A STUDY OF THE LONGTERM DYNAMICS OF A DISCRETIZATION OF A LIGHT VISCOELASTIC ROD CARRYING A HEAVY BLOCK

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A STUDY OF THE LONGTERM DYNAMICS OF A DISCRETIZATION OF A LIGHT VISCOELASTIC ROD CARRYING A HEAVY BLOCK

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ABSTRACT

A heavy rigid body attached to a relatively light deformable body is a common structure in many mechanical systems. The longterm dynamics of this structure characterize the true features of the structure response to external excitation. Therefore, the study of the longterm dynamics can provide fundamental guidance for engineering design and production. In this thesis, we study a system that describes the longitudinal motion of a light viscoelastic rod carrying a heavy particle. We discretize this problem by replacing the rod with $K$ light particles connected by massless springs. The discretized problem is governed by a system of coupled nonlinear ordinary differential equations. We let $\epsilon$ denote the ratio of the mass of one of the light particles to the mass of the end particle. This introduces a small parameter into the problem. We use a singular perturbation approach developed by O’Malley and Hoppensteadt to analyze this problem. We focus on the $K = 1$ case; and the longterm dynamics of this case is reduced to the longterm dynamics of the order 1 outer problem based on the matching condition and the approximate solutions. The longterm dynamics of the order 1 outer problem depends on whether the restoring force $f$ of the springs in the discretized system is monotone or non-monotone. For monotone restoring forces, we show that there exists an invariant manifold $\mathcal{M}$ that attracts all solutions to a
system equivalent to the order 1 outer problem. The dynamics on $M$ is governed by a classical second-order ordinary differential equation. We also illustrate that the longterm dynamics on $M$ determines the longterm dynamics of the order 1 outer problem. For non-monotone restoring forces, the invariant manifold $M$ still exists. We show by considering a specific example that $M$ fails to attract all solutions.
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CHAPTER I
INTRODUCTION

1.1 Motivation and Problem Statement

A common structure in many mechanical systems is that of a heavy rigid body attached to a relatively light deformable body. Examples include the weight attached to a cable on a crane boom, an automotive engine supported by rubber engine mounts, or a heavy garage door attached to springs. A less familiar example is a rigid silicon tip attached to a flexible arm in a atomic force microscopic. Therefore, we need to study the dynamics of coupled systems. Basic engineering issues can often be understood by formulating idealized or toy problems. Idealized versions of the real-world engineering problems mentioned above include a particle attached to a light deformable rod with longitude motion, and a flywheel on crankshaft with torsional motion. These idealized examples describe essential features of coupled systems, and hence their study can provide fundamental guidance for engineering design and production.

The static analysis of these problems is often not enough, especially for systems subject to external excitation or forcing. Techniques from dynamical system theory can be used to investigate the long-term dynamics of the system under external excitation. The long-term dynamics describe the steady state of a system after
transient oscillation. In this sense, it characterizes the true features of the system response to external excitation.

In this paper, we study a system that describes the longitudinal motion of a light viscoelastic rod carrying a heavy particle. We discretize this problem by replacing the rod with $K$ light particles connected by massless springs. See Figure 1.1. The discretized problem is governed by a system of coupled nonlinear ordinary differential equations. We let $\epsilon$ denote the ratio of the mass of one of the light particles to the mass of the end particle. This introduces a small parameter into the problem. We use a singular perturbation approach developed by O’Malley and Hoppensteadt to analyze this problem.

![Discretized Model](image)

Figure 1.1: Discretized Model.
1.2 Literature Review

In this section, we briefly review some literature relevant to our problem. The dynamics of systems of rigid bodies coupled to light deformable bodies has attracted increased study in recent years. As an example, we mention a paper by Georgiou and Schwartz [1]. In [1], the authors studied the dynamics of the conservative flexible spherical pendulum. They used the method of geometric singular perturbations to compute analytically a slow invariant manifold of motion, which determines the reduced dynamics. They applied the proper orthogonal decomposition method to compute the shapes and amplitudes of proper orthogonal modes for the dynamics on the slow invariant manifold and showed this manifold is characterised by proper orthogonal modes.

The dynamics of collections of particles coupled by nonlinear springs has been studied by numerous authors in recent years. Many of these papers focus on using the discrete system to understand some aspect of phase transitions in crystalline solids. We mention several examples. In [2], Cardin and Favretti studied a discretized model of an elastic bar in a hard device, so that one end of the chain is fixed and the length of the chain is a parameter. They applied ideas from relaxation oscillation theory to understand the dynamics of the system. In [3], Truskinovsky and Vainchtein explicitly computed an appropriate closing relation by replacing a continuum model of a viscoelastic body with its natural discrete prototype. They modelled phase boundaries by travelling wave solutions of a fully inertial discrete
model for a bi-stable lattice with harmonic long-range interactions. As a result, they showed the effect of nonlocality of the lattice model on kinetic relation and kinetics in the near-sonic region, and the stability of some of the travelling waves by numerical results. In [4], Trofimov and Vainchtein studied dynamics of phase boundaries in a bistable one-dimensional lattice with harmonic long-range interactions. They formulated the discrete model for the one-dimensional lattice and constructed travelling wave solutions that represent both subsonic phase boundaries (kinks) and intersonic phase boundaries (shocks). They showed that the model parameters have a significant effect on the existence, structure and stability of the travelling waves as well as their behaviour near the sonic limit.

Of direct relevance for our study is a series of papers by Antman and Wilber. This series of papers began with [5], in which Antman studied the motion of a particle on a spring. The spring is modeled as a continuum, and the problem is studied in the limit as the mass density of the spring goes to zero. The author showed that for viscoelastic springs, the limiting problem is typically an ordinary functional differential equation with memory, even though the constitutive function for the spring does not depend on the past history of the deformation. In [6], Antman and Wilber studied the same problem. These authors considered the longterm dynamics of a reduced problem that is obtained by setting the ratio of the inertia of the viscoelastic spring to the inertia of the attached particle equal to zero. Using dynamical systems theory, they proved the existence of an attractor that is contained in an invariant two-dimensional manifold for the degenerate partial differential equation of the re-
duced problem. In [7], Yip et al. studied the longitudinal motion of a nonlinearly viscoelastic rod with one end fixed and the other end attached to a heavy particle. Using the same asymptotic techniques as used in this paper though in a much more technical setting, they constructed expansions for solutions of this problem and justified the asymptotic validity by error estimation. In [8], Wilber studied how the long-term dynamics of the reduced problem determine the long-term dynamics of the full problem when $0 < \epsilon << 1$. In [8], it is assumed that the restoring force of the spring is monotone. To solve the problem, it is natural to transform this problem to a lower dimensional equivalent problem, which is easier to analyze. Wilber studied the problem using geometric singular perturbation theory.

1.3 Summary of Main Results

In this thesis we study the same problem that Wilber studied in [8]. The governing equations for this problem contain a nonlinear function $f$ that describes the restoring force of the springs connecting the discrete particles. The main differences are that here we study both monotone and non-monotone restoring forces and we use the same asymptotic technique as applied in [7] instead of geometric singular perturbation theory. In this thesis, our goal is to understand the longterm dynamics of the simplest case of the discretized system, with $K = 1$. See Figure 2.1 below. We construct the approximate solutions using O’Malley/Hoppensteadt method, and focus on a system equivalent to the order 1 outer problem since the longterm dynamics of the outer problem determines the longterm dynamics of the full system. After presenting the
asymptotics we present some background material on thermodynamics and solids that undergo phase transformations in order to motivate our study on the cases of monotone and non-monotone restoring forces.

For the monotone case, we prove the existence of invariant manifold $\mathcal{M}$ and its attractivity for all solutions to the equivalent system in Theorem 4.1; we also study the dynamics of the order 1 outer problem on the invariant manifold and prove the existence of a trapping set in Theorem 4.2. To show the consistency of these two theorems, we define a class of monotone functions that satisfy both the hypotheses on the restoring force $f$ in Theorem 4.1 and the hypotheses in Theorem 4.2. Lastly, we briefly study a specific example with $f$ monotone to illustrate numerically the results of our theorems.

For the non-monotone case, we first comment on whether the proofs of the results for the monotone case still work for the case in which $f$ is non-monotone. We find that the invariant manifold $\mathcal{M}$ still exists whether $f$ is monotone or not. But $\mathcal{M}$ fails to attract all solutions to the equivalent system since it appears that some type of monotonicity hypothesis is essential to Theorem 4.1. Then we study a specific case with $f = x^3 - x$ and without a forcing term. We study the fixed-point structure and perform the stability analysis of the fixed points. The result shows that some solutions to the equivalent system starting a little bit off of $\mathcal{M}$ will move away from $\mathcal{M}$ and approach a fixed point not on $\mathcal{M}$ as $t \to \infty$.

The structure of this thesis is as follows. In Chapter 2, we introduce the discrete model and the governing equations. In Chapter 3, we present some background
material on continuum thermodynamics. In Chapter 4, we present results that de-
scribe the longterm dynamics of the order 1 outer problem for the case in which
the restoring force $f$ is monotone. In Chapter 5, we study the longterm dynamics
of the order 1 outer problem for a specific example in which the restoring force is
non-monotone.
CHAPTER II
GOVERNING EQUATIONS

In this chapter, we present the governing equations for the discretized model in Figure 1.1. Using O’Malley/Hoppensteadt method [9], we construct the approximate solutions for the simplest case of discretization, $K = 1$; and get the outer system and the boundary-layer correction system. Lastly, we verify that the solutions provided by O’Malley/Hoppensteadt construction provide uniform approximations to the exact solution of our problem.

2.1 Governing Equations

To analyze the forced longitudinal motion of a light viscoelastic rod carrying a heavy particle, we consider the discretized model, shown in Figure 1.1. The light rod is replaced by $K$ light particles connected by massless springs. The heavy particle is joined to the right-most light particle by a massless spring. Using Newton’s Second
Law, we see that the governing equations of the discretized problem are

\[\begin{align*}
\epsilon m \ddot{x}_1 &= -f(x_1) + f(x_2 - x_1) - \eta \dot{x}_1 + \eta (\dot{x}_2 - \dot{x}_1), \\
\epsilon m \ddot{x}_2 &= -f(x_2 - x_1) + f(x_3 - x_2) - \eta (\dot{x}_2 - \dot{x}_1) + \eta (\dot{x}_3 - \dot{x}_2), \\
&\quad \vdots \\
\epsilon m \ddot{x}_K &= -f(x_K - x_{K-1}) + f(w - x_K) - \eta (\dot{x}_K - \dot{x}_{K-1}) + \eta (\dot{w} - \dot{x}_K), \\
m \ddot{w} &= -f(w - x_K) - \eta (\dot{w} - \dot{x}_K) + F(t),
\end{align*}\]

in which \(x_i\) is the displacement of the \(i\)th light particle from its equilibrium position, \(w\) is the displacement of the heavy particle from its equilibrium position, \(f\) is a possibly nonlinear function describing the restoring force of each spring, \(\eta\) is a linear damping constant associated with each spring, \(\epsilon\) is a small positive parameter, \(m\) is the mass of the heavy particle, \(\epsilon m\) is the mass of each light particle, and \(F\) is a function describing an external force on the heavy particle.

