DYNAMIC RESPONSE OF FOAM-CORE COMPOSITE SANDWICH PANELS
UNDER PRESSURE PULSE LOADING

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DYNAMIC RESPONSE OF FOAM-CORE COMPOSITE SANDWICH PANELS
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Thesis

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ABSTRACT

A new analytical technique is presented for obtaining the large amplitude, damped response of a crushable foam core composite sandwich panel subjected to pressure pulse loading. A simply-supported sandwich panel of E-Glass/Vinyl Ester facing and PVC foam core is considered in particular. The equations of motion for the composite sandwich panel were developed by using the Reissner’s Variational Theorem and Hamilton’s Principle. Core plasticity was introduced when the transverse shear stresses exceeded the transverse shear yield strength in the foam. Finite element analysis was also done using ABAQUS Explicit to simulate the dynamic response of the composite sandwich panel subjected to pressure pulse loading. The analytical solution for the panel transient response was found to be in good agreement with the finite element analysis. Analytical solutions were developed for the facesheet and core failure. The sandwich panel failure was either due to facesheet failure or core shear failure. The analytical solutions developed for the panel response and failure analysis were used for the parametric study of panel with different aspect ratios and core materials (Divinycell PVC H30, H100, H200 and HCP100 foams, and Klegecell R300 foams). The sandwich panel blast resistance increased with the increasing core density for the same facesheet material. It was also found that the lower density foam core have more energy absorption capability when compared to the foam core of higher density.
ACKNOWLEDGEMENTS

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I owe thanks to the staff and faculty in the Department of Mechanical Engineering for their help during my study at The University of Akron. Lastly, but importantly I would like to express my gratitude towards my parents, brother, sister and all my friends for their faith and for always being there for me through the progress of my thesis.
<table>
<thead>
<tr>
<th>TABLE OF CONTENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>LIST OF TABLES... viii</td>
</tr>
<tr>
<td>LIST OF FIGURES... ix</td>
</tr>
<tr>
<td>CHAPTERS</td>
</tr>
<tr>
<td>I. INTRODUCTION... 1</td>
</tr>
<tr>
<td>II. LITERATURE REVIEW... 4</td>
</tr>
<tr>
<td>2.1 Composite Sandwich Panels under Blast Load... 4</td>
</tr>
<tr>
<td>2.2 Failure Modes in Composite Sandwich Panels... 9</td>
</tr>
<tr>
<td>III. PROBLEM FORMULATION... 13</td>
</tr>
<tr>
<td>IV. TRANSIENT PANEL RESPONSE... 16</td>
</tr>
<tr>
<td>4.1 Sandwich Plate theory... 16</td>
</tr>
<tr>
<td>4.2 Reissner’s Variational Theorem... 23</td>
</tr>
<tr>
<td>4.3 Force Elastic Response... 30</td>
</tr>
</tbody>
</table>
4.4 Forced Elastic-Plastic Response .......................................................... 31

4.4.1 Plastic yielding ............................................................... 34

4.4.2 Elastic unloading .............................................................. 39

4.4.3 Reverse yielding ............................................................... 41

4.4.4 Consecutive unloading/yielding .............................................. 43

V. FINITE ELEMENT ANALYSIS .......................................................... 49

5.1 Finite Element Analysis Model .................................................. 49

5.2 Comparison of FEA and Analytical Results .................................... 53

5.2.1 Purely Elastic Response ....................................................... 53

5.2.2 Elastic-Plastic Response ....................................................... 54

VI. FAILURE MODES ................................................................. 57

6.1 Facesheet Failure ................................................................. 57

6.2 Core Failure ................................................................. 61

VII. PARAMETRIC STUDY ............................................................. 63

7.1 Aspect Ratios ................................................................. 64

7.2 Core Foam Properties ............................................................. 66
### LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Material properties of 0/90 woven roving E-glass/Vinyl Ester and different foams</td>
<td>15</td>
</tr>
<tr>
<td>7.1</td>
<td>Shear strain values for yield and fracture for different foams</td>
<td>64</td>
</tr>
<tr>
<td>7.2</td>
<td>List of the maximum values of $\zeta_0, \eta_0$</td>
<td>65</td>
</tr>
<tr>
<td>E.1</td>
<td>First and second derivates for normal stress at different panel locations</td>
<td>106</td>
</tr>
<tr>
<td>E.2</td>
<td>Normal stress at different panel locations</td>
<td>107</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>3.1</td>
<td>Rectangular composite sandwich panel with crushable foam core</td>
<td>13</td>
</tr>
<tr>
<td>3.2</td>
<td>Isotropic hardening</td>
<td>14</td>
</tr>
<tr>
<td>4.1</td>
<td>Ply height numbering in sandwich panel</td>
<td>20</td>
</tr>
<tr>
<td>4.2</td>
<td>Plastic dissipation regions in the core of sandwich plate</td>
<td>33</td>
</tr>
<tr>
<td>4.3</td>
<td>Transverse shear force resultants during plastic yielding</td>
<td>33</td>
</tr>
<tr>
<td>4.4</td>
<td>Transverse shear force resultants during elastic unloading</td>
<td>40</td>
</tr>
<tr>
<td>4.5</td>
<td>New plastic regions in core during reverse yield</td>
<td>43</td>
</tr>
<tr>
<td>4.6</td>
<td>Transverse shear force resultants during reverse yielding</td>
<td>42</td>
</tr>
<tr>
<td>4.7</td>
<td>Transverse shear force resultants during consecutive yielding/unloading and yielding/reloding</td>
<td>44</td>
</tr>
<tr>
<td>5.1</td>
<td>Finite element model of composite sandwich panel</td>
<td>50</td>
</tr>
<tr>
<td>Section</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>5.2</td>
<td>Composite sandwich panel with load and boundary conditions</td>
<td>51</td>
</tr>
<tr>
<td>5.3</td>
<td>Meshed finite element model composite sandwich panel</td>
<td>52</td>
</tr>
<tr>
<td>5.4</td>
<td>Comparison of FEA and analytical panel center deflection</td>
<td>53</td>
</tr>
<tr>
<td>5.5</td>
<td>Comparison of FEA and analytical transverse shear strains</td>
<td>54</td>
</tr>
<tr>
<td>5.6</td>
<td>FEA model showing the plastic region</td>
<td>55</td>
</tr>
<tr>
<td>5.7</td>
<td>Comparison of FEA and analytical panel center deflection</td>
<td>55</td>
</tr>
<tr>
<td>5.8</td>
<td>Comparison of FEA and analytical transverse shear strains</td>
<td>56</td>
</tr>
<tr>
<td>6.1</td>
<td>Zone of maximum normal and transverse shear stresses</td>
<td>60</td>
</tr>
<tr>
<td>7.1</td>
<td>Failure pressure for H100 foam and various aspect ratios</td>
<td>65</td>
</tr>
<tr>
<td>7.2</td>
<td>Energy absorbed by H100 and various aspect ratios</td>
<td>66</td>
</tr>
<tr>
<td>7.3</td>
<td>Failure pressure for the panel with different foam cores</td>
<td>67</td>
</tr>
<tr>
<td>7.4</td>
<td>Energy absorbed by panel with various core materials</td>
<td>68</td>
</tr>
<tr>
<td>7.5</td>
<td>Failure pressure /areal weight density for panel with various foam cores</td>
<td>69</td>
</tr>
</tbody>
</table>
B.1 Yield surface (a) transverse shear stresses and (b) transverse shear resultant forces ................................. 83

C.1 Plastic regions overlap .............................................................. 85

C.2 $f_1$ vs. $\eta_0$ ............................................................................. 89

C.3 $f_3$ vs. $\eta_0$ ............................................................................. 94

C.4 Plastic shear regions dominated by $Q_{dx}$ and $Q_{dy}$ .................. 95

C.5 $Q_{dx}/Q_0$ vs. $\eta_0$ ................................................................. 97

C.6 $p_d$ vs. $\eta_0$ ........................................................................... 101
CHAPTER I
INTRODUCTION

Sandwich structures consist of two stiff and strong thin facesheets separated by a thick core of low density. This configuration gives the sandwich structure high stiffness and strength-to-weight ratio when compared to thick monolithic structure. Sandwich structures are widely used in aerospace, marine and transportation industries because of this. Crushable foam-core sandwich panels in particular, have the ability to absorb energy when they are subjected to impact and shock loading. This is due to the micro-inertial resistance and collapse of cellular walls in the foam during panel deformation under such intense loading. The cellular microstructure of the foam allows it to attenuate forces and absorb energy as it crushes at almost constant flow stress. It is therefore important to understand how the core influences the shock response of sandwich panels under external pressure pulse loading.

The purpose of this study is to develop both the analytical solution and finite element analysis (FEA) solution for composite sandwich panels subjected to a blast load. A rectangular composite sandwich panel made of E-Glass/Vinyl Ester facesheets and Divinycell foam core is considered in this study. The E-Glass/Vinyl Ester composite facesheet has the ability to withstand water absorption and is corrosion resistant,
and is widely used in naval and civil infrastructure. Divinycell PVC H100 polymeric foams have the ability to absorb energy and are also corrosion resistant. This foam is a low cost and maintenance free material when compared to other core materials, such as, metallic foam and balsa wood.

When the composite sandwich panel is subjected to the blast load characterized by a pressure pulse of short duration, a shock wave is transmitted through the thickness of the foam. The shock wave generates compression in the foam core and transmits compressive stress waves that travel back and forth through it, while the facesheets deform [1]. Whether the core behaves as an elastic material or an elastic-plastic material depends on the magnitude of the pressure pulse. If the pressure pulse amplitude is low the core behaves as all elastic, while the core behaves as elastic-plastic material when the pressure pulse amplitude is large.

Most of the previous work done on blast load response of composite sandwich panels considered the core as an elastic material [2-6]. This thesis covers the elastic and plastic response of a composite sandwich panel during blast loading. A new analytical technique is presented for obtaining the large amplitude, damped response of a composite sandwich panel subjected to pressure pulse loading. The previous solution developed by Wu and Vinson [7] on the nonlinear oscillations of plates composed of composite materials is extended and modified to get the equations of motion for the large amplitude, damped response of a composite sandwich panel subjected to pressure pulse loading. The failure modes in the sandwich panel due to facesheet and core fracture are also
considered in this thesis. The analytical solution obtained is used to perform parametric studies on the blast resistance and energy absorption of the panel with different polymeric foams (Divinycell PVC H30, H100, H200 and HCP100, and Klegecell R300 foams) and panel aspect ratios.

Finite element analysis (FEA) using ABAQUS Explicit is also used to simulate the blast load response of flat sandwich panel and the FEA results are used to corroborate the analytical solutions. Both the analytical and FEA solutions are developed for flat rectangular plates. Chapter II provides the review of previous work done on rectangular flat panel and the failure modes they encounter due to blast. The problem formulation is given in Chapter III. Chapter IV presents the elastic and elastic-plastic analytical solution for the sandwich plate. Finite element analysis is given in Chapter V. The failure modes and failure criteria for the sandwich plate are covered in Chapter VI. Parametric studies on core geometry and material properties are covered in Chapter VII. Finally, Chapter VIII provides concluding remarks of the study.
CHAPTER II
LITERATURE REVIEW

A great amount of work is being done on the dynamic response of the blast load on composite and sandwich structure. Most of these studies are on large amplitude dynamic response of composite and composite sandwich panels, but there are very few studies on the dynamic, damped response of sandwich panels. Much of the literature concerning the dynamic response of composite sandwich panels under shock does not address the plastic deformation and energy absorption that take place in the sandwich core. This chapter provides a review of the response of sandwich panels under a blast load. It also covers different failure modes criterion for sandwich panels. This chapter is divided into two parts: the first part covers the work done on composite sandwich panels under blast load and the second part covers the failure modes of sandwich panels.

2.1 Composite Sandwich Panels under Blast Load

Much research has been done on the dynamic response of sandwich panels under blast load. In most of the research studies the foam core is treated as a linear elastic material. Very few studies address the elastic-plastic behavior of the foam core material. This section provides a review of these studies.
Wu and Vinson [7] studied nonlinear dynamic oscillation of composite sandwich plate under free vibration. They used the Berger’s approach [8] and modified Reissner’s variational principle to approximate equations of motion for large amplitude vibrations of simply-supported orthotropic rectangular plates. They also took into consideration the effects of rotary inertia and transverse shear deformation. They carried out numerical calculations for the large amplitude lateral oscillations of simply-supported plates composed of a typical composite material. They concluded that for many fiber-reinforced composite materials transverse shear deformation effects must be included. The nonlinearity that occurs for the orthotropic plate is the hardening type and was very important. When the plate is longer in the stiffer direction (the direction of the filaments), the natural frequencies were significantly higher than when the reverse was true.

Xue and Hutchinson [9] compared the sustainability of sandwich plates to monolithic solid plates of same material and total mass subjected large blast loads. They focused on polymer matrix fiber-reinforced sandwich plates with tetragonal truss core. They considered the material to be elastic. Their research demonstrated superior strength and energy absorption capacity of the sandwich plates when compared with solid plates having the same mass. They also highlighted the importance of both strength and energy absorbing capacity of the core for superior blast resistance.
Gdoutos and Daniel [10] investigated the nonlinear behavior of composite sandwich beams made of unidirectional carbon/epoxy facings and PVC foam core under bending. The facing response after an initial linear response exhibited a stiffening nonlinearity in tension and a softening nonlinearity in compression with the longitudinal strength in tension being higher than in compression. They used a simple equation in conjunction with the stress-strain behavior of the facing material in tension and compression to determine the position of the neutral axis of the beam. The experimental results for the displacements of the core material compared well with the analytical predictions. They found higher compressive and smaller tensile stresses in the core and the variation of the normal stresses along the height of the beam was nonlinear and should be calculated by taking the nonlinear stress-strain behavior of the core material in tension and compression.

Li et al. [11] analyzed the nonlinear impulse response of a simply-supported composite sandwich plate exposed to a sudden point-wise transverse loading on the top facesheet. They also took into consideration the core compressibility effects. They modeled the nonlinearity arising from the core compressibility in the thickness direction and incorporated it into the constitutive relations explicitly. Hamilton’s Principle was used to formulate the equations of motion. The numerical results for the dynamic response in terms of the transverse deformation and stresses in the composite sandwich plate were presented. Parametric studies on the effects of the variations of the geometrical parameters of the structures subjected to blast load were also done. It was found that the top face, core and the bottom face behave differently in the transient response. The
transverse stress profiles in the core showed high nonlinearity with maximum amplitudes at the interface between the core and top facesheet on which the blast loading impacted. They concluded that debonding could initiate at that interface as was observed in preliminary experiments.

Hoo Fatt and Palla [12] derived analytical solutions for the transient response and damage initiation of a foam-core composite sandwich panel subjected to blast load. Sandwich panels with E-Glass/Vinyl Ester face sheet and H100 foam core were considered for the study. They modeled the panel response in two consecutive phases: (1) a through thickness wave propagation leading to permanent core crushing deformations and (2) a transverse shear wave propagation phase resulting in global panel deflections. The analytical solution for the deformation of a sandwich panel compared well with the FEA results. The damage initiations in several foam sandwich panels also correlated well with FEA results.

Balkan and Mecitoglu [13] studied the transient dynamic behavior of sandwich composite plates with viscoelastic core under blast loading. Clamped boundary conditions were considered for all edges and the virtual work principle was used to derive the equations of motion. Unlike classical plate theory a shear deformation was considered and the plate deflections were large. Hence, they considered the shear effects and the geometrical nonlinearities in the derivation of the equations of motion. The Kelvin-Voigt linear visoelastic theory was used to model the visoelastic behavior. The displacement field in the space domain was approximated by trial functions to solve the
governing equations and the equations of motion were reduced into time domain using Galerkin method. The viscoelastic behavior of the sandwich composite plate obtained analytically compared well with the FEA solution obtained by ANSYS.

Tagarielli et al. [14] predicted the dynamic response of composite sandwich beams under shock loading. They presented the dynamic shock response of fully clamped monolithic and sandwich beams, with elastic facesheets and a compressible elastic-plastic core. The results for mid-span deflections and deflected shapes obtained from finite element predictions were compared to previously reported response of end-clamped sandwich beams, made from face sheets of glass fiber reinforced vinyl ester and a core of PVC foam or balsa wood. They predicted the sustainable impulse by assuming a tensile failure criterion for the face sheets.

Carrera and Brischetto [15] described and assessed a large variety of plate theory to evaluate the bending and vibration of sandwich structures. At first they presented a brief survey of available works and latest developments on sandwich structure modeling. Closed-form solutions were provided for simply supported panels made of orthotropic layers. Displacements, stresses (both in-plane and out-of-plane components), and the free vibration response were taken into consideration. They carried out a parametric study on length-to-thickness ratio (geometrical parameters) and face-to-core-stiffness ratio (mechanical parameters). They concluded that the higher-order theories could be conveniently used to reduce the error due to the length-to-thickness ratio in thick plate
cases, but they are not effective in increasing the accuracy of the classical theory analysis whenever the error is caused by increasing face-to-core-stiffness ratio.

2.2 Failure Modes in Composite Sandwich Panels

An ample amount of research has been done on the failure modes in composite sandwich panels under different loading conditions. The failure mode map provides insight on how the panel failure depends on different variables such as loading, geometry of the panel and material properties. This section provides a review of failure modes on composite sandwich panels.

Petras and Sutcliffe [16] investigated the failure modes for sandwich beams of GFRP laminate skins and Nomex honeycomb core. They described honeycomb mechanics and classical beam theory. A failure mode map for loading under three-point bending was constructed, showing the dependence of failure mode and load on the ratio of skin thickness to span length and honeycomb relative density. The experimental data agreed satisfactorily with the analytical predictions. The concept of a failure map was extended to give a useful design tool for sandwich panel manufacturers and their customers.

McCormack et al. [17] studied the failure of sandwich beams with metallic foam cores. They stated that sandwich beam can fail by several modes: face yielding, face wrinkling, core yielding and indentation. The initial failure load, corresponding to the first deviation from linearity in the load deflection curve as well as the peak load for each
mode was estimated. Failure mode maps, which illustrated the dominant failure mode for practical beam designs, were constructed. Results from their analysis were compared to experimental results obtained on sandwich beams with aluminum cores in three-point bending. The peak loads and the failure modes were found to be in good agreement.

Lim and Lee [18] investigated the static failure modes and load capabilities of foam core composite sandwich beams composed of E-glass/Epoxy and PVC foam by both experimental and analytical methods. The static failure mode map was constructed using experimental and analytical results. The static load capabilities and failure modes predicted by the theory of beam on an elastic foundation showed good agreement with the three-point bending test results.

Steeves and Fleck [19] predicted the three-point collapse strength of sandwich beams with composite faces and polymer foam cores. They stated failure could be by competing modes of facesheet microbuckling, plastic shear of the core, and facesheet indentation beneath the loading rollers. Particular attention was paid to the development of an indentation model for elastic faces and an elastic-plastic core. Failure mechanism maps were constructed to reveal the operative collapse mode as function of geometry of sandwich beam, and minimum weight designs were obtained as a function of an appropriate structural load index.
Konsta-Gdoutos and Gdoutos [20] investigated the facing compressive failure, facing wrinkling and core shear failure for simply-supported and cantilever beam with facings made of carbon/epoxy composites and cores made of aluminum honeycomb and polymeric foam (H100 and H250). The beams were loaded under different load condition namely concentrated, uniform and triangular. It was found that in beams with foam core facing wrinkling and core shear failure occurred, whereas in beams with honeycomb core facing compressive failure and core shear crimping took place. The critical beam spans for failure mode transition from core shear to wrinkling failure were established. They concluded that initiation of a particular failure mode depends on the properties of the facing and core materials, the geometrical configuration, the type of support and loading of sandwich beams.

Andrews and Moussa [21] presented failure mode maps for sandwich panels with composite facesheets and lightweight core. To include dynamic effects in the problem the sandwich panel was modeled as a single-degree-of-freedom mass-spring system. They presented various modes of failure: facesheet failure, face sheet wrinkling and core failure (shear stress). A comparison was done with some quasi-static test results and it was found that the experimental data were consistent with the analysis.

