AN EFFECTIVE DAMPING MEASURE: EXAMPLES USING A NONLINEAR ENERGY SINK

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AN EFFECTIVE DAMPING MEASURE: EXAMPLES USING A NONLINEAR ENERGY SINK

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Thesis

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ABSTRACT

Analysis of a harmonically forced linear system is completed as a basis for the entire work. The analytic and numerical results are compared for accuracy. Then the development of an instantaneous, cumulative, and global effective damping measure is completed. The effective damping serves as a measure of energy dissipation from the primary system. The aim is to represent the energy dissipation by its equivalent linear damping. The effective damping is then compared to a known linear system’s damping, both analytically and numerically, to ensure reliability. From there a nonlinear energy sink is added to a forced single-degree of freedom primary system and the analysis completed. A more complex forced two-degree of freedom primary system with a nonlinear attachment is then analyzed. Modal analysis is used to decompose the primary structure and calculate the modal effective damping for each mode. Finally, a nonlinear beam shell serves as the last example. The beam shell’s analytic analysis is completed for an arbitrarily positioned attachment and then numerically simulated for an attachment located in the middle of the beam. Galerkin reduction is used to approximate a single mode solution for the nonlinear beam shell numerically and then used to determine the effective damping of the system.
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CHAPTER I

INTRODUCTION

1.1 Overview

Nonlinear energy sinks are attachments that increase energy dissipation from a system. They are of interest in many applications; structures being one such example. Many times the application of a load causes vibration in a structure. Consequently, vibration mitigation is used as a way to sink energy from the system. Vibration based mitigation systems are designed in one of two ways. If a linear sink is used, it often offers the highest energy reduction for a single mode, but at the cost of its effective range. A nonlinear sink on the other hand, can provide energy reduction and vibration mitigation over a much wider bandwidth and consequently can be more efficient in its implementation. In many cases a nonlinear sink is often attached to a linear or approximately linear primary structure and used to increase the damping. The overall effect that a sink has on a system can be measured by calculating an effective damping. The primary linear structure’s damping can then be compared to the effective damping with the sink in place to show just how useful these sinks can be.

The first objective of this thesis is to define a measure of effective damping for
forced systems and numerically implement it on modal approximations of a primary system coupled to a nonlinear energy sink. The second objective is to investigate how effective a nonlinear energy sink is at pulling energy from the primary system. The energy sinks effectiveness will be determined by comparing the effective damping measure of the system with the sink to that of the system with out the sink. One way this can be completed by a simple ratio of the effective damping of the primary system with the attachment in place to that of the primary system without the attachment. Another way is by simple inspection and direct comparison of the two effective damping values.

1.2 Motivation

Every day natural phenomena like wind blowing on a bridge or building, cause structural degradation. Other phenomena like an earthquake might cause structural failure and collapse. In each case they all have a common problem: loading that leads to structural vibration. Significant time and money is spent in the development of better materials and design methods to ensure the long term dependability of these structures, however sometimes that isn’t enough to stop failure from happening. Even with cutting edge materials in place, there are times at which these materials fail naturally or fail because of poor installation. Not to mention how many structures are dated and how expensive it would be to update these with new materials.

Nonlinear energy sinks can be used to help mitigate vibration and energy from the loaded structure. Being mechanically based in nature, they can be designed to
attach to existing structures quite easily. Because they attach to an already existing structure, they are much more cost effective than updating old materials within a structure, or reconstruction. Therefore nonlinear energy sink technology has the potential to improve long term dependability of structures and prevent structural failure due to extreme loads by simply placing an attachment on an already existing structure, all while being relatively cost effective.

1.3 Effective Damping

Energy dissipation can occur in numerous ways. Friction, for example, is a natural phenomena that decreases the energy in a system. Effective damping is a measure that determines how much energy loss occurs in a primary system in question. Physically it is analogous to a decay rate, that is energy reduction per time. For more complex systems, such as nonlinear ones, the effective damping is an equivalent linear damping model that best fits that of the primary system’s energy dissipation, regardless of how the energy is pulled from the system. This work will use nonlinear energy sinks as a way to increase the effective damping of primary systems, however an increase in this measure is not limited to an energy sink. In general any energy dissipation that occurs will be accounted for in this effective damping measure.
1.4 Nonlinear Energy Sink

The nonlinear energy sink that will be considered is a single degree of freedom oscillator with a cubic stiffness and linear damper. The force response for these two components is approximately given by $F = k x^3 + b \ddot{x}$. The attachment will be modeled as an applied force to the primary system.

1.5 Thesis Overview

The first chapter introduces the reader to nonlinear energy sinks and their application. The second discusses previous work done on system identification and effective measure calculation. The majority of the work in this thesis occurs in chapters three through seven. In chapter three a single degree of freedom primary structure will be analyzed. Chapter four shows the development of a forced effective damping parameter. In chapters five and six the analysis of two discrete systems with a nonlinear energy sink is investigated. First, in chapter five a single degree of freedom primary system with a nonlinear sink attached, and then in chapter six a two degree of freedom primary system with a nonlinear sink attached. A Galerkin reduction technique is then used to model a nonlinear beam shell with a nonlinear energy sink attached in chapter seven. Chapter eight concludes the thesis by critiquing the effective damping we defined in chapter four and discussing future work to be completed.
CHAPTER II

LITERATURE REVIEW

2.1 Forced Oscillation

Analysis of linear oscillatory motion is very well developed and understood. What is presented here within only describes what was needed for the analysis that was completed. Numerous references exist that detail the analysis in much more depth [1], [25], [26].

2.2 System Identification and Effective Damping

System identification of parameters play a large roll in engineering fields. Many times there are situations that the structure or dynamic system has known and unknown parameters. Least squares estimation is one way to identify parameters [21]. General linear damping and the dissipation-matrix can be used to determine system parameters and transfer functions [14]. A Hilbert transform can be used to set up the framework to use a least squares estimation method based on instantaneous frequencies and amplitudes [11]. Fractional calculus can be developed for use on nonlinear energy sinks in vibration absorption as well [27].
Wavelet transforms provide another method in parameter estimation. This type of estimation method is found in many references throughout the literature. A Cauchy wavelet transform is applied to free-decay responses of a linear system with non-proportional viscous damping so that modal damping ratios can complex mode shapes can be identified from output-only free vibration signals [24]. Three other wavelet techniques include, wavelet transform cross-section procedure, the impulse response recovery procedure based on wavelet domain filtering, and the wavelet ridge detection procedure [30]. These three methods are applied to multi-degree of freedom systems to estimate the damping in the system.

Structural parametric identification by way of bounded mismatches can also be completed [5]. This approach is a method based on the recurrent use of the methods of parametric estimation of the modal coefficients. Another method that uses a traditional least squares method at its core, but takes advantage of a special objective function is also a viable option [6]. A different approach uses an approximation function built from the response function in the time and frequency domains along with the approximation of the frequency response function and a Discrete Fourier Transform to identify approximate system parameters [18].

The spatial location and correct mechanism of damping can be found within reasonable limits by obtaining a viscous damping matrix for complex modes and the complex natural frequencies using a perturbation method [22]. Non-viscous damping approximations also exist. This technique is further developed by making use of experimentally identified complex modes and complex natural frequencies and an
exponentially decaying relaxation function to identify the exponential time constant and spatial distribution of damping [23]. In reality the nature of damping, be it viscous or non-viscous, is to decrease energy in a system and so categorizing it in one way or another makes little difference in nature, but rather in how it is represented mathematically.

The literature also provides sources that deal with the effective damping, damping enhancement, the dynamics of nonlinear systems, and identifying effective parameters. Vibration absorbers attached to a linear multi-degree of freedom system can induce energy pumping [2]. Energy can also be used to develop an identification method for modal damping and frequencies of multi-degree of freedom systems [8]. This energy based approximation is revisited in another work that considers the dynamics of multi-degree of freedom systems with strongly nonlinear attachments [9]. Experimental passive damping enhancement by way of broadband targeted energy transfer is also of question in the literature [19]. Several other techniques are available in the literature for parameter approximation [15]. Methods are also compared for efficiency and performance as well [10].

2.3 Nonlinear Energy Sinks and System Dynamics

The specific example of an energy sink used here is only one example of these mechanisms [32]. Vibro-impact systems and their dynamics based on numerical results are also considered in the literature [16]. Other designs are also available to consider [28].
The dynamics of these types of sinks are also discussed in numerous articles in the literature. The dynamics of a two-degree of freedom nonlinear system which consists of a linear grounded oscillator coupled to a nonlinear oscillator is one such exampe [31]. Energy pumping also exists in these type of systems [20]. A substructure that is weakly coupled to an essentially nonlinear attachment shown to resonate leading to energy pumping is also considered [3].

2.4 Modal Analysis

Modal analysis is a widely known method used to simplify multi-degree of freedom systems. The technique is used to decouple multi-degree of freedom systems into a set of single degree of freedom modes [13].

2.5 Nonlinear Beam Shell Theory

An introduction to basic beam theory including Euler Bernoulli theory can be found in many sources [7]. The theory used here in 7 however was based on a more complex beam shell theory [4].

2.6 Galerkin Approximation

The Method of Weighted Residuals is a very widely known technique [12]. The Galerkin approximation or method is a special case of the Method of Weighted Resid-
uals [17]. The method also shows up in the literature. One such example deals with deformable mirrors [29].
CHAPTER III

SINGLE DEGREE OF FREEDOM ANALYSIS

In order to represent energy dissipation with linear damping it is important to first understand how a single degree of freedom system behaves. Once understood, the knowledge of the system and damping in a linear system can be used to define an effective measure for damping based on other system quantities. Essentially the linear system is a model for the work to be completed throughout. Knowing the intricacies of a linear system gives background knowledge on more difficult nonlinear problems.

The model considered is a single degree of freedom system with a spring, mass, and damper. The spring and damper are in parallel attached between the mass and ground and there is an applied force acting on the mass itself. The equations of motion for this particular system are

\[ m\ddot{x} + b\dot{x} + kx = f(t), \]

where \( m \) is the mass, \( b \) the damping constant, \( k \) the spring constant, and \( f(t) \) the applied force. The applied force is taken to be harmonic

\[ f(t) = f_0 \sin(\omega t). \]

Eqn. 3.1 can be scaled and written as

\[ \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{f_0}{m} \sin(\omega t), \]

where \( \zeta \) is the damping ratio, \( \omega_n \) is the natural frequency, and \( \frac{f_0}{m} \) is the amplitude of the applied force.
where
\[
\zeta = \frac{b}{2\sqrt{km}},
\]
and
\[
\omega_n^2 = \sqrt{\frac{k}{m}}.
\]
\(\zeta\) is known as the non-dimensional damping ratio and \(\omega_n^2\) as the square of the natural frequency of oscillation. \(\lambda\), an alternative damping parameter that will be used later considerations is defined as
\[
\lambda = 2\zeta \omega_n = \frac{b}{m}.
\]  

Eqn. 3.3 must now be solved for the system response. It is an ordinary, inhomogenous differential equation, thus the solution is a linear combination of the homogenous and particular solutions. The homogenous solution to this particular differential equation is found by solving the characteristic polynomial generated after the general solution of \(x = Ae^{\alpha x}\) has been substituted in and factored. The characteristic polynomial is
\[
\alpha^2 + 2\zeta \omega_n \alpha + \omega_n^2 = 0,
\]
and has roots
\[
\alpha_{+, -} = -\omega_n \left( \zeta \pm \sqrt{\zeta^2 - 1} \right),
\]
thus the solution to the homogenous problem is
\[
x_H(t) = C_1 e^{\alpha_+ t} + C_2 e^{\alpha_- t},
\]

(3.7)
where $C_1$ and $C_2$ are arbitrary constants that are found when initial conditions are applied. Eqn. 3.6 has three distinct cases. The discriminate of the square root can either be greater than zero, the over damped case, less than zero, the underdamped case, or exactly zero, the critically damped case. Only the underdamped case will be considered. Eqn. 3.7 can be written as

$$x_H(t) = e^{-\zeta \omega_n t} \left(C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)\right),$$  

(3.8)

where $\omega_d$ is the damped natural frequency, given by $\omega_d = \sqrt{1 - \zeta^2}$, for $0 < \zeta < 1$. Eqn. 3.8 can be rewritten in a more useful form by combining the sine and cosine terms into a single sine term with a phase shift. Doing so gives

$$x_H(t) = A_1 e^{-\zeta \omega_n t} \sin(\omega_d t + \phi),$$  

(3.9)

where

$$A_1 = \sqrt{C_1^2 + C_2^2},$$  

(3.10)

and $\phi$ is a phase angle given by

$$\tan(\phi) = \frac{C_1}{C_2}.$$  

(3.11)

The method of undetermined coefficients can be used to solve for the particular solution. The right hand side of Eqn. 3.3 is used to construct an attempted solution of

$$x_P(t) = B_1 \sin(\omega t) + B_2 \cos(\omega t),$$  

(3.12)

which is substituted into Eqn. 3.3. After setting like parts equal, $B_1$ and $B_2$ are determined to give equality between the left hand side and right hand side of Eqn.
3.3. The particular solution, written in the same useful form as Eqn. 3.9, is found to be

\[ x_P(t) = A_2 \sin(\omega t - \psi), \]  

where

\[ A_2 = \frac{f_0}{m\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}, \]  

and \( \psi \) is another phase angle given by

\[ \tan(\psi) = \frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}. \]  

The response of Eqn. 3.3 is thus

\[ x(t) = A_1 e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) + A_2 \sin(\omega t - \psi). \]  

Initial conditions can now be applied to determine the unknowns \( C_1 \) and \( C_2 \).

The initial conditions are

\[ x(0) = x_0 \]
\[ \dot{x}(0) = \dot{x}_0. \]  

Using Eqn. 3.8 and the initial conditions we find

\[ C_1 = x_0 + \frac{2f_0\zeta\omega_n\omega}{m\left[(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2\right]}, \]  

and

\[ C_2 = \frac{\dot{x}_0}{\omega_d} + \frac{\zeta\omega_n}{\omega_d} \left[x_0 + \frac{2f_0\zeta\omega_n\omega}{m\left[(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2\right]} - \frac{f_0\omega(\omega_n^2 - \omega^2)}{m\omega_d\left[(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2\right]}\right]. \]
The dynamics of a linear system are now known. This information will be used in helping to develop an effective damping measure and will serve as the model for the forced systems to come. For convenience the response is summarized here:

\[
x(t) = A_1 e^{-\zeta \omega_n t} \sin(\omega_d t + \phi) + A_2 \sin(\omega t - \psi),
\]

\[
A_1 = \sqrt{C_1^2 + C_2^2},
\]

\[
C_1 = x_0 + \frac{2f_0 \zeta \omega_n \omega}{m \left( \omega_n^2 - \omega^2 \right)^2 + (2 \zeta \omega_n \omega)^2},
\]

\[
C_2 = \frac{\ddot{x}_0}{\omega_d} + \frac{\zeta \omega_n}{\omega_d} \left[ x_0 + \frac{2f_0 \zeta \omega_n \omega}{m \left( \omega_n^2 - \omega^2 \right)^2 + (2 \zeta \omega_n \omega)^2} \right] - \frac{f_0 \omega (\omega_n^2 - \omega^2)}{m \omega_d \left( \omega_n^2 - \omega^2 \right)^2 + (2 \zeta \omega_n \omega)^2},
\]

(3.20)

\[
A_2 = \frac{f_0}{m \sqrt{\left( \omega_n^2 - \omega^2 \right)^2 + (2 \zeta \omega_n \omega)^2}},
\]

\[
\phi = \arctan \frac{C_1}{C_2},
\]

\[
\psi = \arctan \frac{2 \zeta \omega_n \omega}{\omega_n^2 - \omega^2}.
\]
CHAPTER IV
EFFECTIVE DAMPING DEVELOPMENT

4.1 Mechanical Energy

Energy is important to consider in any system, how it is dissipated or added to a system over time can give some indication of the system dynamics and evolution, especially in more complex problems that can not be solved analytically. Knowing the mechanical energy and power of a linear system serves as the basis for the development of an effective damping measure.