For the simplest case of discretization, namely \(K = 1\), which is shown in Figure 2.1, the governing system is

\[\begin{align*}
\epsilon m \ddot{x} &= -f(x) + f(w - x) - \eta \dot{x} + \eta (\dot{w} - \dot{x}), \\
m \ddot{w} &= -f(w - x) - \eta (\dot{w} - \dot{x}) + F(t),
\end{align*}\]

in which \(x\) is the displacement of the light particle from its equilibrium position, which corresponds to the natural length of the first spring, \(w\) is the displacement of the heavy particle from its equilibrium position, which corresponds to the sum of the natural lengths of both springs, and \(\epsilon, m, f, \eta\) and \(F\) are the same as in (2.1).
After rewriting (2.2) as a first-order system and assuming, without loss of generality, that \( m = 1 \), we get

\[
\dot{x} = y, \\
\epsilon \dot{y} = -f(x) + f(w - x) - \eta y + \eta(u - y), \\
\dot{w} = u, \\
\dot{u} = -f(w - x) - \eta(u - y) + F(t),
\]

where \( \dot{x} \) denotes \( dx/dt \), etc.

We assume that \( f \) is continuously differentiable on \( \mathbb{R} \) and that \( F \) is continuous and bounded on \( \mathbb{R} \). Additional assumptions about these functions will be stated later.

2.2 Expansions

To construct approximate solutions to this problem, we apply an asymptotic technique known as the O’Malley/Hoppensteadt construction [9]. We apply this method on (2.3), the case \( K = 1 \), which is the case on which this thesis focuses. We introduce
the fast time scale $\tau$ as

$$\tau = \frac{t}{\epsilon}.$$  \hspace{1cm} (2.4)

System (2.3) can be reformulated as the following fast system

$$\begin{align*}
\frac{dx}{d\tau} &= \epsilon y, \\
\frac{dy}{d\tau} &= -f(x) + f(w - x) - \eta y + \eta(u - y), \\
\frac{dw}{d\tau} &= \epsilon u, \\
\frac{du}{d\tau} &= \epsilon[-f(w - x) - \eta(u - y) + F(t)].
\end{align*}$$  \hspace{1cm} (2.5)

According to (2.5), for an initial condition in $\mathbb{R}^4$ which is far from the set of points where the right-hand side of the second equation vanishes, $\frac{dy}{d\tau}$ would be much larger than $\frac{dx}{d\tau}$, $\frac{dw}{d\tau}$ and $\frac{du}{d\tau}$, which all would be essentially zero. Hence $y$ should change rapidly near $t = 0$ with $x$, $w$, $u$ essentially constant. This continues until the solution gets close to set of points where $-f(x) + f(w - x) - \eta y + \eta(u - y) = 0$. So there is a small boundary-layer region near $t = 0$ to describe the rapid change of the solution.

Following the O’Malley/Hoppensteadt method, we capture this behavior by seeking asymptotic representations of the solution $x$, $y$, $w$, and $u$ of (2.3) in the form

$$\begin{align*}
x(t, \epsilon) &\sim X(t, \epsilon) + \epsilon X^*(\tau, \epsilon), \\
y(t, \epsilon) &\sim Y(t, \epsilon) + Y^*(\tau, \epsilon), \\
w(t, \epsilon) &\sim W(t, \epsilon) + \epsilon W^*(\tau, \epsilon), \\
u(t, \epsilon) &\sim U(t, \epsilon) + \epsilon U^*(\tau, \epsilon).
\end{align*}$$  \hspace{1cm} (2.6)
for suitable outer functions $X(t, \epsilon), Y(t, \epsilon), W(t, \epsilon), U(t, \epsilon)$ and suitable boundary-layer correction functions $X^*(\tau, \epsilon), Y^*(\tau, \epsilon), W^*(\tau, \epsilon), U^*(\tau, \epsilon)$. A suitable approximation to the solution of (2.3) consists of the outer approximation, which approximates the solution outside the boundary-layer region, plus the boundary-layer correction term, which is negligible outside the boundary-layer region, but which is required for approximation of the solution inside the boundary-layer region.

Now we introduce the asymptotic expansions

$$\begin{bmatrix} X(t, \epsilon) \\ Y(t, \epsilon) \\ W(t, \epsilon) \\ U(t, \epsilon) \end{bmatrix} \sim \sum_{k=0}^{\infty} \begin{bmatrix} X_k(t) \\ Y_k(t) \\ W_k(t) \\ U_k(t) \end{bmatrix} \epsilon^k \quad (2.7)$$

and

$$\begin{bmatrix} X^*(\tau, \epsilon) \\ Y^*(\tau, \epsilon) \\ W^*(\tau, \epsilon) \\ U^*(\tau, \epsilon) \end{bmatrix} \sim \sum_{k=0}^{\infty} \begin{bmatrix} X_k^*(\tau) \\ Y_k^*(\tau) \\ W_k^*(\tau) \\ U_k^*(\tau) \end{bmatrix} \epsilon^k. \quad (2.8)$$

The boundary-layer correction functions $X^*(\tau, \epsilon), Y^*(\tau, \epsilon), W^*(\tau, \epsilon), U^*(\tau, \epsilon)$ must be negligible when $\tau \to \infty$, so we impose the matching conditions

$$\lim_{\tau \to \infty} \begin{bmatrix} X_k^*(\tau) \\ Y_k^*(\tau) \\ W_k^*(\tau) \\ U_k^*(\tau) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{for } k = 0, 1, 2, \ldots. \quad (2.9)$$
Following [9], we require that the outer solution functions $X, Y, W, U$ satisfy the full system (2.3) for $t > 0$,

\[
\begin{align*}
\frac{dX}{dt} &= Y, \\
\epsilon \frac{dY}{dt} &= -f(X) + f(W - X) - \eta Y + \eta(U - Y), \\
\frac{dW}{dt} &= U, \\
\frac{dU}{dt} &= -f(W - X) - \eta(U - Y) + F(t).
\end{align*}
\]

(2.10)

From (2.3), (2.6), and (2.10), we require that the boundary-layer correction functions $X^*, Y^*, W^*, U^*$ satisfy the system

\[
\begin{align*}
\frac{dX^*}{d\tau} &= Y^*, \\
\frac{dY^*}{d\tau} &= -f(X + \epsilon X^*) + f(W + \epsilon W^* - X - \epsilon X^*) + f(X) - f(W - X) + \eta \epsilon U^* - 2\eta Y^*, \\
\frac{dW^*}{d\tau} &= \epsilon U^*, \\
\frac{dU^*}{d\tau} &= -f(W + \epsilon W^* - X - \epsilon X^*) + f(W - X) - \eta \epsilon U^* + \eta Y^*.
\end{align*}
\]

(2.11)

Then the expansions (2.7) are used in the outer system (2.10), from which we can get the following problem

\[
\begin{align*}
\frac{dX_0}{dt} &= Y_0, \\
0 &= -f(X_0) + f(W_0 - X_0) + \eta(U_0 - 2Y_0), \\
\frac{dW_0}{dt} &= U_0, \\
\frac{dU_0}{dt} &= -f(W_0 - X_0) - \eta(U_0 - Y_0) + F(t).
\end{align*}
\]

(2.12)
for the outer coefficients $X_0$, $Y_0$, $W_0$ and $U_0$, and

\[
\begin{align*}
\frac{dX_1}{dt} &= Y_1, \\
\frac{dY_0}{dt} &= -X_1 f'(X_0) - (W_1 - X_1) f'(W_0 - X_0) + \eta(U_1 - 2Y_1), \\
\frac{dW_1}{dt} &= U_1, \\
\frac{dU_1}{dt} &= -(W_1 - X_1) f'(W_0 - X_0) - \eta(U_1 - Y_1),
\end{align*}
\]

for $X_1$, $Y_1$, $W_1$ and $U_1$. In general,

\[
\begin{align*}
\frac{dX_k}{dt} &= Y_k, \\
\frac{dY_{k-1}}{dt} &= -X_k f'(X_0) - (W_k - X_k) f'(W_0 - X_0) + \eta(U_k - 2Y_k) + A_{k-1} \\
\frac{dW_k}{dt} &= U_k, \\
\frac{dU_k}{dt} &= (W_k - X_k) f'(W_0 - X_0) - \eta(U_k - Y_k) + B_{k-1},
\end{align*}
\]

for $k = 2, 3, \ldots$, with

\[
\begin{align*}
A_1 &= -\frac{1}{2} X_1^2 f''(X_0) + \frac{1}{2} (W_1 - X_1)^2 f''(W_0 - X_0), \\
B_1 &= -\frac{1}{2} (W_1 - X_1)^2 f''(W_0 - X_0),
\end{align*}
\]

and in general, with $A_{k-1}$ and $B_{k-1}$ are defined in terms of $X_j$, $W_j$ for $j \leq k - 1$. Similarly, the expansions of (2.7) and (2.8) are plugged into the boundary-layer correction system (2.11) to get the following problem for the boundary-layer correction
coefficient $X_0^*, Y_0^*, W_0^*$ and $U_0^*$:

\[
\begin{align*}
\frac{dX_0^*}{d\tau} &= Y_0^*, \\
\frac{dY_0^*}{d\tau} &= -2\eta Y_0^*, \\
\frac{dW_0^*}{d\tau} &= 0, \\
\frac{dU_0^*}{d\tau} &= \eta Y_0^*,
\end{align*}
\]  

(2.16)

and for $X_1^*, Y_1^*, W_1^*$ and $U_1^*$

\[
\begin{align*}
\frac{dX_1^*}{d\tau} &= Y_1^*, \\
\frac{dY_1^*}{d\tau} &= -X_0^* f'(X_0) + (W_0^* - X_0^*) f'(W_0 - X_0) + \eta U_0^* - 2\eta Y_1^*, \\
\frac{dW_1^*}{d\tau} &= U_0^*, \\
\frac{dU_1^*}{d\tau} &= -(W_0^* - X_0^*) f'(W_0 - X_0) - \eta U_0^* + \eta Y_1^*.
\end{align*}
\]  

(2.17)

One could compute the equations for the higher-order coefficients in a similar way.

Next, we verify the order 1 outer problem (2.12) satisfies assumptions A.1 and A.2 in Section 6.2 of [9]. These hypotheses are necessary for the proof of Theorem 6.3.1 in [9], which shows that truncating the expansions formed by (2.6), (2.7) and (2.8) provides an approximation to the solution of (2.3) that is uniform for $\epsilon$ small and uniform on bounded time intervals. To verify A.1, we define

\[
\phi(X, W, U) = \frac{1}{2\eta}[-f(X) + f(W - X) + \eta U].
\]  

(2.18)

We denote the right-hand side of the second equation of (2.12) as $V(X, Y, W, U)$, then

\[
V(X, W, U, \phi(X, W, U)) = 0
\]  

(2.19)
if and only if \( Y = \phi(X, W, U) \).

