EFFECT OF CONICITY AND PLY STEER ON LONG COMBINATION VEHICLE YAW PLANE MOTION

A Dissertation
Presented to
The Graduate Faculty of The University of Akron

In Partial fulfillment of the requirements for the Degree Doctor of Philosophy

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August 2011
EFFECT OF CONICITY AND PLY STEER ON
LONG COMBINATION VEHICLE YAW PLANE MOTION

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Dissertation

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The demand for heavy trucking capacity in the United States continues to grow, but the interstate highway infrastructure and the legislation regarding vehicle size and weight have not change for several decades. One possible solution is Long Combination Vehicles (LCV), which can carry more freight using fewer resources than a traditional tractor-trailer. Concerns regarding the safety and lateral stability of LCV’s exist, in particular with respect to the tires, which are highly non-linear and load-dependent components. Tires can also exhibit conicity and ply steer, which are residual lateral forces and aligning moments at zero slip angle. This implies a possible lateral motion in straight line driving, raising a concern regarding units of the vehicle leaving the intended lane of travel. A system of differential equations was derived for the triple trailer LCV system. These equations considered the mass and dimensions of the vehicle units, as well as simplified tire stiffness coefficients. The eigenvalues and eigenvectors of the system were determined, allowing a general assessment of the vehicle stability. A forced vibration analysis was then performed, applying the conicity forces and moments as a loading vector, with the lateral displacement at the end of the last trailer as the response of interest. Parametric studies were performed to assess the impact of variations of the force/moment location, the vehicle forward velocity, the vehicle mass, and the tire properties. The lateral displacement was effected by each of these, but the response for the cases examined was small relative to the width of the travel lane.
DEDICATION

Melissa, the love of my life. Where to start? Every horizontal surface in the house covered with books, paper, and drawings, and you just nod and smile. Giving me a kiss and hug every night before you went to bed, and encouraging me to finish. Driving while I type on the laptop in the passenger seat. Thank you so much for all the encouragement over the years, and for many times carrying more than your share of the load. I love you.

My parents, James and Phylis Patterson. I would never have been able to even think about attempting to do some of things I have done in my life, including this work, without the gifts you gave me as parents. You both showed me determination, perseverance, honesty, and the rewards of hard work. These are traits that allow both your children to be as successful as they are. Thank you, and I love you both.
ACKNOWLEDGMENT

The author gratefully acknowledges financial support from Mr. Joseph A. Petrolino at the National Transportation Research Center, Inc., under Grant NTRCI-11-40-001 (US Department of Transportation Award DTRT06G0043 is associated with this award).

There are several people I would like to acknowledge for their help in this research, and in the completion of my education.

Dr. Michelle Hoo Fatt served as my dissertation advisor, and also helped with my research with the National Transportation Research Center, Inc. (NTRCI). I also had the opportunity to take several of Dr. Hoo Fatt’s classes during my graduate studies. Thank you for your help and patience. Dr. S. Graham Kelly, who inspired my interest in mechanical vibrations. I took every course with your name next to it. Thank you for your patience and replies to my endless email questions. The remaining members of my Interdisciplinary Committee: Dr. Dane Quinn, Dr. Craig C. Menzemer, and Dr. Ali Hajjafar. I know all of you are busy, so I thank you for your participation and inputs.

Hendrickson Trailer Suspension Systems, The Boler Company, and Mr. John Boler, my employer for the last 24 years. Thank you for understanding the importance of continuous learning, for allowing scheduling flexibility so employees can get to class, and for providing an educational assistance program. There is no way I could get this far without those things you have provided.
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CHAPTER I

INTRODUCTION

In North America, a large portion of the economy is driven by heavy trucks. According to the Federal Highway Administration, in 2006 the U.S. trucking industry was used to ship almost 13 billion tons of cargo valued at nearly $9.8 trillion dollars. Close to nine million people are employed in this industry, which annually generates $650 billion dollars in revenues. This contributes to around five percent of the total annual US Gross Domestic Product (GDP). Although there are other shipping options, such as air, rail, and ship, 80% of the communities in the United States are served by trucks alone.

Although the use of heavy vehicles has continued to climb, the infrastructure and government regulation required to support it has not kept pace. Since 1980, the number of interstate lane miles has increased just 18%. However, in the last 20 years, the size of the US economy doubled. The rules regulating vehicle size and weight limits are also decades old, and are stricter than most western countries.

Many different combinations of trucks, tractors, and trailers are utilized to fulfill shipping needs, with many of the configurations determined by federal, state, and local requirements. The most basic of these is the federal “Bridge Law,” which is, among
other things, a regulation specifying the maximum load a specific axle on a vehicle may legally carry. This law has been a major driver behind the design of tractor-trailer combination vehicles on the road today, influencing many parameters, such as trailer length and axle spacing.

![Typical Tractor-Trailer](image)

Figure 1-1 Typical Tractor-Trailer.

For a typical long haul vehicle in the United States, a three axle tractor with a tandem rear suspension is used, as shown in Figure 1-1. A tandem suspension is a pair of axles grouped closely together, and is considered to be a single load carrying unit. This is usually combined with a 53 ft semi-trailer, also equipped with a tandem suspension. The trailer may be of the van, reefer, tanker, or flatbed variety, depending on the intended use and cargo needs. According to the Bridge Law, this vehicle is allowed 12,000 lb on the
steer axle of the tractor, and 34,000 lb on each tandem, resulting in an 80,000 lb Gross Vehicle Weight (GVW).

Regarding vehicle suspension and axle design, from the viewpoint of both the original equipment manufacturer (OEM) as well as tier one suppliers, the legal tandem load is not a primary concern. In fact, many suspensions, brakes, tires, and other components may be rated for use at 20,000 lb or more, which is above the requirements of the individual unit in a legal tandem suspension. This is partially due to the fact that many tandem suspensions systems are simply two single axle suspensions mounted close together on the vehicle, and that there are single axle tractors and trailers. This is particularly true for trailing arm air and mechanical spring suspensions, both of which are in wide spread use.

Industry tendency to design systems based on the capacity of a single suspension leads to different vehicle designs, as well as different applications of the Bridge Law. A configuration used in many western states is the “Long Combination Vehicle”, or LCV, as shown in Figure 1-2. This vehicle is comprised of a tractor utilizing the familiar 12,000 lb steer suspension, coupled with either a single or tandem rear suspension. If a single axle rear suspension is used on the tractor, it can have a legal capacity of 20,000 lb. Of more interest is the trailer configuration. Instead of the typical 53 ft semi-trailer, often two or three short “pup” trailers are used. These units are allowed a maximum length of 28.5 ft, and are most typically of the van type construction. In order to support the load at the kingpin of the second and third trailers, as well as to attach them to the leading units, a converter dolly is employed. A typical converter dolly is shown in Figure 1-3. Both the pup trailers and converter dollies may also be rated for 20,000 lb. Therefore, a
“doubles” LCV may have a legal GVW of 92,000 lb, while a “triples” LCV may have a legal GVW of 132,000 lb.

Figure 1-2 Long Combination Vehicles.

Figure 1-3 Converter Dolly.
The obvious advantage, in the eyes of the fleet operator, is the ability to carry more freight per vehicle employing the LCV configuration. This equates to less fuel being used, and also to fewer drivers being on the payroll. Both of these are large expenses to the fleet owner, and are typical targets of cost reduction efforts. Traffic congestion could also be reduced if LCV’s, with a higher legal load capacity, were allowed on more highways. Another benefit to using less fuel to move the same amount of freight is a reduction in vehicle emissions, including greenhouse gases.

There are, however, issues that need to be understood when considering these vehicles. The first is the fact that LCV’s are, by geometric definition, long vehicles. This means that these vehicles exhibit dynamic behavior different from a typical tractor semitrailer combination. One consequence, which may not be immediately obvious, is the difficulty in driving an LCV in reverse. This hinders their ability to utilize a traditional loading dock, or to operate in an urban environment.

Of more importance is the dynamic behavior of LCV’s on the highway. Due to their unique combination of independent units, and in particular the dollies, which employ a pintle hook connection, LCV’s will respond in a very different manner than standard tractor-trailers. This unique dynamic behavior has led to many studies on LCV’s over the years by private industry, academic institutions, and various government entities. It has also led to regional legislation regarding the use of LCV’s. In the Eastern United States, LCV’s are limited to use on interstate highways, with the triples version restricted to major routes such as Interstate 80. In Western states, triples LCV’s are more common, possibly due to the greater distance between major urban areas. Some of this government
regulation has been put in place due to concerns regarding the safe operation of LCV’s, many of which have a reputation for instability, even in straight line driving.

With the ever-increasing pressure to conserve and minimize the use of fossil fuels, coupled with the increased demand for trucking capacity, the federal government is revisiting the rules regarding the use of LCV’s. The ability to carry more cargo with fewer vehicles, therefore raising overall efficiency, is attractive to all parties. This has led to new investigations regarding the safe operation of LCV’s, and the desire to utilize new tools in the study of their performance on the highway. The results of these efforts, besides the obvious goal of developing safe vehicles, would be to educate the fleet owners and drivers on the proper operation of LCV’s, and to provide design guidance for the original equipment manufacturers. In addition, this information could provide scientific evidence to lawmakers regarding LCV safety, and would raise the comfort level of the general public.

One of the more recent endeavors to quantify the dynamic behavior of LCV’s has been undertaken by the National Transportation Research Center, Incorporated (NTRCI), which is affiliated with Oak Ridge National Laboratory (ORNL) located in Knoxville, Tennessee. The NTRCI has formed, and subsequently led the efforts of, a consortium of private industry and academic members. Over the last several years, the group has studied several topics in vehicle dynamics, including Heavy Truck Rollover and Electronic Stability Control, but all with traditional tractors in combination with a single trailer.

In 2011, the consortium reconvened with the goal of studying LCV’s. This decision was made in light of the renewed interest of government and industry. It also added two
new academic members: The University of Akron and Auburn University. The objective of the group was to establish baseline dynamic behavior for an LCV, and this was accomplished through the completion of two major tasks.

The first task was a data acquisition effort to measure the response of an LCV on the test track and on the highway. The location of this testing was the Auburn University National Center for Asphalt Testing (AU-NCAT) in Auburn, Alabama (Figure 1-4). The primary use of this facility is in the study of pavement construction methods and their response to fatigue loading from heavy trucks. However, its two-mile oval track configuration, allowing near highway speeds, is ideally suited for studying both straight line driving as well as steady state turns and lane change maneuvers. In addition, AU-NCAT employs triples LCV’s to apply the fatigue loads to the asphalt test sections.

Figure 1-4 AU-NCAT Triple LCV.

Although the trailers are actually intermodal containers coupled with container chassis, they were deemed representative and acceptable for the study by the consortium members. The on-highway portion of the data acquisition was performed on local
interstates with special permission of the Alabama Department of Transportation, and was intended to verify if the test track data was representative of typical usage.

The second major task was the development of analytical models of the subject vehicle. The purpose of this task was to create tools to help understand the vehicle’s behavior, and to investigate potential design changes. The consortium chose to approach this task utilizing two commercially available pieces of software.

The first model, shown in Figure 1-5, was created in the multi-body dynamics software “ADAMS” by consortium members from the University of Akron. ADAMS is a general purpose program allowing the user to model any dynamic system. It also has the ability to model non-linear entities, and to incorporate flexible components. However, since it is a general code, it has a fairly steep learning curve, especially when the system is as complicated as a full vehicle.

![Figure 1-5 ADAMS Model of AU-NCAT Triple LCV.](image)

The second model was built in the program “TruckSim” by consortium members from Clemson University’s International Center for Automotive Research (CU-ICAR). The TruckSim program is specifically designed to only model heavy trucks, and was initially developed at the University of Michigan Transportation Research Institute.
(UMTRI) located in Ann Arbor, Michigan. Although TruckSim allows the use of non-linear entities, it is strictly a rigid-body code. Although TruckSim is easier for the casual user to employ, it is restricted by the assumptions of the original programmers.

The NTRCI consortium chose to support both of the modeling efforts. This was decided in order to take advantage of the capabilities of each program, and to provide options for future modeling efforts.

The purpose of this research is to study the contribution of residual lateral forces and residual aligning moments to the straight line driving response of a triple trailer LCV. Chapter II provides the background for the research. It discusses past work in the vehicle dynamics field, as well as sources of the information required for the calculations. In particular, it will refer to the previous work which became the basis for this study. The methods used to formulate the problem, and perform the calculations, are developed in Chapter III. This chapter includes discussions on free-body diagrams, modeling assumptions, the development of the equations of motion, and their use in the frequency domain. For the sake of brevity, the mathematics are developed for a four degree of freedom system consisting of a tractor and single trailer. These ideas are then expanded to the full, twelve degrees-of-freedom system of the LCV in Chapter IV. The vehicle dimensions and parameters used in the initial matrix development are based on the vehicle tested by NTRCI at the Auburn test track. Chapter V applies the model of the triple trailer LCV in several parametric studies. The response of the system is calculated while varying the magnitude of the vehicle forward velocity, the load configuration in the trailers, the total mass, and the linear tire properties. The final chapter presents concluding remarks, and will also contain recommendations for future work.
CHAPTER II

BACKGROUND

The dynamics of heavy vehicles has been studied for many decades. Government, academia, and private industry have all contributed to the wealth of knowledge on this subject, often forming consortiums across many disciplines. The National Transportation Research Center, Incorporated (NTRCI), is one such body dedicated to these efforts. The University of Michigan Transportation Research Center, Incorporated (UMTRI) is another that has been a major source of vehicle dynamics research, including the development and commercialization of computer simulation software [15].

2.1 LCV Stability

Several studies have been performed on double trailer LCV’s [1-3, 5, 6, 10]. Many of these have focused on the dynamics of the pintle hook type connection between vehicle units. In fact, research regarding the replacement of this connection with a non-roll compliant system is common [17]. Studies by Fancher and Winkler [8], as well as Ervin et al. [9], have been published with compilations of the response of these vehicles. Generally speaking, the bulk of the research has focused on rollover and braking
performance. In the literature review of lateral stability research performed by Vlk [8], the vast majority of the studies covered these topics. This is due to the safety issues regarding rollover, which comprises the majority of fatal heavy vehicle accidents, and the substantial government regulation relative to braking performance.

One stability issue for LCV’s occurs in straight line driving. At increased speed, lateral oscillations of the vehicle trailing units can occur. Typically, the rearmost trailing unit will exhibit the highest magnitude of oscillation [12]. However, depending on the modes of vibration that are excited, any of the trailers or dollies could exhibit oscillations. This has resulted in the derogatory nickname “wiggle-wagons” for these vehicles. Figures 2-1 and 2-2 display two of these potential modes of vibration. The first shows a mode where the entire vehicle oscillates. The second is a mode where only the second dolly is excited.

Figure 2-1 Trailer Oscillations.

Figure 2-2 Dolly Oscillations.
The concern with these modes of vibration is that they will grow, and ultimately lead to instability and loss of control of the vehicle. This is the rearward amplification effect studied by Winkler at UMTRI [12] which shows that the lateral accelerations experienced at the tractor are amplified at the trailers. This effect is more pronounced with LCV’s. In extreme cases, this can lead to jack-knifing or rollover of the vehicle. Although the amplitude of these vibrations usually stay relatively low, the visual effect can be unnerving to the general public traveling on the highway near the vehicle. It can also contribute to the shifting of cargo within the Trailer itself, possibly resulting in damaged goods.

2.2 Yaw Plane Models

Lateral stability in the yaw plane has also been a research subject. The rearward amplification phenomena associated with tractor-trailers, and in particular LCV’s, is initially a yaw plane based event leading to rollover consideration [12]. Mallikarjunarao and Fancher [1], El-Gindy et al. [2], Jindra [3] and Hazemoto [10] have all developed mathematical models to study the lateral dynamic behavior of heavy vehicles, including double trailer LCV’s.
An example set of free-body diagrams, derived from the tractor-dolly-trailer vehicle configuration studied by El-Gindy et al. [2], is illustrated in Figure 2-3.

![Free Body Diagram of Tractor-Dolly-Trailer](image)

Figure 2-3 Free Body Diagram of Tractor-Dolly-Trailer [2].

The studies referenced above follow the same general analysis steps. First, a vehicle configuration is selected, which could include up to four units. Next, an appropriate set of degrees of freedom are selected to represent the response of interest. A free-body diagram of each unit of the vehicle is then developed, with two equations resulting from each diagram. One equation is a summation of forces, considering the mass, acceleration and tire and connection forces of each unit. The other is from a summation of moments, applying the inertia, yaw rates, and aligning moments at the tires. The number of equations is reduced by eliminating the forces at the vehicle connection points, and equations are added to the system relating the articulation angle between each vehicle unit. The equations are then expressed in terms of the selected degrees-of-freedom, like terms are collected, and the system is written in a matrix form. The eigenvalues of the
system are calculated and plotted in the complex plane, where an assessment of the stability of the system may be made by inspection.

Many of the researchers in the works cited earlier also calculated a time domain response. This was done through the numerical integration of the system equations, using the steer angle as the input to the system. Specific details regarding the method used for the integration were rarely specified.

2.3 Component Properties

When considering the individual units of the vehicle system, the mass, inertia, and dimensions of each relative to their center of gravity are important values. Many of the review papers listed these value in tabular form for use in the calculations, but not a detailed explanation of their source. Unless a 3D CAD model of the vehicle unit is available, these values can be difficult to obtain, especially inertia, which is the most difficult to measure. However, the researchers at UMTRI [22] published a document with mass, inertia, and dimensional information for tractors and trailers, which will prove useful in this study.

Tire properties also present a challenge. Even though almost all of the sources cited used a linear tire response to slip angle, the source was typically not documented. One of the few documented sources of tire data was developed by Tickling et al. [23]. Although this information is almost 40 years old, it provides a consistent set of data with which to compare.
2.4 Residual Lateral Forces and Aligning Moments

A potential contributor to this dynamic behavior could be a feature of the pneumatic tires on the vehicle. Traditionally, a “duals” tire configuration is employed at each axle end, except the tractor steer axle, resulting in up to 26 tires in the vehicle train. Tires are a complex, non-linear entity exhibiting load dependent behavior in three translational and three rotational degrees-of-freedom.

A common area of study for tires is in their generation of lateral forces and aligning moments (these are also referred to as torques, and the terms will be used interchangeably). The lateral direction is defined as the vector perpendicular to the direction of vehicle travel, and parallel to the ground plane. Aligning torques occur about an axis passing vertically through the tire. These forces and moments, which are generally studied when examining vehicle handling behavior, are generated when the tire experiences “slip” or when a “slip angle” is generated. The slip angle is the difference between the direction the vehicle is traveling and the direction in which the tire is being steered, and is illustrated in Figure 2-4. The slip angle is also a different value at each tire to ground interface, further complicating the study of their impact on vehicle behavior.
For small slip angles, the lateral force and aligning moment are linear and proportional, and intuitively would be zero when the tire is going “straight.” They are also dependent on the vertical load on the tire, which results in the generation of a family of curves for these properties. Similar families of curves would be generated for the two other forces and moments generated by the tire. Figures 2-5 and 2-6 illustrate how lateral force and aligning moment vary with both slip angle and vertical load.
Due to construction methods, and normal manufacturing variability, the tire data curves may not pass through the origin. Two phenomena can occur with tires that will cause a lateral force or aligning moment to be generated at zero slip: conicity and ply steer.

Conicity is caused by the tire being shaped like a cone instead of a true cylinder, and is illustrated in Figure 2-7. A tire with substantial conicity, if rolled freely on a flat surface, would not travel in a straight line, but instead on a curved path. This phenomena is caused by a residual lateral force or aligning moment. Conicity is easily identified on a plot of lateral force versus slip angle. Figure 2-8 is an exaggerated illustration where the lateral force versus slip angle curve, at a fixed vertical tire load, is essentially shifted.
On the other hand, ply steer is caused by the relative translation and rotation of individual layers of the tire construction materials, such as the steel belts. It also results in a lateral force or aligning torque being generated. These effects can also be detected on a plot of aligning torque vs. slip angle. In the literature, these effects are often referred to as residual lateral force (RLF) and residual aligning moment (RAM).

In the passenger car industry, tires are often sorted and installed on a vehicle such that the residual lateral forces and aligning moments will “balance” or “cancel out”
across a vehicle. The tires will be labeled appropriately by the manufacturer, either “white-wall side” or “serial side,” to assist in this effort. Figure 2-9 indicates some of the labels on a passenger car tire. In the test laboratory, care will be taken to note the orientation of the tire using the same identification labels, so that these effects can be quantified and assessed relative to the direction the tire is rolling. If care is not taken, the vehicle may exhibit poor handling characteristics such as “dog-tracking,” “drift,” “steering wheel pull” or “steering wheel torque.” All of these responses lead to customer dissatisfaction. Poor vehicle handling performance can also lead to concerns regarding potential safety risks. An example of the “dog-tracking” effect, in this case caused by conicity at the rear axle, is shown in Figure 2-10.

![Passenger Car Tire Labeling](image)

Figure 2-9 Passenger Car Tire Labeling [20].

On heavy trucks, especially trailers and dollies, these concerns are usually not taken into consideration. In fact, the trailers and dollies often have the poorest tires on the vehicle. Mismatched brands, or even worn steer axles tires, may end up in these positions on the vehicle. In addition, the use of retread tires, where a new tread is installed on a used tire carcass, can add additional irregularity to the residual lateral force
balance situation. One other result of this practice is the appearance of tire treads that have separated from the carcass causing a hazard on the highway. Treads in this state are referred to as “alligators,” and are shown in Figure 2-11.

Figure 2-10 The Effect of Rear Axle Conicity [16].

Figure 2-11 Truck Tire "Alligator."
CHAPTER III

PROBLEM FORMULATION

The dynamics of the LCV in the yaw plane can be described by a set of differential equations. This chapter will discuss the assumptions made in this type of analysis, and will describe the development of the equations of motion for a tractor and single trailer system. The dimensions of this vehicle were based on the triple trailer LCV as tested at AU-NCAT. The next chapter will discuss the application of these same methods to the LCV.

3.1 Notation

For both systems, the individual trailers, dollies, or the tractor may be referred to as a “unit” of the vehicle train. Each unit was given a number as shown in Table 3-1. Only Units 1 and 2, the tractor and first trailer, are relevant to this chapter.