The mechanical energy of the system is composed of the kinetic and potential energy. The kinetic energy arises from the motion of the system and the potential from the spring force acting on the mass. The total mechanical energy can be written as

\[ E = \frac{1}{2} \dot{x}^2 + \frac{1}{2} \omega_n^2 x^2. \]  

(4.1)

4.2 Effective Damping

Determining the damping in the linear model system is very simple, however when more complex nonlinear systems come into question, it becomes difficult to gauge the value of damping. Often energy means are used to help determine decay rates and
other system parameters. The work here is no different, Eqn. 4.1 gives the mechanical
energy of a linear forced system, the goal is to define an effective damping measure
based on the mechanical energy. Damping is no different than energy dissipation and
thus can be thought of as the time rate of change of energy in a system. Thus the
first step in defining an effective measure is to start by differentiating Eqn. 4.1 to
obtain the power. The power is
\[
\dot{E}_m = \dot{x}(\ddot{x} + \omega_n^2 x).
\] (4.2)

Eqn. 4.2 can be rewritten in a more useful way by employing equation 3.3. Solving
Eqn. ?? for \( \ddot{x} \) and substituting the result into Eqn. 4.2, we obtain
\[
\dot{E}_m = f(t)\dot{x} - 2\zeta\omega_n \dot{x}^2,
\] (4.3)

which can alternatively be written in terms of \( \lambda \), defined in Eqn. 3.4, as
\[
\dot{E}_m = f(t)\dot{x} - \lambda \dot{x}^2.
\] (4.4)

Solving for \( \lambda \) the damping is obtained as a function of the power. The effective
damping is given by
\[
\lambda_{eff} = \frac{f(t)\dot{x} - \dot{E}_m}{\dot{x}^2}.
\] (4.5)

Eqn. 4.5 can be used to determine the effective damping as a function of time. It
is also convenient to define the cumulative effective damping and the global effective
damping for the system as well. The cumulative effective damping is given by
\[
\lambda_{cum} = \frac{\int_0^t f(\tau)\dot{x}(\tau) - \frac{\dot{E}(\tau)}{m} d\tau}{\int_0^t \dot{x}^2(\tau) d\tau},
\] (4.6)
while the global effective damping is given by

$$\lambda_{glb} = \frac{\int_0^\infty f(\tau)\dot{x}(\tau) - \frac{\dot{E}(\tau)}{m} \, d\tau}{\int_0^\infty \dot{x}^2(\tau) \, d\tau}. \quad (4.7)$$

The cumulative and global effective damping are time averaged quantities weighted by the energy present in the system. Defining the measures in this way eliminates those times where the effective damping blows up due to the velocity vanishing. In most oscillatory systems there is a fast time scale and slow time scale, just as in the linear model. The slow time scale is the decay rate the amplitude falls off by, where as the fast time scale is the frequency of oscillation. The hope is to capture the slow time scale in some way. One common way is to use an envelope function, that is fitting a curve to the peaks of the signal and using the result as a way to describe the system. However, when dealing with a forced system the symmetry needed to define an envelope may not exist. Filtering the signal however, mimics the behavior of the system, while eliminating the fast oscillatory time scale. Even when filtering the system’s energy and power the linear system’s damping should be equivalent to the effective damping in the steady state limit. Showing this is straight forward. Start by calculating the different terms of the right hand side of Eqn. 4.5. \( \dot{x} \) requires differentiation of Eqn. 3.16 with respect to time,

$$\dot{x}(t) = A_1 e^{-\zeta \omega_n t} \left[ \omega_d \cos(\omega_d t + \phi) - \zeta \omega_n \sin(\omega_d t + \phi) \right] + A_2 \omega \cos(\omega t - \psi). \quad (4.8)$$

Next \( f \dot{x} \),

$$f \dot{x} = \frac{f_0}{m} \sin(\omega t) \left[ A_1 e^{-\zeta \omega_n t} (\omega_d \cos(\omega_d t + \phi) - \zeta \omega_n \sin(\omega_d t + \phi)) + A_2 \omega \cos(\omega t - \psi) \right]. \quad (4.9)$$
which can be written in a more useful form by taking advantage of the product to sum trigonometric identities. After some manipulation,

\[
f \dot{x} = \frac{f_0 \omega_d A_1}{2m} e^{-\zeta \omega_n t} \left[ \sin[(\omega + \omega_d)t + \phi] + \sin[(\omega - \omega_d)t - \phi] \right] 
- \frac{f_0 \zeta \omega_n A_1}{2m} e^{-\zeta \omega_n t} \left[ \cos[(\omega - \omega_d)t - \phi] - \cos[(\omega + \omega_d)t + \phi] \right]
+ \frac{f_0 \omega A_2}{2m} \left[ \sin(2\omega t - \psi) + \sin(\psi) \right],
\]

where

\[
\sin(\psi) = \frac{2\zeta \omega_n \omega}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta \omega_n \omega)^2}}.
\]

Following the same process \(\dot{x}^2, \frac{E}{m}, \) and \(\dot{\frac{E}{m}}\) are obtained. To filter the analytic descriptions of the system requires dropping the oscillatory pieces of \(f \dot{x}, \dot{x}^2, \frac{E}{m}, \) and \(\dot{\frac{E}{m}}\). Doing so yields

\[
< f \dot{x} > = \frac{f_0 \omega A_2}{2m} \sin(\psi),
\]
\[
< \dot{x}^2 > = \frac{1}{2} (A_1 \omega_n)^2 e^{-2\zeta \omega_n t} + \frac{1}{2} (A_2 \omega)^2,
\]
\[
< \frac{E}{m} > = \frac{1}{2} (A_1 \omega_n)^2 e^{-2\zeta \omega_n t} + \frac{1}{4} A_2^2 (\omega_n^2 + \omega^2),
\]
\[
< \dot{\frac{E}{m}} > = - (\omega_n A_1)^2 (\zeta \omega_n) e^{-2\zeta \omega_n t},
\]

where the angled brackets denote a filtered value. Eqns. 4.12, 4.13, and 4.15 can then be substituted into Eqn. 4.5 to obtain the effective damping for the system. In the steady state limit \(\lambda_{eff}\) should be equivalent if not equal to \(\lambda\). After making the appropriate substitutions,

\[
\lambda_{eff} = \frac{f_0 \omega A_2 \sin(\psi) + (\omega_n A_1)^2 (\zeta \omega_n) e^{-2\zeta \omega_n t}}{\frac{1}{2} (A_1 \omega_n)^2 e^{-2\zeta \omega_n t} + \frac{1}{2} (A_2 \omega)^2}.
\]
In the steady state limit Eqn. 4.16 becomes

$$\lambda_{eff} = \frac{f_0 \omega A_2 \sin(\psi)}{\frac{1}{2}(A_2 \omega)^2},$$

(4.17)

which can be further simplified by Eqns. 3.20 and 4.11 producing

$$\lambda_{eff} = \frac{f_0^2 \omega^2 \omega_n \zeta}{m^2((\omega_n^2 - \omega^2)^2 + (2\zeta \omega_n \omega)^2)} \frac{f_0^2 \omega^2}{\frac{1}{2} m^2((\omega_n^2 - \omega^2)^2 + (2\zeta \omega_n \omega)^2)}$$

$$= 2\zeta \omega_n,$$

(4.18)

just as defined in Eqn. 3.4. Thus the filtered effective damping, filtered cumulative effective damping, and filtered global effective damping are defined as

$$\lambda_f = \frac{\langle f \dot{x} \rangle - \langle \frac{\dot{E}}{m} \rangle}{\langle \dot{x}^2 \rangle},$$

(4.19)

$$\lambda_c = \frac{\int_0^t < f(\tau) \dot{x}(\tau) > - \langle \frac{\dot{E}(\tau)}{m} \rangle }{\int_0^t \langle \dot{x}^2(\tau) \rangle } \ d\tau,$$

(4.20)

$$\lambda_g = \frac{\int_0^\infty < f(\tau) \dot{x}(\tau) > - \langle \frac{\dot{E}(\tau)}{m} \rangle }{\int_0^\infty \langle \dot{x}^2(\tau) \rangle } \ d\tau.$$

(4.21)

From this point forward any reference to the effective damping referrs to the filtered effective damping, filtered cumulative effective damping, or filtered global effective damping as given in Eqns. 4.19 to 4.21.

4.3 Numerical Effective Damping

Numerical simulation is a key tool in the solution of many complex systems. Numerical code was developed to calculate the effective damping based on numerical simulations of a system. Satisfying the main objective of the thesis, the code can be used on simulations both with and without a nonlinear energy sink. This allows
for comparison between the two results. MATLAB was used for the coding and the final product entitled \textit{Femeasures.m}. Appendix 8.2 provides the source code for \textit{Femeasures}.

4.3.1 Fourth Order Low-Pass Butterworth Filter

Filtering ensures elimination of the fast time scale and any potential problems that may arise in the effective damping from it. A fourth order low-pass Butterworth filter is used in the numerical work to filter the signal and attenuate the fast time scale of oscillation. A Butterworth filter was chosen because it best approximates the passband compared to other filter types. The stopband for Butterworth filters fall off much slower in comparison, thus a higher order filter had to be used. The cut off frequency used by the filter was set to 0.15 Hz, a value well below both the natural frequency and forcing frequencies that will be sampled. The filter was designed so that the cut off frequency can be varied based on the system parameters and needs of the user.

When the actual signals were filtered, care had to be taken at the end points to ensure preservation. The MATLAB command \texttt{filtfilt} that was used works by first filtering forward from the beginning to the end and then starts at the end of the signal and filters back to the beginning. The command tries to match the initial conditions at each end point as well as both filtering pass values. To aid in getting a better match the data was flipped symmetrically with respect to the vertical axis to generate an even function using the \texttt{flipud} command in MATLAB and then filtered.
by \textit{filtfilt}. By doing this the initial conditions at $t = 0$ are left intact. Once the filtering process is complete the artificial data for negative time is removed and the approximated response for $t \geq 0$ is used.

4.3.2 Analytical and Numerical Comparison

Sec. 4.2 demonstrated that an exact match of the effective damping to the known linear damping can be achieved by filtering the response. Numerically nothing less is expected. To ensure accuracy, numerical simulation of a linear system will be compared to that of the exact analytic solution obtain previously. Once the simulated response is determined to be valid, it will be used to calculate the effective damping of the linear system, which should reproduce the damping, $\lambda$, for the system. MATLAB’s \textit{ode45} was used to simulate the system response. The system parameters of the simulated system are $m = 1$, $k = 10$, $f_0 = 5$, $b = 0.2$, and $\omega = 2$. The initial conditions are $x(0) = \dot{x}(0) = 0$. Fig. 4.1 shows the simulated response and analytic response. There is very little difference between the exact and numerical solution.

Next apply the algorithm for the effective damping. Each term in Eqn. 4.19 is compared to its analytic counterpart in Eqns. 4.12 through 4.15 to ensure a valid result. Fig. 4.2 shows the comparison of each filtered term to its numerically simulated value. The effective damping for this system is shown as a function of time in Fig. 4.3. The global effective damping was found to be $\lambda_g = 0.2020$ as compared to the expected value of $\lambda_{lin} = 0.2$. 

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Figure 4.1: Numerical and Exact Analytic Response Comparison. System Parameters: $m = 1$, $k = 10$, $f_0 = 5$, $b = 0.2$, and $\omega = 2$. The initial conditions are $x(0) = \dot{x}(0) = 0$. The numerical response is indicated by a dotted line and the exact by the solid curve. The bottom plot zooms in on a peak to demonstrate the accuracy of the simulation.

Initially the effective damping and cumulative effective damping curves are far from the expected value $\lambda = 0.2$ in Fig. 4.3. As time goes on however, the curves approach the expected value. Early in time there are low energy levels in the system and a small change in energy leads to a dramatic change in the identified damping. As time goes on, the applied force inputs energy to the system and so the energy level rises. Now at a higher energy level, small changes in the system’s energy have
Figure 4.2: Term by Term Comparison Linear Model. System Parameters: $m = 1$, $k = 10$, $f_0 = 5$, $b = 0.2$, and $\omega = 2$. The initial conditions are $x(0) = \dot{x}(0) = 0$. The three plots show each term in Eqn. 4.19 compared to their numerically simulated results.

less and less an effect on the identified damping. This is confirmed by investigating a system with a higher initial energy. Leaving all system parameters the same and changing only the initial conditions to $x(0) = 7$ and $\dot{x}(0) = 2$, Fig. 4.4 shows the damping as a function of time. With more energy initially in the system the identified damping is much closer to the expected value early in time. The identified damping also reaches the expected value much more rapidly than in the previous case. In this case the global effective damping was found to be $\lambda_g = 0.1998$ as compared to the expected value of $\lambda_{lin} = 0.2$. 

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Figure 4.3: Damping Linear Model. System Parameters: $m = 1$, $k = 10$, $f_0 = 5$, $b = 0.2$, and $\omega = 2$. The initial conditions are $x(0) = \dot{x}(0) = 0$. The effective damping, cumulative effective damping, and global effective damping are shown as functions of time.

The objective of the chapter was to develop an effective damping measure that would allow comparison between different systems. The instantaneous effective damping, cumulative effective damping, and global effective damping given in Eqns. 4.19, 4.20, and 4.21 respectively, allow for such comparisons. The global effective damping provides a way to compare systems based on an overall effective damping value. This measure will be put into practice in the coming chapters and used to compare a primary system with no energy sink to the same system with an energy
Figure 4.4: Damping For a System with Higher Initial Energy. System Parameters: $m = 1$, $k = 10$, $f_0 = 5$, $b = 0.2$, and $\omega = 2$. The initial conditions are $x(0) = 7$ and $\dot{x}(0) = 2$. The effective damping, cumulative effective damping, and global effective damping are shown as functions of time.

sink attached. This analysis will demonstrate the usefulness of nonlinear energy sinks.
CHAPTER V
SINGLE DEGREE OF FREEDOM PRIMARY SYSTEM

5.1 General Forcing

The system in question for this chapter will be a single degree of freedom primary system with a nonlinear energy sink attached. The primary system consists of a linear spring and damper in parallel, attached to the primary mass with time dependent force acting on the mass. Also attached to the primary mass is the attachment, which is composed of a cubically nonlinear spring, linear damper, and the attachment mass. The system’s equations of motion are given by:

\[
m\ddot{x}_1 + b\dot{x}_1 + kx_1 = f_{app}(\tau) + b_a(\dot{x}_2 - \dot{x}_1) + k_a(x_2 - x_1)^3
\]

\[
m_a\ddot{x}_2 + b_a(\dot{x}_2 - \dot{x}_1) + k_a(x_2 - x_1)^3 = 0,
\]

where \( \dot{x}_1 \) is the displacement of the primary system relative to ground and \( \dot{x}_2 \) the displacement of the attachment mass relative to ground. The relative displacement between the primary system and the attachment mass is defined as \( \dot{y}(\tau) = x_2 - x_1 \).

