysteresis, as displayed for the $\theta = 45^\circ$ test in Figures 4.1(a) and (b). Rate-dependent behavior was observed so that the elastic is really viscoelastic and the plastic is really viscoplastic. In addition to this, damage and plasticity occurs simultaneously, and damage was of a Mullins type in that damage increased with increasing displacement amplitude. Figures 4.2 (a) and (b) summarize observed material behavior in all the experiments. Elastic, plastic and damage phenomena are explained separately in Figure 4.2(a), where the damage is expressed by reduced moduli, $\bar{E}_{01}$ and $\bar{E}_{02}$. A lower modulus signifies more damage, and
$E_{02} < E_{01}$ because of higher deformations or strains. Modulus and flow stress increase with increasing strain rate as shown in Figure 4.2(b).

Figure 4.1 Stress-strain responses of PVC H100 foam at compression displacement = 2.54 mm and DR = 0.015 mm/s for $\theta = 45^\circ$: (a) Compression and (b) Shear.
This is typical behavior of a viscoelastic and viscoplastic material. Unloading and reloading produces hysteresis, which could be either due to viscoelasticity or viscoplasticity if the foam is allowed to undergo reversed yielding.

Figure 4.2 Stress-strain behavior of PVC H100 foam: (a) Elastic-plastic, damage response and (b) Viscoelastic, viscoplastic, hysteresis response.
Figure 4.3 Mechanical analogs: (a) Before yielding/damage and (b) After yielding/damage.
A material model that can be used to predict the observed phenomena is shown in Figures 4.3 (a) and (b). A yield or damage initiation criterion separates the two distinct mechanical analogs. Before initial yielding and damage, the standard linear viscoelastic model in Figure 4.3(a) is used to predict rate-dependent behavior. This consists of an equilibrium linear spring with modulus $E$ in parallel with a Maxwell element, a linear viscous damper with viscosity $\eta$ in series with an intermediate linear spring with modulus $E_v$.

Yielding and plastic flow ensues after the initial yield criterion is met and the loading is able to support continued plastic flow. The plastic flow stress is determined by an appropriate plasticity or viscoplasticity law. A viscoplastic element is added in series to the standard linear viscoelastic element in Figure 4.3(b). Here the yield stress is given by $\sigma_y$, and the viscous overstress is determined by plastic viscosity coefficient $\mu$. Unlike metal plasticity, damage occurs simultaneously with plasticity in foams because their elastic properties are determined from a microstructure that is permanently changed after yielding. The equilibrium and intermediate spring moduli and damper viscosity are shown with overhead bars ($E, E_v, \eta$) to exemplify that they are properties which are dependent upon damage.

The following sections describe constitutive equations to predict viscoelastic, viscoplastic, damage and hysteresis of PVC H100 foam. The derivations of these equations are separated into three main sections: behavior before yielding/damage, yielding and
damage initiation and behavior after yielding/damage. Parameter identification, i.e., material property selection from tests, is done under uniaxial stress or compression and shear behavior of the foam. Transversely isotropic conditions are used to develop the three-dimensional constitutive equations, which are based only on compression and shear behavior of the foam. Foam tensile properties, although important, were not measured in this research. Tensile yield strength properties are taken from the literature and the viscoelastic, viscoplastic and damage behavior in tension is assumed to be the same as those observed in compression.

4.2 Behavior before Yielding/Damage Initiation

Uniaxial compression tests in the 3- and 1-directions of the foam as well simple shear tests in the 1-3 and 1-2 plane from Chen [3] were used to obtain directional moduli and viscosity. The test results are summarized in Figures 4.4 to 4.7. Equations that were used to do this are derived next.
Figure 4.4 Uniaxial compression test in the out-of-plane direction of PVC H100 foam.

Figure 4.5 Uniaxial compression test in the in-plane direction of PVC H100 foam.
Figure 4.6 Simple shear test in the out-of-plane direction of PVC H100 foam.

Figure 4.7 Simple shear test in the in-plane direction of PVC H100 foam.
4.2.1 One-dimensional model

The mechanical analog for the foam before yielding and damage initiation is the standard viscoelastic model as shown in Figure 4.3(a). The total stress before yielding / damage response is given by

\[ \sigma = \sigma_{eq} + \sigma_{ov} \] (4.1)

where the equilibrium stress \( \sigma_{eq} \) is

\[ \sigma_{eq} = E_0 \varepsilon \] (4.2)

and the overstress \( \sigma_{ov} \) is

\[ \sigma_{ov} = E_{ev} \varepsilon_{ev} \] (4.3)

Compatibly of strains requires

\[ \varepsilon = \varepsilon_v + \varepsilon_{ev} \] (4.4)

Upon substitution of Equation (4.4) into Equation (4.3), one gets

\[ \sigma_{ov} = E_{ev} (\varepsilon - \varepsilon_v) \] (4.5)

For a linear viscous damper

\[ \sigma_{ov} = \eta \dot{\varepsilon}_v \] (4.6)

Combining Equations (4.5) and (4.6) gives

\[ \eta \dot{\varepsilon}_v = E_{ev} (\varepsilon - \varepsilon_v) \] (4.7)

From Equation (4.7), the evolution equation for the viscous strains may be determined by

\[ \dot{\varepsilon}_v = \frac{E_{ev}}{\eta} (\varepsilon - \varepsilon_v) \] (4.8)

Note that \( \frac{\eta}{E_{ev}} \) is the characteristic relaxation time. Assume \( \varepsilon = \alpha t \), where \( \alpha \) is a constant
strain rate during loading. Since $\dot{\varepsilon}_v = \alpha d\varepsilon_v/d\varepsilon$, Equation (4.8) may be re-written in terms of strains only:

$$\frac{d\varepsilon_v}{d\varepsilon} = \frac{E_{ev}}{\eta} (\varepsilon - \varepsilon_v)$$

(4.9)

With initial condition $\varepsilon_v = 0$ when $\varepsilon = 0$, Equation (4.14) may be solved explicitly to give

$$\varepsilon_v = \frac{\alpha \eta}{E_{ev}} \left[ e^{\left(\frac{-E_{ev}\varepsilon}{\alpha \eta}\right)} - 1\right] + \varepsilon$$

(4.10)

The above expression is substituted into Equation (4.5) to explicitly solve for the overstress in terms of applied strain:

$$\sigma_{ov} = \alpha \eta \left[ 1 - e^{\left(\frac{-E_{ev}\varepsilon}{\alpha \eta}\right)} \right]$$

(4.11)

Finally, the total stress in uniaxial compression is given by

$$\sigma = E_0 \varepsilon + \alpha \eta \left[ 1 - e^{\left(\frac{-E_{ev}\varepsilon}{\alpha \eta}\right)} \right]$$

(4.12)

The same procedure is used for simple shear. The shear stress-strain governing viscoelastic behavior is

$$\tau = G_0 \gamma + \alpha \eta \left[ 1 - e^{\left(\frac{-G_{ev}\gamma}{\alpha \eta}\right)} \right]$$

(4.13)

Equations (4.12) and (4.13) were used to curve fit rate-dependent curves in out-of-plane compression ($\sigma_{33} - \varepsilon_{33}$), in-plane compression ($\sigma_{11} - \varepsilon_{11}$), out-of-plane shear ($\tau_{13} - \gamma_{13}$), and in-plane shear ($\tau_{12} - \gamma_{12}$). A nonlinear regression software program, OriginLab Pro 8 [34], was used to curve-fit Equations (4.12) and (4.13) to the experimental
data. Tables 4.1, 4.2 and 4.3 summarize values for directional modulus and viscosity. The Poisson’s ratios were obtained from Ref. [3], and they are assumed to be the same for all the springs and even the damper.

<table>
<thead>
<tr>
<th>Table 4.1 Properties of Equilibrium Spring.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{11} = E_{22}$ (MPa)</td>
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<tr>
<td>46</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>Table 4.2 Properties of Intermediate Spring.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{ev11} = E_{ev22}$ (MPa)</td>
</tr>
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<td>10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 4.3 Properties of Damper.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta_{11} = \eta_{22}$ (MPa.s)</td>
</tr>
<tr>
<td>1.46</td>
</tr>
</tbody>
</table>

Figures (4.8)-(4.11) show the predicted response using the elastic and viscous properties. Rate-dependent behavior is observed and the results compare reasonably well with test results in Figures (4.4)-(4.7).
Figure 4.8 Predicted uniaxial compression test in the out-of-plane direction of PVC H100 foam.

Figure 4.9 Predicted uniaxial compression test in the in-plane direction of PVC H100 foam.
Figure 4.10 Predicted simple shear test in the out-of-plane direction of PVC H100 foam.

Figure 4.11 Predicted simple shear test in the in-plane direction of PVC H100 foam.
4.2.2 General three-dimensional model

The total stress is given by

$$\sigma = \sigma_{eq} + \sigma_{ov}$$  \hspace{1cm} (4.14)

where bold face is used to denote a tensor quantity. The equilibrium stress $\sigma_{eq}$ is

$$\sigma_{eq} = C_0 \omega$$  \hspace{1cm} (4.15)

where $C_0$ is the stiffness matrix of the equilibrium spring and the equilibrium stress and total strain are given, respectively, in Voigt notation as

$$\sigma = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \tau_{23}, \tau_{13}, \tau_{12}]^T$$ and $$\omega = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \gamma_{23}, \gamma_{13}, \gamma_{12}]^T.$$  

The stiffness matrix of the equilibrium spring is given by

$$C_0 = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66} \\
\end{bmatrix}$$  \hspace{1cm} (4.16)

where

$$C_{11} = \frac{(E_{22} - \nu_{23} E_{33}) E_{11}^2}{\Omega}$$  \hspace{1cm} $$C_{12} = \frac{(\nu_{12} E_{22} + \nu_{13} \nu_{23} E_{33}) E_{11} E_{22}}{\Omega}$$

$$C_{13} = \frac{(\nu_{12} E_{11} + \nu_{13} \nu_{12} E_{22}) E_{22} E_{33}}{\Omega}$$

$$C_{23} = \frac{(\nu_{23} E_{11} + \nu_{13} \nu_{23} E_{22}) E_{22} E_{33}}{\Omega}$$

$$C_{44} = G_{23}$$  \hspace{1cm} $$C_{55} = G_{13}$$  \hspace{1cm} $$C_{66} = G_{12}$$

$$C_{12} = \frac{(E_{11} - \nu_{13} E_{33}) E_{22}^2}{\Omega}$$  \hspace{1cm} $$C_{22} = \frac{(E_{11} - \nu_{13} E_{33}) E_{22}^2}{\Omega}$$
and \( \Omega = E_{11}E_{22} - v_{12}^2E_{22}^2 - v_{13}^2E_{22}E_{33} - v_{23}^2E_{11}E_{33} - 2v_{12}v_{13}v_{23}E_{22}E_{33} \).