Valenza et al. [22] analyzed the failure mechanisms of GFRP/PVC foam core sandwich structures subjected to three-point bending. They carried experimental tests by varying the skin thickness and the span length between supports in order to find the relationship between the geometrical configuration of the sandwiches and the failure
mechanism. By plotting failure mechanism on a graph of span length against skin thickness, a failure map was created identifying the three typical failure mode regions of the sandwiches. The three typical failure modes were (1) Indentation Type I (the core yields under the localized load without the skin failure), (2) Indentation Type II (the locally applied load significantly deforms the upper skin until it eventually collapses) and (3) Tensile Skin Failure. Theoretical failure mode maps were constructed, and they found to be were consistent with the experimental ones. They concluded that the theoretical model was a reliable predictor of failure mechanisms in sandwiches with defined geometry.
Consider a simply-supported, rectangular composite sandwich panel of geometry as shown in Figure 3.1. The composite sandwich consists of two orthotropic face sheets of thickness $h$ and density $\rho_f$. The core is made of crushable polymeric foam of thickness $H$ and density $\rho_c$. Notation is given in Appendix A.

![Figure 3.1 Rectangular composite sandwich panel with crushable foam core.](image)

Let the Cartesian coordinate system $(x, y, z)$ be on the middle plane of the foam core as shown in Figure 3.1 with the coordinate $x$ and $y$ in the in-plane directions and $z$
along the through-thickness direction. The rectangular sandwich panel is subjected to uniformly-distributed pressure pulse $p$ as described by

$$p(t) = \begin{cases} p_0 \left(1 - \frac{t}{\Delta T}\right), & t \leq \Delta T \\ 0, & t > \Delta T \end{cases}$$

(3.1)

where $p_0$ is the peak pressure, $\Delta T$ is the load duration and $t$ is time.

The facesheets are assumed to be elastic, whereas the core is assumed to be elastic-perfectly plastic. Isotropic hardening plastic rule is used for the crushable foam and the analysis is limited to low strain rate (less than 20%). At this low strain rate the density of the foam does not change substantially. The stress-strain response for the crushable foam is elastic-perfectly plastic as shown in Figure 3.2.

![Isotropic hardening](image)

Figure 3.2 Isotropic hardening.

Material properties for the woven roving E-Glass/Vinyl Ester and various foams are given in Table 3.1. For the E-Glass/Vinyl Ester, material properties were taken from Boh et al. [23]. The materials properties for the Divinycell H30, H100, H200, HCP100 and Klegecell R300 foams were taken from References [23-28].
Table 3.1 Material properties of 0/90 woven roving E-glass/Vinyl Ester and different foams.

<table>
<thead>
<tr>
<th></th>
<th>E-Glass/ Vinyl Ester</th>
<th>Divinycell H30</th>
<th>Divinycell H100</th>
<th>Divinycell H200</th>
<th>Klegecell R300</th>
<th>Divinycell HCP100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density(kg/m³)</td>
<td>1391.3</td>
<td>36</td>
<td>100</td>
<td>200</td>
<td>300</td>
<td>400</td>
</tr>
<tr>
<td>(E_{11}) (+) (GPa)</td>
<td>17</td>
<td>0.044</td>
<td>0.149</td>
<td>0.277</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>(E_{22}) (+) (GPa)</td>
<td>17</td>
<td>0.044</td>
<td>0.149</td>
<td>0.277</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>(E_{33}) (+) (GPa)</td>
<td>7.48</td>
<td>0.044</td>
<td>0.149</td>
<td>0.277</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>(E_{11}) (-) (GPa)</td>
<td>19</td>
<td>0.027</td>
<td>0.105</td>
<td>0.293</td>
<td>0.338</td>
<td>0.340</td>
</tr>
<tr>
<td>(E_{22}) (-) (GPa)</td>
<td>19</td>
<td>0.027</td>
<td>0.105</td>
<td>0.293</td>
<td>0.338</td>
<td>0.340</td>
</tr>
<tr>
<td>(E_{33}) (-) (GPa)</td>
<td>--</td>
<td>0.027</td>
<td>0.105</td>
<td>0.293</td>
<td>0.338</td>
<td>0.340</td>
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<tr>
<td>(\nu_{12} \neq \nu_{21})</td>
<td>0.13</td>
<td>0.25</td>
<td>0.31</td>
<td>0.3</td>
<td>0.23</td>
<td>0.3</td>
</tr>
<tr>
<td>(\nu_{13} \neq \nu_{23})</td>
<td>0.28</td>
<td>0.25</td>
<td>0.31</td>
<td>0.3</td>
<td>0.23</td>
<td>0.3</td>
</tr>
<tr>
<td>(\nu_{31} \neq \nu_{32})</td>
<td>0.12</td>
<td>0.25</td>
<td>0.31</td>
<td>0.3</td>
<td>0.23</td>
<td>0.3</td>
</tr>
<tr>
<td>(G_{12} = G_{21}) (GPa)</td>
<td>4.0</td>
<td>0.013</td>
<td>0.0438</td>
<td>0.110</td>
<td>0.123</td>
<td>0.131</td>
</tr>
<tr>
<td>(G_{23} = G_{32}) (GPa)</td>
<td>1.73</td>
<td>0.013</td>
<td>0.0438</td>
<td>0.110</td>
<td>0.123</td>
<td>0.131</td>
</tr>
<tr>
<td>(G_{13} = G_{31}) (GPa)</td>
<td>1.73</td>
<td>0.013</td>
<td>0.0438</td>
<td>0.110</td>
<td>0.123</td>
<td>0.131</td>
</tr>
<tr>
<td>q (MPa)</td>
<td>--</td>
<td>0.3</td>
<td>1.66</td>
<td>4.35</td>
<td>7.8</td>
<td>10.3</td>
</tr>
<tr>
<td>(\varepsilon_D)</td>
<td>--</td>
<td>0.85</td>
<td>0.8</td>
<td>0.7</td>
<td>0.285</td>
<td>0.15</td>
</tr>
<tr>
<td>(\sigma_{1f}) (+) (MPa)</td>
<td>270</td>
<td>0.57</td>
<td>3.2</td>
<td>6.4</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>(\sigma_{1f}) (-) (MPa)</td>
<td>200</td>
<td>0.29</td>
<td>1.53</td>
<td>4.36</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>(\sigma_{2f}) (+) (MPa)</td>
<td>270</td>
<td>0.57</td>
<td>3.5</td>
<td>6.4</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>(\sigma_{2f}) (-) (MPa)</td>
<td>200</td>
<td>0.29</td>
<td>1.53</td>
<td>4.36</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>(\tau_{12f} = \tau_{21f}) (MPa)</td>
<td>40</td>
<td>0.35</td>
<td>1.47</td>
<td>3.86</td>
<td>--</td>
<td>7.44</td>
</tr>
<tr>
<td>(\tau_{13f} = \tau_{21f}) (MPa)</td>
<td>31.6</td>
<td>0.35</td>
<td>1.47</td>
<td>3.86</td>
<td>--</td>
<td>7.44</td>
</tr>
<tr>
<td>(\tau_{32f} = \tau_{31f}) (MPa)</td>
<td>31.6</td>
<td>0.35</td>
<td>1.47</td>
<td>3.86</td>
<td>--</td>
<td>7.44</td>
</tr>
</tbody>
</table>
4.1 Sandwich Plate Theory

Consider the forced vibration of a rectangular composite sandwich plate composed of orthotropic facesheets and a crushable foam core. The plate has geometry as shown in Figure 3.1. The plate equations developed by Wu and Vinson [7] are extended in this analysis as they include the effects of large amplitude transverse deflection and transverse shear deformations. While these effects are important in sandwich panels, other aspects such as plasticity during foam crushing are rarely addressed in traditional sandwich panel analysis. The elastic-plastic response of the foam core is incorporated in the sandwich panel equations of motion in this chapter.

The assumed displacement field is given by

\[ u = u_0(x, y, t) + z\bar{\alpha}(x, y, t) \]  
\[ v = v_0(x, y, t) + z\bar{\beta}(x, y, t) \]  
\[ w = w(x, y, t) \] (4.1) (4.2) (4.3)
where $u_0$ and $v_0$ are the in-plane deformation, $w$ is the transverse deflection at the plate mid-plane, and $\alpha$ and $\beta$ and are the shear rotations associated with the $x$- and $y$-directions. In the above equations the in-plane deformations take the usual form of being the sum of a translation and the rotation of a linear element through the thickness.

The tensor expression of components of strain in the Lagrangian form for a three dimensional body is the following [29]:

$$2\varepsilon_{jk} = u_{j,k} + u_{k,j} + u_{i,j}u_{i,k} \quad (4.4)$$

where commas denote partial differentiation with respect to the following subscripted symbol $i, j, k = x, y, z$.

Substituting Equations (4.1) to (4.3) into (4.4) gives the explicit form of the strain-displacement relations as follows:

$$\varepsilon_x = \frac{\partial u_0}{\partial x} + z \frac{\partial \alpha}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2$$

$$\varepsilon_y = \frac{\partial v_0}{\partial y} + z \frac{\partial \beta}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2$$

$$\varepsilon_{xy} = \frac{1}{2} \left( \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + z \frac{\partial \alpha}{\partial y} + z \frac{\partial \beta}{\partial x} \right) + \frac{1}{2} \left( \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right)$$

$$\varepsilon_{xz} = \frac{1}{2} \left( \frac{\partial w}{\partial x} + \alpha \right); \quad \varepsilon_{yz} = \frac{1}{2} \left( \frac{\partial w}{\partial y} + \beta \right); \quad \varepsilon_z = 0$$
For a single ply orthotropic layer there are nine independent elastic constants for possessing three planes of elastic symmetry at a point. The strain-stress relations are given by

\[
\begin{align*}
\varepsilon_x &= \frac{1}{E_x} \sigma_x - \frac{v_{yx}}{E_y} \sigma_y - \frac{v_{zx}}{E_z} \sigma_z \\
\varepsilon_y &= -\frac{v_{xy}}{E_x} \sigma_x + \frac{1}{E_y} \sigma_y - \frac{v_{zy}}{E_z} \sigma_z \\
\varepsilon_z &= -\frac{v_{xz}}{E_x} \sigma_x - \frac{v_{yz}}{E_y} \sigma_y + \frac{1}{E_z} \sigma_z \\
\varepsilon_{xy} &= \frac{1}{2G_{xy}} \sigma_{xy}; \varepsilon_{xz} = \frac{1}{2G_{xz}} \sigma_{xz}; \varepsilon_{yz} = \frac{1}{2G_{yz}} \sigma_{yz}
\end{align*}
\]

The relation for matrix of elastic coefficients due to the symmetry is given by

\[
E_{ij} = E_{ji} (i, j = x, y, z) \quad (4.7)
\]

which reduce the number of independent elastic constants to six.

The transverse normal stress is negligible compared to the in-plane normal stress and shear stresses. Hence, \( \sigma_z = 0 \). Also, \( \varepsilon_z = 0 \) because the lateral deformation is independent of \( z \). Equation (4.6) then reduces to
\[ \varepsilon_x = \frac{1}{E_x} \sigma_x - \frac{v_{xy}}{E_y} \sigma_y \]
\[ \varepsilon_y = -\frac{v_{xy}}{E_x} \sigma_x + \frac{1}{E_y} \sigma_y \]  
(4.8)
\[ \varepsilon_z = 0 \]
\[ \varepsilon_{xy} = \frac{1}{2G_{xy}} \sigma_{xy} \varepsilon_{xz} = \frac{1}{2G_{xz}} \sigma_{xz} \varepsilon_{yz} = \frac{1}{2G_{yz}} \sigma_{yz} \]

The stress-strain relation for an orthotropic ply can also be written as

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{12} & Q_{22} & 0 \\
0 & 0 & Q_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
\]
(4.9)
\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_{xz}
\end{bmatrix} =
\begin{bmatrix}
Q_{44} & 0 \\
0 & Q_{55} \\
0 & 0 & Q_{66}
\end{bmatrix}
\begin{bmatrix}
\gamma_{xz} \\
\gamma_{yz}
\end{bmatrix}
\]

where \( Q_{11} = E_{11}/(1-v_{12}v_{21}) \), \( Q_{22} = E_{22}/(1-v_{12}v_{21}) \), \( Q_{12} = v_{12}E_{22}/(1-v_{12}v_{21}) \), \( Q_{44} = G_{23} \), \( Q_{55} = G_{13} \) and \( Q_{66} = G_{12} \).

In accordance to the classical sandwich plate theory, additional expressions are used to approximate the stress at a point. The in-plane stress resultants (\( N_{ij} \)), transverse shear resultants (\( Q_x, Q_y \)) and the bending moment resultants (\( M_{ij} \)) for the laminae are given by
where $k$ is the ply number, $n$ is the total number of layer and $h_k$ is defined in the Figure 4.1 for a three-layer system.

![Figure 4.1 Ply height numbering in sandwich panel.](image)

Substituting Equations (4.5) in Equations (4.9) and integrating through the thickness of the plate gives
where $\varepsilon_{x_0}, \varepsilon_{y_0}$ are normal strains of the mid-plane in the $x$- and $y$-direction, respectively,

$\varepsilon_{xy_0}$ is the mid-plane shear strain, and $\overline{Q}_{44}, \overline{Q}_{45}, \overline{Q}_{55}$ are the transformed transverse shear stiffness. The mid-plane strains can be written as

$$
\varepsilon_{x_0} = \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2
$$

$$
\varepsilon_{y_0} = \frac{\partial v_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2
$$
\[
\varepsilon_{x_0y_0} = \frac{1}{2} \left( \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right) + \frac{1}{2} \left( \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right)
\]

The in-plane stress resultants \(N_{ij}\), transverse shear resultants \((Q_x, Q_y)\) and the bending moment resultants \((M_{ij})\) can also be written as

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\
A_{12} & A_{22} & A_{26} \\
A_{16} & A_{26} & A_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{x_0} \\
\varepsilon_{y_0} \\
\varepsilon_{x_0y_0}
\end{bmatrix}
+ \begin{bmatrix} B_{11} & B_{12} & B_{16} \\
B_{12} & B_{22} & B_{26} \\
B_{16} & B_{26} & B_{66}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \alpha}{\partial x} \\
\frac{\partial \beta}{\partial y} \\
\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x}
\end{bmatrix}
\]

\(4.16\)

\[
Q_x = 2(A_{55} \varepsilon_{xz} + A_{45} \varepsilon_{yz})
\]

\(4.17\)

\[
Q_y = 2(A_{45} \varepsilon_{xz} + A_{44} \varepsilon_{yz})
\]

\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\
B_{12} & B_{22} & B_{26} \\
B_{16} & B_{26} & B_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{x_0} \\
\varepsilon_{y_0} \\
\varepsilon_{x_0y_0}
\end{bmatrix}
+ \begin{bmatrix} D_{11} & D_{12} & D_{16} \\
D_{12} & D_{22} & D_{26} \\
D_{16} & D_{26} & D_{66}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \alpha}{\partial x} \\
\frac{\partial \beta}{\partial y} \\
\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x}
\end{bmatrix}
\]

\(4.18\)

where \(A_{ij}\) is membrane stiffness matrix, \(D_{ij}\) is the bending stiffness matrix, \(B_{ij}\) is the bending-membrane coupling matrix, and \(A_{44}, A_{45}\) and \(A_{55}\) are transverse shear stiffness. They are given by
\[ B_{ij} = \frac{1}{2} \sum_{k=1}^{n} (\overline{Q}_{ij})_k (h_k^2 - h_{k-1}^2) \quad i, j = 1, 2, 6 \]

\[ D_{ij} = \sum_{k=1}^{n} (\overline{Q}_{ij})_k (h_k^3 - h_{k-1}^3) \quad i, j = 1, 2, 6 \]

\[ A_{ij} = \sum_{k=1}^{n} (\overline{Q}_{ij})_k (h_k^3 - h_{k-1}^3) \quad i, j = 1, 2, 6 \]

\[ A_{ij} = \frac{5}{4} \sum_{k=1}^{n} (\overline{Q}_{ij})_k \left[ h_k - h_{k-1} - \frac{4}{3} (h_k^3 - h_{k-1}^3) \frac{1}{h_0^2} \right] i, j = 4, 5 \]

For a symmetric sandwich plate \( B_{ij} = 0 \) and \( A_{45} = 0 \). The sandwich plate is also specially orthotropic so that \( A_{16} = A_{26} = 0 \) and \( D_{16} = D_{26} = 0 \). The transverse shear stiffness for a sandwich panel are \( A_{55} = G_{zz}H \) and \( A_{44} = G_{11}H \) [30].

4.2 Reissner’s Variational Theorem

Reissner’s Variational Theorem is first applied to a thick orthotropic plate in order to obtain the equation of motion. The corresponding sandwich plate equations of motions are then derived from it. In applying the Reissner’s Variational Theorem [31], a function \( F \) is defined to be

\[ F = \sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + 2\sigma_{xy} \varepsilon_{xy} + 2\sigma_{xz} \varepsilon_{xz} + 2\sigma_{yz} \varepsilon_{yz} - W(\sigma_y) \quad (4.19) \]

where

\[
W(\sigma_y) = \frac{1}{2} \left[ \frac{1}{E_x} \sigma_x^2 + \frac{1}{E_y} \sigma_y^2 + \frac{1}{E_z} \sigma_z^2 - 2 \left( \frac{v_{xy}}{E_x} \sigma_x \sigma_y + \frac{v_{yz}}{E_y} \sigma_y \sigma_z + \frac{v_{xz}}{E_z} \sigma_z \sigma_x \right) + \frac{1}{G_{yz}} \sigma_{yz}^2 + \frac{1}{G_{xz}} \sigma_{xz}^2 + \frac{1}{G_{xy}} \sigma_{xy}^2 \right]
\]
The Reissner Functional $\psi$ is then written as

$$\psi = \int_F FdV - \int_S pwdS \quad (4.20)$$

where $p$ is a uniformly applied pressure pulse load, $dV = dx dy dz$ is a differential volume and $dS = dx dy$ is a differential surface area. The strain energy can be obtained by substituting Equations (4.5) and (4.10) to (4.15) in (4.20) and integrating through the thickness $(h_0)$ of the plate:

$$\Psi = \int_0^a \int_0^b \left\{ \frac{E_y h_0}{2(1-v_{xy} v_{yx})} \left[ \gamma^2 - 2(\gamma - v_{yx}) \Pi \epsilon \right] + M_x \frac{\partial \epsilon}{\partial x} + M_y \frac{\partial \epsilon}{\partial y} + M_{xy} \left( \frac{\partial \epsilon}{\partial y} + \frac{\partial \beta}{\partial x} \right) + Q_x \left( \frac{\partial w}{\partial x} + \bar{\alpha} \right) + Q_y \left( \frac{\partial w}{\partial y} + \bar{\beta} \right) - \frac{6M_x^2}{E_y h_0^2} - \frac{6M_y^2}{E_x h_0^2} - \frac{6M_{xy}^2}{E_x h_0^2} + \frac{12v_{xy} M_x M_y}{E_x h_0^2} - \frac{3Q_x^2}{5G_x h_0} - \frac{3Q_y^2}{5G_y h_0} - pw \right\} dx dy \quad (4.21)$$

where $\gamma^2 = E_y / E_x$ and $\Pi \epsilon$ and $\Pi \epsilon$ are the first and second invariants of the mid-plane strains of the composite sandwich plate [32] defined as

$$\bar{\Pi} \epsilon = \varepsilon_{xy} + \gamma \varepsilon_{yy}$$

$$\Pi \epsilon = \varepsilon_{xy} \varepsilon_{yy} - \frac{2G_{xy} (1 - v_{xy} v_{yx})}{E_x (\gamma - v_{yx})} e_{xy}^2$$

24
The kinetic energy of the plate after integrating through the thickness of the sandwich plate is

\[
T = \int_0^b \int_0^a \left[ \frac{M}{2} \left( \left( \frac{\partial u_0}{\partial t} \right)^2 + \left( \frac{\partial v_0}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \right) + \frac{I}{2} \left( \left( \frac{\partial \alpha}{\partial t} \right)^2 + \left( \frac{\partial \beta}{\partial t} \right)^2 \right) \right] \, dx \, dy
\]  

(4.22)

where \( \overline{M} \) is the effective mass and \( \overline{I} \) is the effective rotary inertia. For the orthotropic plate,

\[
\overline{M} = \rho h_0; \quad \overline{I} = \frac{\rho h_0^3}{12}
\]  

(4.23)

where \( \rho \) is the density of the plate.