Eqn. 5.1 can be non-dimensionalized. The first step is to define a characteristic length, \( L \), taken to be the initial length of the unstretched spring in the primary system. Thus all lengths scale as \( x_i = Lx_i \). The characteristic time is defined as \( t_i^2 = \frac{1}{\omega_i^2} = \frac{m}{k} \). Substitution of these parameters into Eqn. 5.1, dividing by the mass,
and some manipulation gives

\[
\ddot{x}_1 + \left( \frac{b c_t}{m} \right) \dot{x}_1 + \left( \frac{k c_t^2}{mL} \right) x_1 = \left( \frac{c_t^2}{mL} \right) \tilde{f}_{app}(t) + \left( \frac{b_a c_t}{m} \right) (\dot{x}_2 - \dot{x}_1) + \left( \frac{k_a c_t^2 L^2}{m} \right) (x_2 - x_1)^3
\]

(5.2)

\[
\frac{m_a}{m} \ddot{x}_2 + \left( \frac{b_a c_t}{m} \right) (\dot{x}_2 - \dot{x}_1) \left( \frac{k_a c_t^2 L^2}{m} \right) (x_2 - x_1)^3 = 0.
\]

The mass ratio is then defined as, \( M_r = \frac{m_a}{m} \), and non-dimensional force as \( F(t) = \gamma \tilde{f}_{app}(t) \), where \( \gamma = \frac{c_t^2}{mL} \). Using these non-dimensional parameters Eqn. 5.2 is written as

\[
\ddot{x}_1 + 2\zeta \dot{x}_1 + x_1 = \gamma \tilde{f}_{app}(t) + M_r \left[ B(\dot{x}_2 - \dot{x}_1) + K(x_2 - x_1)^3 \right]
\]

(5.3)

\[
\ddot{x}_2 + B(\dot{x}_2 - \dot{x}_1) + K(x_2 - x_1)^3 = 0,
\]

where \( \lambda = \frac{b}{m} \) and \( \omega_n = \sqrt{\frac{k}{m}} \) are the damping and natural frequency defined previously, and \( B = \frac{b_a}{M_r \sqrt{km}} \), \( K = \frac{k_a L^2}{k M_r} \), and \( \gamma = \frac{k}{L} \).

5.2 Harmonic Forcing

In order to analyze the nonlinear energy sink’s performance, a force must be chosen. For generality take \( \tilde{f}_{app} = A \sin(\omega t) \). MATLAB can now be used to determine the response of the primary system and attachment. Once those are known, Femeasures.m can be used to determine the effective damping of the primary system. For simplicity the characteristic length, \( L \), will be taken to be unitary for the work until the end of the chapter.
5.2.1 Varying Attachment Damping

Consider the case where the primary parameters are fixed and the attachment stiffness and mass ratio are fixed. The values of these parameters are given in Table 5.1. The initial conditions are $x_1(0) = 0$, $\dot{x}_1(0) = 1$, and $x_2(0) = \dot{x}_2(0) = 0$. Fig. 5.1 shows several important items. It shows the response and energy of the primary system both when the attachment is in place and when it is not. The behavior of the response in both cases is exactly what is to be expected, transience early in time giving way to a steady state response that is dictated by the applied force, a sinusoid in this case. The energy in both cases starts off relatively high and decreases toward a steady state value. The curves for which the attachment in place show the response and energy for $b_a = 9.060\frac{Ns}{m}$ which translates to the non-dimensional value of $B = 8.271$.

Table 5.1: Fixed Parameters: Varied Attachment Damping.
Initial Conditions $x_1(0) = 0$, $\dot{x}_1(0) = 1$, and $x_2(0) = \dot{x}_2(0) = 0$.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primary Mass</td>
<td>$m = 1$ kg</td>
</tr>
<tr>
<td>Primary Stiffness</td>
<td>$k = 30\frac{N}{m}$</td>
</tr>
<tr>
<td>Primary Damping</td>
<td>$b = 1\frac{Ns}{m}$</td>
</tr>
<tr>
<td>Forcing Frequency</td>
<td>$\omega = 7$ Hz</td>
</tr>
<tr>
<td>Forcing Amplitude</td>
<td>$A = 1$ N</td>
</tr>
<tr>
<td>Attachment Mass</td>
<td>$m_a = 0.2$ kg</td>
</tr>
<tr>
<td>Attachment Stiffness</td>
<td>$k_a = 15\frac{N}{m}$</td>
</tr>
</tbody>
</table>
Figs. 5.2 and 5.3 show the ratio of global effective damping for the system with the attachment to the same system without the attachment. Initially all curves are near unity. When the attachment damping is zero, it does not contribute to the effective damping, and so the ratio of the effective damping of the system with an attachment to that of just the primary system is very near one. As the attachment damping increases so to does the effective damping, until it reaches a maximum and then decreases toward unity again. The maximum corresponds to the optimum attachment damping value for that particular parameter set.

Figure 5.1: Varied Attachment Damping System Response. System Parameters are given in Table 5.1. The attachment damping value of this response is \( b_a = 9.0604 \frac{N}{m} \) which is translates to the non-dimensional value of \( B = 8.2710 \).
The curve’s shape has a physical meaning. As the value of the attachment damping varies so to does the viscosity in the damper. Once the maximum damping value and corresponding viscosity is reached, the effective damping decays. For larger values of damping and consequently viscosity, the damper acts more as a solid connecting the primary system and attachment. In the limit that the attachment damping approaches infinity the damper becomes a rigid body connecting the two masses and contributes no additional damping to the primary system, thus the ratio is unity.

Fig. 5.2 shows the global effective damping ratio for different initial conditions. Increasing the energy in the system does not move the optimum damping left or right and so has little affect on what value of damping is used. The only influence increasing the energy in the primary system via the initial conditions has, is to affect the amplitude of the peak. What’s more interesting is that the amplitude decreases as the energy increases. However, when the energy input is increased via the applied force, the opposite occurs. The effective damping increases as the forcing amplitude increases. Fig. 5.3 demonstrates this.

The difference in behavior is a consequence of the non-dimensionalization. When the initial conditions and forcing amplitude are varied an energy level is chosen and the non-dimensional damping and stiffness set based on those. However, when the non-dimensional damping or stiffness are chosen the energy level in the system is set based on those.
5.2.2 Varying Attachment Stiffness

Now consider the case where the attachment stiffness is varied. The primary system parameters, attachment damping, and mass ratio are fixed. The values for which are given in Table 5.2. The initial conditions are the same as those in the previous case, \(x_1(0) = 0\), \(\dot{x}_1(0) = 1\), and \(x_2(0) = \dot{x}_2(0) = 0\).

Fig. 5.4 shows the response of the primary system with and without the attachment, both as functions of time. The figure also shows the energy for both cases as functions of time. In this example the attachment stiffness is \(k_a = 4.8658 \frac{N}{m}\).
which is $K = 1.622$ in its non-dimensional form. Again there is transience early on giving way to the steady state sinusoid behavior expected. The energy behaves similar to the case of the varied damping. Initially there is a high energy level in the primary system and then it decreases rapidly and reaches a steady state limit dictated by the forcing. The ratio of global effective damping for the system with the attachment compared to that without it is $\lambda_g = 1.183$ for this example.
Table 5.2: Fixed Parameters: Varied Attachment Stiffness. Initial Conditions $x_1(0) = 0$, $\dot{x}_1(0) = 1$, and $x_2(0) = \dot{x}_2(0) = 0$.

<table>
<thead>
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<th>Parameter</th>
<th>Value</th>
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</thead>
<tbody>
<tr>
<td>Primary Mass</td>
<td>$m = 1 \text{ kg}$</td>
</tr>
<tr>
<td>Primary Stiffness</td>
<td>$k = 30 \frac{N}{m}$</td>
</tr>
<tr>
<td>Primary Damping</td>
<td>$b = 2 \frac{Ns}{m}$</td>
</tr>
<tr>
<td>Forcing Frequency</td>
<td>$\omega = 3 \text{ Hz}$</td>
</tr>
<tr>
<td>Forcing Amplitude</td>
<td>$A = 1 \text{ N}$</td>
</tr>
<tr>
<td>Attachment Mass</td>
<td>$m_a = 0.1 \text{ kg}$</td>
</tr>
<tr>
<td>Attachment Damping</td>
<td>$b_a = 3 \frac{N}{m}$</td>
</tr>
</tbody>
</table>

Figs. 5.5 and 5.6 shows the global effective damping as a function of attachment stiffness. The curve looks very similar to Figs. 5.2 and 5.3. Initially starting near unity the curve peaks and then decays there after. The peak represents the optimum attachment stiffness for the parameter set being studied. The decay after the peak corresponds to the system approaching the limit of attachment stiffness becoming infinite. In the limit the attachment spring acts as a rigid rod and the attachment does not contribute to the effective damping of the system. As evidenced by the ratio becoming unity in this limit.
Figure 5.4: Varied Attachment Stiffness System Response. System Parameters are given in Table 5.2. The attachment stiffness at this instant is $k_a = 4.8658 \, \text{N/m}$, or $K = 1.622$.

Fig. 5.5 shows how the ratio of global effective damping as a function of attachment stiffness varies as the initial conditions vary. Just as in the case of varied attachment damping increasing the energy input from the initial conditions leads to a decrease in performance of the attachment.

Fig. 5.6 shows the ratio of global effective damping as a function of attachment stiffness for varied forcing amplitude. Again as the energy of the primary system is increased the maximum effective damping increases. This difference in behavior is
a result of the non-dimensionalization. Fig. 5.6 also illustrates that a small change in forcing amplitude shifts the peak location. As the forcing amplitude increases the peak shifts to the left. This implies that a softer spring works better for larger amplitude forces.

Figure 5.5: Global Effective Damping as a Function of Attachment Stiffness. System Parameters are given in Table 5.2. Each curve corresponds to different initial conditions.

5.2.3 Varying Attachment Mass to Primary System Mass Ratio

Lastly the mass ratio is varied. Although lower mass ratios are typically used, the analysis will cover a wide range of possible ratios so to determine the behavior of the
Figure 5.6: Global Effective Damping as a Function of Attachment Stiffness. System Parameters are given in Table 5.2. Each curve corresponds to a different forcing amplitude.

The system and attachment parameters that will be held constant are given in Table 5.3. The initial conditions are $x_1(0) = 0$, $\dot{x}_1(0) = 3$, and $x_2(0) = \dot{x}_2(0) = 0$. Fig. 5.7 shows the response of the primary system both with and without the attachment as well as the energy for both cases. The mass ratio for this particular example is $M_r = 0.3268$ and the global effective damping is $\lambda_g = 8.1011$. As in the previous cases the response undergoes transience early in time approaching a steady state response dictated by the applied force. Notice the response amplitude of oscillation is greatly reduced when the attachment is in place. The energy follows the same trend as in
previous cases; initially there is a high energy level which drops off and approaches a much lower steady state value there after. The primary system energy amplitude in the steady state is also significantly lower when the attachment is present as well.

Figs. 5.8 and 5.9 show the ratio of global effective damping as a function of the mass ratio. In both cases the curves follow the expected pattern, they reach a maximum and then decay as the mass ratio increases. For larger mass ratio values the energy transferred to the attachment decreases; the inertia of the heavier attachment impedes the energy transfer. As the mass of the attachment increases so to does the power required to induce motion, and past the optimum value, the interplay between the inertia of the mass and the transfer of energy between the primary system and attachment work to reduce the effective damping. In the limit that the mass ratio goes to infinity the ratio of effective damping of the system with the attachment to that with out will go to unity, in this limit the primary system transfers no energy to the attachment and the attachment does nothing to amplify the system damping.

Fig. 5.8 shows how the effective damping curve varies as the initial conditions are changed. As the initial velocity increases the peak stays in the same location, however the amplitude of the peak decreases. As the energy in the primary system increases the effective damping decreases. The same behavior is shown in Fig. 5.9. As the amplitude of the applied force is increased the maximum value of effective damping decreases. At the same time, the optimum mass ratio increases, that is the peak shifts to the right. Thus for higher energy applied loads, it is beneficial to use a heavier attachment.
Table 5.3: Fixed Parameters: Varied Mass Ratio.
Initial Conditions $x_1(0) = 0$, $\dot{x}_1(0) = 3$, and $x_2(0) = \dot{x}_2(0) = 0$.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primary Mass</td>
<td>$m = 1$ kg</td>
</tr>
<tr>
<td>Primary Stiffness</td>
<td>$k = 60 \frac{N}{m}$</td>
</tr>
<tr>
<td>Primary Damping</td>
<td>$b = 2 \frac{Ns}{m}$</td>
</tr>
<tr>
<td>Forcing Frequency</td>
<td>$\omega = 6$ Hz</td>
</tr>
<tr>
<td>Forcing Amplitude</td>
<td>$A = 1$N</td>
</tr>
<tr>
<td>Attachment Damping</td>
<td>$b_a = 1 \frac{N}{m}$</td>
</tr>
<tr>
<td>Attachment Stiffness</td>
<td>$k_a = 45$ kg</td>
</tr>
</tbody>
</table>

5.2.4 Summary

The analysis for a single degree of freedom primary system with an attached nonlinear energy sink provides a basis from which to compare results in the next chapter where a two degree of freedom primary system will be analyze. Once modal decomposition is completed each mode should behave in much the same way as the single degree of system primary system.
Figure 5.7: Varied Mass Ratio System Response. System Parameters are given in Table 5.3. The mass ratio for this particular case is $M_r = 0.3268$. 
Figure 5.8: Global Effective Damping as a Function of Mass Ratio with Varied Initial Conditions. System Parameters are given in Table 5.3.
Figure 5.9: Global Effective Damping as a Function of Mass Ratio with Varied Forcing Strength. System Parameters are given in Table 5.3.
CHAPTER VI
TWO DEGREE OF FREEDOM PRIMARY SYSTEM

6.1 Modal Analysis

Multi-degree of freedom system analysis is simplified by the technique of modal analysis. This technique is essentially a linear transformation between two basis sets of generalized coordinates. The transformation maps a given set of coordinates to a set of principal coordinates that decouples the equations of motion.