Similarly, the overstress \( \Psi_{ov} \) is
\[
\Psi_{ov} = C_{ev} \psi_{ov} \tag{4.17}
\]
where \( C_{ev} \) is the stiffness matrix of the intermediate spring and the overstress and intermediate elastic strain are given, respectively, as
\[
\Psi_{ov} = [\sigma_{ov11} \sigma_{ov22} \sigma_{ov33} \tau_{ov23} \tau_{ov13} \tau_{ov12}]^T \quad \text{and} \quad
\psi_{ov} = [\epsilon_{ev11} \epsilon_{ev22} \epsilon_{ev33} \gamma_{ev23} \gamma_{ev13} \gamma_{ev12}]^T.
\]
The stiffness matrix of the intermediate spring is given by
\[
C_{ev} = \begin{bmatrix}
C_{ev11} & C_{ev12} & C_{ev13} & 0 & 0 & 0 \\
C_{ev12} & C_{ev22} & C_{ev23} & 0 & 0 & 0 \\
C_{ev13} & C_{ev23} & C_{ev33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{ev44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{ev55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{ev66}
\end{bmatrix} \tag{4.18}
\]
where
\[
C_{ev11} = \frac{(E_{ev22} - v_{23}^2E_{ev33})E_{ev11}}{\Phi} \quad \quad C_{ev12} = \frac{(v_{12}E_{ev22} + v_{13}v_{23}E_{ev33})E_{ev11}E_{ev22}}{\Phi}
\]
\[
C_{ev13} = \frac{(v_{12}v_{23} + v_{13})E_{ev11}E_{ev22}E_{ev33}}{\Phi} \quad \quad C_{ev22} = \frac{(E_{ev11} - v_{23}^2E_{ev33})E_{ev22}}{\Phi}
\]
\[
C_{ev23} = \frac{(v_{23}E_{ev11} + v_{12}v_{13})E_{ev22}E_{ev33}}{\Phi} \quad \quad C_{ev33} = \frac{(E_{ev11} - v_{12}^2E_{ev22})E_{ev22}E_{ev33}}{\Phi}
\]
\[
C_{ev44} = G_{ev23} \quad \quad C_{ev55} = G_{ev13} \quad \quad C_{ev66} = G_{ev12}
\]
and \( \Phi = E_{ev11}E_{ev22} - v_{12}^2E_{ev22} - v_{13}^2E_{ev22}E_{ev33} - v_{23}^2E_{ev11}E_{ev33} - 2v_{12}v_{13}v_{23}E_{ev22}E_{ev33} \).
Compatibility of strain requires that

\[ \omega = \varepsilon_v + \psi \]  

(4.19)

where \( \psi = [\varepsilon_{v11} \quad \varepsilon_{v22} \quad \varepsilon_{v33} \quad \gamma_{v23} \quad \gamma_{v13} \quad \gamma_{v12}]^T \) are viscous strains. Substituting Equation (4.19) into (4.17), one gets

\[ \sigma_{ov} = C_{ev}(\omega - \psi) \]  

(4.20)

The overstress is also governed by a linear viscosity law:

\[ \sigma_{ov} = \mathbf{V} \psi \]  

(4.21)

where the dot denotes time derivative and \( \mathbf{V} \) is the viscosity matrix. The viscosity matrix is

\[ \mathbf{V} = \begin{bmatrix}
V_{11} & V_{12} & V_{13} & 0 & 0 & 0 \\
V_{12} & V_{22} & V_{23} & 0 & 0 & 0 \\
V_{13} & V_{23} & V_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & V_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & V_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & V_{66}
\end{bmatrix} \]  

(4.22)

where

\[ V_{11} = \frac{(\eta_{22} - \nu_{23}^2 \eta_{33}) \eta_{11}}{\Psi} \quad V_{12} = \frac{(\nu_{12} \eta_{22} + \nu_{13} \nu_{23} \eta_{33}) \eta_{11} \eta_{22}}{\Psi} \]

\[ V_{13} = \frac{(\nu_{12} \nu_{23} + \nu_{13}^2 \eta_{33}) \eta_{11} \eta_{22} \eta_{33}}{\Psi} \quad V_{22} = \frac{(\eta_{11} - \nu_{12}^2 \eta_{33}) \eta_{22}^2}{\Psi} \]

\[ V_{23} = \frac{(\nu_{23} \eta_{11} + \nu_{12} \nu_{13} \eta_{22} \eta_{33}) \eta_{22} \eta_{33}}{\Psi} \quad V_{33} = \frac{(\eta_{11} - \nu_{12} \eta_{22}) \eta_{22} \eta_{33}}{\Psi} \]

\[ V_{44} = \eta_{23} \quad V_{55} = \eta_{13} \quad V_{66} = \eta_{12} \]

and

\[ \Psi = \eta_{11} \eta_{22} - \nu_{12}^2 \eta_{22}^2 - \nu_{13} \eta_{22} \eta_{33} - \nu_{23}^2 \eta_{11} \eta_{33} - 2 \nu_{12} \nu_{13} \nu_{23} \eta_{22} \eta_{33} . \]
Combining Equations (4.20) and (4.21) give an evolution equation for $\psi$

$$V \psi = C_{ev}(\omega \cdot \psi) \quad (4.23)$$

From Equations (4.14), (4.15) and (4.20), the total stress is represented as

$$\sigma = C_{0} \cdot \omega \cdot C_{ev}(\omega \cdot \psi) \quad (4.24)$$

Hence the total stress is calculated once a solution for $\psi$ is known.
4.3 Yield and Damage Initiation

Two yield criteria, Tsai-Wu failure [35] and isotropic crushable foam [18], were examined and compared to test results under compression-shear loading at different angles and a displacement rate of 0.015 mm/s. Figures 4.12(a) and (b) show the stress-strain response in compression and shear from each experiment. The stress-strain diagrams for $\theta = 15^\circ$, $\theta = 30^\circ$, and $\theta = 45^\circ$ have approximately linear elastic, perfectly-plastic behaviors. In contrast, the stress-strain curves for $\theta = 60^\circ$ and $\theta = 75^\circ$ have linear elastic, strain-hardening response. Method B in Appendix A (proportional limit), was used to determine the yield points for each test. After selecting the yield points for several angles in compression and shear, a yield surface is created in Figures 4.13(a) and (b).

In order to compare the test yield surface to one predicted by the Tsai–Wu criterion or the isotropic crushable foam criterion, one must consider constraints on the foam due to bonding. Cells near the bond line are constrained by $e_{11} = e_{22} = 0$ and $\gamma_{12} = \gamma_{23} = 0$. We refer to this condition as constrained compression/shear. Cells far from the bond line (near middle of specimen) are unconstrained. We refer to this state as unconstrained compression-shear. The average stresses measured in the test are due to a combination of these two limiting cases. The following sections give predicted yield surfaces under these two limiting cases.
Yield strengths for PVC H100 foam are given in Table 4.4. The tensile yield strengths $X_t, Y_t, Z_t$ for in-plane and out-of-plane directions of the PVC H100 foam were obtained from Deshpande and Fleck [11]. All other mechanical properties of the Divinycell PVC H100 foam were taken from Chen and Hoo Fatt [14].

Table 4.4 Yield strengths of Divinycell PVC H100 foam [14,11].

<table>
<thead>
<tr>
<th>$X_c = Y_c$ (MPa)</th>
<th>$S_{12}$ (MPa)</th>
<th>$X_t = Y_t$ (MPa)</th>
<th>$S_{13} = S_{23}$ (MPa)</th>
<th>$Z_c$ (MPa)</th>
<th>$Z_t$ (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.56</td>
<td>0.64</td>
<td>0.7</td>
<td>0.99</td>
<td>1.53</td>
<td>1.7</td>
</tr>
</tbody>
</table>
Figure 4.12 Stress-strain responses under mixed mode when displacement rate is 0.015mm/s: (a) Compression and (b) Shear.
4.3.1 Tsai-Wu yield criterion

For general 3D states of stress, the Tsai – Wu failure criterion [35] is given by

\[
X_1\sigma_{11} + X_2\sigma_{22} + X_3\sigma_{33} + X_1\sigma_{11}^2 + X_2\sigma_{22}^2 + X_3\sigma_{33}^2 + X_{44}\tau_{23}^2 + X_{55}\tau_{13}^2 + X_{66}\tau_{12}^2 \\
+ 2X_{12}\sigma_{12} + 2X_{13}\sigma_{13} + 2X_{23}\sigma_{23} \leq 1
\]  

(4.25)

where

\[
X_1 = \frac{1}{X_t} - \frac{1}{X_c}, \quad X_{11} = \frac{1}{X_tX_c}
\]

\[
X_2 = \frac{1}{Y_t} - \frac{1}{Y_c}, \quad X_{22} = \frac{1}{Y_tY_c}
\]

\[
X_3 = \frac{1}{Z_t} - \frac{1}{Z_c}, \quad X_{33} = \frac{1}{Z_tZ_c}
\]

\[
X_{44} = \left(\frac{1}{S_{23}}\right)^2, \quad X_{55} = \left(\frac{1}{S_{13}}\right)^2, \quad X_{66} = \left(\frac{1}{S_{12}}\right)^2
\]

\[
X_{12} = -\frac{1}{2}\sqrt{X_{11}X_{22}}, \quad X_{13} = -\frac{1}{2}\sqrt{X_{11}X_{33}}, \quad X_{23} = -\frac{1}{2}\sqrt{X_{22}X_{33}}
\]
The Tsai-Wu yield criterion under two limiting cases of unconstrained and constrained compression-shear are determined below.

**Unconstrained compression/shear**

The Tsai – Wu failure criterion is reduced for a 2D state of stress under the constraints $\sigma_{11} = \sigma_{22} = 0$ and $\tau_{12} = \tau_{23} = 0$ as follows:

$$X_{33}\sigma_{33} + X_{33}\sigma_{33}^2 + X_{55}\tau_{13}^2 \leq 1 \quad (4.26)$$

**Constrained compression/shear**

The Tsai – Wu failure criterion is reduced for a 2D state of stress under the constraints $\varepsilon_{11} = \varepsilon_{22} = 0$ and $\gamma_{12} = \gamma_{23} = 0$ in Appendix B. The resulting yield surface is as follows:

$$F_1\sigma_{33} + F_2\sigma_{33}^2 + X_{55}\tau_{13}^2 = 1 \quad (4.27)$$

where

$$F_1 = 2 \left( \frac{X_1C_{13}}{C_{33}} \right) + X_3 \quad (4.28)$$

and

$$F_2 = 2 \left( \frac{(X_{11} + X_{12})C_{13}^2}{C_{33}^2} \right) + X_{33} + 4 \left( \frac{X_{13}C_{13}}{C_{33}} \right) \quad (4.29)$$

To create the yield surface for compression stresses in 3- direction and shear stresses with respect to 1-3 direction, mechanical properties in Table 4.4 were used and the results are shown in Figure 4.9(a) for constrained and unconstrained compression/shear.
4.3.2 Isotropic crushable foam yield criterion

The isotropic crushable foam yield criterion [18] is given by

\[ \Phi = \hat{\sigma} - Y \]  \hspace{1cm} (4.30)

where \( Y = Z_c \) is the yield strength and \( \hat{\sigma} \) is an equivalent stress. The equivalent stress is defined by

\[ \hat{\sigma}^2 = \frac{1}{1 + \left( \frac{\alpha}{3} \right)^2} + \left[ \sigma^2_e + \alpha^2 \sigma_m^2 \right] \] \hspace{1cm} (4.31)

where \( \sigma_m \) is the mean stress, \( \sigma_e \) is the von Mises or effective stress and \( \alpha \) is a parameter to define the shape of the yield surface. The mean stress is

\[ \sigma_m = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} \] \hspace{1cm} (4.32)

The von Mises stress is

\[ \sigma_e = \frac{1}{\sqrt{2}} \left( (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\tau_{12}^2 + \tau_{23}^2 + \tau_{13}^2) \right)^{1/2} \] \hspace{1cm} (4.33)

The \( \alpha \) is parameter depends on the plastic Poisson’s ratio \( \nu_p \) and is given by

\[ \alpha = 3 \left( \frac{1 - \nu_p}{2 + \nu_p} \right)^2 \] \hspace{1cm} (4.34)

During plasticity, the plastic Poisson’s ratio is nearly zero because the foam crushes without a change in length of transverse sections. Setting \( \nu_p = 0 \) gives \( \alpha = 3/\sqrt{2} \). The isotropic crushable foam criterion under two limiting cases of unconstrained and constrained compression-shear are determined below.
**Unconstrained compression/shear**

For isotropic crushable foam, a 2D state of stress under the constraints $\sigma_{11} = \sigma_{22} = 0$ and $\tau_{12} = \tau_{23} = 0$ gives the mean stress as

$$\sigma_m = \frac{\sigma_{33}}{3}$$  \hspace{1cm} (4.35)

and the von Mises stress as

$$\sigma_e = \left(\sigma_{33}^2 + 3\tau_{13}^2\right)^{\frac{1}{2}}$$  \hspace{1cm} (4.36)

Substituting these expressions into $\dot{\sigma}$ and setting $\dot{\sigma} = Z_c$ give

$$Z_c^2 = \sigma_{33}^2 + 2\tau_{13}^2$$  \hspace{1cm} (4.37)

**Constrained compression/shear**

Derivation of this yield surface under conditions $\varepsilon_{11} = \varepsilon_{22} = 0$ and $\gamma_{12} = \gamma_{23} = 0$, is given in Appendix B. The von Mises $\sigma_e$ and mean stress $\sigma_m$ are given by

$$\sigma_e = \left[\frac{(2\nu-1)^2}{(1-\nu)^2} \sigma_{33}^2 + 3\tau_{13}^2\right]^{\frac{1}{2}}$$  \hspace{1cm} (4.38)

and

$$\sigma_m = \frac{1}{3} \frac{1}{(1-\nu)} \sigma_{33}$$  \hspace{1cm} (4.39)

Substituting these expressions into $\dot{\sigma}$ and setting $\dot{\sigma} = Z_c$ give
\[
Z_c^2 = 2 \left[ \frac{3v^2 - 2v + 1}{(v-1)^2} \sigma_{33}^2 \right] + 2\tau_{12}^2 
\] (4.40)

Using that \( Z_c = 1.53 \text{ MPa} \) from Table 3.2, a predicted yield surface is shown in Figure 4.13 (b) for constrained and unconstrained compression/shear.