Now we consider the initial-value problem

\[
\begin{align*}
\frac{dX}{dt} &= \phi(X, W, U), \\
\frac{dW}{dt} &= U, \\
\frac{dU}{dt} &= -f(W - X) - \eta[U - \phi(X, W, U)] + F(t),
\end{align*}
\]

for \( t > 0 \) and \( X = X(0), W = W(0), U = U(0) \). Assuming that \( f \) is continuously differentiable on \( \mathbb{R} \) and that \( F \) is continuous and bounded on \( \mathbb{R} \), standard results for ordinary differential equations tell us that (2.20) has a solution \( X = X(t), W = W(t), U = U(t) \) on some interval \( 0 \leq t \leq T \). Then we compute

\[
V_Y(X(t), W(t), U(t), Y(t)) = -2\eta \leq -\kappa < 0 \quad \text{for} \quad 0 \leq t \leq T
\]

for some fixed positive constant \( \kappa > 0 \), where

\[
Y(t) = \phi(X(t), W(t), U(t)) \quad \text{for} \quad 0 \leq t \leq T.
\]

This shows that A.1 is valid.

Also, because

\[
V_Y = -2\eta < 0,
\]

A.2 is valid.

Now by Theorem 6.3.1 in [9], we get

\[
|x(t, \epsilon) - x^N(t, \epsilon)|, |y(t, \epsilon) - y^N(t, \epsilon)|, |w(t, \epsilon) - w^N(t, \epsilon)|, |u(t, \epsilon) - u^N(t, \epsilon)| \leq C_N\epsilon^{N+1}
\]

(2.24)
uniformly for all $0 \leq t \leq T$, $0 < \epsilon \leq \epsilon_1$, where $x^N, y^N, w^N, u^N$, the O’Malley/Hoppensteadt approximate functions considering the truncated expansions, are in the form of

\begin{align}
    x^N(t, \epsilon) &= \sum_{k=0}^{N} \left[ X_k(t) + X^*_k(t, \epsilon) \right] \epsilon^k, \\
y^N(t, \epsilon) &= \sum_{k=0}^{N} \left[ Y_k(t) + Y^*_k(t, \epsilon) \right] \epsilon^k, \\
w^N(t, \epsilon) &= \sum_{k=0}^{N} \left[ W_k(t) + W^*_k(t, \epsilon) \right] \epsilon^k, \\
u^N(t, \epsilon) &= \sum_{k=0}^{N} \left[ U_k(t) + U^*_k(t, \epsilon) \right] \epsilon^k,
\end{align}

where $X^*_{k-1}(\tau) = 0, W^*_{k-1}(\tau) = 0, U^*_{k-1}(\tau) = 0$. This tells us that the O’Malley/Hoppensteadt functions provide uniform approximations to the exact solution of (2.3).

2.3 Longterm Dynamics

Our goal is to describe the longterm dynamics of (2.3) within the framework of the O’Malley/Hoppensteadt construction. Based on the matching conditions (2.9), the longterm dynamics of (2.3) is determined by the longterm dynamics of the outer problem (2.10). Solutions to (2.10) are approximated by the order 1 outer problem (2.12). Hence we focus on results for (2.12) in Chapters IV and V below.

What we can prove about (2.12) depends on whether the function $f$, which describes the restoring force of the springs in our discretization, is monotone or non-monotone. The next chapter gives some background motivating our consideration of non-monotone restoring forces.
In this chapter we present some background material on continuum thermodynamics. Our goal is to briefly discuss materials that undergo phase transitions, which motivates our study of monotone versus non-monotone restoring forces in later chapters. First, we need to develop a theory to describe the motion of the thermo-elastic bar. The material in this chapter is drawn from Chapters 2 and 3 of [10].

We note that in Section 3.4 below we introduce the constitutive function $\hat{\sigma}$ for the contact force. As described in Section 3.4, the assumption that $\hat{\sigma}$ is non-monotone in the stretch is often used as a simple model of materials that undergo phase transformations. In the mechanical discretization we study in this thesis, the restoring force $f$ of the spring plays the role of the contact force. This explains why we associate the case of $f$ non-monotone with phase transformations.

3.1 Thermodynamic Properties

In this section, we consider the one-dimensional theory of heat transfer in a rigid, stationary bar to illustrate some general thermodynamics ideas. Mathematically, we define a bar as a subset of $\mathbb{R}$ of the form

$$\{x \in \mathbb{R} : 0 \leq x \leq L\},$$
where $L$ is the length of the bar and the points $x$ in the bar are called material points.

Then, we define two functions of $x$ and time $t$. The function $\epsilon(x, t)$ is the internal energy density of the bar at the material point $x$ at time $t$. For $0 \leq x_1 < x_2 \leq L$, $\int_{x_1}^{x_2} \epsilon(x, t)dx$ is the internal energy of the part of the bar $(x_1, x_2)$ at time $t$. Also, we define $q(x, t)$ as the heat flux. It is the rate at which heat is transferred from right to left at the point $x$ at time $t$. In this setting, the First Law of Thermodynamics says

$$\frac{d}{dt} \int_{x_1}^{x_2} \epsilon(x, t)dx = q(x_2, t) - q(x_1, t),$$

(3.1)

which describes the rate of change of total energy between points $x_1$ and $x_2$ in the bar.

If we assume $\epsilon$ and $q$ are smooth functions, we have

$$\frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \frac{\partial}{\partial t} \epsilon(x, t)dx = \frac{q(x_2, t) - q(x_1, t)}{x_2 - x_1}. $$

(3.2)

Applying the Mean Value Theorem for integrals to (5.16), we get

$$\frac{\partial}{\partial t} \epsilon(\bar{x}, t) = \frac{q(x_2, t) - q(x_1, t)}{x_2 - x_1},$$

(3.3)

for some $\bar{x} \in (x_1, x_2)$. Now we take the limit as $x_1 \to x_2$, we can get the classical form or local form of the first law

$$\frac{\partial \epsilon}{\partial t} = \frac{\partial q}{\partial x}.$$  

(3.4)
3.2 Constitutive Assumptions

We let $\theta(x, t)$ denotes temperature of the bar at position $x$ at time $t$. Then we assume that

$$\epsilon(x, t) = \hat{\epsilon}(\theta(x, t)), \quad (3.5)$$

which says the internal energy at a point depends on temperature at that point.

The function $\hat{\epsilon}$ is called a constitutive function. Its exact form depends on the material of which the bar is made. One often assumes that $\frac{d\hat{\epsilon}}{d\theta} > 0$, which means that the internal energy is an increasing function of temperature. Likewise, one often assumes that

$$q(x, t) = \hat{q}(\theta(x, t), \frac{\partial \theta}{\partial x}(x, t)). \quad (3.6)$$

The constitutive function $\hat{q}$ also depends on the particular material under study.

Fourier’s Law of Heat Conduction corresponds to the choice

$$\hat{q}(\theta, \frac{\partial \theta}{\partial x}) = \kappa \frac{\partial \theta}{\partial x}, \quad (3.7)$$

where $\kappa$ is a positive constant.

Now, by putting the constitutive function (3.5) and Fourier’s law (3.7) into the classical form of the first law (3.4), we have

$$\frac{d\hat{\epsilon}}{d\theta} \frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial x^2} \quad (3.8)$$

as the temperature equation. If we consider the case in which the internal energy depends linearly on the temperature, we obtain the heat equation

$$C_0 \frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial x^2}, \quad (3.9)$$
with the constant $C_0 = \frac{d\tilde{c}}{d\tilde{\sigma}}$.

### 3.3 Mechanics

Here we generalize the ideas in the previous subsections, allowing the bar to stretch or compress. As before, we identify points in the bar with the interval $[0, L]$. We can think of this as the configuration of the bar when no forces are acting on the bar, and we call this configuration the reference configuration of the bar. We let $\rho(x)$ be the mass per unit reference length of the bar at the point $x$, i.e., $\rho$ is the density. So the mass of the part of the bar $(x_1, x_2)$ is $\int_{x_1}^{x_2} \rho(x)dx$. Suppose that we apply forces to the bar and the bar starts to move. We describe the motion by letting $y(x, t)$ denote the position of the point $x$ at time $t$.

Two related quantities of interest are the velocity

$$\dot{y}(x, t) = \frac{\partial}{\partial t} y(x, t) \quad (3.10)$$

and the stretch

$$\lambda = \frac{\partial}{\partial x} y(x, t). \quad (3.11)$$

Note that $\frac{\partial}{\partial x} y(x, t) \approx \frac{y(x+h, t) - y(x, t)}{h}$, which is the ratio of the deformed length to the reference length. So the stretch $\lambda$ is a dimensionless measure of changes of length.

We let $\sigma(x, t)$ denote the contact force that the part of the bar $(x, L]$ exerts on the part $[0, x]$ at time $t$. All other forces acting on the bar are ignored. Newton’s
Second Law yields the equation of motion for the part of the bar \((x_1, x_2)\)
\[
\frac{d}{dt} \int_{x_1}^{x_2} \rho(x) \dot{y}(x,t) dx = \sigma(x_2, t) - \sigma(x_1, t). \tag{3.12}
\]

Just as we did for the classical form of the First Law of Thermodynamics above, we can derive the following classical form of equation of motion
\[
\rho \ddot{y} = \frac{\partial \sigma}{\partial x}. \tag{3.13}
\]

Next, we write down the First Law of Thermodynamics when the bar is in motion. First, the energy of the part of the bar \((x_1, x_2)\) at time \(t\) now looks like
\[
\int_{x_1}^{x_2} \left( \epsilon + \frac{1}{2} \rho \dot{y}^2 \right) dx. \tag{3.14}
\]
Here \(\epsilon\) is the internal energy stored in the bar, and \(\frac{1}{2} \rho \dot{y}^2\) represents the kinetic energy. To the right-hand side of the first law, we add the mechanical power \(\sigma(x_2, t) \dot{y}(x_2, t) - \sigma(x_1, t) \dot{y}(x_1, t)\).

Now we can get the First Law of Thermodynamics for the bar
\[
\frac{d}{dt} \int_{x_1}^{x_2} \left( \epsilon + \frac{1}{2} \rho \dot{y}^2 \right) dx = q(x_2, t) - q(x_1, t) + \sigma(x_2, t) \dot{y}(x_2, t) - \sigma(x_1, t) \dot{y}(x_1, t), \tag{3.14}
\]
which implies
\[
\dot{\epsilon} + \rho \dot{y} \ddot{y} = \frac{\partial q}{\partial x} + \frac{\partial (\sigma \dot{y})}{\partial x}. \tag{3.15}
\]

Then applying the classical form of equation of motion (3.13), we can derive the classical form of (3.14)
\[
\dot{\epsilon} = \frac{\partial q}{\partial x} + \sigma \frac{\partial \dot{y}}{\partial x}. \tag{3.16}
\]

Next, we assume the contact force is given by
\[
\sigma(x, t) = \partial_{\theta} \frac{\partial y}{\partial x}, \tag{3.17}
\]
which says that the contact force depends on the stretch and temperature. If we put
the constitutive functions (3.5) and (3.6) into the classical form of the equation of
motion (3.13) and the classical form of the first law (3.4), we get a system of partial
differential equations for \( \theta \) and \( y \)
\[
\frac{d\varepsilon}{d\theta} \frac{\partial \theta}{\partial t} = \frac{\partial q}{\partial x} + \sigma \frac{\partial y}{\partial x},
\]  
(3.18)
\[
\rho \ddot{y} = \frac{\partial \sigma}{\partial x}.
\]  
(3.19)
If one can solve the system of partial differential equations with appropriate boundary
conditions and initial conditions, one gets the position \( y(x, t) \) and temperature \( \theta(x, t) \)
at the point \( x \) at time \( t \).