Each axle of each unit was given a number, with the number 1 given to the axle closest to the most positive x-axis position on that vehicle unit. This resulted in a numbering system where particular distances, forces, and moments could be referenced
by a dual subscript first denoting the vehicle unit and then axle, i.e., the $j^{th}$ axle on the $i^{th}$ unit. For example, $F_{12}$ refers to the lateral tire force at the second axle of the first unit.

Table 3-1 Vehicle Unit Numbering.

<table>
<thead>
<tr>
<th>Unit</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tractor</td>
<td>1</td>
</tr>
<tr>
<td>Trailer 1</td>
<td>2</td>
</tr>
<tr>
<td>Dolly 1</td>
<td>3</td>
</tr>
<tr>
<td>Trailer 2</td>
<td>4</td>
</tr>
<tr>
<td>Dolly 2</td>
<td>5</td>
</tr>
<tr>
<td>Trailer 3</td>
<td>6</td>
</tr>
</tbody>
</table>

There were several simplified joints representing the connections between each vehicle unit. These were given alphabetic designations as shown in Table 3-2. Only Joint A is used in this present chapter.

Table 3-2 Connection Joint Alphabetic Designation.

<table>
<thead>
<tr>
<th>Designation</th>
<th>First Unit</th>
<th>Second Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Tractor</td>
<td>Trailer 1</td>
</tr>
<tr>
<td>B</td>
<td>Trailer 1</td>
<td>Dolly 1</td>
</tr>
<tr>
<td>C</td>
<td>Dolly 1</td>
<td>Trailer 2</td>
</tr>
<tr>
<td>D</td>
<td>Trailer 2</td>
<td>Dolly 2</td>
</tr>
<tr>
<td>E</td>
<td>Dolly 2</td>
<td>Trailer 3</td>
</tr>
</tbody>
</table>
By utilizing this notation, a logical system could be created for the designation of the forces, angles, and distances to these joints from each vehicle unit center of gravity. For example, $x_{6E}$ is the distance from the center of gravity on Unit 6 to Joint A.

A Cartesian coordinate system was used in the model. The positive $x$-axis of this system was aligned with the forward velocity of the vehicle. The positive $y$-axis was directed from the driver side of the vehicle toward the curb side, and was considered the lateral direction. In order to follow the right hand rule, the positive $z$-axis is therefore directed toward the ground.

3.2 Assumptions

Several assumptions were needed to be made in order to develop the equations of motion. These reflect simplifications to the mathematics involved, as well as to the physics of the system. The modeling assumptions were as follows:

1. The model is linear. Therefore, the input forces and moments stored in the load vector may be scaled.
2. The forward velocity, designated as “$u$,” is constant, and is the same for each unit in the vehicle.
3. All motion of the vehicle is restricted to the yaw plane, i.e., a plane parallel to the $x$-$y$ plane in the defined global coordinate system.
4. All units of the vehicle are considered to be rigid bodies.
5. The joints connecting the vehicle units are frictionless.
6. The vertical load at each axle is constant. There is no load transfer across a particular axle, or between units of the vehicle.

7. The tire cornering stiffness and aligning moment are linearly proportional to the slip angle.

8. There are no other significant tire forces in any other direction, such as from fore/aft driving or braking.

9. All distances are measured relative to the center of gravity of each unit.

10. Angular displacements are considered to be “small,” allowing the usual trigonometric simplifications.

It is also understood that some of the required properties will be estimated due to their lack of availability or proprietary nature. This introduces the potential for significant variation in results, and will be considered in detail in later chapters.

3.3 Equations of Motion

A set of free-body diagrams representing each unit of the vehicle were then developed. These diagrams represented the forces and moments acting on the vehicle unit, as well as the translational and rotational velocities and accelerations. Forces were summed relative to the lateral velocity of the unit, and moments were summed about a vertical axis through the center of gravity. The free-body diagrams, with the relevant quantities of interest, are shown in Figure 3-1. Each quantity is listed and described in Appendix A.
The result is two equations of motion per vehicle unit, for a total of four equations:

Tractor (Unit 1)

\[ m_1 \left( v_1 + u \cdot r_1 \right) = F_A + F_{11} + F_{12} + F_{13} + P_{11} + P_{12} + P_{13} \]  \hspace{1cm} (3.1a)

\[ I_1 \cdot r_1' = -F_A \cdot x_{1A} + F_{11} \cdot x_{11} - F_{12} \cdot x_{12} - F_{13} \cdot x_{13} \]
\[ + P_{11} \cdot x_{11} - P_{12} \cdot x_{12} - P_{13} \cdot x_{13} \]
\[ + M_{11} + M_{12} + M_{13} + T_{11} + T_{12} + T_{13} \]  \hspace{1cm} (3.1b)

Trailer (Unit 2)

\[ m_2 \left( v_2 + u \cdot r_2 \right) = -F_A + F_{21} + P_{21} \]  \hspace{1cm} (3.2a)
where $\dot{}'$ denotes the time derivative.

Through consideration of the kinematics of the trailer, Unit 2, expressions can be developed for its lateral velocity and acceleration

\[ v_2 = u \cdot \theta_A + v_1 - x_{1A} \cdot r_1 - x_{2A} \cdot r_2 \]  

(3.3)

\[ v_2' = u \cdot \theta_A' + v_1' - x_{1A} \cdot r_1' - x_{2A} \cdot r_2' \]  

(3.4)

The rate of change of the angle between the units relative to the yaw rates of each unit is given by

\[ \theta_A' = r_1 - r_2 \]  

(3.5)

The force at Joint A is found from Equation (3.2a) and substituted into the remaining equations. Then $v_2'$ is eliminated using Equation (3.4). A set of generalized coordinates were chosen to describe the yaw plane response of the vehicle. These are listed in Table 3-3, along with their units and symbol assignment. Only a subset of these were considered in this chapter. For the tractor single trailer system the generalized coordinates become

\[ x^T = \begin{pmatrix} v_1 & r_1 & r_2 & \theta_A \end{pmatrix} \]  

(3.6)
Table 3-3 Generalized Coordinates.

<table>
<thead>
<tr>
<th>Description</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tractor Lateral Velocity</td>
<td>$v_1$</td>
</tr>
<tr>
<td>Tractor Yaw Rate</td>
<td>$r_1$</td>
</tr>
<tr>
<td>Trailer 1 Yaw Rate</td>
<td>$r_2$</td>
</tr>
<tr>
<td>Dolly 1 Yaw Rate</td>
<td>$r_3$</td>
</tr>
<tr>
<td>Trailer 2 Yaw Rate</td>
<td>$r_4$</td>
</tr>
<tr>
<td>Dolly 2 Yaw Rate</td>
<td>$r_5$</td>
</tr>
<tr>
<td>Trailer 3 Yaw Rate</td>
<td>$r_6$</td>
</tr>
<tr>
<td>Articulation Angle at Joint A</td>
<td>$\theta_A$</td>
</tr>
<tr>
<td>Articulation Angle at Joint B</td>
<td>$\theta_B$</td>
</tr>
<tr>
<td>Articulation Angle at Joint C</td>
<td>$\theta_C$</td>
</tr>
<tr>
<td>Articulation Angle at Joint D</td>
<td>$\theta_D$</td>
</tr>
<tr>
<td>Articulation Angle at Joint E</td>
<td>$\theta_E$</td>
</tr>
</tbody>
</table>

The goal was then to write the equations of motion in terms of the generalized coordinates. This would then allow the equations to be expressed in the following matrix form:

$$[M]\{x^f\}=[S]\{x\}+[F]$$  \hspace{1cm} (3.8)
where \([M]\) and \([S]\) are coefficient matrices, \(\{x'\}\) is the vector of the time derivatives of the generalized coordinates, \(\{x\}\) is vector of the generalized coordinates, and \(\{F\}\) is the load vector. This is analogous to a second-order system reduced to a first-order system.

In order to do this, the force, moment, velocity, and acceleration terms for each unit needed to be defined. For the lateral forces, a linear relationship relative to the slip angle is defined. Recalling the unit and axle naming convention described above, and referring to Appendix A for a description of the remaining variable definitions, a general tire lateral force expression is

\[
F_{ij} = -C_{Fi} \cdot \alpha_{ij} 
\]

where \(C_{Fi}\) is the cornering force coefficient and \(\alpha_{ij}\) is slip angle.

The aligning moment is comprised of two components. The first is also a linear relationship relative to the slip angle, while an additional term exists on axles with dual tires. For axles with single tires,

\[
M_{ij} = C_{Mij} \cdot \alpha_{ij} 
\]

(3.10)

For axles with dual tires,

\[
M_{ij} = C_{Mij} \cdot \alpha_{ij} - \frac{D_{ij}}{u} \cdot S_{ij} \cdot r_i 
\]

(3.11)
where $C_{Mij}$ is the aligning moment coefficient, $D_{ij}$ is the dual tire spacing, and $C_{Sij}$ is the circumferential stiffness coefficient. The coefficients $C_{Fij}$, $C_{Mij}$, and $C_{Sij}$ represent the tire properties. These are linear simplifications of complex, non-linear behavior.

The slip angles, which are used in the above relationships, are expressed in terms of the velocities, yaw rates, and articulation angle. For the steer axle on the tractor, this results in

$$\alpha_{11} = \left( \frac{v_1 + x_{11} \cdot r_1}{u} \right)$$

(3.12)

For the remaining axles on the vehicle, the relationship is

$$\alpha_{ij} = \left( \frac{v_i - x_{ij} \cdot \Gamma_i}{u} \right)$$

(3.13)

The above equations will also require the lateral velocity and acceleration of the trailer, which is Unit 2. These quantities are then substituted into the system Equations (3.1a), (3.1b), (3.2b), and (3.5). Upon rearranging these expressions, like terms, relative to the generalized coordinates, may now be collected. In addition, terms for the vector of the time derivative of the generalized coordinates are moved to the left hand side of each equation. The terms for the matrix $M$ become
and the individual coefficients of the matrix $S$ become

\[
S_{1,1} = \left( -\frac{C_{F11}}{u} \right) - \frac{C_{F12}}{u} - \frac{C_{F21}}{u} - \frac{C_{F13}}{u}
\]

\[
S_{1,2} = \left( \frac{C_{F12} \cdot x_{12}}{u} \right) - \frac{C_{F11} \cdot x_{11}}{u} + \frac{C_{F13} \cdot x_{13}}{u} + \frac{C_{F21} \cdot x_{1A}}{u} - m_1 \cdot u
\]

\[
S_{1,3} = \left[ \frac{C_{F21} \cdot (x_{21} + x_{2A})}{u} \right] - m_2 \cdot u
\]

\[
S_{1,4} = \left( -C_{F21} \right)
\]

\[
S_{2,1} = \left( \frac{C_{M11}}{u} \right) + \frac{C_{M12}}{u} + \frac{C_{M13}}{u} - \frac{C_{F11} \cdot x_{11}}{u}
\]

\[
+ \left( \frac{C_{F12} \cdot x_{12}}{u} + \frac{C_{F13} \cdot x_{13}}{u} + \frac{C_{F21} \cdot x_{1A}}{u} \right)
\]

\[
S_{2,2} = \left( \frac{C_{M11} \cdot x_{11}}{u} \right) - \frac{C_{F12} \cdot x_{12}^2}{u} - \frac{C_{F13} \cdot x_{13}^2}{u}
\]

\[
+ \left( -\frac{C_{F21} \cdot x_{1A}^2}{u} - \frac{C_{F11} \cdot x_{11}^2}{u} - \frac{C_{M12} \cdot x_{12}}{u} \right)
\]

\[
+ \left( -\frac{C_{M13} \cdot x_{13}}{u} - \frac{C_{S12} \cdot D_{12}^2}{u} - \frac{C_{S13} \cdot D_{13}^2}{u} \right)
\]
The force vector containing the residual lateral force is

\[
F = \begin{bmatrix}
(P_{11} + P_{12} + P_{21} + P_{13}) \\
p_{11}x_{11} - p_{12}x_{12} - p_{13}x_{13} - p_{21}x_{1A} - T_{12} - T_{13} - T_{11} \\
-T_{21} - p_{21}(x_{21} + x_{2A}) \\
0
\end{bmatrix}
\]
At this point, appropriate values for the vehicle mass properties and dimensions, as well as the tire properties, are specified. These values were defined utilizing the parameters from the AU-NCAT vehicle for the mass, and the previously mentioned tires from UMTRI data [23] that were scaled to match the AU-NCAT tire loads. This scaling will be considered in detail in a later chapter. In addition, the forward velocity, $u$, is specified, for this example, as 50 mph or 880 in/sec. The following matrices are then calculated:

$$\mathbf{M} = \begin{pmatrix} 211 \text{ lb s}^2 \text{ in} & -10767 \text{ lb s}^2 & -9105 \text{ lb s}^2 & 110918 \text{ lb s} \\ -10767 \text{ lb s}^2 \text{ in} & 1238510 \text{ lb s}^2 & 777785 \text{ lb s}^2 & -9475203 \text{ lb s} \\ -9105 \text{ lb s}^2 \text{ in} & 777785 \text{ lb s}^2 & 1311392 \text{ lb s}^2 & -8012300 \text{ lb s} \\ 0 \text{ lb s}^2 \text{ in} & 0 \text{ lb s}^2 & 0 \text{ lb s}^2 & 1 \text{ lb s} \end{pmatrix}$$

(3.17)

$$\mathbf{S} = \begin{pmatrix} -445 \text{ lb s} \text{ in} & -45706 \text{ lb s} & -85455 \text{ lb s} & -107400 \text{ lb s} \\ 29498 \text{ lb s} \text{ in} & -4109376 \text{ lb s} & 7299984 \text{ lb s} & 9174664 \text{ lb s} \\ 25723 \text{ lb s} \text{ in} & -2197359 \text{ lb s} & 2612034 \text{ lb s} & 22635953 \text{ lb s} \\ 0 \text{ lb s} \text{ in} & 1 \text{ lb s} & -1 \text{ lb s} & 0 \text{ lb s} \end{pmatrix}$$

(3.18)
3.4 Free Vibration Analysis

When \( \{F\}=0 \), the solutions represent the free vibration portion of the analysis. With them, the eigenvalues and eigenvectors of the system may be calculated. This is done by solving the following:

\[
(M^{-1} S) x = \lambda x
\]

where \( \lambda \) are the eigenvalues.

Although there are several ways to accomplish this task, the software package MATHCAD was used to directly calculate the eigenvalues

\[
\lambda = \begin{bmatrix}
-3.0109 - 4.4006i \\
-3.0109 + 4.4006i \\
-2.2560 - 2.3299i \\
-2.2560 + 2.3299i
\end{bmatrix} \frac{1}{s^2}
\]

The eigenvalues in this case are found to be complex conjugate pairs. The plot for these eigenvalues is shown in Figure 3-2, which generates several topics of interest.
First, complex conjugate eigenvalues suggest the system will experience a damped, oscillatory response. This is regardless of the fact that physical viscous damping was not considered. This damping effect corresponds to the negative real part of the complex eigenvalues.

The second item of interest has to do with the assessment of the stability of the system. When these eigenvalues are plotted on the complex plane, they all lie to the left of the imaginary (vertical) axis. If all the eigenvalues for a system are located on this half of the complex plane it is considered an indicator of a stable system. Systems with values on the right-hand side indicate an unstable system. It is possible that a particular combination of mass and stiffness could lead to an unstable system [1].

Figure 3-2 Eigenvalues of Tractor-Trailer Model.
Finally, the proximity of the eigenvalues to the imaginary axis is also an indication of the damping level for that mode. Values close to the axis will experience more oscillation and a longer decay, while values further away from the axis will damp out quickly. Figure 3-3 shows the eigenvalues for an example system where the trailer mass was increased by a factor of ten, while the tire stiffness was decreased by a factor of ten. The system has one mode which is undamped and another that is unstable.

Figure 3-3 Example of Unstable and Undamped Modes

It is important to note that all of this information regarding the response of the system is available before attempting to solve the system of equations in the time domain. This fact makes this a powerful technique that can yield useful information at a minimum of computational effort.
Referring back to Figure 3-2, the plot would indicate that the system is stable, since all the real components of the eigenvalues are negative, and that the system will exhibit some damping since the values are not near the vertical axis. Note that conicity and ply steer do not affect the stability or damping of the system.

3.5 Forced Vibration Analysis

At this point, the analysis diverged from the previous research [1, 2]. Instead of proceeding to numerical integration of the system equations, a forced vibration problem was formulated. By utilizing modal analysis, the system response could be calculated using the frequency domain information: namely the eigenvalues and corresponding eigenvectors. The residual lateral force and aligning moment were considered to be the sole contributors to the load vector, allowing the potential assessment of different configurations of these forces.

To pursue this analysis, the eigenvectors of the corresponding eigenvalues were required. Since the eigenvectors are ratios that can only be calculated to a multiplicative constant, they also require normalization. This was accomplished by normalizing to the mass matrix, which is defined mathematically as

\[ X_i^T \cdot M \cdot X_i = 1 \]  \hspace{1cm} (3.21)

The resulting eigenvectors are
The system response is determined through the appropriate scaling and combination of the eigenvectors, which are also known as mode shapes. This was done through the use of the expansion theorem and the concept of principal coordinates. The mathematical definition of principal coordinates states that they are the coordinates in which the system is statically and dynamically uncoupled [19]. This fact aids in finding the system response via the following relations:

\[
X_1 = \begin{pmatrix}
-0.0152 - 0.0113i \\
0.0005 - 0.0010i \\
-0.0026 + 0.0008i \\
-0.0000 + 0.0007i
\end{pmatrix}
\quad (3.22)
\]

\[
X_2 = \begin{pmatrix}
-0.0152 + 0.0113i \\
0.0005 + 0.0010i \\
-0.0026 - 0.0008i \\
-0.0000 - 0.0007i
\end{pmatrix}
\quad (3.23)
\]

\[
X_3 = \begin{pmatrix}
0.0194 + 0.0038i \\
0.0001 + 0.0006i \\
-0.0001 + 0.0010i \\
0.0000 + 0.0001i
\end{pmatrix}
\quad (3.24)
\]

\[
X_4 = \begin{pmatrix}
0.0194 - 0.0038i \\
0.0001 - 0.0006i \\
-0.0001 - 0.0010i \\
0.0000 - 0.0001i
\end{pmatrix}
\quad (3.25)
\]
\[ x(t) = \sum_i \left( c_i(t) \cdot X_i \right) \]  \hfill (3.26)

where \( x(t) \) represents one of the desired generalized coordinates and

\[ c_i'(t) - \lambda_i c_i(t) = g_i(t) \]  \hfill (3.27)

\[ g_i(t) = X_i^T F(t) \]  \hfill (3.28)

These relations can also be used to develop the mode shapes for this system as shown Figures 3-4 through 3-5. These were calculated by applying the eigenvectors to Equation (3.26) with the principal coordinates specified as

\[ c_i(t) = 1 \cdot \exp(\lambda t) \]  \hfill (3.29)

where the “1” is replaced with an arbitrary scale factor. In these images, the displacement and rotation of the tractor is ignored to emphasize the motion of the trailer.

Figure 3-4 Tractor Trailer Mode Shape 1.
In a forced vibration problem, the principal coordinates are a function of time. These functions may be calculated using the convolution integral. The convolution integral is a closed-form solution to the differential equations of a system subjected to a general excitation. It computes the homogeneous and particular solutions simultaneously, and the initial conditions are incorporated in its derivation. The correct form of the convolution integral for this type of analysis is

\[
c_i(t) = \int_0^t \left( g_t f(\tau) \right) \exp \left[ \lambda (t - \tau) \right] d\tau
\]  

(3.30)

In Equation (3.30), there is a function \( f(\tau) \). In traditional applications of the convolution integral, this is often a sine function. Since a constant force is being applied, this becomes

\[
f(\tau) = \begin{cases} 
1, & \tau \geq 0 \\
0, & \tau < 0
\end{cases}
\]  

(3.31)
The convolution integral is then reduced to

\[ c_i(t) = \int_0^t (g_i) \cdot \exp[\lambda_i(t - \tau)] \, d\tau \]  

(3.32)

3.6 Load Vector

Equations (3.26) - (3.32) require the evaluation of the load vector. The loading in this analysis will result from considering the residual lateral forces and aligning moments caused by the conicity and ply steer of the tires only. Therefore, the load vector is

\[
F := \begin{bmatrix}
(P_{11} + P_{12} + P_{21} + P_{13}) \\
(P_{11} \cdot x_{11} - P_{12} \cdot x_{12} - P_{13} \cdot x_{13} - P_{21} \cdot x_{1A}) \\
+ (-T_{12} - T_{13} - T_{11}) \\
-T_{21} - P_{21} \cdot x_{2A} \\
0
\end{bmatrix}
\]  

(3.33)

where \( P_{ij} \) is the residual lateral force and \( T_{ij} \) is the residual aligning moment.

As stated in the introduction, these residual forces and moments may be balanced or unbalanced. If they are balanced, the vehicle should have a net zero force or moment summation. For example, Figure 3-6 shows a plan view of the vehicle. The red arrows represent the direction the total residual lateral force or moment is acting at each tire. In
this configuration, the forces are balanced, and would represent a situation where the tires were carefully sorted and installed.

Figure 3-7 shows the simplest possible situation for imbalance, which is having all the residual lateral force pointed in the same direction. The residual forces that do not change direction from the balanced case are shown as dimmed in order to highlight the proposed change.

Of more interest would be cases where the imbalance is focused on the rearmost unit, as in Figure 3-8. Recalling the rearward amplification effect exhibited by these vehicles, this should be the mode that is most likely to be excited.

![Figure 3-6 Residual Lateral Force, Balanced.](image-url)
As with other tire properties, the magnitude of the residual lateral force and aligning moment varies with vertical load. For this analysis, these values were assumed to be on the order of 1% of the vertical tire load. The tire loads are based on measurements taken of the AU-NCAT vehicle. In addition, recall that the model is linear. Therefore, once the system matrices are defined, and a particular load vector is established, the response may be directly scaled to investigate other residual force and aligning moment magnitudes.