According to Hamilton’s principle,

\[
\delta \Phi = \delta \int_{t_1}^{t_2} (T - \psi) \, dt = 0
\]  

(4.24)

The second invariant of the mid-plane of the composite sandwich plate can be neglected by the virtue of Berger hypothesis [8]. Then, substituting Equations (4.20) and (4.22) into (4.24) with \( \overline{I}_c = 0 \) gives

\[
\partial \Phi = \int_0^b \int_0^a \left[ \overline{I} \left( \frac{\partial \alpha}{\partial t} \delta \alpha + \frac{\partial \beta}{\partial t} \delta \beta \right) + \overline{M} \left( \frac{\partial u_0}{\partial t} \delta u_0 + \frac{\partial v_0}{\partial t} \delta v_0 + \frac{\partial w}{\partial t} \delta w \right) \right] \, dx \, dy
\]

\[
- \int_{t_1}^{t_2} \int_0^b \int_0^a M_x \delta \alpha + M_{xy} \delta \beta + \left[ Q_x + \frac{E_x h_0}{1 - v_{xy} v_{yx}} I_c \frac{\partial w}{\partial x} \right] \delta v + \frac{E_x h_0}{1 - v_{xy} v_{yx}} I_c \delta u_0 \, dt \, dy
\]  

25
The Euler-Lagrange equations of motions are written as

\[
- \int_{t_0}^{t_0} \left[ M_x \delta \ddot{x} + M_y \delta \ddot{y} + \left( Q_x + \frac{E_x h_0}{1 - v_{xy} v_{yx}} \tilde{I}_x \frac{\partial \ddot{v}}{\partial y} \right) \delta v + \frac{E_x h_0}{1 - v_{xy} v_{yx}} \tilde{I}_y \delta \theta \right] dt dx \\
+ \int_{t_0}^{t_0} \left[ \frac{\partial \alpha}{\partial y} + \frac{12}{E_x h_0} M_x + \frac{12 v_{xy}}{E_x h_0} M_y \right] \delta M_x + \left[ \frac{\partial \beta}{\partial y} + \frac{12}{E_y h_0} M_y + \frac{12 v_{xy}}{E_y h_0} M_x \right] \delta M_y \\
+ \left[ \frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + Q_y + \tilde{I} \frac{\partial^2 \alpha}{\partial t^2} \right] \delta \beta \plus \left[ \frac{\partial Q_x}{\partial x} - \frac{\partial Q_y}{\partial y} + \frac{E_x h_0}{1 - v_{xy} v_{yx}} \frac{\partial \tilde{I}_x}{\partial y} \right] \delta \theta \\
- \frac{E_x h_0}{1 - v_{xy} v_{yx}} \frac{\partial}{\partial y} \left( \tilde{I}_x \frac{\partial \tilde{w}}{\partial y} \right) + \tilde{M} \frac{\partial^2 \tilde{w}}{\partial t^2} + Q_x + \tilde{I} \frac{\partial^2 \alpha}{\partial t^2} \\
+ \left[ - \frac{E_x h_0}{1 - v_{xy} v_{yx}} \frac{\partial \tilde{I}_x}{\partial x} + \tilde{M} \frac{\partial^2 u_0}{\partial t^2} \right] \delta \theta \plus \left[ \frac{E_x h_0 \gamma}{1 - v_{xy} v_{yx}} \frac{\partial \tilde{I}_x}{\partial y} + \tilde{M} \frac{\partial^2 v_0}{\partial t^2} \right] \delta \theta \right] dt dx dy = 0
\]

(4.25)

The Euler-Lagrange equations of motions are written as

\[
- \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} + Q_x + \tilde{I} \frac{\partial^2 \alpha}{\partial t^2} = 0
\]

(4.26)

\[
- \frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + Q_y + \tilde{I} \frac{\partial^2 \beta}{\partial t^2} = 0
\]

(4.27)

\[
- \frac{\partial Q_x}{\partial x} - \frac{\partial Q_y}{\partial y} - \frac{E_x h_0}{1 - v_{xy} v_{yx}} \left[ \frac{\partial}{\partial x} \left( \tilde{I}_x \frac{\partial \tilde{w}}{\partial x} \right) + \gamma \frac{\partial}{\partial y} \left( \tilde{I}_x \frac{\partial \tilde{w}}{\partial y} \right) \right] + \tilde{M} \frac{\partial^2 \tilde{w}}{\partial t^2} = p
\]

(4.28)

\[
\frac{\partial^2 \alpha}{\partial x^2} - \frac{12}{E_x h_0} M_x + \frac{12 v_{xy}}{E_x h_0} M_y = 0
\]

(4.29)
Equations (4.26) through (4.33) except (4.28) have the same form as those of the linear theory [31, 33]. These equations involve the membrane effects as well as the nonlinear strains. By means of Berger’s hypothesis Equations (4.34) and (4.35) are the simplified equations for in-plane force equilibrium.

The inertia effect of the in-plane motion is assumed insignificantly small and is neglected [7]. This enables $u_0$ and $v_0$ to be eliminated from the equations. Hence, setting the right-hand side of Equations (4.34) and (4.35) equal to zero.

$$\frac{\partial \bar{I}_x}{\partial x} = \frac{\partial \bar{I}_x}{\partial y} = 0$$ (4.36)
It is seen from Equation (4.34), that \( T_e \) is independent of the space coordinates \( x \) and \( y \).

Hence, it is assumed

\[
T_e = \frac{\partial u_0}{\partial x} + \gamma \frac{\partial v_0}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 + \frac{\gamma}{2} \left( \frac{\partial w}{\partial y} \right)^2 = \frac{\Delta^2 h_0^2}{12}
\]  

(4.37)

where \( \Delta^2 \) is a coupling parameter. Eliminating \( M_x, M_y, M_{xy}, Q_x, Q_y, \alpha \) and \( \beta \) from Equations (4.26) through (4.33) and using Equation (4.37) yields

\[
\frac{5G_x h_0}{6} \left( \frac{\partial w}{\partial x} + \alpha \right) - \left[ D_x \frac{\partial^2 \alpha}{\partial x^2} + D_{xy} \frac{\partial^2 \alpha}{\partial y^2} + (H - D_{xy}) \frac{\partial^2 \beta}{\partial x \partial y} - I \frac{\partial^2 \alpha}{\partial t^2} \right] = 0
\]  

(4.38)

\[
\frac{5G_x h_0}{6} \left( \frac{\partial w}{\partial y} + \beta \right) - \left[ D_y \frac{\partial^2 \beta}{\partial y^2} + D_{xy} \frac{\partial^2 \beta}{\partial x^2} + (H - D_{xy}) \frac{\partial^2 \alpha}{\partial x \partial y} - I \frac{\partial^2 \beta}{\partial t^2} \right] = 0
\]  

(4.39)

\[
-\frac{5G_x h_0}{6} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial \alpha}{\partial x} \right) - \frac{5G_x h_0}{6} \left( \frac{\partial^2 w}{\partial y^2} + \frac{\partial \beta}{\partial y} \right) - \frac{E h_0}{1 - \nu_{xy}} \frac{\Delta h_0}{12} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + \frac{\Delta}{\partial x^2} = p
\]  

(4.40)

where \( H = D_{xx} y_x^2 + 2D_{xy} \).

For a simply-supported plate the boundary conditions are

\[
\beta = u_0 = w = M_x = 0 \quad \text{along} \ x = 0, a
\]

\[
\alpha = v_0 = w = M_y = 0 \quad \text{along} \ y = 0, b
\]  

(4.41)
Using these boundary conditions for \( u_0 \) and \( v_0 \), the integration of the Equation (4.41) over the plate gives

\[
\Delta^2 = \frac{6}{h_0^2 ab} \int_0^a \int_0^b \left[ \left( \frac{\partial w}{\partial x} \right)^2 + \gamma \left( \frac{\partial w}{\partial y} \right)^2 \right] dxdy
\]  

(4.42)

The corresponding equations of motion for a symmetric sandwich panel are

\[
A_{55} \left( \frac{\partial w}{\partial x} + \alpha \right) - \left[ D_{11} \frac{\partial^2 \alpha}{\partial x^2} + D_{66} \frac{\partial^2 \alpha}{\partial y^2} + (D_{12} + D_{66}) \frac{\partial^2 \beta}{\partial x \partial y} - I \frac{\partial^2 \alpha}{\partial t^2} \right] = 0
\]  

(4.43)

\[
A_{44} \left( \frac{\partial w}{\partial y} + \beta \right) - \left[ D_{11} \frac{\partial^2 \beta}{\partial y^2} + D_{12} \frac{\partial^2 \beta}{\partial x^2} + (D_{12} + D_{66}) \frac{\partial^2 \alpha}{\partial x \partial y} - I \frac{\partial^2 \beta}{\partial t^2} \right] = 0
\]  

(4.44)

\[
-A_{55} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial \alpha}{\partial x} \right) - A_{44} \left( \frac{\partial^2 w}{\partial y^2} + \frac{\partial \beta}{\partial y} \right) - \frac{A_1 h_0^2}{12} \Delta^2 \left( \frac{\partial^2 w}{\partial x^2} + \gamma \frac{\partial^2 w}{\partial y^2} \right) + M \frac{\partial^2 w}{\partial t^2} - p = 0
\]  

(4.45)

where \( M = \sum_{k=1}^{n} \int_{y_k}^{y_{k+1}} \rho_k dz \) and \( I = \sum_{k=1}^{n} \int_{y_k}^{y_{k+1}} \rho_k z^2 dz \). Equations (4.43) through (4.45) describe fully elastic response of the sandwich panel.

To satisfy the boundary conditions, the solutions are assumed to be of the form

\[
w = \sum_{m} \sum_{n} W_{mn} \sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{m\pi y}{b} \right)
\]
\[ \overline{\alpha} = \sum_m \sum_n \Gamma_{mn} \cos \left( \frac{n \pi x}{a} \right) \sin \left( \frac{m \pi y}{b} \right) \]  

\[ \overline{\beta} = \sum_m \sum_n \Lambda_{mn} \sin \left( \frac{n \pi x}{a} \right) \cos \left( \frac{m \pi y}{b} \right) \]  

where \( W_{mn}, \Gamma_{mn} \) and \( \Lambda_{mn} \) are time-varying amplitudes of the transverse deflection and shear rotations. Only the first term of each expression given by Equation (4.46) is used as an approximation to the true solution. For \( m=n=1 \) in Equation (4.46), the assumed one-term approximations are given by

\[ w = W_{11} \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right) \]  

\[ \overline{\alpha} = \Gamma_{11} \cos \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right) \]  

\[ \overline{\beta} = \Lambda_{11} \sin \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi y}{b} \right) \]  

4.3 Forced Elastic Response

Substituting Equation (4.45) into Equations (4.41) to (4.44) gives

\[ A_{55} \left( \lambda_i W_{11} + \Gamma_{11} \right) + \lambda_i^2 D_{11} \Delta_1 + \delta_1^2 D_{66} \Gamma_{11} + \lambda_i \delta_1^3 (D_{12} + D_{66}) \Lambda_{11} + \overline{\Gamma}_{11} = 0 \]  

\[ A_{44} \left( \delta_1^2 W_{11} + \Lambda_{11} \right) + \delta_1^2 D_{22} \Lambda_{11} + \lambda_i^2 D_{66} \Lambda_{11} + \lambda_i \delta_1^3 (D_{12} + D_{66}) \Gamma_{11} + \overline{\Lambda}_{11} = 0 \]  

\[ A_{55} \left( \lambda_i W_{11} + \lambda_i \Gamma_{11} \right) + A_{44} \left( \delta_1^2 W_{11} + \delta_1 \Lambda_{11} \right) + \frac{1}{12} A_{44} \lambda_i^2 \delta_1^3 \left[ \hat{\gamma}_{11} W_{11} + \gamma_1^3 \Delta_{11} \right] + \overline{W}_{11} - \overline{\lambda}_{11} = 0 \]
where  \( \Delta^2 = \frac{3W_{11}^2}{2h_0^2} \left( \lambda_1^2 + \gamma \delta^2 \right) \), \( \lambda_1 = \pi/a \), \( \delta_1 = \pi/b \) and for uniformly distributed pressure pulse is  \( \bar{p}_{11} = \frac{16p(t)}{\pi^2} \)  \( \) (4.49) \( \)

where  \( p(t) \) is defined in Equation (3.1).

4.4 Forced Elastic – Plastic Response

The PVC foam core in the analysis has an elastic-perfectly plastic behavior. Due to the elastic-plastic nature of the core, plastic regions develop in the core and plastic work is dissipated every time there is core yielding. Following Equation (4.3), the transverse shear strains are given by

\[
\gamma_{xc} = \gamma_{13} \cos \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right)
\]

\( \) (4.50) \( \)

\[
\gamma_{yc} = \gamma_{23} \sin \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi y}{b} \right)
\]

\( \) (4.51) \( \)

where  \( \gamma_{13} = \pi/a W_{11} + \Gamma_{11} \) and  \( \gamma_{23} = \pi/b W_{11} + \Lambda_{11} \). The transverse shear strains are maximum at the panel edges, i.e.,  \( \gamma_{xc} \) is maximum at  \( x = 0, a \) and  \( \gamma_{yc} \) is maximum at  \( y = 0, b \). Plastic yielding of the core takes place when the transverse shear strains near the simply-supported edges exceed the transverse shear yield strains. The zones of the plastic region are approximately elliptical in nature. Four plastic regions (half ellipses)
develop along the sides of the plate as shown in Figure 4.2. The areas of the ellipses are
given by

\[ A_1 = \pi \xi_0 \eta_0 \]  \hspace{1cm} (4.52)  
\[ A_2 = \pi \xi_0 \eta_0' \]  \hspace{1cm} (4.53)  

where \( A_1 \) and \( A_2 \) are the area of the ellipse along the \( x \)- and \( y \)-axis, respectively, and \( \xi_0, \eta_0, \xi_0' \) and \( \eta_0' \) are their respective minor and major axes. To evaluate the minor and
major axes, the amplitude of transverse shear strains are set equal to the transverse shear
yield strains to give

\[ \xi_0 = \frac{a}{\pi} \cos^{-1} \left( \frac{\gamma_{130}}{\gamma_{13}} \right), \quad \eta_0 = \frac{b}{\pi} \cos^{-1} \left( \frac{\gamma_{130}}{\gamma_{13}} \right) \]  \hspace{1cm} (4.54)  
\[ \xi_0' = \frac{b}{\pi} \cos^{-1} \left( \frac{\gamma_{230}}{\gamma_{23}} \right), \quad \eta_0' = \frac{a}{\pi} \cos^{-1} \left( \frac{\gamma_{230}}{\gamma_{23}} \right) \]  \hspace{1cm} (4.55)  

where \( \gamma_{130} = \tau_0 / G_{13} \) and \( \gamma_{230} = \tau_0 / G_{23} \) are the transverse shear strain at yield, as shown in
Figure 4.3, and \( \tau_0 \) is the shear plastic yield strength derived in Appendix B.
Figure 4.2 Plastic dissipation regions in the core of the sandwich plate.

Figure 4.3 Transverse shear force resultants during plastic yielding.
The total dissipation energy by the four semi-ellipses is approximately given by

\[ D_p = Q_0 \left( \frac{\gamma_{13} - \gamma_{130}}{2} \right) \pi \xi_0 \eta_0 + Q_0 \left( \frac{\gamma_{23} - \gamma_{230}}{2} \right) \pi \xi_0 \eta' \]  \quad (4.56)

4.4.1 Plastic yielding

The transverse shear resultant forces \((Q_x, Q_y)\) for the forced elastic-plastic response of the sandwich are given as

\[
Q_x = \begin{cases} 
A_{55} \left( \frac{\partial w}{\partial x} + \alpha \right) & \text{in } S_1 \\
Q_{x_0} & \text{in } S_2
\end{cases}
\]  \quad (4.57)

\[
Q_y = \begin{cases} 
A_{44} \left( \frac{\partial w}{\partial y} + \beta \right) & \text{in } S_1 \\
Q_{y_0} & \text{in } S_2
\end{cases}
\]  \quad (4.58)

where \(Q_{x_0}\) and \(Q_{y_0}\) are the transverse shear resultant forces at yield. From Appendix B, one finds that during yielding

\[ Q_x^2 + Q_y^2 = Q_0^2 \]  \quad (4.59)

In general, \(Q_{x_0}\) and \(Q_{y_0}\) vary with \(x\) and \(y\). At the point of yielding, the transverse shear force resultants are related to transverse shear strains and

\[
Q_{x_0} = A_{55} \gamma_{130} \cos \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right) \]  \quad (4.60)

\[
Q_{y_0} = A_{44} \gamma_{230} \sin \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi y}{b} \right) \]  \quad (4.61)
Therefore,

\[
\frac{Q_{x0}}{Q_{y0}} = \frac{A_{55}\gamma_{130} \tan \left( \frac{\pi y}{b} \right)}{A_{44}\gamma_{230} \tan \left( \frac{\pi x}{a} \right)} \quad (4.62)
\]

\[
\frac{Q_{y0}}{Q_{x0}} = \frac{A_{44}\gamma_{230} \tan \left( \frac{\pi x}{a} \right)}{A_{55}\gamma_{130} \tan \left( \frac{\pi y}{b} \right)} \quad (4.63)
\]

Substituting Equations (4.62) and (4.63) into Equation (4.59) gives

\[
Q_{x0} = \frac{Q_0 A_{55}\gamma_{130} \tan \left( \frac{\pi y}{b} \right)}{\sqrt{(A_{44}\lambda_{230})^2 \tan^2 \left( \frac{\pi x}{a} \right) + (A_{55}\lambda_{130})^2 \tan^2 \left( \frac{\pi y}{b} \right)}} \quad (4.64)
\]

\[
Q_{y0} = \frac{Q_0 A_{44}\gamma_{230} \tan \left( \frac{\pi x}{a} \right)}{\sqrt{(A_{44}\lambda_{230})^2 \tan^2 \left( \frac{\pi x}{a} \right) + (A_{55}\lambda_{130})^2 \tan^2 \left( \frac{\pi y}{b} \right)}} \quad (4.65)
\]

Equations (4.57) and (4.58) are substituted into Equations (4.26)-(4.28) to give the following differential equations in the plastic region ($S_2$):

\[
Q_{x0} + \ddot{I} \frac{\partial^2 \bar{a}}{\partial t^2} - \left[ D_{11} \frac{\partial^2 \bar{a}}{\partial x^2} + D_{66} \frac{\partial^2 \bar{a}}{\partial x^2} + (D_{12} + D_{66}) \frac{\partial^2 \bar{B}}{\partial x \partial y} \right] = 0 \quad (4.66)
\]

\[
Q_{y0} + \ddot{I} \frac{\partial^2 \bar{B}}{\partial t^2} - \left[ D_{66} \frac{\partial^2 \bar{B}}{\partial x^2} + D_{22} \frac{\partial^2 \bar{B}}{\partial y^2} + (D_{12} + D_{66}) \frac{\partial^2 \bar{a}}{\partial x \partial y} \right] = 0 \quad (4.67)
\]
In the elastic region \((S_1)\) shown in Figure 4.2, the differential equations of motion are specified by Equations (4.43) to (4.45). The combined elastic-plastic equations of motion are found by integrating these differential equations with weighting functions over the surface area of the area of the panel.