If \( \sigma \) does not depend on temperature, then (3.17) becomes
\[
\sigma(x, t) = \dot{\sigma}(y_x(x, t)),
\]  
(3.20)
which says that the contact force is a function of just the stretch \( \lambda \). Hence (3.19)
becomes
\[
\rho \ddot{y} = \frac{\partial \sigma}{\partial x} \dot{y}_x = \dot{\sigma}'(y_x)y_{xx},
\]  
(3.21)
which is a nonlinear wave equation.

3.4 Assumptions on \( \sigma \)

We return to the general case where the contact force depends on both the stretch
and the temperature. Since \( \sigma(x, t) \) is the contact force that the part of the bar \( (x, L] \)
exerts on the part \( [0, x) \) across the point \( x \) at time \( t \), the most natural assumption
about \( \hat{\sigma} \) is that it is monotone increasing in the stretch for a fixed temperature \( \theta \).

See Figure 3.1. From Figure 3.1, we see that when the stretch is 1, the contact force equals 0. When the deformed length equals the reference length, no contact force acts across that point of the bar. When the stretch is greater than 1, the contact force is positive. Because the deformed length is greater than the reference length, a positive force is necessary to stretch the bar. And the more the bar is stretched, the more force is required. When the stretch is less than 1, the contact force is negative. A stretch less than 1 means the bar is compressed near the point \( x \). The contact force is negative because the part \( (x, L] \) is pushing to the left to compress the bar near \( x \). And the more the bar is compressed, the more force is required.

![Figure 3.1: The graph of \( \hat{\sigma} \) as a monotone function of the stretch \( \lambda \).](image-url)
Another possibility is that $\sigma$ is non-monotone as a function of the stretch. See Figure 3.2. Such constitutive laws are used to describe solids that undergo phase transformations. We mention a standard example of solid phase transformations, the martensitic transformation [11]. Some alloys and metals can be in one of two phases, a high symmetry phase called austenite and a low symmetry phase called martensite. In these two different phases, the material has different crystal structures and hence different mechanical responses. The phase of the material is determined by temperature and stress. Below a certain critical temperature, the martensite phase is stable, and above that temperature, the austenite phase is stable. So phase transformations occur when heating or cooling the material. If a load is applied, the critical temperature depends on the state of stress in the material, with higher values of stress leading to a higher critical temperature. See, for example, the problem discussed in [12]. Hence, the martensite phase can be stabilized by stressing the material above the critical temperature.

The graph of $\sigma$ is shown in Figure 3.2. In Figure 3.2, the two monotone increasing pieces of the curve, $\lambda < \lambda_1$ and $\lambda > \lambda_2$, describe the stress-strain relations of the two separate phases. The piece between, $\lambda_1 < \lambda < \lambda_2$, describes the transformation between phases.

3.5 A Simple Example

In this section, we consider a specific example. We model the following experiment. We clamp the end of the bar $x = 0$ and apply a prescribed force of size $k$ at the end.
Figure 3.2: The graph of $\tilde{\sigma}$ as a non-monotone function of the stretch $\lambda$.

$x = L$. The bar is kept at a constant temperature $\bar{\theta}$. Then we can solve (3.19) with the boundary conditions $y(0, t) = 0$ and $\sigma(L, t) = k$. We look for the equilibrium solution. So we assume $\ddot{y} = 0$. Then (3.19) becomes $0 = \tilde{\sigma}'(y_x, \bar{\theta})y_{xx}$. By assuming $\tilde{\sigma}'(y_x, \bar{\theta}) \neq 0$, we have $y_{xx} = 0$. so the solution is $y = Ax + B$. Applying the boundary conditions yields $B = 0$ and $A$ is a constant that satisfies $\tilde{\sigma}(A, \bar{\theta}) = k$.

If $\tilde{\sigma}$ is monotone in the stretch, then the equation $\tilde{\sigma}(A, \bar{\theta}) = k$ has a unique solution. On the other hand, $\tilde{\sigma}$ is non-monotone in the stretch, then the equation $\tilde{\sigma}(A, \bar{\theta}) = k$ may fail to have a unique solution. This is the case, for example, if the graph of $\tilde{\sigma}$ as a function of the stretch looks as in Figure 3.2 and $\sigma_2 < k < \sigma_1$.

Now we consider the following thought experiment in which we increase $k$ by small increments starting from 0. During the experiment the bar is kept at a
fixed temperature $\bar{\theta}$. If $\hat{\sigma}$ is monotone in the stretch, then as $k$ is increased in small increments, the solution of $\hat{\sigma}(A, \bar{\theta}) = k$ increases in small increments and nothing too interesting occurs. If $\hat{\sigma}$ is non-monotone in the stretch, like in Figure 3.2, we will reach a point $\lambda_3 < \lambda_1$ where adding a small load produces a sudden increase in the stretch to reach a point $\lambda_4 > \lambda_2$. A phase transformation has occurred in the material. We do not get any data points for an interval of values of $\lambda$. 
CHAPTER IV
RESULTS FOR MONOTONE RESTORING FORCE

In this Chapter we present results that describe the longterm dynamics of the order 1 outer problem for the case in which the restoring force $f$ is monotone. Our study on this case is motivated by the discussion of monotone restoring force in page 24.

In Section 4.1, we show the existence of an invariant manifold $\mathcal{M}$ for a system equivalent to the order 1 outer problem. Then we prove that the invariant manifold $\mathcal{M}$ attracts all solutions of the equivalent system in Section 4.2. In Section 4.3, we study the dynamics on the invariant manifold $\mathcal{M}$. The two theorems in this chapter, Theorems 4.1 and 4.2, have different hypotheses on the restoring force $f$. In Section 4.4 we describe a class of monotone functions that satisfy both sets of hypotheses. In the final section, Section 4.5, we present some numerical work that illustrates the results of this chapter.
4.1 Existence of Invariant Manifold

Recall the order 1 outer problem

\[
\frac{dX_0}{dt} = Y_0, \quad (4.1)
\]
\[
0 = -f(X_0) + f(W_0 - X_0) + \eta(U_0 - 2Y_0), \quad (4.2)
\]
\[
\frac{dW_0}{dt} = U_0, \quad (4.3)
\]
\[
\frac{dU_0}{dt} = -f(W_0 - X_0) - \eta(U_0 - Y_0) + F(t). \quad (4.4)
\]

We can solve (4.2) for \(Y_0\) and plug it into (4.1) and (4.4). This yields the equivalent system

\[
\frac{dX_0}{dt} = \frac{1}{2\eta}(-f(X_0) + f(W_0 - X_0) + \eta U_0), \quad (4.5)
\]
\[
\frac{dW_0}{dt} = U_0, \quad (4.6)
\]
\[
\frac{dU_0}{dt} = -f(W_0 - X_0) - \eta U_0 + \frac{1}{2}(-f(X_0) + f(W_0 - X_0) + \eta U_0) + F(t). \quad (4.7)
\]

Note that (4.5)-(4.7) is a non-autonomous system of ordinary differential equations whose solutions lie in \(\mathbb{R}^3\).

We show that the 2-dimensional linear manifold

\[
\mathcal{M} = \{(x, w, u) \in \mathbb{R}^3 : w = 2x\} \quad (4.8)
\]

is invariant for (4.5)-(4.7), which means any solution that starts on \(\mathcal{M}\) stays on \(\mathcal{M}\) for all \(t > 0\) where the solution exists. To show \(\mathcal{M}\) is invariant, first we introduce the
system

\[
\begin{align*}
\frac{dW_0}{dt} &= U_0, \\
\frac{dU_0}{dt} &= -f(\frac{W_0}{2}) - \frac{1}{2} \eta U_0 + F(t).
\end{align*}
\]

(4.9)

Then we pick \((x_0, w_0, u_0) \in \mathcal{M}\). Hence \(w_0 = 2x_0\). Let \((\hat{W}_0, \hat{U}_0)\) be the solution to (4.9) satisfying \(\hat{W}_0(0) = w_0, \hat{U}_0(0) = u_0\). Define \(W_0(t) = \hat{W}_0(t), X_0(t) = \frac{\hat{W}_0(t)}{2},\) and \(U_0(t) = \hat{U}_0(t)\). By definition, \((X_0(t), W_0(t), U_0(t)) \in \mathcal{M}\) for all \(t > 0\) where the solution \((\hat{W}_0, \hat{U}_0)\) exists. Because \(w_0 = 2x_0\), it follows that \((X_0(0), W_0(0), U_0(0)) = (x_0, w_0, u_0)\). Now we check that \((X_0, W_0, U_0)\) satisfies the system (4.5)-(4.7).

\[
\begin{align*}
\frac{dX_0}{dt} &= \frac{1}{2} \frac{d\hat{W}_0}{dt} = \frac{\hat{U}_0}{2} \\
&= \frac{1}{2\eta}(-f(X_0) + f(X_0) + \eta U_0) \\
&= \frac{1}{2\eta}(-f(X_0) + f(W_0 - X_0) + \eta U_0), \\
\frac{dW_0}{dt} &= \frac{d\hat{W}_0}{dt} = U_0, \\
\frac{dU_0}{dt} &= \frac{d\hat{U}_0}{dt} \\
&= -f(\frac{W_0}{2}) - \frac{1}{2} \eta U_0 + F(t) = -f(W_0 - X_0) - \eta U_0 + \frac{1}{2} \eta U_0 + F(t) \\
&= -f(W_0 - X_0) - \eta U_0 + \frac{1}{2}(-f(X_0) + f(W_0 - X_0) + \eta U_0) + F(t).
\end{align*}
\]

(4.10) (4.11) (4.12)

By standard uniqueness results for ordinary differential equations, we know that \((X_0, W_0, U_0)\) is the solution to (4.5)-(4.7) starting at \((x_0, w_0, u_0)\). So we have shown that any solution to (4.5)-(4.7) that starts on \(\mathcal{M}\) stays on \(\mathcal{M}\) for all \(t > 0\) where the solution exists. Therefore, \(\mathcal{M}\) is invariant for (4.5)-(4.7). And this argument does not depend on whether \(f\) is monotone.

30
4.2 Attractivity of the Invariant Manifold $\mathcal{M}$

In this subsection we show that the invariant manifold $\mathcal{M}$ defined in Section 4.1 attracts all solutions of (4.5)-(4.7).