With these values in place, the load vector was calculated as
The coefficients required by the convolution integral can now be determined as

\[
F = \begin{pmatrix} 195 \\ -1385 \\ -3576 \\ 0 \end{pmatrix} \text{ lb}
\]  

(3.34)

The convolution integrals may now be calculated to determine the generalized coordinates as functions of time, and the expansion theorem may be applied. This results in the following expressions for the response of each generalized coordinate:

\[
g_1 := X_1^T \cdot F = 5.695 - 3.845i \text{ lb}
\]  

(3.35)

\[
g_2 := X_2^T \cdot F = 5.695 + 3.845i \text{ lb}
\]  

(3.36)

\[
g_3 := X_3^T \cdot F = 4.013 - 3.559i \text{ lb}
\]  

(3.37)

\[
g_4 := X_4^T \cdot F = 4.013 + 3.559i \text{ lb}
\]  

(3.38)

The convolutions integrals may now be calculated to determine the generalized coordinates as functions of time, and the expansion theorem may be applied. This results in the following expressions for the response of each generalized coordinate:

\[
v_1(t) = c_1(t) \cdot X_{11} + c_2(t) \cdot X_{21} + c_3(t) \cdot X_{31} + c_4(t) \cdot X_{41}
\]  

(3.39)

\[
r_1(t) = c_1(t) \cdot X_{12} + c_2(t) \cdot X_{22} + c_3(t) \cdot X_{32} + c_4(t) \cdot X_{42}
\]  

(3.40)

\[
r_2(t) = c_1(t) \cdot X_{13} + c_2(t) \cdot X_{23} + c_3(t) \cdot X_{33} + c_4(t) \cdot X_{43}
\]  

(3.41)

\[
\theta_A(t) = c_1(t) \cdot X_{14} + c_2(t) \cdot X_{24} + c_3(t) \cdot X_{34} + c_4(t) \cdot X_{44}
\]  

(3.42)

A plot of each response is shown in Figures 3.9 – 3.12.
Figure 3-9 Tractor Lateral Velocity – Time Response.

Figure 3-10 Tractor Yaw Rate – Time Response.
Figure 3-11 Trailer 1 Yaw Rate – Time Response.

Figure 3-12 Articulation Angle A – Time Response.
Once the generalized coordinates are found, other quantities may be calculated. By utilizing the system dimensions, and the calculated articulation angle at Joint A, the lateral displacement at the end of the trailer can be determined. This relationship is given by

\[ y_2(t) = (x_{2A} + x_{2B}) \cdot \sin(\theta_A(t)) \]  

(3.43)

A plot of this response is shown in Figure 3-13.

![Figure 3-13 Trailer 1 Lateral Displacement – Time Response.](image)

This response of the vehicle is of interest because it provides an indication of how far out of its lane of travel the trailer may move. This is both a safety issue and a perception
Drivers in adjacent lanes may be intimidated by such a response, and this contributes to a poor image for these vehicles.

It should be noted that the response shown in Figure 3-13 was a typical result in this study. The first part of the chart shows an oscillatory transient. As will be shown in a later chapter, as the parameters of the vehicle, the tires, and the forward velocity were changed, the damped response also changed. This should not be surprising, as all of those factors are found in the coefficients defining the system matrices, which subsequently govern the eigenvalues and eigenvectors. This response is followed by a behavior that reduces to a steady-state offset. This should also be expected, since the forcing function is a constant value.

In this system, with these vehicle parameters, both the transient and steady-state responses are small. In general, single trailer vehicles do not exhibit the magnitude of rearward amplification experienced by double or triple trailer LCV’s. The analysis for this vehicle would therefore conclude that the residual lateral forces and aligning moments are not of a magnitude significant enough to develop an unsafe response as measured by lateral displacement of the vehicle.
CHAPTER IV

TRIPLE TRAILER LCV MODEL

The analysis methods developed in the previous chapter will now be extended to a model of a triple trailer configuration LCV. The dimensions and physical properties used for the parameters describing the system will be based on the vehicle tested at AU-NCAT by the NTRCI consortium in May of 2011. This consists of a tandem drive axle tractor, two converter dollies, and three intermodal containers attached to individual container chassis trailers. The intermodal containers are loaded with “jersey barriers,” which are large concrete structures used to control traffic flow in construction zones.

It is understood that some of the required properties will be estimated due to their lack of availability or proprietary nature. This introduces the potential for significant variation in results, and will be considered in detail in later chapters.

4.1 Notation

The vehicle unit, connection point, and axle number assignment system explained in the previous chapter is also applied here. Tables 3-1 and 3-2 list these designations. The
Cartesian coordinate system described earlier will also be used. In addition, the same set of assumptions regarding the physics and mathematics of the system will be applied.

4.2 Equations of Motion

The differential equations describing the system are again written based on free-body diagrams of each unit in the vehicle train. These diagrams represent the forces and moments acting on the vehicle unit, as well as the translational and rotational velocities and accelerations. Forces are summed relative to the lateral velocity of the unit, and moments are summed about a vertical axis through the center of gravity. The free-body diagrams, with the relevant dimension and quantities of interest, are shown in Figure 4-1. A description of each quantity can be found in Appendix A.
Figure 4-1 Triple Trailer LCV Free-Body Diagram.
The result is two equations per vehicle unit, for a total of 12 equations:

Tractor (Unit 1)

\[ m_1 (v_1' + u r_1) = F_A + F_{11} + F_{12} + F_{13} + P_{11} + P_{12} + P_{13} \]  
\[ (4.1a) \]

\[ I_1 r_1' = -F_A \cdot x_{1A} + F_{11} \cdot x_{11} - F_{12} \cdot x_{12} - F_{13} \cdot x_{13} \]
\[ + P_{11} \cdot x_{11} - P_{12} \cdot x_{12} - P_{13} \cdot x_{13} \]
\[ + M_{11} + M_{12} + M_{13} + T_{11} + T_{12} + T_{13} \]  
\[ (4.1b) \]

Trailer 1 (Unit 2)

\[ m_2 (v_2' + u r_2) = -F_A + F_B + F_{21} + P_{21} \]  
\[ (4.2a) \]

\[ I_2 r_2' = -F_{21} \cdot x_{21} - F_A \cdot x_{2A} - F_B \cdot x_{2B} + M_{21} - P_{21} \cdot x_{21} + T_{21} \]  
\[ (4.2b) \]

Dolly 1 (Unit 3)

\[ m_3 (v_3' + u r_3) = -F_B + F_C + F_{31} + P_{31} \]  
\[ (4.3a) \]

\[ I_3 r_3' = -F_{31} \cdot x_{31} - F_B \cdot x_{3B} - F_C \cdot x_{3C} + M_{31} - P_{31} \cdot x_{31} + T_{31} \]  
\[ (4.3b) \]

Trailer 2 (Unit 4)

\[ m_4 (v_4' + u r_4) = -F_C + F_D + F_{41} + P_{41} \]  
\[ (4.4a) \]

\[ I_4 r_4' = -F_{41} \cdot x_{41} - F_C \cdot x_{4C} - F_D \cdot x_{4D} + M_{41} - P_{41} \cdot x_{41} + T_{41} \]  
\[ (4.4b) \]

Dolly 2 (Unit 5)

\[ m_5 (v_5' + u r_5) = -F_D + F_E + F_{51} + P_{51} \]  
\[ (4.5a) \]
\[
I_{5}r_{5} = -F_{51}x_{51} - F_{D}x_{5D} - F_{E}x_{5E} + M_{51} - P_{51}x_{51} + T_{51} \quad (4.5b)
\]

**Trailer 3 (Unit 6)**

\[
m_{6}(v_{6} + u_{r_{6}}) = -F_{E} + F_{61} + P_{61} \quad (4.6a)
\]

\[
I_{6}r_{6} = -F_{61}x_{61} - F_{E}x_{6E} + M_{61} - P_{61}x_{61} + T_{61} \quad (4.6b)
\]

Through consideration of the kinematics of the trailers and dollies, the lateral velocity and acceleration of each unit is given by

\[
v_{2} = u\theta A + v_{1} - x_{1A}r_{1} - x_{2A}r_{2} \quad (4.7a)
\]

\[
v_{3} = u(\theta A + \theta B) + v_{1} - x_{1A}r_{1} - (x_{2A} + x_{2B})r_{2} - x_{3B}r_{3} \quad (4.7b)
\]

\[
v_{4} = u(\theta A + \theta B + \theta C) + v_{1} - x_{1A}r_{1} - (x_{2A} + x_{2B})r_{2} - (x_{3B} + x_{3C})r_{3} - x_{4C}r_{4} \quad (4.7c)
\]

\[
v_{5} = u(\theta A + \theta B + \theta C + \theta D) + v_{1} - x_{1A}r_{1} - (x_{2A} + x_{2B})r_{2} - (x_{3B} + x_{3C})r_{3} - (x_{4C} + x_{4D})r_{4} - x_{5D}r_{5} \quad (4.7d)
\]

\[
v_{6} = u(\theta A + \theta B + \theta C + \theta D + \theta E) + v_{1} - x_{1A}r_{1} - (x_{2A} + x_{2B})r_{2} - (x_{3B} + x_{3C})r_{3} - (x_{4C} + x_{4D})r_{4} - (x_{5D} + x_{5E})r_{5} - x_{6E}r_{6} \quad (4.7e)
\]

\[
v_{2}' = u\theta A' + v_{1}' - x_{1A'}r_{1}' - x_{2A'}r_{2}' \quad (4.8a)
\]

\[
v_{3}' = u(\theta A' + \theta B') + v_{1}' - x_{1A'}r_{1}' - (x_{2A} + x_{2B})r_{2}' - x_{3B}r_{3}' \quad (4.8b)
\]
The rate of change of the angle between the units relative to the yaw rates of each unit is given by

\[ \theta_A' = r_1 - r_2 \]  
(4.9a)

\[ \theta_B' = r_2 - r_3 \]  
(4.9b)

\[ \theta_C' = r_3 - r_4 \]  
(4.9c)

\[ \theta_D' = r_4 - r_5 \]  
(4.9d)

\[ \theta_E' = r_5 - r_6 \]  
(4.9e)

The forces at the connection points are found from Equations (4.2a), (4.3a), (4.4a), (4.5a), and (4.6a), and substituted into the remaining equations. Then \( v_2', v_3', v_4', v_5', \) and \( v_6' \), are eliminated using Equations (4.7a) through (4.8e). A set of generalized coordinates were chosen to describe the yaw plane response of the vehicle. These are
listed in Table 3-3, along with their units and symbol assignment. For the triple trailer LCV system the generalized coordinates become

\[
x^T = \left( v_1 \ r_1 \ r_2 \ r_3 \ r_4 \ r_5 \ r_6 \ \theta_A \ \theta_B \ \theta_C \ \theta_D \ \theta_E \right)
\]  
(4.10)

\[
x'^T = \left( v_1' \ r_1' \ r_2' \ r_3' \ r_4' \ r_5' \ r_6' \ \theta_A' \ \theta_B' \ \theta_C' \ \theta_D' \ \theta_E' \right)
\]  
(4.11)

As in Chapter III, the equations of motion are expressed in the following matrix form:

\[
[M] \{x\}' = [S] \{x\} + \{F\}
\]  
(4.12)

These quantities are then substituted into the system. Upon rearranging these expressions, like terms, relative to the generalized coordinates, are collected. In addition, terms for the vector of the derivatives of the generalized coordinates are moved to the left hand side of each equation. The terms are then arranged in the correct locations in each matrix. Due to the length of these terms, the resulting [M], [S], and \{F\} matrices are listed in Appendix E.

At this point, appropriate values for the vehicle mass properties and dimensions, as well as the tire properties, are specified. These values were defined utilizing the parameters from the AU-NCAT vehicle, and the previously mentioned tires from UMTRI data [22] that were scaled to match the AU-NCAT tire loads. In addition, the forward velocity \(u\) is specified for this example as 50 mph or 880 in/sec. The terms for the
matrices are then calculated. Due to the length of the terms, the matrices are listed in Appendix E.

4.3 Free Vibration Analysis

When \{F\}=0, the results represent the free vibration portion of the analysis. With them, the eigenvalues and eigenvectors of the system may be calculated. This is done by solving the equation

\[(M^{-1} S) x = \lambda x \quad (4.13)\]

Again, MATHCAD was used to directly calculate the eigenvalues as follows:

\[
\lambda = \begin{bmatrix}
-3.648 + 7.278i \\
-3.648 - 7.278i \\
-2.931 - 4.406i \\
-2.931 + 4.406i \\
-2.694 + 6.472i \\
-2.694 - 6.472i \\
-2.254 - 2.415i \\
-2.254 + 2.415i \\
-1.701 - 3.521i \\
-1.701 + 3.521i \\
-1.303 + 3.062i \\
-1.303 - 3.062i
\end{bmatrix}
\frac{1}{s^2} \quad (4.14)
The plot for these eigenvalues is shown in Figure 4-2. The plot would indicate that the system is stable, since all the real components of the eigenvalues are negative, and the system will exhibit some damping since the values are not near the vertical axis.

![Eigenvalues of Triple Trailer LCV at 50mph.](image)

Figures 4-3 through 4-8 show the mode shapes for this system. These were calculated by applying the system eigenvectors to Equation (3.26) with the principal coordinates specified as

\[ c_i(t) = 1 \cdot \exp(\lambda t) \]  

(4.15)
where the “1” is replaced with an arbitrary scale factor. In these images, the displacement and rotation of the tractor is ignored to emphasize the motion of the trailers and dollies. Note that in all cases the lateral displacement of Trailer 3 is the largest.

Figure 4-3 Triple Trailer LCV - Mode Shape 1.

Figure 4-4 Triple Trailer LCV - Mode Shape 2.

Figure 4-5 Triple Trailer LCV - Mode Shape 3.
4.4 Forced Vibration Analysis

Modal analysis, as described in Chapter III, is used to calculate the system response using the eigenvalues and corresponding eigenvectors. The normalized eigenvectors are listed in Appendix E. The system response is again determined through the appropriate scaling and combination of the eigenvectors and principal coordinates, which are calculated using the convolution integral.
The loading in this analysis will result from considering the residual lateral forces and aligning moments caused by the conicity and ply steer of the tires only. Therefore, the load vector is

\[
F := \begin{bmatrix}
P_{11} + P_{12} + P_{21} + P_{13} + P_{31} + P_{41} + P_{51} + P_{61} \\
T_{11} + T_{12} + T_{13} - x_{1A}(P_{21} + P_{31} + P_{41} + P_{51} + P_{61}) + P_{11}\cdot x_{11} - P_{12}\cdot x_{12} - P_{13}\cdot x_{13} \\
T_{21} - x_{2B}(P_{31} + P_{41} + P_{51} + P_{61}) - x_{2A}(P_{21} + P_{31} + P_{41} + P_{51} + P_{61}) - P_{21}\cdot x_{21} \\
T_{31} - x_{3B}(P_{31} + P_{41} + P_{51} + P_{61}) - x_{3A}(P_{21} + P_{31} + P_{41} + P_{51} + P_{61}) - P_{31}\cdot x_{31} - x_{3C}(P_{41} + P_{51} + P_{61}) \\
T_{41} - x_{4D}(P_{51} + P_{61}) - P_{41}\cdot x_{41} - x_{4C}(P_{41} + P_{51} + P_{61}) \\
T_{51} - x_{5D}(P_{51} + P_{61}) - P_{51}\cdot x_{51} - P_{61}\cdot x_{5E} \\
T_{61} - P_{61}\cdot x_{61} - P_{61}\cdot x_{6E} \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

(4.16)

where \(P_{ij}\) is the residual lateral force and \(T_{ij}\) is the residual aligning moment.

As stated in the introduction, these residual forces and moments may be balanced or unbalanced. If they are balanced, the vehicle should have a net zero force or moment summation. For example, Figure 4-9 shows a plan view of the vehicle. The red arrows represent the direction the total residual lateral force or moment is acting at each tire. In this configuration, the forces are balanced, and would represent a situation where the tires were carefully sorted and installed.

Figure 4-10 shows the simplest possible situation for imbalance, which is having all the residual lateral force pointed in the same direction. The residual forces that do not
change direction from the balanced case are shown as dimmed in order to highlight the proposed change.

Of more interest would be cases where the imbalance is focused on the rearmost unit, as in Figure 4-11. Recalling the rearward amplification effect exhibited by these vehicles, this should be the mode that is most likely to be excited.

Figure 4-9 Residual Lateral Force, Balanced.

Figure 4-10 Residual Lateral Force, Unbalanced Across Vehicle.
As with other tire properties, the magnitude of the residual lateral force and aligning moment varies with vertical load. For this analysis, these values were assumed to be on the order of 1% of the vertical tire load. The tire loads are based on measurements taken of the AU-NCAT vehicle. In addition, recall that the model is linear. Therefore, once the system matrices are defined, and a particular load vector is established, the response may be directly scaled to investigate other residual force and aligning moment magnitudes. For this analysis, the load was considered at all three trailer axles. With these values in place, the load vector was calculated as
The coefficients required by the convolution integral can now be determined as follows:

\[
F = \begin{pmatrix}
423 \\
-3008 \\
-9156 \\
-2744 \\
-8021 \\
-1354 \\
-3414 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\text{lb}
\hspace{1cm} (4.17)
\]

\[
\begin{align*}
g_1 &= X_1^T F = 8.103 + 3.580i \text{ lb} \\
g_2 &= X_2^T F = 8.103 - 3.580i \text{ lb} \\
g_3 &= X_3^T F = 7.468 - 3.398i \text{ lb} \\
g_4 &= X_4^T F = 7.468 + 3.398i \text{ lb} \\
g_5 &= X_5^T F = -8.045 - 2.797i \text{ lb} \\
g_6 &= X_6^T F = -8.045 + 2.797i \text{ lb}
\end{align*}
\hspace{1cm} (4.18-4.23)
The convolution integrals may now be calculated to determine the generalized coordinates as functions of time, and the expansion theorem may be applied. This results in the following expressions for the response of each generalized coordinate:

\[ v_1(t) := c_1(t) \cdot X_{11}^T + c_2(t) \cdot X_{12} + c_3(t) \cdot X_3 + c_4(t) \cdot X_4 + c_5(t) \cdot X_5 + c_6(t) \cdot X_6 + c_7(t) \cdot X_7 + c_8(t) \cdot X_8 + c_9(t) \cdot X_9 + c_{10}(t) \cdot X_{101} + c_{11}(t) \cdot X_{111} + c_{12}(t) \cdot X_{121} \]  

\[ r_1(t) := c_1(t) \cdot X_{11}^T + c_2(t) \cdot X_{12} + c_3(t) \cdot X_3 + c_4(t) \cdot X_4 + c_5(t) \cdot X_5 + c_6(t) \cdot X_6 + c_7(t) \cdot X_7 + c_8(t) \cdot X_8 + c_9(t) \cdot X_9 + c_{10}(t) \cdot X_{102} + c_{11}(t) \cdot X_{112} + c_{12}(t) \cdot X_{122} \]  

\[ g_7 := X_7^T F = -8.698 + 4.489i \text{ lb} \] (4.24)

\[ g_8 := X_8^T F = -8.698 - 4.489i \text{ lb} \] (4.25)

\[ g_9 := X_9^T F = -7.240 + 4.979i \text{ lb} \] (4.26)

\[ g_{10} := X_{10}^T F = -7.240 - 4.979i \text{ lb} \] (4.27)

\[ g_{11} := X_{11}^T F = 7.488 + 6.042i \text{ lb} \] (4.28)

\[ g_{12} := X_{12}^T F = 7.488 - 6.042i \text{ lb} \] (4.29)

The convolution integrals may now be calculated to determine the generalized coordinates as functions of time, and the expansion theorem may be applied. This results in the following expressions for the response of each generalized coordinate:
\( r_2(t) := c_1(t) \cdot X_{13} + c_2(t) \cdot X_{23} + c_3(t) \cdot X_{33} + c_4(t) \cdot X_{43} \\
+ c_5(t) \cdot X_{53} + c_6(t) \cdot X_{63} + c_7(t) \cdot X_{73} + c_8(t) \cdot X_{83} \\
+ c_9(t) \cdot X_{93} + c_{10}(t) \cdot X_{103} + c_{11}(t) \cdot X_{113} + c_{12}(t) \cdot X_{123} \)  

(4.32)

\( r_3(t) := c_1(t) \cdot X_{14} + c_2(t) \cdot X_{24} + c_3(t) \cdot X_{34} + c_4(t) \cdot X_{44} \\
+ c_5(t) \cdot X_{54} + c_6(t) \cdot X_{64} + c_7(t) \cdot X_{74} + c_8(t) \cdot X_{84} \\
+ c_9(t) \cdot X_{94} + c_{10}(t) \cdot X_{104} + c_{11}(t) \cdot X_{114} + c_{12}(t) \cdot X_{124} \)  

(4.33)

\( r_4(t) := c_1(t) \cdot X_{15} + c_2(t) \cdot X_{25} + c_3(t) \cdot X_{35} + c_4(t) \cdot X_{45} \\
+ c_5(t) \cdot X_{55} + c_6(t) \cdot X_{65} + c_7(t) \cdot X_{75} + c_8(t) \cdot X_{85} \\
+ c_9(t) \cdot X_{95} + c_{10}(t) \cdot X_{105} + c_{11}(t) \cdot X_{115} + c_{12}(t) \cdot X_{125} \)  

(4.34)

\( r_5(t) := c_1(t) \cdot X_{16} + c_2(t) \cdot X_{26} + c_3(t) \cdot X_{36} + c_4(t) \cdot X_{46} \\
+ c_5(t) \cdot X_{56} + c_6(t) \cdot X_{66} + c_7(t) \cdot X_{76} + c_8(t) \cdot X_{86} \\
+ c_9(t) \cdot X_{96} + c_{10}(t) \cdot X_{106} + c_{11}(t) \cdot X_{116} + c_{12}(t) \cdot X_{126} \)  

(4.35)

\( r_6(t) := c_1(t) \cdot X_{17} + c_2(t) \cdot X_{27} + c_3(t) \cdot X_{37} + c_4(t) \cdot X_{47} \\
+ c_5(t) \cdot X_{57} + c_6(t) \cdot X_{67} + c_7(t) \cdot X_{77} + c_8(t) \cdot X_{87} \\
+ c_9(t) \cdot X_{97} + c_{10}(t) \cdot X_{107} + c_{11}(t) \cdot X_{117} + c_{12}(t) \cdot X_{127} \)  

(4.36)

\( \theta_A(t) = c_1(t) \cdot X_{18} + c_2(t) \cdot X_{28} + c_3(t) \cdot X_{38} + c_4(t) \cdot X_{48} \\
+ c_5(t) \cdot X_{58} + c_6(t) \cdot X_{68} + c_7(t) \cdot X_{78} + c_8(t) \cdot X_{88} \\
+ c_9(t) \cdot X_{98} + c_{10}(t) \cdot X_{108} + c_{11}(t) \cdot X_{118} + c_{12}(t) \cdot X_{128} \)  

(4.37)

\( \theta_B(t) = c_1(t) \cdot X_{19} + c_2(t) \cdot X_{29} + c_3(t) \cdot X_{39} + c_4(t) \cdot X_{49} \\
+ c_5(t) \cdot X_{59} + c_6(t) \cdot X_{69} + c_7(t) \cdot X_{79} + c_8(t) \cdot X_{89} \\
+ c_9(t) \cdot X_{99} + c_{10}(t) \cdot X_{109} + c_{11}(t) \cdot X_{119} + c_{12}(t) \cdot X_{129} \)  

(4.38)
\[ \theta_C(t) = c_1(t)X_{110} + c_2(t)X_{210} + c_3(t)X_{310} + c_4(t)X_{410} \]
\[ + c_5(t)X_{510} + c_6(t)X_{610} + c_7(t)X_{710} + c_8(t)X_{810} \]
\[ + c_9(t)X_{910} + c_{10}(t)X_{1010} + c_{11}(t)X_{1110} + c_{12}(t)X_{1210} \]  

\[ \theta_D(t) = c_1(t)X_{111} + c_2(t)X_{211} + c_3(t)X_{311} + c_4(t)X_{411} \]
\[ + c_5(t)X_{511} + c_6(t)X_{611} + c_7(t)X_{711} + c_8(t)X_{811} \]
\[ + c_9(t)X_{911} + c_{10}(t)X_{1011} + c_{11}(t)X_{1111} + c_{12}(t)X_{1211} \]  

\[ \theta_E(t) = c_1(t)X_{112} + c_2(t)X_{212} + c_3(t)X_{312} + c_4(t)X_{412} \]
\[ + c_5(t)X_{512} + c_6(t)X_{612} + c_7(t)X_{712} + c_8(t)X_{812} \]
\[ + c_9(t)X_{912} + c_{10}(t)X_{1012} + c_{11}(t)X_{1112} + c_{12}(t)X_{1212} \]

A plot of each response is shown in Figures 4.12 – 4.14.
Figure 4-12 Tractor Lateral Velocity – Time Response.