The experimental yield stresses for different angles are superimposed on the theoretical yield surfaces assuming unconstrained and constrained conditions in Figures 4.13 (a) and (b). Test values should lie somewhere between these two limiting cases. The experimental results for different angles compared slightly better with the theoretical Tsai–Wu criterion yield surfaces than the isotropic crushable foam yield surfaces. With the Tsai–Wu criterion, one sees relatively good agreements for \( \theta = 15^\circ, \theta = 30^\circ, \theta = 45^\circ \) tests, but the tests at \( \theta = 60^\circ \) and \( \theta = 75^\circ \) are lower than the theory predicted. It was noted in the experiments that cracks and debonding occurred near the edges of the foam and this may mean that the data for such high angles are unreliable. The tests do not appear to follow the trend-line of the isotropic crushable foam criterion. Under constrained conditions, the isotropic yield surface deviates substantially from the Tsai-Wu criterion. The constrained yield surfaces in isotropic yielding moves inward towards the left, while the shift occurs in the opposite direction with the Tsai-Wu yield criterion.
Figure 4.13 Comparison of yield surfaces of PVC H100 foam predicted by Tsai–Wu and isotropic yield criteria with experimental results for different angles at displacement rate of 0.015 mm/s: (a) Tsai–Wu criterion and (b) Isotropic criterion.
4.4 Behavior after Yield/Damage Initiation

In order to predict behavior after initial yielding and damage initiation, both plastic flow and viscoelastic damage response or hysteresis must be considered together. As shown in Figure 4.14, the plastic strain during yielding is determined from the damage modulus of the equilibrium spring. It cannot be found directly from the unloading curve because of the viscoelastic overstress, which produces hysteresis.

![Diagram of plastic flow and viscoelastic damage](image)

Figure 4.14 Plastic flow and viscoelastic damage.

The following section outlines equations that can be used to obtain plastic flow curves and damage parameters from uniaxial compression and shear tests in the out-of-plane and in-plane directions. This is followed by Tsai-Wu plasticity and general three-dimensional equations for viscoelastic hysteresis. Finally, a viscoplastic model is developed in the last section.
4.4.1 One-dimensional model

The plastic stress-strain curves and hysteresis for PVC H100 foam in uniaxial compression and simple shear are shown in Figures 4.15-18. These are used to obtain plasticity and viscoelastic damage properties. Equations used to determine plastic strains, which are needed to derive a plastic flow rule, and damage material properties are presented in this section.

Figure 4.15 Out-of-plane compression stress-strain curve with hysteresis for PVC H100 foam.
Figure 4.16 In-plane compression stress-strain curve with hysteresis for PVC H100 foam.

Figure 4.17 Out-of-plane shear stress-strain curve with hysteresis for PVC H100 foam.
Figure 4.18 In-plane shear stress-strain curve with hysteresis for PVC H100 foam.

Loading and unloading stress-strain curves for the foam after initial yield and damage are shown schematically in Figure 4.19. These are said to occur at constant strain rate $\alpha$, as shown in Figure 4.20. The hysteresis produced is viscoelastic but with damage properties.

Figure 4.19 Stress-strain responses for hysteresis in PVC H100 foam.
In order to model strain rate-dependency and hysteresis of the PVC H100 foam during damage case, the model shown in Figure 4.3(b) is utilized. During plasticity,

\[ \sigma = \sigma_\gamma \]  \hspace{1cm} (4.41)

where \( \sigma_\gamma \) may increase with plastic strain \( \varepsilon_p \), i.e., strain-hardening may occur. During viscoelastic unloading and reloading, the total stress is

\[ \sigma = \sigma_{eq} + \sigma_{ov} \]  \hspace{1cm} (4.42)

where the equilibrium stress is

\[ \sigma_{eq} = \overline{E}_0 \varepsilon_e \]  \hspace{1cm} (4.43)

and the overstress is

\[ \sigma_{ov} = \overline{E}_m \varepsilon_{ev} \]  \hspace{1cm} (4.44)
The overhead bar denotes stiffness after damage and plastic flow. Strain compatibility gives

\[ \varepsilon = \varepsilon_e + \varepsilon_p \]  

(4.45)

where also

\[ \varepsilon_e = \varepsilon_v + \varepsilon_{ev} \]  

(4.46)

Equations (4.45) and (4.46) may be used to eliminate \( \varepsilon_{ev} \), and the total stress is simplified by

\[ \sigma = \bar{E}_0 (\varepsilon - \varepsilon_p) + \bar{E}_{ev}(\varepsilon - \varepsilon_p - \varepsilon_v) \]  

(4.47)

The overstress is

\[ \sigma_{ov} = \bar{E}_{ov}(\varepsilon - \varepsilon_p - \varepsilon_v) \]  

(4.48)

The overstress is also given by

\[ \sigma_{ov} = \bar{\eta} \dot{\varepsilon}_v \]  

(4.49)

The viscoelastic unloading response is denoted in Figure 4.19 by \( \sigma_{UL} \) and it is equal to

\[ \sigma_{UL} = \bar{E}_0 (\varepsilon - \varepsilon_p) + \bar{E}_{ev}(\varepsilon - \varepsilon_p - \varepsilon_{v1}) \]  

(4.50)

where \( \varepsilon_{v1} \) is the viscous strain during unloading. To find \( \varepsilon_{v1} \), Equations (4.48) and (4.49) are set equal to give

\[ \bar{\eta} \dot{\varepsilon}_v = \bar{E}_{ev}(\varepsilon - \varepsilon_p - \varepsilon_{v1}) \]  

(4.51)
where \( \dot{\varepsilon}_v = -\alpha \frac{d\varepsilon_v}{d\varepsilon} \) for unloading (see Figure 4.20). The above differential equation is also expressed as
\[
\varepsilon - \varepsilon_p - \varepsilon_v = -\frac{\eta\alpha}{E_{ev}} \frac{d\varepsilon_v}{d\varepsilon} \quad (4.52)
\]
Equation (4.52) is solved by using an initial condition \( \varepsilon_{v_1} = \varepsilon_{\text{v1}} \) when \( \varepsilon = \varepsilon_{\text{max}} \). Here \( \varepsilon_{v1} \) is an unknown parameter. As a result, the viscous strain for unloading is
\[
\varepsilon_{v_1} = (\varepsilon_{\text{v1}} - \frac{\eta\alpha}{E_{ev}} + \varepsilon_p - \varepsilon_{\text{max}}) \varepsilon + \frac{\eta\alpha}{E_{ev}} - \varepsilon_p + \varepsilon \quad (4.53)
\]
The two limits of the hysteresis, \( (\varepsilon_{\text{max}}, \sigma_{\text{max}}) \) and \( (\varepsilon_{\text{res}}, \sigma_{\text{res}}) \), may be used to obtain additional equations to solve for \( \varepsilon_{v1} \) and other unknowns. In terms of \( \sigma_{\text{max}}, \varepsilon_{\text{max}} \) and \( \varepsilon_{v1} \), Equation (4.50) gives
\[
\sigma_{\text{max}} = E_0 (\varepsilon_{\text{max}} - \varepsilon_p) + E_{ev} (\varepsilon_{\text{max}} - \varepsilon_p - \varepsilon_{v1}) \quad (4.54)
\]
Simplifying the above equation for \( \varepsilon_{v1} \), one gets
\[
\varepsilon_{v1} = \frac{E_0 (\varepsilon_{\text{max}} - \varepsilon_p) - \sigma_{\text{max}}}{E_{ev}} + \varepsilon_{\text{max}} - \varepsilon_p \quad (4.55)
\]
For reloading, the total stress \( \sigma_{RL} \) becomes
\[
\sigma_{RL} = E_0 (\varepsilon - \varepsilon_p) + E_{ev} (\varepsilon - \varepsilon_p - \varepsilon_{v2}) \quad (4.56)
\]
where \( \varepsilon_{v2} \) is the viscous strain during unloading. Again Equations (4.48) and (4.49) are set equal to find \( \varepsilon_{v2} \),
\[
\bar{\eta} \dot{\varepsilon}_v = E_{ev} (\varepsilon - \varepsilon_p - \varepsilon_{v2}) \quad (4.57)
\]
The resulting differential equation for reloading is
This differential equation is solved with initial condition $\varepsilon_{v2} = \bar{\varepsilon}_{v2}$ when $\varepsilon = \varepsilon_{res}$. The resulting viscous strain during reloading is

$$\varepsilon_{v2} = (\bar{\varepsilon}_{v2} + \frac{\bar{\eta} \alpha}{E_{ev}} + \varepsilon_p - \varepsilon_{res})e^{-\frac{\bar{\eta} \alpha (\varepsilon - \varepsilon_{max})}{E_{ev}}} - \frac{\bar{\eta} \alpha}{E_{ev}} - \varepsilon_p + \varepsilon$$  

(4.59)

where $\bar{\varepsilon}_{v2}$ is found at the end of unloading in Equation (4.53). Setting $\varepsilon = \varepsilon_{res}$ in Equations (4.59), we get

$$\bar{\varepsilon}_{v2} = (\varepsilon_{v1} + \frac{\bar{\eta} \alpha}{E_{ev}} + \varepsilon_p - \varepsilon_{max})e^{\frac{\bar{\eta} \alpha (\varepsilon - \varepsilon_{max})}{E_{ev}}} + \frac{\bar{\eta} \alpha}{E_{ev}} - \varepsilon_p + \varepsilon_{res}$$  

(4.60)

If $\varepsilon_{res} = 0$, Equations (4.59) and (4.60) give

$$\varepsilon_{v2} = (\bar{\varepsilon}_{v2} + \frac{\bar{\eta} \alpha}{E_{ev}} + \varepsilon_p - \varepsilon_{max})e^{-\frac{\bar{\eta} \alpha \varepsilon}{E_{ev}}} - \frac{\bar{\eta} \alpha}{E_{ev}} - \varepsilon_p + \varepsilon$$  

(4.61)

and

$$\bar{\varepsilon}_{v2} = (\varepsilon_{v1} - \frac{\bar{\eta} \alpha}{E_{ev}} + \varepsilon_p - \varepsilon_{max})e^{-\frac{\bar{\eta} \alpha \varepsilon_{max}}{E_{ev}}} + \frac{\bar{\eta} \alpha}{E_{ev}} - \varepsilon_p$$  

(4.62)

The plastic strain $\varepsilon_p$ is still unknown, so it needs to be derived from algebra. Equation (4.60) is used at the coordinate $\varepsilon_{res}, \sigma_{res}$:

$$\sigma_{res} = E_0 (\varepsilon_{res} - \varepsilon_p) + E_{ev} (\varepsilon_{res} - \varepsilon_p - \bar{\varepsilon}_{v2})$$  

(4.63)

Simplifying the above equation for $\bar{\varepsilon}_{v2}$, one finds
\[ \varepsilon_{v2} = \frac{E_0 (\varepsilon_{res} - \varepsilon_p) - \sigma_{res}}{E_{ev}} + \varepsilon_{res} - \varepsilon_p \]  

(4.64)

Combining expressions for \( \varepsilon_{v2} \) in Equations (4.62) and (4.64), one gets

\[ \varepsilon_{v1} = \frac{\eta \alpha}{E_{ev}} - \varepsilon_p + \varepsilon_{max} + \left( \frac{E_0 (\varepsilon_{res} - \varepsilon_p) - \sigma_{res}}{E_{ev}} - \frac{\eta \alpha}{E_{ev}} \right) \left( \frac{-E_0 (\varepsilon_{res} - \varepsilon_{max})}{\eta \alpha} \right) \]  