To illustrate basic ideas, we start with the special case of $f$ linear. So we assume $f(x) = kx$ for some positive constant $k$. The order 1 outer problem (4.1)-(4.4) becomes

\[
\frac{dX_0}{dt} = Y_0, \quad (4.13)
\]

\[
0 = kW_0 - 2kX_0 + \eta(U_0 - 2Y_0), \quad (4.14)
\]

\[
\frac{dW_0}{dt} = U_0, \quad (4.15)
\]

\[
\frac{dU_0}{dt} = kX_0 - kW_0 - \eta(U_0 - Y_0) + F(t). \quad (4.16)
\]

Let $(X_0, Y_0, W_0, U_0)$ be a solution to (4.13)-(4.16). We solve (4.14) for $Y_0$ and get

\[
Y_0 = \frac{k}{2\eta}W_0 - \frac{k}{\eta}X_0 + \frac{1}{2}U_0. \quad (4.17)
\]

Using (4.17) to rewrite (4.13) as

\[
\frac{dX_0}{dt} = \frac{k}{2\eta}W_0 - \frac{k}{\eta}X_0 + \frac{1}{2}U_0. \quad (4.18)
\]

Then we use (4.15) to rewrite (4.18) as

\[
\frac{d}{dt}(X_0 - \frac{1}{2}W_0) = -\frac{k}{\eta}(X_0 - \frac{1}{2}W_0). \quad (4.19)
\]

The solution to (4.19) can be written as

\[
X_0 - \frac{1}{2}W_0 = e^{-\frac{k}{\eta}t}(X_0(0) - \frac{1}{2}W_0(0)), \quad (4.20)
\]
which implies that $W_0 - 2X_0 \to 0$ as $t \to \infty$. Note that (4.20) also tells us that $\mathcal{M}$ is invariant.

For any point $(\bar{x}, \bar{w}, \bar{u}) \in \mathbb{R}^3$, we define the distance between that point and $\mathcal{M}$ by $\text{dist}((\bar{x}, \bar{w}, \bar{u}), \mathcal{M}) = \inf\{\|(\bar{x}, \bar{w}, \bar{u}) - (x, w, u)\| : (x, w, u) \in \mathcal{M}\}$, where $\| \cdot \|$ denotes the usual Euclidean norm in $\mathbb{R}^3$. By elementary geometry, one can check that $\text{dist}((\bar{x}, \bar{w}, \bar{u}), \mathcal{M}) \leq |\bar{x} - \frac{1}{2} \bar{w}|$. Hence (4.20) implies that given any solution to (4.13)-(4.16), the distance between that solution and $\mathcal{M}$ goes to 0 as $t \to \infty$.

If the restoring force is not linear, we cannot get the explicit solution to the order 1 outer problem. But we still have the same result by proving the following theorem. Specifically, we prove the same result about the attractivity of $\mathcal{M}$ but without assuming that the restoring force $f$ is linear.

**Theorem 4.1:** Consider the order 1 outer problem (4.1)-(4.4). Suppose that for any $c > 0$, there exists $\mu_c > 0$ such that $y \geq x + c$ implies that $f(y) - f(x) \geq \mu_c(y - x)$.

Let $(X_0, Y_0, W_0, U_0)$ be a solution to (4.1)-(4.4). Then $W_0 - 2X_0 \to 0$ as $t \to \infty$.

**Proof.** Let $(X_0, Y_0, W_0, U_0)$ be a solution to (4.1)-(4.4). We solve (4.2) for $Y_0$ and obtain

$$Y_0 = \frac{1}{2\eta}(-f(X_0) + f(W_0 - X_0) + \eta U_0). \quad (4.21)$$

Then, plugging (4.21) into (4.1) and substituting $\frac{dW_0}{dt}$ for $U_0$, we get

$$\frac{dX_0}{dt} - \frac{1}{2} \frac{dW_0}{dt} = -\frac{1}{2\eta}(f(X_0) - f(W_0 - X_0)). \quad (4.22)$$

Let $c$ be a positive constant. Consider the region $\Omega_c$ in the $xw$-plane defined by the inequality $(w - 2x)^2 < c^2$. See Figure 4.1. Consider the function
Figure 4.1: The region $\Omega_c$.

\[ F(t) = (W_0(t) - 2X_0(t))^2. \] We see that

\[ F' = \frac{2}{\eta} (W_0 - 2X_0)(f(X_0) - f(W_0 - X_0)). \] (4.23)

Suppose the solution $(X_0,W_0)$ lies to the right of the line $w = 2x - c$ over some interval $(t_1, t_2)$. Then $2X_0 - W_0 \geq c$ on this interval. By hypothesis, we have

\[ f(X_0) - f(W_0 - X_0) \geq \mu_c (X_0 - (W_0 - X_0)) \geq \mu_c c \] (4.24)

for some $\mu_c > 0$. Hence from (4.23) we see that

\[ F'(t) < -\frac{2\mu_c c^2}{\eta} < 0 \] (4.25)

for $t \in (t_1, t_2)$, which implies that

\[ 0 \leq F(t) < F(t_1) + \left( -\frac{2\mu_c c^2}{\eta} \right)(t - t_1). \] (4.26)
We can apply a similar argument for the case that the solution \((X_0, W_0)\) lies to the left of the line \(w = 2x + c\), and get the same inequality (4.26). This inequality implies that given any \(c > 0\) and given an arbitrary solution to (4.1)-(4.4), there exists a positive time \(t\) at which \((X_0, W_0)\) lies inside \(\Omega_c\).

Next, suppose that at some time \(\bar{t}\), \((X_0(\bar{t}), W_0(\bar{t}))\) lies inside \(\Omega_c\) for some \(c > 0\). We show that \((X_0, W_0)\) cannot leave \(\Omega_c\) by showing that \((X_0, W_0)\) cannot cross either of the lines \(w = 2x \pm c\). To show that, we consider the function \(F(t)\) again. Suppose the solution \((X_0, W_0)\) reaches the right boundary line \(w = 2x - c\) at some time \(t' > \bar{t}\). Then

\[
F'(t') = -\frac{2c}{\eta}(f(X_0) - f(W_0 - X_0)) < 0, \tag{4.27}
\]

where the last inequality follows because \(X_0(t') > W_0(t') - X_0(t')\) and \(f\) is an increasing function. To see this latter fact, suppose \(y > x\). Then we can write \(y \geq x + (y - x)\), where \(y - x > 0\), which by our basic hypothesis on \(f\) implies that there is a constant \(\mu_{y-x}\) such that \(f(y) - f(x) > \mu_{y-x}(y - x) > 0\). Thus, the solution \((X_0, W_0)\) cannot leave \(\Omega_c\) through the line \(w = 2x - c\). If the solution \((X_0, W_0)\) reaches the left boundary of \(\Omega_c\), we can apply a similar argument.

Therefore, we have proved that for any solution to (4.1)-(4.4) and any \(c > 0\), \((X_0, W_0)\) will enter \(\Omega_c\) in finite time and is then trapped in this region. So we can say \(W_0 - 2X_0 \rightarrow 0\) as \(t \rightarrow \infty\). \(\Box\)
4.3 Dynamics on the Invariant Manifold

In this section we consider the dynamics of (4.1)-(4.4) on the invariant manifold \( \mathcal{M} \) defined in Section 4.1. Observe that the dynamics on \( \mathcal{M} \) is governed by (4.9). We can rewrite (4.9) as the single second-order equation

\[
\frac{d^2 X_0}{dt^2} + \frac{1}{2} \eta \frac{dX_0}{dt} + \frac{1}{2} f(X_0) = \frac{1}{2} F(t).
\]  

(4.28)

This is in the form of a classical equation that has been studied extensively. Specific results about the dynamics of (4.28) depend on \( f \) and \( F \). See, for example, [13]. In order to illustrate the types of results known for (4.28), we prove the following basic theorem on the existence of a trapping set for (4.28).

**Theorem 4.2:** Consider the second-order ordinary differential equation

\[
\ddot{x} + \eta \dot{x} + f(x) = F(t).
\]

(4.29)

Suppose \( xf(x) \geq 0 \) for any \( x \) and suppose there exist \( c > 0, \bar{x} > 0 \) such that \( xf(x) \geq cx^2 \) for any \( x \) satisfying \( |x| \geq \bar{x} \). Also, suppose \( |F(t)| \leq M \) for any \( t \). Define the function \( E \) on the \( xy \)-plane by

\[
E(x, y) := \frac{1}{2} y^2 + \int_0^x f(\zeta)d\zeta + \nu xy + \frac{1}{2} \nu x^2.
\]

(4.30)

If \( \nu > 0 \) is sufficiently small, then (i) \( E \geq 0 \) and (ii) there exists a constant \( R \) such that \( E \) is decreasing along any solution \( (x, y) \) to (4.29) if \( \|(x, y)\| \geq R \).

**Proof.** We recall the inequality \( 2|AB| \leq A^2 + B^2 \) for any two real numbers \( A, B \). More generally, for any \( \gamma > 0 \), letting \( A = u/\sqrt{\gamma}, B = \sqrt{\gamma}v \), we have

\[
2|uv| \leq \frac{u^2}{\gamma} + \gamma v^2.
\]

(4.31)
(i) From (4.30) and (4.31) with $\gamma = 1$, we have
\[
E(x, y) \geq \frac{1}{2}y^2 + \int_0^x f(\zeta)d\zeta - \frac{\nu}{2}(x^2 + y^2) + \frac{1}{2}\nu x^2 \\
= \frac{1}{2}(1 - \nu)y^2 + \int_0^x f(\zeta)d\zeta.
\] (4.32)
Since $xf(x) \geq 0$, we have $\int_0^x f(\zeta)d\zeta \geq 0$. If we choose $0 < \nu < 1$, then $E(x, y) \geq 0$.

