MODELING COMPLEX CONTACT PHENOMENA WITH NONLINEAR BEAMSHELS

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# Modeling Complex Contact Phenomena with Nonlinear Beamshells

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ABSTRACT

As built-up engineering structures (i.e. structures consisting of many individual pieces connected together) become more complicated and expensive, the need to accurately model their response to dynamic events increases. Take for example electronics mounted to a satellite via a bolted connection. Without the proper understanding of how the electronics will react during launch, the connection will either be over designed, resulting in excess weight, or under designed, resulting in possible damage to the unit. In bolted connections, energy dissipation due to micro-slip (partial slipping of an elastic body in contact that occurs prior to slipping of the entire contact patch) is often the dominating damping mechanism. Capturing this type of nonlinear damping is often challenging within a simulation of large-scale structures. Finite element falls short due to the small element size required to achieve a converged solution in the contact patch. Researchers have been developing reduced order models that capture the micro-slip phenomenon without the numerical penalty associated with finite element analysis. This work shows that nonlinear beamshells could be used as a reduced order model for elastic bodies connected with a frictional connection which exhibits energy dissipation due to micro-slip.

In this work we consider the energy dissipated from an elastic shell on a rigid foundation and focus on two unique contact phenomenon: the effect of shear leading
to load transfer beyond the slip zone, and the effect of compressive material loads that can give rise to receding contact areas. Both phenomenon are investigated using the nonlinear geometrically exact shell theory. It is concluded that edge shearing effects serve to reduce the energy dissipated from the system. This is studied with nonlinear shell theory and validated with finite element analysis. Likewise, it has been postulated that changing contact patch areas during oscillations effects the energy dissipated per load cycle. This work expands on current nonlinear shell theory to account for through thickness compressive stresses as applied to a Cosserat surface. Several examples are solved to show the effects of the expanded nonlinear beamshell theory.
DEDICATION

To my father and mother, Robert and Linda Brink, who taught me that with hard work I could accomplish anything.
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CHAPTER I

INTRODUCTION

The impetus of this research is to determine if the theory of nonlinear shells can be exploited to create a reduced order model for joint dynamics. A reduced order model or model order reduction is a mathematical model of a system or phenomenon which reduces the complexity while preserving certain important characteristics. The need for these types of models arises from complexity that is difficult to capture numerically, such as large system dynamics, modern controls or highly nonlinear systems. They work by reducing the degrees of freedom or state spaces by making approximations on how the system should respond based on mathematical or physical insight and experimentation. For example, as discussed later, Iwan developed a reduced order model for modeling plasticity, including the Bauschinger effect (later applied to joint dynamics) which reduces the response to a two parameter break-free force while preserving the overall response of the structure. His model has been applied to the modeling joints and other frictional interfaces subjected to tangential oscillatory loads. It is discussed at length in Chapter 2.

The term joints, as used in this research, refers to connections between components of a larger engineering structure. A common example of a joint, as shown in Figure 1.1, is a lap joint. This type of joint consists of two or more components held together with a pre-loaded bolt. Pre-loading the bolt creates a squeezing force
between the components which allows friction to carry load in the direction tangent to the joint. The lap joint, as well as other types of frictional joints, are common in engineering structures such as aircraft bodies, satellites, launch vehicles, and bridges, to name a few. Part of the reason they are so commonly used is their ease of assembly, disassembly and manufacturability. However, the same features that make them attractive for manufacturing make them difficult to analyze; e.g. the structure being several components instead of monolithic. If the structure is monolithic the flow of force through it is relatively easy to understand and the dynamic response, assuming a linear elastic material, is linear in nature. However, once a frictional contact interface is added between the components, nonlinear conditions can dominate the systems’ response. One major nonlinearity derives from the nonlinear nature of friction itself. Assuming a Coulomb friction model, the transverse stress that each point on the contact patch carries prior to slipping is proportional to the magnitude of the normal stress at the same point. As seen in Figure 1.2, these types of joints tend to have uneven normal loads at the contact interface; highest near the bolt and decaying quickly moving away from it. This type of force distribution means
that each point carries different transverse loads prior to slip, thus complicating the problem. Additionally, since the normal stress is small at the leading edge of the contact patch, relative *slipping* between the components occurs, dissipating energy from the system. This type of dissipation (known as micro-slip) is a significant source of nonlinear damping to the system and is the primary focus point of this research. Micro-slip, as defined by [1], is the relative elastic and/or plastic deformation that occurs in two bodies which are in contact before the entire contact patch slips. This energy dissipation plays an important role in the overall dynamic response of a built-up structure, contributing up to 90% of the total system damping [2]. For long-life structures, this is detrimental in the form of fretting fatigue [3].

To understand the importance of properly capturing the effect of frictional joints on a system, consider the financial impact of *not* properly modeling them. Brake et. al. [4] detail several examples of mechanical engineering failures which
derive directly from the lack of understanding of mechanical joints. One example they discuss is the grounding of the Airbus A380 European fleet. Cracks in the ribs which hold the wing together developed as a direct results of fretting fatigue in the joint. Brake shows the financial impact of this single failure to be $1.22 billion dollars in lost profit and maintenance cost. Brake also considers the cost of launching a payload into geosynchronous orbit. The average cost to per pound to reach geosynchronous orbit is approximately $16,000. With the typical launch vehicle, crew module, payload and payload interface structures containing hundreds if not thousands of frictional joints each, the potential cost savings in optimizing each joint can be significant. The lack of understanding of joint dynamics in critical applications, such as launching items into space, often leads to them being over designed and hence adds additional weight.

While the dynamic response of joints and other types of frictional interfaces (i.e. any structure which transmits force between two constitutive components of a larger system via friction) are not difficult to capture numerically, they become troublesome when they are to be included in the analysis of large-scale structure. Per the mesh density study shown in Chapter 3, an appropriately mesh length scale to accurately capture micro-slip for a hexahedral element is approximately $5 \times 10^{-5} \ m$. To achieve a numerically stable solution for an explicit direct integration time history solution with this mesh size requires a time step size of $9.6 \times 10^{-9} \ s$ (for steel, this gives a Courant number of 1). This time step size would be prohibitive when studying a larger system such as a Saturn V rocket, whose length is 42.1 $\ m$ and has an average diameter of 10.1 $\ m$. The small time step size makes solving any significant loading event on this rocket impractical and/or possibly impossible.
First studied by Cattaneo in 1938 [5] and Mindlin in 1949 [6], proper modeling techniques for micro-slip continue to elude researchers. Goodman [7] shows that if the amplitude of the tangential forcing function is small, the theoretical energy dissipated from the system is proportional to the cube of the tangential forcing functions’ amplitude. The system under consideration, which is shown in Figure 1.3, consisted of two spheres or cylinders, pressured together to create a Hertzian reaction distribution at the contact interface. The system was then subjected to a oscillatory load tangent to the contact patch. Physical experimentation shows that energy dissipation for small oscillations tends to be proportional to a power between 2.5 and 3 of the forcing amplitude [8]. Johnson attributes this difference from the theoretical value
to internal material hysteresis, variations of the coefficient of friction and effects of surface roughness [9]. Segalman partially attributes the discrepancy to bending of the overall joint structure [10]. When a load is applied to the structure, it can create a moment in the joint which changes the shape of the contact patch, as shown in Figure 1.4. Many other authors attribute it to the inadequacy of current friction models to capture this effect [11, 12, 13]. In Chapter 3 it is postulated that shearing of the elastic body above the contact patch influences the energy dissipation. A finite element study of an elastic continuum contacting a rough, rigid foundation subjected to an oscillating tangential load shows that as the height of the body increases, the energy dissipated per cycle falls away from Goodman’s theoretically predicted value. It also shows that as the amplitude of the oscillating end load approaches the level required to produce macro-slip, the energy dissipated per cycle approaches the theoretical value for all heights studied. This reveals an underlying problem with current reduced order joint models (such as Iwan elements) in that they assume a one-dimensional geometry which is incapable of capturing this effect [14].

In addition to the gross structural bending described by [10], the distribution of the compressive normal load, also known as the squeezing force, shown in Figure 1.2, tends to create receding contact patches. This effect, described in detail later in this research, reduces the original contact patch size. As shown by Dundurs [15],
receding contact occurs discontinuously upon the application of a squeezing load. The shape of the resulting receding contact patch is only a function of the normal load distribution and not the magnitude of the normal load. Although not transient in nature, receding contact must still be accounted for to accurately model joint dynamics since it affects the characteristics of the frictional interface.

An effective reduced order joint model is capable of capturing the effects mentioned above, is easily inserted into existing analysis techniques, and greatly reduces the computational cost over traditional analysis methods. It is postulated that nonlinear shells can be used for this application. Shells have a long history of use and have been thoroughly investigated, both theoretically and experimentally (as far back as the 14th century with Leonardo Da Vinci [16]). They have proven a reliable and effective tool in studying the dynamic response of structures both with direct time integration and modal analyses. It is shown in this paper that nonlinear shells are capable of capturing the complex contact phenomena described above. Thus, with the obvious reduction shells offer in computational cost over three-dimensional analysis methods, this paper shows that nonlinear shells are a viable candidate as a reduced order joint model.
CHAPTER II
LITERATURE REVIEW

This research investigates two unique contact phenomena as they relate to the dissipation in mechanical joints, using nonlinear shell theory. Since shell theory is the overarching theme, a brief history of its evolution is presented followed by the foundations of nonlinear shell theory. This is followed by a literature review of energy dissipation in mechanical joints and receding contact.

2.1 A Brief History of Beam and Shell Theory

Beam and shell theory has been studied since as early as the 14th century starting with Leonardo da Vinci (1452-1519) [16]. While he did not officially publish his experiments nor his findings, historians found work in his personal notebooks. In 1638 Galileo laid out the first officially published work relating to beam theory in his work *Two New Sciences* [17]. While not a mathematically rigorous investigation, he poses and answers several important questions. First he considers the resistance to fracture of a prism whose width is greater than its thickness. He concludes that when acting under its own weight that it offers greater resistance when standing on edge. He also concludes that a cantilever beam acted on by its own weight will see its bending

---

1The history of beam theory presented here is derived from Timoshenko’s *History of Strength of Materials* [16]. References to the original papers are added for the convenience of the reader.
moment increase by the square of its length. Taking an interest in the work of Galileo, Mariotte (1620-1684) worked experimentally and found Galileo’s failure theory for a cantilever beam to be exaggerated [16]. He introduced material deformation into the his beam formulation. Furthermore, he is the first to suggest that fibers in a cantilever beam subjected to a moment are in compression in the lower portion of the beam and in tension in the upper portion. Looking for an application for infinitesimal calculus, Jacob Bernoulli (1654-1705) defined the shape of a cantilever beam acted upon by an end load. Using Mariotte’s assumption of fiber deformation, and employing Hooke’s law, he finds that the tensile force acting on all fibers in an infinitesimal section are described by,

\[ P = \frac{1}{2} m \Delta ds \frac{A}{ds}, \]  

(2.1)

where \( P \) is the tensile force, \( A \) is the cross-sectional area, \( m \) is a material constant and \( ds \) is the infinitesimal arc of the beam. This lead him to the final form,

\[ \frac{C}{r} = Px, \]  

(2.2)

where \( C \) represents the beam’s resistance to bending and \( r \) is the radius of curvature at a point. While the form of this equation is correct, he incorrectly calculates \( C \) because he assumes the beams neutral bending axis is located on the bottom of the concave side of the beam. Jacob Bernoulli’s grandson, Daniel Bernoulli (1700-1782), made two important contributions to beam theory. First, he is the first mathematician to formulate the differential equations of motion for a laterally vibrating, prismatic
beam. Second, he suggested to Leonard Euler that he should study the deformation of an elastic beam using the newly developed field of mathematics, variational calculus. He verified his equations of motion with experimentation. Leonard Euler (1707-1783) was the first to integrate the differential equations of motion derived by Daniel Bernoulli. Published in 1744 [18], he used variational calculus to minimize the strain energy in an initially straight elastic beam of uniform cross-section. This minimization yielded a form of Bernoulli’s Equation (2.2),

$$C \frac{y''}{(1 + y'^2)^{3/2}} = Px,$$  \hspace{1cm} (2.3)

where $y$ is the lateral displacement. Notice that Equation (2.3) does not depend on a small deformation assumption and is nonlinear in form. By limiting his discussion to small deflections, he defines the curvature of the beam as $d^2y/dx^2$ and arrives at the familiar form of the Euler-Bernoulli beam,

$$C \frac{d^4y}{dx^4} = \frac{wy}{l},$$  \hspace{1cm} (2.4)

where the fourth derivative is due to the fact that the second derivative of the moment is equal to the intensity of the lateral load. Additionally, in studying the lateral deformation of two perpendicular strings, Euler derived the differential equation for a membrane,

$$\frac{\partial^2 w}{\partial t^2} = A \frac{\partial^2 w}{\partial x^2} + B \frac{\partial^2 w}{\partial y^2},$$  \hspace{1cm} (2.5)
where \( w \) is the lateral displacement of a material point, \( t \) is time, \( x \) and \( y \) are perpendicular material planes, and \( A \) and \( B \) are constants. This approximation of a membrane led Jacques Bernoulli (1759-1789) to his approximate solution of a plate in bending [19],

\[
D \left( \frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} \right) = q. \tag{2.6}
\]

C. A. Coulomb (1736-1806) found the correct solutions to many important beam problems in his landmark memoir [20]. In addition to several experiments related to beam theory he gives a detailed theoretical discussion on beams in bending. He concludes that the axis of rotation of a cross-section, which as mentioned earlier was assumed to be at the bottom of the concave surface, must be further off of the surface.

In 1809 Napoleon, then the emperor of France, became interested in the theory of the bending of plates and had the French Academy propose a contest to develop a mathematical theory of plate vibration and compare them to experimental results. Sophie Germain (1776-1831), with assistance from Lagrange, found the satisfactory form of the equation and won the prize (after three attempts),

\[
k \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + \frac{\partial^2 w}{\partial t^2} = 0, \tag{2.7}
\]

where \( k \) is a constant. The next breakthroughs in beam theory came from Navier (1785-1836) in his published mechanics lectures [21]. In these lectures he proves that for materials following Hooke’s law, the neutral axis of a beam must be located at the centroid of the cross-section. The main body of his work, however, is concerned
with the solution of statically indeterminate problems. He states that problems are only indeterminate in as much as the beam is considered absolutely rigid. He shows that by introducing the equations of elasticity into the problem that they become solvable. In a similar manner he expanded upon Euler’s work with initially curved beams. Also, building on the work of Germain, Navier develops a similar equation for the flexural lateral displacements of a plate using a particle theory proposed by Poisson. For statics he recovers Equation (2.7), but with the proper elastic constants.

The theories thus far have assumed that forces and displacements due to shear in beams and plates are negligible. Barre de Saint-Venant (1797-1886) is the first to consider shear force in determining the total deformation [22]. Jacques Antoine Charles Bresse (1822-1883) was the first to include the effect of rotary inertia of a beam [23]. It is interesting to note that including shear is often attributed to Stephen Timoshenko and including the rotary inertia term to Lord Rayleigh, but Bresse discussed both of these nearly 60 years earlier. Probably the most important publication to modern plate theory comes from Gustave Robert Kirchhoff (1824-1887). By assuming that (1) all lines initially normal to a cross-section remain normal during the deformation and (2) stretching does not occur, he developed the correct expression for the strain energy of a bent plate [24],

\[
V = \frac{1}{2} D \int \int \left[ \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 + 2\mu \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \\
+ 2(1-\mu) \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \, dx \, dy, \quad (2.8)
\]
where $V$ is the strain energy, $D$ is the flexural rigidity and $\mu$ is a material constant.