Equations (4.43) and (4.66) are first multiplied by \(\cos(\pi x/a)\sin(\pi y/b)\) (the shape function for \(\alpha\)) before integration over the plate surface area:

\[
\begin{align*}
\int_0^a \int_0^b D_{11} \left( \frac{\pi^2}{a^2} \Gamma_{11} + D_{66} \left( \frac{\pi}{b} \right)^2 \Gamma_{11} + (D_{12} + D_{66}) \left( \frac{\pi}{a} \right) \left( \frac{\pi}{b} \right) \Lambda_{11} + \Gamma_{11} + \bar{I}_{11} \right) \cos \left( \frac{\pi x}{a} \right) \sin^2 \left( \frac{\pi y}{b} \right) dxdy &= 0
\end{align*}
\]

(4.69)

Equations (4.44) and (4.67) are pre-multiplied by \(\sin(\pi x/a)\cos(\pi y/b)\) (the shape function for \(\beta\)) before integration over the plate surface area:

\[
\begin{align*}
\int_0^a \int_0^b D_{22} \left( \frac{\pi^2}{b^2} \Lambda_{11} + D_{66} \left( \frac{\pi}{a} \right)^2 \Lambda_{11} + (D_{12} + D_{66}) \left( \frac{\pi}{a} \right) \left( \frac{\pi}{b} \right) \Lambda_{11} + \bar{I}_{11} \right) \sin \left( \frac{\pi x}{a} \right) \cos^2 \left( \frac{\pi y}{b} \right) dxdy &= 0
\end{align*}
\]

(4.70)
Equations (4.45) and (4.68) are pre-multiplied by $\sin(\pi x/a)\sin(\pi y/b)$ (the shape function for $w$) before integration over the plate surface area:

$$\begin{align*}
\iint_{S_1} A_{35} \left( \frac{\pi^2}{a} W_{11} + \frac{\pi}{a} \Gamma_{11} \right) \sin^2 \left( \frac{\pi x}{a} \right) \sin^2 \left( \frac{\pi y}{b} \right) dxdy + \iint_{S_1} A_{44} \left( \frac{\pi^2}{b} W_{11} + \frac{\pi}{b} \Lambda_{11} \right) \sin^2 \left( \frac{\pi x}{a} \right) \sin^2 \left( \frac{\pi y}{b} \right) dxdy \\
- \iint_{S_1} \left[ \frac{\partial Q_{y_0}}{\partial x} + \frac{\partial Q_{y_0}}{\partial y} \right] \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right) dxdy \\
\iint_{0}^{a} \left\{ \frac{A_3}{8} \left( \frac{\pi}{a} \right)^2 + \gamma \left( \frac{\pi}{b} \right)^2 \right\} W_{11}^3 + \bar{M} \bar{\dot{W}}_{11} - p_{11} \right\} \sin^2 \left( \frac{\pi x}{a} \right) \sin^2 \left( \frac{\pi y}{b} \right) dxdy = 0
\end{align*}$$

(4.71)

Integrating and solving Equations (4.69) to (4.71) lead to the following:

$$\begin{align*}
f_1 A_{35} \left( \frac{\pi}{a} W_{11} + \Gamma_{11} \right) + D_{11} \left( \frac{\pi}{a} \right)^2 \Gamma_{11} + D_{66} \left( \frac{\pi}{b} \right)^2 \Gamma_{11} + (D_{12} + D_{66}) \frac{\pi}{a} \frac{\pi}{b} \Lambda_{11} + \bar{f}_{\Gamma_{11}} &= -Q_{dx} \\
f_2 A_{44} \left( \frac{\pi}{b} W_{11} + \Lambda_{11} \right) + D_{22} \left( \frac{\pi}{a} \right)^2 \Lambda_{11} + D_{66} \left( \frac{\pi}{b} \right)^2 \Lambda_{11} + (D_{12} + D_{66}) \frac{\pi}{a} \frac{\pi}{b} \Gamma_{11} + \bar{f}_{\Lambda_{11}} &= -Q_{dy} \\
f_3 A_{55} \left( \frac{\pi^2}{a} W_{11} + \frac{\pi}{a} \Gamma_{11} \right) + \frac{A_3}{8} \left( \frac{\pi}{a} \right)^2 W_{11} + \frac{\pi}{a} \Lambda_{11} + \frac{A_3}{8} \left( \frac{\pi}{b} \right)^2 W_{11}^3 + \bar{M} \bar{\dot{W}}_{11} &= \bar{p}_{11} - p_d
\end{align*}$$

(4.72) (4.73) (4.74)

where $f_i = \frac{4}{ab} \int_{S_1} \cos^2 \left( \frac{\pi x}{a} \right) \sin^2 \left( \frac{\pi y}{b} \right) dxdy$

$$f_2 = \frac{4}{ab} \int_{S_1} \sin^2 \left( \frac{\pi x}{a} \right) \cos^2 \left( \frac{\pi y}{b} \right) dxdy$$

$$f_3 = \frac{4}{ab} \int_{S_1} \sin^2 \left( \frac{\pi x}{a} \right) \sin^2 \left( \frac{\pi y}{b} \right) dxdy$$

37
\[ Q_{dx} = \frac{4}{ab} \int_{S_2} Q_{x_0} \cos \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right) dxdy \]

\[ Q_{dy} = \frac{4}{ab} \int_{S_2} Q_{y_0} \sin \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi y}{b} \right) dxdy \]

\[ p_d = -\frac{4}{ab} \int_{S_2} \left( \frac{\partial Q_{x_0}}{\partial x} + \frac{\partial Q_{y_0}}{\partial y} \right) \sin \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi y}{b} \right) dxdy \]

For a square sandwich panel with \( D_{22} = D_{11} \) and \( A_{44} = A_{55} \):

\[ b = a, \quad \zeta_0' = \eta_0' = \zeta_0 = \eta_0, \quad \text{and} \quad f_1 = f_2. \]

Also,

\[ Q_{x_0} = \frac{Q_0 \tan \left( \frac{\pi y}{a} \right)}{\sqrt{\tan^2 \left( \frac{\pi x}{a} \right) + \tan^2 \left( \frac{\pi y}{a} \right)}} \]  

(4.75)

\[ Q_{y_0} = \frac{Q_0 \tan \left( \frac{\pi x}{a} \right)}{\sqrt{\tan^2 \left( \frac{\pi x}{a} \right) + \tan^2 \left( \frac{\pi y}{a} \right)}} \]  

(4.76)

and \( Q_{dx} = Q_{dy} \). Substituting these conditions and integrating the above coefficients in the equations of motion yields

\[ f_1 = f_2 = 1 - 0.38687 \bar{\eta}_0 + 0.73854 \bar{\eta}_0^2 - 11.97946 \bar{\eta}_0^3 + 18.38428 \bar{\eta}_0^4 - 7.75357 \bar{\eta}_0^5 \]  

(4.77)

\[ f_3 = 1 - 0.38802 \bar{\eta}_0 + 3.94158 \bar{\eta}_0^2 - 11.54681 \bar{\eta}_0^3 + 7.01261 \bar{\eta}_0^4 \]  

(4.78)

\[ Q_{dx} = Q_{dy} = Q_0 \left( 0.46085 \bar{\eta}_0 - 1.74869 \bar{\eta}_0^2 + 16.22462 \bar{\eta}_0^3 - 23.01037 \bar{\eta}_0^4 + 9.27461 \bar{\eta}_0^5 \right) \]  

(4.79)
where \( \eta_0 = 2\eta_0/a \) is a normalized plastic region. The above integration was performed numerically and is only valid when \( 0 < \eta_0 < 1 \).

4.4.2 Elastic unloading

Eventually the panel reaches a peak displacement and begins to rebound. This rebound leads to elastic unloading in the plastic regions, as depicted in Figure 4.4. The transverse shear force resultants are given by

\[
Q_x = \begin{cases} 
    A_{55} \left( \frac{\partial w}{\partial x} + \bar{\alpha} \right) & \text{in } S_1 \\
    Q_{x0} - A_{55} \gamma_{xzm}^{(1)} + A_{55} \left( \frac{\partial w}{\partial x} + \bar{\alpha} \right) & \text{in } S_2 
\end{cases}
\]

\[
Q_y = \begin{cases} 
    A_{44} \left( \frac{\partial w}{\partial y} + \bar{\beta} \right) & \text{in } S_1 \\
    Q_{y0} - A_{44} \gamma_{yzm}^{(2)} + A_{44} \left( \frac{\partial w}{\partial y} + \bar{\beta} \right) & \text{in } S_2 
\end{cases}
\]

where in a one-term approximation \( \gamma_{xzm}^{(1)} = \gamma_{13m}^{(1)} \cos(\pi x/a) \sin(xy/b) \) and

\( \gamma_{yzm}^{(1)} = \gamma_{23m}^{(1)} \sin(\pi x/a) \cos(xy/a) \) are the distributed maximum transverse shear strain before unloading. Substituting these expressions into Equations (4.26)-(4.28) and solving with weighted integrals as in Section 4.4.1 give
\[
A_{55} \left( \frac{\pi}{a} W_{11} + \Gamma_{11} \right) + D_{11} \left( \frac{\pi}{a} \right)^2 \Gamma_{11} + D_{66} \left( \frac{\pi}{b} \right)^2 \Gamma_{11} + (D_{12} + D_{66}) \frac{\pi}{a} \frac{\pi}{b} \Lambda_{11} + \bar{f}\Gamma_{11}
= -Q_{dx}^{(1)} + A_{55} \gamma_{13m}^{(1)}(1 - f_{1}^{(1)})
\]

\[
A_{44} \left( \frac{\pi}{b} W_{11} + \Lambda_{11} \right) + D_{22} \left( \frac{\pi}{a} \right)^2 \Lambda_{11} + D_{66} \left( \frac{\pi}{b} \right)^2 \Lambda_{11} + (D_{12} + D_{66}) \frac{\pi}{a} \frac{\pi}{b} \Gamma_{11} + \bar{f}\Lambda_{11}
= -Q_{dy}^{(1)} + A_{55} \gamma_{23m}^{(1)}(1 - f_{2}^{(1)})
\]

\[
A_{55} \left( \frac{\pi}{a} W_{11} + \Gamma_{11} \right) + A_{44} \left( \frac{\pi}{b} \right)^2 W_{11} + \frac{A_{14}}{8} \left( \frac{\pi}{a} \right)^2 + \gamma \left( \frac{\pi}{b} \right)^2 \right)^2 W_{11} + \bar{M}\bar{W}_{11}
= \bar{p} - p_{d}^{(1)} + A_{55} \frac{\pi}{a} \gamma_{13m}^{(1)}(1 - f_{3}^{(1)}) + A_{44} \frac{\pi}{b} \gamma_{23m}^{(1)}(1 - f_{3}^{(1)})
\]

where \( Q_{dx}^{(1)}, Q_{dy}^{(1)}, p_{d}^{(1)}, f_{1}^{(1)}, f_{2}^{(1)} \) and \( f_{3}^{(1)} \) are the respective values at the start of unloading.

Figure 4.4 Transverse shear force resultants during elastic unloading.
4.4.3 Reverse yielding

The panel could have enough residual energy during the rebound to cause reverse yielding in the core. When this happens, another plastic region ($S_3$) will emerge within the previous region ($S_2$), which is unloading (see Figure 4.5). The transverse shear force resultant in $S_3$ is shown in Figure 4.6. The size of this new plastic zone is given by

\[
\xi_0 = \frac{a}{\pi} \cos^{-1} \left( \frac{2\gamma_{130}}{\gamma_{13}^{(1)} - \gamma_{13}} \right), \quad \eta_0 = \frac{b}{\pi} \cos^{-1} \left( \frac{2\gamma_{130}}{\gamma_{13}^{(1)} - \gamma_{13}} \right) \quad (4.86)
\]

\[
\xi_0' = \frac{b}{\pi} \cos^{-1} \left( \frac{2\gamma_{230}}{\gamma_{23}^{(1)} - \gamma_{23}} \right), \quad \eta_0' = \frac{a}{\pi} \cos^{-1} \left( \frac{2\gamma_{230}}{\gamma_{23}^{(1)} - \gamma_{23}} \right) \quad (4.87)
\]

Figure 4.5 New plastic regions in core during reverse yielding.
The transverse shear force resultants in the core are now described by

\[
Q_x = \begin{cases} 
    A_{55} \left( \frac{\partial w}{\partial x} + \bar{\alpha} \right) & \text{in } S_1 \\
    Q_{x_0} - A_{55} \gamma_{yzm}^{(1)} + A_{55} \left( \frac{\partial w}{\partial x} + \bar{\alpha} \right) & \text{in } S_2 \\
    -Q_{x_0} & \text{in } S_3 
\end{cases} 
\]  \quad (4.88)

\[
Q_y = \begin{cases} 
    A_{44} \left( \frac{\partial w}{\partial y} + \bar{\beta} \right) & \text{in } S_1 \\
    Q_{y_0} - A_{44} \gamma_{yzm}^{(1)} + A_{44} \left( \frac{\partial w}{\partial y} + \bar{\beta} \right) & \text{in } S_2 \\
    -Q_{y_0} & \text{in } S_3 
\end{cases} 
\]  \quad (4.89)

Substituting these expressions into Equations (4.26)-(4.28) and integrating with weighted functions give

\[
f_1 A_{55} \left( \frac{\pi}{a} W_{11} + \Gamma_{11} \right) + D_{11} \left( \frac{\pi}{a} \right)^2 \Gamma_{11} + D_{66} \left( \frac{\pi}{b} \right)^2 \Gamma_{11} + (D_{12} + D_{66}) \frac{\pi}{a} \frac{\pi}{b} \Lambda_{11} + \bar{\Gamma}_{11} = 2Q_{dx} - Q_{dx}^{(1)} + A_{45} \gamma_{13m}^{(1)} (f_1 - f_1^{(1)}) \]  \quad (4.90)

\[
f_2 A_{44} \left( \frac{\pi}{b} W_{11} + \Lambda_{11} \right) + D_{22} \left( \frac{\pi}{a} \right)^2 \Lambda_{11} + D_{66} \left( \frac{\pi}{b} \right)^2 \Lambda_{11} + (D_{12} + D_{66}) \frac{\pi}{a} \frac{\pi}{b} \Gamma_{11} + \bar{\Lambda}_{11} = 2Q_{dy} - Q_{dy}^{(1)} + A_{55} \gamma_{23m}^{(1)} (f_2 - f_2^{(1)}) \]  \quad (4.91)
\[
f_3A_3\left(\frac{\pi}{a}W_{11} + \frac{\pi}{a}A_{11}\right) + f_3A_{44}\left(\frac{\pi}{b}W_{11} + \frac{\pi}{b}A_{11}\right) + \frac{A_{11}}{8}\left(\frac{\pi}{a} + \gamma\left(\frac{\pi}{b}\right)^2\right)^2W_{11}^2 + \tilde{M}\tilde{W}_{11}^1
\]

\[
= \bar{p}_{11} + 2p_d - p_d^{(i)} + A_{35}\left(\gamma_{13m} - f_3^{(i)}\right) + A_{44}\left(\gamma_{23m} - f_3^{(i)}\right)
\]

Figure 4.6 Transverse shear force resultants during reverse yield.

4.4.4 Consecutive unloading/yielding

The first reverse yielding may be followed by several cycles of unloading and yielding. The transverse shear force resultant shown in Figure 4.7 indicates that each cycle ends with maximum transverse shear strain \(\gamma_{13m}^{(k)}\) or \(\gamma_{23m}^{(k)}\), where \(k = 3\). Each time the core yields a new plastic zone develops, smaller than the previous one. In general the plastic zone size is given by

\[
\xi_0 = \frac{a}{\pi}\cos^{-1}\left(\frac{2\gamma_{13m}^{(k)}}{\gamma_{13} - \gamma_{13m}^{(k)}\text{sgn}(\tilde{\gamma}_1)}\right), \quad \eta_0 = \frac{b}{\pi}\cos^{-1}\left(\frac{2\gamma_{13m}^{(k)}}{\gamma_{13} - \gamma_{13m}^{(k)}\text{sgn}(\tilde{\gamma}_1)}\right)
\]

(4.93)
\[ \xi_0' = \frac{b}{\pi} \cos^{-1}\left( \frac{2\gamma_{23m}}{(\gamma_{23} - \gamma_{23m}) \text{sgn}(\dot{\gamma}_{23})} \right), \quad \eta_0' = \frac{a}{\pi} \cos^{-1}\left( \frac{2\gamma_{23m}}{(\gamma_{23} - \gamma_{23m}) \text{sgn}(\dot{\gamma}_{23})} \right) \]  

(4.94)

where \( \text{sgn}(\dot{\gamma}_{13}) \) and \( \text{sgn}(\dot{\gamma}_{23}) \) are negative for unloading and positive for reloading.

Figure 4.7 Transverse shear force resultants during consecutive yielding/unloading and yielding/reloading.
During unloading/reloading, the transverse shear force resultants are

\[
Q_x = \begin{cases} 
A_{55} \left( \frac{\partial w}{\partial x} + \bar{\alpha} \right) & \text{in } S_1 \\
Q_{x0} - A_{55} \gamma_{yzm}^{(1)} + A_{55} \left( \frac{\partial w}{\partial x} + \bar{\alpha} \right) & \text{in } S_2 \\
-Q_{x0} - A_{55} \gamma_{yzm}^{(2)} + A_{55} \left( \frac{\partial w}{\partial x} + \bar{\alpha} \right) & \text{in } S_3 \\
Q_{x0} - A_{55} \gamma_{yzm}^{(3)} + A_{55} \left( \frac{\partial w}{\partial x} + \bar{\alpha} \right) & \text{in } S_4 \\
\vdots \\
(-1)^{k} Q_{x0} - A_{55} \gamma_{yzm}^{(k-1)} + A_{55} \left( \frac{\partial w}{\partial x} + \bar{\alpha} \right) & \text{in } S_k 
\end{cases}
\]

(Eq. 4.95)

\[
Q_y = \begin{cases} 
A_{44} \left( \frac{\partial w}{\partial y} + \bar{\beta} \right) & \text{in } S_1 \\
Q_{y0} - A_{44} \gamma_{yzm}^{(1)} + A_{44} \left( \frac{\partial w}{\partial y} + \bar{\beta} \right) & \text{in } S_2 \\
-Q_{y0} - A_{44} \gamma_{yzm}^{(2)} + A_{44} \left( \frac{\partial w}{\partial y} + \bar{\beta} \right) & \text{in } S_3 \\
Q_{y0} - A_{44} \gamma_{yzm}^{(3)} + A_{44} \left( \frac{\partial w}{\partial y} + \bar{\beta} \right) & \text{in } S_4 \\
\vdots \\
(-1)^{k} Q_{y0} - A_{44} \gamma_{yzm}^{(k-1)} + A_{44} \left( \frac{\partial w}{\partial y} + \bar{\beta} \right) & \text{in } S_k 
\end{cases}
\]

(Eq. 4.96)
The equations of motion during unloading/reloading are

\[
A_{55} \left( \frac{\pi}{a} W_{11} + \Gamma_{11} \right) + D_{11} \left( \frac{\pi}{a} \right)^2 \Gamma_{11} + D_{66} \left( \frac{\pi}{b} \right)^2 \Gamma_{11} + (D_{12} + D_{66}) \frac{\pi}{a} \frac{\pi}{b} \Lambda_{11} + \bar{I} \bar{\Gamma}_{11} = -Q_{dx}^{(1)} + 2Q_{dx}^{(2)} - 2Q_{dx}^{(3)} + \ldots + (-1)^k 2Q_{dx}^{(k)} + A_{55} \gamma_{13m}^{(1)} (f_1^{(2)} - f_1^{(1)}) + A_{55} \gamma_{13m}^{(2)} (f_1^{(3)} - f_1^{(2)}) + \ldots + A_{55} \gamma_{13m}^{(k-1)} (1 - f_1^{(k-1)})
\]