Figure 4-13 Yaw Rates – Time Response.
By utilizing the system dimensions, and the calculated articulation angle at the connection points, the lateral displacement at the end of each unit can be determined. These relationships are given by

\[ y_2(t) = (x_{2A} + x_{2B}) \cdot \sin(\theta_A(t)) \]  

(4.42)

\[ y_3(t) = y_2(t) + (x_{3B} + x_{3C}) \cdot \sin(\theta_A(t) + \theta_B(t)) \]  

(4.43)

\[ y_4(t) = y_3(t) + (x_{4C} + x_{4D}) \cdot \sin(\theta_A(t) + \theta_B(t) + \theta_C(t)) \]  

(4.44)

\[ y_5(t) = y_4(t) + (x_{5D} + x_{5E}) \cdot \sin(\theta_A(t) + \theta_B(t) + \theta_C(t) + \theta_D(t)) \]  

(4.45)
Equation (4.46) represents the lateral displacement of the last unit of the vehicle system, Trailer 3. This should be the trailer that experiences the greatest lateral motion. A plot of the response of all three trailers is shown in Figure 4-15. This plot shows the typical situation where Trailer 3 dominates the response.

In this system, and with these vehicle parameters, both the transient and steady-state responses are again relatively small. The analysis for this vehicle would therefore conclude that the residual lateral forces and aligning moments are not of a magnitude significant enough to develop an unsafe response as measured by lateral displacement of the last unit of the LCV.

![Figure 4-15 Trailer 3 Lateral Displacement – Time Response.](image)
4.5 Model Correlation

The vehicle model in this chapter was based on the triple trailer LCV in use at the AU-NCAT test track. Physical testing of the vehicle was carried out in May of 2011 by the consortium led by NTRCI. The University of Akron is a member of the group, and therefore has access to the test data collected. This affords an opportunity to attempt to correlate the model predictions to the physical measurements. This is always an important step when developing simplified mathematical models of complex systems. The researcher, as well as other potential end users of the model, must understand its limitations and applicability.

The bulk of the on-track testing was focused on handling maneuvers. This included steady state turns at a constant radius, as well as various forms of obstacle avoidance maneuvers, typically referred to as lane changes. Fortunately, a large portion of the test track, which is oval in shape, consists of long straight-aways. Data was also collected on these portions of the track, providing the straight line driving information required.

The vehicle was also driven at various forward speeds. As will be discussed thoroughly in the next chapter, this has a sizeable impact on the response of the system. This should be expected behavior, since many of the coefficients in the system matrices developed in this work contain forward velocity terms.

The vehicle was heavily instrumented to measure various responses during the testing. Displacements, velocities, and acceleration from each unit were measured using string potentiometers and accelerometers. A pair of string potentiometers were also installed to assist in calculating the relative angle between Trailer 2 and Dolly 1 as well
as Trailer 3 and Dolly 2. These values correspond to the generalized coordinates $\theta_C$ and $\theta_E$ of the system. After the completion of the vehicle testing, a problem with the data collected by the string potentiometers measuring the angle between Trailer 3 and Dolly 2 was discovered. At the time of the completion of this research, the issue had not been resolved. Therefore, the correlation effort was made using the angle between Trailer 2 and Dolly 1, or the generalized coordinate $\theta_C$.

The vehicle was configured in its normal, fully loaded condition, resulting in a gross vehicle weight of 156,300 lb. This is well above any legal GVW allowed in the US, but is the desired loading condition for the daily pavement testing at AU-NCAT. Before the testing period began, new tires were installed at every position on the vehicle. Frequently during the testing, the air pressure in the tires was checked and adjusted if needed. Property data for these specific tires, however, was not available.

While considering these facts, a data set was selected. The average forward velocity on the straight-away section of the track for the data set was 18.5 mph. This was not a constant as assumed in the model. This generated an average angle between Trailer 2 and Dolly 1 of 0.40 degrees. This became the target for the model to calculate.

Therefore, the model was set to the mass and velocity of the vehicle as tested. With the lack of knowledge regarding the exact tire properties, the data from UMTRI [23] was used. The conicity and ply steer forces were applied at all the trailer and dolly axles, assuming this could be a likely scenario. Figure 4-16 shows the relationship between the test data and the model results.

Comparing the calculated steady state value and the measured data, the differences is around 25%. The response of the model also shows an over-damped condition.
Several observations can be made regarding this result. The first issue is the magnitude of the number being evaluated, which is very small. The measurements for this value were performed with simple string potentiometers, which were mounted to the vehicle on a hand built frame of unknown stiffness. This arrangement, which also utilized pulleys to allow for direction change, is shown in Figure 4-17. The subsequent calculations using this data are also approximate, with assumptions of planar motion, etc. These factors contribute to errors in the reported values which may be sizeable relative to the response of interest. The model developed also is limited by simplifications. Although the values it calculates are mathematically “exact,” the assumptions made in its generation omit potentially important effects of unknown magnitude. Put another way, the model may ignore some “higher order terms” that could be contributors. This effect...
may be compounded by the lack of information regarding the actual tire data. These
difficulties do not imply the model is useless, only that its underlying assumptions must be considered when reviewing results. Finally, the response measured on the vehicle is a total response to any input applied. At this point, there is no way to discern what portion of that response is due solely to conicity effects, and what portion may be due to suspension misalignment, payload offset, road crown, or any other possible source.

These issues regarding the correlation effort contributed to the interest in performing a study of the effect of various parameters on the ability of the model to predict vehicle behavior. This parametric study begins in the next chapter.

Figure 4-17 Dolly to Trailer String Potentiometer Arrangement
CHAPTER V

PARAMETRIC STUDY

Several parameters are used to define the triple trailer LCV, and their values influence the response of the vehicles to conicity and ply steer forces. In order to understand the impact of varying these values, a parametric study was performed. The range of values used for some of the parameters was determined from past research [23]. This was done in an attempt to assure consistent sets of data were used, especially relative to the tire properties. Other values were varied based on regulations regarding axle loads, and past engineering experience.

Four studies were performed. A minimum of five variations were considered for each one, in order to assure any non-linear relationships would be captured. A table will be provided as each study is discussed noting the configuration of the conicity force location, the vehicle forward velocity, the vehicle mass, and the tire properties. For each investigation, one parameter was changed per simulation. This was done in order to provide a means of quantifying the impact of each individual change. The response which was considered the output of each simulation was the lateral displacement of the end of the last trailer. This was chosen due to the interest in understanding if the vehicle was forced to leave its intended lane of travel. The ending time for the simulations was 1 second, which allowed the system to find a steady state condition.
5.1 Initial Model Configuration

A base configuration was chosen for the model before beginning the parametric studies. These became the default settings for the system at the beginning of each study. The baseline forward velocity was set to 50 mph.

The initial mass for the system was based on the vehicle measured at AU-NCAT. This was done in order to maintain the connection to the NTRCI research, and due to the availability of test data for potential comparison. This was done even though the vehicle GVW is well above the legal maximum. The values for the axle loads, mass, and mass moment of inertia are listed in Appendix B.

The default tire properties were based on data provided in the research by Fancher et al. at UMTRI [23]. This data was chosen because it provided a consistent set of properties at various tire loads. By curve fitting these properties as a function of applied vertical load, the range of values required for the simulations could be determined. Appendix C lists the various sets of data applied in the parametric studies.

As the studies progressed, new values became the default. For example, once the loading with the highest lateral displacement response was found, it became the default for the subsequent parametric studies. This same logic was applied after the forward velocity study.
5.2 Influence of Force/Moment Location

The first step in the parametric study was to determine the load application that generated the largest response. This was done by varying the axle that was subjected to the conicity forces. Several cases were also generated considering multiple axles. This study takes place completely within the forced vibration portion of the analysis; there are no changes to the system matrices for these simulations. Table 5-1 lists the vehicle configurations for this portion of the research.

<table>
<thead>
<tr>
<th>Run</th>
<th>Conicity Force/Moment Location</th>
<th>Mass</th>
<th>Tires</th>
<th>Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Trailer 1</td>
<td>AU-NCAT</td>
<td>UMTRI</td>
<td>50 mph</td>
</tr>
<tr>
<td>2</td>
<td>Trailer 2</td>
<td>AU-NCAT</td>
<td>UMTRI</td>
<td>50 mph</td>
</tr>
<tr>
<td>3</td>
<td>Trailer 3</td>
<td>AU-NCAT</td>
<td>UMTRI</td>
<td>50 mph</td>
</tr>
<tr>
<td>4</td>
<td>Trailer 2 &amp; 3</td>
<td>AU-NCAT</td>
<td>UMTRI</td>
<td>50 mph</td>
</tr>
<tr>
<td>5</td>
<td>Trailer 1, 2, &amp; 3</td>
<td>AU-NCAT</td>
<td>UMTRI</td>
<td>50 mph</td>
</tr>
<tr>
<td>6</td>
<td>Trailer 3, Dolly 2</td>
<td>AU-NCAT</td>
<td>UMTRI</td>
<td>50 mph</td>
</tr>
<tr>
<td>7</td>
<td>Trailer 2 &amp; 3, Dolly 1 &amp; 2</td>
<td>AU-NCAT</td>
<td>UMTRI</td>
<td>50 mph</td>
</tr>
<tr>
<td>8</td>
<td>Trailer 3, Dolly 1 &amp; 2</td>
<td>AU-NCAT</td>
<td>UMTRI</td>
<td>50 mph</td>
</tr>
</tbody>
</table>

The results are plotted in Figures 5-1 through 5-3. In Figure 5-1, the total response for all eight simulation is plotted, showing both the transient and steady state responses.
In order to understand the details of the two portions of the response, Figures 5-2 and 5-3 have their time scale set to show each individually. The numbers in the charts denote the peak response magnitude, and match the color code of the appropriate simulation. It is also clear that the transient response is much higher than the steady state response, and that the response is damped.
Figure 5-2 Trailer 3 Lateral Displacement – Transient Time Response for Various Conicity Force/Moment Locations.

Figure 5-3 Trailer 3 Lateral Displacement – Steady State Time Response for Various Conicity Force/Moment Locations.
Figure 5-4 shows the maximum transient response and the maximum steady state response for the conicity force/moment location study. The values are ranked by steady state response, and the plot shows that the greatest lateral displacement occurs when the forces and moments are applied to Trailer 2 and 3. This loading became the baseline configuration, and was used for the remainder of the study. Application of the load to only the first or second trailer generates a small response. The greatest impact on the results is determined by whether the loading considers Trailer 3. This is logical since the last trailer in the system should see the greatest rearward amplification. Also, as shown in the previous chapter, the response of Trailer 3 was always larger than Trailer 1 or Trailer 2.

![Figure 5-4 Effect of Conicity Force/Moment Location on Maximum Lateral Displacement of Trailer 3.](image-url)
5.3 Influence of Vehicle Speed

With the loading selected, the next parametric study was with regard to forward velocity. To this point, all the simulations had been run at 50 mph. For this study, the maximum speed was increased to 70 mph. This was chosen as it is currently the maximum on many interstate highways. The speed was then decreased in 10 mph increments down to 30 mph. Since the forward velocity is contained in the system matrices, a new set of eigenvalues and eigenvectors are generated for each simulation. The configuration for each run is listed in Table 5-2.

<table>
<thead>
<tr>
<th>Run</th>
<th>Force Location</th>
<th>Mass</th>
<th>Tires</th>
<th>Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Trailer 2 &amp; 3</td>
<td>AU-NCAT</td>
<td>UMTRI</td>
<td>70 mph</td>
</tr>
<tr>
<td>2</td>
<td>Trailer 2 &amp; 3</td>
<td>AU-NCAT</td>
<td>UMTRI</td>
<td>60 mph</td>
</tr>
<tr>
<td>3</td>
<td>Trailer 2 &amp; 3</td>
<td>AU-NCAT</td>
<td>UMTRI</td>
<td>50 mph</td>
</tr>
<tr>
<td>4</td>
<td>Trailer 2 &amp; 3</td>
<td>AU-NCAT</td>
<td>UMTRI</td>
<td>40 mph</td>
</tr>
<tr>
<td>5</td>
<td>Trailer 2 &amp; 3</td>
<td>AU-NCAT</td>
<td>UMTRI</td>
<td>30 mph</td>
</tr>
</tbody>
</table>

Figure 5-5 shows the eigenvalues for each vehicle configuration in this series plotted on the complex plane. Recalling previous discussions of these plots, several pieces of information can be deduced. First, all of the systems are stable, since all of the eigenvalues, which are complex conjugate pairs, have negative real components. All of
the systems will also exhibit a damped response, because the eigenvalues have negative real parts.

Figure 5-5 Eigenvalues for Various Forward Velocities.

Figure 5-6 is limited to show only the upper left quadrant of the complex plane, and is plotting the same sets of eigenvalues. The black dashed line in the plot is connecting values from Mode 7, which assumes the modes are in the same order for each run, and shows a steady and predictable change with speed. As the forward velocity decreases, the real component of the eigenvalues becomes more negative. This implies that the lower speed systems will exhibit a relatively higher level of damping.
Figure 5-6 Upper Quadrant Eigenvalues for Various Forward Velocities.

Figure 5-7 shows the total response for each simulation in this series. This plot confirms the assumption regarding the eigenvalues plots. The response for the 30 mph curve damps out the fastest, although all of the configurations display relatively high damping. The lower speed simulation also has the highest transient and steady state value.
The lateral displacement response is sorted by steady state amplitude in Figure 5-8. It shows a linear increase in displacement as the forward velocity decreases. This trend is illustrated in Figure 5-9, which shows a linear curve fit through this data.

The forward velocity contributes to many of the coefficients contained in the system matrices, appearing in both numerators and denominators of terms. It therefore has a significant effect on the results.
Figure 5-8 Effect of Forward Velocity on Lateral Displacement of Trailer 3.

Figure 5-9 Variation of Lateral Displacement of Trailer 3 with Forward Velocity.
5.4 Influence of Vehicle Mass

With the lowest forward velocity generating the largest lateral displacement, 30 mph became the new default for the vehicle mass variation portion of the parametric study. The mass chosen for each simulation was developed by considering reasonable potential loading conditions for a triple trailer LCV. The baseline continued to be the mass as measured on the AU-NCAT vehicle, which is known to be very high. Two other obvious conditions would be the maximum legal GVW, and also an empty trailer condition. The final two conditions involved a combination of empty and loaded trailers. Therefore, “Loaded, Loaded, Empty” indicates the last trailer is unloaded, while “Empty, Empty, Loaded” designates Trailer 1 and Trailer 2 are unloaded. See the listings in Appendix B for the actual values that were applied.

As was discussed earlier, tires are highly non-linear entities. In addition, their response is dependent on the applied vertical load. This parametric study is the first where the loaded condition of the tires is changing. Therefore, appropriately scaled properties, based on the data listed in the research by UMTRI [22], are used for each simulation. The magnitude of the scaling was set by the changing axle loads, which determined the GVW for the configuration. The axle loads for each configuration are listed in Appendix B, while the tire scaling method and resulting properties are explained in Appendix C. Table 5-3 shows the resulting configurations.
### Table 5-3 Vehicle Mass Variation.

<table>
<thead>
<tr>
<th>Run</th>
<th>Force Location</th>
<th>Mass</th>
<th>Tires</th>
<th>Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Trailer 2 &amp; 3</td>
<td>AU-NCAT</td>
<td>UMTRI</td>
<td>30 mph</td>
</tr>
<tr>
<td>2</td>
<td>Trailer 2 &amp; 3</td>
<td>Legal Max</td>
<td>UMTRI</td>
<td>30 mph</td>
</tr>
<tr>
<td>3</td>
<td>Trailer 2 &amp; 3</td>
<td>Loaded, Loaded, Empty</td>
<td>UMTRI</td>
<td>30 mph</td>
</tr>
<tr>
<td>4</td>
<td>Trailer 2 &amp; 3</td>
<td>Empty, Empty, Loaded</td>
<td>UMTRI</td>
<td>30 mph</td>
</tr>
<tr>
<td>5</td>
<td>Trailer 2 &amp; 3</td>
<td>Empty</td>
<td>UMTRI</td>
<td>30 mph</td>
</tr>
</tbody>
</table>

Figure 5-10 shows the eigenvalues for each vehicle configuration in this series plotted on the complex plane. Recalling previous discussions of these plots, several pieces of information can be deduced. First, all of the systems are stable, since all of the eigenvalues, which are complex conjugate pairs, have negative real components. All of the systems will also exhibit a damped response, because the eigenvalues are “away” from the vertical, imaginary axis. The “Empty” and “Empty, Empty, Loaded” configurations each have eigenvalues to the far left of the plot. They also have eigenvalues which are purely real, indicating un-damped modes. This is the influence of the lower mass and subsequent lower tire properties.
Figure 5-10 Eigenvalues for Various Vehicle Masses.

Figure 5-11 is limited to show only the upper left quadrant of the complex plane, and is plotting the same sets of eigenvalues. The black dashed line in the plot is again attempting to connect values from Mode 7, which assumes the modes are in the same order for each run. However, as opposed to the forward velocity study, there is no discernable pattern. The change in mass and the resulting change in tire properties are both causing this effect.
Figure 5-11 Upper Quadrant Eigenvalues for Various Vehicle Masses.

Figure 5-12 shows the total response for each simulation in this series. This plot shows that the mass has a small effect on the damping of the initial oscillatory response. Referring back to the eigenvalues plot, this would appear logical, as there is no clear “order” for the systems relative to their distance from the vertical axis. The cases where the last trailer is loaded, regardless of the condition of the other trailers, show the highest response. This again points out the importance of Trailer 3.
Figure 5-12 Trailer 3 Lateral Displacement Time Response for Various Masses.

The lateral displacement response is sorted by steady state amplitude in Figure 5-13. In general it shows an increase in lateral displacement of Trailer 3 as the mass increases.

Figure 5-14 shows a plot of the steady state response as a function of mass. The effect of moving the loaded trailer position is apparent in this plot, with the curve showing a discontinuity when shifting from Trailer 1 to Trailer 3. The mass and inertia contributes to many of the coefficients contained in the system matrices. It therefore has a significant effect on the results. A curve fit was not attempted or deemed appropriate.
Figure 5-13 Effect of Vehicle Mass on Lateral Displacement of Trailer 3.

Figure 5-14 Variation of Lateral Displacement of Trailer 3 with Vehicle Mass.
5.5 Influence of Tire Properties

The properties of the tires have a large effect on the response of the vehicle model, as these stiffness coefficients dominate the terms in the system matrices. In order to study these effects, a series of tire data sets was generated based on the stiffness coefficients found in the work by UMTRI [22]. Once the baseline values were set by designation of the appropriate axle loads, a scale factor of +/-20% was determined and applied based on the UMTRI research listed in [23]. In order to generate the required five variations, this value was then doubled to generate extreme responses at both ends of the spectrum. Details regarding the general scaling of the tire data is shown in Appendix C. The settings for each configuration of this portion of the study are listed in Table 5-4.

Table 5-4 Tire Property Variation.

<table>
<thead>
<tr>
<th>Run</th>
<th>Force Location</th>
<th>Mass</th>
<th>Tires</th>
<th>Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Trailer 2 &amp; 3</td>
<td>AU-NCAT</td>
<td>0.6x</td>
<td>30 mph</td>
</tr>
<tr>
<td>2</td>
<td>Trailer 2 &amp; 3</td>
<td>AU-NCAT</td>
<td>0.8x</td>
<td>30 mph</td>
</tr>
<tr>
<td>3</td>
<td>Trailer 2 &amp; 3</td>
<td>AU-NCAT</td>
<td>1x</td>
<td>30 mph</td>
</tr>
<tr>
<td>4</td>
<td>Trailer 2 &amp; 3</td>
<td>AU-NCAT</td>
<td>1.2x</td>
<td>30 mph</td>
</tr>
<tr>
<td>5</td>
<td>Trailer 2 &amp; 3</td>
<td>AU-NCAT</td>
<td>1.4x</td>
<td>30 mph</td>
</tr>
</tbody>
</table>
Figure 5-15 shows the eigenvalues for each vehicle configuration in this series plotted on the complex plane. Recalling previous discussions of these plots, several pieces of information can be deduced. First, all of the systems are stable, since all of the eigenvalues have negative real components. All of the systems will also exhibit a damped response, because the real component of the eigenvalues are negative. The system using the tires with a 40% increase also exhibits an un-damped mode. The imaginary components of the eigenvalues show a large range for these systems. The change in tire properties is producing this effect.