Using the principle of virtual work he arrives at the equation for bending of a plate,

$$D \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) = q,$$

where $q$ is an applied load. Kirchhoff, in conjunction with A.E.H. Love, developed the base assumptions for modern-day small deflection shell theory [25]:

1. The plate is composed of linear-elastic, isotropic material
2. The plate’s initial configuration is flat
3. The thickness of the plate is small compared to its other dimensions
4. Deflections are small compared to the plate thickness
5. The slope of the deflected surface is small compared to unity
6. Ignoring shear, straight lines normal to the surface remain straight and normal after deformation
7. The plate is deformed by displacing material points comprising a middle surface
8. Through thickness stresses, normal to the surface, are small

Kirchhoff’s work was extended by A. Clebsch (1833-1872) and von Kármán when they allowed for large deformations. Additionally, Clebsch suggested that stress should be averaged through the thickness of the beam. A collaboration between Kirchhoff and Clebsch led to the addition of stretching into the equations of deformation [26].
Turning now to more modern beam theories, Stephen P. Timoshenko (1878-1972) developed the first beam theory that fully contained rotary inertia terms, as well as displacement due to shearing [27]. While these terms are important to the determining a more exact beam displacement, the real benefit of including these additional terms was discussed by W. Flugge [28]. Flugge noted that without the addition of these terms that travelling waves propagate at an infinite velocity. By adding shear and rotary inertia, these waves take on a finite velocity. Timoshenko, in conjunction with S. Woinowsky-Kreiger, also wrote a landmark textbook for plates and shells [29]. In this text they not only methodically derive the equations of motion, but also solve many useful examples. R.D. Mindlin (1906-1987) wrote several important papers expounding upon the principles layed out by Timoshenko [30, 31]. His paper on rotary inertia and shear in plates is considered a landmark paper in the theory of bending of plates. This paper gives a complete and thorough investigation into high-order plate theory.

This is obviously not a complete history on the subject, and many more great minds have contributed to the field (A.E.H. Love, Reissner, etc...), but it is sufficient to bring the discussion to nonlinear shell theory, which is presented in the next section.

2.2 Nonlinear Shell Theory

Using the definition of nonlinearity often used in continuum mechanics, a shell is said to be nonlinear if the deflections of a point in the shell is not linearly proportional to
the magnitude of the applied load. Libai [32] identifies the two sources of nonlinearity to be:

1. Geometric – The strain-displacement relationship is nonlinear.


Unlike the research presented in the historical section of this chapter in which most shells are assumed to follow Hooke’s law and are restricted to small rotations, most nonlinear shell theories have geometrically exact equations of motion. Libai [32] suggests that two approaches exist when deriving the equations of nonlinear shells: direct and derived.

The direct method was first investigated by the Cosserat brothers in 1908 [33], hence the bodies in this method are often called Cosserat surfaces or Cosserat continua. Essentially this technique ignores the thickness of the body entirely and treats the shell as existing in two-dimensions only. Each point along the surface defining the shell has a stiffness and mass associated with it. This approach is nearly isolated from three-dimensional continuum mechanics. In addition to Cosserat’s original work, a detailed explanation of this method is provided by Ericksen and Truesdale [34]. They start with a mathematically rigorous derivation of the motion of a thin body. In this they give an explanation of the bodies intrinsic strain based on the curvature of the differential geometries. The paper’s main focus is on the strain of orientation. First suggested by Duhem [35] in 1893, the body is not only a collection of points, but also a collection of location vectors associated with each point. Known as directors, these vectors have stretch and rotations associated with them which are independent of the
body’s particles. A bulk of the Erickson and Truesdale paper is spent explaining the strains associated with these directors. The rest of the paper discusses stresses in the shells. This is the basis of the Libai and Simmonds formulation given later in this section.

The derived approach is considerably more complicated than the direct approach. In this formulation a distribution of stress is assumed through the thickness of the shell. Then, decomposing this stress field using asymptotic expansions, the form of the shell’s equations of motion is found. The earliest work in this field was conducted by Goodier [36]. A bulk of the work in this field is conducted by John [37, 38, 39]. This type of shell theory is not the focus of this research.

The shell theory proposed by Libai and Simmonds [32] is used to research the contact problems presented later in this paper. In their formulation, the equation of motion are derived directly from three-dimensional continuum mechanics. As such, these equations are exact. Only after these exact equations are formulated, assumptions on conservation of energy and material constitutive laws are made. Their shell theory is outlined in the next section for a beamshell.

2.2.1 Formulation of a Beamshell using the Methods of Libai and Simmonds

Following the derivation of Libai and Simmonds [32], a geometrically exact, nonlinear beamshell is formulated. A beamshell is a deformable line extruded to form a surface. This shell is only capable of cylindrical motion in one plane, i.e. uniform deformation along its length. This derivation assumes that the shell can undergo large deformations. To accomplish this an orthogonal triad (also known as a director) is
attached to each material point along the shell surface. The strains are described in a Lagrangian framework in reference to these coordinates. In addition to displacements and rotations associated with each material point, the triad is also able to displace and rotate. This makes the formulation of displacements for the problem mathematically simplistic (but, as noted by Simo [40], makes the kinetic portion of the problem cumbersome).

Newton’s Third Law for a continuum states that the surface tractions acting over the surface of a closed volume plus the total body forces acting inside of that same closed volume is equal to the total impulse of that body [41],

$$
\int_A \mathbf{S}_t \, dA + \int_V \rho \mathbf{S}_b \, dV = \frac{d}{dt} \int_V \rho \mathbf{v} \, dV,
$$

where \( \mathbf{S}_t \) is the vector of tractions acting on the surface bounding the volume, \( \mathbf{S}_b \) is the vector of body forces acting on the volume, \( \rho \) is the mass density of the volume and \( \mathbf{v} \) is the vector of the material particles’ velocity. The beamshell domain is described by the \((\sigma, \zeta)\) coordinate system, where \( \sigma \) is tangent to the beamshell reference curve and \( \zeta \) is normal to the reference curve. The equations of motion for the beamshell are developed on a differential continuum element spanning the domain \( a \leq \sigma \leq b \) and \( -H \leq \zeta \leq H \). An example of this differential element in the undeformed and deformed configurations is shown in Figure 2.1. Since, as stated earlier, the beamshell only allows cylindrical deformation in a plane, Equation (2.10) is reduced to tractions acting on the boundaries and body forces acting on an area. Working on a differential
Figure 2.1: Beamshell Geometry and Force Definition
element,

\[
\int_{-t}^{t} \left[ \int_{-H}^{H} S^\sigma \mu d\zeta \right]_a^b + \int_{a}^{b} \left( \int_{-H}^{H} f \mu d\zeta + S^\zeta \mu \right) \right|_{-H}^{H} d\sigma \right] dt \\
- \int_{a}^{b} \left( \int_{-H}^{H} \rho \dot{\mathbf{x}} \mu \zeta \right) \right|_{-t}^{t} d\sigma = 0, \quad (2.11)
\]

where \(S^\sigma\) and \(S^\zeta\) are the components of stress acting on the edge and face of the body, respectively, and \(\mu\) is a geometric scale factor. For future references, \emph{edge} is defined as \(\sigma = a, b\) and \emph{face} is defined as \(\zeta = \pm H\). A physical interpretation of the integral equations of motion in Equation (2.11), is that the stress is integrated through the thickness of the continuum to achieve a resultant internal force. This force, in conjunction with the tractions applied to the faces, when imposed on the final beamshell, act to produce the same net translational motion as the actual three-dimensional stress state. As such, let the internal force vector be defined as

\[
\mathbf{F} = \int_{-H}^{H} S^\sigma \mu d\zeta, \quad (2.12)
\]

which is defined per unit width of the beamshell. Similarly, let the external force vector be

\[
\mathbf{p} = \int_{H}^{H} f \mu d\zeta + S^\zeta \mu \right|_{-H}^{H}, \quad (2.13)
\]

which is defined per unit width, per unit length of the deformed reference curve, \(C\).
Thus, the final integral equation of linear momentum is

\[
\int_{-t}^{t} \left( F^b_a + \int_a^b p \, d\sigma \right) \, dt - \int_a^b m v^t_{-t} \, d\sigma = 0. \tag{2.14}
\]

To form the differential equations of motion, take the derivative of the internal force vector along the beamshell length (along \( \sigma \)). This allows the entire equation to be expressed inside the same integrand. This yields the differential equation of linear motion,

\[
F' + p - m \dot{v} = 0. \tag{2.15}
\]

Note that throughout this paper a prime symbol \((f'(x,t))\) indicates a spatial derivative and a dot over a function \((\dot{f}(x,t))\) indicates a derivative with respect to time.

The rotational equations of motion are derived in the same manner. In a continuum the derivative of the moment of momentum is equal to the vector sum of the external moments [41],

\[
\int_A (r \times S_t) \, dA + \int_V (r \times \rho S_b) \, dV = \frac{d}{dt} \int_V (r \times \rho v) \, dV, \tag{2.16}
\]

where \( r \) is the vector from the undeformed to deformed material point in the continuum. Working with the same static differential beamshell element as before, this
becomes the scalar equation

$$\int_{-t}^{t} \left( [e_z \cdot (\bar{y} \times F) + M]^b_a + \int_a^b [e_z \cdot (\bar{y} \times p) + l] \, d\sigma \right) \, dt$$

$$- \int_a^b [e_z \cdot (\bar{y} \times m\nu) + I\omega] |_{-t}^t \, d\sigma = 0, \quad (2.17)$$

where $e_z$ is the unit vector acting along the depth of the beamshell (the direction of extrusion). The other terms require more background to describe. The undeformed beamshell reference curve is described by the parametric curve $y(\sigma)$. The deformed projection of this curve is $\bar{y}(\sigma)$. It is important to note that $\bar{y}$ is not simply the deformed position of a material point lying on the beamshell, but is rather the center of mass of the deformed configuration. The internal moment per unit width, $M$ is defined as

$$e_z \cdot \int_{-H}^{H} \bar{z} \times S^\sigma \mu \, d\zeta, \quad (2.18)$$

where $\bar{z}$ is the deformed position of a material point normal to the beamshell reference curve. In conjunction with the center of mass location, a material point is located by

$$\bar{r} = \bar{z} + \bar{y}. \quad (2.19)$$

The external couple per unit length of the beamshell reference curve $C$, per unit width is

$$l = e_z \cdot \left( \int_{-H}^{H} \bar{z} \times f\mu \, d\zeta + S^\zeta \mu |_{-H}^{H} \right). \quad (2.20)$$

As with the linear momentum balance, all terms are brought within the same inte-
gral to form the differential equation of angular momentum. Hence, the complete differential equations of motion are

\[
\text{Linear Momentum Balance: } F' + P - m\dot{v} = 0, \quad (2.21)
\]
\[
\text{Angular Momentum Balance: } M' + m \cdot F + l - I\dot{\omega} = 0, \quad (2.22)
\]

where \(m\) is the rational normal defined as

\[
m = e_z \times \bar{y}'. \quad (2.23)
\]

The beamshell is described in one of four coordinate systems shown in Figure 2.2: undeformed shell coordinates \((t, b)\), deformed shell coordinates \((\bar{t}, \bar{b})\), shell cross-section coordinates \((T, B)\), or Cartesian coordinates \((e_x, e_y)\). Working in the cross-section coordinates, the strain normal to the cross-section is denoted \(e\) and the strain tangent to the cross-section is denoted \(g\). The bending strain is the same for all coordinate systems and is defined as the change in the angle between the undeformed reference curve \(C\) and the vector normal to the cross-section \((T)\) per unit reference curve length \((d\sigma)\), which is denoted \(k = \beta'\). The internal forces and moments conjugate to these strains are \(N, Q\) and \(M\), respectively. Therefore, the internal force vector, defined earlier, is

\[
F = NT + QB. \quad (2.24)
\]

To find the derivative of the internal force vector, a relationship is first developed between the cross-section coordinate system and the Cartesian coordinate system.
Figure 2.2: Coordinate System Definition: \((T, B)\) - Cross-Section Coordinates, \((t, b)\) - Undeformed Shell Coordinates, \((\bar{t}, \bar{b})\) - Deformed Shell Coordinates, \((e_x, e_y)\) - Cartesian Coordinates, \(\alpha\) - Angle Between Undeformed Shell and \(e_x\), \(\beta\) - Angle Between Undeformed Shell and Normal to the Cross-Section, \(\gamma\) - Angle Between Normal to the Cross-Section and the Deformed Shell, \(\alpha\) - Angle Between \(e_x\) and the Deformed Shell
This relationship is:

\[
T = \cos (\alpha + \beta) e_x + \sin (\alpha + \beta) e_y, \quad (2.25)
\]

\[
B = -\sin (\alpha + \beta) e_x + \cos (\alpha + \beta) e_y. \quad (2.26)
\]

It is clear from Equations (2.25) and (2.26) that

\[
T' = (\alpha' + k)B, \quad (2.27)
\]

\[
B' = - (\alpha' + k)T. \quad (2.28)
\]

With the derivative of the coordinate system, the derivative of the internal force vector is

\[
F' = [N' - (\alpha' + k)Q] T + [Q' + (\alpha' + k)N] B. \quad (2.29)
\]

Substituting Equation (2.29) into Equation (2.21) yields the two linear momentum balance equations which are written as

\[
N' - (\alpha' + k) Q + P_T - m\dot{v}_T = 0, \quad (2.30)
\]

\[
Q' + (\alpha' + k) N + P_B - m\dot{v}_B = 0. \quad (2.31)
\]

Turning now to the angular momentum balance, the internal force vector creates a moment, described by \( \mathbf{m} \cdot \mathbf{F} \). In terms of the cross-section coordinate system, the
unit vector tangent to the beamshell is

$$\mathbf{y'} = (1 + e)\mathbf{T} + g\mathbf{B}.$$  \hspace{1cm} (2.32)

With Equation (2.32), the moment created by the internal force vector is

$$\mathbf{m} \cdot \mathbf{F} = (1 + e)Q - gN.$$ \hspace{1cm} (2.33)

Inserting Equation (2.33) into Equation (2.22) yields

$$M' + (1 + e)Q - gN + l - I\ddot{\omega} = 0.$$ \hspace{1cm} (2.34)

Equations (2.30), (2.31), and (2.34) constitute the differential equations of motion of the nonlinear beamshell. Note that to this point, the equations are exact.