The equations of motion for an \( n \)-degree of freedom undamped system are given as

\[
M \ddot{\vec{x}} + K \vec{x} = \vec{F},
\]

where \( M \), \( K \), and \( \vec{F} \) are the mass matrix, stiffness matrix, and forcing vector respectively. The initial conditions are

\[
\vec{x}(0) = \vec{x}_0 \quad \text{and} \quad \dot{\vec{x}}(0) = \dot{\vec{x}}_0.
\]

Determination of the modal matrix is the next step. Doing so requires solving the unforced free vibration problem to obtain the natural frequencies of each mode. This is done by assuming a normal mode solution, substituting in to Eqn. 6.1 and setting up the eigenvalue problem

\[
M^{-1}K \vec{x}_i = \omega_i^2 \vec{x}_i,
\]
where $\omega_i^2$ are the eigenvalues and are also the square of the natural frequencies of the modes. The corresponding eigenvector for each eigenvalue, $\vec{x}_i$, is known as the mode shape. Just as in any eigenvalue problem the eigenvectors or mode shapes need to be normalized with respect to some inner product. In the case of vibratory systems the kinetic energy inner product,

$$(\vec{x}_i, \vec{x}_i)_M = \vec{x}_i^T M \vec{x}_i = 1,$$

is used to normalize the mode shapes. Consequently the modal mass matrix is transformed to the identity matrix. Choosing this normalization scheme has one additional advantage. The potential energy inner product,

$$(\vec{x}_i, \vec{x}_i)_K = \vec{x}_i^T K \vec{x}_i = \omega_i^2,$$

(6.5)

diagonalizes the modal stiffness matrix with the values on the diagonal becoming the eigenvalues or natural frequencies. $\Omega$ is defined as that matrix, thus $\Omega = \text{diag}(\omega_i^2)$. The natural frequencies must be ordered properly,

$$\omega_1 \leq \omega_2 \leq \cdots \leq \omega_n,$$

(6.6)

while keeping their respective mode shapes in the same order. The modal matrix is formed by the column mode shape vectors as

$$P = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix}.$$

(6.7)

$P$ is the modal matrix that is used in the linear transformation between coordinates sets. The principal coordinates are given by

$$\vec{x} = P \vec{q},$$

(6.8)
thus the equations of motion are

\[ MP\ddot{\vec{q}} + KP\vec{q} = \vec{F}. \quad (6.9) \]

Multiplying Eqn. 6.9 by the transpose of the modal matrix causes the system to reduce to a convenient form. The mass matrix is transformed to the identity matrix and the stiffness matrix to a matrix whose diagonal elements are the square of the natural frequencies and zero elsewhere. Thus

\[ P^T MP\ddot{\vec{q}} + P^T KP\vec{q} = P^T \vec{F} \]

or

\[ \ddot{\vec{q}} + \Omega \vec{q} = \vec{G}, \quad (6.10) \]

where \( \vec{G} \) is the modal forcing vector given by \( \vec{G} = P^T \vec{F} \).

Although the analysis completed is done so for an undamped system, damping can be added simply by assuming it is an applied force and transforming it the same way the forcing vector is transformed. With damping present the equations of motion are

\[ \ddot{\vec{q}} + (P^T CP)\dot{\vec{q}} + \Omega \vec{q} = \vec{G}. \quad (6.11) \]
In both cases the initial conditions transform as

\[ \tilde{x}(0) = P \tilde{q}(0) \]

\[ P^T M \tilde{x}(0) = P^T M P \tilde{q}(0) \]

or

\[ \tilde{q}(0) = P^T M \tilde{x}_0 = \tilde{q}_0, \]

similiary

\[ \tilde{q}(0) = P^T M \tilde{x}_0 = \tilde{q}_0. \] (6.13)

6.2 Two-Degree of Freedom Primary System Analysis

Using modal analysis an \( n \)-degree of freedom primary system with an attachment in place can now be analyzed. A two-degree of freedom primary system with an attachment on the top mass and an applied force acting on it will now be analyzed. The set up is similar to the previous example of a one-degree of freedom primary system. The bottom mass has a spring and damper in parallel attached to ground, with the top mass attached to the bottom mass in the same manner. The same attachment that was used in the one-degree of freedom case is used again here, that is another mass connected to the top mass via a damper and cubic spring in parallel. There is also an applied force acting on the second mass.
The equations of motion for this system are

\[
\begin{align*}
m_1 \ddot{x}_1 + (c_1 + c_2) \dot{x}_1 + (k_1 + k_2) x_1 &= c_2 \dot{x}_2 + k_2 x_2 \\
m_2 \ddot{x}_2 + c_2 \dot{x}_2 + k_2 x_2 &= c_2 \dot{x}_1 + k_2 x_1 + b_N (\dot{x}_3 - \dot{x}_2) + k_N (x_3 - x_2)^3 + f(t) \\
m_N \ddot{x}_3 + b_N (\dot{x}_3 - \dot{x}_2) + k_N (x_3 - x_2)^3 &= 0,
\end{align*}
\]

where \( x_1, x_2, \) and \( x_3 \) track the absolute displacement of the first, second, and attachment’s mass respectively. Modal analysis can be used to uncouple the primary system, determine the natural frequencies, and the modal matrix. The modal response of the system can then be calculated and used to calculate the effective damping of each mode both with and with out the attachment in place. MATLAB was used to determine the natural frequencies and modal matrix. Appendix 8.2 shows the code that was developed to perform modal analysis numerically. The parameter set used takes \( m_1 = m_2 = 1, \ k_1 = 30, \ k_2 = 50, \ c_1 = 3, \ c_2 = 2, \) and \( f(t) = A \sin(\omega_f t) \) having \( A = 2 \) and \( \omega_f = 8. \) The initial conditions were taken as \( x_1(0) = \dot{x}_1(0) = x_2(0) = 0 \) and \( \dot{x}_2(0) = 1. \) For this set of parameters the natural frequencies are \( \omega_1 = 3.577 \) and \( \omega_2 = 10.826 \) and the modal matrix is

\[
P = \begin{bmatrix}
-0.5969 & -0.8023 \\
-0.8023 & 0.5969
\end{bmatrix}.
\]

MATLAB was also used to simulate the equations of motion numerically. A derivative file was created and \textit{ode45} was used to complete the simulation. Appendix 8.2 shows the derivative file for the two degree of freedom system. Once the response was generated the transformation to modal coordinates was completed. Figs. 6.1 and
6.2 show the modal response for the first and second modes in the primary system as well as their corresponding energies for the case when there is no attachment in place, $m_a = k_a = c_a = 0$.

![Figure 6.1: Mode One Response and Energy. System Parameters: $m_1 = m_2 = 1$, $k_1 = 30$, $k_2 = 50$, $c_1 = 3$, $c_2 = 2$, $f(t) = A \sin(\omega_f t)$, $A = 2$ and $\omega_f = 8$. The natural frequency for this mode is $\omega_1 = 3.577$. The response looks reasonable transience giving way to a forced steady state sinusoid.](image)
Figure 6.2: Mode Two Response and Energy. System Parameters: $m_1 = m_2 = 1$, $k_1 = 30$, $k_2 = 50$, $c_1 = 3$, $c_2 = 2$, $f(t) = A \sin(\omega_f t)$, $A = 2$ and $\omega_f = 8$. The natural frequency is $\omega_2 = 10.826$. Transience early in time again giving way to the forced steady state sinusoid response.

Just as in the case of the one-degree of freedom primary system, the attachment stiffness, damping and mass ratio will be varied to determine the modal behavior of the effective damping as functions of those three parameters. The applied force is taken to be harmonic for the analysis, $f(t) = A \sin(\omega_f t)$.

6.2.1 Varied Attachment Damping

First consider the behavior of the ratio of global effective damping for the system with no attachment to that of the system with an attachment as the attachment damping.
is varied. The primary system parameters and fixed attachment parameters are given in Table 6.1. The modal natural frequencies are determined to be $\omega_1 = 3.577$ and $\omega_2 = 10.826$ and the initial conditions are $q_1(0) = q_2(0) = 0$ and $\dot{q}_1(0) = \dot{q}_2(0) = 1$.

Table 6.1: Fixed Parameters: Varied Attachment Damping.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primary Masses</td>
<td>$m_1 = m_2 = 1$ kg</td>
</tr>
<tr>
<td>Primary Stiffnesses</td>
<td>$k_1 = 30$, $k_2 = 50 \frac{N}{m}$</td>
</tr>
<tr>
<td>Primary Damping</td>
<td>$c_1 = 2$, $c_2 = 3 \frac{Ns}{m}$</td>
</tr>
<tr>
<td>Forcing Frequency</td>
<td>$\omega_f = 5$ Hz</td>
</tr>
<tr>
<td>Forcing Amplitude</td>
<td>$A = 3$ N</td>
</tr>
<tr>
<td>Attachment Mass</td>
<td>$m_a = 0.1$ kg</td>
</tr>
<tr>
<td>Attachment Stiffness</td>
<td>$k_a = 50 \frac{N}{m}$</td>
</tr>
</tbody>
</table>

Figs. 6.3 and 6.4 show the modal effective damping for the first and second modes as functions of attachment damping respectively. The curves follow the same trend as in the single degree of freedom primary case. Both curves peak and then decay. The first mode decays much more rapidly then the second though. In both cases the decay approaches unity for larger attachment damping values. The shape of the modal effective damping curves have physical significance just as in the single degree of freedom system. As the attachment damping increases so to does the viscosity of the fluid in the damper. In the limit that the attachment damping
goes to infinity the curve goes to unity. It is in this limit the effective damping of
the primary system with no attachment is the same as that of the system with the
attachment in place, thus a ratio of unity.

Figs. 6.5 and 6.6 show the response of the first and second mode as functions of time respectively at each modal effective damping curves’ respective peak. Both figures show the response of each mode both with and without the attachment enabled. The response amplitude of the system with the attachment in place
Figure 6.4: Global Effective Damping Ratio as a Function of Attachment Damping. System Parameters are given in Table 6.1. The natural frequency is $\omega_2 = 10.826$.

is decreased in both cases. The responses shown for the first mode peak in effective damping however, have a much smaller amplitudes than their no attachment counterparts. In the case of the response for the peak effective damping of the second mode, there is a decrease in amplitude as well, but not as significant. This difference in amplitude is accounted for in how the energy is dissipated. In Fig. 6.4 there is not only a global maximum to the curve, but a local maximum near $c_a \approx 0.2$. This local maximum lines up very closely with the global max effective damping for the
first mode. With both modes having a maximum at this point, we can infer most of the energy dissipation will be due to the attachment. Fig. 6.7 shows the energy dissipation by the first and second mode as well as the attachment, all as functions of time, and for $c_a \approx 0.2$. More energy is dissipated by the attachment compared to that of the individual modes. This is further supported by Fig. 6.8, which shows the same information as Fig. 6.7 but for $c_a \approx 1.5$. In this case however more energy is dissipated via the modes than the attachment, which is due to only the second mode effective damping peaking near $c_a \approx 1.5$. 
Figure 6.5: Modal Response at Maximum Effective Damping of First Mode. The response amplitude for the system with the attachment is less than that of the system without the attachment.

Figs. 6.3 and 6.4 complement each other. That is Fig. 6.3 peaks for small values of $c_a$ while Fig. 6.4 peaks for larger values. The peak in the second mode’s modal effective damping decays very slowly. Thus for any reasonable given value of $c_a$, the attachment will have a positive influence on the modal effective damping for either the first mode, second mode, or both. Even in the case that the attachment damping value is not optimized, the attachment is robust enough to span a wide range
of attachment damping values and still increase the modal effective damping. Thus even when attachment degradation occurs, the primary system will still be positively influenced by the presence of the attachment.

6.2.2 Varied Attachment Stiffness

Now consider the behavior of each modes modal effective damping as a function of attachment stiffness. Table 6.2 give the primary system parameter and the fixed
attachment parameter values. The natural frequencies for the first and second mode were found to be $\omega_1 = 3.627$ and $\omega_2 = 11.698$ respectively. The initial conditions are $q_1(0) = q_2(0) = 0$ and $\dot{q}_1(0) = \dot{q}_2(0) = 1$. Here again the ratio of the effective damping of the system with the attachment to that of the system without the attachment is used to analyze the attachment’s performance.

Figs. 6.9 and 6.10 show the first and second mode global effective damping as a function of attachment stiffness. In both cases a maximum is reached and then
the curve falls as the stiffness increases. As in the case of the single-degree of freedom primary system the curve decays toward unity. As the attachment stiffness increases the more the spring acts as a rigid rod, so the modal effective damping approaches unity.

The second mode decays much slower than the first. This is an advantageous feature in designing the attachment. The second mode is much more robust than the first in terms of the range of attachment stiffnesses for which the effective damping
Table 6.2: Fixed Parameters: Varied Attachment Stiffness.
Initial Conditions $q_1(0) = q_2(0) = 0$ and $\dot{q}_1(0) = \dot{q}_2(0) = 1$.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primary Masses</td>
<td>$m_1 = m_2 = 1$ kg</td>
</tr>
<tr>
<td>Primary Stiffnesses</td>
<td>$k_1 = 30$, $k_2 = 60 \frac{N}{m}$</td>
</tr>
<tr>
<td>Primary Damping</td>
<td>$c_1 = 3$, $c_2 = 2 \frac{Ns}{m}$</td>
</tr>
<tr>
<td>Forcing Frequency</td>
<td>$\omega_f = 5$ Hz</td>
</tr>
<tr>
<td>Forcing Amplitude</td>
<td>$A = 1.5$ N</td>
</tr>
<tr>
<td>Attachment Mass</td>
<td>$m_a = 0.1$ kg</td>
</tr>
<tr>
<td>Attachment Damping</td>
<td>$c_a = 1 \frac{Ns}{m}$</td>
</tr>
</tbody>
</table>

experiences significant increases.

Figs. 6.11 and 6.12 show the modal responses for $k_a \approx 576.9$ and $k_a \approx 1641$ respectively. Fig. 6.11 shows the response for the peak of the modal effective damping in the first mode and Fig. 6.12 shows the response for the peak of the modal effective damping in the second mode. In both cases the amplitude of the response is slightly smaller than that of the case when there is no attachment.

6.2.3 Varied Mass Ratio

Lastly consider the system dynamics and modal effective damping for varied mass ratios of the attachment as compared to the mass of the primary system. The primary system’s mass is taken in a lumped approximation. That is the sum of the first and
The second mass is used as the primary system mass and the ratio of the attachment based from that. The mass of each of the two blocks in the primary system are set to be $m_1 = m_2 = 1$. The primary system parameters and fixed attachment parameters are given in Table 6.3. The initial conditions are $q_1(0) = q_2(0) = 0$ and $\dot{q}_1(0) = \dot{q}_2(0) = 1$. The natural frequencies were calculated to be $\omega_1 = 4.059$ for the first mode and $\omega = 11.555$ for the second mode.

Figs. 6.13 and 6.14 show the modal global effective damping as a function
Figure 6.10: Mode Two Global Effective Damping as a Function of Attachment Stiffness. The system parameters are given in Table 6.2. The natural frequency is $\omega_2 = 11.698$.

of mass ratio for the first and second mode respectively. The second mode modal effective damping curve peaks and decays as it has in previous cases. Past the maximum the added inertia of the attachment mass decreases the modal effective damping because it inhibits energy transfer to the attachment.

The first mode’s modal effective damping behaves quite differently than what has been seen previously. Rather than reach a maximum and decrease the curve approaches $m_a = 1.6$ asymptotically as the mass ratio is increased. Fig. 6.15 shows the
Figure 6.11: Modal Response for $k_a \approx 576.9$. The modal response at the peak of the first mode’s effective damping curve.