(4.65)

Likewise setting expressions for \( \varepsilon_{v1} \) in Equations (4.65) and (4.55) one may solve for \( \varepsilon_p \) as

\[ \varepsilon_p = \sigma_{max} - \frac{E_0 \varepsilon_{max}}{\eta \alpha} + \eta \alpha + \left( \frac{E_0 \varepsilon_{res} - \sigma_{res} - \eta \alpha}{\eta \alpha} \right) \left( \frac{-E_0 (\varepsilon_{res} - \varepsilon_{max})}{\eta \alpha} \right) \]  

\[ -E_0 + \frac{E_0 \varepsilon_{max}}{\eta \alpha} \]  

(4.66)

If \( \varepsilon_{res} = 0 \) the plastic strain from above equation becomes as the following:

\[ \varepsilon_p = \frac{\sigma_{max} - E_0 \varepsilon_{max} + \eta \alpha - (\sigma_{res} + \eta \alpha) e}{E_0 + E_0 e} \]  

(4.67)

In summary, the unloading stress is given by Equation (4.50) and the reloading stress is given by Equation (4.56). Each equation has five independent unknowns: \( E_0, E_{ev} \), \( \eta \), \( \varepsilon_p \), and \( \varepsilon_{v1} \). Two of these unknowns, \( \varepsilon_p \) and \( \varepsilon_{v1} \), may be determined from the coordinates \( (\varepsilon_{max}, \sigma_{max}) \) and \( (\varepsilon_{res}, \sigma_{res}) \). Data from uniaxial compression and simple shear tests were used in a curve-fitting exercise to determine \( E_0, E_{ev}, \) and \( \eta \) under the constraint that \( \varepsilon_p < \varepsilon_{max} \) and \( E_0 < E_0 \). Tables 4.5-4.8 give the results from this exercise. Figures 4.21 to 4.22 show how the analytical predictions using these values compared to actual hysteresis loops in the test.
Table 4.5 Plasticity and damage properties for out-of-plane compression.

<table>
<thead>
<tr>
<th>$\varepsilon_{\text{max}}$</th>
<th>$\varepsilon_p$</th>
<th>$E_{033}$ (MPa)</th>
<th>$E_{\text{ev}33}$ (MPa)</th>
<th>$\bar{\eta}_{33}$ (MPa.s)</th>
<th>$E_{033}$ (MPa)</th>
<th>$E_{\text{ev}33}$ (MPa)</th>
<th>$\eta_{33}$ (MPa.s)</th>
<th>d$_{33}$</th>
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<td>20</td>
<td>950</td>
<td>89</td>
<td>9</td>
<td>3.4</td>
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<td>0.1</td>
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<td>89</td>
<td>9</td>
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<td>0.831461</td>
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Table 4.6 Plasticity and damage properties for in-plane compression.

<table>
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<tr>
<th>$\varepsilon_{\text{max}}$</th>
<th>$\varepsilon_p$</th>
<th>$E_{011}$ (MPa)</th>
<th>$E_{\text{ev}11}$ (MPa)</th>
<th>$\bar{\eta}_{11}$ (MPa.s)</th>
<th>$E_{011}$ (MPa)</th>
<th>$E_{\text{ev}11}$ (MPa)</th>
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<td>0.782609</td>
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Table 4.7 Plasticity and damage properties for out-of-plane shear.

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<th>$\gamma_{\text{max}}$</th>
<th>$\gamma_p$</th>
<th>$G_{013}$ (MPa)</th>
<th>$G_{\text{ev}13}$ (MPa)</th>
<th>$\bar{\eta}_{13}$ (MPa.s)</th>
<th>$G_{013}$ (MPa)</th>
<th>$G_{\text{ev}13}$ (MPa)</th>
<th>$\eta_{13}$ (MPa.s)</th>
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<td>0.877551</td>
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Table 4.8 Plasticity and damage properties for in-plane shear.

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<tr>
<th>$\gamma_{\text{max}}$</th>
<th>$\gamma_p$</th>
<th>$G_{012}$ (MPa)</th>
<th>$G_{\text{ev}12}$ (MPa)</th>
<th>$\bar{\eta}_{12}$ (MPa.s)</th>
<th>$G_{012}$ (MPa)</th>
<th>$G_{\text{ev}12}$ (MPa)</th>
<th>$\eta_{12}$ (MPa.s)</th>
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<td>7.5</td>
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<td>0.126668</td>
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<td>8</td>
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<td>16</td>
<td>1.8</td>
<td>9.83</td>
<td>0.875</td>
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135
Figure 4.21 Out-of-plane compression hysteresis of PVC H100 foam.

Figure 4.22 In-plane compression hysteresis of PVC H100 foam.
Figure 4.23 Out-of-plane shear hysteresis of PVC H100 foam.

Figure 4.24 In-plane shear hysteresis of PVC H100 foam.
Using mechanical properties in Tables 4.5 to 4.8, damage functions versus plastic strain curves were obtained based on the formulas $d_{ii} = 1 - (E_{ii} / E_{ii})$ and $d_{ij} = 1 - (G_{ij} / G_{ij})$ for the equilibrium spring, and these are shown in Figures 4.25 (a)-(d). Ratios of the damage to undamaged modulus of the intermediate spring $E_{evij} / E_{evij}$ and $G_{evij} / G_{evij}$ versus plastic strain are plotted in Figures 4.26(a)-(d). Ratios of $\eta_{ij} / \eta_{ij}$ versus plastic strain are shown in Figures 4.27(a)-(d). Functions for the quantities shown in Figures 4.25-4.27 are listed in Appendix E.

Figure 4.25 Damage function of equilibrium spring versus plastic strain response: (a) Out-of-plane compression, (b) In-plane compression, (c) Out-of-plane shear and (d) In-plane shear.
Figure 4.26 Ratio of damage to undamage intermediate spring modulus versus plastic strain response: (a) Out-of-plane compression, (b) In-plane compression, (c) Out-of-plane shear and (d) In-plane shear.
Figure 4.27 Ratio of damage to undamage viscosity versus plastic strain response: (a) Out-of-plane compression, (b) In-plane compression, (c) Out-of-plane shear and (d) In-plane shear.

The plastic flow stress in the out-of-plane compression and shear are shown in Figures 4.28 and 4.29, respectively. Notice the difference in hardening encountered under compression and shear. Hardening is very slight in compression but steep in shear. These curves will be used to determine plasticity properties in the next section.
Figure 4.28 Compression stress-plastic strain curve for compression only to show ideal type of plastic hardening.

Figure 4.29 Shear stress-plastic strain curve for shear only to display nonlinear plastic hardening response.
4.4.2 Three-dimensional model

Plasticity and viscoelastic damage/hysteresis are presented in separate sections.

Tsai-Wu Plasticity

The Tsai-Wu yield criterion comes from a special class of quadratic yield criteria for anisotropic material [36]. The Tsai-Wu plastic potential function is given by

\[ \Phi = \frac{1}{2} \Phi T \Phi + \Phi T q - \sigma^2 = 0 \]  

(4.68)

where

\[
\begin{bmatrix}
X_{11} & X_{12} & X_{13} & 0 & 0 & 0 \\
X_{12} & X_{22} & X_{23} & 0 & 0 & 0 \\
X_{13} & X_{23} & X_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & X_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & X_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & X_{66}
\end{bmatrix}
\]

\( q = \begin{bmatrix} X_1 & X_2 & X_3 & 0 & 0 \end{bmatrix} T \), \( \bar{\sigma} = \Phi(\varepsilon_p) \) and \( \varepsilon_p \) is the equivalent plastic strain. Here isotropic hardening is assumed. The plastic strain rate \( \dot{\varepsilon}_p \) is given by an associate flow rule

\[ \dot{\varepsilon}_p = \lambda \frac{\partial \Phi}{\partial \bar{\sigma}} \]  

(4.69)

where \( \lambda \) is the plastic multiplier. In terms of the \( P \) and \( q \), Equation (4.69) reduces to

\[ \dot{\varepsilon}_p = \dot{\lambda}(P \Phi + q) \]  

(4.70)

The rate of accumulated equivalent plastic strain may be expressed as

\[ \dot{\varepsilon}_p = \dot{\lambda} \sqrt[3]{2 \varepsilon : \varepsilon} \]  

(4.71)
or

\[ \dot{\varepsilon} = \dot{\lambda} \sqrt{\frac{2}{3}} \varepsilon \]

(4.72)

where

\[ Z = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/2 \\
\end{bmatrix} \]

Substituting Equation (4.70) into (4.72) gives

\[ \dot{\varepsilon} = \dot{\lambda} \sqrt{\frac{2}{3}} (P \circ q)^T Z (P \circ q) \]

(4.73)

Equation (4.73) offers a convenient way to obtain the equivalent plastic strain in terms of the stress, and it is often used in computational plasticity schemes. The time derivative in Equation (4.73) may be removed to express the increment in equivalent plastic strain:

\[ d \varepsilon = d \sqrt{\frac{2}{3}} (P \circ q)^T Z (P \circ q) \]

(4.74)

Equation (4.74) must be solved numerically such that the yield criterion, Equation (4.68), is satisfied during plastic flow. Various implicit integration schemes can be used to ensure this for general three-dimensional plasticity [37].
The Tsai-Wu plasticity yield criterion may also be expressed in an alternate form as follows:

\[ \Phi = \sigma^2_* - \bar{\sigma}^2 = 0 \]  

(4.75)

where \( \bar{\sigma} \) is a normalized flow stress and the equivalent stress \( \sigma_* \) is

\[ \sigma_*^2 = X_1\sigma_{11} + X_2\sigma_{22} + X_3\sigma_{33} + X_{11}\sigma_{11}^2 + X_{22}\sigma_{22}^2 + X_{33}\sigma_{33}^2 + X_{44}\tau_{23}^2 + X_{55}\tau_{13}^2 \]

\[ + X_{66}\tau_{12}^2 + 2X_{12}\sigma_{11}\sigma_{22} + 2X_{13}\sigma_{11}\sigma_{33} + 2X_{23}\sigma_{22}\sigma_{33} \]  

(4.76)

Initial yielding occurs when \( \bar{\sigma} = 1 \), but continued plastic flow will occur with strain hardening. There are several types of hardening relations, including linear and power-law hardening functions of equivalent plastic strain.

The plasticity curves in the previous sections are used to determine a yield surface for unconstrained out-of-plane compression and shear. The equivalent stress under these conditions is defined as

\[ \sigma_* = \sqrt{X_3\sigma_{33}^2 + X_{33}\sigma_{33}^2 + X_{55}\tau_{13}^2} \]  

(4.77)

The incremental equivalent plastic strain can be expressed by

\[ d\varepsilon_p = \sqrt{(2/3)(de_{p33}^2 + de_{p13}^2 + de_{p31}^2)} \]  

(4.78)

Where tensorial shear plastic strains are defined as \( \varepsilon_{p13} = d\gamma_{p13}/2 \) and \( \varepsilon_{p31} = d\gamma_{p31}/2 \). The flow stress in compression only and shear only shown in Figures 4.28 and 4.29, respectively, are used to determine equivalent stress and equivalent plastic strain in Figure 4.30. The compression curve is projected (red dashed line) because the data was so limited in range after converting to equivalent plastic strains. It is clearly seen that hardening is more prevalent in shear. Differences in hardening are due to the different micro-
mechanisms causing plastic flow. In the case of compression, cells have buckled and are collapsing with compressive strain. Eventually strain hardening in compression will be due to foam densification or compaction. This occurs at about 0.4 equivalent plastic strain and is not shown in Figure 4.30. In shear, cells bend and elongate. Strain hardening is due to this.

![Hardening Functions](image)

Figure 4.30 Mixed hardening curves for PVC H100 foam.

Not accounting for densification, the two hardening curves are expressed below

\[
\sigma = \begin{cases} 
1 + 0.0852[1 - e^{-128.58\varepsilon_p}], & \text{compression only} \\
1 + 0.7059[1 - e^{-32.128\varepsilon_p}], & \text{shear only} 
\end{cases} 
\] (4.79)

Hardening surfaces for combined compression and shear are shown in Figure 4.31. The surfaces were created for fixed equivalent plastic strain at \( \varepsilon_p = 0.01 \) and \( \varepsilon_p = 0.02 \). These curves were created by setting the yield strength in Equations (4.75) and (4.77) equal to
the flow stress. Hardening of the yield strength in tension $Z_t$ is assumed to be the same as for $Z_c$. Hardening in shear is always higher compared with uniaxial compression. The hardening surface is neither isotropic nor kinematic, but it could be due to a combination of both, i.e., mixed hardening.