The weak form of the equations of motions are found by taking the dot products of Equations (2.21) and (2.22) by the test functions \( \mathbf{V}(\sigma,t) \) and \( \Omega(\sigma,t) \), respectively, and integrate over the differential element \((a,b)\). This is written as

$$\int_{t_1}^{t_2} \int_{a}^{b} \left[ (\mathbf{F'} + \mathbf{p}) \cdot \mathbf{V} + (M' + \mathbf{m} \cdot \mathbf{F} + l) \Omega \right] d\sigma dt.$$ \hspace{1cm} (2.35)

Let the test functions be the translation velocity, \( \mathbf{v} \), and the angular velocity, \( \omega \).
Integrating Equation (2.35) by parts yields

\[
\int_{t_1}^{t_2} (\mathbf{F} \cdot \mathbf{v} + M \omega) \big|_a^b \, dt + \int_{t_1}^{t_2} \int_a^b (\mathbf{p} \cdot \mathbf{v} + l \omega) \, d\sigma \, dt \ldots
\]

- \int_{t_1}^{t_2} \int_a^b [\mathbf{F} \cdot (\mathbf{v}' - \mathbf{m} \omega) + M \omega'] \, d\sigma \, dt = 0. \quad (2.36)

The spatial derivative of the velocity in Equation (2.36) is resolved using the spatial derivative of the deformed tangent vector defined in Equation (2.32) as

\[
\mathbf{v}' = \dot{\mathbf{y}}' = \dot{\mathbf{e}} \mathbf{T} + \dot{\mathbf{g}} \mathbf{B} + \omega \mathbf{e}_z \times \dot{\mathbf{y}}'
\]

\[
= (\dot{\mathbf{e}} - \omega g) \mathbf{T} + (\dot{\mathbf{g}} + \omega (1 + e)) \mathbf{B}. \quad (2.37)
\]

Plugging these new relations into the deformation power defined in Equation (2.36) gives

\[
\int_{t_1}^{t_2} \int_a^b \left[ \mathbf{F} \cdot (\dot{\mathbf{e}} \mathbf{T} + \dot{\mathbf{g}} \mathbf{B}) + M \dot{k} \right] \, d\sigma \, dt. \quad (2.39)
\]

Since the dot vector product is commutative and distributive, this is rewritten as

\[
\int_{t_1}^{t_2} \int_a^b \left[ (\mathbf{F} \cdot \mathbf{T}) \dot{e} + (\mathbf{F} \cdot \mathbf{B}) \dot{g} + M \dot{k} \right] \, d\sigma \, dt. \quad (2.40)
\]

It is known from Equation (2.24) that the internal force vector \( \mathbf{F} \) dotted with the beamshell unit vectors yields the component of force in that direction. Hence, the deformation power is

\[
\int_{t_1}^{t_2} \int_a^b \left[ N \dot{e} + Q \dot{g} + M \dot{k} \right] \, d\sigma \, dt. \quad (2.41)
\]
To this point the equations have been exact, derived solely from the three-dimensional
equations of equilibrium. Libai and Simmonds introduce the first approximation when
postulating a form of the conservation of energy principle. Their suggestion is that
the internal energy of the beamshell is a functional of the strains \(e, g, \text{ and } k\) only.
This admits a stress for every strain with no extraneous strain terms. With this
assumption, for a static beamshell the conservation of mechanical energy is written
as
\[
\int_{t_1}^{t_2} W \, dt = \mathcal{E}(e, g, k)|_a^b \tag{2.42}
\]
where \(W\) is the apparent external power defined in Equation (2.36). Assuming an
elastic material whose strain-energy density, \(\Phi\), is differentiable in \(e, g, \text{ and } k\), Equa-
tion (2.41) becomes
\[
\int_a^b \left[ (N - \Phi_{,e}) \dot{e} + (Q - \Phi_{,g}) \dot{g} + (M - \Phi_{,k}) \dot{k} \right] \, d\sigma = 0 \tag{2.43}
\]
For Equation (2.43) to be identically zero for all possible displacements and rotations,
every parenthetical term be identically zero. Thus, the relationships between the
internal forces and their conjugate strains are
\[
N = \Phi_{,e} \tag{2.44} \\
Q = \Phi_{,g} \tag{2.45} \\
M = \Phi_{,k} \tag{2.46}
\]
Each particular system being analyzed will be best served with a particular strain energy density function. It is up to the user to define an strain-energy density that is appropriate for the system being examined.

2.3 Energy Dissipation due to Micro-Slip in Contact Patches

Micro-slip, as defined by [1], is the relative elastic and/or plastic deformation that occurs at the interface between two bodies prior to gross-slip initiation. This phenomenon is important in the study of energy dissipated from jointed systems, fretting fatigue [3], the study of geological phenomena [42] and engineering control theory [43]. Some of the earliest observations of this phenomenon were made by Stevens [44]. While physically measuring the displacement of two beams subjected to tangential loads, he noted that some points of the bodies experienced motion relative to each other prior to gross-slip occurring.

Cattaneo [5] and Mindlin [6] were the first to provide an in-depth investigation of energy dissipated by micro-slip. They examine two elastic cylinders pressed against each other subject to Hertzian contact. Assuming a constant parabolic distribution of normal stress at the contact interface, $P$, and an increasing tangential traction, $Q$, they determine that slip occurs at the boundaries of the contact patch immediately, but not over the entire contact patch. If the force of friction is limited by $\mu P$, per the standard Coulomb friction model, then it is evident that slipping at the edges of the contact patch should occur first as $P \to 0$ at the edges of the contact patch. With the total contact patch defined by $2a$ and the stuck domain $2c$, they define the ratio
of stuck-to-total contact patch lengths as

\[
\frac{c}{a} = \left(1 - \frac{Q}{\mu P}\right)^{\frac{1}{3}}.
\]  \hspace{1cm} (2.47)

Treating spheres in a similar fashion, the ratio is the same parenthetical term raised to the 1/3 power. Gaining in complexity, Mindlin et al. [45, 46] investigated an oscillating tangential traction, \(Q\), for two elastic spheres in contact. It is found that the energy dissipated per cycle is

\[
\Delta W = \frac{9\mu^2 P^2}{10a} \left(\frac{2 - \nu_1}{G_1} + \frac{2 - \nu_2}{G_2}\right) \\
\times \left[1 - \left(1 - \frac{Q}{\mu P}\right)^5 - \frac{5Q}{6\mu P} \left(1 - \left(1 - \frac{Q}{\mu P}\right)^{2/3}\right)\right],
\]  \hspace{1cm} (2.48)

where \(\Delta W\) is the energy dissipated from the system, \(\nu_i\) and \(G_i\) are the Poisson’s ratio and shear constant of each sphere’s material, respectively. Goodman and Brown validated this experimentally [47]. Goodman [7] shows that if the amplitude of the tangential forcing function is small, the energy dissipated from the system is proportional to the cube of the force amplitude, such as

\[
\Delta W = \frac{1}{36a\mu P} \left(\frac{2 - \nu_1}{G_1} + \frac{2 - \nu_2}{G_2}\right) Q^3.
\]  \hspace{1cm} (2.49)

Physical experimentation shows that energy dissipation for small oscillations tends to be proportional to a power between 2.5 and 3 of the forcing amplitude [8]. Johnson attributes this difference from the theoretical value to internal material hysteresis,
variations of the coefficient of friction and effects of surface roughness [9]. Segalman partially attributes the discrepancy to bending of the overall joint structure [10]. When a load is applied to the structure, it can create a moment in the joint which changes the shape of the contact patch, as shown in Figure 1.4. Many other authors attribute it to the inadequacy of current friction models to capture this effect [11, 12, 13]. The work of Cattaneo and Mindlin was extended from cylinders and spheres to a more general half space method by Ciavarella [48] and Jager [49].

Large engineering structures often consist of many constitutive components joined together by a mechanical joint. Commonly, these joints rely on friction to transmit load across the joint body, and as such, micro-slip occurs. It is estimated that up to 90% of the damping in one of these structures comes from micro-slip in the joints [2]. These built-up structures are often complex, expensive, and have designs which are highly dependant on a sound understanding of their dynamic response. Some examples include aircraft frames, launch vehicles and high-precision robotics. The most commonly used tool of current engineers, finite element method, cannot be directly employed to capture the micro-slip phenomenon without special purpose elements being developed.
The problem with traditional finite element analyses is that the length scale required to capture micro-slip accurately is in the order of $5 \times 10^{-5}$ m, which makes the time scale necessary for a stable solution extremely small as well [50]. While these scales may not seem detrimental for analyzing a single simple joint, if the structure is large, as the examples mentioned earlier, the entire structural response is dictated by this time scale, hence, making a solution computationally intensive. One approach to this problem is to develop reduced order models that accurately capture micro-slip without the prohibitive time and length scales.

In 1966 and 1967 W.D. Iwan developed a one-dimensional model to capture yielding behavior of material and structures which present the Bauschinger effect of the Massing type [51, 52]. He based his model on physical observations that hysteretic materials behave like a system of elastic elements having different yield points. His model consisted of series and parallel Jenkin’s elements (see Figure 2.3) assembled in either parallel or series. As seen in Figure 2.3, a Jenkin’s element is an elastic element coupled with a frictional damper. An example of a parallel-series model is shown in Figure 2.3. The frictional component of his model had a two parameter break-free force determined by a band-limited probability density function. From Iwan’s initial postulation of this model, many researchers have worked to improve it. Quinn and Miller [53] used two sets of parallel-series Iwan models to incorporate the effect of
both sides of the joint interface. Segalman [54] made a significant contribution in formulating the four-parameter Iwan model. While Iwan elements are predictive in nature, they still require parameters to be entered as collected by physical testing.

Segalman proposed the four parameters needed are:

- $F_S$ is the force needed to initiate macro-slip
- $K_T$ is the joint’s stiffness
- $\chi$ is related to the slope of the log energy dissipation versus log modal force during micro-slip
- $\beta$ is related to the level of energy dissipated and the shape of the energy dissipation curve near macro-slip

Quinn and Segalman [14] explored using a series-series Iwan model to capture micro-slip effects. Song et al. [55] use the same parameters used by Segalman but use a neural network to solve for their values. Some detailed overviews of Iwan models are found in [56, 12] and [50]. Mohanty and Nanda [57] study energy dissipation in mechanical joints due to micro-slip using a theoretical beam formulation. However, it is different than the study presented in this work because their beam formulation is geometrically linear and does not include shear. They compare their numerical results to the experimental and show good correlation.

Chen and Deng [58] conduct a finite element study of two plane strain geometries clamped together, as in a two component lap joint. The purpose of their study was to validate under what conditions the Metherell-Diller and Goodman-Klumpp models are appropriate. Metherell and Diller [59] derived a load-deflection expression
Figure 2.4: Iwan Proposed an Assembly of Series Jenkin’s Elements in Parallel to Model Plasticity in Metals
for energy dissipated per cycle based on the instantaneous rate of energy dissipation in a lap joint. Goodman and Klumpp [60] derived theory based on two cantilever beams clamped together subjected to an oscillatory end load. Chen and Deng show that as the lap joint members either get taller or have a larger Young’s modulus the energy dissipation falls away from the theoretically predicted values of Metherell and Diller. They show that the Metherell-Diller expressions work well when the aspect ratio of height to length is between 1/5000 and 1/2500.

Jang and Barber [61] investigate energy dissipated due to friction from a system subjected to harmonically oscillating loads in both the normal and tangential directions. Their system consisted of a single friction node attached to a spring in the normal direction and tangent direction. They find that the energy dissipation is most significantly effected when the normal and tangential loads are applied $\pi/2$ out of phase. However, the largest overall impact occurs when the normal load is large enough to bring the point out of contact. The authors suggest that this is realized in physical systems in the form of Hertzian contact.

Guthrie and Kammer [62] develop a reduced order physically motivated model to capture micro-slip in a 1-D frictional interface, similar to that developed by Quinn and Segalman [14]. Using a Guyan reduction, Guthrie and Kammer remove all degrees of freedom from the system except those at the boundary of the problem. His model offers greatly reduced computational cost and shows good correlation with the results found by Quinn [14].
2.4 Receding Contact Surfaces

When an elastic body is pressed against a rigid surface, the contact tends to behave in one of three fashions. If the elastic body is initially curved and it is pressed into a flat rigid surface, then the contact patch length increases as the load is applied (see (a) in Figure 2.5). If the elastic body is short compared to its height and is flat, then as load is applied, the contact patch will consume the entire bottom of the elastic body and remain unchanged (see (b) in Figure 2.5). If the elastic body is flat and long compared with its height then as load is applied the contact patch will recede (see (c) in Figure 2.5) [15]. Receding contact was first predicted by Filon [63] in 1903 who was studying an elastic continuum squeezed by two equal and opposite forces. He found that if the continuum was sufficiently long compared to its height that at some point along its length the stress changes from compressive to tensile. Later, Coker and Filon [64] found that the ratio of contact length to height for an elastic strip on an elastic foundation, subjected to a point load, is 1.35 using photo elasticity. Dundurs et al. found that while the stresses in the body are proportional to the load, the shape of the contact patch is discontinuous, forming immediately at load application and unchanging during an increase of load [65]. Treating the receding elastic layer as a beam on an elastic half-space, Gladwell [66] found the ratio of the length of contact patch to height is

$$\frac{c}{h} = \left[ \frac{1.845 E_1 (1 - \nu_2)}{E_2 (1 - \nu_1)} \right]^{1/3}, \tag{2.50}$$

where $\nu_i$ and $E_i$ are the Poisson ratio and Young’s modulus of the elastic layer and elastic half-space, respectively. Ahn and Barber [67] examined Dundur’s postulations
Figure 2.5: Types of Contact: (a) - Advancing Contact, (b) - Constant Contact, (c) - Receding Contact. $L_i$ is the length of the contact patch prior to load application and $L_f$ is the length of the contact patch after load application.

including Coulomb friction. They found that under monotonic loading the original work holds. However, if the load is oscillatory the shape and extent of the contact patch changes.
CHAPTER III

EDGE SHEARING EFFECTS ON ENERGY DISSIPATED FROM AN ELASTIC BEAM RESTING ON A RIGID FOUNDATION

3.1 Introduction

Micro-slip, as defined by [1], is the relative elastic and/or plastic deformation that occurs in contact between two bodies prior to gross-slip initiation. While studying energy dissipated from spheres and cylinders in contact, as shown in Figure 1.3, Goodman [7] shows that if the amplitude of the tangential forcing function is small, the energy dissipated from the system is proportional to the cube of the amplitude, such as

$$
\Delta W = \frac{1}{36a\mu P} \left( \frac{2-\nu_1}{G_1} + \frac{2-\nu_2}{G_2} \right) Q^3,
$$

(3.1)

where $\Delta W$ is the energy dissipated per cycle, $\nu_i$ and $G_i$ are the Poisson ratio and shear modulus of the materials in contact, respectively, $\mu$ is the coefficient of friction, $Q$ is the amplitude of the forcing function, $P$ is the normal stress at the contact interface, and $a$ is the total length of the contact patch. It is important to note that energy dissipation is defined by Goodman such that it is dependent on the geometric characteristic $a$. 
Physical experimentation on flat plates in contact shows that energy dissipation due to small tangential oscillations tends to be proportional to a power between 2.5 and 3 of the forcing amplitude [8]. Johnson attributes this difference from the theoretical value to internal material hysteresis, variations of the coefficient of friction and effects of surface roughness [9]. Segalman partially attributes the discrepancy to bending of the overall joint structure [10]. When a load is applied to the structure, it can create a moment in the joint that changes the shape of the contact patch, as shown in Figure 1.4. Many other authors attribute it to the inadequacy of current friction models to capture this effect [11, 12, 13].