The modal response at the peak of the first mode’s effective damping for a wider range of mass ratio values. It demonstrates that the curve does indeed approach $m_a = 1.6$ asymptotically. Apparently once a value of around $m_a \approx 0.6$ is reached, the modal effective damping starts to level off and an increase in the mass ratio has little effect on the modal global effective damping. Physically this means that the added inertia from increasing attachment mass ratio has little effect on the first mode’s effective damping and has little to no interference with the energy transfer out of the first mode. The exact opposite
Figure 6.12: Modal Response for $k_a \approx 1641$. The modal response at the peak of the second mode’s effective damping curve.

affect increasing the mass ratio has for the second mode past it’s maximum. Fig. 6.16 shows the modal response for $m_a \approx 0.75$, which is near the roll off point where the effective damping starts to approach $m_a = 1.6$ asymptotically. Fig. 6.17 shows the modal response for $m_a \approx 0.13$, the peak of the second mode’'s effective damping curve.
Table 6.3: Fixed Parameters: Varied Attachment Mass Ratio.
Initial Conditions $q_1(0) = q_2(0) = 0$ and $\dot{q}_1(0) = \dot{q}_2(0) = 1$.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primary Masses</td>
<td>$m_1 = m_2 = 1 \text{ kg}$</td>
</tr>
<tr>
<td>Primary Stiffnesses</td>
<td>$k_1 = 40, \ k_2 = 55 \ \frac{N \text{ m}}{m}$</td>
</tr>
<tr>
<td>Primary Damping</td>
<td>$c_1 = 3, \ c_2 = 2 \ \frac{Ns}{m}$</td>
</tr>
<tr>
<td>Forcing Frequency</td>
<td>$\omega_f = 7 \ \text{Hz}$</td>
</tr>
<tr>
<td>Forcing Amplitude</td>
<td>$A = 2 \ \text{N}$</td>
</tr>
<tr>
<td>Attachment Stiffness</td>
<td>$k_a = 30 \ \frac{N}{m}$</td>
</tr>
<tr>
<td>Attachment Damping</td>
<td>$c_a = 1 \ \frac{Ns}{m}$</td>
</tr>
</tbody>
</table>

6.2.4 Summary

The analysis of the two-degree of freedom primary system with an attached nonlinear energy sink on the second mass is complete. Although similarities between the single degree of freedom primary system exist, there were some subtle differences between that primary system and the modal systems here. Clearly the more degrees of freedom that are added the more complex the analysis becomes.
Figure 6.13: First Mode Global Effective Damping as a Function of Attachment to Primary System Mass Ratio. The system parameters are given in Table 6.3. The natural frequency is $\omega_1 = 4.059$. 
Figure 6.14: Second Mode Global Effective Damping as a Function of Attachment to Primary System Mass Ratio. The system parameters are given in Table 6.3. The natural frequency is $\omega_2 = 11.555$. 
Figure 6.15: Wide Range First Mode Global Effective Damping as a Function of Attachment to Primary System Mass Ratio. The modal response near the roll off point of the first mode’s modal effective damping curve.
Figure 6.16: Modal Response for $m_a \approx 0.75$. The modal response near the roll off point of the first mode’s effective damping curve.
Figure 6.17: Modal Response for $m_a \approx 0.13$. The modal response near the peak of the second mode’s effective damping curve.
7.1 Background and Preliminary Assumptions

The previous two examples have concentrated on energy dissipation in a single-degree of freedom primary system and a two-degree of freedom primary system. Although the analysis of this type of system is important and interesting, it should be made known that the effective damping that is calculated in Femeasures.m can be used for any single mode approximation, with or without a force acting on it. It this chapter, rather than look at a primary system composed of discrete blocks, a continuous beam that undergoes free vibration with a nonlinear energy sink attached is considered. The same type of analysis preformed previously will take place here, however different approximation methods will be used.

The goal of the first chapter will be state the assumptions the analysis will be constructed under. The construction starts by assuming both ends of the beam are fixed and unable to undergo motion.

Next, assume a quadratic strain energy density of the form

\[
\psi = \frac{1}{2} \left[ - \frac{N^2}{E_e h} - \frac{Q^2}{E_s h} + E_b h^3 k^2 \right],
\] (7.1)

68
where $E_b$, $E_e$, and $E_s$ are bending, elastic, and shear moduli respectfully, $h$ is a unit length, $N(\sigma,t)$ is the tensile force, and $Q(\sigma,t)$ is the shear force. These quantities are denoted by

$$c_1 = \frac{1}{E_e h},$$
$$c_2 = \frac{1}{E_s h},$$
$$c_3 = E_b h^3.$$

The units of $\psi$ should be noted; $\psi$ has units of mass per time squared, while $c_1$ and $c_2$ have units of meters per Newton, and $c_3$ has units of Newton meters. It can be shown that the strains are related to the strain energy density by the following relations

$$e = -\frac{d\psi}{dN},$$
$$g = -\frac{d\psi}{dQ},$$
$$M = \frac{d\psi}{dk},$$

(7.2)

where $e$ is the extensional strain, $g$ is the shearing strain, and $k$ is the bending strain.

Next, assume an unshearable beam, $g = 0$, which implies the strain energy density is constant in $Q$. Consequently,

$$\psi = -\frac{1}{2}c_1 N^2 + \frac{1}{2}c_3 k^2.$$  

(7.3)

One can also show $k = \frac{d\beta}{d\sigma}$, where $\sigma$ is the spatial variable relating the origin to a material point, and where $\beta$ orients the tangent of the shell, $\hat{T}$, relative to the reference configuration, assumed to be in the $\hat{e}_x$ direction initially. This implies,
\[ T = \cos \beta \dot{e}_x + \sin \beta \dot{e}_y. \] Thus,

\[ \psi = -\frac{1}{2} c_1 n^2 + \frac{1}{2} c_3 (\beta')^2, \] (7.4)

where the prime notation indicates a spatial derivative. Later a dot will be used to indicate a time derivative.

The final assumption made that the primary system is a thin beam; that the moment of inertia, \( J \), of the beam, is negligible, \( J \approx 0 \).

7.2 Non-dimensionalization

The attention shifts to deriving and non-dimensionalizing the beam shell equations. To non-dimensionalize, \( c_x \), \( c_t \), and \( c_f \) will be used to eliminate any spatial, time, and force dimensions.

7.2.1 Strain Energy Density

Start by non-dimensionalizing Eqn. 7.4. Define

\[ N = \frac{c_f}{c_x} n, \]
\[ Q = \frac{c_f}{c_x} q, \]
\[ \beta = \beta', \] (7.5)
\[ t = c_t \tau, \]
\[ \sigma = c_x s. \]

Direct substitution gives

\[ \psi = \frac{c_3}{2 c_x^2} \left[ -\frac{c_1 c_f}{c_3} n^2 + (\beta')^2 \right]. \] (7.6)
Now define the new non-dimensional strain energy density, \( \phi \), as

\[
\phi = \frac{c_2}{c_3} \psi
\]

\[
= \frac{1}{2} \left[ -\epsilon n^2 + (\beta')^2 \right],
\]  

(7.7)

where \( \epsilon = \frac{c_1 c_f}{c_3} \), and physically represents the ratio of the bending stiffness to the longitudinal stiffness.

### 7.2.2 Beam Shell Equations

The derivation to follow is taken from Libai and Simmonds [4].

A Brief Derivation: Linear Momentum Balance

The linear momentum balance in the most general sense is given by

\[
F' + p = m_b \ddot{r},
\]

(7.8)

where \( m_b \) is the mass of the beam per unit length squared, \( F \) is the internal forces acting on the beam, \( p \) is the outside force on the beam, and \( r \) is the position vector. In order to put this in a mixed strain energy form, we must take a spatial derivative of the equation. This gives

\[
F'' + p' = m_b (\dddot{r}).
\]  

(7.9)
The terms reduce to

\[ \mathbf{F}'' = [N\dot{T} + Q\dot{B}]'' \quad (7.10) \]

\[ = \left[ N'' - N(\beta')^2 - 2Q'\beta' - Q'' \right] \dot{T} + \left[ Q'' - (\beta')^2Q + 2N'\beta' + N\beta'' \right] \dot{B}, \]

\[ p' = \left( p_T\dot{T} + p_B\dot{B} \right) ', \quad (7.11) \]

\[ = \left( [p_y\sin\beta + p_x\cos\beta] \dot{T} + [p_y\sin\beta - p_x\cos\beta] \dot{B} \right) ', \]

\[ = p_y'\sin\beta \dot{T} + p_y'\cos\beta \dot{B}, \]

\[ \ddot{\beta} = \frac{d^2}{dt^2} \left( (1 + e)\dot{T} + g \dot{B} \right) \quad (7.12) \]

\[ = \left[ \ddot{\beta} - (1 + e)\dot{\beta}^2 \right] \dot{T} + \left[ 2\dot{\beta}\dot{\beta} + (1 + e)\dot{\beta} \right] \dot{B}. \]

Using Eqns. 7.2, 7.10, 7.11, 7.12 and breaking the linear momentum balance into components two equations are obtained

\[ \dot{T} : \quad N'' + N(\beta')^2 - 2Q'\beta' - Q'' + p'\sin\beta = m_b \left[ \ddot{e} - (1 + e)\dot{\beta}^2 \right], \quad (7.13) \]

\[ \dot{B} : \quad Q'' - Q(\beta')^2 + 2N'\beta' + N'' + p'y\cos\beta = m_b \left[ 2\dot{\beta}\dot{\beta} + (1 + e)\dot{\beta} \right]. \quad (7.14) \]

A Brief Derivation: Angular Momentum Balance

The third and final equation of motion is found by using the angular momentum balance. In its most general form angular momentum balance is given by

\[ \mathbf{M}' + \mathbf{m} \cdot \mathbf{F} = J\ddot{\omega}, \quad (7.15) \]

where \( \mathbf{M} \) is related to the bending strain via Eqn. 7.2, \( \mathbf{m} \) is related to the shear, \( J \) is the moment of inertia, and \( \ddot{\omega} = \ddot{\beta} \). The second term, \( \mathbf{m} \cdot \mathbf{F} \) describes the moment
produced by the internal shearing of the beam. It is given by

\[ \mathbf{m} \cdot \mathbf{F} = \left( -g \hat{T} + (1 + e) \hat{B} \right) \cdot \left( N \hat{T} + Q \hat{B} \right) \]

\[ = -gN + (1 + e)Q. \]

Using Eqn. 7.16 and the assumptions stated, the angular momentum equation becomes

\[ \mathcal{M}' + (1 + e)Q = 0. \] (7.17)

Equations of Motion

The dimensional equations of motion are

\[ N'' + N(\beta')^2 - 2Q'\beta' - Q\beta'' + p'_y \sin \beta = m_b \left[ \ddot{e} - (1 + e)\dot{\beta}^2 \right], \]

\[ Q'' - Q(\beta')^2 + 2N'\beta' + N\beta'' + p'_y \cos \beta = m_b \left[ 2\dot{e}\dot{\beta} + (1 + e)\dot{\beta} \right], \] (7.18)

\[ \mathcal{M}' + (1 + e)Q = 0. \]

A word on notation, both dimensional and non-dimensional quantities will use primes to denote spatial derivatives and dots to denote time derivatives. The context and variables used should make obvious which each derivative refers to, be it the dimensional or non-dimensional set of variables.

The next objective is to non-dimensionalize Eqn. 7.18. Direct substitution
of Eqn. 7.5 into Eqn. 7.18 yields

\[
\frac{c_f}{c_x^3} n'' + \frac{c_f}{c_x^3} n(\beta')^2 - 2 \frac{c_f}{c_x^3} q' \beta' - \frac{c_f}{c_x^3} q \beta'' + \left( \frac{c_f}{c_x^3} \right) \left( \frac{c_x^2 p'_q}{c_f} \right) \sin \beta = \frac{m_b}{c_f^2} \left[ \dot{\varepsilon} - (1 + e) \dot{\beta}^2 \right],
\]

\[
\frac{c_f}{c_x^3} q'' - \frac{c_f}{c_x^3} q(\beta')^2 + 2 \frac{c_f}{c_x^3} n' \beta' + \frac{c_f}{c_x^3} n \beta'' + \left( \frac{c_f}{c_x^3} \right) \left( \frac{c_x^2 p'_g}{c_f} \right) \cos \beta = \frac{m_b}{c_f^2} \left[ 2 \dot{\varepsilon} \dot{\beta} + (1 + e) \ddot{\beta} \right],
\]

(7.19)

\[
\frac{c_f c_x}{c_x^2} m' + (1 + e) \frac{c_f}{c_x} q = 0.
\]

The non-dimensional force is defined as

\[
w_y = \frac{c_x^2}{c_f} p_y.
\]

(7.20)

Using Eqn. 7.20 and multiplying by \( \frac{c_x^3}{c_f} \) gives

\[
n'' + n(\beta')^2 - 2 q' \beta' - q \beta'' + w'_y \sin \beta = \frac{m_b c_x^3}{c_f^2} \left[ \dot{\varepsilon} - (1 + e) \dot{\beta}^2 \right],
\]

\[
q'' - q(\beta')^2 + 2 n' \beta' + n \beta'' + w'_y \cos \beta = \frac{m_b c_x^3}{c_f^2} \left[ 2 \dot{\varepsilon} \dot{\beta} + (1 + e) \ddot{\beta} \right],
\]

(7.21)

\[
m' + (1 + e) q = 0.
\]

Using the strain energy density a substitution for \( e \) can now be made. Eqn. 7.2 gives the relationships between the strains and the strain energy density. The definitions in Eqn. 7.2 can be extended easily to their non-dimensional counter parts given in Eqn. 7.5. Their non-dimensional counter parts are given by

\[
e = - \frac{d\phi}{dn},
\]

\[
g = - \frac{d\phi}{dq},
\]

\[
m = \frac{d\phi}{dk}.
\]

(7.22)
Thus the non-dimensional beam shell equations of motion are

\[ n'' + n(\beta')^2 - 2q'\beta' - q\beta'' + w'_y\sin\beta = \frac{mbc^3}{cf^2} \left[ \epsilon \ddot{n} - (1 + \epsilon n)\dot{\beta}^2 \right], \]

\[ q'' - q(\beta')^2 + 2n'\beta' + n\beta'' + w'_y\cos\beta = \frac{mbc^3}{cf^2} \left[ 2\epsilon \dot{n}\dot{\beta} + (1 + \epsilon n)\ddot{\beta} \right], \]  

(7.23)

\[ \beta'' + (1 + \epsilon n)q = 0. \]

Eqn. 7.23 is used to describe the dynamics of a beam fixed at both ends with an applied force \( w'_y \). The non-dimensional applied force will also allow for the incorporation of a nonlinear attachment. The attachment, although physically attached to the system, can be viewed as an outside force acting on the primary system. This allows for determination of the attachment’s equations of motion separate from the beams, while keeping Eqn. 7.23 as general as possible. Do note that any force within the assumptions of the model can be applied to the beam in the same manner as the application of the energy sink.