![Tsai-Wu Yield and Hardening Surfaces](image)

Figure 4.31 Hardening curves for PVC H100 foam.

One way to address the type of hardening encountered in PVC H100 foam is to make hardening dependent on mode-mixity in addition to plastic strain. First assume strain hardening is of a general form:

$$
\bar{\sigma} = 1 + k \left( 1 - e^{-b\epsilon_p} \right)
$$

(4.80)
where $k$ and $b$ parameters depend on mode-mixity. Each parameter lies between values for compression only and shear only, i.e. $k_c < k < k_y$ and $b_c < b < b_y$. Suitable functions for them are

\[
k = k_c + \left[ 1 - e^{-\frac{\gamma_p}{\varepsilon_{pm}}} \right] (k_y - k_c) \tag{4.81}
\]

and

\[
b = b_c + \left[ 1 - e^{-\frac{\gamma_p}{\varepsilon_{pm}}} \right] (b_y - b_c) \tag{4.82}
\]

where

\[
\varepsilon_{pm} = \max \left( \varepsilon_{p1}, \varepsilon_{p2}, \varepsilon_{p3} \right)
\]

and

\[
\gamma_p = \max \left( \gamma_{p12}, \gamma_{p23}, \gamma_{p13} \right)
\]

and $\beta_1$ and $\beta_2$ are material parameter to be determined. The above expression for $k$ and $b$ tends to the correct value in pure compression when $\frac{\gamma_p}{\varepsilon_{pm}}$ approaches zero, and it tends to the correct value in pure shear when $\frac{\gamma_p}{\varepsilon_{pm}}$ is a large number. Note that the hardening equation is only valid when the normal plastic strain is in compression. If the plastic strain is in tension, then the shear hardening function should be used.
**Viscoelastic Damage and Hysteresis**

These equations are similar to those presented in Section 4.2.2, but with damage material properties and strains modified by plastic deformation. The total stress is given again as the sum of equilibrium stress and overstress:

\[
\sigma = \sigma_{eq} + \sigma_{ov}
\]  

(4.83)

The equilibrium stress \( \sigma_{eq} \) is

\[
\sigma_{eq} = \overline{C}_0 (\omega \ \gamma)
\]

(4.84)

where \( \overline{C}_0 \) is the damage stiffness matrix of the equilibrium spring. This is given by

\[
\overline{C}_0 = \begin{bmatrix}
C_{11} \ C_{12} \ C_{13} & 0 & 0 & 0 \\
C_{12} \ C_{22} \ C_{23} & 0 & 0 & 0 \\
C_{13} \ C_{23} \ C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & \overline{C}_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & \overline{C}_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & \overline{C}_{66}
\end{bmatrix}
\]

(4.85)

where

\[
\overline{C}_{11} = \frac{(E_{11} - v_{23}E_{33})E_{11}^2}{\Omega} \quad \overline{C}_{12} = \frac{(v_{12}E_{22} + v_{13}E_{33})E_{11}E_{22}}{\Omega}
\]

\[
\overline{C}_{13} = \frac{(v_{12}E_{23} + v_{13}E_{33})E_{11}E_{22}E_{33}}{\Omega} \quad \overline{C}_{22} = \frac{(E_{11} - v_{13}E_{33})E_{22}^2}{\Omega}
\]

\[
\overline{C}_{23} = \frac{(v_{23}E_{11} + v_{12}v_{13}E_{22})E_{22}E_{33}}{\Omega} \quad \overline{C}_{33} = \frac{(E_{11} - v_{12}E_{22})E_{22}E_{33}}{\Omega}
\]

\[
\overline{C}_{44} = \overline{C}_{23} \quad \overline{C}_{55} = \overline{G}_{13} \quad \overline{C}_{66} = \overline{G}_{12}
\]

and \( \Omega = E_{11}E_{22} - v_{12}^2E_{22}^2 - v_{13}^2E_{22}E_{33} - v_{23}^2E_{11}E_{33} - 2v_{12}v_{13}v_{23}E_{22}E_{33} \).
Similarly, the overstress \( \sigma_{ov} \) is

\[
\sigma_{ov} = \mathbf{C}_{ev} \mathbf{e}_{ov}
\]  

(4.86)

where \( \mathbf{C}_{ev} \) is the damage stiffness matrix of the intermediate spring. This is given by

\[
\mathbf{C}_{ev} = \begin{bmatrix}
\mathbf{C}_{ev11} & \mathbf{C}_{ev12} & \mathbf{C}_{ev13} & 0 & 0 & 0 \\
\mathbf{C}_{ev12} & \mathbf{C}_{ev22} & \mathbf{C}_{ev23} & 0 & 0 & 0 \\
\mathbf{C}_{ev13} & \mathbf{C}_{ev23} & \mathbf{C}_{ev33} & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbf{C}_{ev44} & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{C}_{ev55} & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{C}_{ev66}
\end{bmatrix}
\]  

(4.87)

where

\[
\mathbf{C}_{ev11} = \left( \frac{E_{ev22} - \nu_{23}^2 E_{ev33}}{\Phi} \right) E_{ev11}^2
\]

\[
\mathbf{C}_{ev12} = \left( \frac{(\nu_{12} E_{ev22} + \nu_{13} \nu_{23} E_{ev33})}{\Phi} \right) E_{ev11} E_{ev22}
\]

\[
\mathbf{C}_{ev13} = \left( \frac{(\nu_{23} E_{ev11} + \nu_{12} \nu_{13} E_{ev22})}{\Phi} \right) E_{ev11} E_{ev33}
\]

\[
\mathbf{C}_{ev22} = \left( \frac{(E_{ev11} - \nu_{13}^2 E_{ev33})}{\Phi} \right) E_{ev22}^2
\]

\[
\mathbf{C}_{ev23} = \left( \frac{(\nu_{23} E_{ev11} + \nu_{12} \nu_{23} E_{ev22})}{\Phi} \right) E_{ev22} E_{ev33}
\]

\[
\mathbf{C}_{ev33} = \left( \frac{(E_{ev11} - \nu_{13}^2 E_{ev33})}{\Phi} \right) E_{ev33}^2
\]

\[
\mathbf{C}_{ev44} = \mathbf{C}_{ev23} \\
\mathbf{C}_{ev55} = \mathbf{C}_{ev13} \\
\mathbf{C}_{ev66} = \mathbf{G}_{ev22}
\]

and \( \Phi = E_{ev11} E_{ev22} - \nu_{12}^2 E_{ev22} - \nu_{13}^2 E_{ev33} - \nu_{23}^2 E_{ev11} E_{ev33} - 2\nu_{12} \nu_{13} \nu_{23} E_{ev22} E_{ev33} \).

Compatibility of strain requires that

\[
\omega = \mathbf{e}_{ov} + \mathbf{e}_{ov} + \mathbf{e}_{ov}
\]  

(4.88)

Substituting Equation (4.57) into (4.55)

\[
\sigma_{ov} = \mathbf{C}_{ev} (\omega - \mathbf{e}_{ov})
\]  

(4.89)
The overstress is also governed by a linear viscosity law:

$$\sigma_{ov} = \mathbf{V} \mathbf{\Psi}$$  \hspace{1cm} (4.90)

where $\mathbf{V}$ is the damage viscosity matrix. The damage viscosity matrix is given as

$$
\mathbf{V} = \begin{bmatrix}
V_{11} & V_{12} & V_{13} & 0 & 0 & 0 \\
V_{12} & V_{22} & V_{23} & 0 & 0 & 0 \\
V_{13} & V_{23} & V_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & \bar{V}_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & \bar{V}_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & \bar{V}_{66}
\end{bmatrix}
$$  \hspace{1cm} (4.91)

where

$$
\bar{V}_{11} = \left( \bar{\eta}_{22} - \nu_{23}^2 \bar{\eta}_{33} \right) \bar{\eta}_{11}^2 \\
\bar{V}_{12} = \frac{\left( v_{12} \bar{\eta}_{22} + v_{13} \nu_{23} \bar{\eta}_{33} \right) \bar{\eta}_{11} \bar{\eta}_{22}}{\Psi} \\
\bar{V}_{13} = \frac{\left( v_{12} \bar{\eta}_{23} + v_{13} \nu_{23} \bar{\eta}_{33} \right) \bar{\eta}_{11} \bar{\eta}_{23}}{\Psi} \\
\bar{V}_{23} = \frac{\left( v_{23} \bar{\eta}_{23} \bar{\eta}_{11} \bar{\eta}_{22} \bar{\eta}_{33} \right) \bar{\eta}_{22} \bar{\eta}_{33}}{\Psi} \\
\bar{V}_{33} = \frac{\left( v_{11} \nu_{13} \bar{\eta}_{13} \bar{\eta}_{22} \bar{\eta}_{33} \right) \bar{\eta}_{22} \bar{\eta}_{33}}{\Psi} \\
\bar{V}_{44} = \bar{\eta}_{23} \\
\bar{V}_{55} = \bar{\eta}_{13} \\
\bar{V}_{66} = \bar{\eta}_{12}
$$

and $\Psi = \bar{\eta}_{11} \bar{\eta}_{22} - \nu_{12}^2 \bar{\eta}_{22}^2 - \nu_{13}^2 \bar{\eta}_{23}^2 \bar{\eta}_{33}^2 - \nu_{23}^2 \bar{\eta}_{11} \bar{\eta}_{22} \bar{\eta}_{33}^2 - 2 \nu_{12} \nu_{13} \nu_{23} \bar{\eta}_{22} \bar{\eta}_{33} \bar{\eta}_{33} \bar{\eta}_{33}$.

Combining Equations (4.89) and (4.90) give an evolution equation for $\mathbf{\Psi}$

$$
\mathbf{V} \cdot \mathbf{\Psi} = \mathbf{C}_{ev} \left( \mathbf{\omega} \cdot \mathbf{\Psi} - \mathbf{\Psi} \right)
$$  \hspace{1cm} (4.92)

The total stress is represented as

$$
\mathbf{\sigma} = \mathbf{C}_0 \mathbf{\omega} \mathbf{C}_{ev} \left( \mathbf{\omega} \cdot \mathbf{\Psi} - \mathbf{\Psi} \right)
$$  \hspace{1cm} (4.93)
4.4.3 Viscoplasticity

Material rate-dependency during plastic flow can be described by names such as viscoplasticity or rate-dependent plasticity. The PVC H100 foam exhibits viscoplasticity behavior as discussed in Chapter III. This can be introduced by defining the plastic multiplier in Equation (4.69) in terms of viscoplastic law. Equation (4.69) is re-written in terms of plastic strain rate components:

\[ \dot{\varepsilon}_{ij} = \dot{\lambda} \frac{\partial \Phi}{\partial \sigma_{ij}} \]  
(4.94)

The viscoplastic law by Perić [37] is introduced for \( \dot{\lambda} \) as

\[ \dot{\lambda}(\sigma_*, \sigma) = \begin{cases} 
\frac{1}{\mu} \left( \frac{\sigma_*}{\sigma} \right)^\frac{1}{\varepsilon} - 1 & \text{if } \Phi(\sigma_*, \sigma) \geq 0 \\
0 & \text{if } \Phi(\sigma_*, \sigma) < 0 
\end{cases} \]  
(4.95)

where \( \mu \) is the viscosity parameter and \( \varepsilon \) is a non-dimensional, rate-sensitively parameter.

Both of these material constants are strictly positive numbers.