In this study it is postulated that shearing, which occurs most significantly at the leading edge of the contact patch, causes the energy dissipation to deviate from the cubic relation established by Goodman. Two studies are provided to show the impact that these edge-effects have on energy dissipation. First, a finite element (FE) study is conducted. This serves to introduce the topic as well as illustrate the shearing mechanism in a graphical manner. The second study is conducted using geometrically exact, nonlinear shell theory using the same parameters as the FE study. Presented in a non-dimensional framework, this study reveals the parameters that are important to these shearing effects. Additionally, this shows nonlinear shell theory as a likely reduced order model to accurately capture micro-slip type effects.
3.2 Physical Model Under Consideration

The model considered for both studies consists of a plane-strain continuum resting on a rough-rigid foundation, as shown in Figure 3.1. The rough-rigid foundation does not allow penetration into the continuum, but does allow separation. The parameters that describe the continuum are the Young’s modulus, \( E \), the cross-sectional area, \( A \), the height, \( h \), and the Poisson’s ratio, \( \nu \). Both studies use a Coulomb friction law where \( \mu \) is the coefficient of friction. No distinction is drawn between static and dynamic friction coefficient values. A uniform normal pressure is applied to the top of the beam, \( F_N(x) \), which in turn creates a clamping force with the foundation, \( R_N(x,t) \). A force is applied at one end of the continuum, tangent to the foundation, \( F(t) \). In this paper, this force is applied harmonically over a range of amplitudes. The \( \hat{e}_x \)-axis is tangential to the continuum’s center-line and the positive \( \hat{e}_y \)-axis is pointing away from the foundation.
3.3 One-Dimensional Beam Formulation

The purpose of this study is to determine what the effect is of adding beam height to the energy dissipation calculation has. First a one-dimensional beam is studied. As a consequence reducing the system in Figure 3.1 to one-dimension, the reaction force, \( R_N(x, t) \), is equal to the applied force, \( F_N(x, t) \), because there is no mechanism built into the one-dimensional beam to alter the reaction force. Also, the stiffness of the beam is defined entirely by the Young’s modulus times the area of the beam, \( EA \). Reducing the system in Figure 3.1 to a one-dimensional system has the potential energy given by

\[
V = \frac{1}{2} \int_0^L E(x)A(x) \left( \frac{\partial u}{\partial x} \right)^2 dx + \frac{1}{2} \int_0^L E(x)A(x) \left( \frac{\partial u}{\partial x} \right)^2 dx,
\]  

(3.2)

where the second integral represents the strain-energy stored in the stuck-region of the bar. This is based on the strain energy given by linear elasticity of \( \Phi = \frac{1}{2} \sigma \epsilon = \frac{1}{2} E \epsilon \left( \frac{\partial u}{\partial x} \right)^2 \). The kinetic energy is

\[
T = \frac{1}{2} \rho \int_0^L A(x) \left( \frac{\partial u}{\partial t} \right)^2 dx.
\]  

(3.3)

Substituting these into Hamilton’s principle and accounting for the non-conservative forces yields

\[
\int_{t_1}^{t_2} \left[ \delta T - \delta V + F(t) \delta u(L, t) + \mu N(x) \delta u(x, t) \right]_{L}^{1} dt.
\]  

(3.4)
Integrating by parts and using the Reymond-DuBois lemma gives the equation of motion as

\[ \rho A(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left( E(x)A(x) \frac{\partial u}{\partial x} \right) + G(x, t), \quad (3.5) \]

The function \( G(x, t) \) relates to the force of friction in the slip zone and/or the elastic potential energy stored in the stuck zone. Quinn and Segalman define this in [14] as

\[ G(x, t) = \begin{cases} 
-\mu N(x) \text{sgn}(\dot{u}(x, t)) & \text{if } \dot{u}(x, t) \neq 0 \\
-\min( |G^{eq}(x, t)| , \mu N(x)) \text{sgn}(G^{eq}(x, t)) & \text{if } \dot{u}(x, t) = 0.
\end{cases} \quad (3.6) \]

\( G^{eq}(x, t) \) represents the stored strain potential in the stuck zone and is defined as

\[ G^{eq}(x, t) = -\frac{\partial}{\partial x} \left( E(x)A(x) \frac{\partial u}{\partial x} \right) \quad (3.7) \]

To solve Equation (3.5) in closed form, the time-dependent term will be dropped and it will be solved quasi-statically. Also, \( E(x) \) and \( A(x) \) will be held constant. Also, the interface will be reduced to one-stick and one-slip domain. To eliminate the inhomogeneous boundary conditions for the slip domain, a new function is introduced,

\[ w(x) = u(x) - \frac{F(t)}{2EA(L - l)} (x - l)^2. \quad (3.8) \]
Transforming the dependent variable in Equation (3.5) from \( u(x) \) to \( w(x) \) gives the inhomogeneous ODE

\[
\frac{d^2 w(x)}{dx^2} = \frac{(L - l) \mu N \text{sgn}(\dot{w}) - F(t)}{EA(L - l)}.
\] (3.9)

Solving this simple ordinary differential equations gives a solution of

\[
w(x) = \frac{(L - l) \mu N - F(t)}{2EA(L - l)} x^2 + \frac{F(t) - \mu N(L - l)}{EA(L - l)} Lx - \frac{F(t) l (2L - l) + (L - l) \mu N l (l - 2L)}{2EA(L - l)}. \] (3.10)

Transforming Equation (3.10) back to the original dependent variable \( u(x) \),

\[
u(x) = \frac{(L - l) \mu N - F(t)}{2EA(L - l)} x^2 + \frac{F(t)}{2EA(L - l)} (x - l)^2 + \frac{F(t) - \mu N(L - l)}{EA(L - l)} Lx - \frac{F(t) l (2L - l) + (L - l) \mu N l (l - 2L)}{2EA(L - l)}. \] (3.11)

In the solution of Equation (3.11), a slip length is determined from the sum of external forces,

\[
\sum F_x = 0 = F(t) - \int_l^L \text{sgn}(\dot{u}(x)) \mu N(x) \, dx. \] (3.12)

For ease of computation, only a monotonic pull of the bar will be calculated from Equation (3.11). Using Masing’s hypothesis, the hysteresis loop can be calculated from the monotonic pull. Masing developed a relationship between a monotonic pull test and a hysteresis loop for non-conservative systems [68]. His hypothesis,
specifically how it relates to energy dissipated from contact patches is described in detail in [69]. As it is applied here, Masing’s hypothesis is summarized as

\[ F_d(f) = F_0 - 2G\left(\frac{u_0 - u}{2}\right), \quad (3.13) \]

where \( F_d \) is the unloading force, \( F_0 \) is the force at reversal, and \( G \) is a function relating force and displacement. For this analysis, \( G \) is generated from a regression curve of the monotonic pull from Equation (3.11),

\[ u = (8 \times 10^{-9}) f^2 \quad (3.14) \]

The monotonic pull and associated hysteresis loop generated from Masing’s hypothesis is shown in Figure 3.2. The energy dissipated by this system is given as

\[ \Delta W = 1.33 \times 10^{-8} f^{2.96}. \quad (3.15) \]

The most important thing to note about this analysis is that the displacement of points along the beam in the slip zone is a function of the axial stiffness, \( EA \), the applied load, \( F(t) \) and the friction load at the interface \( \mu N(x) \). This implies that if two beams with different heights have the same axial stiffness, they will dissipate the same amount of energy per cycle. As is shown in the following sections, if height and shear edge effects are accounted for, two beams with an equivalent axial stiffness but different heights will dissipate different amounts of energy per cycle.
3.4 Finite Element Study

As stated in the literature review of the micro-slip phenomenon, finite element analysis is prohibitive when studying large scale engineering structures that include frictional joint interfaces. The FE model presented here does not seek a solution to a larger scale problem, but rather displays the shear effects acting on an elastic body interacting with a frictional interface representative of a single interface. Mostly it is of interest to determine the effect that the height has on the energy dissipated and how closely a 1-D model matches a 2-D representation.
3.4.1 Model Setup

The FE model is set up as a 50.8 mm long elastic bar resting on a rigid surface with a coefficient of friction of $\mu = 0.1$, as shown in Figure 3.1. A uniform pressure of $F_N(x) = 1.4 \text{ MPa}$ is applied to the top, and a cyclic load, $F(t) = A \sin(\omega t)$, is applied tangentially to the bar at $\omega = 25.13 \text{ (rad/sec)}$. The heights evaluated are 0.254, 0.635, 1.27, 2.54, 3.81, and 5.08 millimeters. The axial stiffness of the rod, $E \times Area$, is held constant for all heights, at 207 GPa. By holding the axial stiffness constant, the only effect evaluated is the height change.

The FE model is analyzed using ABAQUS/Standard version 6.11. The bar is a plane-strain geometry with $\nu = 0.3$. The surface is an analytically rigid surface with no inherent material properties. The tangential behaviour of the contact surface is realized with a surface-to-surface penalty method with finite sliding. The default penalty method parameters are used which set the allowable elastic slip (small displacement allowed when a node is stuck) to 0.005 times the characteristic element length. Hard contact is enforced in the normal direction, which does not allow the two bodies in contact to overlap. The loading is applied in two steps:

**Step 1** The pressure is applied to the top surface using an Abaqus pressure load. During the application of this pressure, the coefficient of friction is set to $\mu = 0$ to avoid locking in residual shear stress and the bottom right-most node is constrained from translation to avoid rigid body motion. This step uses Abaqus’ implicit, static solver with default solution tolerances and parameters. The time step length is arbitrarily set to one second.
Step 2  Once the pressure load is stable, the coefficient of friction is set to $\mu = 0.1$ and a force is applied to the left end of the continuum, which oscillates over a period of one second according to $F = A \sin(\omega t)$, where $\omega = 25.13$ rad/sec. The force is applied as an Abaqus traction load on the left end of the beam. The magnitude of the traction multiplied by the height of the beam gives the total force acting on the beam end. This step uses Abaqus’ implicit, dynamic solver with default solution tolerances and parameters.

To capture the energy dissipated due to micro-slip, an appropriately sized fine mesh is needed. A 4 node plane-strain element is selected and a global element size of $m = 0.0762$ mm, $m = 0.127$ mm, $m = 0.635$ mm and $m = 1.27$ mm are considered. A mesh convergence study is conducted in which displacement along the beam is extracted at an instance in time during the oscillation. A properly sized mesh gives a smooth displacement function, as described below. Figure 3.3 shows the displacements at an instant in time and indicates that $m = 0.0762$ mm is a fine enough mesh to capture the micro-slip phenomenon. This mesh size is used for the remainder of this study.
Figure 3.3: Mesh Convergence Study
3.4.2 FEA Results

Each section height is evaluated at 10 different tangential load amplitudes. These forces are selected to be evenly spaced on a logarithmic scale and range from 1% to 90% of the load that would achieve macro-slip of the bar, \( \mu \cdot F_N(X) L \). Upon convergence, each case is evaluated at the third cycle with the energy dissipation defined by

\[
\Delta W = \int_0^t \int_0^L \tau \frac{\partial u(x,t)}{\partial t} dx \, dt,
\]

where \( \tau \) is the shear stress at the interface and \( u(x,t) \) is the displacement, tangent to the contact surface, of a point lying on the contact interface. This is calculated at the third cycle because any locked in residual stress are shaken down and the system has reached steady-state. This value is extracted from Abaqus as a history output.

Figure 3.4 shows the energy dissipation per cycle results for this study. The dashed line represents the nearly theoretical cubic power law solution found in the 1-D solution. The graph shows that the theoretical cubic power law solution of Goodman holds true for all height values forced at the top 3 loads; 33.11%, 54.59%, and 90% of the load needed to induce macro-slip. However, for the loads analyzed below the 33.11% value, the power law solution deteriorates. It is also seen that the taller the beam, the more quickly the solution falls away from the cubic power law. For example, the \( h = 0.254 \) mm solution tracks closely to the theoretical solution, only deviating at the smaller forcing amplitudes. However, the \( h = 5.08 \) mm solution deviates from the theoretical solution significantly after the third highest force. The graph stops after the sixth highest forcing amplitude because energy is no longer being dissipated.
Figure 3.4: Energy Dissipated per Cycle

beyond this. The reason for this deviation comes from an edge shear effect that becomes evident at the load reversal (i.e. the forcing function changes from pulling on the leading edge to pushing). For a continuum with inherent height, the slip length is a nonlinear function. The reason this occurs is shown graphically in Figures 3.5 and 3.6. Snapshots of cross-sections along the beam during oscillation are shown in Figure 3.5. In this figure, the black and grey cross-section indicate if that particular cross-section is slipping or sticking, respectively, and the light grey line connecting the cross-sections indicates the motion of the contact point during the oscillation. This
figure shows that beyond the slip zone, shear stress is propagating and causing the cross-sections to tilt. Also it shows the tilting of the leading cross-sections during load reversal. At the load reversal the contact point of the leading edge does not slip, and this interface does not dissipate energy. Instead shear dominates the deformation. Figure 3.6 shows the propagation of shear stresses along the length of the continuum during the load cycle (each snapshot is at progressing times in the cycle). The green color indicates little shear occurring at that location, blue is shearing acting to the left and red is acting to the right. The intensity of the blue and red indicate the magnitude of the shear. Of particular interest in this plot are the third through fifth and eighth through tenth snap shops, as these are at load reversals. The third snap shot is at the time when the load is just starting to reverse. The shear stress propagates from the leading edge back along the entire length of the slip zone. The fourth snap shot is at the point where the applied load is near zero. At this time, the cross-section is nearly vertical and no shear exists at the leading edge. However, shear is locked in further along the continuum. This shear is trapped from the previous pull. During the fifth snap shot, the cross-section still has not undergone slip, however, it is seen that at the leading edge the shear stress is now acting in the opposite direction. The larger the height, the further into the stuck zone that this shear propagates, which in turn allows the cross-section to rotate more prior to slipping. Snap shots eight through ten show the same phenomenon, but the shears are acting in the opposite directions as those in snap shots three through five. If this were a 1-D system (i.e. no height), the blue locked in shear shown in snap shots four and five still exist, however, the shear stress propagation prior to slipping in five does not. At load reversal the leading cross-
Figure 3.5: History Response of System. Black Cross-Section - Slipping, Grey Cross-Section - Stuck.
section would immediately slip and dissipate energy. An idealization of a 1-D system is shown in Figure 3.7. This figure shows the change in shear stress, from positive to negative, occurs like a step function, as opposed to the smooth change permitted by a continuum with height. The ability for the stuck zone to carry shear and allow the leading cross-sections to rotate contribute to the degradation from the power law prediction and neither effect is predicted by a 1-D formulation of the system, such as a series-series Iwan [14]. Figure 3.8 ($A = 21.48 \text{ N}$), shows the energy dissipated during the cycle, as a function of time. The points of no energy dissipation discussed above (i.e. when the cross-section is only rotating) are clearly seen on this graph as flat spots. The slope of the lines during periods of energy dissipation remains relatively constant for all heights, however, it is clear that as the continuum increases in height that the intervals of zero dissipation become longer. This study makes clear several important points. First it helps to illustrate the problem with using finite element analysis when analyzing a large jointed structure. The $m = 0.0762 \text{ mm}$ mesh characteristic length requires a maximum time step of $7.41 \mu\text{s}$ for a stable, explicit finite element solution (for a Courant number of 1). This is obviously prohibitive for large scale structures (i.e. total model more than several meters large). Second it shows that by not including the height above the contact patch in a reduced model formulation, the energy dissipation, and consequently the structural damping, is over predicted. This could result in damage to sensitive equipment during a vibration event.
Figure 3.6: Shear Stress Propagating into Stuck Zone
Figure 3.7: Shear Distribution in Idealized 1D Model
Figure 3.8: Energy Dissipated Over Cycle
3.5 Nonlinear Beamshell Study

To further investigate the edge shearing effect on energy dissipation, a nonlinear beamshell is formulated. The methodology of Libai and Simmonds [32], as outlined in 2 of this document, is employed. For the current problem being investigated, friction forces at the interface are inserted into the shell equations via the external force and couple terms. A Coulomb friction formulation is used.