7.2.3 NES Equation of Motion

The same parameters used to non-dimensionalize the beam shell equations can be used to non-dimensionalize the attachment’s equation of motion. The dimensional equation of motion is

\[ m_N \ddot{D} = -f_{NES}, \]  

(7.24)

\[ = - \left( b_N \dot{Z} + k_N Z^3 \right), \]

where \( m_N \) is the attachment’s mass, \( k_N \) and \( b_N \) are the attachment’s spring and damping constants respectively, and \( Z(n, \beta, t) \) is the relative displacement of the attachment.
tached mass to the beam shell. $Z(n, \beta, t)$ is related to $D(n, \beta, t)$ by the coordinate transformation, $D(n, \beta, t) = Z(n, \beta, t) + Y(n, \beta, t)$. $D(n, \beta, t)$ serves to measure the absolute displacement of the attachment’s mass relative to the ground. As deformation occurs the attachment’s force is dependent on not only the stretch in the spring and damper, but also the vertical displacement of the beam at $\sigma = A$, where the NES is attached. $Y(n, \beta, t)$ measures the deflection of the beam. Special attention will be given to $Y(n, \beta, t)$ shortly. As an example, when the beam is at rest and in it’s undisturbed, undeformed state $Y(n, \beta, t) = 0$ and $D(n, \beta, t) = Z(n, \beta, t)$.

Using Eqn. 7.5 the attachment equation of motion is

$$\frac{m_N c_x}{c_t^2} \ddot{d} = -f_{NES}, \quad (7.25)$$

which can be written as

$$\ddot{d} = -\left( \frac{c_t^2}{m_N c_x} \right) f_{NES}$$

$$= -\left( \frac{c_x^2 c_t^2}{m_N c_x^3} \right) \left[ k_N c_x z^3 + b_N \frac{c_x}{c_t} \dot{z} \right]. \quad (7.26)$$

The non-dimensional attachment force can be defined as

$$G_N = \frac{c_x^2 c_t^2}{m_N c_x^3} \left[ k_N c_x z^3 + b_N \frac{c_x}{c_t} \dot{z} \right]. \quad (7.27)$$

Furthermore the non-dimensional stiffness and damping are defined as

$$K = \frac{k_N c_x^2 c_t^2}{m_N}, \quad (7.28)$$

and

$$B = \frac{b_N c_t}{m_N}. \quad (7.29)$$
Thus the non-dimensional equation of motion governing the NES is

$$\ddot{d} = -Kz^3 - B\dot{z}. \quad (7.30)$$

7.2.4 Attachment Force On Beam Shell

The dimensional forces acting on the beam shell are given by

$$p_y = \left[ k_N Z^3 + b_N \dot{Z} \right] \delta(\sigma - A), \quad (7.31)$$

which can be related to the non-dimensional force on the beam, $w_y$, by Eqn. 7.20. The delta function serves to localize the force on the beam shell to the specific location of attachment. Using of Eqn. 7.20 we see

$$w_y = \frac{c_x^2}{c_f} p_y$$

$$= \frac{c_x^2}{c_f} \left[ k_N Z^3 + b_N \dot{Z} \right] \delta(\sigma - A)$$

$$= \frac{c_x^2}{c_f} \left[ k_N c_x^3 z^3 + b_N \frac{c_x \dot{z}}{c_t} \right] \left( \frac{1}{c_x^2} \right) \delta(s - a)$$

$$= \frac{1}{c_f} \left[ k_N c_x^3 z^3 + b_N \frac{c_x \dot{z}}{c_t} \right] \delta(s - a),$$

which can be rewritten using Eqn. 7.27 as

$$w_y = \left( \frac{m_N c_x^3}{c_f c_t^2} \right) G_N \delta(s - a). \quad (7.32)$$

Eqn. 7.33 is the non-dimensional force acting on the beam shell at $s = a$.

7.2.5 The Physical Deflection of the Beam

The physical deflection of the beam is given by the integral equation

$$PD = \int_0^\sigma (1 + e) \cos(\beta) \, d\sigma \, \dot{e}_x + \int_0^\sigma (1 + e) \sin(\beta) \, d\sigma \, \dot{e}_y, \quad (7.34)$$
which has a very similar form when non-dimensionalized. The non-dimensional physical beam deflection is

\[
pd = \int_0^s (1 + \epsilon n) \cos(\beta) \, ds \, \hat{e}_x + \int_0^s (1 + \epsilon n) \sin(\beta) \, ds \, \hat{e}_y.
\]  
(7.35)

Vertical Deflection, \( Y \)

As promised in section 7.2.3 special attention is given to the vertical deflection, \( Y \). Calculating \( Y \) is very similar to calculating the physical deflection of the beam, however the integration need not happen over the entire length of the beam. Instead it needs only to occur up to the point where the attachment is located, \( s = a \). Moreover, only the vertical displacement needs to be determined. The vertical deflection, \( Y \), is calculated by

\[
Y = \int_0^A (1 + \epsilon) \sin(\beta) \, d\sigma,
\]  
(7.36)

or in its non-dimensional form

\[
y = \int_0^a (1 + \epsilon n) \sin(\beta) \, ds.
\]  
(7.37)

7.2.6 Boundary Conditions

The beam is fixed at both ends and undergoes no motion at those points. As a result

\[
\beta(0,t) = \beta(l,t) = 0,
\]  
(7.38)

or in their non-dimensional form

\[
\beta(0,t) = \beta(\pi,t) = 0,
\]  
(7.39)

where \( c_x \) is defined to be \( c_x = \frac{\pi}{l} \). The choice of this scaling is chosen to simplify the integration to come.
The ends of the beam are fixed, as such there are two fixed displacement conditions that need to be considered. These restrictions are additional constraints placed on the mathematical model to ensure it matches the physical system it is describing. The horizontal fixed displacement condition is

\[
\int_0^l (1 + e)\cos(\beta) \, d\sigma = H, \tag{7.40}
\]

or in it’s non-dimensional form

\[
\int_0^\pi (1 + \epsilon n)\cos(\beta) \, ds = c_x H \tag{7.41}
\]

\[= \mu \pi,\]

where \(\mu\) is the ratio of the length of the deformed beam to the original undeformed length. In the model the beam is an elastica and as such it is subject to shrinkage and elongation depending on the initial conditions and/or forces applied to it. Since the end points are fixed the length between the end points cannot change. The length of the beam though can change, so this condition serves to account for these phenomena.

The vertical fixed displacement condition is

\[
\int_0^l (1 + e)\sin(\beta) \, d\sigma = 0, \tag{7.42}
\]

or in it’s non-dimensional form

\[
\int_0^\pi (1 + \epsilon n)\sin(\beta) \, ds = 0. \tag{7.43}
\]
7.3 The Galerkin Approximation

The Galerkin approximation is a means of reducing a partial differential equation to a system of ordinary differential equations by restricting possible solutions to a smaller space than the original solution. This makes the system easier to solve, but simultaneously introduces error in the solution.

This method approximates the solution of a boundary value problem by using a linear combination of some set of basis trial functions, say $\phi_i(x)$. The choice of basis is determined by requiring that the residual be orthogonal to each homogeneous basis function and that the boundary conditions are satisfied. The residual is defined as

$$R(U, x) = D\mathcal{U} - F, \quad (7.44)$$

where $D$ is some differential operator, $F$ is the forcing function, and $U$ is yet to be solved for. The basis set is chosen so that

$$\int_{0}^{L} \phi_i(x)R(U, x) \, dx = 0, \quad i = 0, 1, 2, 3, \ldots \quad (7.45)$$

Finally, the boundary conditions are guaranteed to be satisfied by the choice of trial function.

Assume a solution set of the form

$$n = n(t),$$

$$q = q(t)\sin(2s), \quad (7.46)$$

$$\beta = \beta(t)\sin(2s),$$
substituting these into Eqn. 7.23, computing the residual, and integrating with the appropriate trail function, or mode shape in this case, gives

$$
\int_0^\pi \left[ -4n\beta^2 \cos^2(2s) - 8q\beta \cos^2(2s) + 4q\beta \sin(2s) + w_y' \sin(\beta \sin(2s)) - \epsilon \ddot{n} 
+ (1 + \epsilon n) \dot{\beta} \sin^2(2s) \right] \, ds = 0,
$$

(7.47)

$$
\int_0^\pi \sin(2s) \left[ -4q \sin(2s) - 4q\beta^2 \sin(2s) \cos^2(2s) - 4n\beta \sin(2s) + w_y' \cos(\beta \sin(2s)) - 2\epsilon \dot{n} \dot{\beta} 
- (1 + \epsilon n) \ddot{\beta} \sin(2s) \right] \, ds = 0,
$$

(7.48)

$$
\int_0^\pi \sin(2s) \left[ -4\beta \sin(2s) + (1 + \epsilon n)q \sin(2s) \right] \, ds = 0.
$$

(7.49)

Carrying out the integration gives

$$
-2n\beta^2 - 4q\beta + \int_0^\pi \frac{w_y'}{\pi} \sin(\beta \sin(2s)) \, ds = \epsilon \ddot{n} - \frac{(1 + \epsilon n)}{2} \dot{\beta}^2,
$$

(7.50)

$$
-2(q + n\beta) - \frac{q\beta^2}{2} + \int_0^\pi \frac{w_y'}{\pi} \cos(\beta \sin(2s)) \sin(2s) \, ds = \epsilon \dot{n} \dot{\beta} + \frac{(1 + \epsilon n)}{2} \ddot{\beta},
$$

(7.51)

$$
-4\beta + (1 + \epsilon n)q = 0.
$$

(7.52)

Eqns. 7.50 to 7.52 can further be simplified by assuming \( w_y' \), \( q \), and \( \beta \) are small. Mathematically this is

$$
w_y \rightarrow \epsilon w_y
$$

$$
q \rightarrow \epsilon q
$$

(7.53)

$$
\beta \rightarrow \epsilon \beta,
$$
which implies that the amplitude of the forcing and the response are limited. Keeping up to second order terms in $\epsilon$

$$-2\epsilon^2 n \beta^2 - 4\epsilon^2 q \beta + \int_0^{\pi} \frac{\epsilon w_y'}{\pi} \sin(\epsilon \beta \sin(2s)) \, ds = \epsilon \ddot{n} - \frac{\epsilon^2}{2} \beta^2,$$  \hspace{1cm} (7.54)

$$-2\epsilon (q + n\beta) + \int_0^{\pi} \frac{\epsilon w_y'}{\pi} \cos(\epsilon \beta \sin(2s)) \sin(2s) \, ds = \epsilon^2 \dot{n} \beta + \frac{(1 + \epsilon n)}{2} \epsilon \beta,$$  \hspace{1cm} (7.55)

$$-4\epsilon \beta + (1 + \epsilon n) \epsilon q = 0.$$  \hspace{1cm} (7.56)

Keep in mind that Eqns. 7.54, 7.55, and 7.56 are for any applied force within our previous assumptions. The equations do not become specific until substitution of the attachment force is applied.

This set of PDEs is only a restricted version of the general shell equations given earlier. That is to say they are valid for any unshearable system with limited amplitude and forcing. The Galerkin approximation does nothing to limit the system the equations describe, the approximation simply gives one possible form of the solution based on a modal approximation that satisfy the boundary conditions. In most cases the approximation’s modes are chosen because they fit the expected system dynamics as well as the boundary conditions, but that is not a requirement, and so guessing the mode shapes do not limit possible solutions, it simply restricts solutions to be of the form chosen.

7.3.1 Boundary and Fixed Displacement Conditions

The boundary conditions are satisfied immediately by the choice of mode shapes. A simple substitution into Eqn. 7.39 is all that is needed to show this. The fixed
displacement conditions take slightly more work due to the nested functions. The horizontal condition in Eqn. 7.41 becomes

$$\int_0^\pi (1 + \epsilon n) \cos(\epsilon \beta \sin(2s)) \, ds = \mu \pi. \quad (7.57)$$

Taylor expanding the cosine gives

$$\int_0^\pi (1 + \epsilon n) \left[ 1 - \frac{(\epsilon \beta)^2}{2} \sin^2(2s) \right] \, ds = \mu \pi. \quad (7.58)$$

Integrating and keeping only second order terms in $\epsilon$ yields

$$1 + \epsilon n - \frac{1}{4} \epsilon^2 \beta^2 = \mu. \quad (7.59)$$

The vertical condition given in Eqn. 7.43 is satisfied by the symmetry of the sine function. The symmetry guarantees zero deflection at the end points.

### 7.3.2 Vertical Deflection $Y$

Recall in section 7.2.5 the vertical deflection of a point on the beam was discussed. Using Eqn. 7.53 and Eqn. 7.37 the limited amplitude assumption is applied. This implies

$$y = \int_0^a (1 + \epsilon n) \sin(\epsilon \beta \sin(2s)) \, ds. \quad (7.60)$$

Taylor expanding the sine, integrating, and keeping second order terms in $\epsilon$, gives

$$y = (1 + \epsilon n) \epsilon \beta \sin^2(a). \quad (7.61)$$

### 7.3.3 Obtaining a Differential Equation in $\beta$

Eqns. 7.54, 7.55, and 7.56 are left to be solved under the constraint of Eqn. 7.59. Four equations of three unknowns. The four equations are not independent of each
other and as such, there is no loss of information about the system when one is discarded. The algebraic constraint in Eqn. 7.59 and Eqns. 7.54 and 7.56 are kept for simplicity.