![Figure 5-15 Eigenvalues for Various Tire Properties.](image)

Figure 5-16 is limited to show only the upper left quadrant of the complex plane, and is plotting the same sets of eigenvalues. The black dashed line in the plot is again attempting to connect values from Mode 7, which assumes the modes are in the same
order for each run. A consistent, predictable pattern is shown until the 40% increase configuration is reached. This is due to the realization of the un-damped modes.

Figure 5-16 Upper Quadrant Eigenvalues for Various Tire Properties.

Figure 5-17 shows the total response for each simulation in this series. This plot shows that, as expected, the tire properties have a large effect on the response. The configurations where the tire properties were scaled by +/-20% show a predictable difference in lateral displacement response. The extreme settings of these properties display exaggerate response, especially the 40% increase. Accurate tire properties are therefore a key contributor to the response.
Figure 5-17 Trailer 3 Lateral Disp. – Time Response for Various Tire Properties.

The lateral displacement response is sorted by steady state amplitude in Figure 5-18. It shows an increase in steady state lateral displacement as the tire properties are scaled. The anomaly is again the 40% increase. This would indicate a transition of behavior at this scaling. Figure 5-19 shows a curve fit of the steady state response as a function of the tire property scaling. The response reasonably follows a polynomial fit.
Figure 5-18 Effect of Tire Properties on Lateral Displacement of Trailer 3.

Figure 5-19 Variation of Lateral Displacement of Trailer 3 with Tire Properties.
5.6 Discussion of Results

Four parametric studies have been performed with the model, varying the configuration of the vehicle across a range of parameters. Conicity force and moment location, vehicle forward velocity, vehicle total mass, and the tire properties all had an effect on the response. The effect could be detected by noting changes in the eigenvalues, the damping behavior, and the transient and steady state lateral displacement response.

Figure 5-20 shows the results of the steady state lateral response from all of the simulations as a change in value relative to the baseline of each parametric study. Changes with a positive value indicate a response greater than the baseline for that study. For the conicity force and moment location study, the values are normalized to the “Trailer 2 & 3” run. For the forward velocity, the values are normalized to the 50 mph run, which was the starting value for that sequence. For the mass, the AU-NCAT load is considered the baseline, and for the tire properties, the un-scaled values. The figure shows what variation in each study caused the greatest change. For the conicity force and moment location, the shift of the load from Trailer 3 had the largest impact. For the velocity, the slowest speed caused the greatest change, while for the mass study, it was the lightest vehicle. Finally, the softest tires had the largest contribution to the lateral displacement of Trailer 3.
Regardless of the parameter studied, the steady state lateral displacement values were all of the same order of magnitude. They were all also relatively small, with the largest value around 8.5 in. This was achieved with the tires scaled to an extreme value. When considering whether the tire conicity or ply steer is an effect that could cause a safety concern, the answer is that it appears unlikely for the range of values considered in this study. A typical lane on a highway is 12 ft (144 in) wide, and a wide trailer has a width dimension of 102 in. That leaves 21 in of lateral travel per side of the vehicle before it could intrude into another lane. This is a positive result. It says that the current level of care taken with regard to tires on trailers and dollies is acceptable from a conicity and ply steer standpoint.

However, if parameter values outside the assumed range are applied to the model, this may not be the case. For example, consider a case where the vehicle is overloaded
by a factor of 2, and the tires are at 10% stiffness. In addition, also consider applying the maximum conicity value, which is around 5% of the tire vertical load. The response is shown in Figure 5-21, which compares this system to the baseline. The last trailer for this vehicle would exhibit enough lateral displacement to encroach into the lane parallel to its travel lane, causing an unsafe condition. Although this is an exaggerated condition, it shows the importance of the parameter values used in the model, and that the range of the numbers is important.

![Exaggerated LCV Response](image)

Figure 5-21 Exaggerated LCV Response.
CHAPTER VI

CONCLUSIONS

The demand for heavy trucking capacity in the United States continues to grow. However, the interstate highway infrastructure has not increased significantly relative to that demand, and the legislation regarding vehicle size and weight has not change for several decades. One possible solution is Long Combination Vehicles (LCV), which can carry more freight using fewer resources than a traditional tractor-trailer.

Concerns regarding the safety and stability of LCV’s exist. Lateral stability can be of concern when considering vehicle systems with multiple trailers and converter dollies. One of the major contributors to lateral vehicle dynamics are the tires, which are highly non-linear and load dependent components. Tires can also exhibit conicity and ply steer, which are residual lateral forces and aligning moments at zero slip angle. This implies a possible lateral motion in straight line driving, raising a concern regarding units of the vehicle leaving the intended lane of travel.

To study this phenomena, a system of differential equations were derived for the triple trailer LCV system. These equations considered the mass and dimensions of the vehicle units, as well as simplified tire stiffness coefficients. The eigenvalues and eigenvectors of the system were determined, allowing a general assessment of the vehicle stability. A forced vibration analysis was then performed, applying the conicity forces
and moments as a loading vector. These forces and moments were considered to be in a balanced or unbalanced state at each axle of the vehicle. The lateral displacement at the end of the last trailer was the quantity of interest.

Four parametric studies were performed to develop an understanding of the sensitivity of the system response to various parameters. The range for the parameters was selected based on the available data and engineering judgment. For each simulation, a time history of the lateral displacement was plotted. This plot showed the initial transient, the damping behavior, and the final steady state value. The conclusion from the parametric study were as follows:

1. The first study, performed with a forward velocity of 50 mph, examined the effect of changing the location of the conicity force and moment application. This was done considering both single and multiple axles in a search for the combination which produced the highest response. The configurations which loaded Trailer 3, in any combination with other axles, tended to generate a large response. This is logical, as the last trailer in these vehicle systems tends to have its lateral displacement amplified.

2. The next study examined the effect of forward velocity on the response. The speed was set to 70 mph, an interstate maximum, and dropped in 10 mph increments down to 30 mph. As the speed decreased, the steady state lateral displacement increased in a non-linear fashion. This would suggest that at higher forward velocities, such as would be experienced on the highway, conicity and ply steer are not as significant.
3. The total vehicle mass was the subject of the third parametric study. Configurations were selected based on legal maximum loads, and conditions with various trailers empty. In this case, the lateral displacement increased as the total mass increased. This could also be attributed to the rearward amplification effect, with the mass of Trailer 3 being key.

4. The last study considered the tire properties. Data from the literature was selected and scaled to generate new tire properties. The scaling was significant in order to exaggerate the response. This study showed that the system with the tire properties scaled to the smallest stiffness values exhibited the highest response.

The value of the lateral response of each system was the quantity of interest. With the reputation LCV’s have of being “wiggle-wagons”, the goal of this research was to determine if the residual forces and moments typically found in tires through normal manufacturing variation were a contributor to this behavior. The answer to that question, when considering the range of parameter values included in this study, is probably not. The steady state responses being calculated by the model are around 8.5 in at maximum when considering an overloaded vehicle with tires scaled to an extreme value. When considering the 12 ft (144 in) width of typical highway lanes, as well as the maximum width of 102 in for a trailer body, it would appear unlikely these values would be reached.

This results in some conclusions regarding these vehicles. First, conicity and ply steer do not appear to be a significant concern if the vehicle is configured with
appropriate tires, and is loaded to a reasonable level. In these cases, the sorting of tires to help balance these forces is not required. In addition, any apprehension regarding the current practice of using mismatched tires on the trailers and dollies is unwarranted in regard to these forces. However, it has also been shown that if the vehicle is severely overloaded, and if the tires are not adequate, or perhaps poorly maintained, large lateral displacement of the last trailer is a possibility. In North America, where the trucking industry is regulated by the Bridge Law and vehicles are regularly inspected, this should not be the norm.

Several aspects of this research could be extended in the future. The primary portion to consider would be in regard to tire data properties. Collecting tire data is a time consuming and expensive process, and therefore the resulting data tends to become proprietary and well guarded. This makes research difficult. Contributing to this problem is the fact that trailer tires tend to generate the least interest among the tire original equipment manufacturers and truck fleet owners. In that field, the tractor and its tires tend to generate the most interest, partially due to the impact of ride on the driver. In addition, extra care must be taken during testing to determine the conicity and ply steer. The 1% applied in this research is based on a single data set, and although representative, the range of possible values is unknown.

An industry trend that could generate interest in tire data generation is the increased use of wide base single tires. This is a single large tire that replaces the traditional dual tire arrangement found on most heavy vehicles. Industry acceptance of new technology can be slow, so there could be potential collaboration between industry, academia, and the government to pursue this topic.
Better quality test data to be used for correlation would be a useful contribution. The data available for this research regarding the articulation angle was generated in a course manner due to budget and time constraints. With the small magnitude of the numbers being calculated, the potential for errors to overwhelm the quantity being measured could be high.

Another potential research effort would be through continued work with NTRCI. During the Spring and Summer of 2011, this consortium focused on developing baseline LCV response to standard handling events, including constant radius cornering, single and double lane changes, and on-highway driving. Future work with this group will focus on understandings methods of improving the response of the vehicle. Modeling tools, such as the one presented in this research, can be a useful first step in meeting that goal.
REFERENCES


APPENDICES
APPENDIX A

NOTATION

\( C_{Fi} \) Lateral stiffness coefficient of the tires on the \( j^{th} \) axle of the \( i^{th} \) unit
\( C_{Mij} \) Aligning moment coefficient of the tires on the \( j^{th} \) axle of the \( i^{th} \) unit
\( C_{Si} \) Longitudinal stiffness of the tires on the \( i^{th} \) unit
\( D_{ij} \) Lateral spacing between dual tires on the \( j^{th} \) axle of the \( i^{th} \) unit
\( F_{ij} \) Lateral force proportional to the slip angle at the \( j^{th} \) axle of the \( i^{th} \) unit
\( F_{P} \) Lateral force at connection Point P
\( F_{ij} \) Lateral force proportional to the slip angle at the \( j^{th} \) axle of the \( i^{th} \) unit
\( I_{i} \) Mass moment of inertia, about a vertical axis, of the \( i^{th} \) unit
\( M_{ij} \) Aligning moment proportional to the slip angle at the \( j^{th} \) axle of the \( i^{th} \) unit
\( m_{i} \) Mass of the \( i^{th} \) unit
\( P_{ij} \) Lateral force due to conicity and/or ply steer at the \( j^{th} \) axle of the \( i^{th} \) unit
\( r_{i} \) Yaw rate of the \( i^{th} \) unit
\( T_{ij} \) Aligning moment due to conicity and/or ply steer at the \( j^{th} \) axle of the \( i^{th} \) unit
\( t \) Time
\( u \) Velocity in the forward direction
\( v_{i} \) Lateral velocity of the \( i^{th} \) unit
\( X_{i} \) \( i^{th} \) eigenvector
x Vector of generalized coordinates

x' Vector of time derivatives of the generalized coordinates

x_{ij} Distance from the center of gravity of the i^{th} vehicle to its j^{th} axle

x_{ip} Distance from the center of gravity of the i^{th} vehicle to connection Point P (P ∈ A, B, C, D, E) on that unit

\lambda_i i^{th} eigenvalues

\Theta_P Articulation angle at connection Point P
APPENDIX B

VEHICLE DIMENSIONS

The tables in this appendix list the geometric dimensions used to represent the various vehicle configurations. Table B.1 lists the values of the LCV as tested at AU-NCAT in May of 2011, which is considered the baseline. These values should be used unless otherwise specified.
Table B.1 Dimensions for AU-NCAT Vehicle.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{1A}$</td>
<td>Tractor C.G. to Joint A</td>
<td>85.42 in</td>
</tr>
<tr>
<td>$X_{2A}$</td>
<td>Trailer 1 C.G. to Joint A</td>
<td>72.24 in</td>
</tr>
<tr>
<td>$X_{2B}$</td>
<td>Trailer 1 C.G. to Joint B</td>
<td>187.83 in</td>
</tr>
<tr>
<td>$X_{3B}$</td>
<td>Dolly 1 C.G. to Joint B</td>
<td>77.92 in</td>
</tr>
<tr>
<td>$X_{3C}$</td>
<td>Dolly 1 C.G. to Joint C</td>
<td>0 in</td>
</tr>
<tr>
<td>$X_{4C}$</td>
<td>Trailer 2 C.G. to Joint C</td>
<td>104.68 in</td>
</tr>
<tr>
<td>$X_{4D}$</td>
<td>Trailer 2 C.G. to Joint D</td>
<td>155.28 in</td>
</tr>
<tr>
<td>$X_{5D}$</td>
<td>Dolly 2 C.G. to Joint D</td>
<td>77.92 in</td>
</tr>
<tr>
<td>$X_{5E}$</td>
<td>Dolly 2 C.G. to Joint E</td>
<td>0 in</td>
</tr>
<tr>
<td>$X_{6E}$</td>
<td>Trailer 3 C.G. to Joint E</td>
<td>104.68 in</td>
</tr>
<tr>
<td>$X_{11}$</td>
<td>Tractor C.G. to Axle 11</td>
<td>90.45 in</td>
</tr>
<tr>
<td>$X_{12}$</td>
<td>Tractor C.G. to Axle 12</td>
<td>70.55 in</td>
</tr>
<tr>
<td>$X_{13}$</td>
<td>Tractor C.G. to Axle 13</td>
<td>122.55 in</td>
</tr>
<tr>
<td>$X_{21}$</td>
<td>Trailer 1 C.G. to Axle 21</td>
<td>136.40 in</td>
</tr>
<tr>
<td>$X_{31}$</td>
<td>Dolly 1 C.G. to Axle 31</td>
<td>0 in</td>
</tr>
<tr>
<td>$X_{41}$</td>
<td>Trailer 2 C.G. to Axle 41</td>
<td>103.84 in</td>
</tr>
<tr>
<td>$X_{51}$</td>
<td>Dolly 2 C.G. to Axle 51</td>
<td>0 in</td>
</tr>
<tr>
<td>$X_{61}$</td>
<td>Trailer 3 C.G. to Axle 61</td>
<td>103.84 in</td>
</tr>
</tbody>
</table>
Table B.2 Dimensions for Max Legal Load Vehicle.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>X_{1A}</td>
<td>Tractor C.G. to Joint A</td>
<td>85.42 in</td>
</tr>
<tr>
<td>X_{2A}</td>
<td>Trailer 1 C.G. to Joint A</td>
<td>72.24 in</td>
</tr>
<tr>
<td>X_{2B}</td>
<td>Trailer 1 C.G. to Joint B</td>
<td>187.83 in</td>
</tr>
<tr>
<td>X_{3B}</td>
<td>Dolly 1 C.G. to Joint B</td>
<td>77.92 in</td>
</tr>
<tr>
<td>X_{3C}</td>
<td>Dolly 1 C.G. to Joint C</td>
<td>0 in</td>
</tr>
<tr>
<td>X_{4C}</td>
<td>Trailer 2 C.G. to Joint C</td>
<td>104.68 in</td>
</tr>
<tr>
<td>X_{4D}</td>
<td>Trailer 2 C.G. to Joint D</td>
<td>155.28 in</td>
</tr>
<tr>
<td>X_{5D}</td>
<td>Dolly 2 C.G. to Joint D</td>
<td>77.92 in</td>
</tr>
<tr>
<td>X_{5E}</td>
<td>Dolly 2 C.G. to Joint E</td>
<td>0 in</td>
</tr>
<tr>
<td>X_{6E}</td>
<td>Trailer 3 C.G. to Joint E</td>
<td>104.68 in</td>
</tr>
<tr>
<td>X_{11}</td>
<td>Tractor C.G. to Axle 11</td>
<td>90.45 in</td>
</tr>
<tr>
<td>X_{12}</td>
<td>Tractor C.G. to Axle 12</td>
<td>70.55 in</td>
</tr>
<tr>
<td>X_{13}</td>
<td>Tractor C.G. to Axle 13</td>
<td>122.55 in</td>
</tr>
<tr>
<td>X_{21}</td>
<td>Trailer 1 C.G. to Axle 21</td>
<td>136.40 in</td>
</tr>
<tr>
<td>X_{31}</td>
<td>Dolly 1 C.G. to Axle 31</td>
<td>0 in</td>
</tr>
<tr>
<td>X_{41}</td>
<td>Trailer 2 C.G. to Axle 41</td>
<td>103.84 in</td>
</tr>
<tr>
<td>X_{51}</td>
<td>Dolly 2 C.G. to Axle 51</td>
<td>0 in</td>
</tr>
<tr>
<td>X_{61}</td>
<td>Trailer 3 C.G. to Axle 61</td>
<td>103.84 in</td>
</tr>
</tbody>
</table>
Table B.3 Dimensions for Loaded, Loaded, Empty Vehicle.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>X_{1A}</td>
<td>Tractor C.G. to Joint A</td>
<td>85.42 in</td>
</tr>
<tr>
<td>X_{2A}</td>
<td>Trailer 1 C.G. to Joint A</td>
<td>72.24 in</td>
</tr>
<tr>
<td>X_{2B}</td>
<td>Trailer 1 C.G. to Joint B</td>
<td>187.83 in</td>
</tr>
<tr>
<td>X_{3B}</td>
<td>Dolly 1 C.G. to Joint B</td>
<td>77.92 in</td>
</tr>
<tr>
<td>X_{3C}</td>
<td>Dolly 1 C.G. to Joint C</td>
<td>0 in</td>
</tr>
<tr>
<td>X_{4C}</td>
<td>Trailer 2 C.G. to Joint C</td>
<td>104.68 in</td>
</tr>
<tr>
<td>X_{4D}</td>
<td>Trailer 2 C.G. to Joint D</td>
<td>155.28 in</td>
</tr>
<tr>
<td>X_{5D}</td>
<td>Dolly 2 C.G. to Joint D</td>
<td>77.92 in</td>
</tr>
<tr>
<td>X_{5E}</td>
<td>Dolly 2 C.G. to Joint E</td>
<td>0 in</td>
</tr>
<tr>
<td>X_{6E}</td>
<td>Trailer 3 C.G. to Joint E</td>
<td>134.68 in</td>
</tr>
<tr>
<td>X_{11}</td>
<td>Tractor C.G. to Axle 11</td>
<td>90.45 in</td>
</tr>
<tr>
<td>X_{12}</td>
<td>Tractor C.G. to Axle 12</td>
<td>70.55 in</td>
</tr>
<tr>
<td>X_{13}</td>
<td>Tractor C.G. to Axle 13</td>
<td>122.55 in</td>
</tr>
<tr>
<td>X_{21}</td>
<td>Trailer 1 C.G. to Axle 21</td>
<td>136.40 in</td>
</tr>
<tr>
<td>X_{31}</td>
<td>Dolly 1 C.G. to Axle 31</td>
<td>0 in</td>
</tr>
<tr>
<td>X_{41}</td>
<td>Trailer 2 C.G. to Axle 41</td>
<td>103.84 in</td>
</tr>
<tr>
<td>X_{51}</td>
<td>Dolly 2 C.G. to Axle 51</td>
<td>0 in</td>
</tr>
<tr>
<td>X_{61}</td>
<td>Trailer 3 C.G. to Axle 61</td>
<td>73.84 in</td>
</tr>
</tbody>
</table>
Table B.4 Dimensions for Empty, Empty, Loaded Vehicle.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{1A}$</td>
<td>Tractor C.G. to Joint A</td>
<td>85.42 in</td>
</tr>
<tr>
<td>$X_{2A}$</td>
<td>Trailer 1 C.G. to Joint A</td>
<td>102.24 in</td>
</tr>
<tr>
<td>$X_{2B}$</td>
<td>Trailer 1 C.G. to Joint B</td>
<td>157.83 in</td>
</tr>
<tr>
<td>$X_{3B}$</td>
<td>Dolly 1 C.G. to Joint B</td>
<td>77.92 in</td>
</tr>
<tr>
<td>$X_{3C}$</td>
<td>Dolly 1 C.G. to Joint C</td>
<td>0 in</td>
</tr>
<tr>
<td>$X_{4C}$</td>
<td>Trailer 2 C.G. to Joint C</td>
<td>134.68 in</td>
</tr>
<tr>
<td>$X_{4D}$</td>
<td>Trailer 2 C.G. to Joint D</td>
<td>125.28 in</td>
</tr>
<tr>
<td>$X_{5D}$</td>
<td>Dolly 2 C.G. to Joint D</td>
<td>77.92 in</td>
</tr>
<tr>
<td>$X_{5E}$</td>
<td>Dolly 2 C.G. to Joint E</td>
<td>0 in</td>
</tr>
<tr>
<td>$X_{6E}$</td>
<td>Trailer 3 C.G. to Joint E</td>
<td>104.68 in</td>
</tr>
<tr>
<td>$X_{11}$</td>
<td>Tractor C.G. to Axle 11</td>
<td>90.45 in</td>
</tr>
<tr>
<td>$X_{12}$</td>
<td>Tractor C.G. to Axle 12</td>
<td>70.55 in</td>
</tr>
<tr>
<td>$X_{13}$</td>
<td>Tractor C.G. to Axle 13</td>
<td>122.55 in</td>
</tr>
<tr>
<td>$X_{21}$</td>
<td>Trailer 1 C.G. to Axle 21</td>
<td>106.40 in</td>
</tr>
<tr>
<td>$X_{31}$</td>
<td>Dolly 1 C.G. to Axle 31</td>
<td>0.00 in</td>
</tr>
<tr>
<td>$X_{41}$</td>
<td>Trailer 2 C.G. to Axle 41</td>
<td>73.84 in</td>
</tr>
<tr>
<td>$X_{51}$</td>
<td>Dolly 2 C.G. to Axle 51</td>
<td>0 in</td>
</tr>
<tr>
<td>$X_{61}$</td>
<td>Trailer 3 C.G. to Axle 61</td>
<td>103.84 in</td>
</tr>
</tbody>
</table>
Table B.5 Dimensions for Empty Vehicle

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>X_{1A}</td>
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<td>85.42 in</td>
</tr>
<tr>
<td>X_{2A}</td>
<td>Trailer 1 C.G. to Joint A</td>
<td>102.24 in</td>
</tr>
<tr>
<td>X_{2B}</td>
<td>Trailer 1 C.G. to Joint B</td>
<td>157.83 in</td>
</tr>
<tr>
<td>X_{3B}</td>
<td>Dolly 1 C.G. to Joint B</td>
<td>77.92 in</td>
</tr>
<tr>
<td>X_{3C}</td>
<td>Dolly 1 C.G. to Joint C</td>
<td>0 in</td>
</tr>
<tr>
<td>X_{4C}</td>
<td>Trailer 2 C.G. to Joint C</td>
<td>134.68 in</td>
</tr>
<tr>
<td>X_{4D}</td>
<td>Trailer 2 C.G. to Joint D</td>
<td>125.28 in</td>
</tr>
<tr>
<td>X_{5D}</td>
<td>Dolly 2 C.G. to Joint D</td>
<td>77.92 in</td>
</tr>
<tr>
<td>X_{5E}</td>
<td>Dolly 2 C.G. to Joint E</td>
<td>0 in</td>
</tr>
<tr>
<td>X_{6E}</td>
<td>Trailer 3 C.G. to Joint E</td>
<td>134.68 in</td>
</tr>
<tr>
<td>X_{11}</td>
<td>Tractor C.G. to Axle 11</td>
<td>90.45 in</td>
</tr>
<tr>
<td>X_{12}</td>
<td>Tractor C.G. to Axle 12</td>
<td>70.55 in</td>
</tr>
<tr>
<td>X_{13}</td>
<td>Tractor C.G. to Axle 13</td>
<td>122.55 in</td>
</tr>
<tr>
<td>X_{21}</td>
<td>Trailer 1 C.G. to Axle 21</td>
<td>106.40 in</td>
</tr>
<tr>
<td>X_{31}</td>
<td>Dolly 1 C.G. to Axle 31</td>
<td>0 in</td>
</tr>
<tr>
<td>X_{41}</td>
<td>Trailer 2 C.G. to Axle 41</td>
<td>73.84 in</td>
</tr>
<tr>
<td>X_{51}</td>
<td>Dolly 2 C.G. to Axle 51</td>
<td>0 in</td>
</tr>
<tr>
<td>X_{61}</td>
<td>Trailer 3 C.G. to Axle 61</td>
<td>73.84 in</td>
</tr>
</tbody>
</table>
APPENDIX C

MASS AND AXLE LOAD CONFIGURATIONS

The axle load, tire load, mass, and inertia values used for each configuration of the vehicle is listed in Tables C.1 – C.5. Where the data was available, the properties were based on testing. Other sources included data from the research conducted and published by UMTRI [22]. Since there was no direct correlation between the vehicles being modeled and the data, this too could be a source of error in the model. In some instances, primarily the components of the AU-NCAT vehicle, mass properties from a 3D solid CAD model were used.
Table C.1 AU-NCAT Mass Properties.