3.5.1 Definition of Strains

Let \( \bar{y}' \) represent the vector tangent to the deformed beamshell reference curve. In terms of the coordinate system whose unit normal vectors are perpendicular and parallel to the beam’s cross-section (\( \hat{T}, \hat{B} \)), \( \bar{y}' \) is written as

\[
\bar{y}' = (1 + e)\hat{T} + g\hat{B},
\]  

(3.17)

where \( e \) and \( g \) represent strains in the \( \hat{T} \) and \( \hat{B} \) directions, respectively. While the problem is defined in terms of this coordinate system, kinematic constraints on the motion of the beamshell are more conveniently written in the Cartesian coordinate system (\( \hat{e}_x, \hat{e}_y \)). In terms of the Cartesian coordinate system, the beamshell tangent becomes

\[
\bar{y}' = (1 + u'_x)\hat{e}_x + u'_y\hat{e}_y,
\]  

(3.18)

where \( u'_x \) and \( u'_y \) are strains in the \( \hat{e}_x \) and \( \hat{e}_y \) directions, respectively. The coordinate transformation between the two systems is defined using the angle formed between
the unit vectors $\hat{T}$ and $\hat{e}_x$, denoted as $\beta$, such that,

$$\hat{T} = \cos \beta \hat{e}_x + \sin \beta \hat{e}_y,$$  \hfill (3.19)

$$\hat{B} = -\sin \beta \hat{e}_x + \cos \beta \hat{e}_y.$$ \hfill (3.20)

This allows Equations (3.17) and (3.18) to be combined to form

$$\bar{y}' = [(1 + e) \cos \beta - g \sin \beta] \hat{e}_x + [(1 + e) \sin \beta + g \cos \beta] \hat{e}_y.$$ \hfill (3.21)

Setting Equation (3.21) equal to (3.18) allows the strains $(e, g)$ to be written in terms of the Cartesian strains $(u'_x, u'_y)$ as

$$e = (1 + u'_x) \cos \beta + u'_y \sin \beta - 1,$$ \hfill (3.22)

$$g = u'_y \cos \beta - (1 + u'_x) \sin \beta.$$ \hfill (3.23)

### 3.5.2 Differential Equations of Motion

For a statically loaded beamshell, the vector differential equations of motion are

$$F' + p = 0,$$ \hfill (3.24)

$$M' + m \cdot F + l = 0,$$ \hfill (3.25)
where $\mathbf{F}$ is the internal force vector per unit height, $\mathbf{p}$ is the external force vector per unit length per unit width, $M$ is the internal moment per unit height, $\mathbf{m}$ is the rational normal vector, which is defined in Equation (3.27), and $l$ is the external couple per unit length, which is defined in Equation (3.32). The internal force vector is decomposed into the cross-sectional coordinate system such that

$$\mathbf{F} = N\hat{T} + Q\hat{B}. \quad (3.26)$$

The rational normal is the cross product of $\hat{e}_z$ and $\vec{y}'$, written as

$$\mathbf{m} = \begin{vmatrix} \hat{T} & \hat{B} & \hat{e}_z \\ 0 & 0 & 1 \end{vmatrix} = (1 + e)\hat{B} - g\hat{T}. \quad (3.27)$$

Therefore, the second term in Equation (3.25) becomes

$$\mathbf{m} \cdot \mathbf{F} = (1 + e)Q - gN. \quad (3.28)$$

To find the spatial derivative of $\mathbf{F}$, first the derivative of the unit vectors defining the cross-section coordinate system is found. Using the transformations defined in Equations (3.19) and (3.20), the relation is written as

$$\hat{T}' = \beta'\hat{B}. \quad (3.29)$$
\[ \hat{B}' = -\beta' \hat{T}. \]  

(3.30)

This results in the derivative of the force becoming

\[ \mathbf{F}' = \left[ N' - \beta'Q \right] \hat{T} + \left[ Q' + \beta'N \right] \hat{B}. \]  

(3.31)

For a homogeneous, continuous beamshell, the external couple is defined as the total cross product of the vector going from the beamshell reference curve to the outer fiber, normal to the beamshell \( \bar{z} \) and the externally applied tractions \( \mathbf{S}^\xi \). This is written mathematically as

\[ l = \hat{e}_z \cdot \left( \bar{z} \times \mathbf{S}^\xi |_{H} \right). \]  

(3.32)

### 3.5.3 Elastic Beamshell on a Rigid Foundation

The specific problem for which a solution is being sought is that of an elastic beamshell resting on a rough, rigid surface, subjected to an oscillating end load acting in the \( \hat{e}_x \) direction, denoted \( F(t) \). In addition to the applied oscillatory load, a static traction is applied on the top surface of the beamshell, in the \( \hat{e}_y \) direction, denoted \( F_N(x) \). Since the beamshell is constrained by the rough rigid surface, two additional traction loads are generated by the applied forces; a traction acting in the \( \hat{e}_x \) direction arising from friction at the contact interface, denoted \( F_t(x) \) and a traction acting in the \( \hat{e}_y \) direction arising from normal contact with the rigid surface, denoted \( R_N(x) \), as shown in Figure 3.5.3. In Cartesian coordinates these forces define \( \mathbf{p} \) from Equation (3.24) such that

\[ \mathbf{p} = F_t(x) \hat{e}_x + [R_N(x) - F_N(x)] \hat{e}_y. \]  

(3.33)
As we are working in the cross section coordinates, these external forces are transformed from Cartesian using Equations (3.19) and (3.20. This allows $p$ to be written as

$$p = \left[ F_t \cos \beta + (R_N - F_N) \sin \beta \right] \hat{T} + \left[-F_t \sin \beta + (R_N - F_N) \cos \beta \right] \hat{B}. \quad (3.34)$$

To find the external couple, let the beamshell be a height $2H$. With the definition of $p$ from Equation (3.34), Equation (3.32) becomes

$$l = F_t H \cos \beta + (R_N + F_N) H \sin \beta. \quad (3.35)$$
With all terms defined, the equations of motion are:

Linear Momentum Balance in $\hat{T}$ —

$$N' - \beta'Q + F_t \cos \beta + (R_N - F_N) \sin \beta = 0 \quad (3.36)$$

Linear Momentum Balance in $\hat{B}$ —

$$Q' + \beta'N - F_t \sin \beta + (R_N - F_N) \cos \beta = 0 \quad (3.37)$$

Angular Momentum Balance —

$$M' + (1 + e)Q - gN + F_t H \cos \beta + H (R_N + F_N) \sin \beta = 0 \quad (3.38)$$

Assuming that the beamshell remains in contact with the rigid surface (i.e. no impingement into the surface and no separation) and assuming that plane sections remain plane, the displacement of the beamshell, in Cartesian coordinates, is defined with geometric arguments

$$u_x = -H \sin \beta + v, \quad (3.39)$$

$$u_y = H (\cos \beta - 1). \quad (3.40)$$

The displacement in $x$ is decomposed into a component due to the rotation of the cross section ($H \sin \beta$) and a component due to slipping of a point on the contact patch ($v$), as shown in Figure 3.5.3. Taking the derivative gives the strains as a
function of $\beta$ and $v$ such that

\begin{align*}
 u_x' &= -H\beta' \cos \beta + v', \quad \text{(3.41)} \\
 u_y' &= -H\beta' \sin \beta. \quad \text{(3.42)}
\end{align*}

Substituting these strains into Equations (3.22) and (3.23) gives the strains, $g$ and $e$, as functions of $\beta$ and $v$ only. Letting the force and moment set $(N, Q, M)$ be conjugate to the strain set $(e, g, \beta')$, the constitutive law exists such that

\[ N = \Phi_e = 2EHc_1e, \quad \text{(3.43)} \]
\[ Q = \Phi_{,g} = 2EHc_4g, \quad (3.44) \]
\[ M = \Phi_{,\beta'} = \frac{1}{2}EHc_3H^2\beta'. \quad (3.45) \]

The constants are found by relating the total three-dimensional strain energy back to the strain energy in the shell. For a homogeneous, isotropic material, assume the three-dimensional strain energy is of the form

\[ W = \frac{E\epsilon_x}{2(1-\nu^2)}, \quad (3.46) \]

where \( E \) is the Young’s modulus, \( \nu \) is the Poisson ratio and \( \epsilon_x \) is the strain in the \( x \)-direction as a function of \( s \). For a shell-like body assume that \( x \) is along the length and \( s \) is the coordinate pointing from the neutral axis to the outer fibers of the body. Assuming that plane-sections remain plane, \( \epsilon_x \) is described in beamshell strains as

\[ \epsilon_x(s) = e + sk. \quad (3.47) \]

Substituting Equation (3.47) into Equation (3.46) and integrating through the thickness ensures that the total strain energy in the beamshell is equivalent to that of the continuous body. Thus the equation is

\[ \Phi = \frac{E}{2(1-\nu^2)} \int_{-h}^{h} [e + sk]^2 ds. \quad (3.48) \]
Equation (3.48) is then set equal to strain energy function used for the beamshell. In this example a quadratic strain energy is used such that

\[ \Phi = Eh \left( c_1 e + 4c_3 h^2 \kappa^2 + c_4 g^2 \right). \]  

(3.49)

Solving for \( c_1 \) and \( c_3 \) yields

\[ c_1 = \frac{1}{1 - \nu^2}, \]  

(3.50)

\[ c_3 = \frac{1}{12 (1 - \nu^2)}. \]  

(3.51)

The three-dimensional strain energy for shear is defined as

\[ W = G\kappa \gamma_{xy}^2, \]  

(3.52)

where \( G \) is the shear modulus, \( \gamma_{xy} \) is the shear strain and \( \kappa \) is the shear coefficient in Timoshenko’s beam theory. For this paper, the shear coefficient derived by Cowper [70] is used, and is defined by

\[ \kappa = \frac{10 (1 + \nu)}{12 + 11\nu}. \]  

(3.53)

This yields a beamshell shear constant of

\[ c_4 = \frac{5}{12 + 11\nu}. \]  

(3.54)
3.5.4 Stuck Solution

If the applied force $F(t)$ is not large enough to overcome the frictional force at a point on the contact interface, then only motion due to rotation of the cross-section occurs (i.e. $v$ is constant). With this stuck constraint, the forces and moments become functions of $\beta$ only, and the unknowns in the linear and angular momentum balances are $F_t$, $R_N$ and $\beta''$.

3.5.5 Slipping Solution

If the applied force $F(t)$ is large enough to overcome the force of friction at a point on the contact interface, then that point displaces relative to its initial point on the interface. Using a Coulomb friction law introduces a fourth equation to the system such that

$$F_t(x) = \mu R_N(x). \quad (3.55)$$

With these four equations the four unknowns are $F_t$, $R_N$, $\beta''$ and $v''$.

3.5.6 Non-Dimensionalization

Let a non-dimensional length parameter be introduced, which is defined as

$$x^* = \frac{x}{a}. \quad (3.56)$$

The parameter $a$ is defined as the predicted slip zone length of an infinitely thin beamshell subjected to the amplitude of the oscillatory force, $F(t)$, and is written
mathematically as

\[ a = \frac{F_{\text{max}}}{\mu \bar{F}_N}, \]  

(3.57)

where \( \bar{F}_N \) is the average of the applied normal force. The non-dimensional equations of motion thus become:

Linear Momentum Balance in \( \hat{T} \) —

\[ 2c_1\phi \frac{d}{dx^*}(e) - 2c_4\phi g \frac{d}{dx^*}(\beta) + F_t^* \cos \beta + (R_N^* - F_N^*) \sin \beta = 0 \]  

(3.58)

Linear Momentum Balance in \( \hat{B} \) —

\[ 2c_4\phi \frac{d}{dx^*}(g) + 2c_1\phi e \frac{d}{dx^*}(\beta) - F_t^* \sin \beta + (R_N^* - F_N^*) \cos \beta = 0 \]  

(3.59)

Angular Momentum Balance —

\[ \frac{1}{2} \eta \phi c_3 \frac{d^2}{dx'^2}(\beta) + \frac{\phi}{\eta} [(1 + e)(2c_4g) - 2c_1eg] + F_t^* \cos \beta + (R_N^* + F_N^*) \sin \beta = 0 \]  

(3.60)

Here the non-dimensional parameters are defined as

\[ \phi = \frac{EH}{F_{\text{max}}} \rightarrow \text{Controls the Axial Displacement}, \]  

(3.61)

\[ \eta = \frac{H}{a} \rightarrow \text{Ratio of Height to Slip Zone Length}. \]  

(3.62)
The external forces are normalized by $\mu \tilde{F}_N$.

As a check, consider the effect on these equations of motion if the height approaches zero ($H \to 0$). If this is the case it is clear that $\beta$ approaches zero as well. With $\beta$ being zero, there is no mechanism to produce strain in the $y$-direction, and all strain in the $x$-direction arises from slip (i.e. $u'_x = v'$). Thus only the linear momentum balance in the $x$-direction has a non-trivial solution, and it takes the well known form of the 1D wave equation, as expected. The one-dimensional form is written as

$$\frac{\partial^2 v}{\partial x^2} + F^*_t = 0. \quad (3.64)$$

This corresponds to the 1-D equation of motion (Equation (3.5)), as derived earlier.

3.5.7 Results

A study is conducted similar to the finite element study presented earlier. Here the non-dimensional height $\eta$, is varied, and several forcing amplitudes are applied. The beamshell’s non-dimensional length is $L^* = 2$, the coefficient of friction is $\mu = 0.1$ and the applied normal traction is $F^*_N = 10$. The applied tangential load’s amplitude is varied from 0.1 to 1.9, where an amplitude of 2.0 would cause macro-slip. A fixed boundary condition is applied at $x^* = 0$, which restrains both $v$ and $\beta$ from motion. As shown in the similarities between Figures 3.11 and 3.12 for the beamshell and Figures 3.8 and 3.4 for the FE study presented earlier, the shear effect is shown to have a large effect on the energy dissipated from the system. Figure 3.11 shows that as the beamshell height approaches zero, a power law relationship between the forcing amplitude and the energy dissipated per cycle is recovered. As the height
increases, the power law relationship no longer holds. Figure 3.12 shows the reason for the departure from the power law is the interval of zero dissipation that occurs at the load reversals. As previously discussed, during these intervals the cross-section is rotating about the fixed contact point, and no energy is dissipated. It is noted that the energy dissipation shown in Figure 3.11 does not converge to a single point at the largest forcing function as is the case with the FE study. This is due to the difference in boundary conditions applied to the two studies. In the FE study the cross-section that is farthest away from the forcing amplitude has its node at the contact interface fixed from displacement. This still allows the cross-section to rotate freely. The beamshell fixes this cross-section from all motion including rotation. By fixing all degrees of freedom this cross-section carries load along its entire length, and not just at the contact interface. The taller the beamshell the cross-section carries more load away from the contact interface. This in turn changes the shear distribution and effects the energy dissipation. The experimental work conducted by Mohanty and Nanda [57], which employs a similar fixed boundary condition, shows the same spreading in energy dissipation with different heights at the largest forcing amplitude. This comparison reinforces the validity of this beamshell model. Figure 3.13 shows the force required to initiate slip in an undeformed, virgin beamshell. The relationship between the non-dimensional force and non-dimensional height is perfectly linear. Figure 3.14 shows the friction force along the beamshell’s length for a single pull (non-oscillatory) into an undeformed, virgin bar. This figure shows that as the height increases, the length of the stuck zone capable of carrying load increases. This is the shearing mechanism that allows for the cross-section to rotate
Non-Dimensional Forcing Amplitude

Non-Dimensional Energy Dissipated Per Cycle

Figure 3.11: Energy Dissipated Over Cycle
Figure 3.12: Energy Dissipation Showing Intervals of Zero Energy Dissipation at Load Reversals
Figure 3.13: Force Required to Initiate Slip in a Virgin Undeformed Beamshell
and not dissipate energy from the system. As the beamshell height approaches zero, the curve in Figure 3.14 approaches a step function, where the stuck zone carries no load. Figure 3.16 shows that for the small heights ($\eta = 0.01$) the hysteresis predicted by one-dimensional reduced order models is recovered (such as Quinn and Segalman’s [14]). Figure 3.17 shows the same set of curves for $\eta = 0.030526$. While the curves are similar, it is observed that there are horizontal flat spots at the turning spots of the harmonic oscillation. These flat spots are most evident for the smaller forcing function amplitudes and become less noticeable as the forcing function amplitude increases. This observation matches those seen earlier when examining Figure 3.11 and 3.8. Figure 3.15 shows several values of $\eta$ for the forcing amplitude $F = 0.7855$. Again the horizontal flat spots are evident, particularly for larger values of $\eta$. With results extracted from a single pull analysis, Figure 3.13 shows that the relationship between the beamshell height and the force to initiate slip into an undeformed, virgin beamshell is linear, and described by the equation

$$F_{\text{slip}} = 1.767 \eta - 0.0013. \quad (3.65)$$

Equation 3.65 can be used as a design basis when designing systems involving beams on frictional foundations. Any force applied below that predicted by Equation 3.65 will not slip and thus will not dissipate energy. If damping is desired for the system, then the load path should be designed such that a force above the force to slip predicted by Equation 3.65 should be selected. It is noted that this equation is only valid for the stiffness and friction parameters used in this study. If either the stiffness
or friction parameters are different, than a similar study needs to be conducted to
determine the force to slip. For a linear elastic material, with different parameters,
a linear relationship between force to slip and the non-dimensional height will be
recovered, but the constants in Equation 3.65 will be different.
Figure 3.14: Shear Traction at the Contact Interface
Figure 3.15: Hysteresis Curves for Various $\eta$ Values with $F = 0.7855$
Figure 3.16: Hysteresis Curves for Various Forces with $\eta = 0.01$
Figure 3.17: Hysteresis Curves for Various Forces with $\eta = 0.030526$
In the interest of validating the beamshell model, an FE model is constructed with the same parameters as the beamshell investigated here. Figure 3.19 shows that for a single pull into undeformed, virgin material, the displacement of the FE and beamshell models are nearly identical. However, note that the FE model was constructed with 2D continuum elements, which introduce a Poisson effect into the system that is not present in the beamshell formulation. To construct this graph, the displacement after application of the normal force is subtracted from the displacement after the application of the tangential load. Figure 3.18 shows that the energy dissipated per cycle matches closely between the two formulations as well. At the lower end of the forcing amplitudes the FE analysis dissipates less energy than the beamshell. At these lower amplitudes the stiffening provided by the Poisson effect in the FE model becomes more evident. Also, the fact that beamshell’s cross sections remain plain makes the beamshell formulation stiffer than the FE formulation. This variation in stiffness also accounts for this difference in energy dissipated per cycle between the two formulations.
Figure 3.18: Comparison of FE to Beamshell Energy Dissipation Results
Figure 3.19: Comparison of FE to Beamshell Displacement Results
This nonlinear beamshell study reinforces the importance of considering the deformation above the contact patch when calculating energy dissipation due to micro-slip. The deformation above the contact patch dictates the shear distribution in the beamshell stuck zone significantly and cannot be neglected. This is demonstrated through the energy per cycle and hysteresis plots for various heights and forcing functions.