Before further work can be completed, the integral in the forcing term of Eqn. 7.54 must be eliminated, doing so will allow the equation to be written in a much more convenient form. The forcing term is

\[ F_{att} = \int_0^\pi \frac{\epsilon w_y'}{\pi} \sin(\epsilon \beta \sin(2s)) \, ds, \]  

(7.62)

Taylor expanding the sine gives

\[ \int_0^\pi \frac{\epsilon^2 w_y'}{\pi} \beta \sin(2s) \, ds. \]  

(7.63)

Eqn. 7.63 can be integrated by parts

\[ \frac{\epsilon^2 \beta}{\pi} \int_0^\pi w_y' \sin(2s) = \frac{\epsilon^2 \beta}{\pi} \left( w_y \sin(2s) \bigg|_0^\pi - 2 \int_0^\pi w_y \cos(2s) \, ds \right) \]  

(7.64)

\[ = - \frac{2\epsilon^2 \beta}{\pi} \int_0^\pi w_y \cos(2s) \, ds. \]

Substituting in Eqn. 7.33 we get

\[ -\frac{2\epsilon^2 M_R}{\pi} \left[ K\dot{z}^3 + B\dot{z} \right] \beta \int_0^\pi \delta(s-a) \cos(2s) \, ds. \]  

(7.65)

Integration yields

\[ F_{att} = -\frac{2\epsilon^2 M_R}{\pi} \left[ K\dot{z}^3 + B\dot{z} \right] \beta \cos(2a). \]  

(7.66)
The system that remains to be solved is

\[-2\epsilon^2 n \beta^2 - 4\epsilon^2 q \beta - \frac{2\epsilon^2 M_R}{\pi} [Kz^3 + B\dot{z}] \beta \cos(2a) = \epsilon \ddot{n} - \frac{\epsilon^2}{2} \beta^2, \]  \tag{7.67}

\[-4\epsilon \beta + (1 + \epsilon n) \epsilon q = 0, \]  \tag{7.68}

\[1 + \epsilon n - \frac{\epsilon^2}{4} \beta^2 = \mu. \]  \tag{7.69}

Each equation is dependent on \( \beta \). In fact, solving Eqn. 7.67 for \( \beta \) causes \( n \) and \( q \) to fall out algebraically thereafter. Solving for \( q \) in Eqn. 7.68 and \( \epsilon n \) in Eqn. 7.69 yields

\[q = 4\beta (1 + \epsilon n)^{-1}, \]  \tag{7.70}

\[n = \frac{\mu - 1}{\epsilon} + \frac{\epsilon}{4} \beta^2. \]  \tag{7.71}

\( q \) can be written explicitly in terms of \( \beta \) by substituting Eqn. 7.71 into Eqn. 7.70 and then performing a binomial expansion of the bracketed term in Eqn. 7.70. Keeping only second order terms in \( \epsilon \) yields

\[q = \frac{4\beta}{\mu} \left( 1 - \frac{\epsilon^2 \beta^2}{4\mu} \right). \]  \tag{7.72}

Then eliminating the \( n \) and \( q \) dependence in Eqn. 7.67 and writing it solely in terms of \( \beta \) is completed by simply substituting Eqns. 7.71 and 7.72 into Eqn. 7.67. Doing so leaves an ODE in the unknown \( \beta \) of the form

\[\ddot{\beta} = -\frac{4(\mu - 1)}{\epsilon} - \frac{32}{\mu} \beta - \frac{2M_R}{\pi} [Kz^3 + B\dot{z}] \cos(2a). \]  \tag{7.73}

There is however, one last subtly. The beam is coupled to the nonlinear attachment whose response depends on that of the beams. Thus the relationship between \( z(t) \) and \( \beta(t) \) must be determined explicitly. The coordinate transformation in section
7.2.3 shows that $z(t)$ depends on the vertical deflection of the beam at the point of attachment. Eqn. 7.61 shows how the vertical deflection depends on $\beta$, combining it with the coordinate transformation of section 7.2.3 and use the resulting relation in Eqns. 7.30 and 7.73 gives

$$\ddot{\beta} = -\frac{4(\mu - 1)}{\epsilon} - \frac{32}{\mu} - \frac{2M_R}{\pi} \left[ K(d - \mu \epsilon \beta \sin^2(a))^3 + B(d - \mu \epsilon \dot{\beta} \sin^2(a)) \right] \cos(2a)$$

and

$$\ddot{d} = -K(d - \mu \epsilon \beta \sin^2(a))^3 - B(d - \mu \epsilon \dot{\beta} \sin^2(a)),$$

two coupled ODEs that when solved, give the response as a function of time for the beam and the nonlinear attachment located at $s = a$ on the beam.

7.4 MATLAB Simulation

MATLAB will be used to simulate Eqn. 7.74 and solve for $\beta(t)$ and $d(t)$ to determine the response of the beam and attachment. Once $\beta(t)$ is known, $n(t)$ and $q(t)$ remain. Eqns. 7.71 and 7.72 can be used to solved for $n(t)$ and $q(t)$ respectively. Once these three functions are known the approximate response of the beam can be determined by using Eqn. 7.46. Eqn. 7.35 is then used to determine the physical deflection of the beam as it evolves in time. The ultimate goal is to determine how effective the nonlinear energy sink is at dissipating energy relative to the beam with out the sink in place, thus the system must be solved twice, once without the attachment and once with it. This will not only allow for determination of how effective the attachment is
at energy dissipation, but also it will lend support to the model by allowing for the inspection of the systems free vibration behavior.

7.4.1 No Attachment

MATLAB’s ode45 was used to solve for the time dependance in Eqn. 7.74. An m-file was created to describe the system of equations and the solver used to obtain the time series response. The equation of motion for this case is given by

$$\ddot{\beta} = -\frac{4(\mu - 1)}{\epsilon} - \frac{32}{\mu} \beta.$$  (7.75)

Notice there is no $\dot{\beta}$ term, this indicates the damping coefficient is zero. The stiffness coefficient is $\frac{32}{\mu}$, and the beam’s mass was initially taken as unitary. Physically the response should oscillate in time with a frequency of $\sqrt{\frac{32}{\mu}}$ and an amplitude based on the initial condition applied. For the case $\epsilon = 0.1$, $\mu = 1.001$, $\beta(0) = 0.3$, and $\dot{\beta}(0) = 0$, Fig. 7.1 shows the first period of oscillation the actual beam would undergo. Fig. 7.2 shows $\beta(t)$ and the energy in the beam as a function of time. The energy and amplitude of the oscillation do decrease, but do so very slowly. Fig. 7.3 shows the energy and response amplitude over a long period of time. It’s obvious just how slow it decreases. Although there is no damping term present in the equations of motion the constant term serves to slowly pull energy out of the system. One reason for this energy loss is the internal heat of the beam undergoing deformation. Recall $\mu$ represents the relative length of the beam after deformation. Deformation takes energy to occur and thus so long as the beam is oscillating and deforming, energy will be slowly pulled from the system causing the amplitude of beam to decrease slowly.
as well. This is confirmed by simply changing the value of $\mu$, the more deformation, the quicker the energy dissipation occurs. Fig. 7.4 demonstrates this for $\mu = 1.1$. The rate of energy dissipation almost doubles. This is confirmed by simply fitting a best fit line through the oscillating energy and comparing the slopes. In the case the of $\mu = 1.001$ the slope of the best fit line is approximately $3.5 \times 10^{-4}$ \(^1\) where as in the case that $\mu = 1.1$ the slope of the best fit line is approximately $6.15 \times 10^{-4}$ \(^2\).

Effective Damping with No Attachment

The effective damping of the beam with no attachment was calculated. The values shown in Table 7.1 will serve as a benchmark for the calculation when the attachment is added and allow us to compare just how effective the attachment is at damping out the excitation.

Table 7.1: Effective Damping of Beam. Initial Conditions $\beta(0) = 0.3$, $\dot{\beta}(0) = 0$, $\epsilon = 0.1$, $\mu = 1.001$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>Effective Damping, $\lambda_{eff}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.001</td>
<td>$3.24 \times 10^{-4}$</td>
</tr>
<tr>
<td>1.1</td>
<td>$9.84 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

\(^1\)Calculation: \(\frac{\sqrt{2} - 0.75}{2000} \approx 3.5 \times 10^{-4}\)

\(^2\)Calculation: \(\frac{3 - 1.77}{2000} \approx 6.15 \times 10^{-4}\)
7.4.2 Attachment

Eqn. 7.74 describes the system with an attachment located at position $s = a$. Taking $a = \frac{\pi}{2}$ Eqn. 7.74 becomes

$$\ddot{\beta} = -\frac{4(\mu - 1)}{\epsilon} \frac{32}{\mu} \beta + \frac{2MR}{\pi} \left[ K(d - \mu \epsilon \beta)^3 + B(d - \mu \epsilon \dot{\beta}) \right]$$

and

$$\ddot{d} = -K(d - \mu \epsilon \beta)^3 - B(d - \mu \epsilon \dot{\beta}).$$
Figure 7.2: $\beta(t)$ and Beam Energy as Functions of Time. System Parameters: $\epsilon = 0.1$, $\mu = 1.001$ Initial Conditions: $\beta(0) = 0.3$ and $\dot{\beta}(0) = 0$.

Again using the MATLAB solver *ode45* an approximate solution is obtained. The response should oscillate in time, and die out more rapidly compared to that of just beam. A comparison between the two will be given shortly, first though, appropriate system response needs to be ensured. Fig. 7.5 shows $\beta(t)$ and the energy in the beam as a function of time. Both follow the expected trend and both decay much more rapidly than those in Fig. 7.1, the case of the beam with no attachment. The system parameters for the beam itself remain unchanged, those being $k_{\text{beam}} = \frac{32}{\mu}$, $\epsilon = 0.1$, $\mu = 1.001$, $\beta(0) = 0.3$, and $\dot{\beta}(0) = 0$. The attachment parameters are set at $K = 100$,
Figure 7.3: $\beta(t)$ and Beam Energy as Functions of Time. System Parameters: $\epsilon = 0.1$, $\mu = 1.001$ Initial Conditions: $\beta(0) = 0.3$ and $\dot{\beta}(0) = 0$.

$B = 10$, and $M_R = 0.2$. The increase in dissipation must be due to the attachment. The system itself is unchanged otherwise, thus an increase in effective damping is expected.

Effective Damping

The effective damping for the case shown in Fig. 7.5 was found to be $\lambda_{eff} = 3.08 \times 10^{-2}$, two orders of magnitude higher than that of the system without the attachment.
Figure 7.4: $\beta(t)$ and Beam Energy as Functions of Time. System Parameters: $\epsilon = 0.1$, $\mu = 1.1$ Initial Conditions: $\beta(0) = 0.3$ and $\dot{\beta}(0) = 0$.

Just as in the previous two examples the mass ratio, stiffness, and damping will be varied to see how the effective damping depends on each. Fig. 7.6 shows the effective damping as a function of varied mass ratio of the primary system to the attachment while holding the attachment stiffness and damping constant at $K = 100$ and $B = 10$. As the mass ratio increases so too does the effective damping. Comparing the effective damping of the beam with no attachment, $\lambda_g = 3.24 \times 10^{-4}$, to that for varied mass ratio, the global effective damping as a function of mass ratio is more than 100 times higher for most values of the mass ratio.
Figure 7.5: $\beta(t)$ and Beam Energy as Functions of Time. Attachment in Place.
System Parameters: $\epsilon = 0.1$, $\mu = 1.001$ Initial Conditions: $\beta(0) = 0.3$ and $\dot{\beta}(0) = 0$,
Attachment Parameters: $K = 100$, $B = 10$, $M_R = .2$.

The curve in Fig. 7.6 differs from the previous two examples general shape for varied mass ratio. It almost seems to be linear for small values of mass ratio. The difference in behavior could potentially be due to the assumption that the beam’s moment of inertia is negligible. Even to this end, a reasonable physical explanation exists. The kinetic energy in any system in general is half the mass times the square of the velocity, so if one wanted to increase the kinetic energy, the velocity needs to increase, or the mass does. So as the mass ratio increases the more massive the
attachment and thus the more kinetic energy in the attachment. The attachment’s only source of energy is the beam and the initial velocity is the same for each different mass ratio so as the mass ratio increases the more energy the attachment pulls from the beam earlier in time, thus the more effective the attachment is at damping the primary system. A smaller mass ratio will pull energy from the beam and act as an additive damping force, however it takes much longer for the energy dissipation to occur.

![Effective Damping as a Function of Attachment to Beam Mass Ratio](image)

Figure 7.6: Effective Damping as a function of Attachment to Beam Mass Ratio. System Parameters: $\epsilon = 0.1$, $\mu = 1.001$ Initial Conditions: $\beta(0) = 0.3$ and $\dot{\beta}(0) = 0$, Attachment Parameters: $K = 100$ and $B = 10$.

Figs. 7.7 and 7.8 show $\beta(t)$, energy, attachment relative response, and at-
attachment absolute response for $M_R = 0.08$ and $M_R = 0.64$ respectively. Notice in Fig. 7.8, $\beta(t)$ dies off much more quickly than in Fig. 7.7. Table 7.2 shows the effective damping of the two cases.

![Figure 7.7: $\beta(t)$, Energy, Attachment Relative Response, and Attachment Absolute Response vs Time, $M_R = 0.08$. System Parameters: $\epsilon = 0.1$, $\mu = 1.001$ Initial Conditions: $\beta(0) = 0.3$ and $\dot{\beta}(0) = 0$, Attachment Parameters: $K = 100$ and $B = 10$.](image-url)
Figure 7.8: $\beta(t)$, Energy, Attachment Relative Response, and Attachment Absolute Response vs Time, $M_R = 0.64$. System Parameters: $\epsilon = 0.1$, $\mu = 1.001$ Initial Conditions: $\beta(0) = 0.3$ and $\dot{\beta}(0) = 0$, Attachment Parameters: $K = 100$ and $B = 10$. 
Table 7.2: Effective Damping for Varied Mass Ratio.
Initial Conditions $\beta(0) = 0.3, \dot{\beta}(0) = 0,$ $\epsilon = 0.1, \mu = 1.001$

<table>
<thead>
<tr>
<th>$M_R$</th>
<th>Effective Damping, $\lambda_{eff}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.08</td>
<td>$1.29 \times 10^{-2}$</td>
</tr>
<tr>
<td>0.64</td>
<td>$9.56 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

Fig. 7.9 shows the effective damping as a function of the attachment stiffness while holding the damping and mass ratio constant at $B = 10$ and $M_R = 0.2$. The stiffness rises to a maximum and then falls as it approaches infinity, just as in the case of the previous two examples. Physically this indicates an optimum stiffness for the system. It follows a predictable trend; at $K = 0$ there is no attachment stiffness which means no displacement across the spring and thus no energy transfer to the attachment via the spring. There is however still a viscous damping element present in the attachment, so there is some energy transfer, thus the nonzero effective damping for $K = 0$. 
Figure 7.9: Effective Damping as a function of Attachment Stiffness. System Parameters: $\epsilon = 0.1$, $\mu = 1.001$ Initial Conditions: $\beta(0) = 0.3$ and $\dot{\beta}(0) = 0$, Attachment Parameters: $M_R = .2$ and $B = 10$.

Once the maximum is reached, the spring stiffness continues to increase and the effective damping decreases. Just as before this physically indicates less energy transfer to the attachment, due to less stretch across the spring. Physically the spring starts to act more and more like a rigid rod. Theoretically a rigid rod connecting the attachment mass and beam together would have an infinite stiffness. Numerically however, this is observed as a decay towards the theoretical limit, which accounts for
the seemingly exponential decay toward zero that occurs after the optimum stiffness is reached.

Just as in the case of varying attachment mass ratio the global effective damping as a function of varying attachment stiffness is 100 times larger than the global effective damping in the case of just the beam itself. Eventually as curve continues to decay it will reach the same small values as the beam’s global effective damping, however that is in the limit of the spring being a rigid rod, a design that wouldn’t be used in any practical application.