(4.97)

\[
A_{44} \left( \frac{\pi}{b} W_{11} + \Lambda_{11} \right) + D_{22} \left( \frac{\pi}{a} \right)^2 \Lambda_{11} + D_{66} \left( \frac{\pi}{b} \right)^2 \Lambda_{11} + (D_{12} + D_{66}) \frac{\pi}{a} \frac{\pi}{b} \Gamma_{11} + \bar{I} \bar{\Lambda}_{11} = -Q_{dy}^{(1)} + 2Q_{dy}^{(2)} - 2Q_{dy}^{(3)} + \ldots + (-1)^k 2Q_{dy}^{(k)} + A_{44} \gamma_{23n}^{(1)} (f_2^{(2)} - f_2^{(1)}) + A_{44} \gamma_{23n}^{(2)} (f_2^{(3)} - f_2^{(2)}) + \ldots + A_{44} \gamma_{23n}^{(k-1)} (1 - f_2^{(k-1)})
\]

(4.98)

\[
A_{1} \left( \frac{\pi}{a} W_{11} + \pi \Gamma_{11} \right) + A_{4} \left( \frac{\pi}{b} W_{11} + \pi \Lambda_{11} \right) + A_{1} \left( \frac{\pi}{a} \right)^2 W_{11} + \pi \Lambda_{11} + A_{4} \left( \frac{\pi}{b} \right)^2 W_{11} + \pi \Lambda_{11} + \bar{\pi} \bar{W}_{11} = -\rho_1 - \rho_3 2\rho_2 3\rho_2 4\rho_2 \ldots + (-1)^k 2\rho_2 3\rho_2 4\rho_2 5\rho_2 + A_{55} \gamma_{13m}^{(1)} (f_3^{(2)} - f_3^{(1)}) + A_{44} \gamma_{23n}^{(2)} (f_3^{(3)} - f_3^{(2)}) + \ldots + A_{55} \gamma_{13m}^{(k-1)} (1 - f_3^{(k-1)}) + A_{44} \gamma_{23n}^{(k-2)} (1 - f_3^{(k-2)})
\]

(4.99)
During yielding, the transverse shear force resultants are

\[
Q_x = \begin{cases} 
A_{s5} \left( \frac{\partial w}{\partial x} + \bar{\alpha} \right) & \text{in } S_1 \\
Q_{x0} - A_{s5} \gamma_{x2m}^{(1)} + A_{s5} \left( \frac{\partial w}{\partial x} + \bar{\alpha} \right) & \text{in } S_2 \\
- Q_{x0} - A_{s5} \gamma_{x2m}^{(2)} + A_{s5} \left( \frac{\partial w}{\partial x} + \bar{\alpha} \right) & \text{in } S_3 \\
Q_{x0} - A_{s5} \gamma_{x2m}^{(3)} + A_{s5} \left( \frac{\partial w}{\partial x} + \bar{\alpha} \right) & \text{in } S_4 \\
\vdots \\
(-1)^{(k)} Q_{x0} - A_{s5} \gamma_{x2m}^{(k-1)} + A_{s5} \left( \frac{\partial w}{\partial x} + \bar{\alpha} \right) & \text{in } S_k \\
(-1)^{(k+1)} Q_{x0} & \text{in } S_{k+1} 
\end{cases} 
\tag{4.100}
\]

\[
Q_y = \begin{cases} 
A_{t4} \left( \frac{\partial w}{\partial y} + \bar{\beta} \right) & \text{in } S_1 \\
Q_{y0} - A_{t4} \gamma_{y2m}^{(1)} + A_{t4} \left( \frac{\partial w}{\partial y} + \bar{\beta} \right) & \text{in } S_2 \\
- Q_{y0} - A_{t4} \gamma_{y2m}^{(2)} + A_{t4} \left( \frac{\partial w}{\partial y} + \bar{\beta} \right) & \text{in } S_3 \\
Q_{y0} - A_{t4} \gamma_{y2m}^{(3)} + A_{t4} \left( \frac{\partial w}{\partial y} + \bar{\beta} \right) & \text{in } S_4 \\
\vdots \\
(-1)^{(k)} Q_{y0} - A_{t4} \gamma_{y2m}^{(k-1)} + A_{t4} \left( \frac{\partial w}{\partial y} + \bar{\beta} \right) & \text{in } S_k \\
(-1)^{(k+1)} Q_{y0} & \text{in } S_{k+1} 
\end{cases} 
\tag{4.101}
\]
The equations of motion during yielding are

\[ f_1 A_{kk} \left( \frac{\pi}{a} W_{11} + \alpha_1 \right) + D_{11} \left( \frac{\pi}{a} \right)^2 G_{11} + D_{66} \left( \frac{\pi}{b} \right)^2 G_{11} + (D_{12} + D_{66}) \frac{\pi}{a} \alpha_1 + I_1 = \]
\[ -Q_{dx}^{(1)} + 2Q_{dx}^{(2)} + 2Q_{dx}^{(3)} + \ldots + (-1)^i 2Q_{dx} + A_{55} \gamma_{13m} \left( f_1^{(2)} - f_1^{(1)} \right) + A_{55} \gamma_{13m} \left( f_1^{(3)} - f_1^{(2)} \right) \]
\[ + A_{55} \gamma_{13m} \left( f_1^{(4)} - f_1^{(3)} \right) \ldots + A_{55} \gamma_{13m} \left( f_1^{(k)} - f_1^{(k-1)} \right) \]

\[ (4.102) \]

\[ f_2 A_{kk} \left( \frac{\pi}{b} W_{11} + \alpha_1 \right) + D_{22} \left( \frac{\pi}{a} \right)^2 G_{11} + D_{66} \left( \frac{\pi}{b} \right)^2 G_{11} + (D_{12} + D_{66}) \frac{\pi}{a} \alpha_1 + I_1 = \]
\[ -Q_{dy}^{(1)} + 2Q_{dy}^{(2)} + 2Q_{dy}^{(3)} + \ldots + (-1)^i 2Q_{dy} + A_{44} \gamma_{23m} \left( f_2^{(2)} - f_2^{(1)} \right) + A_{44} \gamma_{23m} \left( f_2^{(3)} - f_2^{(2)} \right) \]
\[ + A_{44} \gamma_{23m} \left( f_2^{(4)} - f_2^{(3)} \right) \ldots + A_{44} \gamma_{23m} \left( f_2^{(k)} - f_2^{(k-1)} \right) \]
\[ (4.103) \]

\[ f_3 A_{kk} \left( \frac{\pi}{a} W_{11} + \alpha_1 \right) + f_4 A_{kk} \left( \frac{\pi}{b} W_{11} + \alpha_1 \right) + \frac{A_4}{8} \left( \frac{\pi}{a} \right)^2 + \frac{\pi}{b} G_{11} = \]
\[ -p_{11}^{(1)} + 2p_{11}^{(2)} - 2p_{11}^{(3)} + 2p_{11}^{(4)} \ldots + (-1)^i 2p_{11} + A_{55} \gamma_{13m} \left( f_3^{(2)} - f_3^{(1)} \right) + A_{55} \gamma_{13m} \left( f_3^{(3)} - f_3^{(2)} \right) \]
\[ + A_{55} \gamma_{13m} \left( f_3^{(4)} - f_3^{(3)} \right) \ldots + A_{55} \gamma_{13m} \left( f_3^{(k)} - f_3^{(k-1)} \right) \]
\[ + A_{55} \gamma_{13m} \left( f_3^{(k)} - f_3^{(k-1)} \right) + A_{44} \gamma_{23m} \left( f_3^{(2)} - f_3^{(1)} \right) + A_{44} \gamma_{23m} \left( f_3^{(3)} - f_3^{(2)} \right) \]
\[ + A_{44} \gamma_{23m} \left( f_3^{(4)} - f_3^{(3)} \right) \ldots + A_{44} \gamma_{23m} \left( f_3^{(k)} - f_3^{(k-1)} \right) \]
\[ (4.104) \]

where \( Q_{dx}^{(k)}, Q_{dy}^{(k)}, f_1^{(k)}, f_2^{(k)} \) and \( f_3^{(k)} \) are the respective values at the start of unloading or reloading.
CHAPTER V
FINITE ELEMENT ANALYSIS

Finite element analysis using ABAQUS Explicit was carried out on the composite sandwich panel subjected to pressure pulse loading in order to gage the accuracy of the analytical solution. Geometry, material properties, element type, mesh, load, boundary conditions and analysis are discussed in this chapter.

5.1 Finite Element Analysis Model

A simply-supported rectangular composite sandwich panel of length $a=740$ mm, breadth $b=740$ mm, facesheet thickness $h=6$ mm, and core thickness $H=25$ mm is considered. Due to the symmetry of the problem, only a quarter section of the composite sandwich panel was modeled in ABAQUS. The finite element model for the rectangular composite sandwich panel is as shown in Figure 5.1.
Figure 5.1 Finite element model of composite sandwich panel.

The facesheet is E-Glass/Vinyl Ester, while the core is Divinycell PVC H100. The material properties of the facesheet and core are given in Table 3.1. The material properties for the composite sandwich panel were assigned in a local rectangular Cartesian coordinate system (XYZ) according to the ABAQUS 6.9 User Manual [34]. The orthotropic properties of the facesheet were specified using *Elastic and type=ENGINEERING CONSTANTS and the isotropic properties of the core were specified using *Crushable Foam, hardening=ISOTROPIC for elastic-plastic analysis and *Elastic and type=ENGINEERING CONSTANTS for elastic analysis. Plasticity curve for Divinycell PVC H100 foam was taken from Mines et al. [35]. For the polymeric foam, the plastic Poisson’s ratio was taken to be zero.

The composite sandwich panel was subjected to a uniformly distributed pressure pulse load of magnitude $p_0$ (0.25 MPa) and with a load duration of $\Delta T$ (4ms). The pressure time history is shown in Equation (3.1).
A pinned boundary condition was used on the mid-plane of the two edges of the quarter sandwich panel problem constraining the translational motion and X-SYMM and Y-SYMM symmetry condition were used to constraint the symmetry planes. A pinned boundary condition could not be assigned directly to the foam. When the boundary condition was assigned directly to the mid-surface plane of the foam, there were very large stresses concentrated in localized regions of the foam. Hence, the sandwich was modeled with frictionless, rigid (non-deformable) rollers along the edges as shown in Figure 5.2. The rollers were attached to the upper and lower facesheet using *Interaction, surface to surface discretization method. The interaction properties was assigned between the rollers and the facesheet using *Interaction, tangential behavior, friction formulation, frictionless. A small overhang was allowed beyond the rollers to prevent slippage.

![Composite sandwich panel with load and boundary conditions.](image)

Figure 5.2 Composite sandwich panel with load and boundary conditions.
Dynamic, Explicit analysis was performed in finite element analysis. Dynamic Implicit analysis, although more accurate than Explicit analysis, was not used due to the long computational time for the 3D geometry. The facesheets, both the top and bottom, were rigidly tied to the core using *Tie constraint, surface to surface discretization method. In the tie constraint, each node of the Slave surface (core) moves similar to the node of the Master surface (facesheet).

The mesh was generated with the help of the ABAQUS CAE software. A mesh convergence study was done to find the mesh size. An eight-node linear brick element (C3D8) was used for both the facesheet and core, respectively. A full-integration, second-order accuracy and the default distortion control was chosen for the integration type. The meshed composite quarter, sandwich panel is as shown in Figure 5.3.

Figure 5.3 Meshed finite element model composite sandwich panel.
5.2 Comparison of FEA and Analytical Results

A comparison of FEA and analytical results was done for both the elastic and elastic-plastic response of the sandwich panel.

5.2.1 Purely Elastic Response

The purely elastic response of the sandwich panel for the panel center deflection and shear strain at the edges are given in Figures 5.4 and 5.5, respectively. The core had no plastic dissipation and the sandwich panel behaved as perfectly elastic. The analytical solution presented in Chapter IV is also shown in Figures 5.5 and 5.6 for comparison. The FEA results for the panel center deflection and the shear strains at the edges are in good agreement with the analytical results. However, the two solutions deviated somewhat at longer times. This is because of accumulated numerical errors in the FEA and MATLAB programs.

![Comparison of FEA and analytical panel center deflection.](image)

Figure 5.4 Comparison of FEA and analytical panel center deflection.
5.2.2 Elastic-Plastic Response

The PVC H100 foam was modeled with crushable foam and isotropic hardening, compressive yield strain ratio (=1.732). The plastic dissipation in the core occurs at the edges as shown in Figure 4.1 in Chapter IV, where the transverse shear strains are high. The ABAQUS solution also shows the same type of plastic region as in Figure 5.6. The foam core exhibited plastic dissipation when the critical shear strain value was reached. The FEA and analytical comparisons for the panel center deflection and transverse shear strains are compared in Figures 5.7 and 5.8. The analytical and FEA results were in good agreement up to the first peak, but both deviated at later times as was the case in the purely elastic response. Again the discrepancy at later times is attributed to accumulated numerical errors.
Figure 5.6 FEA model showing the plastic region.

Figure 5.7 Comparison of FEA and analytical panel center deflection.
Figure 5.8 Comparison of FEA and analytical transverse shear strains.
CHAPTER VI
FAILURE MODES

Analytical models provide simple design tools for determining the survivability of the panel when it is subjected to an intense pressure pulse load. The fracture of the woven E-Glass/Vinyl Ester facesheet and PVC foam core are examined in this chapter.

6.1 Facesheet Failure

The facesheet failure may be either compressive or tensile failure in the top or bottom facesheet. The failure of the facesheet is predicted using a modified Hashin-Rotem failure criterion for the woven composites [36]. To determine the failure of the facesheet the principal deformation strains in the facesheet are first calculated by

\[
\varepsilon_x = \frac{\partial u_0}{\partial x} + z \frac{\partial \alpha}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2
\]

\[
\varepsilon_y = \frac{\partial v_0}{\partial y} + z \frac{\partial \beta}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2
\]

\[
\varepsilon_{xy} = \frac{1}{2} \left( \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + z \frac{\partial \alpha}{\partial y} + z \frac{\partial \beta}{\partial x} \right) + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right) \left( \frac{\partial w}{\partial y} \right)
\]

(6.1)
To evaluate these strains, a one-term approximation for the in-plane deformations is assumed as

\[
 u_0 = U_{11} \cos \left( \frac{2\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right) 
\]

(6.2)

\[
 v_0 = V_{11} \sin \left( \frac{\pi x}{a} \right) \cos \left( \frac{2\pi y}{b} \right) 
\]

(6.3)

The time-varying amplitudes \( U_{11} \) and \( V_{11} \) are approximated in Appendix D.

Substituting deformations in Equation (6.1) gives

\[
 \varepsilon_x = U_{11} \frac{2\pi}{a} \cos \left( \frac{2\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right) - z\Gamma_{11} \frac{\pi}{a} \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right) + \frac{1}{2} \left( \frac{\pi}{a} \right)^2 \left( W_{11} \cos \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right) \right)^2 
\]

(6.4)

\[
 \varepsilon_y = V_{11} \frac{2\pi}{b} \sin \left( \frac{\pi x}{a} \right) \cos \left( \frac{2\pi y}{b} \right) - z\Lambda_{11} \frac{\pi}{b} \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right) + \frac{1}{2} \left( \frac{\pi}{b} \right)^2 \left( W_{11} \sin \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi y}{b} \right) \right)^2 
\]

\[
 \varepsilon_{xy} = \frac{1}{2} \left[ U_{11} \frac{\pi}{b} \cos \left( \frac{2\pi x}{a} \right) \cos \left( \frac{\pi y}{b} \right) + V_{11} \frac{\pi}{a} \sin \left( \frac{2\pi y}{b} \right) \cos \left( \frac{\pi x}{a} \right) + z\Gamma_{11} \frac{\pi}{a} \cos \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi y}{b} \right) \right] 
\]

In the orthotropic facesheet with fibers in 0 and 90 degrees parallel to the x- and y-axes, the relationship between principal stresses and strains is given by

\[
 \begin{bmatrix}
 \sigma_x \\
 \sigma_y \\
 \tau_{xy}
 \end{bmatrix} =
 \begin{bmatrix}
 Q_{11} & Q_{12} & 0 \\
 Q_{12} & Q_{22} & 0 \\
 0 & 0 & Q_{66}
 \end{bmatrix}
 \begin{bmatrix}
 \varepsilon_x \\
 \varepsilon_y \\
 2\varepsilon_{xy}
 \end{bmatrix} 
\]

(6.5)
where $\overline{Q}_{ij}$ is the transformed stiffness matrix, given by $\overline{Q}_{11} = E_{11} / (1 - \nu_{12} \nu_{21})$, $\overline{Q}_{22} = E_{22} / (1 - \nu_{12} \nu_{21})$, $\overline{Q}_{12} = \nu_{12} E_{22} / (1 - \nu_{12} \nu_{21})$ and $\overline{Q}_{66} = G_{12}$.

According to the modified Hashin-Rotem failure criteria, the failure of the composite occurs when

$$\frac{|\sigma_x|}{X_T} = 1 \quad \text{if} \quad \sigma_x > 0 \quad \text{or} \quad \frac{|\sigma_x|}{X_C} = 1 \quad \text{if} \quad \sigma_x < 0$$

(6.6)

$$\frac{|\sigma_y|}{X_T} = 1 \quad \text{if} \quad \sigma_y > 0 \quad \text{or} \quad \frac{|\sigma_y|}{X_C} = 1 \quad \text{if} \quad \sigma_y < 0$$

(6.7)

$$\frac{|\tau_{xy}|}{S_L} = 1$$

(6.8)

where $X_T$, $X_C$ and $S_L$ are the maximum strength in tension, compression and shear respectively.

Solving Equation (6.5) yields

$$\sigma_x = \overline{Q}_{11} \varepsilon_x + \overline{Q}_{12} \varepsilon_y$$

(6.9)

$$\sigma_y = \overline{Q}_{12} \varepsilon_x + \overline{Q}_{22} \varepsilon_y$$

(6.10)

$$\tau_{xy} = 2 \overline{Q}_{66} \varepsilon_{xy}$$

(6.11)

In the above equations $\sigma_x$ and $\sigma_y$ are equal since the plate is square, $a=b$. The normal stresses $\sigma_x$ and $\sigma_y$ are also greater than the in-plane shear stress of the facesheet.
Now, substituting Equation (6.4) into Equation (6.9) yields

\[
\sigma_x = \overline{Q}_{11} \left[ U_{11} \frac{2\pi}{a} \cos\left(\frac{2\pi}{a}\right) \sin\left(\frac{\pi y}{b}\right) - z \Gamma_{11} \frac{\pi}{a} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) + \frac{1}{2} \left(\frac{\pi}{a}\right)^2 \left( W_{11} \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) \right)^2 \right] + \\
\overline{Q}_{12} \left[ V_{11} \frac{2\pi}{a} \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{2\pi y}{b}\right) - z \Lambda_{11} \frac{\pi}{b} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) + \frac{1}{2} \left(\frac{\pi}{b}\right)^2 \left( W_{11} \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi y}{b}\right) \right)^2 \right]
\]

(6.12)

The principal stress components (\(\sigma_x\) or \(\sigma_y\)) are greatest at the center of the panel (\(x=a/2,\ y=b/2\)) for \(z=\pm(H/2+h)\) (see Figure 6.1). It was found that the bottom facesheet failed in tension after applying material properties for the E-Glass Vinyl/Ester in Table 3.1. The analysis of the maxima and minima of the stress are provided in Appendix E.