This study also shows that nonlinear shell theory is capable of accurately capturing this complex contact phenomenon. By using kinematic constraints in conjunction with the base theory developed by Libai and Simmonds [32], micro-slip and its effect on energy dissipated from the system are easily introduced, with the added advantage over 1D regularizations of implicitly incorporating geometric effects.
CHAPTER IV
MODELING RECEDING CONTACT WITH NON-LINEAR BEAMSHHELLS

4.1 Introduction

When an elastic body is pressed against a rigid surface, the contact tends to behave in one of three fashions. If the elastic body is initially curved and it is pressed into a flat rigid surface, then the contact patch length increases as the load is applied (see (a) in Figure 4.1). If the elastic body is short compared to its height and is flat, then as load is applied, the contact patch will consume the entire bottom of the elastic body and remain unchanged (see (b) in Figure 4.1). If the elastic body is flat and long compared with its height, and the applied load does not span the entire length of the body, then as load is applied the contact patch will recede (see (c) in Figure 4.1).

This chapter presents two methods of predicting the response of an elastic body undergoing receding contact. First a method is presented that uses the existing nonlinear shell theory derived by Libai and Simmonds. This method relies on kinematic relationships that exist when analyzing an elastic continuum on a frictionless rigid foundation. The second method seeks to generalize the mechanism which causes receding contact and make it integral to the beamshell formulation. By adding a state variable to the Libai and Simmonds derivation which represents a pseudo squeezing
strain, the effect is captured in a more general sense. This allows receding contact, as well as the transition from compressive to tensile strain observed by Filon [63] (see Chapter 2), to be directly incorporated into the theory.

4.2 Prediction of Receding Contact using Existing Nonlinear Shell Theory

As a starting point for this study, kinematic constraints and enforced squeezing strain values are used to solve the problem using Libai and Simmonds formulation directly. This has several drawbacks. First, the squeezing strain is directly input and not
determined as the solution to the mechanics of the problem. This serves the Poisson couple that should exist between the squeezing strain and the axial strain. Second, the derivation depends directly on assumptions of the through thickness state of strain. An ultimate solution is sought which does not require such assumptions and can be treated as a Cosserat surface, as described in detail in Chapter 2. However, it is a convenient way to present the mechanism behind receding contact.

First a form of the transverse hydrostatic stress (squeezing stress) is assumed. An elastic continuum pressed into a rigid foundation by a pressure $F_N(x)$ develops a reaction pressure at the contact interface $R_N(x)$. A reasonable assumption for the hydrostatic stress is the arithmetic mean of the two pressures and for the strain to be this stress divided by the Young’s modulus, such as,

$$\sigma_y = \frac{F_N(x) + R_N(x)}{2},$$

(4.1)

$$\epsilon_y = \frac{F_N(x) + R_N(x)}{2E}.$$

(4.2)

Shear strain is the mechanism responsible for Filon’s observation that stress at a body’s center line switches from compression to tension when it is acted upon by equal and opposite squeezing forces which do not encompass the entire length of the body. The material directly under the squeezing load application displaces towards the body’s center line. In the region along the body’s length where the load no longer acts, the material points want to return to their original, undeformed location. This creates a shear stress at this location. Since the material does not return to its original
configuration in a discontinuous manner, some span is spent returning to this state, which infers that in this region compressive stress still exists at the center line. This is shown graphically in Figure 4.2. In this figure a normal force $F_t(x)$ is applied over a small span of the beam and then is released. The black cross-section is a constant height in the area that this load is applied. After the point of load application it is seen that the cross-section height slowly increases which is resisted by the shear stress, which is indicated by the red arrows. A tensile load must eventually act to create a balanced system. An elastic continuum pressed into a rigid foundation acts similar to the elastic continuum pressed equally on both sides by equal and opposite forces studied by Filon. Therefore, the rigid surface is thought of as a symmetry plane. However, in the absence of adhesive forces, the contact patch cannot handle a tensile load. Thus, when a tensile load would be developed in Filon’s model, the continuum in contact must lift away from the rigid surface. The displacement of the reference curve is found by integrating Equation (4.2) from $-H$, where the contact patch is, to 0, where the reference curve is, then differentiating with respect to $x$. 

Figure 4.2: Shear Stress Predicted by Filon
Expand the shear strain definition to be

\[ g = \tilde{g} + g_\nu, \quad (4.3) \]

where \( \tilde{g} \) is the strain tangent to the beamshell cross-section caused by the difference in the forces acting in the same direction and \( g_\nu \) is the shear strain caused by the change in displacement in this direction with respect to \( x \) resulting from hydrostatic portion of the forces.

With this relationship, the differential equations of motion defined by Equations (2.30), (2.31), and (2.34), are expanded to be

\[ \begin{align*}
Ehc_1(e') - Ehc_4 (\alpha' + k) (\tilde{g} + g_\nu) + P_T = mv_T', \\
Ehc_4 (\tilde{g}' + g_\nu') + Ehc_1 (\alpha' + k) e + P_B = mv_B, \\
Eh^3 c_3 k' + Ehc_4 (1 + e) (\tilde{g} + g_\nu) - Ehc_1 g(e) + l = I\dot{\omega}.
\end{align*} \quad (4.4, 4.5, 4.6) \]

For this example, the assumed and enforced squeezing strain is

\[ g_\nu = \frac{R'_N(x) + F'_N(x)}{2E}. \quad (4.7) \]

For the purposes of this example, let \( F_N \) act in the negative \( y \)-direction and \( R_N \) act in the positive \( y \)-direction. With these directions determined, the external forces are
defined as

\[ P = (R_N - F_N) \hat{e}_y \]  
\[ = \sin \beta (R_N - F_N) \hat{T} + \cos \beta (R_N - F_N) \hat{B}. \]  

Inserting the definitions of the external forces, \( g_{\nu} \), \( e \) and the internal forces into Equations (4.4), (4.5) and (4.6) gives the complete equations of motion for the system shown in Figure 4.3. The unknowns in system are: \( e', \tilde{g}', R''_N \), and \( \beta'' \). To reduce the number of unknowns, a kinematic relationship is developed. The conditions at the contact interface prohibit the beamshell from impinging into the rigid surface. Therefore, assuming plane-sections remain plane, a relationship is developed between the total shear strain \( g \) and bending strain \( k \). Using the geometry shown in Figure 4.4 the displacement and derivative of the displacement in the \( y \)-direction is

\[ u_y = -\tilde{H} (1 - \cos \beta) - (H - \tilde{H}), \]  

Figure 4.3: Example Receding Contact System
Figure 4.4: Beamshell Displacement in $y$ for a Rotating Cross-Section

where $\bar{H}$ is the cross-section height after the hydrostatic stress, $\sigma_y$, acts on it,

$$\bar{H} = H \left( 1 - \frac{R_N + F_N}{2E} \cos \beta \right).$$  \hspace{1cm} (4.11)

This yields

$$u_y' = \bar{H}' (\cos \beta - 1) - \bar{H}' \beta' \sin \beta - \frac{(R'_N + F'_N) H}{2E} \cos \beta + \frac{(R_N + F_N) H \beta'}{2E} \sin \beta$$

$$= \left( \bar{H}' - \frac{(R'_N + F'_N) H}{2E} \right) \cos \beta + \left( \frac{(R_N + F_N) H}{2E} \beta' + \bar{H}' \beta' \right) \sin \beta - \bar{H}'. \hspace{1cm} (4.13)$$
The unit vector tangent to the deformed beamshell reference curve, \( \bar{y}' \) is defined as

\[
\bar{y}' = \lambda \cos \bar{\alpha} e_x + \lambda \sin \bar{\alpha} e_y,
\]  
(4.14)

where \( \lambda \) is the axial stretch of the beamshell. To transform from cross-section coordinate strains \((e, g)\) to deformed beamshell strains \((\lambda, \gamma)\) use the relation

\[
(1 + e) = \lambda \cos \gamma.
\]  
(4.15)

The slope of the beamshell for the system being studied is

\[
\tan \bar{\alpha} = \frac{u_y'}{\bar{y}' \cdot e_x} 
= \left( \frac{\bar{H}' - (R'_N + F'_N) H}{2E(1 + e)} \cos \beta + \left( \frac{(R_N + F_N) H}{2E} \beta' + \bar{H}' \beta' \right) \sin \beta - \bar{H}' \right) \cos \gamma
\]

\[
(1 + e) \cos \bar{\alpha}.
\]

(4.16)

Using the trigonometric relationship that

\[
\sin (\beta + \gamma) = \sin \beta \cos \gamma + \sin \gamma \cos \beta,
\]  
(4.17)

Equation (4.16) becomes

\[
\tan \gamma = \frac{2E \bar{H}' - (R'_N + F'_N) H}{2E(1 + e)} - \frac{\bar{H}'}{(1 + e) \cos \beta} 
+ \tan \beta \left[ \frac{(R_N + F_N) H \beta' + 2E \bar{H} \beta}{2E(1 + e)} - 1 \right].
\]  
(4.18)
To transform the shear stress, \( g \), into deformed beamshell coordinates, the relationship is

\[
g = \lambda \sin \gamma = (1 + e) \tan \gamma.
\]  

(4.19)

The relationship in conjunction with Equation (4.18) gives the shear strain \( g \) in terms of \( R'_N, e, \beta, \) and \( \beta' \), thus reducing the number of unknowns.

4.2.1 Example Problem

This example applies the method described above to an elastic continuum resting on a rigid, frictionless surface. The elastic continuum is a plane-strain geometry with a total length of 10 inches and a total height of 0.1 inches, with only half of the continuum length modeled \((0 \leq x \leq L/2)\). The material has a Young’s modulus of \( E = 30 \times 10^6 \) psi and Poisson’s ratio of \( \nu = 0.3 \). The beamshell constant, \( c_1, c_3, \) and \( c_4 \), are the same values derived in Section 3.5. A load is applied to the top surface with a magnitude of 1,000 pounds force. This load is constant until \( x = 2.405 \) inches at which point it decreases linearly from 1,000 to 0 pounds force at \( x = 2.500 \) inches. Since the solution depends on the derivative of the applied force, the load is varied to avoid numerical issues. The problem is solved using a shooting method in Matlab, and results are shown in Figure 4.5. The reaction forces normal to the contact patch are compared to a finite element model, analyzed in Abaqus version 6.11, with the same parameters as the beamshell. Both methods predict an increase in reaction force before the applied load magnitude decreases. After the point where the load magnitude has decreased to zero, reaction forces still exist. This leads to a force imbalance at this location which in turn causes a moment which lifts the
body away from the contact surface. The scallops that are observed in the beamshell solution are artifacts of the numerical solver. They occur when the shooting method boundary value problem solver starts to exceed the numerical tolerances, at which point it adjusts the solver and continues.
Figure 4.5: Comparison of FE to Beamshell Reaction Contact Forces
4.3 A Modification to Nonlinear Shell Theory

The results of the previous section depended upon an assumed state of strain through the thickness of the beamshell. As discussed in the review of nonlinear shells in Chapter 2, there is a derived approach to shell formulation and a direct approach. The assumed states of the prior section is along the lines of a derived approach. However, a solution is sought which adds information about the through thickness stress state of the shell to a direct (Cosserat) shell. As seen in the derivation of the Libai and Simmonds shell in Chapter 2, the stress state is integrated through the entire thickness of the shell, which results in a loss of data about the through thickness state, aside from the shear stress generated by force imbalances \( Q \). The modification of Libai and Simmonds theory developed in this section adds the through thickness stress state back into the shell without assumptions and in a mathematically rigorous way.