<table>
<thead>
<tr>
<th>AU-NCAT</th>
<th>Axle Load</th>
<th>Tire Load</th>
<th>Mass</th>
<th>Inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(lb)</td>
<td>(lb)</td>
<td>(lb s²/in)</td>
<td>(lb in s²)</td>
</tr>
<tr>
<td>Tractor Steer</td>
<td>11,900</td>
<td>5,950</td>
<td>85</td>
<td>318,715</td>
</tr>
<tr>
<td>Tractor Tandem</td>
<td>41,550</td>
<td>5,194</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Trailer1</td>
<td>19,450</td>
<td>4,863</td>
<td>126</td>
<td>653,691</td>
</tr>
<tr>
<td>Dolly1</td>
<td>21,000</td>
<td>5,250</td>
<td>11</td>
<td>2,454</td>
</tr>
<tr>
<td>Trailer2</td>
<td>21,400</td>
<td>5,350</td>
<td>86</td>
<td>359,607</td>
</tr>
<tr>
<td>Dolly2</td>
<td>20,150</td>
<td>5,038</td>
<td>11</td>
<td>2,454</td>
</tr>
<tr>
<td>Trailer3</td>
<td>20,850</td>
<td>5,213</td>
<td>86</td>
<td>359,607</td>
</tr>
<tr>
<td>GVW</td>
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<td></td>
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</tbody>
</table>

Table C.2 Legal Max Load Mass Properties.

<table>
<thead>
<tr>
<th>Max Legal Load</th>
<th>Axle Load</th>
<th>Tire Load</th>
<th>Mass</th>
<th>Inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(lb)</td>
<td>(lb)</td>
<td>(lb s²/in)</td>
<td>(lb in s²)</td>
</tr>
<tr>
<td>Tractor Steer</td>
<td>12,000</td>
<td>6,000</td>
<td>57</td>
<td>318,715</td>
</tr>
<tr>
<td>Tractor Tandem</td>
<td>27,000</td>
<td>3,375</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Trailer1</td>
<td>17,000</td>
<td>4,250</td>
<td>88</td>
<td>1,211,054</td>
</tr>
<tr>
<td>Dolly1</td>
<td>21,000</td>
<td>5,250</td>
<td>10</td>
<td>2,454</td>
</tr>
<tr>
<td>Trailer2</td>
<td>17,000</td>
<td>4,250</td>
<td>88</td>
<td>1,211,054</td>
</tr>
<tr>
<td>Dolly2</td>
<td>21,000</td>
<td>5,250</td>
<td>10</td>
<td>2,454</td>
</tr>
<tr>
<td>Trailer3</td>
<td>17,000</td>
<td>4,250</td>
<td>88</td>
<td>1,211,054</td>
</tr>
<tr>
<td>GVW</td>
<td>132,000</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Table C.3 Loaded, Loaded, Empty Mass Properties.

<table>
<thead>
<tr>
<th>Loaded, Loaded, Empty</th>
<th>Axle Load</th>
<th>Tire Load</th>
<th>Mass</th>
<th>Inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(lb)</td>
<td>(lb)</td>
<td>(lb s^2/in)</td>
<td>(lb in s^2)</td>
</tr>
<tr>
<td>Tractor</td>
<td>12,000</td>
<td>6,000</td>
<td>57</td>
<td>318,715</td>
</tr>
<tr>
<td>Tractor</td>
<td>27,000</td>
<td>3,375</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Trailer1</td>
<td>17,000</td>
<td>4,250</td>
<td>88</td>
<td>1,211,054</td>
</tr>
<tr>
<td>Dolly1</td>
<td>21,000</td>
<td>5,250</td>
<td>10</td>
<td>2,454</td>
</tr>
<tr>
<td>Trailer2</td>
<td>17,000</td>
<td>4,250</td>
<td>88</td>
<td>1,211,054</td>
</tr>
<tr>
<td>Dolly2</td>
<td>8,050</td>
<td>2,013</td>
<td>10</td>
<td>2,454</td>
</tr>
<tr>
<td>Trailer3</td>
<td>4,050</td>
<td>1,013</td>
<td>21</td>
<td>475,519</td>
</tr>
<tr>
<td>GVW</td>
<td>106,100</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table C.4 Empty, Empty, Loaded Mass Properties.

<table>
<thead>
<tr>
<th>Empty, Empty, Loaded</th>
<th>Axle Load</th>
<th>Tire Load</th>
<th>Mass</th>
<th>Inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(lb)</td>
<td>(lb)</td>
<td>(lb s^2/in)</td>
<td>(lb in s^2)</td>
</tr>
<tr>
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<td>4,500</td>
<td>44</td>
<td>125,143</td>
</tr>
<tr>
<td>Tractor</td>
<td>12,050</td>
<td>1,506</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Trailer1</td>
<td>4,050</td>
<td>1,013</td>
<td>21</td>
<td>475,519</td>
</tr>
<tr>
<td>Dolly1</td>
<td>8,150</td>
<td>2,038</td>
<td>10</td>
<td>2,454</td>
</tr>
<tr>
<td>Trailer2</td>
<td>4,050</td>
<td>1,013</td>
<td>21</td>
<td>475,519</td>
</tr>
<tr>
<td>Dolly2</td>
<td>21,100</td>
<td>5,275</td>
<td>10</td>
<td>2,454</td>
</tr>
<tr>
<td>Trailer3</td>
<td>17,000</td>
<td>4,250</td>
<td>88</td>
<td>1,211,054</td>
</tr>
<tr>
<td>GVW</td>
<td>75,400</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table C.5 Empty Mass Properties.

<table>
<thead>
<tr>
<th>Empty</th>
<th>Axle Load (lb)</th>
<th>Tire Load (lb)</th>
<th>Mass (lb s²/in)</th>
<th>Inertia (lb in s²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tractor</td>
<td>9,000</td>
<td>4,500</td>
<td>44</td>
<td>125,143</td>
</tr>
<tr>
<td>Tractor</td>
<td>12,050</td>
<td>1,506</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Trailer1</td>
<td>4,050</td>
<td>1,013</td>
<td>21</td>
<td>475,519</td>
</tr>
<tr>
<td>Dolly1</td>
<td>8,150</td>
<td>2,038</td>
<td>10</td>
<td>2,454</td>
</tr>
<tr>
<td>Trailer2</td>
<td>4,050</td>
<td>1,013</td>
<td>21</td>
<td>475,519</td>
</tr>
<tr>
<td>Dolly2</td>
<td>8,150</td>
<td>2,038</td>
<td>10</td>
<td>2,454</td>
</tr>
<tr>
<td>Trailer3</td>
<td>4,050</td>
<td>1,013</td>
<td>21</td>
<td>475,519</td>
</tr>
<tr>
<td>GVW</td>
<td>49,500</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX D

TIRE PROPERTY SCALING AND CONFIGURATIONS

Figures D.1 – D.3 represent curve fits from tire data developed by UMTRI [22]. The data represents the relevant tire stiffness coefficients as functions of applied vertical load for an 11-22.5 tire, which is a typical size found on heavy vehicles. These plots were used to determine the tire properties at the specific vertical loads.

Figure D.1 UMTRI Cornering Force vs. Vertical Load.
Figure D.2 UMTRI Aligning Moment vs. Vertical Load.

Figure D.3 UMTRI Aligning Moment vs. Vertical Load.
Tables D.1 through D.5 list the resulting tire stiffness coefficients used in the model.

The values listed are per tire. They were multiplied by the number of tires per axle for the calculations.

**Table D.1 AU-NCAT Tire Properties.**

<table>
<thead>
<tr>
<th>AU-NCAT</th>
<th>Axle Load (lb)</th>
<th>Tire Load (lb)</th>
<th>Cornering Force Coefficient (lb/rad)</th>
<th>Aligning Moment Coefficient (in lb/rad)</th>
<th>Circumferential Stiffness (lb/rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tractor Steer</td>
<td>11,900</td>
<td>5,950</td>
<td>30,322</td>
<td>73,452</td>
<td></td>
</tr>
<tr>
<td>Tractor Tandem</td>
<td>41,550</td>
<td>5,194</td>
<td>27,998</td>
<td>62,052</td>
<td>54,145</td>
</tr>
<tr>
<td>Trailer1</td>
<td>19,450</td>
<td>4,863</td>
<td>26,850</td>
<td>57,024</td>
<td>51,205</td>
</tr>
<tr>
<td>Dolly1</td>
<td>21,000</td>
<td>5,250</td>
<td>28,185</td>
<td>62,916</td>
<td>54,607</td>
</tr>
<tr>
<td>Trailer2</td>
<td>21,400</td>
<td>5,350</td>
<td>28,511</td>
<td>64,428</td>
<td>55,397</td>
</tr>
<tr>
<td>Dolly2</td>
<td>20,150</td>
<td>5,038</td>
<td>27,466</td>
<td>59,676</td>
<td>52,804</td>
</tr>
<tr>
<td>Trailer3</td>
<td>20,850</td>
<td>5,213</td>
<td>28,060</td>
<td>62,340</td>
<td>54,300</td>
</tr>
</tbody>
</table>

**Table D.2. Max Legal Load Tire Properties.**

<table>
<thead>
<tr>
<th>Max Legal Load</th>
<th>Axle Load (lb)</th>
<th>Tire Load (lb)</th>
<th>Cornering Force Coefficient (lb/rad)</th>
<th>Aligning Moment Coefficient (in lb/rad)</th>
<th>Circumferential Stiffness (lb/rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tractor Steer</td>
<td>12,000</td>
<td>6,000</td>
<td>30,461</td>
<td>74,184</td>
<td></td>
</tr>
<tr>
<td>Tractor Tandem</td>
<td>27,000</td>
<td>3,375</td>
<td>20,680</td>
<td>34,980</td>
<td>34,989</td>
</tr>
<tr>
<td>Trailer1</td>
<td>17,000</td>
<td>4,250</td>
<td>24,514</td>
<td>47,760</td>
<td>44,953</td>
</tr>
<tr>
<td>Dolly1</td>
<td>21,000</td>
<td>5,250</td>
<td>28,185</td>
<td>62,916</td>
<td>54,607</td>
</tr>
<tr>
<td>Trailer2</td>
<td>17,000</td>
<td>4,250</td>
<td>24,514</td>
<td>47,760</td>
<td>44,953</td>
</tr>
<tr>
<td>Dolly2</td>
<td>21,000</td>
<td>5,250</td>
<td>28,185</td>
<td>62,916</td>
<td>54,607</td>
</tr>
<tr>
<td>Trailer3</td>
<td>17,000</td>
<td>4,250</td>
<td>24,514</td>
<td>47,760</td>
<td>44,953</td>
</tr>
</tbody>
</table>
Table D.3 Loaded, Loaded, Empty Tire Properties.

<table>
<thead>
<tr>
<th>Loaded, Loaded, Empty</th>
<th>Axle Load (lb)</th>
<th>Tire Load (lb)</th>
<th>Cornering Force Coefficient (lb/rad)</th>
<th>Aligning Moment Coefficient (in lb/rad)</th>
<th>Circumferential Stiffness (lb/rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tractor</td>
<td>12,000</td>
<td>6,000</td>
<td>30,461</td>
<td>74,184</td>
<td></td>
</tr>
<tr>
<td>Tractor</td>
<td>27,000</td>
<td>3,375</td>
<td>20,680</td>
<td>34,980</td>
<td>34,989</td>
</tr>
<tr>
<td>Trailer1</td>
<td>17,000</td>
<td>4,250</td>
<td>24,514</td>
<td>47,760</td>
<td>44,953</td>
</tr>
<tr>
<td>Dolly1</td>
<td>21,000</td>
<td>5,250</td>
<td>28,185</td>
<td>62,916</td>
<td>54,607</td>
</tr>
<tr>
<td>Trailer2</td>
<td>17,000</td>
<td>4,250</td>
<td>24,514</td>
<td>47,760</td>
<td>44,953</td>
</tr>
<tr>
<td>Dolly2</td>
<td>8,050</td>
<td>2,013</td>
<td>13,496</td>
<td>17,292</td>
<td>20,001</td>
</tr>
<tr>
<td>Trailer3</td>
<td>4,050</td>
<td>1,013</td>
<td>7,234</td>
<td>7,056</td>
<td>12,004</td>
</tr>
</tbody>
</table>

Table D.4 Empty, Empty, Loaded Tire Properties.

<table>
<thead>
<tr>
<th>Empty, Empty, Loaded</th>
<th>Axle Load (lb)</th>
<th>Tire Load (lb)</th>
<th>Cornering Force Coefficient (lb/rad)</th>
<th>Aligning Moment Coefficient (in lb/rad)</th>
<th>Circumferential Stiffness (lb/rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tractor</td>
<td>9,000</td>
<td>4,500</td>
<td>25,501</td>
<td>51,528</td>
<td></td>
</tr>
<tr>
<td>Tractor</td>
<td>12,050</td>
<td>1,506</td>
<td>10,433</td>
<td>11,760</td>
<td>15,477</td>
</tr>
<tr>
<td>Trailer1</td>
<td>4,050</td>
<td>1,013</td>
<td>7,234</td>
<td>7,056</td>
<td>12,004</td>
</tr>
<tr>
<td>Dolly1</td>
<td>8,150</td>
<td>2,038</td>
<td>13,641</td>
<td>17,580</td>
<td>20,246</td>
</tr>
<tr>
<td>Trailer2</td>
<td>4,050</td>
<td>1,013</td>
<td>7,234</td>
<td>7,056</td>
<td>12,004</td>
</tr>
<tr>
<td>Dolly2</td>
<td>21,100</td>
<td>5,275</td>
<td>28,267</td>
<td>63,288</td>
<td>54,808</td>
</tr>
<tr>
<td>Trailer3</td>
<td>17,000</td>
<td>4,250</td>
<td>24,514</td>
<td>47,760</td>
<td>44,953</td>
</tr>
</tbody>
</table>
Table D.5 Empty Tire Properties.

<table>
<thead>
<tr>
<th>Empty</th>
<th>Axle Load (lb)</th>
<th>Tire Load (lb)</th>
<th>Cornering Force Coefficient (lb/rad)</th>
<th>Aligning Moment Coefficient (in lb/rad)</th>
<th>Circumferential Stiffness (lb/rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tractor</td>
<td>9,000</td>
<td>4,500</td>
<td>25,501</td>
<td>51,528</td>
<td></td>
</tr>
<tr>
<td>Tractor</td>
<td>12,050</td>
<td>1,506</td>
<td>10,433</td>
<td>11,760</td>
<td>15,477</td>
</tr>
<tr>
<td>Trailer1</td>
<td>4,050</td>
<td>1,013</td>
<td>7,234</td>
<td>7,056</td>
<td>12,004</td>
</tr>
<tr>
<td>Dolly1</td>
<td>8,150</td>
<td>2,038</td>
<td>13,641</td>
<td>17,580</td>
<td>20,246</td>
</tr>
<tr>
<td>Trailer2</td>
<td>4,050</td>
<td>1,013</td>
<td>7,234</td>
<td>7,056</td>
<td>12,004</td>
</tr>
<tr>
<td>Dolly2</td>
<td>8,150</td>
<td>2,038</td>
<td>13,641</td>
<td>17,580</td>
<td>20,246</td>
</tr>
<tr>
<td>Trailer3</td>
<td>4,050</td>
<td>1,013</td>
<td>7,234</td>
<td>7,056</td>
<td>12,004</td>
</tr>
</tbody>
</table>
The system representing the triple trailer LCV has 12 degrees-of-freedom. This resulted in large expressions for each of the 288 matrix coefficients. Those coefficients, the final matrices, the normalized eigenvectors, and the eigenvalues are presented here. Each coefficient is referred to by its position in its respective matrix, and due to their large number they will not be assigned equation numbers. The \([M]\) and \([S]\) matrices have a vector of units beneath each, which lists the units for the column directly above it.

The numerical results are for the vehicle configuration with the AU-NCAT mass, the UMTRI [23] scaled tires, the conicity force applied at Trailers 2 and 3, and the forward velocity set at 50 mph. Components of the \([S]\) matrix are

\[
S_{1,1} = \left( -\frac{C_{F11}}{u} - \frac{C_{F12}}{u} - \frac{C_{F21}}{u} - \frac{C_{F13}}{u} - \frac{C_{F31}}{u} - \frac{C_{F41}}{u} - \frac{C_{F51}}{u} - \frac{C_{F61}}{u} \right)
\]

\[
S_{1,2} = \left( \frac{C_{F12}x_{12}}{u} - \frac{C_{F11}x_{11}}{u} + \frac{C_{F13}x_{13}}{u} + \frac{C_{F21}x_{1A}}{u} + \frac{C_{F31}x_{1A}}{u} \right) + \left( \frac{C_{F41}x_{1A}}{u} + \frac{C_{F51}x_{1A}}{u} + \frac{C_{F61}x_{1A}}{u} \right) - m_1 u
\]

\[
S_{1,3} = \left[ \frac{C_{F21}(x_{2A} + x_{2B})}{u} + \frac{C_{F31}(x_{2A} + x_{2B})}{u} + \frac{C_{F41}(x_{2A} + x_{2B})}{u} \right]
\]

...
\[ S_{1,4} = \left[ \frac{C_{F31}(x_{31} + x_{3B})}{u} + \frac{C_{F41}(x_{3B} + x_{3C})}{u} + \frac{C_{F51}(x_{3B} + x_{3C})}{u} + \frac{C_{F61}(x_{3B} + x_{3C})}{u} - m_3u \right] \]

\[ S_{1,5} = \left[ \frac{C_{F41}(x_{41} + x_{4C})}{u} + \frac{C_{F51}(x_{4C} + x_{4D})}{u} + \frac{C_{F61}(x_{4C} + x_{4D})}{u} - m_4u \right] \]

\[ S_{1,6} = \left[ \frac{C_{F51}(x_{51} + x_{5D})}{u} + \frac{C_{F61}(x_{5D} + x_{5E})}{u} - m_5u \right] \]

\[ S_{1,7} = \left[ \frac{C_{F61}(x_{61} + x_{6E})}{u} - m_6u \right] \]

\[ S_{1,8} = (-C_{F21} - C_{F31} - C_{F41} - C_{F51} - C_{F61}) \]

\[ S_{1,9} = (-C_{F31} - C_{F41} - C_{F51} - C_{F61}) \]

\[ S_{1,10} = (-C_{F41} - C_{F51} - C_{F61}) \]

\[ S_{1,11} = (-C_{F51} - C_{F61}) \]

\[ S_{1,12} = (-C_{F61}) \]

\[ S_{2,1} = \left( \frac{C_{M11}}{u} + \frac{C_{M12}}{u} + \frac{C_{M13}}{u} - \frac{C_{F11}x_{11}}{u} + \frac{C_{F12}x_{12}}{u} + \frac{C_{F13}x_{13}}{u} \right) \]

\[ S_{2,2} = \left( \frac{C_{M11}x_{11}}{u} - \frac{C_{F12}x_{12}}{u} - \frac{C_{F13}x_{13}}{u} - \frac{C_{F21}x_{1A}}{u} - \frac{C_{F31}x_{1A}}{u} \right) \]

\[ S_{2,3} = \left( \frac{C_{F21}x_{1A}(x_{21} + x_{2A})}{u} - \frac{C_{F31}x_{1A}(x_{2A} + x_{2B})}{u} - \frac{C_{F41}x_{1A}(x_{2A} + x_{2B})}{u} \right) \]

\[ 125 \]
\[
S_{2,4} = \left[ \frac{C_{F31}x_{1A}(x_{31} + x_{3B})}{u} - \frac{C_{F41}x_{1A}(x_{3B} + x_{3C})}{u} - \frac{C_{F51}x_{1A}(x_{3B} + x_{3C})}{u} \right] \\
+ \frac{-C_{F61}x_{1A}(x_{3B} + x_{3C})}{u} + m_3 u \cdot x_{1A}
\]

\[
S_{2,5} = \left[ \frac{-C_{F41}x_{1A}(x_{41} + x_{4C})}{u} - \frac{C_{F51}x_{1A}(x_{4C} + x_{4D})}{u} - \frac{C_{F61}x_{1A}(x_{4C} + x_{4D})}{u} + m_4 u \cdot x_{1A} \right]
\]

\[
S_{2,6} = \left[ \frac{-C_{F51}x_{1A}(x_{51} + x_{5D})}{u} - \frac{C_{F61}x_{1A}(x_{5D} + x_{5E})}{u} + m_5 u \cdot x_{1A} \right]
\]

\[
S_{2,7} = \left[ \frac{-C_{F61}x_{1A}(x_{61} + x_{6E})}{u} + m_6 u \cdot x_{1A} \right]
\]

\[
S_{2,8} = (C_{F21}x_{1A} + C_{F31}x_{1A} + C_{F41}x_{1A} + C_{F51}x_{1A} + C_{F61}x_{1A})
\]

\[
S_{2,9} = (C_{F31}x_{1A} + C_{F41}x_{1A} + C_{F51}x_{1A} + C_{F61}x_{1A})
\]

\[
S_{2,10} = (C_{F41}x_{1A} + C_{F51}x_{1A} + C_{F61}x_{1A})
\]

\[
S_{2,11} = (C_{F51}x_{1A} + C_{F61}x_{1A})
\]

\[
S_{2,12} = (C_{F61}x_{1A})
\]

\[
S_{3,1} = \left[ \frac{C_{M21} + C_{F21}(x_{21} + x_{2A})}{u} + \frac{C_{F31}(x_{2A} + x_{2B})}{u} + \frac{C_{F41}(x_{2A} + x_{2B})}{u} \right] \\
+ \frac{C_{F51}(x_{2A} + x_{2B})}{u} + \frac{C_{F61}(x_{2A} + x_{2B})}{u}
\]

\[
S_{3,2} = \left[ \frac{-C_{M21}x_{1A}}{u} - \frac{C_{F21}x_{1A}(x_{21} + x_{2A})}{u} - \frac{C_{F31}x_{1A}(x_{2A} + x_{2B})}{u} - \frac{C_{F41}x_{1A}(x_{2A} + x_{2B})}{u} \right] \\
+ \frac{-C_{F51}x_{1A}(x_{2A} + x_{2B})}{u} - \frac{C_{F61}x_{1A}(x_{2A} + x_{2B})}{u}
\]

\[
S_{3,3} = \left[ \frac{-C_{M21}(x_{21} + x_{2A})}{u} - \frac{C_{S21}D_{21}^2}{u} - \frac{C_{F21}(x_{21} + x_{2A})^2}{u} - \frac{C_{F31}(x_{2A} + x_{2B})^2}{u} \right] \\
+ \frac{-C_{F41}(x_{2A} + x_{2B})^2}{u} - \frac{C_{F51}(x_{2A} + x_{2B})^2}{u} - \frac{C_{F61}(x_{2A} + x_{2B})^2}{u} + m_2 u \cdot x_{2A}
\]