In order to find \( \mu \) and \( \varepsilon \), we consider rate-dependent plastic flow curves from the out-of-plane or transverse shear tests in Figure 4.32.
The transverse shear plastic strain rate in Equation (4.94) is given explicitly by

$$
\dot{\gamma}_{p13} = \frac{1}{\mu} \left[ \left( \frac{\sigma_*}{\sigma} \right)^\frac{1}{\mu} - 1 \right] \frac{\partial \Phi}{\partial \tau_{13}} \tag{4.96}
$$

From Equation (4.68), one gets

$$
\frac{\partial \Phi}{\partial \tau_{13}} = 2X_{55}\tau_{13} = 2 \frac{\tau_{13}}{S_{13}} \tag{4.97}
$$

The equivalent stress is

$$
\sigma_* = \sqrt{X_{55}\tau_{13}} = \frac{\tau_{13}}{S_{13}} \tag{4.98}
$$

The hardening curve for simple out-of-plane shear

$$
\bar{\sigma} = \frac{\tau_{13y}}{S_{13}} \tag{4.99}
$$
where \( \tau_{13Y} \) is the flow stress at the equilibrium response. Substituting these expressions into Equation (4.96) gives the final formula of plastic multiplier in simple shear as

\[
\dot{\gamma}_{p13} = \frac{2}{\mu \cdot S_{13}^2} \left( \frac{1}{\tau_{13}^{1/\varepsilon}} - 1 \right) \tau_{13}
\]

(4.100)

In order to find \( \mu \) and \( \varepsilon \), the shear flow stress is plotted against shear plastic strain at various strain rates, as shown in Figure 4.33. Each of these tests were done at constant shear strain rate and has a unique plastic shear strain history, which could be curved fit to estimate plastic train rate by time derivative. An example of the plastic shear strain history for the test at strain rate 0.05 \( s^{-1} \) is given in Figure 4.34.

Nonlinear regression analysis was used to fit the plastic strain versus time with a cubic polynomial function. Good prediction with this polynomial function was found with test data, as shown in Figure 4.34. Time derivative of the polynomial gives \( \dot{\gamma}_{p13} \). Shear plastic strain rates and the corresponding values of shear flow stress at shear plastic strains of 0.075, 0.12 and 0.23 are plotted in Figure 4.35. It is remarkable that they follow a similar trend, which is given by Equation (4.100). Once again, nonlinear regression analysis was used to find viscoplastic material constants \( \mu \) and \( \varepsilon \). Table 4.9 shows the values of viscoplastic material constants. The curve fit or predicted response using these two values are shown in Figure 4.35. A very good comparison was found between tests and predicted values using these constants.
Figure 4.33 Out-of-plane shear stress-plastic strain curve of PVC H100 foam at various strain rates.

Figure 4.34 Shear plastic strain history for test at 0.05 s$^{-1}$. 
Table 4.9 Viscoplastic material constants.

<table>
<thead>
<tr>
<th>$\gamma_{p13}$</th>
<th>$\mu$ (Sec/MPa)</th>
<th>$\varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.075, 0.12, 0.23</td>
<td>353.6001246</td>
<td>0.054576212</td>
</tr>
</tbody>
</table>

Figure 4.35 Shear strain rate v. shear stress at plastic strains 0.075, 0.12 and 0.23.
4.5 Finite Element Analysis

Finite element analysis (FEA) using ABAQUS Explicit Version 6.13-2 [81] was performed to simulate the PVC H100 foam in the test. The model, mesh and results are discussed in this section.

4.5.1 Model

The PVC foam specimen of length 25.4 mm, width 25.4 mm, and thickness 25.4 mm was modeled in ABAQUS with displacement boundary conditions as specified in the test. As shown in Figure 4.36, the z-faces were tied to Reference Points RP-1 and RP-2. The Reference Points both at the top and bottom in z-direction were constrained to the foam using *MPC Beam constraint. In the MPC constraint, the Slave surface is on the foam, and the Master surface is the Reference Point. Point RP-1 was given displacements $U_1 = \delta \sin \theta$ and $U_3 = \delta \cos \theta$, where $\delta$ is the time-varying displacement provided to the angled specimens. The other Displacement/Rotation conditions at RP-1 were set equal to zero, $U_2 = U_1 = U_3 = 0$. The displacement condition for Point RP-2 was fixed ($U_1 = U_2 = U_3 = U_1 = U_2 = U_3 = 0$). Dynamic Explicit analysis was run with these displacement loadings.
4.5.2 Material properties

An ABAQUS Explicit user-defined material subroutine (VUMAT) was written to describe viscoelastic response before yielding/damage and plastic flow based on the 3D constitutive model in the previous section.

4.5.3 Mesh

The mesh was generated with the help of ABAQUS CAE software. As shown in Figure 4.37, a total of 15,625 linear brick elements were used to model the specimen. The element type was Continuum 3D, 8-node linear brick element with reduced integration (C3D8R). Distortion Control and Enhanced hourglass control were chosen for the C3D8R elements.
4.5.4 Results before yield/damage

Stress distributions in the foam just before yield in the tests at $\theta = 15^\circ$, $45^\circ$ and $75^\circ$ are shown in Figures 4.38, 4.39 and 4.40, respectively. These figures show stresses are not uniform because the specimen surfaces were glued. Constraining effect of the glue gives higher $S_{33}$ ($\sigma_{33}$) near the bond line, and there are also higher shear stresses near the corners due to stress concentration effects. However, in the center of the specimen these stresses $S_{33}$ and $S_{13}$ ($\tau_{13}$) are more or less uniform. Average stresses were calculated for the specimen by using the reaction force from the RP-2 which is at the bottom surface of the polymeric foam. Average strains were calculated by using the displacement from the RP-1, which is at the top surface of the PVC H100 foam.

Stress-strain curves in compression and shear are shown for $\theta = 15^\circ$, $45^\circ$ and $75^\circ$ in Figures 4.41, 4.42 and 4.43, respectively. Only results of the experiments at lowest displacement rate 0.015 mm/s and highest displacement rate 150 mm/s were compared to
FEA in these graphs for clarity. There was too little variation between the two limits to show all the different displacement rates. The experiments and FEA results are in good agreement.

Figure 4.38 Stresses in PVC H100 foam just before yield in $\theta = 15^\circ$ test: (a) Compression and (b) Shear.

Figure 4.39 Stresses in PVC H100 foam just before yield in $\theta = 45^\circ$ test: (a) Compression and (b) Shear.
Figure 4.40 Stresses in PVC H100 foam just before yield in $\theta = 75^\circ$ test: (a) Compression and (b) Shear.
Figure 4.41 Viscoelasticity stress-strain curves comparison between the experiments and FEA for $\theta = 15^\circ$: (a) compression and (b) Shear.
Figure 4.42 Viscoelasticity stress-strain curves comparison between the experiments and FEA for $\theta = 45^\circ$: (a) compression and (b) Shear.
Figure 4.43 Viscoelasticity stress-strain curves comparison between the experiments and FEA for $\theta = 75^\circ$: (a) compression and (b) Shear.
4.5.5 Results after yield/damage

Finite element analysis was also used to simulate the experiments of PVC H100 foam during plasticity at a displacement rate 0.015 mm/s. Figures 4.44 (a) and (b) show the results between FEA and test at $\theta = 15^\circ$. The FEA follows the elastic-plastic response of PVC H100 foam in compression and shear. For $\theta = 30^\circ$, the stress-strain diagram in compression and shear from FEA and experiments are presented in Figures 4.45 (a) and (b). At this angle, the FEA and test have almost the same response during elastic and plastic behavior. At $\theta = 45^\circ$, stress-strain curve in compression and shear for FEA and test are displayed in Figures 4.46 (a) and (b). They have the same behavior in elastic region, but the FEA has slightly higher flow stress compared with the test.

The stress-strain response as predicted by FEA and test for $\theta = 60^\circ$ in compression and shear are shown in Figures 4.47 (a) and (b). The FEA does not follow the test behavior in the plasticity region. This may be because fracture took place early in the corners of the specimen during the test. For $\theta = 75^\circ$, the FEA and test results in compression and shear stress-strain curves are presented in Figures 4.48 (a) and (b). Responses in elastic region of the FEA and test curves follow each other, but they again are different in the plastic region. Again, this variation could be because of fracture near the edges of specimen during the test.
Figure 4.44 Elastic-plastic stress-strain curves comparison between the experiments and FEA for $\theta = 15^\circ$: (a) compression and (b) Shear.
Figure 4.45 Elastic-plastic stress-strain curves comparison between the experiments and FEA for $\theta = 30^\circ$: (a) compression and (b) shear.
Figure 4.46 Elastic-plastic stress-strain curves comparison between the experiments and FEA for $\theta = 45^\circ$: (a) compression and (b) Shear.
Figure 4.47 Elastic-plastic stress-strain curves comparison between the experiments and FEA for $\theta = 60^\circ$: (a) compression and (b) shear.
Figure 4.48 Elastic-plastic stress-strain curves comparison between the experiments and FEA for $\theta = 75^\circ$: (a) compression and (b) Shear.
The yield points for the 60 degrees and 70 degrees test were taken from FEA and added to the Tsai – Wu initial yield surface in Figure 4.49. Now the Tsai – Wu yield criterion could be seen as a very good yield criterion for PVC H100 foam.

![Tsai-Wu criterion and Test Yield Surfaces](image)

Figure 4.49 Comparison of yield surfaces for PVC H100 foam predicted by the Tsai-wu criterion from both test and FEA.
CHAPTER V

CONCLUDING REMARKS

The objective of this research was to investigate the mechanical properties of Divinycell PVC H100 foam under combined cyclic compression-shear loading. A second objective was to develop constitutive models to predict multiaxial behavior of PVC H100 foam. Experiments were designed to obtain out-of-plane mechanical properties of PVC H100 foam under cyclic compression-shear loading. The tests were conducted under various combinations of compression and shear, displacement amplitude and displacement rates.

The PVC H100 foam displayed viscoelastic response before yielding. Yielding and damage then occurred simultaneously because of permanent changes in cell structure. After yielding/damage, PVC H100 foam exhibited plastic flow followed by viscoelastic/viscoplastic damage and hysteresis. Energy dissipation or hysteresis was due to viscoelastic damage and/or viscoplastic damage after initial yielding/damage. Yielding and damage initiation was predicted better by the Tsai – Wu failure criterion than the isotropic crushable foam criterion.
A phenomenological model was developed to describe the behavior of PVC H100 foam before and after damage. Before yielding/damage, the model consisted of a standard linear viscoelastic model, an equilibrium spring in parallel with a Maxwell element. After yielding/damage, a plastic Prandtl element with a viscoplastic overstress damper was placed in series with the standard linear viscoelastic model in order to simulate plastic flow. Spring and damping properties in the standard linear viscoelastic elements were controlled by the amount of damage the foam underwent. Extensive parameter identification for these material properties was undertaken in this research. In most cases, predictions from theoretical models compared very well with test data.

One of the drawbacks of this study was that the specimen lacked uniformity in the distribution of compression and shear stresses, and that stresses and strains in the specimen had to be determined on an average sense. Finite element analysis had to be used to compare the predicted stress-strain response of the specimen after incorporating the constitutive model in an ABAQUS user-material subroutine (VUMAT). Another problem encountered in the experiments was that data obtained for specimens with high amount of shear were questionable. Failure was observed at the edges of these specimens and such failure may have occurred unnoticed even earlier on in the tests. Predicted stresses at yielding/damage initiation from the FEA were higher than data from these experiments. Future work should be done in addressing these problems by designing Arcan or butterfly type specimens with fixtures that would allow them to carry both uniformly-distributed tension-shear or compression-shear loading with no restrictions on the magnitude of shear
strains. Experiments should also be designed to test the PVC H100 foam under full three-dimensional or multiaxial loading.
REFERENCES


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[33] Henkel, Technical Data Sheet

http://www.originlab.com


APPENDIX A

DETERMINATION OF YIELD POINT
Determination of Yield Point

The stress-strain curve for metal foam specimen under uniaxial compression is assumed to show onset of failure by an apparent yield point. There are three ways to determine yield point in this case [80]:

1. The intersection point by the tangent lines between the elastic and plateau regions (e.g., Peroni et al., 2008) as explained in Figure 2.23 (a).

2. The crossing point between the elastic part and the slop tangent line where the curve starts deviate from elastic region (e.g., Doyoyo and Wierzicki, 2003) as displayed in Figure 2.23 (b).

3. The point is determined by insert a line parallel to the elastic section to fined offset plastic strain (e.g., Kushch et al., 2008) as exhibited in Figure 2.23 (c).