![Figure 6.1 Zone of maximum normal and transverse shear stresses.](image)

The criterion for the bottom facesheet fracture is
\[
\bar{Q}_{11} \left[ \frac{2\pi^2}{a^2} \frac{W_{11}^2}{8+2\sqrt{2}} + \frac{\pi}{a} \frac{(H+2h)}{2} \Gamma_{11} \right] + \bar{Q}_{12} \left[ \frac{2\pi^2}{b^2} \frac{W_{11}^2}{8+2\sqrt{2}} + \frac{\pi}{b} \frac{(H+2h)}{2} \Lambda_{11} \right] = X_f \quad (6.13)
\]

6.2 Core Failure

The core failure is due to transverse shear failure. To determine the failure of the core the shear strain deformations on the core are evaluated from

\[
\gamma_{xz} = \frac{\partial w}{\partial x} + \bar{\alpha}
\quad (6.14)
\]

\[
\gamma_{yz} = \frac{\partial w}{\partial y} + \bar{\beta}
\quad (6.15)
\]

Substituting deformations and shear rotations in Equations (6.14) and (6.15) one gets the shear strains as

\[
\gamma_{xz} = \left[ W_{11} \left( \frac{\pi}{a} \right) + \Gamma_{11} \right] \cos \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right)
\quad (6.16)
\]

\[
\gamma_{yz} = \left[ W_{11} \left( \frac{\pi}{b} \right) + \Lambda_{11} \right] \sin \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi y}{b} \right)
\quad (6.17)
\]

In the above equations both shear strains are equal because the sandwich panel is square. From the above equations it can be seen that the shear strains are maximum at the edge of the panel and zero at the center of the panel (see Figure 6.1). Hence, the core fails due to shear strains at the edge. The core shear failure criteria are
where $\gamma_{13f}$ and $\gamma_{23f}$ are transverse shear fracture strains.
CHAPTER VII
PARAMETRIC STUDY

Blast resistance and energy absorption are very important attributes in the design of sandwich panels. Sandwich panels are used to manufacture lightweight aerospace, marine and transportation vehicles, which may be exposed to blast. The blast resistance of these panels is measured by its ability to survive the highest pressure pulse from an incident shock wave. In addition to this, sandwich panels can be used as cladding, a sacrificial covering, to protect infrastructure from blast. In the later application, the sandwich panel energy absorption before failure becomes the more important parameter. The analytical solutions developed in Chapters IV and VI are used to understand the mechanisms and parameters that are responsible for blast resistance and energy absorption. A parametric study is carried out on various square panel aspect ratios \( (a/(2h + H)) \) of 10, 20 and 40 for PVC H100 foam core. Then several foams (Divinycell PVC H30, H100, H200 and HCP100, and Klegecell R300) are considered for aspect ratio of 40. In these parametric studies the load duration is kept at 4 ms and the peak pressure to failure is examined. The panel failure may be either due to core shear failure or facesheet failure depending on the combination of aspect ratio and core properties. The failure is due to tensile failure when the normal stress reaches a critical value that is
\(\sigma_{i1}\) reaches \(X_f\). Failure is due to transverse shear fracture when the transverse shear strains exceeded the transverse shear fracture strain. The transverse shear yield strain \((\gamma_{130})\) and fracture strain \((\gamma_{13f})\) are as shown in Table 7.1. Transverse shear fracture strains were only found for PVC H30 \([24]\) and H100 \([37]\). According to References \([38, 39]\) the critical shear strain for failure increases with increase in the density of the core. It was assumed that the transverse shear fracture strain for the PVC H200 was greater than that for the PVC H100 \([37]\). The transverse shear fracture strain for the R300 and HCP100 were taken to be greater than their respective yield strains.

Table 7.1 Shear strain values for yield and fracture for different foams.

<table>
<thead>
<tr>
<th></th>
<th>H30</th>
<th>H100</th>
<th>H200</th>
<th>R300</th>
<th>HCP100</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\gamma_{130}, \gamma_{230})</td>
<td>0.016</td>
<td>0.029</td>
<td>0.028</td>
<td>0.045</td>
<td>0.056</td>
</tr>
<tr>
<td>(\gamma_{13f}, \gamma_{23f})</td>
<td>0.09</td>
<td>0.16</td>
<td>&gt;0.16</td>
<td>&gt;0.045</td>
<td>&gt;0.056</td>
</tr>
</tbody>
</table>

7.1 Aspect Ratio

A parametric study was done on different panel aspect ratio for constant facesheet (6mm) and core thickness (25mm). The foam core of PVC H100 was chosen to carry the parametric study for various panel aspect ratios. For all the aspect ratios it was found that there was a core shear fracture of the foam core before any facesheet damage. From Figure 7.1 it can be seen that panels with smaller aspect ratios required higher loads for the core shear failure. This is because the smaller aspect ratio panels had higher shear and bending stiffness than the larger aspect ratio panels.
Figure 7.1 Failure pressure for H100 foam and various aspect ratios.

The energy absorption by the core is also an important parameter for the design of sandwich panel. The amount of energy absorbed by the core is calculated from Equation (4.56). The amount of energy absorbed by the PVC H100 for different panel aspect ratios is shown in Figure 7.2. The energy absorbed by the panel with larger aspect ratio is greater when compared to that of smaller aspect ratio panels. This is because the plastic region \( S_2 \) for the larger aspect ratio panels is bigger than the smaller aspect ratio panels. Table 7.2 list the maximum values of \( \zeta_0 \) or \( \eta_0 \) for the plastic zones.

Table 7.2 List of the maximum values of \( \zeta_0 \), \( \eta_0 \)

<table>
<thead>
<tr>
<th>Aspect Ratio</th>
<th>10</th>
<th>20</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \zeta_0, \eta_0 )</td>
<td>0.162</td>
<td>0.326</td>
<td>0.659</td>
</tr>
</tbody>
</table>
7.2 Core Foam Properties

A parametric study was done for the sandwich panel with aspect ratio of 40 and different core materials (PVC H30, H100, H200 and HCP100 and Klegecell R300). In this analysis it was seen that panels with lower core densities (PVC H30 and H100) failed due to core shear failure while the panels with higher core densities (PVC H200, HCP100, and Klegecell R300) failed due to facesheet failure. The panels with the highest core densities (R300 and HCP100) did not even yield (no plasticity) before the normal tensile stress in the facesheet reached the tensile strength of the facesheet. The panels behaved elastically and failed due to facesheet rupture. The H200 foam panel just yielded before there was a facesheet failure.

Figure 7.2 Energy absorbed by H100 and various aspect ratios.
Figure 7.3 shows the failure pressure for panel of aspect ratio of 40 with different foam core materials. It can be seen that the panel with HCP100 foam as a core material is the most blast resistant. Further, the blast resistance of the panel increases with increasing foam core density.

![Failure pressure for panel with different foam cores.](image)

Figure 7.3 Failure pressure for panel with different foam cores.

Figure 7.4 shows the energy absorbed by the panel of aspect ratio of 40 with different foam core. The panel with H100 foam core has the capability to absorb the most energy compared to other foam-core sandwich panels. The panel with H200 core absorbs very little energy, whereas the panels with R300 and HCP 100 foams absorb no energy before failure.
The failure pressure was normalized by the areal weight density $\rho_w$ of the panel. The areal weight density is given by

$$\rho_w = (\rho_c H + \rho_f 2h)g \quad (7.1)$$

where $\rho_c$ and $\rho_f$ are the core and facesheet density, respectively, and $g$ is the acceleration due to gravity. The blast resistance of panel with H200 foam core has the highest failure pressure per unit areal weight density as shown in Figure 7.5.
Figure 7.5 Failure pressure/areal weight density for panel with various foam cores.
CHAPTER VIII

CONCLUDING REMARKS

An analytical solution for the large amplitude, damped response of a simply-supported sandwich panel with E-Glass/Vinyl Ester facing and PVC foam core under pressure pulse loading has been presented in this study. The composite facesheets and the crushable PVC foam core were considered to be orthotropic elastic and isotropic elastic-plastic material, respectively. The elastic and elastic-plastic response of the sandwich panel were developed using the Reissner’s Variational Method and Hamilton’s Principle. Finite element analysis was also done using ABAQUS Explicit to gage the accuracy of the analytical solution. The analytical results for the panel center deflection and the shear strains at the edges compared well with the FEA results. Failure of the sandwich panel was either due to the rupture of the bottom facesheet at the center of the panel or due to the transverse shear fracture of the foam core at the edges of the panel.

Parametric studies were done on different panel aspect ratios and foam core materials. Three panel aspect ratios of 10, 20 and 40 and five core materials namely, Divinycell PVC H30, H100, H200 and HCP100, and Klegercell R300 foams were used in the study. The sandwich panels with higher core densities had more blast resistance capability. The blast resistance capacity of the sandwich panel increased with increasing core density. When compared on a per unit areal weight density basis, the panel with
H200 foam core was found to be the most blast resistant of all panels. Furthermore, foam cores with lighter densities absorbed more energy compared to those with higher densities. The sandwich panels with the highest densities, R300 and HCP100, exhibited no plasticity or energy absorption before failure. The sandwich panel with H30 and H100 foam cores failed due to core shear fracture for all the aspect ratios, while the panels with H200, R300 and HCP 100 foams failed due to tensile failure of the facesheet for the aspect ratio of 40.

The parametric studies were limited because of insufficient data on the core transverse shear fracture strain for the Divinycell H200 and HCP 100 foams and the Klegecell R300 foam. Experimental tests are underway to measure the transverse shear fracture strain of the above-mentioned foams. Once these have been done, parametric studies will be done for the other panel aspect ratios. In addition to this, the analytical model should be verified with blast experiments.
REFERENCES


APPENDICES
APPENDIX A

LIST OF NOTATIONS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>length of the panel</td>
</tr>
<tr>
<td>$b$</td>
<td>width of the panel</td>
</tr>
<tr>
<td>$A_i, A_2$</td>
<td>area of the ellipses in $x$- and $y$- directions</td>
</tr>
<tr>
<td>$A_{ij}, i, j = 1,2,6$</td>
<td>membrane stiffness matrix</td>
</tr>
<tr>
<td>$A_{44}, A_{55}$</td>
<td>transverse shear stiffness</td>
</tr>
<tr>
<td>$B_{ij}$</td>
<td>bending-membrane coupling matrix</td>
</tr>
<tr>
<td>$D_{ij}$</td>
<td>bending stiffness matrix</td>
</tr>
<tr>
<td>$D_p$</td>
<td>total dissipation energy per unit area</td>
</tr>
<tr>
<td>$E$</td>
<td>Young’s modulus of the foam</td>
</tr>
<tr>
<td>$E_{ij}$</td>
<td>Young’s modulus of facesheet</td>
</tr>
<tr>
<td>$G$</td>
<td>shear modulus of the foam</td>
</tr>
<tr>
<td>$G_{ij}$</td>
<td>shear modulus of facesheet</td>
</tr>
<tr>
<td>$h$</td>
<td>facesheet thickness</td>
</tr>
</tbody>
</table>
\( h_k \)  
ply height numbering of the sandwich

\( h_0 \)  
plate thickness

\( H \)  
core thickness

\( \bar{I} \)  
effective rotary inertia

\( \bar{I}_e \)  
first invariant

\( \bar{I}_e \)  
second invariant

\( k \)  
ply number

\( M_{ij} \)  
bending moment resistance

\( \bar{M} \)  
effective mass of the sandwich

\( N_{ij} \)  
in-plane stress resistance

\( n \)  
total number of layers

\( p \)  
pressure pulse

\( p_0 \)  
peak pressure

\( \bar{p} \)  
one-term approximation for pressure pulse

\( p_d \)  
plastic damping pressure

\( Q_{ij} \)  
stiffness matrix

\( Q_{x, y} \)  
transverse shear force resistance
\( Q_{xu}, Q_{yu} \) transverse shear force at yield

\( Q_{xu}, Q_{dy} \) plastic damping transverse shear force

\( Q_0 = \tau_0 H \) transverse shear yield strength

\( S_L \) in-plane shear strength

\( T \) kinetic energy

\( u, v \) in-plane deformations

\( U_{ij}, V_{ij} \) amplitude of in-plane deformations

\( w \) transverse deflection

\( W_g \) amplitude of transverse deflection

\( x, y \) in-plane direction coordinates

\( X_T, X_c \) tensile (compressive) strength

\( z \) through thickness direction coordinate

\( \bar{\alpha}, \bar{\beta} \) shear rotations associated with \( x \)- and \( y \)-directions

\( \delta_1 = \frac{\pi}{b} \)

\( \Delta T \) load duration

\( \varepsilon_{ij} \) strains

\( \eta_0 \) major axis of the ellipse in \( y \)-direction
\[ \eta_0 \] major axis of the ellipse in \( x \)-direction

\[ \gamma^2 = E_y / E_x \] ratio of in-plane modulus

\[ \gamma_{13}, \gamma_{23} \] transverse shear strain

\[ \gamma_{130}, \gamma_{230} \] transverse shear strain at yield

\[ \gamma_{13f}, \gamma_{23f} \] transverse shear strain at fracture

\[ \gamma^{(k)}_{13m}, \gamma^{(k)}_{23m} \] maximum transverse shear strain at \( k^{th} \) yield zone

\[ \Gamma_j, \Lambda_j \] amplitude of shear rotations associated with \( x \)- and \( y \)-directions

\[ \lambda_i = \frac{\pi}{a} \]

\[ \psi \] Reissner Functional

\[ \sigma_{ij} \] stresses

\[ \tau_0 \] transverse shear yield strength

\[ \tau_{13}, \tau_{23} \] transverse shear stress

\[ \zeta_0 \] minor axis of the ellipse in \( y \)-direction

\[ \zeta_0' \] minor axis of the ellipse in \( x \)-direction

\[ \mu \] flow potential

\[ \rho \] plate density
\( \rho_c \) core density

\( \rho_f \) facesheet density

\( \rho_w \) areal weight density

\( \sigma_e \) misses effective stress

\( \sigma_m \) mean stress

\( \sigma_0 \) uniaxial tensile or compressive yield strength
APPENDIX B
SHEAR YIELDING

For the shear yielding of the foam core the yield surface according to Deshpande and Fleck [40] is given by

\[ \sigma_0^2 = \frac{1}{[1 + (\mu/3)^2]}[\sigma_e^2 + \mu^2 \sigma_m^2] \]  \hspace{1cm} (B.1)

where \( \sigma_0 \), \( \sigma_e \), \( \sigma_m \), \( \mu \) are the uniaxial yield strength of the foam, the misses effective stress, mean stress and the flow potential, respectively, where

\[ \sigma_m = \frac{\sigma_x + \sigma_y + \sigma_z}{3} \]  \hspace{1cm} (B.2)

\[ \sigma_e = \sqrt{\frac{1}{2} \left[ (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 \right] + 3 (\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2)} \]  \hspace{1cm} (B.3)

\[ \mu = \frac{3}{\sqrt{2}} \sqrt{\frac{1 - 2\nu_p}{1 + \nu_p}} \]  \hspace{1cm} (B.4)

where \( \nu_p \) is the plastic Poisson’s ratio. The stresses in the foam are primarily transverse shear

\[ \sigma = \begin{bmatrix} 0 & 0 & \tau_{xz} \\ 0 & 0 & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & 0 \end{bmatrix} \]  \hspace{1cm} (B.5)

Substituting Equation (B.5) into Equations (B.2) and (B.3) gives
\[ \sigma_m = 0, \sigma_e = \sqrt{3(\tau_{xz}^2 + \tau_{yz}^2)} \]  

(B.6)

During plasticity, the plastic Poisson’s ratio is assumed to be zero. Therefore, Equation (B.4) becomes

\[ \mu = \frac{3}{\sqrt{2}} \]  

(B.7)

Substituting Equations (B.6) and (B.7) into Equation (B.1) yields

\[ \frac{\sigma_0}{\sqrt{2}} = \sqrt{\left(\tau_{xz}^2 + \tau_{yz}^2\right)} \]  

(B.8)

Equation (B.8) can be written as

\[ \tau_{xz}^2 + \tau_{yz}^2 = \tau_0^2 \]  

(B.9)

where \( \tau_0 = \frac{\sigma_0}{\sqrt{2}} \). In terms of transverse shear resultant forces

\[ Q_{x_0}^2 + Q_{y_0}^2 = Q_0^2 \]  

(B.10)

where \( Q_0 = \tau_0 H \).

The yield surface is shown in Figures B.1 (a) and (b).

![Figure B.1 Yield Surface: (a) transverse shear stresses and (b) transverse shear resultant forces.](image)
APPENDIX C

INTEGRATION FUNCTIONS FOR ELASTIC-PLASTIC RESPONSE OF SQUARE SANDWICH PANEL

In a square sandwich panel \( b = a \), \( \xi_0 = \xi_0' \), \( \eta_0 = \eta_0' \), \( f_1 = f_2 \) and \( Q_{dx} = Q_{dy} \).

Function \( f_1 \):

The \( f_1 \) function is given by

\[
f_1 = \frac{4}{a^2} \int \int_{S_1} \cos^2 \left( \frac{\pi x}{a} \right) \sin^2 \left( \frac{\pi y}{b} \right) dx \, dy \tag{C.1}
\]

As shown in Figure C.1, the four half ellipses overlap when \( \eta_0 = a/(2\sqrt{2}) \). The two ellipses overlap at

\[
x_{1,2} = \frac{a}{4} \pm \frac{1}{4} \sqrt{8\eta_0^2 - a^2} \tag{C.2}
\]
Consider when $\eta_0 \leq a/(2\sqrt{2})$ and $\eta_0 > a/(2\sqrt{2})$ separately.