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\[
S_{3,4} = \left[ -\frac{C_{F31}(x_3 + x_B)(x_2 + x_B)}{u} - \frac{C_{F41}(x_2 + x_B)(x_3 + x_C)}{u} + \frac{C_{F51}(x_2 + x_B)(x_3 + x_C)}{u} + m_3(u \cdot x_2 + u \cdot x_B) \right]
\]

\[
S_{3,5} = \left[ -\frac{C_{F41}(x_4 + x_C)(x_2 + x_B)}{u} - \frac{C_{F51}(x_2 + x_B)(x_4 + x_D)}{u} + \frac{C_{F61}(x_2 + x_B)(x_4 + x_D)}{u} + m_4(u \cdot x_2 + u \cdot x_B) \right]
\]

\[
S_{3,6} = \left[ -\frac{C_{F51}(x_5 + x_D)(x_2 + x_B)}{u} - \frac{C_{F61}(x_2 + x_B)(x_5 + x_E)}{u} + m_5(u \cdot x_2 + u \cdot x_B) \right]
\]

\[
S_{3,7} = \left[ -\frac{C_{F61}(x_6 + x_E)(x_2 + x_B)}{u} + m_6(u \cdot x_2 + u \cdot x_B) \right]
\]

\[
S_{3,8} = \left[ C_{M21} + C_{F21}(x_2 + x_A) + C_{F31}(x_2 + x_B) + C_{F41}(x_2 + x_B) + C_{F51}(x_2 + x_B) \right]
\]

\[
S_{3,9} = \left[ C_{F31}(x_2 + x_B) + C_{F41}(x_2 + x_B) + C_{F51}(x_2 + x_B) + C_{F61}(x_2 + x_B) \right]
\]

\[
S_{3,10} = \left[ C_{F41}(x_2 + x_B) + C_{F51}(x_2 + x_B) + C_{F61}(x_2 + x_B) \right]
\]

\[
S_{3,11} = \left[ C_{F51}(x_2 + x_B) + C_{F61}(x_2 + x_B) \right]
\]

\[
S_{3,12} = \left[ C_{F61}(x_2 + x_B) \right]
\]

\[
S_{4,1} = \left[ \frac{C_{M31}(x_1 + x_B)}{u} + \frac{C_{F31}(x_3 + x_B)}{u} + \frac{C_{F41}(x_3 + x_C)}{u} + \frac{C_{F51}(x_3 + x_C)}{u} + \frac{C_{F61}(x_3 + x_C)}{u} \right]
\]

\[
S_{4,2} = \left[ \frac{-C_{M31}x_1A}{u} - \frac{C_{F31}x_1A(x_3 + x_B)}{u} - \frac{C_{F41}x_1A(x_3 + x_C)}{u} - \frac{C_{F51}x_1A(x_3 + x_C)}{u} + \frac{C_{F61}x_1A(x_3 + x_C)}{u} \right]
\]

\[
S_{4,3} = \left[ \frac{-C_{M31}(x_2 + x_B)}{u} - \frac{C_{F31}(x_3 + x_B)(x_2 + x_B)}{u} - \frac{C_{F41}(x_2 + x_B)(x_3 + x_C)}{u} + \frac{C_{F51}(x_2 + x_B)(x_3 + x_C)}{u} - \frac{C_{F61}(x_2 + x_B)(x_3 + x_C)}{u} \right]
\]
\[ S_{4,4} = \left[ \frac{-C_{M31}(x_{31} + x_B)}{u} - \frac{C_{S31}D_{31}}{u} - \frac{C_{F31}(x_{31} + x_B)^2}{u} - \frac{C_{F41}(x_B + x_C)^2}{u} \right] + \frac{-C_{F51}(x_B + x_C)^2}{u} - \frac{C_{F61}(x_B + x_C)^2}{u} + m_3 u x_B \]

\[ S_{4,5} = \left[ \frac{-C_{F41}(x_{41} + x_C)(x_B + x_C)}{u} - \frac{C_{F51}(x_B + x_C)(x_C + x_D)}{u} \right] + \frac{-C_{F61}(x_B + x_C)(x_C + x_D)}{u} + m_4 (u x_B + u x_B) \]

\[ S_{4,6} = \left[ \frac{-C_{F51}(x_{51} + x_D)(x_B + x_C)}{u} - \frac{C_{F61}(x_B + x_C)(x_D + x_E)}{u} + m_5 (u x_B + u x_B) \right] \]

\[ S_{4,7} = \left[ \frac{-C_{F61}(x_{61} + x_E)(x_B + x_C)}{u} + m_6 (u x_B + u x_B) \right] \]

\[ S_{4,8} = [C_{M31} + C_{F31}(x_{31} + x_B) + C_{F41}(x_B + x_C) + C_{F51}(x_B + x_C) + C_{F61}(x_B + x_C)] \]

\[ S_{4,9} = [C_{M31} + C_{F31}(x_{31} + x_B) + C_{F41}(x_B + x_C) + C_{F51}(x_B + x_C) + C_{F61}(x_B + x_C) + C_{F61}(x_B + x_C)] \]

\[ S_{4,10} = [C_{F41}(x_B + x_C) + C_{F51}(x_B + x_C) + C_{F61}(x_B + x_C)] \]

\[ S_{4,11} = [C_{F51}(x_B + x_C) + C_{F61}(x_B + x_C)] \]

\[ S_{4,12} = [C_{F61}(x_B + x_C)] \]

\[ S_{5,1} = \left[ \frac{C_{M41}}{u} + \frac{C_{F41}(x_{41} + x_C)}{u} + \frac{C_{F51}(x_C + x_D)}{u} + \frac{C_{F61}(x_C + x_D)}{u} \right] \]

\[ S_{5,2} = \left[ \frac{-C_{M41} x_{1A}}{u} - \frac{C_{F41} x_{1A}(x_{41} + x_C)}{u} - \frac{C_{F51} x_{1A}(x_C + x_D)}{u} - \frac{C_{F61} x_{1A}(x_C + x_D)}{u} \right] \]

\[ S_{5,3} = \left[ \frac{-C_{M41}(x_{2A} + x_{2B})}{u} - \frac{C_{F41}(x_{2A} + x_{2B})(x_{41} + x_C)}{u} - \frac{C_{F51}(x_{2A} + x_{2B})(x_C + x_D)}{u} \right] + \frac{-C_{F61}(x_{2A} + x_{2B})(x_C + x_D)}{u} \]

\[ S_{5,4} = \left[ \frac{-C_{M41}(x_{3B} + x_{3C})}{u} - \frac{C_{F41}(x_{3B} + x_{3C})(x_{4C} + x_{4D})}{u} - \frac{C_{F51}(x_{3B} + x_{3C})(x_{4C} + x_{4D})}{u} \right] + \frac{-C_{F61}(x_{3B} + x_{3C})(x_{4C} + x_{4D})}{u} \]
\[ S_{5,5} = \left[ -\frac{C_{M41}(x_{41} + x_C) - C_{S41}D_{41}^2}{u} - \frac{C_{F41}(x_{41} + x_C)^2}{u} - \frac{C_{F51}(x_{4C} + x_D)^2}{u} \\
+ \frac{-C_{F61}(x_{4C} + x_D)^2}{u} + m_4 u \cdot x_C \right] \]

\[ S_{5,6} = \left[ -\frac{C_{F51}(x_{51} + x_{5D})(x_{4C} + x_D)}{u} - \frac{C_{F61}(x_{4C} + x_D)(x_{5D} + x_E)}{u} + m_5 u \cdot (x_{4C} + x_D) \right] \]

\[ S_{5,7} = \left[ -\frac{C_{F61}(x_{61} + x_{6E})(x_{4C} + x_D)}{u} + m_6 u \cdot (x_{4C} + x_D) \right] \]

\[ S_{5,8} = [C_{M41} + C_{F41}(x_{41} + x_C) + C_{F51}(x_{4C} + x_D) + C_{F61}(x_{4C} + x_D)] \]

\[ S_{5,9} = [C_{M41} + C_{F41}(x_{41} + x_C) + C_{F51}(x_{4C} + x_D) + C_{F61}(x_{4C} + x_D)] \]

\[ S_{5,10} = [C_{M41} + C_{F41}(x_{41} + x_C) + C_{F51}(x_{4C} + x_D) + C_{F61}(x_{4C} + x_D)] \]

\[ S_{5,11} = [C_{F51}(x_{4C} + x_D) + C_{F61}(x_{4C} + x_D)] \]

\[ S_{5,12} = [C_{F61}(x_{4C} + x_D)] \]

\[ S_{6,1} = \left[ -\frac{C_{M51}}{u} + \frac{C_{F51}(x_{51} + x_{5D})}{u} + \frac{C_{F61}(x_{5D} + x_E)}{u} \right] \]

\[ S_{6,2} = \left[ -\frac{C_{M51}x_{1A}}{u} - \frac{C_{F51}x_{1A}(x_{51} + x_{5D})}{u} - \frac{C_{F61}x_{1A}(x_{5D} + x_E)}{u} \right] \]

\[ S_{6,3} = \left[ -\frac{C_{M51}(x_{2A} + x_{2B})}{u} - \frac{C_{F51}(x_{51} + x_{5D})(x_{2A} + x_{2B})}{u} - \frac{C_{F61}(x_{2A} + x_{2B})(x_{5D} + x_E)}{u} \right] \]

\[ S_{6,4} = \left[ -\frac{C_{M51}(x_{3B} + x_{3C})}{u} - \frac{C_{F51}(x_{51} + x_{5D})(x_{3B} + x_{3C})}{u} - \frac{C_{F61}(x_{3B} + x_{3C})(x_{5D} + x_E)}{u} \right] \]

\[ S_{6,5} = \left[ -\frac{C_{M51}(x_{4C} + x_{4D})}{u} - \frac{C_{F51}(x_{51} + x_{5D})(x_{4C} + x_{4D})}{u} - \frac{C_{F61}(x_{4C} + x_{4D})(x_{5D} + x_E)}{u} \right] \]

\[ S_{6,6} = \left[ -\frac{C_{M51}(x_{51} + x_{5D})}{u} - \frac{C_{S51}D_{51}^2}{u} - \frac{C_{F51}(x_{51} + x_{5D})^2}{u} - \frac{C_{F61}(x_{5D} + x_E)^2}{u} + m_5 u \cdot x_{5D} \right] \]

\[ S_{6,7} = \left[ -\frac{C_{F61}(x_{61} + x_{6E})(x_{5D} + x_E)}{u} + m_6 u \cdot (x_{5D} + x_E) \right] \]
\[ S_{6,8} = \left[ C_{M51} + C_{F51}(x_5 + x_5) + C_{F61}(x_5 + x_5) \right] \]
\[ S_{6,9} = \left[ C_{M51} + C_{F51}(x_5 + x_5) + C_{F61}(x_5 + x_5) \right] \]
\[ S_{6,10} = \left[ C_{M51} + C_{F51}(x_5 + x_5) + C_{F61}(x_5 + x_5) \right] \]
\[ S_{6,11} = \left[ C_{M51} + C_{F51}(x_5 + x_5) + C_{F61}(x_5 + x_5) \right] \]
\[ S_{6,12} = \left[ C_{F61}(x_5 + x_5) \right] \]

\[ S_{7,1} = \left[ \frac{C_{M61}}{u} + \frac{C_{F61}(x_6 + x_6)}{u} \right] \]
\[ S_{7,2} = \left[ \frac{C_{M61}x_{1A}}{u} - \frac{C_{F61}x_{1A}(x_6 + x_6)}{u} \right] \]
\[ S_{7,3} = \left[ \frac{C_{M61}(x_{2A} + x_{2B})}{u} - \frac{C_{F61}(x_6 + x_6)(x_{2A} + x_{2B})}{u} \right] \]
\[ S_{7,4} = \left[ \frac{C_{M61}(x_{3B} + x_{3C})}{u} - \frac{C_{F61}(x_6 + x_6)(x_{3B} + x_{3C})}{u} \right] \]
\[ S_{7,5} = \left[ \frac{C_{M61}(x_{4D} + x_{4E})}{u} - \frac{C_{F61}(x_6 + x_6)(x_{4D} + x_{4E})}{u} \right] \]
\[ S_{7,6} = \left[ \frac{C_{M61}(x_{5D} + x_{5E})}{u} - \frac{C_{F61}(x_6 + x_6)(x_{5D} + x_{5E})}{u} \right] \]
\[ S_{7,7} = \left[ \frac{C_{M61}(x_6 + x_6)}{u} - \frac{C_{S61}D_{61}}{u} - \frac{C_{F61}(x_6 + x_6)^2}{u} + m_6u \cdot x_6E \right] \]
\[ S_{7,8} = \left[ C_{M61} + C_{F61}(x_6 + x_6) \right] \]
\[ S_{7,9} = \left[ C_{M61} + C_{F61}(x_6 + x_6) \right] \]
\[ S_{7,10} = \left[ C_{M61} + C_{F61}(x_6 + x_6) \right] \]
\[ S_{7,11} = \left[ C_{M61} + C_{F61}(x_6 + x_6) \right] \]
\[ S_{7,12} = \left[ C_{M61} + C_{F61}(x_6 + x_6) \right] \]
$S_{8,i} = 0, i = 1, 4-12$

$S_{8,2} = 1$

$S_{8,3} = -1$

$S_{9,i} = 0, i = 1-2, 5-12$

$S_{9,3} = 1$

$S_{9,4} = -1$

$S_{10,i} = 0, i = 1-3, 6-12$

$S_{10,4} = 1$

$S_{10,5} = -1$

$S_{11,i} = 0, i = 1-4, 7-12$

$S_{11,5} = 1$

$S_{11,6} = -1$

$S_{12,i} = 0, i = 1-5, 8-12$

$S_{12,6} = 1$

$S_{12,7} = -1$
Components of the $[M]$ matrix are

\[
\begin{align*}
M_{1,1} &= (m_1 + m_2 + m_3 + m_4 + m_5 + m_6) \\
M_{1,2} &= (-m_2 x_1 A - m_3 x_1 A - m_4 x_1 A - m_5 x_1 A - m_6 x_1 A) \\
M_{1,3} &= [-m_3 (x_2 A + x_2 B) - m_4 (x_2 A + x_2 B) - m_5 (x_2 A + x_2 B) - m_6 (x_2 A + x_2 B) - m_2 x_2 A] \\
M_{1,4} &= [-m_4 (x_3 B + x_3 C) - m_5 (x_3 B + x_3 C) - m_6 (x_3 B + x_3 C) - m_3 x_3 B] \\
M_{1,5} &= [-m_5 (x_4 C + x_4 D) - m_6 (x_4 C + x_4 D) - m_4 x_4 C] \\
M_{1,6} &= [-m_6 (x_5 D + x_5 E) - m_5 x_5 D] \\
M_{1,7} &= (-m_6 x_6 E) \\
M_{1,8} &= (m_2 u + m_3 u + m_4 u + m_5 u + m_6 u) \\
M_{1,9} &= (m_3 u + m_4 u + m_5 u + m_6 u) \\
M_{1,10} &= (m_4 u + m_5 u + m_6 u) \\
M_{1,11} &= (m_5 u + m_6 u) \\
M_{1,12} &= (m_6 u) \\
M_{2,1} &= (-m_2 x_1 A - m_3 x_1 A - m_4 x_1 A - m_5 x_1 A - m_6 x_1 A) \\
M_{2,2} &= (1 + m_2 x_1 A^2 + m_3 x_1 A^2 + m_4 x_1 A^2 + m_5 x_1 A^2 + m_6 x_1 A^2) \\
M_{2,3} &= \left[ m_3 x_1 A (x_2 A + x_2 B) + m_4 x_1 A (x_2 A + x_2 B) + m_5 x_1 A (x_2 A + x_2 B) \right] \\
&\quad + \left[ m_6 x_1 A (x_2 A + x_2 B) + m_2 x_1 A x_2 A \right] \\
M_{2,4} &= \left[ m_4 x_1 A (x_3 B + x_3 C) + m_5 x_1 A (x_3 B + x_3 C) + m_6 x_1 A (x_3 B + x_3 C) + m_3 x_1 A x_3 B \right] \\
M_{2,5} &= \left[ m_5 x_1 A (x_4 C + x_4 D) + m_6 x_1 A (x_4 C + x_4 D) + m_4 x_1 A x_4 C \right] \\
M_{2,6} &= \left[ m_6 x_1 A (x_5 D + x_5 E) + m_5 x_1 A x_5 D \right]
\end{align*}
\]
\[ M_{2,7} = (m_6 \cdot x_1 A \cdot x_6 E) \]
\[ M_{2,8} = (-m_2 u \cdot x_1 A - m_3 u \cdot x_1 A - m_4 u \cdot x_1 A - m_5 u \cdot x_1 A - m_6 u \cdot x_1 A) \]
\[ M_{2,9} = (-m_3 u \cdot x_1 A - m_4 u \cdot x_1 A - m_5 u \cdot x_1 A - m_6 u \cdot x_1 A) \]
\[ M_{2,10} = (-m_4 u \cdot x_1 A - m_5 u \cdot x_1 A - m_6 u \cdot x_1 A) \]
\[ M_{2,11} = (-m_5 u \cdot x_1 A - m_6 u \cdot x_1 A) \]
\[ M_{2,12} = (-m_6 u \cdot x_1 A) \]

\[ M_{3,1} = [-m_3 (x_2 A + x_2 B) - m_4 (x_2 A + x_2 B) - m_5 (x_2 A + x_2 B) - m_6 (x_2 A + x_2 B) - m_2 x_2 A] \]
\[ M_{3,2} = [m_3 x_1 A (x_2 A + x_2 B) + m_4 x_1 A (x_2 A + x_2 B) + m_5 x_1 A (x_2 A + x_2 B) + m_6 x_1 A (x_2 A + x_2 B)] \]
\[ + m_2 x_1 A x_2 A \]
\[ M_{3,3} = [I_2 + m_3 (x_2 A + x_2 B)^2 + m_4 (x_2 A + x_2 B)^2 + m_5 (x_2 A + x_2 B)^2 + m_6 (x_2 A + x_2 B)^2 + m_2 x_2 A^2] \]
\[ M_{3,4} = [m_4 (x_2 A + x_2 B)+ m_5 (x_2 A + x_2 B)(x_3 B + x_3 C) + m_6 (x_2 A + x_2 B)(x_3 B + x_3 C)] \]
\[ + m_3 x_3 B (x_2 A + x_2 B) \]
\[ M_{3,5} = [m_5 (x_2 A + x_2 B)(x_4 C + x_4 D) + m_6 (x_2 A + x_2 B)(x_4 C + x_4 D) + m_4 x_4 C (x_2 A + x_2 B)] \]
\[ M_{3,6} = [m_6 (x_2 A + x_2 B)(x_5 D + x_5 E) + m_5 x_5 D (x_2 A + x_2 B)] \]
\[ M_{3,7} = [m_6 x_6 E (x_2 A + x_2 B)] \]
\[ M_{3,8} = [-m_3 u (x_2 A + x_2 B) - m_4 u (x_2 A + x_2 B) - m_5 u (x_2 A + x_2 B) - m_6 u (x_2 A + x_2 B) - m_2 u x_2 A] \]
\[ M_{3,9} = [-m_3 u (x_2 A + x_2 B) - m_4 u (x_2 A + x_2 B) - m_5 u (x_2 A + x_2 B) - m_6 u (x_2 A + x_2 B)] \]
\[ M_{3,10} = [-m_4 u (x_2 A + x_2 B) - m_5 u (x_2 A + x_2 B) - m_6 u (x_2 A + x_2 B)] \]
\[ M_{3,11} = [-m_5 u (x_2 A + x_2 B) - m_6 u (x_2 A + x_2 B)] \]
\[ M_{3,12} = [-m_6 u (x_2 A + x_2 B)] \]