![Stress-Strain Curve](image)

**Figure A.1** Explanation yield point in compression stress-strain curve for metallic foam as defined in Ref. [80].
APPENDIX B

YIELD CRITERIA UNDER CONSTRAINED COMPRESSION AND SHEAR
Two of the most commonly used yield criteria for PVC foams, the Tsai – Wu and isotropic yield criteria are given in this section. These criteria are used to derive yield surface for the PVC H100 foam under combined transverse compression and shear.

**Tsai – Wu Yield Criterion**

The Tsai – Wu criterion is designed for orthotropic materials [35] with different strengths in tension and compression. An orthotropic material is described by the following stress-strain relation:

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\tau_{23} \\
\tau_{13} \\
\tau_{12}
\end{bmatrix} = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{bmatrix} \begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{23} \\
\gamma_{13} \\
\gamma_{12}
\end{bmatrix}
\]  

(B.1)

where

\[
C_{11} = \frac{(E_{22} - v_{23}^2E_{33})E_{11}^2}{\Omega} \quad C_{12} = \frac{(v_{12}E_{22} + v_{13}v_{23}E_{33})E_{11}E_{22}}{\Omega}
\]

\[
C_{13} = \frac{(v_{12}v_{23} + v_{13})E_{11}E_{22}E_{33}}{\Omega} \quad C_{22} = \frac{(E_{11} - v_{23}^2E_{33})E_{22}^2}{\Omega}
\]

\[
C_{23} = \frac{(v_{23}E_{11} + v_{12}v_{13}E_{22})E_{22}E_{33}}{\Omega} \quad C_{33} = \frac{(E_{11} - v_{12}^2E_{22})E_{22}E_{33}}{\Omega}
\]

\[
C_{44} = G_{23} \quad C_{55} = G_{13} \quad C_{66} = G_{12}
\]

and \(\Omega = E_{11}E_{22} - v_{12}^2E_{22}^2 - v_{13}^2E_{22}E_{33} - v_{23}^2E_{11}E_{33} - 2v_{12}v_{13}v_{23}E_{22}E_{33}\).

For general 3-D states of stress, the Tsai – Wu failure criterion is given as
\[ X_1 \sigma_{11} + X_2 \sigma_{22} + X_3 \sigma_{33} + X_{11} \sigma_{11}^2 + X_{22} \sigma_{22}^2 + X_{33} \sigma_{33}^2 + X_{44} \tau_{23}^2 + X_{55} \tau_{13}^2 + X_{66} \tau_{12}^2 + \\
2X_{12} \sigma_{11} \sigma_{22} + 2X_{13} \sigma_{11} \sigma_{33} + 2X_{23} \sigma_{22} \sigma_{33} \leq 1 \]  
(B.3)

where

\[
X_1 = \frac{1}{X_t} - \frac{1}{X_c} \quad X_{11} = \frac{1}{X_t X_c} 
\]
(B.4)

\[
X_2 = \frac{1}{Y_t} - \frac{1}{Y_c} \quad X_{22} = \frac{1}{Y_t Y_c} 
\]
(B.5)

\[
X_3 = \frac{1}{Z_t} - \frac{1}{Z_c} \quad X_{33} = \frac{1}{Z_t Z_c} 
\]
(B.6)

\[
X_{44} = \left( \frac{1}{S_{23}} \right)^2 \quad X_{55} = \left( \frac{1}{S_{13}} \right)^2 \quad X_{66} = \left( \frac{1}{S_{12}} \right)^2 
\]
(B.7)

\[
X_{12} = \frac{-1}{2} \sqrt{X_{11} X_{22}} 
\]

\[
X_{13} = \frac{-1}{2} \sqrt{X_{11} X_{33}} 
\]
(B.8)

\[
X_{23} = \frac{-1}{2} \sqrt{X_{22} X_{33}} 
\]

and \( \sigma_{iit}, \sigma_{iic} \) (\( i = 1,3 \)) are tensile and compressive yield strengths in the \( i \)-direction, and \( \tau_{ij} \) is the shear yield strengths on plane \( ij \). Stiffness and strength properties of the Divinycell PVC H100 foam are given in Table 3.2. In accordance with the orientation of PVC H100 foam specimen shown in Figure 3.1.

To create a yield surface for combined compression and shear using the Tsai – Wu yield criterion, substitute \( \varepsilon_{11} = \varepsilon_{22} = 0 \) and \( \gamma_{12} = \gamma_{23} = 0 \) into Equation (B.1) to give

\[
\sigma_{11} = \frac{C_{13} \sigma_{33}}{C_{33}} 
\]
(B.9)
\[ \sigma_{22} = \frac{C_{23} \sigma_{33}}{C_{33}} \]  

(B.10)

\[ \tau_{12} = 0 \]  

(B.11)

\[ \tau_{23} = 0 \]  

(B.12)

The Divinycell PVC H100 foam is transversely isotropic properties so that \( C_{23} = C_{13} \) \( X_2 = X_1 \), \( X_{22} = X_{11} \), and \( X_{23} = X_{13} \). Substituting Equations (B.9)-(B.12) into Equation (B.3) and using transverse isotropy, one gets

\[
2 \left( \frac{X_1 C_{13}}{C_{33}} \right) \sigma_{33} + X_3 \sigma_{33} + 2 \left( \frac{X_1 C_{13}^2}{C_{33}^2} \right) \sigma_{33}^2 + X_{33} \sigma_{33}^2 + X_{55} \tau_{13}^2 + 2 \left( \frac{X_1 C_{13}^2}{C_{33}^2} \right) \sigma_{33}^2 + 4 \left( \frac{X_1 C_{13}}{C_{33}} \right) \sigma_{33}^2 \leq 1 \]  

(B.13)

Simplifying Equation (B.13), one gets

\[
F_1 \sigma_{33} + F_2 \sigma_{33}^2 + X_{55} \tau_{13}^2 = 1 \]  

(B.14)

where

\[
F_1 = 2 \left( \frac{X_1 C_{13}}{C_{33}} \right) + X_3
\]

and

\[
F_2 = 2 \left( \frac{(X_{11} + X_{12}) C_{13}^2}{C_{33}^2} \right) + X_{33} + 4 \frac{X_1 C_{13}}{C_{33}}
\]
Isotropic Yield Criterion

A generalized Hooke’s law for a three-dimensional state of stress and strain of an isotropic material [72] is given by

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\tau_{12} \\
\tau_{23} \\
\tau_{13}
\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix}
(1-\nu) & \nu & \nu & 0 & 0 & 0 \\
\nu & (1-\nu) & \nu & 0 & 0 & 0 \\
\nu & \nu & (1-\nu) & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2}
\end{bmatrix} \begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{12} \\
\gamma_{23} \\
\gamma_{13}
\end{bmatrix}
\] (B.15)

where \( E = E_{33} \) is Young’s modulus and \( \nu = \nu_{13} \) is Poisson’s ratio.

An isotropic yield criterion for PVC foam [6] is given by

\[
\Phi \equiv \hat{\sigma} - Y \leq 0
\] (B.16)

where \( Y = Z_e \) is the yield strength and \( \hat{\sigma} \) is an equivalent stress. The equivalent stress is defined by

\[
\hat{\sigma}^2 = \frac{1}{1 + (\frac{\alpha^2}{3} \sigma_m^2)} [\sigma_e^2 + \alpha^2 \sigma_m^2]
\] (B.17)

where \( \sigma_m \) is the mean stress, \( \sigma_e \) is the von Mises or effective stress and \( \alpha \) is a parameter to define the shape of the yield surface. The mean stress is

\[
\sigma_m = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}
\] (B.18)

The von Mises stress is defined by

\[
\sigma_e = \frac{1}{\sqrt{2}} \left( (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\tau_{12}^2 + \tau_{23}^2 + \tau_{13}^2) \right)^{\frac{1}{2}}
\] (B.19)

The \( \alpha \) parameter is
\[ \alpha = 3 \left( \frac{1}{2} - \nu^p \right)^2 \]  
(B.20)

where \( \nu^p \) is the plastic Poisson’s ratio, which is approximately zero because there is no change in the dimensions of the transverse cross-section due to cell compression. This gives \( \alpha = 3/\sqrt{2} \).

To obtain the yield surface for the PVC H100 foam under combined compression and shear, one applies \( \varepsilon_{11} = 0, \varepsilon_{22} = 0, \gamma_{12} = 0 \) and \( \gamma_{23} = 0 \) to the generalized Hooke’s law to eliminate \( \sigma_{11}, \sigma_{22}, \tau_{12} \) and \( \tau_{13} \). Setting \( \varepsilon_{11} = 0 \) and \( \varepsilon_{22} = 0 \) in Equations (B.15) gives

\[ \sigma_{11} = \nu(\sigma_{22} + \sigma_{33}) \]  
(B.21)

and

\[ \sigma_{22} = \nu(\sigma_{11} + \sigma_{33}) \]  
(B.22)

Since \( \sigma_{11} = \sigma_{22} \),

\[ \sigma_{11} = \sigma_{22} = \frac{\nu}{(1-\nu)} \sigma_{33} \]  
(B.23)

Simplifying the mean stress gives

\[ \sigma_m = \frac{1}{3} \left[ \frac{(1+\nu)}{(1-\nu)} \right] \sigma_{33} \]  
(B.24)

Since \( \gamma_{12} = \gamma_{23} = 0 \), one also get \( \tau_{12} = \tau_{23} = 0 \). Therefore, the von Mises stress reduces to the following:

\[ \sigma_v = \left[ \left( \frac{2\nu - 1}{(1-\nu)} \right)^2 \sigma_{33}^2 + 3\tau_{13}^2 \right]^{\frac{1}{2}} \]  
(B.25)
The yield surface for the PVC H100 foam under combined transverse compression and shear is thus given by

\[
Z_c = \sqrt{2 \left[ \frac{(3\nu^2 - 2\nu + 1)\sigma_{33}^2}{(\nu - 1)^2} \right] + 2\tau_{13}^2}
\]  

(B.26)
APPENDIX C

CYCLIC COMPRESSION-SHEAR TEST RESULTS UNDER VARYING DISPLACEMENT AMPLITUDES AND FIXED DISPLACEMENT RATE
Figure C.1 Stress-strain curves with various displacements control for $\theta = 15^\circ$ at DR = 0.015 mm/s: (a) Compression and (b) Shear.
Figure C.2 Stress-strain curves with various displacements control for $\theta = 30^\circ$ at DR = 0.015 mm/s: (a) Compression and (b) Shear.
Figure C.3 Stress-strain curves with various displacements control for $\theta = 45^\circ$ at DR = 0.015 mm/s: (a) Compression and (b) Shear.
Figure C.4 Stress-strain curves with various displacements control for $\theta = 60^\circ$ at DR = 0.015 mm/s: (a) Compression and (b) Shear.
Figure C.5 Stress-strain curves with various displacements control for $\theta = 75^\circ$ at DR = 0.015 mm/s: (a) Compression and (b) Shear.
Figure C.6 Stress-strain curves with various displacements control for $\theta = 15^\circ$ at DR = 0.15 mm/s: (a) Compression and (b) Shear.
Figure C.7 Stress-strain curves with various displacements control for $\theta = 30^\circ$ at DR = 0.15 mm/s: (a) Compression and (b) Shear.
Figure C.8 Stress-strain curves with various displacements control for θ = 45° at DR = 0.15 mm/s: (a) Compression and (b) Shear.
Figure C.9 Stress-strain curves with various displacements control for $\theta = 60^\circ$ at DR = 0.15 mm/s: (a) Compression and (b) Shear.
Figure C.10 Stress-strain curves with various displacements control for $\theta = 75^\circ$ at $\text{DR} = 0.15 \ \text{mm/s}$: (a) Compression and (b) Shear.
Figure C.11 Stress-strain curves with various displacements control for $\theta = 15^\circ$ at $\text{DR} = 1.5 \text{ mm/s}$: (a) Compression and (b) Shear.
Figure C.12 Stress-strain curves with various displacements control for $\theta = 30^\circ$ at DR = 1.5 mm/s: (a) Compression and (b) Shear.
Figure C.13 Stress-strain curves with various displacements control for $\theta = 45^\circ$ at DR = 1.5 mm/s: (a) Compression and (b) Shear.
Figure C.14 Stress-strain curves with various displacements control for $\theta = 60^\circ$ at DR = 1.5 mm/s: (a) Compression and (b) Shear.
Figure C.15 Stress-strain curves with various displacements control for $\theta = 75^\circ$ at $\text{DR} = 1.5 \text{ mm/s}$: (a) Compression and (b) Shear.
Figure C.16 Stress-strain curves with various displacements control for $\theta = 15^\circ$ at DR = 15 mm/s: (a) Compression and (b) Shear.
Figure C.17 Stress-strain curves with various displacements control for θ = 30° at DR = 15 mm/s: (a) Compression and (b) Shear.
Figure C.18 Stress-strain curves with various displacements control for $\theta = 45^\circ$ at DR = 15 mm/s: (a) Compression and (b) Shear.
Figure C.19 Stress-strain curves with various displacements control for $\theta = 60^\circ$ at DR = 15 mm/s: (a) Compression and (b) Shear.
Figure C.20 Stress-strain curves with various displacements control for $\theta = 75^\circ$ at DR = 15 mm/s: (a) Compression and (b) Shear.
Figure C.21 Stress-strain curves with various displacements control for $\theta = 15^\circ$ at DR = 150 mm/s: (a) Compression and (b) Shear.
Figure C.22 Stress-strain curves with various displacements control for $\theta = 30^\circ$ at $\text{DR} = 150 \text{ mm/s}$: (a) Compression and (b) Shear.
Figure C.23 Stress-strain curves with various displacements control for $\theta = 45^\circ$ at DR = 150 mm/s: (a) Compression and (b) Shear.
Figure C.24 Stress-strain curves with various displacements control for $\theta = 60^\circ$ at DR = 150 mm/s: (a) Compression and (b) Shear.
Figure C.25 Stress-strain curves with various displacements control for $\theta = 75^\circ$ at DR = 150 mm/s: (a) Compression and (b) Shear.
APPENDIX D