Case A: $\eta_0 \leq a/2\sqrt{2}$

Let

$$\int \int_{S_1} \cos^2 \left( \frac{\pi x}{a} \right) \sin^2 \left( \frac{\pi y}{a} \right) = \int \int_{S_0} \cos^2 \left( \frac{\pi x}{a} \right) \sin^2 \left( \frac{\pi y}{a} \right) dx dy - \int \int_{S_2} \cos^2 \left( \frac{\pi x}{a} \right) \sin^2 \left( \frac{\pi y}{a} \right) dx dy \quad (C.3)$$

The first term in the right-hand side of Equation (C.3) is

$$\int \int_{S_0} \cos^2 \left( \frac{\pi x}{a} \right) \sin^2 \left( \frac{\pi y}{a} \right) dx dy = \frac{a^2}{4} \quad (C.4)$$

The second term in the right-hand side of Equation (C.3) is
\[
\int \int \cos^2 \left( \frac{\pi x}{a} \right) \sin^2 \left( \frac{\pi y}{a} \right) 
\] 
\[\text{d}x \text{d}y = \int \int \sin^2 \left( \frac{\pi a}{a} \right) \cos^2 \text{d}x \text{d}y
\]

\[\int \int \frac{a_y}{a} \sin^3 \left( \frac{\pi y}{a} \right) \text{d}y \cos^2 \text{d}x + \int \int \frac{a_y}{a} \cos^2 \left( \frac{\pi a}{a} \right) \text{d}x \sin^2 \left( \frac{\pi y}{a} \right) \text{d}y
\]

\[\int \int \frac{a_y}{a} \cos^2 \left( \frac{\pi a}{a} \right) \text{d}x \sin^2 \left( \frac{\pi y}{a} \right) \text{d}y
\]

\[\int \int \frac{a_y}{a} \cos^2 \left( \frac{\pi a}{a} \right) \text{d}x \sin^2 \left( \frac{\pi y}{a} \right) \text{d}y
\]

The above integral in Equation (C.5) cannot be evaluated in closed-form. It must be numerically integrated instead. Each integral term in Equation (C.5) is a function of \( \eta \) and \( a \). The first term in the right-hand side of Equation (C.5) is

\[I_{f1} = \int \int \frac{a_y}{a} \sin \left( \frac{2\pi \eta_0}{a} \sqrt{1 - \left( \frac{x-a/2}{\eta_0} \right)^2} \right) \cos \left( \frac{\pi a}{a} \right) \text{d}x\]

Let \( p = \frac{x-a/2}{\eta_0} \), then

\[I_{f1} = \eta_0 \int \int \frac{a_y}{a} \sin \left( \frac{2\pi \eta_0}{a} \sqrt{1 - p^2} \right) \cos \left( \frac{\pi \eta_0 p}{a} \right) \text{d}p\]

Define \( \eta_0 = \frac{2\eta_0}{a} \) where \( 0 < \eta_0 < 1 \). Then
\[ I_{f_{a}} = a^{2} \int_{-1}^{1} \left[ \frac{\eta_{0}^{2}}{8} \sqrt{1-p^{2}} - \frac{\eta_{0}}{8\pi} \sin \left( \pi \eta_{0} \sqrt{1-p^{2}} \right) \right] \sin^{2} \left( \frac{\pi \eta_{0} z}{2} \right) dp \]  
(C.8)

It can be shown that the second term in the right-hand side of Equation (C.5) is

the same as the first term. The third term in the right-hand side of Equation (C.5) is

\[ I_{f_{a}} = \int_{a_{s}+\eta_{0}}^{a_{s}-\eta_{0}} \frac{a}{4\pi} \sin \left( \frac{2\pi \eta_{0}}{a} \sqrt{1-\left( \frac{y-a/2}{\eta_{0}} \right)^{2}} + \frac{\eta_{0}}{2} \sqrt{1-\left( \frac{y-a/2}{\eta_{0}} \right)^{2}} \right) + \frac{\eta_{0}}{2} \sin^{2} \left( \frac{\pi y}{a} \right) dy 
(C.9)

Let \( p = \frac{y-a/2}{\eta_{0}} \), then

\[ I_{f_{a}} = \eta_{0} \int_{-\eta_{0}}^{\eta_{0}} \frac{a}{4\pi} \sin \left( \frac{2\pi \eta_{0}}{a} \sqrt{1-p^{2}} \right) + \frac{\eta_{0}}{2} \sqrt{1-p^{2}} \right] \cos^{2} \left( \frac{\pi \eta_{0} p}{a} \right) dp \]  
(C.10)

In terms of \( \eta_{0} = \frac{2\eta_{0}}{a} \),

\[ I_{f_{a}} = a^{2} \int_{-1}^{1} \left[ \frac{\eta_{0}^{2}}{8\pi} \sin \left( \pi \eta_{0} \sqrt{1-p^{2}} \right) + \frac{\eta_{0}^{2}}{8} \sqrt{1-p^{2}} \right] \cos^{2} \left( \frac{\pi \eta_{0} p}{2} \right) dp \]  
(C.11)

It can be shown that the fourth term in the right-hand side of Equation (C.5) is the

same as the third term. Therefore,

\[ \int \cos^{2} \left( \frac{\pi x}{a} \right) \sin^{2} \left( \frac{\pi y}{a} \right) dxdy = a^{2} \int_{-1}^{1} \left[ \frac{\eta_{0}^{2}}{4} \sqrt{1-p^{2}} - \frac{\eta_{0}}{4\pi} \sin \left( \pi \eta_{0} \sqrt{1-p^{2}} \right) \right] \sin^{2} \left( \frac{\pi \eta_{0} p}{2} \right) \]

\[ + \left[ \frac{\eta_{0}^{2}}{4\pi} \sin \left( \pi \eta_{0} \sqrt{1-p^{2}} + \frac{\eta_{0}^{2}}{4} \sqrt{1-p^{2}} \right) \right] \cos^{2} \left( \frac{\pi \eta_{0} p}{2} \right) dp \]  
(C.12)

or
\[
\iint_{s_2} \cos^2 \left( \frac{\pi x}{a} \right) \sin^2 \left( \frac{\pi y}{a} \right) \, ds \, dy = a^2 I_{f_1} \tag{C.13}
\]

where \( I_{f_1} = \int_{-1}^1 \left[ \frac{\eta_0^2}{4} \sqrt{1 - p^2} + \frac{\eta_0}{4\pi} \sin \left( \pi \eta_0 \sqrt{1 - p^2} \right) \cos \left( \pi \eta_0 p \right) \right] \, dp \)

From Equations (C.1), (C.3), (C.4) and (C.13), one gets

\[
f_1 = \frac{4}{a^2} \left[ \frac{a^2}{4} - a^2 I_{f_1} \right] \]

\[
f_1 = 1 - 4I_{f_1} \tag{C.14}
\]

Case B: \( \eta_0 > a/2\sqrt{2} \)

The \( f_1 \) function is symmetric about \( x = a/2 \) and \( y = b/2 \). It is directly integrated over the \( S_1 \) region to give

\[
f_1 = \frac{16}{a^2} \left\{ \int_0^{y_1/2} \int_0^{a/2-y_1/2} \sin^2 \left( \frac{\pi y}{b} \right) \, dy \, \cos^2 \left( \frac{\pi x}{a} \right) \, dx + \int_{x_2}^{\eta_0} \int_0^{y_1/2} \sin^2 \left( \frac{\pi y}{b} \right) \, dy \, \cos^2 \left( \frac{\pi x}{a} \right) \, dx \right. \\
\left. + \int_{x_2}^{\eta_0} \int_{y_1/2}^{\eta_0} \sin^2 \left( \frac{\pi y}{b} \right) \, dy \, \cos^2 \left( \frac{\pi x}{a} \right) \, dx + \int_{x_2}^{\eta_0} \int_{y_1/2}^{\eta_0} \sin^2 \left( \frac{\pi y}{b} \right) \, dy \, \cos^2 \left( \frac{\pi x}{a} \right) \, dx \right\} \tag{C.15}
\]

where \( y_1 = \eta_0 \sqrt{1 - \left( \frac{x-a/2}{\eta_0} \right)^2} \) and \( y_3 = \frac{a}{2} - \eta_0 \sqrt{1 - \left( \frac{x}{\eta_0} \right)^2} \)

Let \( p = \frac{x-a/2}{\eta_0} \) and \( \eta_0 = \frac{2\eta_0}{a} \)
\[ f_1 = 2 \int_{-1/\eta_0}^{1} \left( \eta_0 - \eta_0^2 \sqrt{1 - \left( \frac{1}{\eta_0} + p \right)^2} - \frac{\eta_0}{\pi} \sin \left( \frac{\pi \eta_0}{\pi \eta_0^2} \right) \right) \sin^2 \left( \frac{\pi p \eta_0}{2} \right) dp \\
+ 2 \int_{-1/\eta_0}^{1} \left( \frac{1}{\eta_0} - \eta_0^2 \sqrt{1 - \left( \frac{1}{\eta_0} + p \right)^2} - \frac{\eta_0}{\pi} \sin \left( \frac{\pi \eta_0}{\pi \eta_0^2} \right) \right) \sin^2 \left( \frac{\pi p \eta_0}{2} \right) dp \\
+ \frac{\eta_0}{\pi} \sin \left( \frac{\pi \eta_0}{\pi \eta_0^2} \right) \sin^2 \left( \frac{\pi p \eta_0}{2} \right) dp + 2 \int_{-\eta_0}^{\eta_0} \left( \int_{-1/\eta_0}^{1} \sqrt{1 - \left( \frac{1}{\eta_0} + p \right)^2} \right) dp + 2 \int_{-\eta_0}^{\eta_0} \left( \int_{-1/\eta_0}^{1} \sqrt{1 - \left( \frac{1}{\eta_0} + p \right)^2} \right) dp \\
+ \frac{\eta_0}{\pi} \sin \left( \frac{\pi \eta_0}{\pi \eta_0^2} \right) \sin^2 \left( \frac{\pi p \eta_0}{2} \right) dp \]

where \( p_1 = -\frac{1}{2\eta_0} - \frac{1}{2} \left( \frac{1}{2\eta_0} \right)^2 \) and \( p_2 = -\frac{1}{2\eta_0} + \frac{1}{2} \left( \frac{1}{2\eta_0} \right)^2 \). A plot of \( f_1 \) as a function of \( \eta_0 \) given by Equations (C.13) and (C.13) is shown in Figure C.2.

Figure C.2 \( f_1 \) vs. \( \eta_0 \) 

89
Nonlinear regression analysis using Origin Lab [41] is used to curve-fit $f_1$ as

$$f_1 = 1 - 0.38687\eta_0 + 0.73854\eta_0^2 - 11.97946\eta_0^3 + 18.38428\eta_0^4 - 7.75357\eta_0^5$$  \hspace{1cm} (C.17)

Function $f_3$:

The $f_3$ function is given by

$$f_3 = \frac{4}{a^2} \int \int s_i \sin^2\left(\frac{\pi x}{a}\right) \sin^2\left(\frac{\pi y}{a}\right) dx dy$$  \hspace{1cm} (C.18)

Again this is evaluated in two regions.

Case A: $\eta_0 < a/2\sqrt{2}$

Let

$$\int \int s_i \sin^2\left(\frac{\pi x}{a}\right) \sin^2\left(\frac{\pi y}{a}\right) dx dy = \int \int s_i \sin^2\left(\frac{\pi x}{a}\right) \sin^2\left(\frac{\pi y}{a}\right) dx dy - \int \int s_i \sin^2\left(\frac{\pi x}{a}\right) \sin^2\left(\frac{\pi y}{a}\right) dx dy$$  \hspace{1cm} (C.19)

The first term in the right-hand side of the Equation (C.19) is

$$\int \int s_0 \sin^2\left(\frac{\pi x}{a}\right) \sin^2\left(\frac{\pi y}{a}\right) dx dy = \frac{a^2}{4}$$  \hspace{1cm} (C.20)

The second term in the right-hand side of Equation (C.19) is
\[
\int \int \sin^2 \left( \frac{\pi x}{a} \right) \sin^2 \left( \frac{\pi y}{a} \right) dx dy = \int \int \sin^2 \left( \frac{\pi x}{a} \right) dy \sin^2 dx
\]

\[
+ \int \int \frac{a}{\eta} \sin^2 \left( \frac{\pi y}{a} \right) \sin^2 dx + \int \int \frac{a}{\eta} \cos^2 \left( \frac{\pi x}{a} \right) dy \sin^2 \left( \frac{\pi y}{a} \right) dy \quad \text{(C.21)}
\]

The first term in the right-hand side of Equation (C.21) is

\[
I_{f_{II}} = \int \int \eta_0 \left[ \frac{2}{x-a/2} - \frac{2}{x-a/2} \sin \left( \frac{2\pi \eta_0}{a} \sqrt{1 - \left( \frac{x-a/2}{2} \right)^2} \right) \right] \sin^2 \left( \frac{\pi x}{a} \right) dx \quad \text{(C.22)}
\]

Let \( p = \frac{x-a/2}{\eta_0} \), then

\[
I_{f_{II}} = \eta_0 \left[ \frac{2}{x-a/2} \sqrt{1-p^2} - \frac{2}{a} \sin \left( \frac{2\pi \eta_0}{a} \sqrt{1-p^2} \right) \right] \cos^3 \left( \frac{\pi \eta_0 p}{2} \right) dp \quad \text{(C.23)}
\]

In terms of \( \eta_0 = \frac{2\eta_0}{a} \),

\[
I_{f_{II}} = a^2 \int \int \eta_0 \left[ \frac{2}{x-a/2} - \frac{2}{a} \sin \left( \frac{2\pi \eta_0}{a} \sqrt{1-p^2} \right) \right] \cos^3 \left( \frac{\pi \eta_0 p}{2} \right) dp \quad \text{(C.24)}
\]
It can be shown that each term in the right-hand side of Equation (C.21) are the same. Therefore,

$$\int \int \sin^2 \left( \frac{\pi x}{a} \right) \sin^2 \left( \frac{\pi y}{a} \right) \, dx \, dy = a^2 I_{f_3} \quad \text{(C.25)}$$

where

$$I_{f_3} = 4 \int \left[ \frac{\eta_0^2}{8} \sqrt{1 - p^2} - \frac{\eta_0}{8\pi} \sin \left( \pi \eta_0 \sqrt{1 - p^2} \right) \right] \cos^2 dp$$

Then from Equations (C.18), (C.19), (C.20) and (C.25),

$$f_3 = \frac{4}{a^2} \left[ \frac{a^2}{4} - a^2 I_{f_3} \right] = 1 - 4I_{f_3} \quad \text{(C.26)}$$

Case B: $\eta_0 > a/2\sqrt{2}$

As in the case of $f_1$, function $f_3$ is integrated over $S_1$. The $f_3$ function is also symmetric with respect to $x = a/2$ and $y = b/2$. Hence,

$$f_3 = \frac{16}{a^2} \int_0^{\frac{a}{2} - \eta_0} \int_0^{y_1} \sin^2 \left( \frac{\pi y}{b} \right) \, dy \cos^2 \left( \frac{\pi x}{a} \right) \, dx + \int_{\frac{a}{2} - \eta_0}^{\frac{a}{2}} \int_0^{y_1} \sin^2 \left( \frac{\pi y}{b} \right) \, dy \cos^2 \left( \frac{\pi x}{a} \right) \, dx$$

$$+ \int_{\eta_0}^{\eta_0} \int_{y_0}^{y_1} \sin^2 \left( \frac{\pi y}{b} \right) \, dy \cos^2 \left( \frac{\pi x}{a} \right) \, dx + \int_{\eta_0}^{\eta_0} \int_{y_0}^{\eta_0} \sin^2 \left( \frac{\pi y}{b} \right) \, dy \cos^2 \left( \frac{\pi x}{a} \right) \, dx$$

Where $y_1 = \eta_0 \sqrt{1 - \left( \frac{x - a/2}{\eta_0} \right)^2}$ and $y_3 = a/2 - \eta_0 \sqrt{1 - \left( \frac{x}{\eta_0} \right)^2}$

Let $p = \frac{y - b/2}{\eta_0}$ and $\overline{\eta_0} = \frac{2\eta_0}{b}$, then

92
\[ f_3 = 2 \int_{-\sqrt{\eta_0}}^{\eta_0} \left\{ \eta_0 - \eta_0^2 \sqrt{1 - \left( \frac{1}{\eta_0} + p \right)^2} \right\} \cos^2 \left( \frac{\pi p \eta_0}{2} \right) dp \\
+ 2 \int_{-\sqrt{\eta_0}}^{\eta_0} \left\{ \eta_0 - \eta_0^2 \sqrt{1 - \left( \frac{1}{\eta_0} + p \right)^2} \right\} \cos^2 \left( \frac{\pi p \eta_0}{2} \right) dp \\
+ \frac{\eta_0}{\pi} \sin(\pi \eta_0 \sqrt{1 - p^2}) \cos^2 \left( \frac{\pi p \eta_0}{2} \right) dp \]
\( \text{(C.28)} \)

where \( p_1 = -\frac{1}{2\eta_0} - \sqrt{\frac{1}{2} - \left( \frac{1}{2\eta_0} \right)^2} \) and \( p_2 = -\frac{1}{2\eta_0} + \sqrt{\frac{1}{2} - \left( \frac{1}{2\eta_0} \right)^2} \).

A plot of \( f_3 \) as a function of \( \eta_0 \) as defined by Equations (C.26) and (C.26) is shown in Figure C.3. This function was curve fitted with Origin Lab [41] to give

\[ f_3 = 1 - 0.38802\eta_0^2 + 3.94158\eta_0^4 - 11.54681\eta_0^6 + 7.01261\eta_0^8 \]
\( \text{(C.29)} \)
Functions \( Q_{dx} \) and \( Q_{dy} \):

The functions are defined by

\[
Q_{dx} = \frac{4}{a^2} \int \int Q_{x_0} \cos \left( \frac{x}{a} \right) \sin \left( \frac{y}{a} \right) dx dy
\]  

(C.30)

and

\[
Q_{dy} = \frac{4}{a^2} \int \int Q_{y_0} \cos \left( \frac{x}{a} \right) \sin \left( \frac{y}{a} \right) dx dy
\]  

(C.31)

These functions are equal and evaluated in two regions. Only function \( Q_{dy} \) is evaluated in this section.

Case A: \( \eta_0 \leq a/2\sqrt{2} \)

The \( Q_{dy} \) is evaluated over \( S_2 \). Substituting Equation (4.65) into Equation (C.31) gives
\[ Q_{dy} = \frac{8}{a^2} Q_0 \int_{-\eta_0}^{\eta_0} \int_0^{\frac{a-x}{\eta_0}} \tan \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{a} \right) \cos \left( \frac{\pi y}{a} \right) \frac{dy}{\sqrt{\tan^2 \left( \frac{\pi x}{a} \right) + \tan^2 \left( \frac{\pi y}{a} \right)}} \, dx \]  

(C.32)

Symmetry about \( y = a/2 \) is used in the above integral.

Let \( p = \frac{x-a/2}{\eta_0} \) and \( \bar{\eta}_0 = \frac{2\eta_0}{a} \)

\[ Q_{dy} = 2Q_0\bar{\eta}_0^2 \int_{-1}^{1} \int_0^{\sqrt{p^2-1}} \cos \left( \frac{\pi \bar{\eta}_0 p}{2} \right) \cos \left( \frac{\pi \bar{\eta}_0 \bar{y}}{2} \right) \frac{dy}{\sqrt{1 + \tan^2 \left( \frac{\pi \bar{\eta}_0 p}{2} \right) \tan^2 \left( \frac{\pi \bar{\eta}_0 \bar{y}}{2} \right)}} \, dp \]  

(C.33)

Case B: \( \eta_0 > \frac{a}{2\sqrt{2}} \)

When plastic regions overlap, the \( Q_{dx} \) and \( Q_{dy} \) are limited to the shaded zones shown in Figure C.4.

Figure C.4 Plastic shear regions dominated by \( Q_{dx} \) and \( Q_{dy} \).
The $Q_{xy}$ is given by

$$Q_{xy} = \frac{16}{a^2} Q_0 \left\{ x = \int_{\eta_0}^{\eta} \int_{-\eta_0}^{\eta_0} \frac{\tan \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi y}{a} \right)}{\sqrt{\tan^2 \left( \frac{\pi x}{a} \right) + \tan^2 \left( \frac{\pi y}{a} \right)}} \, dx \, dy + \int_{x_2}^{x_1} \frac{\tan \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi y}{a} \right)}{\sqrt{\tan^2 \left( \frac{\pi x}{a} \right) + \tan^2 \left( \frac{\pi y}{a} \right)}} \, dx \, dy \right\} (C.34)$$

Let $p = \frac{x-a}{2\eta_0}$ and $\bar{\eta}_0 = \frac{2\eta_0}{a}$, then

$$Q_{xy} = 4Q_0 \bar{\eta}_0^2 \left\{ \int_{0}^{\bar{\eta}_0} \int_{0}^{\bar{\eta}_1} \frac{\cos \left( \frac{\pi \bar{\eta}_0 p}{2} \right) \cos \left( \frac{\pi \bar{\eta}_0 \bar{y}}{2} \right)}{\sqrt{1 + \tan^2 \left( \frac{\pi \bar{\eta}_0 p}{2} \right) \tan^2 \left( \frac{\pi \bar{\eta}_0 \bar{y}}{2} \right)}} \, d\bar{y} \, dp + \int_{\bar{p}_2}^{\bar{p}_1} \int_{0}^{\bar{p}_1} \frac{\cos \left( \frac{\pi \bar{\eta}_0 p}{2} \right) \cos \left( \frac{\pi \bar{\eta}_0 \bar{y}}{2} \right)}{\sqrt{1 + \tan^2 \left( \frac{\pi \bar{\eta}_0 p}{2} \right) \tan^2 \left( \frac{\pi \bar{\eta}_0 \bar{y}}{2} \right)}} \, d\bar{y} \, dp + \int_{\bar{p}_1}^{1} \int_{0}^{\bar{p}_1} \frac{\cos \left( \frac{\pi \bar{\eta}_0 p}{2} \right) \cos \left( \frac{\pi \bar{\eta}_0 \bar{y}}{2} \right)}{\sqrt{1 + \tan^2 \left( \frac{\pi \bar{\eta}_0 p}{2} \right) \tan^2 \left( \frac{\pi \bar{\eta}_0 \bar{y}}{2} \right)}} \, d\bar{y} \, dp \right\} (C.35)$$

where $\bar{p}_1 = \frac{1}{2\bar{\eta}_0} - \frac{1}{2} \left( \frac{1}{2\bar{\eta}_0} \right)^2$ and $\bar{p}_2 = \frac{1}{2\bar{\eta}_0} + \frac{1}{2} \left( \frac{1}{2\bar{\eta}_0} \right)$. 