The equation of equilibrium in the \( x_2 \)-direction (with \( x_2 \) being defined normal to the beamshell and \( x_1 \) along the beamshell), is written as

\[
\int_{-H}^{H} \tau_{12} \, dx_2 \bigg|_a^b + \int_{-H}^{H} \sigma_{22} \, dx_1 \bigg|_a^H = 0,
\]

(4.20)

where \( 2H \) is the total beam height. Change the limits of integration, which in turn changes the control volume being analyzed, through the thickness to be from a min-
imun of $y$ and a maximum of $H$. Thus the equation of equilibrium becomes

$$
\int_{y}^{H} \tau_{12} \, dx_2 \bigg|_{a}^{b} + \int_{a}^{b} \sigma_{22} \, dx_1 \bigg|_{y}^{H} = 0, \tag{4.21}
$$

which expands to

$$
\int_{y}^{H} \tau_{12} \, dx_2 \bigg|_{a}^{b} + \int_{a}^{b} \left[ \sigma_{22} (H) - \sigma_{22} (y) \right] \, dx_1 = 0. \tag{4.22}
$$

Equation (4.22) describes a control volume the varies from the positive outer-most beam fiber to a point above the beamshell $y$. In a similar fashion, change the limits of integration through the thickness to be from a maximum of $-y$ and a minimum of $-H$. Thus another equation of equilibrium is introduced which is

$$
\int_{-H}^{-y} \tau_{12} \, dx_2 \bigg|_{a}^{b} + \int_{a}^{b} \sigma_{22} \, dx_1 \bigg|_{-H}^{-y} = 0, \tag{4.23}
$$
which expands to
\[ \int_{-\tau_{12}}^{y} \tau_{12} \, dx_2 \bigg|_{a}^{b} + \int_{a}^{b} [\sigma_{22}(-y) - \sigma_{22}(-H)] \, dx_1 = 0. \quad (4.24) \]

Subtracting Equation (4.24) from Equation (4.22) gives
\[ \left[ \int_{y}^{H} \tau_{12} \, dx_2 - \int_{-\tau_{12}}^{-y} \tau_{12} \, dx_2 \right]_{a}^{b} \]
\[ + \int_{a}^{b} \left[ -\sigma_{22}(y) - \sigma_{22}(-y) + \sigma_{22}(H) + \sigma_{22}(-H) \right] \, dx_1 = 0. \quad (4.25) \]

Integrate Equation (4.25) over the entire height of the beam, which gives
\[ \left[ \int_{-\tau_{12}}^{H} \left( \int_{y}^{H} \tau_{12} \, dx_2 - \int_{-\tau_{12}}^{-y} \tau_{12} \, dx_2 \right) \, dy \right]_{a}^{b} \]
\[ + \int_{a}^{b} \left[ 2H (\sigma_{22}(H) + \sigma_{22}(-H)) - \int_{-\tau_{12}}^{H} (\sigma_{22}(y) + \sigma_{22}(-y)) \, dy \right] \, dx_1 = 0. \quad (4.26) \]

The term in Equation (4.25) which represents the squeezing stress is reduced to
\[ \int_{-\tau_{12}}^{H} (\sigma_{22}(y) + \sigma_{22}(-y)) \, dy = 2 \int_{-\tau_{12}}^{H} \sigma_{22}(y) \, dy. \quad (4.27) \]

Divide Equation (4.26) by the two times the total height, and inserting the reduction in Equation (4.27) yields
\[ \frac{1}{4H} \left[ \int_{-\tau_{12}}^{H} \left( \int_{y}^{H} \tau_{12} \, dx_2 - \int_{-\tau_{12}}^{-y} \tau_{12} \, dx_2 \right) \, dy \right]_{a}^{b} \]
\[ + \int_{a}^{b} \left[ \frac{(\sigma_{22}(H) + \sigma_{22}(-H))}{2} \right] \, dx_1 = 0. \quad (4.28) \]
Define shell variable as follows: Average Squeezing Stress (per unit length per unit width) -

\[ \chi = \int_{-H}^{H} \sigma_{22}(y) \, dy. \]

Applied External Squeezing Stress (per unit length per unit width) -

\[ p_z = \frac{(\sigma_{22}(H) + \sigma_{22}(-H))}{2}. \]

Shear Stress Distribution Term -

\[ \xi = \frac{1}{4H} \left[ \int_{-H}^{H} \left( \int_{y}^{H} \tau_{12} \, dx_2 - \int_{-H}^{-y} \tau_{12} \, dx_2 \right) \, dy \right]. \]

Take the derivative of \( \xi \) with respect to \( x_1 \) such that all terms can be brought under the integral from \( a \) to \( b \) along \( x_1 \). This gives the final equation of equilibrium as

\[ \xi' - \frac{1}{2H} \chi + p_z = 0. \] (4.29)

With the addition of Equation (4.29), the equations of motion for the shell are

\[ N' - kQ + P_T = 0, \] (4.30)

\[ Q' + kN + P_B = 0, \] (4.31)

\[ M' + (1 + e)Q - gN + l = 0, \] (4.32)

\[ \xi' - \frac{1}{2H} \chi + P_z = 0. \] (4.33)
It is seen that there are four total equations and five unknowns \((N, Q, k\) (recall \(M = f(k)\)), \(\chi\) and \(\xi\)). However, as is shown in the next section, both \(\xi\) and \(\chi\) are related to the same strain, and therefore the unknowns are reduced to four.

4.4 Dynamic Equilibrium

As with the static case, the beamshell is split into two separate control volumes, such that the two equations of equilibrium for the top and bottom control volumes are

\[
\int_{t_2}^{t_1} \left\{ \int_y^H \tau_{12} \, dx_2 \bigg|_a^b + \int_a^b [\sigma_{22}(H) - \sigma_{22}(y)] \, dx_1 \right\} \, dt \\
= \int_a^b \int_y^H \rho \frac{\partial v}{\partial t} \, dx_2 \, dx_1 \bigg|_{t_2}^{t_1}, \tag{4.34}
\]

\[
\int_{t_2}^{t_1} \left\{ \int_{-H}^{-y} \tau_{12} \, dx_2 \bigg|_a^b + \int_a^b [\sigma_{22}(-y) - \sigma_{22}(-H)] \, dx_1 \right\} \, dt \\
= \int_a^b \int_{-H}^{-y} \rho \frac{\partial v}{\partial t} \, dx_2 \, dx_1 \bigg|_{t_2}^{t_1}, \tag{4.35}
\]

where

\[
v = \frac{\partial x_2}{\partial t}. \tag{4.36}
\]

By definition, the total mass above and below the shell reference curve are equal and that the acceleration of a fiber above the reference curve has an equal and opposite
acceleration of a fiber below the reference curve such that

\[ \tilde{m} = \int_{-H}^{H} \left( \int_{y}^{H} \rho \, dx_2 + \int_{-y}^{-y} \rho \, dx_2 \right) dy, \quad (4.37) \]

\[ \tilde{v} = \frac{1}{2H\tilde{m}} \int_{-H}^{H} \left( \int_{y}^{H} \rho v_2 \, dx_2 + \int_{-y}^{-y} \rho v_2 \, dx_2 \right) dy. \quad (4.38) \]

Proceeding as the static case did, the final equation is

\[ \xi' - \frac{1}{2H} \chi + p_z = \tilde{m} \frac{\partial \tilde{v}}{\partial t}, \quad (4.39) \]

4.5 Weak Form

To put the equations of motion into the weak form, take the dot product of the linear momentum balance equations with a smooth test function \( V \), multiply the angular momentum balance by a smooth test function \( \Omega \) and multiply Equation (4.38) by a smooth test function \( \tilde{V} \). Let \( x_1 = \sigma \), this gives

\[ (F' + p - \tilde{m}\tilde{v}) \cdot V + (M' + m \cdot F + l - I\dot{\omega}) \Omega + (\xi' - \chi + p_z - \tilde{m}\dot{\tilde{v}}) \tilde{V} = 0. \quad (4.40) \]

Integrate Equation (4.40) over some portion of the time and physical domain \([t_1, t_2] \), \([a, b] \). Integrate by parts to remove the spatial and time derivatives from the solution
space. These steps give

\[
\int_{t_1}^{t_2} \left[ \left. \left( \mathbf{F} \cdot \mathbf{V} + M\Omega + \xi \tilde{V} \right) \right|_a^b \right. \\
- \int_a^b \left( \mathbf{F} \cdot \mathbf{V'} + M\Omega' + \xi \tilde{V}' - \mathbf{m} \cdot \mathbf{F} \Omega - \mathbf{p} \cdot \mathbf{V} + \chi \tilde{V} - \mathbf{p}_z \tilde{V} - l\Omega \right) \, d\sigma \right] \, dt \\
- \int_a^b \left( I\omega \Omega + \mathbf{mv} \cdot \mathbf{V} + \tilde{m} \tilde{v} \tilde{V} \right) \bigg|_{t_1}^{t_2} \, d\sigma + \int_{t_1}^{t_2} \int_a^b \left[ \mathbf{m} \dot{v} \cdot \dot{\mathbf{V}} + I\omega \Omega + \tilde{m} \tilde{v} \dot{\tilde{V}} \right] \, d\sigma \, dt = 0.
\]

(4.41)

Let the test functions be \( \mathbf{V} = \mathbf{v}, \Omega = \omega \) and \( \tilde{V} = \tilde{v} \). This allows the weak form to be written as

\[
\int_{t_1}^{t_2} \mathcal{W} \, dt = \mathcal{K}|_{t_1}^{t_2} + \int_{t_1}^{t_2} \mathcal{D} \, dt,
\]

where

\[
\mathcal{W} \equiv \left. \left( \mathbf{F} \cdot \mathbf{v} + M\omega + \xi \tilde{v} \right) \right|_a^b + \int_a^b (\mathbf{p} \cdot \mathbf{v} + l\omega + \mathbf{p}_z \tilde{v}) \, d\sigma,
\]

(4.43)

is the apparent external power,

\[
\mathcal{K} = \frac{1}{2} \left( m\mathbf{v} \cdot \mathbf{v} + I\omega^2 + \tilde{m} \tilde{v}^2 \right),
\]

(4.44)

is the kinetic energy, and

\[
\mathcal{D} = \int_a^b (\mathbf{F} \cdot (\mathbf{v}' - \omega \mathbf{m}) + M\omega' + \xi \tilde{v}' - \chi \tilde{v}) \, d\sigma,
\]

(4.45)

is the deformation power. The kinetic energy term \( \tilde{m} \tilde{v}^2 \) is the energy associated with material points that would be moving towards the beamshell centerline. Thus the correct total kinematic energy is preserved.
As shown in Chapter 2, the strains conjugate to the forces \( N \) and \( Q \) and the moment \( M \) are found using the deformation power. As derived in Chapter 2, the spatial derivative of the velocity in Equation (2.36) is resolved using the spatial derivative of the deformed tangent vector defined in Equation (2.32) such that

\[
\mathbf{v}' = \mathbf{\hat{y}}' = \dot{\mathbf{e}} \mathbf{T} + \dot{\mathbf{g}} \mathbf{B} + \omega \mathbf{e}_z \times \mathbf{y}'
\]

(4.46)

\[
= (\dot{e} - \omega g) \mathbf{T} + (\dot{g} + \omega (1 + e)) \mathbf{B}.
\]

(4.47)

Recall that

\[
\mathbf{m} = \hat{\mathbf{e}}_z \times \mathbf{y}',
\]

(4.48)

which leads to

\[
\mathbf{v}' - \omega \mathbf{m} = \dot{e} \mathbf{T} + \dot{g} \mathbf{B}.
\]

(4.49)

Plugging these new relations into the deformation power gives

\[
\int_{t_1}^{t_2} \int_a^b \left[ \mathbf{F} \cdot (\dot{e} \mathbf{T} + \dot{g} \mathbf{B}) + M \dot{k} \right] d\sigma dt.
\]

(4.50)

Since the dot vector product is commutative and distributive, the equation is rewritten as

\[
\int_{t_1}^{t_2} \int_a^b \left[ (\mathbf{F} \cdot \mathbf{T}) \dot{e} + (\mathbf{F} \cdot \mathbf{B}) \dot{g} + M \dot{k} \right] d\sigma dt.
\]

(4.51)
It is known that the internal force vector $\mathbf{F}$ dotted with the beamshell unit vectors yields the component of force in that direction. Hence, the deformation power is

$$\int_{t_1}^{t_2} \int_a^b \left[ N \dot{\epsilon} + Q \dot{\gamma} + M \dot{k} \right] d\sigma dt. \quad (4.52)$$

The new stress terms, $\chi$ and $\xi$ do not directly effect the motion of the beamshell, and thus the conjugate strain is not resolved as the other strains are. The velocity term, $\bar{v}$ represents the average velocity of material points moving inward towards the beamshell centerline. Therefore, a strain velocity is defined $-\dot{\psi}$ which is equivalent to $\bar{v}$ (an average strain rate through the thickness of the beamshell) and $\dot{\psi}'$ is equivalent to $\bar{v}'$. Thus the deformation power is

$$\mathcal{D} = \int_a^b \tau : \dot{\epsilon}, \quad (4.53)$$

where

$$\tau : \dot{\epsilon} = N \dot{\epsilon} + Q \dot{\gamma} + M \dot{k} + \chi \dot{\psi} + \xi \dot{\psi}'. \quad (4.54)$$

4.6 Strain Energy

Shell theory requires a strain energy be defined such that stress and strains can be related in a proper fashion. A strain energy density is assumed for the continuum, which is then integrated through the thickness. Then the shell quadratic energy density is set equal to the integrated continuum value and the constants which multiply
the strain terms are solved. The ensures that the total strain energy between the continuum and the shell is preserved.

Assume a quadratic strain energy density function for the shell of the form

\[
\Phi = EH \left[ c_1 e^2 + c_2 e \psi + 4c_3 H^2 k^2 + c_4 g^2 + c_5 \psi^2 + 4c_6 H^2 (\psi')^2 \right].
\]

(4.55)

Assuming plane strain conditions, stress is related to strain as

\[
\sigma_x = 2G\epsilon_x + \lambda (\epsilon_x + \epsilon_y),
\]

(4.56)

\[
\sigma_y = 2G\epsilon_y + \lambda (\epsilon_x + \epsilon_y).
\]

(4.57)

Therefore, the strain energy density with only hydrostatic terms is

\[
W = \frac{1}{2} \left[ \sigma_x \epsilon_x + \sigma_y \epsilon_y \right]
\]

(4.58)

\[
= \frac{1}{2} \left[ (2G + \lambda)\epsilon_x^2 + 2\lambda\epsilon_x \epsilon_y + (2G + \lambda)\epsilon_y^2 \right].
\]

(4.59)

Let the continuum strains be defined in terms of the shell strains as

\[
\epsilon_x = e + sk,
\]

(4.60)

\[
\epsilon_y = \psi.
\]

(4.61)

Substituting these into the strain energy density formula and integrating over the
height yields

\[
\int_{-H}^{H} W \, ds = \frac{EH}{1+\nu} \left[ \left( 1 + \frac{\nu}{1-2\nu} \right) e^2 + \frac{1}{3} H^2 \left( 1 + \frac{\nu}{1-2\nu} \right) k^2 + \frac{2\nu}{1-2\nu} e\psi + \left( 1 + \frac{\nu}{1-2\nu} \right) \psi^2 \right].
\]  

(4.62)

The shear portion of the strain energy density is separated into two portions: the shear which results in beamshell motion \( (g) \) and the shear which contributes to squeezing \( (2H\psi') \). Thus the shear energy is written as

\[
W_{\text{shear}} = \frac{1}{2} \kappa G g^2 + \frac{1}{2} G \int_{-H}^{H} (2H\psi')^2 \, ds
\]

(4.63)

\[
= EH \left[ \frac{5}{12 + 11\nu} g^2 + \frac{1}{2(1+\nu)} 4H^2 (\psi')^2 \right].
\]  

(4.64)

4.7 Equation of Motion in Terms of Strain

The forces are related to their conjugate strains by taking the derivative of the strain energy density function such that

\[
N = \Phi_{ee} = 2c_1 EHe + c_2 EH\psi,
\]

(4.65)

\[
Q = \Phi_{eg} = 2c_1 EHg,
\]

(4.66)

\[
M = \Phi_{ek} = 8c_3 E H^3 k,
\]

(4.67)

\[
\chi = \Phi_{e\psi} = c_2 EHe + 2c_5 EH\psi,
\]

(4.68)

\[
\xi = \Phi_{e\psi'} = 8c_6 E H^3 \psi'.
\]

(4.69)
Substituting these relations into the equation of motion yields

\[ EH \left( 2c_1 e' + c_2 \psi' \right) - \beta' \left( 2c_4 EH g \right) + P_T = 0, \quad (4.70) \]

\[ 2c_4 EH g' + \beta' EH \left( 2c_1 e + c_2 \psi \right) + P_B = 0, \quad (4.71) \]

\[ 8EH^3 c_3 \beta'' + (1 + e) \left( 2c_4 EG g \right) - g EH \left( 2c_1 e + c_2 \psi \right) + l = 0, \quad (4.72) \]

\[ 8EH^3 c_6 \psi'' - \frac{1}{2H} \left( c_2 EH e + 2c_5 EH \psi \right) + P_z = 0. \quad (4.73) \]