CYCLIC COMPRESSION-SHEAR TEST RESULTS UNDER VARYING DISPLACEMENT RATES AND FIXED DISPLACEMENT AMPLITUDE
Figure D.1 Stress-strain curves with various displacement rates for $\theta = 15^\circ$ at compression displacement = 12.7 mm: (a) Compression and (b) Shear.
Figure D.2 Stress-strain curves with various displacement rates for $\theta = 15^\circ$ at compression displacement = 10.16 mm: (a) Compression and (b) Shear.
Figure D.3 Stress-strain curves with various displacement rates for $\theta = 15^\circ$ at compression displacement = 7.62 mm: (a) Compression and (b) Shear.
Figure D.4 Stress-strain curves with various displacement rates for $\theta = 15^\circ$ at compression displacement $= 5.08$ mm: (a) Compression and (b) Shear.
Figure D.5 Stress-strain curves with various displacement rates for $\theta = 15^\circ$ at compression displacement = 2.54 mm: (a) Compression and (b) Shear.
Figure D.6 Stress-strain curves with various displacement rates for $\theta = 30^\circ$ at compression displacement = 10.16 mm: (a) Compression and (b) Shear.
Figure D.7 Stress-strain curves with various displacement rates for $\theta = 30^\circ$ at compression displacement = 7.62 mm: (a) Compression and (b) Shear.
Figure D.8 Stress-strain curves with various displacement rates for $\theta = 30^\circ$ at compression displacement = 5.08 mm: (a) Compression and (b) Shear.
Figure D.9 Stress-strain curves with various displacement rates for $\theta = 30^\circ$ at compression displacement = 2.54 mm: (a) Compression and (b) Shear.
Figure D.10 Stress-strain curves with various displacement rates for $\theta = 30^\circ$ at compression displacement = 1.27 mm: (a) Compression and (b) Shear.
Figure D.11 Stress-strain curves with various displacement rates for $\theta = 45^\circ$ at compression displacement = 5.08 mm: (a) Compression and (b) Shear.
Figure D.12 Stress-strain curves with various displacement rates for $\theta = 45^\circ$ at compression displacement = 4.572 mm: (a) Compression and (b) Shear.
Figure D.13 Stress-strain curves with various displacement rates for $\theta = 45^\circ$ at compression displacement = 4.064 mm: (a) Compression and (b) Shear.
Figure D.14 Stress-strain curves with various displacement rates for $\theta = 45^\circ$ at compression displacement = 3.556 mm: (a) Compression and (b) Shear.
Figure D.15 Stress-strain curves with various displacement rates for $\theta = 45^\circ$ at compression displacement = 2.54 mm: (a) Compression and (b) Shear.
Figure D.16 Stress-strain curves with various displacement rates for $\theta = 60^\circ$ at compression displacement = 2.54 mm: (a) Compression and (b) Shear.
Figure D.17 Stress-strain curves with various displacement rates for $\theta = 60^\circ$ at compression displacement = 2.032 mm: (a) Compression and (b) Shear.
Figure D.18 Stress-strain curves with various displacement rates for $\theta = 60^\circ$ at compression displacement = 1.524 mm: (a) Compression and (b) Shear.
Figure D.19 Stress-strain curves with various displacement rates for $\theta = 60^\circ$ at compression displacement = 1.016 mm: (a) Compression and (b) Shear.
Figure D.20 Stress-strain curves with various displacement rates for $\theta = 75^\circ$ at compression displacement = 1.905 mm: (a) Compression and (b) Shear.
Figure D.21 Stress-strain curves with various displacement rates for $\theta = 75^\circ$ at compression displacement = 1.651 mm: (a) Compression and (b) Shear.
Figure D.22 Stress-strain curves with various displacement rates for $\theta = 75^\circ$ at compression displacement = 1.397 mm: (a) Compression and (b) Shear.
Figure D.23 Stress-strain curves with various displacement rates for $\theta = 75^\circ$ at compression displacement = 1.143 mm: (a) Compression and (b) Shear.
APPENDIX E

MODULUS AND VISCOSITY AFTER YIELDING/DAMAGE
Damage Moduli for Equilibrium Spring

The damage functions used to predict damage versus plastic strain in compression and shear are as the following:

\[ d_\mu = \frac{c_1 \epsilon^c_p}{1 + c_1 \epsilon^c_p} \quad (E.1) \]

and

\[ d_\gamma = \frac{c_1 \gamma^c_p}{1 + c_1 \gamma^c_p} \quad (E.2) \]

Nonlinear regression analysis was used to find the constant parameters \( c_1 \) and \( c_2 \). Tables E.1 and E.2 show the values of material constants.

Table E.1 Damage material constants in compression.

<table>
<thead>
<tr>
<th>Constants</th>
<th>Damage 33</th>
<th>Damage 11=22</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_1 )</td>
<td>112.691481677564</td>
<td>108.597222449788</td>
</tr>
<tr>
<td>( c_2 )</td>
<td>0.775985201160373</td>
<td>0.880725302051788</td>
</tr>
</tbody>
</table>

Table E.2 Damage material constants in shear.

<table>
<thead>
<tr>
<th>Constants</th>
<th>Damage 13</th>
<th>Damage 12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_1 )</td>
<td>150.966787693828</td>
<td>51.4140840071072</td>
</tr>
<tr>
<td>( c_2 )</td>
<td>1.33656591878317</td>
<td>0.961561376289476</td>
</tr>
</tbody>
</table>
The damage using the values in Tables E.1 and E.2 are displayed together with test results in Figures E.1 (a)-(d). Good agreement was found between experimental and predicted curves.

Figure E.1 Predicted damage function of equilibrium spring versus plastic strain response: (a) Out-of-plane compression, (b) In-plane compression, (c) Out-of-plane shear and (d) In-plane shear.
Ratio of Damage to Undamage Moduli for Intermediate Spring

Functions that could be used to predict the ratio of the damage to undamage modulus of the intermediate spring are as follows:

\[
\frac{E_{evii}}{E_{evii}} = 1 + \frac{c_1 \epsilon_p + c_2 \epsilon_p^{c_3}}{c_4 + c_2 \epsilon_p^{c_3}}
\]

(E.3)

and

\[
\frac{G_{esij}}{G_{esij}} = 1 + \frac{c_1 \gamma_p + c_2 \gamma_p^{c_3}}{c_4 + c_2 \gamma_p^{c_3}}
\]

(E.4)

Material constants were found by using nonlinear regression analysis. The values are shown in the following Tables E.3 and E.4. Predicted stiffness ratios with test data are shown in Figures E.2 (a)-(d).

Table E.3 Material constants in compression for ratio of intermediate spring.

<table>
<thead>
<tr>
<th>Constants</th>
<th>Ratio of $\frac{E_{ev33}}{E_{ev33}}$</th>
<th>Ratio of $\frac{E_{ev11}}{E_{ev11}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>391.295215576763</td>
<td>239.985441779132</td>
</tr>
<tr>
<td>$c_2$</td>
<td>59.0152417642692</td>
<td>1.77615693929478</td>
</tr>
<tr>
<td>$c_3$</td>
<td>1.02424366584198</td>
<td>1.41051606016680E-17</td>
</tr>
<tr>
<td>$c_4$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Table E.4 Material constants in shear for ratio of intermediate spring.

<table>
<thead>
<tr>
<th>Constants</th>
<th>Ratio of $\overline{G}<em>{ev13}/G</em>{ev13}$</th>
<th>Ratio of $\overline{G}<em>{ev12}/G</em>{ev12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>59.9113583624629</td>
<td>170.133688103477</td>
</tr>
<tr>
<td>$c_2$</td>
<td>271.593280175093</td>
<td>73.7846551526899</td>
</tr>
<tr>
<td>$c_3$</td>
<td>1.80340053495300</td>
<td>1.09476649199507</td>
</tr>
<tr>
<td>$c_4$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Figure E.2 Predicted ratio of damage to undamage intermediate spring modulus versus plastic strain response: (a) Out-of-plane compression, (b) In-plane compression, (c) Out-of-plane shear and (d) In-plane shear.
Ratio of Damage to Undamage Viscosity for Damper

Functions that could be used to predict the ratio of the damage to undamage viscosity of the damper are as follows:

\[
\frac{\bar{\eta}_{ii}}{\eta_{ii}} = 1 + \frac{c_1 \varepsilon_p + c_2 \varepsilon_p^{c_3}}{c_4 + c_2 \varepsilon_p^{c_3}}
\]  
(E.5)

and

\[
\frac{\bar{\eta}_{ij}}{\eta_{ij}} = 1 + \frac{c_1 \gamma_p + c_2 \gamma_p^{c_3}}{c_4 + c_2 \gamma_p^{c_3}}
\]  
(E.6)

Material constants were found by using nonlinear regression analysis. The values are shown in the following Tables E.4 and E.5. Predicted ratios of viscosity are compared to test data in Figures E.3 (a)-(d).

Table E.5 Material constants in compression for ratio of damper.

<table>
<thead>
<tr>
<th>Constants</th>
<th>Ratio of $\bar{\eta}<em>{33}/\eta</em>{33}$</th>
<th>Ratio of $\bar{\eta}<em>{11}/\eta</em>{11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>573100551.950776</td>
<td>401005.356744121</td>
</tr>
<tr>
<td>$c_2$</td>
<td>1335475.75261932</td>
<td>1068.56579050162</td>
</tr>
<tr>
<td>$c_3$</td>
<td>0.922970247462633</td>
<td>0.983794758508493</td>
</tr>
<tr>
<td>$c_4$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Table E.6 Material constants in shear for ratio of damper.

<table>
<thead>
<tr>
<th>Constants</th>
<th>Ratio of $\frac{\bar{\eta}<em>{13}}{\eta</em>{13}}$</th>
<th>Ratio of $\frac{\bar{\eta}<em>{12}}{\eta</em>{12}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>463.532342527545</td>
<td>723790.737926903</td>
</tr>
<tr>
<td>$c_2$</td>
<td>171.238608793725</td>
<td>4816.98365522713</td>
</tr>
<tr>
<td>$c_3$</td>
<td>2.4776239552561</td>
<td>1.15631552868435</td>
</tr>
<tr>
<td>$c_4$</td>
<td>1</td>
<td>169.972381149516</td>
</tr>
</tbody>
</table>

Figure E.3 Predicted ratio of damage to undamage viscosity versus plastic strain response: (a) Out-of-plane compression, (b) In-plane compression, (c) Out-of-plane shear and (d) In-plane shear.