Figs. 7.10 and 7.11 show $\beta(t)$, energy, attachment relative response, and attachment absolute response for $K = 3.467 \times 10^5$ and $K = 1.327 \times 10^6$ respectively. Notice in Fig. 7.10, $\beta(t)$ dies off much more quickly than in Fig. 7.11. Table 7.3 shows the effective damping of the two cases.
Figure 7.10: $\beta(t)$, Energy, Attachment Relative Response, and Attachment Absolute Response vs Time, $K = 346734$. System Parameters: $\epsilon = 0.1, \mu = 1.001$ Initial Conditions: $\beta(0) = 0.3$ and $\dot{\beta}(0) = 0$, Attachment Parameters: $M_R = 0.2$ and $B = 10$. 
Figure 7.11: $\beta(t)$, Energy, Attachment Relative Response, and Attachment Absolute Response vs Time, $K = 1326633$. System Parameters: $\epsilon = 0.1$, $\mu = 1.001$ Initial Conditions: $\beta(0) = 0.3$ and $\dot{\beta}(0) = 0$, Attachment Parameters: $M_R = 0.2$ and $B = 10$. 
Table 7.3: Effective Damping for Varied Stiffness.
Initial Conditions $\beta(0) = 0.3$, $\dot{\beta}(0) = 0$,
$\epsilon = 0.1$, $\mu = 1.001$

<table>
<thead>
<tr>
<th>$K$</th>
<th>Effective Damping, $\lambda_{eff}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3.467 \times 10^5$</td>
<td>0.0376</td>
</tr>
<tr>
<td>$1.327 \times 10^6$</td>
<td>0.0196</td>
</tr>
</tbody>
</table>

Lastly, varied attachment damping is considered. Fig. 7.12 shows the effective damping as a function of the attachment damping. The mass ratio and the stiffness are held fixed at $M_R = 0.2$ and $K = 100$ respectfully. Just as in the previous two examples, when the damping is varied the effective damping rises to a maximum and then falls off once the optimum value is reached. Keep in mind, Fig. 7.12 shows the global effective damping value, and not the ratio of it to the case of just the beam itself. Once again the global effective damping is much higher in the case when the attachment is in place.
Figure 7.12: Effective Damping as a function of Attachment Damping. System Parameters: $\epsilon = 0.1, \mu = 1.001$ Initial Conditions: $\beta(0) = 0.3$ and $\dot{\beta}(0) = 0$, Attachment Parameters: $M_R = .2$ and $K = 100$.

Similar to the previous examples, as the damping increases past the optimum value the effective damping decreases. Physically this corresponds to the viscosity of the damper’s fluid increasing. As the viscosity increases it approaches a solid. At which point there would be no transfer of energy via the damping mechanism because there would be relative motion between the attachment and the beam, which implies no energy dissipation from the damper connecting the two. The nonzero effective
damping value is attributed to the spring portion of the attachment, only when
the attachment damping goes to infinity does there become no effective damping.
Figs. 7.13 and 7.14 show $\beta(t)$, energy, attachment relative response, and attachment
absolute response for $B = 5.705$ and $B = 26.510$ respectively. Fig. 7.13 shows that
$\beta(t)$ dies off much more quickly as compared to Fig. 7.14. Table 7.4 shows the
effective damping of the two cases.
Figure 7.13: $\beta(t)$, Energy, Attachment Relative Response, and Attachment Absolute Response vs Time, $B = 5.705$. System Parameters: $\epsilon = 0.1$, $\mu = 1.001$ Initial Conditions: $\beta(0) = 0.3$ and $\dot{\beta}(0) = 0$, Attachment Parameters: $M_R = 0.2$ and $K = 100$. 
Figure 7.14: $\beta(t)$, Energy, Attachment Relative Response, and Attachment Absolute Response vs Time, $K = 26.510$. System Parameters: $\epsilon = 0.1$, $\mu = 1.001$ Initial Conditions: $\beta(0) = 0.3$ and $\dot{\beta}(0) = 0$, Attachment Parameters: $M_R = 0.2$ and $K = 100$. 
Table 7.4: Effective Damping for Varied Damping. 
Initial Conditions \( \beta(0) = 0.3, \dot{\beta}(0) = 0, \)
\( \epsilon = 0.1, \mu = 1.001 \)

<table>
<thead>
<tr>
<th>( B )</th>
<th>Effective Damping, ( \lambda_{\text{eff}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.705</td>
<td>0.0361</td>
</tr>
<tr>
<td>26.510</td>
<td>0.0146</td>
</tr>
</tbody>
</table>

7.4.3 Summary

Comparing Table 7.1 to Tables 7.2, 7.3, and 7.4 show how dramatic the difference the dissipation behavior is when the attachment is added. The orders of magnitude difference between the two scenarios (with and without the attachment) provides overwhelming support of how well the attachment does at dissipating energy. Optimizing the attachment for the beam may even increase its dissipation properties, as all that has been demonstrated thus far is how useful an attachment can be at increasing the energy dissipation. Table 7.5 summarizes the different cases.
Table 7.5: Summary of Effective Damping for System with Attachment to System without Attachment.
Initial Conditions $\beta(0) = 0.3$, $\dot{\beta}(0) = 0$, $\epsilon = 0.1$, $\mu = 1.001$

<table>
<thead>
<tr>
<th>Case</th>
<th>Overall Effective Damping Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Attachment, $\mu = 1.001$</td>
<td>$3.24 \times 10^{-4}$</td>
</tr>
<tr>
<td>No Attachment, $\mu = 1.1$</td>
<td>$9.84 \times 10^{-5}$</td>
</tr>
<tr>
<td>$M_R = 0.08$</td>
<td>$1.29 \times 10^{-2}$</td>
</tr>
<tr>
<td>$M_R = 0.64$</td>
<td>$9.56 \times 10^{-2}$</td>
</tr>
<tr>
<td>$K = 3.467 \times 10^5$</td>
<td>0.0376</td>
</tr>
<tr>
<td>$K = 1.327 \times 10^6$</td>
<td>0.0196</td>
</tr>
<tr>
<td>$B = 5.705$</td>
<td>0.0361</td>
</tr>
<tr>
<td>$B = 26.510$</td>
<td>0.0146</td>
</tr>
</tbody>
</table>
CHAPTER VIII

CONCLUSIONS

8.1 Summary

This work develops the idea of effective damping and illustrates it on several examples. The examples demonstrate how useful an energy sink can be in helping dissipate unwanted energy in a primary system due to outside forcing. A harmonic force was used in the analysis, in reality though, any force can be accounted for. In many cases a good approximation of a forcing function can be achieved by conducting a Fourier Series expansion and approximating the force by a series of sines. Other times the forces are simply left alone and used in place of the harmonic term. Either way though, the analysis shows the attachment of a nonlinear energy sink will help decrease energy in the primary system more efficiently.

Energy sinks can be used to dissipate energy from an entire structure, as was shown in the case of a single-degree of freedom and a two-degree of freedom primary system, or from components within a structure, the nonlinear beam shell example. In both cases the effect of having the attachments in place was to decrease the energy in the primary system more rapidly and increase the primary systems effective damping. In many cases this also meant a decrease in the response amplitude. Both are positive
consequences of the sinks.

Different systems require different attachment parameters, varying these pa-
rameters can have a huge impact on the attachments effectiveness. Each primary
system had a different attachment parameter set that worked best for it. None of the
systems were optimized. If an optimization were to be performed, one can easily see
that a further increase in effective damping may occur.

8.2 Future work

There is still much that can be done in the way of determining the best attachment
set up for a primary system. In the future an optimization study for each of the
systems within can be performed to ensure the attachment is the best it can possibly
be for the primary structure it is placed on.

In the case of the discrete system examples, it would also be beneficial to
extend the analysis to a general $N$-degree of freedom primary system, rather than
limiting the analysis to only two-degrees of freedom. Obviously, numerical simulations
will need a set degree of freedom to work from, but the basis of the numerics comes
from the analytic side, and once a general code was developed, any degree of freedom
system could be analyzed.

It would also be interesting to move the applied force to different places on
each primary system and see what changes result under the same analysis. Similarly,
applying different forces and multiple forces would be on interest. Further work could
be done investigating how varying the attachment location affects the results and if
attaching more than one energy sink has a positive impact on the effective damping as well.

Finally, in the case of the beam shell, further investigation into the effective damping as a function of the mass ratio needs to be completed. The behavior might be due to the assumption that the moment of inertia for the beam is negligible, investigation of this hypothesis could prove useful in actual application of the attachment. Furthermore, it would be interesting to consider a forced primary system as well as different boundary conditions on the beam.


function [t, qdot2, E, cumave_lambda_eff, avelambda_eff, global_lambda_eff] = Femeasures(t, q, qdot, F, k, m, pltcntrl)

Wmodal = sqrt(k/m);

% pltcntrl=1 plots are on, pltcntrl=0 plots are off

t_all = t;
q_all = q;
qdot_all = qdot;

%% Forth Order Butterworth Filter
fco = 0.15; % Cut-off freq, Hz
fs = 1.0 / (t(2) - t(1)); % Sample rate, Hz
fn = fs / 2.0; % Nyquist freq, Hz
Wn = fco / fn; % Normalized cut-off freq

[bh, ah] = butter(4, Wn, 'high'); % high pass filter
[bl, al] = butter(4, Wn, 'low'); % low pass filter (actually used within)

%% Needed Filter Values Calculated

%%% Time
tau = [-flipud(t_all(2:end));t_all];
tfiltered = filtfilt(bl,al,tau);
clear t
t=tfiltered(length(t_all) end);

%%%Position
%q_flip = [flipud(q_all(2:end));q_all];
%qfiltered = filtfilt(bl,al,q_flip);
%clear q;
%q = qfiltered(length(q_all) end);

%%%qdot2
qdot2_all=qdot.^2;
qdot2_flip = [flipud(qdot2_all(2:end));qdot2_all];
qdot2filtered = filtfilt(bl,al,qdot2_flip);
clear qdot2
qdot2=qdot2filtered(length(qdot2_all:end));

%%%Fqdot
Fqdot_all=F.*qdot;
Fqdot_flip = [flipud(Fqdot_all(2:end));Fqdot_all];
Fqdotfiltered = filtfilt(bl,al,Fqdot_flip);
clear Fqdot
Fqdot = Fqdotfiltered(length(Fqdot_all:end));

%%%Edot Calculation
\( E_{\text{all}} = m \times (0.5 \times (W_{\text{modal}}^2 \times q_{\text{all}}^2 + 0.5 \times (q_{\text{dot2}}_{\text{all}})) \);

\( E_{\text{filtered}} = \text{filtfilt}(bl,al,E_{\text{all}}); \)

\text{clear E}

\( E = E_{\text{filtered}}; \)

\text{\%Power output}

\( E_{\text{dot}} = \text{gradient}(E,t); \)

\( E_{\text{dot\_all}} = \text{gradient}(E_{\text{all}},t_{\text{all}}); \)

\text{\%For damping calculation}

\( \text{top\_all} = F_{\text{qdot\_all}} - E_{\text{dot\_all}}; \)

\( \text{top} = F_{\text{qdot}} - E_{\text{dot}}; \)

\text{\%DAMPING CALCULATED}

\( \text{check2} = \text{length}(\text{top}) - \text{length}(q_{\text{dot2}}); \)

\text{if check2 == 0}

\text{\%instantaneous damping}

\( \lambda_{\text{eff}} = \text{top} ./ q_{\text{dot2}}; \)

\text{\%cumulative average of damping}

\( \text{cumave}_\lambda_{\text{eff}} = \text{cumtrapz}(t,\text{top})./\text{cumtrapz}(t,q_{\text{dot2}}); \)

\text{\%global damping}

\( \text{global}_\lambda_{\text{eff}} = \text{cumave}_\lambda_{\text{eff}}(\text{end}); \)

\text{\%weighted average of the damping (not as useful as the others)}

\( \lambda_{\text{eff}} = \text{trapz}(t,\text{top})/\text{trapz}(t,q_{\text{dot2}}); \)

\text{else}
disp('Error: Vectors not the same length')
end

%%PLOTS
if pltcntrl == 1
    figure(1)
    subplot(2,1,1),plot(t_all,E_all,'black',t,E,'red')
    xlabel('Time')
    ylabel('Energy (red–filtered)')
    title('Energy vs Time')
    subplot(2,1,2),plot(t_all,Edot_all,'black',t,Edot,'red')
    xlabel('Time')
    ylabel('Power (red–filtered)')
    title('Power vs Time')

    figure(2)
    subplot(2,1,1),plot(t_all,q_all,'black')
    xlabel('Time')
    ylabel('q (black)')
    title('Position vs Time')
    subplot(2,1,2),plot(t_all,qdot_all,'black')
    xlabel('Time')
    ylabel('qdot (black)')
    title('Velocity vs Time')
figure(3)

subplot(3,1,1),plot(t_all,qdot2_all,'black',t,qdot2,'red')
xlabel('Time')
ylabel('qdot2 (red–filtered)')
title('qdot2 vs Time')

subplot(3,1,2),plot(t_all,Fqdot_all,'black',t,Fqdot,'red')
xlabel('Time')
ylabel('Fqdot (red–filtered)')
title('Fqdot vs Time')

subplot(3,1,3),plot(t_all,top_all,'black',t,top,'red')
xlabel('Time')
ylabel('Fqdot - Edot (red–filtered)')
title('Fqdot - Edot vs Time')

figure(4)

plot(t,lambda_eff,'black',t,cumave_lambda_eff,'green',t,global_lambda_eff,'red')
xlabel('Time')
ylabel({'lambda_{eff} (black)','Cumulative Average lambda_{eff} (green)','Global lambda_{eff} (red)'})
title('Damping as a Function of Time')

end

end
function [wdot,t] = One_dof_Prim(t,w,zeta,K,B,Mr,OmF,\gamma)
%1dof primary sys with attachment eqn of mot
wdot = zeros(length(w),1);
x1 = w(1);
x1dot = w(2);
x2 = w(3);
x2dot = w(4);
\gamma \sin(\Omega_F t);
F = \gamma \sin(\Omega_F t);
x1ddot = - x1 - 2 \zeta x1dot + Mr*(K*(x2 - x1)^3 + B*(x2dot - x1dot)) + F;
x2ddot = - K*(x2 - x1)^3 - B*(x2dot - x1dot);
wdot = [x1dot;x1ddot;x2dot;x2ddot];
end
function [P,Om2] = modalanalysis(M,K)

%MModal Analysis Function

M_inv = inv(M);

[P,Om2] = eig(M_inv*K);

end
function [wdot,t] = Two_dof_Prim(t,w,m1,k1,b1,m2,k2,b2,A,OmF,ma,ka,ba)

% 2dof Primary derivative file

wdot = zeros(length(w));

x1 = w(1);

x1dot = w(2);

x2 = w(3);

x2dot = w(4);

xa = w(5);

xadot = w(6);

f_app = A*sin(OmF*t); f_NES = (ka)*(xa-x2)^3 + (ba)*(xadot-x2dot);

x1ddot = - (k1/m1)*x1 - (b1/m1)*x1dot + (k2/m1)*(x2-x1) + (b2/m1)*(x2dot-x1dot);

x2ddot = - (k2/m2)*(x2-x1) - (b2/m2)*(x2dot-x1dot) + (1/m2)*f_app + (1/m2)*f_NES;

if ma == 0
    xaddot = - f_NES;
else
    xaddot = - (1/ma)*f_NES;
end

%% This Assumes zero is entered for the other attachment parameters, ka, 
%% and ba also. Used in the case of no attachment.
wdot = [x1dot; x1ddot; x2dot; x2ddot; xadot; xaddot];

end
function [wdot,t] = beam_deriv(t,w,eps,mu,Mr,k,b)

wdot = zeros(length(w),1);

beta = w(1);

betadot = w(2);

D = w(3);

Ddot = w(4);

y = eps*beta*mu; \text{\% Using N Eqn}

ydot = eps*mu*betadot; \text{\% Using N Eqn}

wy = ((2*Mr)/(pi))*(k*(D-y)^3 + b*(Ddot-ydot)); \text{\%Using N Eqn}

Dddot = - k*(D-y)^3 - b*(Ddot - ydot);

betaddot = -(4/eps)*(mu-1) - (32/mu)*beta + wy; \text{\%Using N Eqn}

wdot = [betadot;betaddot;Ddot;Dddot];

end