4.7.1 Example - Uniform Squeezing of a Bi-Material Laminated Beamshell

Consider a beamshell comprised of two different materials, of varying thickness, as shown in Figure 4.7. If the materials have different axial stiffnesses, then a squeezing stress causes the beamshell to bend. This is due to the varying extensional strain that is created in each layer due to the Poisson effect. For example, if the bottom material in Figure 4.7 is less stiff than the top layer, it extends more than the top layer at a cross-section, which creates an internal moment. Traditional beamshell theory does not account for this. The new equations of motion presented in this chapter account for this effect by coupling the extensional strain and bending strain via the strain energy density. The plane stress strain energy density for this problem is

\[ W = \frac{1}{2} \left[ \sigma_x \epsilon_x + \sigma_y \epsilon_y \right], \quad (4.74) \]
where,

\[
\sigma_x = \frac{E}{1 - \nu^2} (\epsilon_x + \nu \epsilon_y), \quad (4.75)
\]

\[
\sigma_y = \frac{E}{1 - \nu^2} (\epsilon_y + \nu \epsilon_x). \quad (4.76)
\]

The plane stress terms written in beamshell variables are

\[
\epsilon_x = e + (s - d)k, \quad (4.77)
\]

\[
\epsilon_y = \psi, \quad (4.78)
\]

where \(d\) is the base-reference deviation, which measures the distance from the base curve (here the interface between the two materials) and the reference curve (the mass center). As before, the strain energy density function is integrated through the
thickness, from $0 \rightarrow h^+$ and $h^- \rightarrow 0$, which gives

$$
\Phi = \frac{1}{2 (1 - \nu^2)} \left\{ E^- \int_{h^-}^{0} [(e + (s - d)k)^2 + 2\nu (e + (s - d)k) \psi + \psi^2] \, ds \\
+ E^+ \int_{0}^{h^+} [(e + (s - d)k)^2 + 2\nu (e + (s - d)k) \psi + \psi^2] \, ds \right\}. \quad (4.79)
$$

This is set equal to the beamshell strain energy density function, which gives

$$
\Phi = \frac{1}{2} Eh \left[ c_1 e^2 + c_2 hek + c_3 h^2 k^2 + c_4 e\psi + c_5 hk\psi + c_6 \psi^2 + c_7 g^2 + c_8 h^2 (\psi')^2 \right], \quad (4.80)
$$

where $E = 0.5 (E^+ + E^-)$ and $h = h^+ + h_-$. This yields the following coefficients:

$$
c_1 = \frac{1}{1 - \nu^2} \left[ \left( \frac{E^+}{E} \right) \left( \frac{h^+}{h} \right) - \left( \frac{E^-}{E} \right) \left( \frac{h^-}{h} \right) \right], \quad (4.81)
$$

$$
c_2 = \frac{1}{1 - \nu^2} \left\{ \frac{E^+}{E} \left[ \left( \frac{h^+}{h} \right)^2 - \frac{h^+ d}{h^2} \right] + \frac{E^-}{E} \left[ \frac{h^- d}{h^2} - \left( \frac{h^-}{h} \right)^2 \right] \right\}, \quad (4.82)
$$

$$
c_3 = \frac{1}{1 - \nu^2} \left\{ \frac{E^-}{E} \left[ \left( \frac{h^-}{h} \right)^2 \frac{d}{h} - \frac{1}{3} \left( \frac{h^-}{h} \right)^3 - \frac{d^2 h^-}{h^2 h} \right] \\
+ \frac{E^+}{E} \left[ - \left( \frac{h^+}{h} \right)^2 \frac{d}{h} + \frac{1}{3} \left( \frac{h^+}{h} \right)^3 + \frac{d^2 h^+}{h^2 h} \right] \right\}, \quad (4.83)
$$
\( c_4 = \frac{2\nu}{1 - \nu^2} \left[ \left( \frac{E^+}{E} \right) \left( \frac{h^+}{h} \right) - \left( \frac{E^-}{E} \right) \left( \frac{h^-}{h} \right) \right], \quad (4.84) \)

\( c_5 = \frac{\nu}{1 - \nu^2} \left\{ \frac{E^+}{E} \left[ \left( \frac{h^+}{h} \right)^2 - 2dh^+ \right] + \frac{E^-}{E} \left[ 2dh^- - \left( \frac{h^-}{h} \right)^2 \right] \right\}, \quad (4.85) \)

\( c_6 = \frac{1}{1 - \nu^2} \left[ \left( \frac{E^+}{E} \right) \left( \frac{h^+}{h} \right) - \left( \frac{E^-}{E} \right) \left( \frac{h^-}{h} \right) \right]. \quad (4.86) \)

As in the previous example, the shear term is split such that

\[
W_{\text{shear}} = \frac{1}{2}\kappa G g^2 + \frac{1}{2} G \int_{-h^-}^{h^+} (h\psi')^2 \, ds \quad (4.87)
\]

\[
= \frac{1}{2} Eh \left[ \frac{5}{12 + 11\nu} g^2 + \frac{1}{2(1 + \nu)} h^2 (\psi')^2 \right]. \quad (4.88)
\]

This gives

\( c_7 = \frac{5}{11 + 12\nu}, \quad (4.89) \)

\( c_8 = \frac{1}{2(1 + 2\nu)}. \quad (4.90) \)
The forces in terms of their conjugate strains are:

\[ N = \frac{1}{2} Eh \left[ 2c_1 e + c_2 hk + c_4 \psi \right], \quad (4.91) \]

\[ Q = Ehc_7g \quad (4.92) \]

\[ M = \frac{1}{2} Eh \left[ c_2 he + 2c_3 h^2 k + c_5 h \psi \right], \quad (4.93) \]

\[ \xi = Ehc_8 h^2 \psi' \quad (4.94) \]

\[ \chi = \frac{1}{2} Eh \left[ c_4 e + c_5 hk + 2c_6 \psi \right]. \quad (4.95) \]

A bi-metallic, 200 mm long, 2 mm thick beamshell, as shown in Figure 4.9, is analyzed. Both layers have a Poisson’s ratio of 0.3 but the top layer is 100 time stiffer than the bottom layer. For this example, both material layers have the same uniform thickness of 1 mm.

As shown in Figure 4.10, the beamshell behaves as predicted by bending toward the stiffer layer. This figure also shows the beamshell and finite element solution correspond closely for this problem. Figure 4.11 shows the angle between the cross-section and the x-axis (\( \beta \)) increases linearly along the beamshell. This
is an obvious consequence of the first derivative of the angle being constant. For this problem, since there is no change to the applied squeeze load along the length of the beamshell, the \( \xi \) term is everywhere equal to zero. Capturing this bending effect, which is not permitted in current shell theory, can be important in modeling composite structures.
Figure 4.10: Displacement in the y-Direction
Figure 4.11: Angle $\beta$ for Beamshell
4.7.2 Example - Uniform Squeezing on a Thin Isotropic, Homogeneous Beamshell

Consider an isotropic homogeneous beam subjected to a uniform squeezing force, as shown in Figure 4.12. The total beamshell is 200 mm long and 2 mm in total height. It is made of a material with a Young’s modulus of $2.09 \times 10^8$ MPa and a Poisson’s ratio of 0.3. A uniform squeezing pressure of 100 MPa is applied to both the top and bottoms surfaces uniformly along the mid-span of the beamshell. Only half of the beamshell is analyzed, which is appropriate due to symmetry. Traditional shell theory produces no strain nor displacement due to this type of load. Additionally, no load is transmitted passed the point of load application. Using the traditional beamshell equations of motion, this beam exhibits no motion and all strains are identically zero. By adding the squeezing strain $\psi$ and the strain energy density derived in Sections 4.6 and 4.7, the squeeze strain couples with the extensional strain to create a Poisson effect in the direction transverse to the applied load.

As shown in Figures 4.13 and 4.14 the beamshell allows the squeeze stress to extend past the point of load application, as suggested by Filon [63]. However, it is seen that the length in which the squeeze stress changes is much longer than that predicted by finite element analysis. This difference indicates that the shear distribution term is over predicting the shear influence zone in the beamshell formulation. Recall that traditional Timoshenko beams have a shear adjustment term included, $\kappa$, to account for the fact that the shear strain is not uniform through the cross-section. A similar adjustment is made to the shear term $\xi$ which is called $\kappa_s$. The shear strain observed in the finite element model used to compare results is concentrated near free
surfaces, dropping to zero at the center plane. With the $\kappa_s$ adjustment, the shear strain energy density becomes

$$W_{\text{shear}} = \frac{1}{2} \kappa G g^2 + \frac{1}{2} \kappa_s G \int_{-H}^{H} (2H \psi')^2 \, ds$$

(4.96)

$$= EH \left[ \frac{5}{12 + 11\nu} g^2 + \frac{\kappa_s}{2(1 + \nu)} 4H^2 (\psi')^2 \right].$$

(4.97)

The value of $\kappa_s A$ is selected to match experimental or finite element data. Figure 4.15 shows a better correlation to the finite element study when using a shear adjustment term of $\kappa_s=0.02$. This reduces the effect of $\xi$. As shown in Figure 4.16, the peak and extent of $\xi$ is adjusted to give a better solution. A check is made to ensure that the applied load magnitude does not change the distribution of the squeezing stress.

In addition to the $F_N = -100$ MPa solution, $F_N = -50$ MPa is considered. As suggested by Dunders [15], if the magnitude of the applied load is changed, the shape and distribution of the squeezing stress remains unchanged. As shown in Figure 4.17, the two different applied load magnitude solutions give an identical solution,
when normalized by the applied load. Therefore, the new beamshell is behaving as expected. Thus, the phenomena is adequately captured.
Figure 4.13: Squeezing Stress with $\kappa_s=1$
Figure 4.14: Squeezing Stress with $\kappa_s=1$, at Transition Zone
Figure 4.15: Squeezing Stress with $\kappa_s=0.02$, at Transition Zone
Figure 4.16: $\xi$ at Transition Zone
Figure 4.17: Force Check at the Transition Zone

\[ F_N = -100 \]
\[ F_N = -50 \]
4.7.3 Example - Receding Contact of a Homogeneous Beamshell on a Frictionless Rigid Foundation

Receding contact occurs as a consequence of the squeezing stress extending past the point of load application. For this example, the beam considered in the previous example is set on a frictionless, rigid surface, as shown in Figure 4.18. The surface is flat, and runs parallel to the x-axis. The same equations of motion and strain energy density used in the previous example are employed for this problem. However, since the reaction force $R_N$ is unknown, the average applied squeeze stress $p_z$ is also unknown, which results in 4 equations and 5 unknowns ($e, g, \beta, R_N$, and $\psi$). However, since the foundation does not permit the beam to impinge through it, a kinematic constraint exists which relates $e$ and $g$ to $\beta$ and $\psi$. As previously defined, the strains are

\begin{align}
  e &= (1 + u'_x) \cos \beta + u'_y \sin \beta, \quad (4.98) \\
  g &= u'_y \cos \beta - (1 + u'_x) \sin \beta. \quad (4.99)
\end{align}

Define the displacements in $x$ and $y$ as

\begin{align}
  u_x &= -H \sin \beta + v + H \psi \sin \beta, \quad (4.100) \\
  u_y &= H (\cos \beta - 1) - H \psi \cos \beta, \quad (4.101)
\end{align}
and the derivatives are

\begin{align}
  u'_x &= v' - H \beta' \cos \beta + H \psi' \sin \beta + H \psi \beta' \cos \beta, \\
  u'_y &= H \psi \beta' \sin \beta - H \beta' \sin \beta - H \psi' \cos \beta, 
\end{align}

(4.102, 4.103)

where $H$ is half of the total height and $v$ is the displacement due to sliding, as opposed to rotation. This is different from the kinematic constraint in Chapter 3 because it includes displacement due to the squeezing strain. It is assumed that $F_N$ always acts in the $\mathbf{B}$ direction (parallel to the cross-section) and the $R_N$ always acts in the $\mathbf{e}_y$ direction. This gives the applied loads in the equations of motion as

\begin{align}
  p_T &= R_N \sin \beta, \\
  p_B &= R_N \cos \beta - F_N, \\
  p_z &= \frac{1}{2} \left( R_N \cos \beta + F_N \right). 
\end{align}

(4.104, 4.105, 4.106)

As shown in Figure 4.19, the beamshell behaves as expected, with the reaction force dropping to zero slightly after the applied load ends. The extension of this reaction force creates and unbalanced load which in turn produces a moment, as shown in
Figure 4.20. The mechanism that allows the beamshell to exhibit this behavior is the addition of the squeezing strain equations of motion. As shown in Figure 4.21, the squeezing stress distribution is similar to that of the previous example, and matches closely with that predicted by finite element analysis. Also, as shown in Figure 4.22, the squeezing stress distribution and shape is insensitive to changes in applied load magnitude, resulting in the same contact patches. This was originally predicted by Dundurs [15] and verifies that the beamshell is behaving properly.

4.7.4 Conclusions

This study shows that by adding a new state variable to the equations of motion defined by Libai and Simmonds, the effect of squeezing stress is considered in the solution of beamshell problems. This is different than previous attempts at this (see Essenburg [71]) in that the transverse squeezing strain is incorporated in a Cosserat surface instead of the derived approach discussed earlier. This makes the equations of motion much more compact and exact in nature.
Figure 4.19: Applied and Reaction Pressures
Figure 4.20: Cross-Section Rotation Angle
Figure 4.21: Squeezing Stress
Figure 4.22: Squeezing Stress
CHAPTER V

CONCLUSIONS

5.1 Summary

The ability of nonlinear, geometrically exact beamshells to model complex contact phenomena is demonstrated in this work. The first study shows that nonlinear beamshells give an accurate representation of energy dissipated by friction in a contact patch. By including shear edge effects in the beamshell formulation, energy dissipated per cycle becomes a function of not only the forcing amplitude, as is the case with most energy dissipation models, but is also a function of the aspect ratio of the shell height above the contact patch to the overall length. This is demonstrated using finite element analysis and nonlinear beamshells. The study shows that as the aspect ratio increases, the zone of the beamshell which has not slipped is able to carry the shear load at the contact interface more efficiently. At certain aspect ratios, the leading cross-section will only rotate and not undergo any slipping. As the aspect ratio approaches zero, the leading edge always slips, despite the magnitude of the applied load. The second study shows that nonlinear beamshells can capture receding contact. However, a new form of nonlinear shell theory is needed which includes the effect of the average squeeze stress through the cross-section. Starting with Cauchy equations of motion, a new equation of motion is derived which includes...
this effect. Adding this equation to the equations of motion already derived by Libai and Simmonds allows for receding contact, and other interesting phenomena which standard shell theory does not permit. First an example with a uniform external applied squeezing stress applied over the length of a bi-metallic beamshell is considered. In traditional shell theory, this type of loading produces no shears nor does it produce any displacements. By including the squeezing stress, the beamshell bends towards the stiffer material. This is because of a Poisson expansion effect which causes the softer layer to expand more than the stiffer layer, producing a net moment at the cross-section, hence bending the beamshell. A second example shows that the squeezing stress behaves properly when the applied external squeezing load does not encompass the entire length of the beamshell. As predicted by a continuum solution, the internal squeezing stress extends passed the point of load application. It is shown that the beamshell displays the same behavior, and as predicted by Filon, the distribution of squeezing stress is independent of the applied external squeezing load’s magnitude. Lastly, it is shown that the new formulation permits receding contact to be modeled using kinematic constraints.
5.2 Future work

The current work proves that nonlinear beamshells are capable of analyzing complex contact phenomena, specifically the type that may exist in mechanical joint contact patches. Future research should extend the nonlinear shell theory developed in this work to become a proper reduced order model of a mechanical joint. The argument for this extension is that shell theory:

1. Is a well vetted reduction from continuum mechanics with a vast amount of experimental and numerical validation.

2. Includes the ability to easily incorporate the bending discussed by Segalman [10], since this bending is a result of couples generated by offsets in the constitutive joint members.

3. Can incorporate any material model through the strain energy density function without alterations to the basic equation of motion.

4. Can incorporate any friction model through the external force and moment vectors.

5. Can intrinsically include transient contact patches in the formulation derived in this work.

6. Intrinsically includes the geometry above the contact patch.