96
A plot of \( Q_{dx} / Q_0 \) as a function of \( \eta_0 \), is shown in Figure C.5. This plot is curve fitted to give

\[
Q_{dx} = Q_{dy} = Q_0 \left( 0.46085 \eta_0 - 1.74869 \eta_0^2 + 16.22462 \eta_0^3 - 23.01037 \eta_0^4 + 9.27461 \eta_0^5 \right) \tag{C.36}
\]

![Figure C.5 \( Q_{dx}/Q_0 \) vs. \( \eta_0 \)](image)

Function \( p_d \):

The \( p_d \) function is defined as

\[
p_d = -\frac{4}{a^2} \int \int_{S_2} \left[ \frac{\partial Q_{su}}{\partial x} + \frac{\partial Q_{tu}}{\partial y} \right] \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{a} \right) dx dy \tag{C.37}
\]

The \( p_d \) function is evaluated over \( S_2 \).
Case A: $\eta_0 \leq a/2\sqrt{2}$

From Equation (4.64)

$$\frac{\partial Q_{x_0}}{\partial x} = -\frac{Q_0 \pi}{a} \frac{\tan \left( \frac{\pi x}{a} \right) \tan \left( \frac{\pi y}{a} \right) \sec^2 \left( \frac{\pi y}{a} \right)}{\left[ \tan^2 \left( \frac{\pi x}{a} \right) + \tan^2 \left( \frac{\pi y}{a} \right) \right]^{1/2}}$$

(C.38)

From Equation (4.65)

$$\frac{\partial Q_{y_0}}{\partial y} = -\frac{Q_0 \pi}{a} \frac{\tan \left( \frac{\pi x}{a} \right) \tan \left( \frac{\pi y}{a} \right) \sec^2 \left( \frac{\pi y}{a} \right)}{\left[ \tan^2 \left( \frac{\pi x}{a} \right) + \tan^2 \left( \frac{\pi y}{a} \right) \right]^{1/2}}$$

(C.39)

Because of symmetry about $x = a/2$ and $y = a/2$,

$$\int \int_{s_2} \frac{\partial Q_{x_0}}{\partial x} \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{a} \right) dx dy = \int \int_{s_2} \frac{\partial Q_{y_0}}{\partial y} \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{a} \right) dx dy$$

(C.40)

Therefore,

$$p_s = \frac{16Q_0 \pi}{a^3} \int_{a/2 - \eta_0}^{a/2 + \eta_0} \int_0^{\eta_0} \frac{\tan \left( \frac{\pi x}{a} \right) \tan \left( \frac{\pi y}{a} \right) \sec^2 \left( \frac{\pi y}{a} \right)}{\left[ \tan^2 \left( \frac{\pi x}{a} \right) + \tan^2 \left( \frac{\pi y}{a} \right) \right]^{1/2}} dy dx$$

(C.41)

Let $p = \frac{x-a/2}{\eta_0}$ and $\overline{\eta}_0 = \frac{2\eta_0}{a}$, then
\[
p_d = \frac{8Q_0\pi}{a} \frac{1}{\bar{\eta}_0^2} \int_0^{1-x^2} \frac{1}{y^2} \tan^2\left(\frac{\pi\bar{\eta}_0 p}{2}\right) \tan^2\left(\frac{\pi\bar{\eta}_0 y}{2}\right) \cos\left(\frac{\pi\bar{\eta}_0 p}{2}\right) \left[1 + \tan^2\left(\frac{\pi\bar{\eta}_0 p}{2}\right) \tan^2\left(\frac{\pi\bar{\eta}_0 y}{2}\right)\right]^{1/2} \cos\left(\frac{\pi\bar{\eta}_0 y}{2}\right) d\bar{y} dp \quad (C.42)
\]

Case B: \( \eta_0 > a/2\sqrt{2} \)

Again using symmetry about \( x = a/2 \) and \( y = a/2 \), one gets

\[
p_d = \frac{32Q_0\pi}{a^3} \left\{ \int_{\bar{x}_1}^{\bar{x}_2} \int_0^{1-x^2} \tan\left(\frac{\pi x}{a}\right) \tan\left(\frac{\pi y}{a}\right) \sec^2\left(\frac{\pi y}{a}\right) \left[\tan^2\left(\frac{\pi x}{a}\right) + \tan^2\left(\frac{\pi y}{a}\right)\right]^{1/2} dy dx \\
+ \int_{\bar{x}_3}^{\bar{x}_4} \int_0^{1-x^2} \tan\left(\frac{\pi x}{a}\right) \tan\left(\frac{\pi y}{a}\right) \sec^2\left(\frac{\pi y}{a}\right) \left[\tan^2\left(\frac{\pi x}{a}\right) + \tan^2\left(\frac{\pi y}{a}\right)\right]^{1/2} dy dx \right\} \quad (C.43)
\]

where \( \bar{x}_1 = \frac{3a}{4} - \frac{1}{4} \sqrt{8\eta_0^2 - a^2} \) and \( \bar{x}_2 = \frac{3a}{4} + \frac{1}{4} \sqrt{8\eta_0^2 - a^2} \).

Let \( p = \frac{x - a/2}{\eta_0} \) and \( \bar{\eta}_0 = \frac{2\eta_0}{a} \), then
\[
p_d = \frac{8Q_o \pi}{a} \frac{\eta_0^2}{\bar{\eta}_0^4} \left[ \int_0^\pi \int_0^{\eta_0^2 - \eta_0^2} \left( \frac{\pi \eta_0 p}{2} \right)^2 \tan^2 \left( \frac{\pi \eta_0 \bar{y}}{2} \right) \cos \left( \frac{\pi \eta_0 p}{2} \right) \cos \left( \frac{\pi \eta_0 \bar{y}}{2} \right) d\bar{y} dp \right] + \int_{\bar{\eta}_1}^{\pi} \int_0^{\eta_0^2 - \eta_0^2} \left( \frac{\pi \eta_0 p}{2} \right)^2 \tan^2 \left( \frac{\pi \eta_0 \bar{y}}{2} \right) \cos \left( \frac{\pi \eta_0 p}{2} \right) \cos \left( \frac{\pi \eta_0 \bar{y}}{2} \right) d\bar{y} dp \]
(C.44)

where \( \bar{\eta}_1 = \frac{1}{2\eta_0} - \sqrt{\frac{1}{2} - \left( \frac{1}{2\eta_0} \right)^2} \) and \( \bar{p}_2 = \frac{1}{2\eta_0} + \sqrt{\frac{1}{2} - \left( \frac{1}{2\eta_0} \right)^2} \).

A plot of \( p_d \) as a function of \( \eta_0 \), defined in Equation (C.39) and (C.41) is shown in Figure C.6. The function is curve fitted to give

\[
p_d = \begin{cases}
\frac{Q_o}{a} \left( 0.00026\eta_0 - 0.01214\eta_0^2 + 0.1944\eta_0^3 - 1.4068\eta_0^4 + 4.574\eta_0^5 \right), \eta_0 \leq 0.25 \\
\frac{Q_o}{a} \left( 5.13 + 52.73\eta_0 - 199.78\eta_0^2 + 346.69\eta_0^3 - 269.6\eta_0^4 + 77.95\eta_0^5 \right), 0.25 < \eta_0 \leq 1
\end{cases}
(C.45)
\]
Figure C.6 $p_d$ vs. $\bar{\eta}_0$
APPENDIX D

IN-PLANE DEFORMATIONS

In Chapter IV it was assumed that the inertia terms governing in-plane motion can be neglected. Hence setting $\frac{\partial^2 u_0}{\partial t^2}$ and $\frac{\partial^2 v_0}{\partial t^2}$ equal to zero in Equations (4.34) and (4.35) yields

$$\frac{\partial I_z}{\partial x} = \frac{\partial I_z}{\partial y} = 0$$

(D.1)

where

$$\frac{\partial I_z}{\partial x} = \frac{\partial \varepsilon_{zz}}{\partial x} + \gamma \frac{\partial \varepsilon_{xy}}{\partial x}$$

Substituting the values for $\varepsilon_{zz}$ and $\varepsilon_{xy}$ in Equation (D.1) gives

$$\frac{\partial I_z}{\partial x} = \frac{\partial^2 u_0}{\partial x^2} + \left( \frac{\partial w}{\partial x} \right) \left( \frac{\partial^2 w}{\partial x^2} \right) + \gamma \left[ \frac{\partial^2 v_0}{\partial x \partial y} + \left( \frac{\partial w}{\partial y} \right) \left( \frac{\partial^2 w}{\partial x \partial y} \right) \right]$$

(D.2)
For a one-term approximation the in-plane deformations and transverse deflection were assumed to be of the form

\[ u_0 = U_{11} \sin \left( \frac{2\pi}{a} \right) \sin \left( \frac{\pi y}{b} \right) \]  
\hspace{1cm} \text{(D.3)}

\[ v_0 = V_{11} \sin \left( \frac{2\pi y}{b} \right) \sin \left( \frac{\pi x}{a} \right) \]  
\hspace{1cm} \text{(D.4)}

\[ w = W_{11} \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right) \]  
\hspace{1cm} \text{(D.5)}

Substituting Equations (D.3), (D.4) and (D.5) into Equation (D.2) and from Equation (D.1) one gets

\[ 0 = -U_{11} \left( \frac{2\pi}{a} \right)^2 \sin \left( \frac{2\pi}{a} \right) \sin \left( \frac{\pi y}{b} \right) + \nu W_{11} \left( \frac{\pi}{a} \right) \left( \frac{2\pi}{b} \right) \cos \left( \frac{2\pi y}{b} \right) \cos \left( \frac{\pi x}{a} \right) \]

\[ + W_{11} \left\{ \left( \frac{\pi}{a} \right)^3 \frac{1}{2} \sin \left( \frac{2\pi}{a} \right) \sin \left( \frac{\pi y}{b} \right) + \nu \left( \frac{\pi}{a} \right) \left( \frac{\pi}{b} \right)^2 \frac{1}{2} \sin \left( \frac{2\pi y}{b} \right) \cos \left( \frac{\pi x}{a} \right) \right\} \]  
\hspace{1cm} \text{(D.6)}

For special case \( \nu = 1 \) (as \( \nu^2 = A_{11}/A_{22} = 1 \), \( a = b \) and \( U_{11} = V_{11} \). Equation (D.6) becomes

\[ 0 = -2 \left( \frac{\pi}{a} \right)^2 U_{11} \left[ 2 \sin \left( \frac{2\pi}{a} \right) \sin \left( \frac{\pi y}{b} \right) - \cos \left( \frac{2\pi}{b} \right) \cos \left( \frac{\pi y}{a} \right) + \left( \frac{\pi}{a} \right)^3 \frac{1}{2} \sin \left( \frac{2\pi}{a} \right) \cos \left( \frac{2\pi y}{b} \right) \right] \]  
\hspace{1cm} \text{(D.7)}

While looking at the distribution, the maximum value of the right-hand side of Equation (D.7) is at \( x = 0 \) and \( y = b/2 \). Hence, evaluating Equation (D.7) at \( x = 0, y = b/2 \) gives
0 = -2 \left( \frac{\pi}{a} \right)^2 U_{11} \left[ 2 + \frac{1}{\sqrt{2}} \right] \left( -\frac{1}{2} \right)^3 W_{11}^2 \quad (D.8)

Solving Equation (D.8) gives $U_{11}$ as

\[ U_{11} = -\left( \frac{\pi}{a} \right) \frac{W_{11}^2}{8 + 2\sqrt{2}} \quad (D.9) \]

Similarly $V_{11}$ can be found as

\[ V_{11} = -\left( \frac{\pi}{b} \right) \frac{W_{11}^2}{8 + 2\sqrt{2}} \quad (D.10) \]
APPENDIX E

MAXIMUM AND MINIMUM NORMAL STRESS IN FACESHEET

In this section the location of the maximum and minimum normal stress in the sandwich panel will be shown. In Chapter VI it was shown that the maximum normal stress was at the center of the panel. The stress is given as

$$\sigma_x = \overline{Q}_{11} \left[ U_{11} \frac{2\pi}{a} \cos \left( \frac{2\pi}{a} \right) \sin \left( \frac{\pi y}{b} \right) - z \Gamma_{11} \frac{\pi}{a} \sin \left( \frac{\pi y}{b} \right) \sin \left( \frac{\pi y}{b} \right) + \frac{1}{2} \left( \frac{\pi}{a} \right)^2 \left( W_{11} \cos \left( \frac{\pi y}{b} \right) \sin \left( \frac{\pi y}{b} \right) \right)^2 \right] +$$

$$- \overline{Q}_{12} \left[ V_{11} \frac{2\pi}{a} \sin \left( \frac{\pi y}{b} \right) \cos \left( \frac{2\pi y}{b} \right) - z \Lambda_{11} \frac{\pi y}{a} \sin \left( \frac{\pi y}{b} \right) \sin \left( \frac{\pi y}{b} \right) + \frac{1}{2} \left( \frac{\pi b}{a} \right)^2 \left( W_{11} \sin \left( \frac{\pi y}{b} \right) \cos \left( \frac{\pi y}{b} \right) \right)^2 \right]$$

(E.1)

To find the maximum and minimum location of the normal stress one takes the first and second derivative of Equation (E.1) with respect to $x$ to give

$$\sigma_x = \overline{Q}_{11} \left[ -\Gamma_{11} \frac{\pi^3}{a} \cos \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi y}{b} \right) \frac{1}{a^2 b} - \frac{\pi^4 W_{11}}{a^3 b} \cos \left( \frac{\pi y}{b} \right) \sin \left( \frac{\pi y}{b} \right) - \frac{4\pi^4 U_{11}}{a^2 b} \cos \left( \frac{\pi y}{b} \right) \sin \left( \frac{\pi y}{b} \right) \right] +$$

$$- \overline{Q}_{12} \left[ -\Lambda_{11} \frac{\pi^3}{ab^2} \cos \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi y}{b} \right) \frac{1}{a b^2} - \frac{\pi^4 W_{11}}{a^3 b} \cos \left( \frac{\pi y}{b} \right) \sin \left( \frac{\pi y}{b} \right) - \frac{4\pi^4 V_{11}}{a b^2} \cos \left( \frac{\pi y}{b} \right) \sin \left( \frac{2\pi y}{b} \right) \right]$$

(E.2)
\[ \sigma_x^* = \frac{1}{2} \bar{Q}_{11} \left[ \pi^6 W_{11}^2 \frac{\pi x}{a} \frac{\sin \left( \frac{\pi y}{b} \right)}{a^2 b^2} + 8 \pi^5 U_{11} \frac{2 \pi x}{a} \frac{\sin \left( \frac{\pi y}{b} \right)}{a^3 b^3} - \frac{\Gamma_{11} \pi^3 z \frac{\pi x}{a} \frac{\sin \left( \frac{\pi y}{b} \right)}{a^3 b^3}}{a^4 b^4} \right] + \frac{1}{2} \bar{Q}_{12} \left[ \pi^6 W_{11}^2 \frac{\pi y}{b} \frac{\sin \left( \frac{\pi x}{a} \right)}{a^2 b^2} + 8 \pi^5 V_{11} \frac{2 \pi y}{b} \frac{\sin \left( \frac{\pi x}{a} \right)}{a^3 b^3} - \frac{\Lambda_{11} \pi^3 z \frac{\pi y}{b} \frac{\sin \left( \frac{\pi x}{a} \right)}{a^3 b^3}}{a^4 b^4} \right] \]

(E.3)

Now finding the value of first and second derivatives of the normal stress at different locations of the panel shows that the global maxima is at the center \((x=a/2, y=b/2)\) of the panel. The values of the first and second derivatives of the normal stress are provided in Table E.1.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
<th>(\sigma'_x)</th>
<th>(\sigma''_x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>(-\frac{\bar{Q}<em>{11} \Gamma</em>{11} \pi^3 z}{a^2 b} - \frac{\bar{Q}<em>{12} \Lambda</em>{11} \pi^3 z}{a b^2})</td>
<td>0</td>
</tr>
<tr>
<td>(a/2)</td>
<td>(b/2)</td>
<td>0</td>
<td>(-8 \frac{\bar{Q}<em>{11} \pi^5 U</em>{11}}{a^3 b} - \frac{8 \bar{Q}<em>{11} \Gamma</em>{11} \pi^5 V_{11}}{a^3 b^2} - \frac{8 \bar{Q}<em>{12} \pi^5 V</em>{11}}{a^3 b^3} - \frac{\bar{Q}<em>{12} \Lambda</em>{11} \pi^3 z}{a^2 b^3})</td>
</tr>
<tr>
<td>0</td>
<td>(b)</td>
<td>(\frac{\bar{Q}<em>{11} \Gamma</em>{11} \pi^3 z}{a^2 b} + \frac{\bar{Q}<em>{12} \Lambda</em>{11} \pi^3 z}{a b^2})</td>
<td>0</td>
</tr>
<tr>
<td>(a)</td>
<td>0</td>
<td>(\frac{\bar{Q}<em>{11} \Gamma</em>{11} \pi^3 z}{a^2 b} + \frac{\bar{Q}<em>{12} \Lambda</em>{11} \pi^3 z}{a b^2})</td>
<td>0</td>
</tr>
<tr>
<td>(a)</td>
<td>(b)</td>
<td>(-\frac{\bar{Q}<em>{11} \Gamma</em>{11} \pi^3 z}{a^2 b} - \frac{\bar{Q}<em>{12} \Lambda</em>{11} \pi^3 z}{a b^2})</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>(b/2)</td>
<td>(-\frac{1}{2} \frac{\bar{Q}<em>{11} \pi^6 W</em>{11}^2}{a b^3})</td>
<td>(\frac{1}{2} \frac{\bar{Q}<em>{11} \pi^6 W</em>{11}^2}{a^2 b^2} + \frac{8 \bar{Q}<em>{11} \pi^5 U</em>{11}}{a^3 b^2})</td>
</tr>
<tr>
<td>(a)</td>
<td>(b/2)</td>
<td>(\frac{1}{2} \frac{\bar{Q}<em>{12} \pi^6 W</em>{11}^2}{a b^3})</td>
<td>(-\frac{1}{2} \frac{\bar{Q}<em>{11} \pi^6 W</em>{11}^2}{a^2 b^2} - \frac{8 \bar{Q}<em>{11} \pi^5 U</em>{11}}{a^3 b^2})</td>
</tr>
</tbody>
</table>
It is clear from above table that the maximum normal stress is at the center of the panel. Hence it can be concluded that the normal stress is maximum at the center of the panel as was found in Table E.1. Table E.2 provides the normal stress values at different panel location.

Table E.2 Normal stress at different panel locations.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>$\sigma_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a/2</td>
<td>b/2</td>
<td>$\overline{Q}<em>{11} \left[ \frac{2\pi^2}{a^2} \frac{W</em>{r_1}^2}{(8+2\sqrt{2})} \pm \frac{\pi h_n}{a} \Gamma_{11} \right] + \overline{Q}<em>{12} \left[ \frac{2\pi^2}{b^2} \frac{W</em>{r_1}^2}{(8+2\sqrt{2})} \pm \frac{\pi h_n}{b} \Lambda_{11} \right]$</td>
</tr>
<tr>
<td>0</td>
<td>b</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>b</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>b/2</td>
<td>$\overline{Q}<em>{11} \left[ \frac{\pi^2}{a^2} \frac{W</em>{r_1}^2}{(8+2\sqrt{2})} - \frac{2}{(8+2\sqrt{2})} + \frac{1}{2} \right]$</td>
</tr>
<tr>
<td>a</td>
<td>b/2</td>
<td>$\overline{Q}<em>{11} \left[ \frac{\pi^2}{a^2} \frac{W</em>{r_1}^2}{(8+2\sqrt{2})} - \frac{2}{(8+2\sqrt{2})} + \frac{1}{2} \right]$</td>
</tr>
</tbody>
</table>

From Table E.2 it can be seen that the maximum normal stress is at the center of the panel at the bottom facesheet. Hence it can be concluded that bottom facesheet fails due to tensile failure.