\[ M_{4,1} = [-m_4 (x_3 B + x_3 C) - m_5 (x_3 B + x_3 C) - m_6 (x_3 B + x_3 C) - m_3 x_3 B] \]
\[ M_{4,2} = [m_4 x_{1A}(x_{3B} + x_{3C}) + m_5 x_{1A}(x_{3B} + x_{3C}) + m_6 x_{1A}(x_{3B} + x_{3C}) + m_3 x_{1A} x_{3B}] \]

\[ M_{4,3} = \left[ m_4 (x_{2A} + x_{2B})(x_{3B} + x_{3C}) + m_5 (x_{2A} + x_{2B})(x_{3B} + x_{3C}) + m_6 (x_{2A} + x_{2B})(x_{3B} + x_{3C}) \right] \]

\[ M_{4,4} = \left[ I_3 + m_4 (x_{3B} + x_{3C})^2 + m_5 (x_{3B} + x_{3C})^2 + m_6 (x_{3B} + x_{3C})^2 + m_3 x_{3B}^2 \right] \]

\[ M_{4,5} = \left[ m_5 (x_{3B} + x_{3C})(x_{4C} + x_{4D}) + m_6 (x_{3B} + x_{3C})(x_{4C} + x_{4D}) + m_4 x_{4C}(x_{3B} + x_{3C}) \right] \]

\[ M_{4,6} = \left[ n_6 (x_{3B} + x_{3C})(x_{5D} + x_{5E}) + m_5 x_{5D}(x_{3B} + x_{3C}) \right] \]

\[ M_{4,7} = \left[ n_6 x_{5E}(x_{3B} + x_{3C}) \right] \]

\[ M_{4,8} = \left[ -m_4 u(x_{3B} + x_{3C}) - m_5 u(x_{3B} + x_{3C}) - m_6 u(x_{3B} + x_{3C}) - m_3 x_{3B} \right] \]

\[ M_{4,9} = \left[ -m_4 u(x_{3B} + x_{3C}) - m_5 u(x_{3B} + x_{3C}) - m_6 u(x_{3B} + x_{3C}) - m_3 x_{3B} \right] \]

\[ M_{4,10} = \left[ -m_4 u(x_{3B} + x_{3C}) - m_5 u(x_{3B} + x_{3C}) - m_6 u(x_{3B} + x_{3C}) \right] \]

\[ M_{4,11} = \left[ -m_5 u(x_{3B} + x_{3C}) - m_6 u(x_{3B} + x_{3C}) \right] \]

\[ M_{4,12} = \left[ -m_6 u(x_{3B} + x_{3C}) \right] \]

\[ M_{5,1} = \left[ -m_5 (x_{4C} + x_{4D}) - m_6 (x_{4C} + x_{4D}) - m_4 x_{4C} \right] \]

\[ M_{5,2} = \left[ m_5 x_{1A}(x_{4C} + x_{4D}) + m_6 x_{1A}(x_{4C} + x_{4D}) + m_4 x_{1A} x_{4C} \right] \]

\[ M_{5,3} = \left[ m_5 (x_{2A} + x_{2B})(x_{4C} + x_{4D}) + m_6 (x_{2A} + x_{2B})(x_{4C} + x_{4D}) + m_4 x_{4C}(x_{2A} + x_{2B}) \right] \]

\[ M_{5,4} = \left[ m_5 (x_{3B} + x_{3C})(x_{4C} + x_{4D}) + m_6 (x_{3B} + x_{3C})(x_{4C} + x_{4D}) + m_4 x_{4C}(x_{3B} + x_{3C}) \right] \]

\[ M_{5,5} = \left[ I_3 + m_5 (x_{4C} + x_{4D})^2 + m_6 (x_{4C} + x_{4D})^2 + m_4 x_{4C}^2 \right] \]

\[ M_{5,6} = \left[ n_6 (x_{4C} + x_{4D})(x_{5D} + x_{5E}) + m_5 x_{5D}(x_{4C} + x_{4D}) \right] \]

\[ M_{5,7} = \left[ n_6 x_{5E}(x_{4C} + x_{4D}) \right] \]

\[ M_{5,8} = \left[ -m_5 u(x_{4C} + x_{4D}) - n_6 u(x_{4C} + x_{4D}) - m_4 u x_{4C} \right] \]

\[ M_{5,9} = \left[ -m_5 u(x_{4C} + x_{4D}) - n_6 u(x_{4C} + x_{4D}) - m_4 u x_{4C} \right] \]
\[ M_{5,10} = [-m_5 u^i (x_{4c} + x_{4d}) - m_6 u^i (x_{4c} + x_{4d}) - m_4 u^i x_{4c}] \]

\[ M_{5,11} = [-m_5 u^i (x_{4c} + x_{4d}) - m_6 u^i (x_{4c} + x_{4d})] \]

\[ M_{5,12} = [-m_6 u^i (x_{4c} + x_{4d})] \]

\[ M_{6,1} = [-m_6 (x_{5d} + x_{5e}) - m_5 x_{5d}] \]

\[ M_{6,2} = [m_6 x_{1a} (x_{5d} + x_{5e}) + m_5 x_{1a} x_{5d}] \]

\[ M_{6,3} = [m_6 (x_{2a} + x_{2b}) (x_{5d} + x_{5e}) + m_5 x_{5d} (x_{2a} + x_{2b})] \]

\[ M_{6,4} = [m_6 (x_{3b} + x_{3c}) (x_{5d} + x_{5e}) + m_5 x_{5d} (x_{3b} + x_{3c})] \]

\[ M_{6,5} = [m_6 (x_{4c} + x_{4d}) (x_{5d} + x_{5e}) + m_5 x_{5d} (x_{4c} + x_{4d})] \]

\[ M_{6,6} = [I_5 + m_6 (x_{5d} + x_{5e})^2 + m_5 x_{5d}] \]

\[ M_{6,7} = [m_6 x_{6e} (x_{5d} + x_{5e})] \]

\[ M_{6,8} = [-m_6 u^i (x_{5d} + x_{5e}) - m_5 u^i x_{5d}] \]

\[ M_{6,9} = [-m_6 u^i (x_{5d} + x_{5e}) - m_5 u^i x_{5d}] \]

\[ M_{6,10} = [-m_6 u^i (x_{5d} + x_{5e}) - m_5 u^i x_{5d}] \]

\[ M_{6,11} = [-m_6 u^i (x_{5d} + x_{5e}) - m_5 u^i x_{5d}] \]

\[ M_{6,12} = [-m_6 u^i (x_{5d} + x_{5e})] \]

\[ M_{7,1} = (-m_6 x_{6e}) \]

\[ M_{7,2} = (m_6 x_{1a} x_{6e}) \]

\[ M_{7,3} = [m_6 x_{6e} (x_{2a} + x_{2b})] \]

\[ M_{7,4} = [m_6 x_{6e} (x_{3b} + x_{3c})] \]

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\begin{align*}
M_{7,5} &= \left[ m_6 x_6 E \left( x_{4C} + x_{4D} \right) \right] \\
M_{7,6} &= \left[ m_6 x_6 E \left( x_{5D} + x_{5E} \right) \right] \\
M_{7,7} &= \left( m_6 x_6 E^2 + l_6 \right) \\
M_{7,8} &= \left( -m_6 u \cdot x_6 E \right) \\
M_{7,9} &= \left( -m_6 u \cdot x_6 E \right) \\
M_{7,10} &= \left( -m_6 u \cdot x_6 E \right) \\
M_{7,11} &= \left( -m_6 u \cdot x_6 E \right) \\
M_{7,12} &= \left( -m_6 u \cdot x_6 E \right) \\
M_{8,i} &= 0, \ i = 1-7, 9-12 \\
M_{8,8} &= 1 \\
M_{9,i} &= 0, \ i = 1-8, 10-12 \\
M_{9,9} &= 1 \\
M_{10,i} &= 0, \ i = 1-9, 11-12 \\
M_{10,10} &= 1 \\
M_{11,i} &= 0, \ i = 1-10, 12 \\
M_{11,11} &= 1 \\
M_{12,i} &= 0, \ i = 1-11 \\
M_{12,12} &= 1
\end{align*}
\[
S = \begin{pmatrix}
-956 & -2130 & 47205 & 30360 & 16594 & 10279 & -49446 & -556291 & -448891 & -336152 & -222107 & -112241 \\
73074 & -7831819 & -4032499 & -2593500 & -1417525 & -878060 & 4223888 & 47521172 & 38346509 & 28715804 & 18973460 & 9588204 \\
158383 & -13529842 & -31888196 & -7895563 & -4315465 & -2673137 & 12859063 & 139376776 & 116740823 & 87421429 & 57762164 & 29190006 \\
40034 & -3419924 & -10415002 & -2423731 & -1293022 & -300939 & 3852898 & 35230124 & 35230124 & 26193654 & 17306994 & 8746058 \\
92928 & -7938344 & -24167228 & -7241108 & -14826666 & -2672037 & 12833772 & 81776327 & 81776327 & 81776327 & 57738399 & 29177996 \\
19938 & -1703231 & -5185258 & -1553633 & -5183124 & -856637 & 3852898 & 17545717 & 17545717 & 17545717 & 17545717 & 8746058 \\
26879 & -2296121 & -6950233 & -2094449 & -6987357 & -2094449 & 2319497 & 23653338 & 23653338 & 23653338 & 23653338 & 23653338
\end{pmatrix}
\]
\[
\mathbf{M} = \begin{pmatrix}
405 & -27353 & -59598 & -15129 & -34281 & -7565 & -9045 & 281777 & 170859 & 161470 & 85409 & 76041 \\
-15129 & 1292407 & 3934560 & 1181347 & 2671281 & 589446 & 704810 & -13313645 & -13313645 & -12582089 & -6656822 & -5925266 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\( \text{lb}^2/\text{in} \quad \text{lb}^2 \quad \text{lb}^2 \quad \text{lb}^2 \quad \text{lb}^2 \quad \text{lb}^2 \quad \text{lb}^2 \quad \text{lb} \quad \text{lb} \quad \text{lb} \quad \text{lb} \quad \text{lb} \quad \text{lb} \quad \text{lb} \quad \text{lb} \quad \text{lb} \)
The eigenvalues are

\[
\lambda = \begin{pmatrix}
-3.648 + 7.278i \\
-3.648 - 7.278i \\
-2.931 - 4.406i \\
-2.931 + 4.406i \\
-2.694 + 6.472i \\
-2.694 - 6.472i \\
-2.254 - 2.415i \\
-2.254 + 2.415i \\
-1.701 - 3.521i \\
-1.701 + 3.521i \\
-1.303 + 3.062i \\
-1.303 - 3.062i \\
\end{pmatrix}
\]

\[\frac{1}{s^2}\]
The eigenvectors are as follows:

\[
X_1 = \begin{pmatrix}
-1.102 \times 10^{-5} - 1.121i \times 10^{-4} \\
-1.033 \times 10^{-5} + 1.016i \times 10^{-6} \\
2.931 \times 10^{-5} - 1.754i \times 10^{-5} \\
3.295 \times 10^{-4} + 1.188i \times 10^{-3} \\
-2.830 \times 10^{-4} - 9.829i \times 10^{-4} \\
7.100 \times 10^{-3} + 3.183i \times 10^{-3} \\
-4.859 \times 10^{-3} - 9.232i \times 10^{-4} \\
4.219 \times 10^{-6} + 3.331i \times 10^{-6} \\
-1.158 \times 10^{-4} + 9.930i \times 10^{-5} \\
2.046 \times 10^{-4} - 1.867i \times 10^{-4} \\
-5.104 \times 10^{-5} + 1.040i \times 10^{-3} \\
-2.074 \times 10^{-4} - 1.539i \times 10^{-3}
\end{pmatrix}
\]

\[
X_2 = \begin{pmatrix}
-1.102 \times 10^{-5} + 1.121i \times 10^{-4} \\
-1.033 \times 10^{-5} - 1.016i \times 10^{-6} \\
2.931 \times 10^{-5} + 1.754i \times 10^{-5} \\
3.295 \times 10^{-4} - 1.188i \times 10^{-3} \\
-2.830 \times 10^{-4} + 9.829i \times 10^{-4} \\
7.100 \times 10^{-3} - 3.183i \times 10^{-3} \\
-4.859 \times 10^{-3} + 9.232i \times 10^{-4} \\
4.219 \times 10^{-6} - 3.331i \times 10^{-6} \\
-1.158 \times 10^{-4} - 9.930i \times 10^{-5} \\
2.046 \times 10^{-4} + 1.867i \times 10^{-4} \\
-5.104 \times 10^{-5} - 1.040i \times 10^{-3} \\
-2.074 \times 10^{-4} + 1.539i \times 10^{-3}
\end{pmatrix}
\]
\[ X_3 = \begin{pmatrix}
-9.819 \times 10^{-4} + 1.988i \times 10^{-3} \\
-1.215 \times 10^{-4} - 3.453i \times 10^{-5} \\
1.428 \times 10^{-4} + 2.892i \times 10^{-4} \\
-1.393 \times 10^{-4} - 4.915i \times 10^{-6} \\
2.153 \times 10^{-4} - 1.145i \times 10^{-3} \\
5.428 \times 10^{-4} + 3.254i \times 10^{-4} \\
-3.194 \times 10^{-3} + 3.060i \times 10^{-3} \\
7.859 \times 10^{-5} - 7.697i \times 10^{-6} \\
-7.579 \times 10^{-5} + 1.359i \times 10^{-5} \\
-1.422 \times 10^{-4} - 1.751i \times 10^{-4} \\
2.656 \times 10^{-4} + 1.024i \times 10^{-4} \\
3.904 \times 10^{-5} + 8.742i \times 10^{-4}
\end{pmatrix} \]

\[ X_4 = \begin{pmatrix}
-9.819 \times 10^{-4} - 1.988i \times 10^{-3} \\
-1.215 \times 10^{-4} + 3.453i \times 10^{-5} \\
1.428 \times 10^{-4} - 2.892i \times 10^{-4} \\
-1.393 \times 10^{-4} + 4.915i \times 10^{-6} \\
2.153 \times 10^{-4} + 1.145i \times 10^{-3} \\
5.428 \times 10^{-4} - 3.254i \times 10^{-4} \\
-3.194 \times 10^{-3} - 3.060i \times 10^{-3} \\
7.859 \times 10^{-5} + 7.697i \times 10^{-6} \\
-7.579 \times 10^{-5} - 1.359i \times 10^{-5} \\
-1.422 \times 10^{-4} + 1.751i \times 10^{-4} \\
2.656 \times 10^{-4} - 1.024i \times 10^{-4} \\
3.904 \times 10^{-5} - 8.742i \times 10^{-4}
\end{pmatrix} \]
\[
X_5 = \begin{pmatrix}
6.525 \times 10^{-6} - 1.128i \times 10^{-4} \\
-8.748 \times 10^{-6} - 5.539i \times 10^{-8} \\
2.548 \times 10^{-5} - 1.548i \times 10^{-5} \\
3.083 \times 10^{-4} + 6.305i \times 10^{-4} \\
-2.472 \times 10^{-4} - 3.995i \times 10^{-4} \\
-5.296 \times 10^{-3} - 2.599i \times 10^{-4} \\
4.729 \times 10^{-3} + 1.382i \times 10^{-3} \\
3.908 \times 10^{-6} + 3.662i \times 10^{-6} \\
-6.957 \times 10^{-5} + 7.265i \times 10^{-5} \\
1.052 \times 10^{-4} - 1.296i \times 10^{-4} \\
-2.951 \times 10^{-4} - 6.572i \times 10^{-4} \\
3.333 \times 10^{-4} + 1.410i \times 10^{-3}
\end{pmatrix}
\]

\[
X_6 = \begin{pmatrix}
6.525 \times 10^{-6} + 1.128i \times 10^{-4} \\
-8.748 \times 10^{-6} + 5.539i \times 10^{-8} \\
2.548 \times 10^{-5} + 1.548i \times 10^{-5} \\
3.083 \times 10^{-4} - 6.305i \times 10^{-4} \\
-2.472 \times 10^{-4} + 3.995i \times 10^{-4} \\
-5.296 \times 10^{-3} + 2.599i \times 10^{-4} \\
4.729 \times 10^{-3} - 1.382i \times 10^{-3} \\
3.908 \times 10^{-6} - 3.662i \times 10^{-6} \\
-6.957 \times 10^{-5} - 7.265i \times 10^{-5} \\
1.052 \times 10^{-4} + 1.296i \times 10^{-4} \\
-2.951 \times 10^{-4} + 6.572i \times 10^{-4} \\
3.333 \times 10^{-4} - 1.410i \times 10^{-3}
\end{pmatrix}
\]
\[
X_7 = \begin{pmatrix}
2.540 \times 10^{-3} - 2.435i \times 10^{-3} \\
9.674 \times 10^{-5} + 5.213i \times 10^{-5} \\
1.489 \times 10^{-4} + 8.611i \times 10^{-5} \\
2.573 \times 10^{-4} - 1.766i \times 10^{-4} \\
4.698 \times 10^{-4} - 8.414i \times 10^{-5} \\
1.059 \times 10^{-4} - 9.852i \times 10^{-4} \\
1.025 \times 10^{-3} - 1.163i \times 10^{-3} \\
1.830 \times 10^{-5} - 4.531i \times 10^{-6} \\
-3.575 \times 10^{-5} - 7.826i \times 10^{-5} \\
6.436 \times 10^{-5} - 2.792i \times 10^{-5} \\
-2.746 \times 10^{-4} - 1.056i \times 10^{-4} \\
1.506 \times 10^{-4} - 2.399i \times 10^{-4}
\end{pmatrix}
\]

\[
X_8 = \begin{pmatrix}
2.540 \times 10^{-3} + 2.435i \times 10^{-3} \\
9.674 \times 10^{-5} - 5.213i \times 10^{-5} \\
1.489 \times 10^{-4} + 8.611i \times 10^{-5} \\
2.573 \times 10^{-4} + 1.766i \times 10^{-4} \\
4.698 \times 10^{-4} + 8.414i \times 10^{-5} \\
1.059 \times 10^{-4} + 9.852i \times 10^{-4} \\
1.025 \times 10^{-3} + 1.163i \times 10^{-3} \\
1.830 \times 10^{-5} + 4.531i \times 10^{-6} \\
-3.575 \times 10^{-5} + 7.826i \times 10^{-5} \\
6.436 \times 10^{-5} + 2.792i \times 10^{-5} \\
-2.746 \times 10^{-4} + 1.056i \times 10^{-4} \\
1.506 \times 10^{-4} + 2.399i \times 10^{-4}
\end{pmatrix}
\]
\[
X_9 = \begin{pmatrix}
2.781 \times 10^{-5} + 4.731i \times 10^{-5} \\
-1.936 \times 10^{-6} + 1.130i \times 10^{-6} \\
4.628 \times 10^{-6} + 7.309i \times 10^{-6} \\
8.447 \times 10^{-5} + 7.912i \times 10^{-5} \\
-3.702 \times 10^{-5} + 9.339i \times 10^{-5} \\
3.209 \times 10^{-6} - 1.480i \times 10^{-3} \\
2.131 \times 10^{-3} - 1.169i \times 10^{-3} \\
2.153 \times 10^{-6} - 8.240i \times 10^{-7} \\
2.542 \times 10^{-5} - 1.039i \times 10^{-5} \\
-1.023 \times 10^{-5} + 2.956i \times 10^{-5} \\
-3.578 \times 10^{-4} - 1.843i \times 10^{-4} \\
3.083 \times 10^{-4} - 4.554i \times 10^{-4}
\end{pmatrix}
\]

\[
X_{10} = \begin{pmatrix}
2.781 \times 10^{-5} - 4.731i \times 10^{-5} \\
-1.936 \times 10^{-6} - 1.130i \times 10^{-6} \\
4.628 \times 10^{-6} - 7.309i \times 10^{-6} \\
8.447 \times 10^{-5} - 7.912i \times 10^{-5} \\
-3.702 \times 10^{-5} - 9.339i \times 10^{-5} \\
3.209 \times 10^{-6} + 1.480i \times 10^{-3} \\
2.131 \times 10^{-3} + 1.169i \times 10^{-3} \\
2.153 \times 10^{-6} + 8.240i \times 10^{-7} \\
2.542 \times 10^{-5} + 1.039i \times 10^{-5} \\
-1.023 \times 10^{-5} - 2.956i \times 10^{-5} \\
-3.578 \times 10^{-4} + 1.843i \times 10^{-4} \\
3.083 \times 10^{-4} + 4.554i \times 10^{-4}
\end{pmatrix}
\]
\[
X_{11} = \begin{pmatrix}
1.609 \times 10^{-5} - 2.329i \times 10^{-5} \\
-8.546 \times 10^{-7} - 4.844i \times 10^{-7} \\
1.337 \times 10^{-6} - 6.089i \times 10^{-6} \\
3.505 \times 10^{-5} - 8.723i \times 10^{-5} \\
-7.649 \times 10^{-5} - 1.928i \times 10^{-4} \\
-4.433 \times 10^{-4} - 1.404i \times 10^{-3} \\
-1.867 \times 10^{-3} - 6.760i \times 10^{-4} \\
1.808 \times 10^{-6} - 5.362i \times 10^{-8} \\
2.640 \times 10^{-5} - 2.260i \times 10^{-7} \\
1.607 \times 10^{-5} - 4.327i \times 10^{-5} \\
2.917 \times 10^{-4} - 2.440i \times 10^{-4} \\
-3.688 \times 10^{-4} - 3.079i \times 10^{-4}
\end{pmatrix}
\]

\[
X_{12} = \begin{pmatrix}
1.609 \times 10^{-5} + 2.329i \times 10^{-5} \\
-8.546 \times 10^{-7} + 4.844i \times 10^{-7} \\
1.337 \times 10^{-6} + 6.089i \times 10^{-6} \\
3.505 \times 10^{-5} + 8.723i \times 10^{-5} \\
-7.649 \times 10^{-5} + 1.928i \times 10^{-4} \\
-4.433 \times 10^{-4} + 1.404i \times 10^{-3} \\
-1.867 \times 10^{-3} + 6.760i \times 10^{-4} \\
1.808 \times 10^{-6} + 5.362i \times 10^{-8} \\
2.640 \times 10^{-5} + 2.260i \times 10^{-7} \\
1.607 \times 10^{-5} + 4.327i \times 10^{-5} \\
2.917 \times 10^{-4} + 2.440i \times 10^{-4} \\
-3.688 \times 10^{-4} + 3.079i \times 10^{-4}
\end{pmatrix}
\]