(ii) Equation (4.29) can be written as a system
\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -\eta y - f(x) + F(t).
\end{align*}
\] (4.33)
Let $(x, y)$ be a solution to (4.33). We compute the derivative $\dot{E} = \frac{d}{dt}E(x(t), y(t))$ along $(x, y)$ and get
\[
\dot{E} = f(x)y + \nu y^2 + \nu xy + y(-\eta y - f(x) + F(t)) + \nu x(-\eta y - f(x) + F(t)) \\
= \nu y^2 + \nu xy - \eta y^2 + yF(t) - \nu xf(x) - \nu xy + \nuxF(t) \\
\leq \nu y^2 + \nu \left(\frac{x^2}{2\gamma} + \frac{\gamma y^2}{2}\right) - \eta y^2 + yF(t) \\
-\nu xf(x) + \nu \eta \left(\frac{x^2}{2\gamma} + \frac{\gamma y^2}{2}\right) + \nu xF(t),
\] (4.34)
where the last inequality follows from (4.31). Hence we have
\[
-\dot{E} \geq -\frac{\nu}{2\gamma} (\eta + 1)x^2 + \nu xf(x) - \nu xF(t) + \left[\eta - \nu - \frac{\nu}{2}(\eta + 1)\gamma\right]y^2 - yF(t). 
\] (4.35)
Since we obtain \(-yF(t) \geq -\frac{\delta y^2}{2} - \frac{F^2(t)}{2\delta}\) for any \(\delta > 0\) using (4.31), and
\(-\nu xF(t) \geq -\frac{\nu \xi x^2}{2} - \frac{\nu F^2(t)}{2\xi}\) for any \(\xi > 0\), inequality (4.35) becomes
\[
-\dot{E} \geq -\nu \left(\frac{\xi}{2} + \frac{1}{2\gamma}(\eta + 1)\right)x^2 + \nu xf(x) + \left[\eta - \nu - \frac{\nu}{2}(\eta + 1)\gamma - \frac{\delta}{2}\right]y^2 - \left(\frac{1}{2\delta} + \frac{\nu}{2\xi}\right)F^2(t). \tag{4.36}
\]
If \(|x| \geq \bar{x}\), then because there exists \(c > 0\) such that \(xf(x) \geq cx^2\), (4.36) can be
rewritten as
\[
-\dot{E} \geq \nu \left[c - \frac{\xi}{2} - \frac{1}{2\gamma}(\eta + 1)\right]x^2 + \left[\eta - \nu - \frac{\nu}{2}(\eta + 1)\gamma - \frac{\delta}{2}\right]y^2 - \left(\frac{1}{2\delta} + \frac{\nu}{2\xi}\right)F^2(t). \tag{4.37}
\]
Now choose \(\xi\) such that \(\frac{\xi}{2} < c\). Then we choose \(\gamma\) sufficiently large so that
\[
\frac{1}{2\gamma}(\eta + 1) < c - \frac{\xi}{2}. \quad \text{These choices ensure that the coefficient of } x^2 \text{ in (4.37) is}
\]
positive. Then we choose \(\nu\) sufficiently small so that \(\nu < \eta/\left(1 + \gamma(\eta + 1)/2\right)\). We
have already chosen \(\nu\) so that \(0 < \nu < 1\); so now we can choose \(\nu\) such that \(\nu < \min\left\{1, \eta/\left(1 + \gamma(\eta + 1)/2\right)\right\}\). Lastly, we choose \(\delta\) so that \(0 < \delta < 2[\eta - \nu - \frac{\nu}{2}(\eta + 1)\gamma]\). These
choices ensure that the coefficient of \(y^2\) in (4.37) is positive. Thus the coefficients of
\(x^2\) and \(y^2\) in (4.37), denoted by \(\hat{A}\) and \(\hat{B}\), respectively, are both positive.

Now suppose \(|x| < \bar{x}\). Then \[\left| -\nu \left(\frac{\xi}{2} + \frac{1}{2\gamma}(\eta + 1)\right)x^2 + \nu xf(x) \right| \leq K \]
for some constant \(K\). Note that at this point \(\nu, \gamma, \text{ and } \xi\) are fixed. Suppose \(x^2 + y^2 \geq R\).
Because \(x^2 \leq \bar{x}^2\), we have \(y^2 \geq R^2 - \bar{x}^2\). By hypothesis, we also have \(|F(t)| \leq M\) for
any $t$. So if we choose $R$ large enough so that

$$R^2 \geq \bar{x}^2 + \left[ \eta - \nu - \frac{\nu}{2} (\eta + 1) \gamma - \frac{\delta}{2} \right]^{-1} \left[ K + \left( \frac{1}{2\delta} + \frac{\nu}{2\xi} \right) M^2 \right],$$

(4.38)

then the right-hand side of (4.36) is non-negative, and we get $-\dot{E} \geq 0$.

Next suppose $x^2 \geq \bar{x}^2$. We also denote the coefficient of $F^2(t)$ as $\hat{C}$. Since $|F(t)| \leq M$, then (4.37) can be rewritten as

$$-\dot{E} \geq \hat{A} x^2 + \hat{B} y^2 - \hat{C} M^2.$$  (4.39)

Now let $R > 0$. Note that if $x^2 + y^2 \geq R^2$, then either

$$x^2 \geq \frac{R^2}{2} \text{ or } y^2 \geq \frac{R^2}{2}.$$  (4.40)

Thus if $x^2 + y^2 \geq R^2$, then either

$$-\dot{E} \geq \frac{\hat{A} R^2}{2} - \hat{C} M^2,$$  (4.41)

or

$$-\dot{E} \geq \frac{\hat{B} R^2}{2} - \hat{C} M^2.$$  (4.42)

We choose $R$ sufficiently large so that the right-hand side of both (4.41) and (4.42) are non-negative. Note that $R$ does not depend on the solution $(x, y)$. Therefore, $\dot{E} \leq 0$ if $x^2 + y^2 \geq R^2$. This implies that $E$ is decreasing along any solution that is sufficiently far from the origin. □

More specific assumptions about $f$ and $F$ would yield more specific results on the long-term dynamics of (4.29). For example, Theorem 15.1 and Corollary 16.2 in Chapter 11 of [13] give additional conditions on $f$ and $F$ so that (4.29) has a periodic solution that attracts all solutions to the equation.
4.4 A Comment on the Consistency of the Hypotheses of Theorems 4.1 and 4.2

In this section we identity a class of functions that satisfy both the hypotheses on the restoring force $f$ in Theorem 4.1 and the hypotheses on $f$ in Theorem 4.2.

For convenience, we list these hypotheses here.

(H$_1$) For any $c > 0$, there exists $\mu_c > 0$ such that $y \geq x + c$ implies $f(y) - f(x) \geq \mu_c(y - x)$;

(H$_2$) $xf(x) \geq 0$ for all $x \in \mathbb{R}$;

(H$_3$) there are constants $c > 0$, $\bar{x} > 0$ such that $xf(x) \geq cx^2$ for all $x$ such that $|x| \geq \bar{x}$.

Before defining a class of functions that satisfy (H$_1$), (H$_2$), and (H$_3$) we prove

**Proposition 4.3:** Let $f$ be a function defined on $\mathbb{R}$ such that $f'(x) > 0$ for any $x \in \mathbb{R}$ such that $x \neq 0$, $f(0) = 0$ and $xf''(x) \geq 0$. Then $f$ satisfies (H$_1$).

**Proof.** Consider the function $x \mapsto \frac{f(x)}{x}$ for $x > 0$. Since $f(0) = 0$ and $xf''(x) \geq 0$, we have $f(x) = \int_0^x f' \leq xf'(x)$ for $x > 0$. Then $\left(\frac{f(x)}{x}\right)' = \frac{f'x - f}{x^2} \geq 0$. So $\frac{f(x)}{x}$ is non-decreasing for $x > 0$. Similarly, we can get that $\frac{f(x)}{x}$ is non-increasing for $x < 0$.

Now let $c > 0$ and assume $y \geq x + c$. To find $\mu_c > 0$ such that $f(y) - f(x) \geq \mu_c(y - x)$, we consider several cases.

If $x = 0$, then $y \geq c$, which implies that $\frac{f(y)}{y} \geq \frac{f(c)}{c}$, or $f(y) - f(0) \geq \frac{f(c)}{c}(y - 0)$.
For \( x > 0 \), we have
\[
f(y) - f(x) \geq \frac{f(x + c)}{x + c} y - \frac{f(x)}{x} x
\]
\[
\geq \frac{f(x + c)}{x + c} y - \frac{f(x + c)}{x + c} x
\]
\[
\geq \frac{f(c)}{c} (y - x).
\] (4.43)

If \( y \leq 0 \), we can use a similar method to prove that \( f(y) - f(x) \geq \mu_1(y - x) \) where \( \mu_1 > 0 \) and \( \mu_1 \) depends only on \( c \).

For \( -\frac{c}{2} \leq x < 0 \), we have \( y \geq x + c \geq \frac{c}{2} \). Since \( xf''(x) \geq 0 \), the function \( f(x) \) is concave up for \( x > 0 \). By the hypotheses that \( f'(x) > 0 \) for \( x \neq 0 \) and that \( f(0) = 0 \), we know that \( f(x) < 0 \) for \( x < 0 \) and \( f(x) > 0 \) for \( x > 0 \). Then
\[
f(y) - f(x) = \int_x^y f' = \int_x^{\frac{c}{2}} f'(x)dx + \int_{\frac{c}{2}}^y f'(x)dx
\]
\[
\geq \frac{f(\frac{c}{2}) - f(x)}{\frac{c}{2} - x} \left( \frac{c}{2} - x \right) + \left( y - \frac{c}{2} \right) f'\left( \frac{c}{2} \right)
\]
\[
\geq \mu_2 (y - x),
\] (4.44)

where \( \mu_2 = \min \{ c^{-1} f\left( \frac{c}{2} \right), f'\left( \frac{c}{2} \right) \} \). For \( x < -\frac{c}{2} \) and \( 0 < y < \frac{c}{2} \), we can use a similar method to prove that \( f(y) - f(x) \geq \mu_3(y - x) \) where \( \mu_3 > 0 \) and \( \mu_3 \) depends only on \( c \).

For \( x < -\frac{c}{2} \) and \( y \geq \frac{c}{2} \), we have \( \frac{f(y)}{y} \geq \frac{f\left( \frac{c}{2} \right)}{\frac{c}{2}} \) and \( \frac{f(x)}{x} \geq \frac{f\left( -\frac{c}{2} \right)}{-\frac{c}{2}} \). Then
\[
f(y) - f(x) = \frac{f(y)}{y} y - \frac{f(x)}{x} x
\]
\[
\geq \frac{f\left( \frac{c}{2} \right)}{\frac{c}{2}} y - \left( f\left( -\frac{c}{2} \right) \right) x
\]
\[
\geq \mu_4 (y - x),
\] (4.45)
where \( \mu_4 = \min \left\{ \frac{f\left(\frac{c}{2}\right)}{\frac{c}{2}}, -\frac{f\left(-\frac{c}{2}\right)}{\frac{c}{2}} \right\} \).

Now we define \( \mu_c = \min \{ \mu_1, \mu_2, \mu_3, \mu_4, c^{-1}f(c) \} \). It is clear that \( \mu_c > 0 \). Then we have

\[
 f(y) - f(x) \geq \mu_c(y - x) \quad \text{if} \quad y \geq x + c \quad \text{for any} \quad c > 0.
\]

So we proved the claim. \( \square \)

Now we define the set \( \mathcal{F} \) of functions by

\[
\mathcal{F} = \{ f : \mathbb{R} \to \mathbb{R} : f(0) = 0, f'(x) > 0 \text{ for } x \neq 0, \text{ and } xf''(x) \geq 0 \text{ for all } x \}. \quad (4.46)
\]

Proposition 4.3 shows that if \( f \in \mathcal{F} \), then \( f \) satisfies (H\(_1\)). It is clear that if \( f \in \mathcal{F} \), then \( f \) satisfies (H\(_2\)). In the first part of the proof of Proposition 4.3, we showed that the function \( x \mapsto x^{-1}f(x) \) is non-decreasing for \( x > 0 \) and non-increasing for \( x < 0 \).

Pick \( \bar{x} > 0 \). Then

\[
\begin{align*}
 x \geq \bar{x} & \quad \text{implies} \quad xf(x) = x^2 \frac{f(x)}{x} \geq x^2 \frac{f(\bar{x})}{\bar{x}}, \quad (4.47) \\
 x \leq \bar{x} & \quad \text{implies} \quad xf(x) = x^2 \frac{f(x)}{x} \geq x^2 \left( \frac{f(-\bar{x})}{-\bar{x}} \right). \quad (4.48)
\end{align*}
\]

Hence \( f \) satisfies (H\(_3\)). Therefore we have shown that any function on \( \mathcal{F} \) satisfies (H\(_1\)), (H\(_2\)), and (H\(_3\)). One checks easily that \( \mathcal{F} \) includes functions of the form \( x \mapsto ax + bx^\alpha \) where \( a \geq 0, b > 0, \) and \( \alpha \) is an odd natural number.

### 4.5 Longterm Dynamics and Numerical Results

The class of functions \( \mathcal{F} \) is a class of monotone functions that satisfy the hypotheses of the two theorems of this chapter. Moreover, the functions in this class include reasonable models of springs with nonlinear restoring force.
Using the results of this section, we can now provide the following description of the longterm dynamics of the order 1 outer problem. A solution starting from any initial condition will be attracted to the invariant manifold $\mathcal{M}$ as $t$ increases. As the solution approaches $\mathcal{M}$, one expects based on continuity in initial conditions that this solution will behave similarly to solutions that start on $\mathcal{M}$. On $\mathcal{M}$, the longterm dynamics are determined by the specific properties of $f$ and $F$. Hence we expect that the longterm dynamics on $\mathcal{M}$ determines the longterm dynamics of the order 1 outer problem. For example, if $f$ and $F$ are such that on $\mathcal{M}$ there is a unique periodic solution that attracts solutions, our results imply that this periodic solution will attract all solutions to the order 1 outer problem.

We close this chapter with some numerical results illustrating the ideas of the previous paragraph. We study (4.5)-(4.7) for $f(x) = x + x^3$ with the forcing term $F(t) = \sin t$. One checks easily that $f \in \mathcal{F}$. We illustrate the behaviour of $W_0 - 2X_0$ as $t \to \infty$ numerically with different initial conditions. The result is shown in Figure 4.2. Figure 4.2 shows that $W_0 - 2X_0 \to 0$ as $t \to \infty$ for a variety of different initial conditions in $\mathbb{R}^3$, which means that the invariant manifold $\mathcal{M}$ attracts the solutions to (4.5)-(4.7) starting from these initial conditions as $t$ increases.

In Figure 4.3, we show a periodic solution to (4.9), or, equivalently, a periodic solution to (4.5)-(4.7) on $\mathcal{M}$. The existence of this periodic solution is in accordance with Theorem 15.1 in Chapter 11 of [13] given our specific choices of $f$ and $F$. Corollary 16.2 in Chapter 11 of [13] tells us that this periodic solution attracts all solutions starting on $\mathcal{M}$. In Figure 4.4, we show a solution that starts off of $\mathcal{M}$
Figure 4.2: The behaviour of $W_0 - 2X_0$ for monotone restoring force with different initial conditions $(X_0(0), W_0(0), U_0(0)) = (0, \pm 0.2, \mp 0.25)$, $(X_0(0), W_0(0), U_0(0)) = (0, \pm 0.4, \mp 0.5)$, $(X_0(0), W_0(0), U_0(0)) = (0, \pm 0.6, \mp 0.75)$, $(X_0(0), W_0(0), U_0(0)) = (0, \pm 0.8, \mp 1)$, $(X_0(0), W_0(0), U_0(0)) = (0, \pm 1, \mp 1.25)$, $(X_0(0), W_0(0), U_0(0)) = (0, \pm 1.2, \mp 1.5)$, $(X_0(0), W_0(0), U_0(0)) = (0, \pm 1.4, \mp 1.75)$, $(X_0(0), W_0(0), U_0(0)) = (0, \pm 1.6, \mp 2)$, $(X_0(0), W_0(0), U_0(0)) = (0, \pm 1.8, \mp 2.25)$, $(X_0(0), W_0(0), U_0(0)) = (0, \pm 2, \mp 2.5)$.

approaches $\mathcal{M}$ as $t$ increases. As this solution approaches $\mathcal{M}$, it is attracted to the unique periodic solution on $\mathcal{M}$, hence illustrating the idea that the longterm dynamics on $\mathcal{M}$ determines the longterm dynamics of the order 1 outer problem.
Figure 4.3: The periodic solution of (4.5)-(4.7) on $\mathcal{M}$.

Figure 4.4: The periodic solution on $\mathcal{M}$ (dashed line) and a solution starting off of $\mathcal{M}$ at $(2.5, -1, 2.5)$ (solid line).
CHAPTER V
RESULTS FOR NON-MONOTONE RESTORING FORCE

In this chapter, we consider the longterm dynamics of the order 1 outer problem for the case of a non-monotone restoring force $f$. In Section 5.1, we make some comments on the results of Chapter 4 for the case in which $f$ is non-monotone. In Section 5.2, we study the specific case with $f(x) = x^3 - x$ and without a forcing term. To understand this case, we study the fixed point structure and the stability of the fixed points for systems (5.2) and (4.5)-(4.7).

5.1 Comments on the Results of Chapter 4 for the Non-monotone Case

In this section, we comment on whether the proofs of the results of Chapter 4 still work for the case in which $f$ is non-monotone.

In previous chapter, we proved the existence of an invariant manifold $\mathcal{M}$ for (4.5)-(4.7). The existence of $\mathcal{M}$ depends only on the structure of the equations and not on whether $f$ is monotone. Then, in Theorem 4.1, we proved that $\mathcal{M}$ attracts all solutions of (4.5)-(4.7). In Theorem 4.1, we suppose that for any $c > 0$, there exists $\mu_c > 0$ such that $y \geq x + c$ implies that $f(y) - f(x) \geq \mu_c(y - x)$, which can not be satisfied if $f$ fails to be monotone on $\mathbb{R}$. This hypothesis or similar monotonicity hypothesis appears to be essential for the proof of Theorem 4.1. Without it one
cannot prove that any solution to (4.1)-(4.4) will enter $\Omega_c$ in finite time for arbitrary $c$. In Theorem 4.2, we suppose $xf(x) \geq 0$ for any $x$ and there exist $c > 0$, $\bar{x} > 0$ such that $xf(x) \geq cx^2$ for any $x$ satisfying $|x| \geq \bar{x}$. These hypotheses are not related to the hypothesis that $f$ is monotone increasing on $\mathbb{R}$. For example,

$$f(x) = (x-a)^3 - (x-a) + d$$

(5.1)

with $a > 0$ sufficiently large and $d = a^3 - a$ is a non-monotone function that satisfies these hypotheses.

5.2 Numerical Results

In this section we study (4.5)-(4.7) for a specific choice of $f$ that is non-monotone. We consider $f(x) = x^3 - x$. For simplicity we set the forcing term $F(t) = 0$. Our goals are to illustrate that Theorem 4.1 is not true for this particular example and to explore the longterm dynamics of this example.

First we illustrate the behaviour of $W_0 - 2X_0$ as $t \to \infty$ numerically with different initial conditions that correspond to the solution starting a little bit off of the invariant manifold $\mathcal{M}$. The result is shown in Figure 5.1. For some initial conditions, $W_0 - 2X_0$ goes to $-2$ or $2$ rather than $0$ as $t \to \infty$. So in this case $\mathcal{M}$ does not attract all solutions of (4.5)-(4.7).

To understand what is happening in Figure 5.1, we study the fixed point structure and the stability of the fixed points of (4.5)-(4.7). We start by consider just the dynamics on the invariant manifold. By substituting $\frac{W_0}{2}$ for $X_0$ in (4.5)-(4.7),
we get the system on $\mathcal{M}$

\begin{align*}
\frac{dW_0}{dt} &= U_0, \\
\frac{dU_0}{dt} &= -\left(\left(\frac{W_0}{2}\right)^3 - \frac{W_0}{2}\right) - \frac{\eta}{2}U_0. \quad (5.2)
\end{align*}

We note that (5.2) is a version of the Duffing equation without forcing. Then setting the right-hand sides of (5.2) equal 0, we get

\begin{align*}
U_0 &= 0, \quad (5.3) \\
-\left(\left(\frac{W_0}{2}\right)^3 - \frac{W_0}{2}\right) - \frac{\eta}{2}U_0 &= 0. \quad (5.4)
\end{align*}

Using (5.3) in (5.4) yields \(\left(\frac{W_0}{2}\right)^3 - \frac{W_0}{2} = 0\), so \(W_0 = 0,\pm 2\). The three fixed points for (5.2) are \((0,0)\), \((2,0)\) and \((-2,0)\). Then we calculate the Jacobian matrix of the
right-hand side of (5.2) as follows

\[
\begin{pmatrix}
0 & 1 \\
-\left[\frac{3}{2} (W_0) - \frac{1}{2}\right] & -\frac{\eta}{2}
\end{pmatrix}.
\] (5.5)

Plugging \((0, 0)\) into (5.5) yields

\[
\begin{pmatrix}
0 & 1 \\
\frac{1}{2} & -\frac{\eta}{2}
\end{pmatrix}.
\] (5.6)

The eigenvalues of (5.6) are

\[
\lambda_{11} = \frac{-\eta + \sqrt{\eta^2 + 8}}{4}, \quad \lambda_{12} = \frac{-\eta - \sqrt{\eta^2 + 8}}{4}.
\] (5.7)

Corresponding eigenvectors are

\[
v_{11} = \begin{pmatrix} \frac{\eta + \sqrt{\eta^2 + 8}}{2} \\ 1 \end{pmatrix}, \quad v_{12} = \begin{pmatrix} \frac{\eta - \sqrt{\eta^2 + 8}}{2} \\ 1 \end{pmatrix}.
\] (5.8)

Since one eigenvalue is positive and one is negative, \((0, 0)\) is a saddle point. \(\text{Span}\{v_{11}\}\) is tangent to the 1 dimensional unstable manifold of \((0, 0)\), and \(\text{span}\{v_{12}\}\) is tangent to the 1 dimensional stable manifold of \((0, 0)\).

Similarly, we can get the eigenvalues for the fixed point \((2, 0)\), which are

\[
\lambda_{21} = \frac{-\eta + \sqrt{\eta^2 - 16}}{4}, \quad \lambda_{22} = \frac{-\eta - \sqrt{\eta^2 - 16}}{4}.
\] (5.9)

Corresponding eigenvectors are

\[
v_{21} = \begin{pmatrix} \frac{\eta - \sqrt{\eta^2 - 16}}{4} \\ 1 \end{pmatrix}, \quad v_{22} = \begin{pmatrix} \frac{\eta + \sqrt{\eta^2 - 16}}{4} \\ 1 \end{pmatrix}.
\] (5.10)
Both eigenvalues have negative real parts, so \((2,0)\) is stable.

Likewise, the eigenvalues for the fixed point \((-2,0)\) are

\[
\lambda_{31} = \frac{-\eta + \sqrt{\eta^2 - 16}}{4}, \quad \lambda_{32} = \frac{-\eta - \sqrt{\eta^2 - 16}}{4}.
\]  

(5.11)

Corresponding eigenvectors are

\[
v_{31} = \begin{pmatrix} \eta - \sqrt{\eta^2 - 16} \\ 4 \\ 1 \end{pmatrix}, \quad v_{32} = \begin{pmatrix} \eta + \sqrt{\eta^2 - 16} \\ 4 \\ 1 \end{pmatrix}.
\]  

(5.12)

Both eigenvalues have negative real parts, so \((-2,0)\) is also stable.

We show the phase plane numerically in Figure 5.2. From Figure 5.2, we see that the numerical result is in accordance with what we discussed above. In particular, \((0,0)\) is a saddle point and \((2,0)\) and \((-2,0)\) are stable spirals.

![Figure 5.2: The phase plane of (5.2) with \(\eta = 1\).](image-url)

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Now we consider the structure of the fixed points for (4.5)-(4.7). We first set the right-hand sides of (4.5)-(4.7) equal 0 and get the system

\[-(X_0^3 - X_0) + (W_0 - X_0)^3 - (W_0 - X_0) + \eta U_0 = 0, \quad (5.13)\]

\[U_0 = 0, \quad (5.14)\]

\[-(W_0 - X_0)^3 + (W_0 - X_0) - \eta U_0 + \]

\[\frac{1}{2}[-(X_0^3 - X_0) + (W_0 - X_0)^3 - (W_0 - X_0) + \eta U_0] = 0. \quad (5.15)\]

Using (5.13), (5.14) in (5.15) gives

\[-(W_0 - X_0)^3 + (W_0 - X_0) = 0. \quad (5.16)\]

Using (5.16) in (5.13) tells us that \(X_0^3 - X_0 = 0\), so \(X_0 = 0, \pm 1\). Putting each of these possibilities for \(X_0\) into (5.16) yields 9 fixed points. By (5.14), we have \(U_0 = 0\) for all the fixed points. The three fixed points \((0, 0, 0), (1, 2, 0)\) and \((-1, -2, 0)\) are on \(M\) and correspond to the 3 fixed points we discovered for (5.2). The other six fixed points, \((0, 1, 0), (0, -1, 0), (1, 1, 0), (1, 0, 0), (-1, -1, 0)\) and \((-1, 0, 0)\), are not on \(M\).

We consider next the stability of the two fixed points \((0, 0, 0)\) and \((1, 0, 0)\). To study the stability of these fixed points, we first calculate the Jacobian matrix of the right-hand side of (4.5)-(4.7) as follows

\[
\begin{pmatrix}
\frac{1}{2\eta} Q & \frac{1}{2\eta} [3(W_0 - X_0)^2 - 1] & \frac{1}{2} \\
0 & 0 & 1 \\
3(W_0 - X_0)^2 + \frac{1}{2} Q & -3(W_0 - X_0)^2 + 1 + \frac{1}{2} [3(W_0 - X_0)^2 - 1] & -\eta/2
\end{pmatrix}, \quad (5.17)
\]
where \( Q = -3X_0^2 + 2 - 3(W_0 - X_0)^2 \). Then we plug \((0, 0, 0)\) into (5.17) and get the following matrix

\[
\begin{pmatrix}
\frac{1}{\eta} & -\frac{1}{2\eta} & \frac{1}{2} \\
0 & 0 & 1 \\
0 & \frac{1}{2} & -\frac{\eta}{2}
\end{pmatrix}.
\] (5.18)

The eigenvalues of (5.18) are

\[
\lambda_1 = \frac{1}{\eta}, \quad \lambda_2 = \frac{-\eta + \sqrt{\eta^2 + 8}}{4}, \quad \lambda_3 = \frac{-\eta - \sqrt{\eta^2 + 8}}{4}.
\] (5.19)

Corresponding eigenvectors are

\[
v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} \frac{\eta + \sqrt{\eta^2 + 8}}{4} \\ 0 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} \frac{\eta - \sqrt{\eta^2 + 8}}{4} \\ 0 \\ 1 \end{pmatrix}.
\] (5.20)

Since \( \lambda_1 \) and \( \lambda_2 \) have positive real parts, the span\{\( v_1, v_2 \)\} is tangent to the unstable manifold of \((0, 0, 0)\). The vector \( v_2 \) is parallel to \( \mathcal{M} \). But \( v_1 \) points away from \( \mathcal{M} \).

Similarly, we can get the eigenvalues for the fixed point \((1, 0, 0)\), which are

\[
\bar{\lambda}_1 = -\frac{2}{\eta}, \quad \bar{\lambda}_2 = \frac{-\eta + \sqrt{\eta^2 - 16}}{4}, \quad \bar{\lambda}_3 = \frac{-\eta - \sqrt{\eta^2 - 16}}{4}.
\] (5.21)

Corresponding eigenvectors are

\[
\bar{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \bar{v}_2 = \begin{pmatrix} \frac{-\eta - \sqrt{\eta^2 - 16}}{4} \\ \frac{-\eta + \sqrt{\eta^2 - 16}}{4} \\ 1 \end{pmatrix}, \quad \bar{v}_3 = \begin{pmatrix} \frac{-\eta + \sqrt{\eta^2 - 16}}{4} \\ \frac{-\eta - \sqrt{\eta^2 - 16}}{4} \\ 1 \end{pmatrix}.
\] (5.22)
Figure 5.3: The solutions of (4.5)-(4.7) with different initial conditions for non-monotone restoring force. $A$ is the initial condition $(0, -0.1, 0.25)$. The black solid line is the solution that starts from $A$ and appears to approach the fixed point $(1, 0, 0)$ as $t \to \infty$.

Since all the eigenvalues have negative real parts, the span{$v_1$, $\tilde{v}_2$, $\tilde{v}_3$} is tangent to the stable manifold of $(1, 0, 0)$. Hence there is an open set in $\mathbb{R}^3$ containing $(1, 0, 0)$ such that all solutions that enter this open set approach $(1, 0, 0)$ as $t \to \infty$.

Because $v_1$ is tangent to the unstable manifold of $(0, 0, 0)$ and $v_1$ is not parallel to $\mathcal{M}$, it follows that we can find a initial condition near $(0, 0, 0)$ but not on $\mathcal{M}$ such that the solution move away from $\mathcal{M}$. To illustrate this, we consider the initial condition $(0, -0.1, 0.25)$. We can see from Figure 5.3 that the solution starting from the initial condition, labelled $A$ in the figure, appears to approach the fixed point $(1, 0, 0)$ as $t \to \infty$. Because $(1, 0, 0)$ is not on $\mathcal{M}$, we see that in this case $\mathcal{M}$ does not attract all solutions or even all solutions that start near $\mathcal{M}$.
CHAPTER VI

CONCLUSION

In this paper, we study a system that describes the longitudinal motion of a light viscoelastic rod carrying a heavy particle. We discretize this problem by replacing the rod with $K$ light particles connected by massless springs. The discretized problem is governed by a system of coupled nonlinear ordinary differential equations. The governing equations for this problem contain a nonlinear function $f$ that describes the restoring force of the springs connecting the discrete particles. We study both monotone and non-monotone restoring forces and we use the asymptotic technique developed by O’Malley and Hoppensteadt to analyze this problem. In this paper, our goal is to understand the longterm dynamics of the simplest case of the discretized model, namely $K = 1$. We construct the approximate solutions using O’Malley and Hoppensteadt method, and focus on a system equivalent to the order 1 outer problem since the longterm dynamics of the order 1 outer problem determines the longter dynamics of the full system. Interest in solids that undergo phase transformations motivates our study of non-monotone restoring forces.

For the monotone case, we prove in Theorem 4.1 that there exists an invariant manifold $\mathcal{M}$ that attracts all solutions to the order 1 outer problem. By studying the dynamics of the order 1 outer problem on the invariant manifold, we prove the
existence of a trapping set in Theorem 4.2. Since these two theorems have different hypotheses on the restoring force, we define a class of functions that satisfy both the hypotheses on the restoring force in Theorem 4.1 and the hypotheses in Theorem 4.2. Then we study a specific example with \( f(x) = x + x^3 \) and \( F(t) = \sin t \) numerically. With these specific choices of \( f \) and \( F \), there exists a periodic solution that attracts all solutions, which illustrates the results of Theorem 15.1 and Corollary 16.2 in [13].

For the non-monotone case, Theorem 4.1 does not hold since the monotonicity hypothesis which is essential to this theorem cannot be satisfied. So the invariant manifold \( \mathcal{M} \) fails to attract all solutions to the equivalent system. But \( \mathcal{M} \) still exists whether \( f \) is monotone or not. Then we study a specific case with \( f(x) = x^3 - x \) and without a forcing term. The numerical result shows that \( W_0 - 2X_0 \) goes to \(-2\) or \(2\) rather than 0 as \( t \to \infty \) for some initial conditions, which proves that \( \mathcal{M} \) does not attract all solutions. By studying the fixed-point structure and performing the stability analysis of the fixed points, we find that some solutions starting a little bit off of \( \mathcal{M} \) will move away from \( \mathcal{M} \) and approach a fixed point not on \( \mathcal{M} \) as \( t \to \infty \).

In this paper, we only study the longterm dynamics of the simplest case of the discretized system, with \( K = 1 \). Future work could study numerically the dynamics of order 1 outer problem when \( K > 1 \). For the monotone case, more work could consider the original system of nonlinear ordinary differential equations and prove that an attractor on \( \mathcal{M} \) attracts all solutions to the system when \( 0 < \epsilon << 1 \). For the non-monotone case, future work could consider the original system and develop results that illustrate the longterm dynamics of this system when \( 0 < \epsilon << 1 \).
BIBLIOGRAPHY


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APPENDIX

Consider the system

\[\begin{align*}
\frac{dx}{dt} &= u(t, x, y, \epsilon), \\
\frac{\epsilon}{dt} &= v(t, x, y, \epsilon) \quad \text{for } t > 0,
\end{align*}\]

(A.1)

and

\[\begin{align*}
x &= \alpha(\epsilon), \\
y &= \beta(\epsilon) \quad \text{at } t = 0.
\end{align*}\]

(A.2)

A.1: There is a continuously differentiable function \( \phi = \phi(t, x) \) such that

\[v_0(t, x, \phi(t, x)) = 0 \quad \text{for all suitable } t \text{ and } x,\]

(A.3)

and such that the initial value problem

\[\begin{align*}
\frac{dX_0}{dt} &= u_0(t, X_0, \phi(t, X_0)) \quad \text{for } t > 0, \\
X_0 &= \alpha_0 \quad \text{at } t = 0,
\end{align*}\]

(A.4)

has a solution \( X_0 = X_0(t) \) on some compact interval, say \( 0 \leq t \leq T \), with

\[v_{0, y}(t, X_0(t), Y_0(t)) \leq -\kappa < 0 \quad \text{for } 0 \leq t \leq T\]

(A.5)

for some fixed positive constant \( \kappa > 0 \), where

\[Y_0(t) := \phi(t, X_0(t)) \quad \text{for } 0 \leq t \leq T.\]

(A.6)
The condition (A.3) need hold only for all \((t, x)\) near \((t, X_0(t))\) for \(0 \leq t \leq T\).

A.2: With the same constant \(\kappa\) of (A.5), there holds

\[
v_{0, y}(0, \alpha_0, \xi) \leq -\kappa < 0 \tag{A.7}
\]

for all values of \(\xi\) between \(\beta_0\) and \(Y_0(0) = \phi(0, \alpha_0)\).