MODELING ALCOHOL ABUSE PATTERNS IN HISPANIC-AMERICAN POPULATIONS USING AN SIR MODEL

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MODELING ALCOHOL ABUSE PATTERNS IN HISPANIC-AMERICAN POPULATIONS USING AN SIR MODEL

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SIR (Susceptible-Infected-Recovered) models may be used to examine the dynamics of socially based phenomena, such as alcohol abuse, by assuming such issues are a product of social contact and interaction. Recent studies have shown that there is an increase in the prevalence of abusive drinking behavior in the Hispanic-Immigrant population due to the increase in interaction with the U.S. population. By altering the governing differential equations of the SIR model, it is possible to model the increase in alcohol abuse in the Hispanic population. The Sanchez et al. [1] model is analyzed by seeking steady state solutions and performing a linear stability analysis. We find that the base state is stable if the rate at which individuals enter the drinking sub-population is less than the rate at which individuals leave the drinking sub-population. For the bifurcating solution, the upper branch is stable and the lower branch is unstable. If the population lies on the upper branch, then there exists a finite population of abusive drinkers in the population. Finally, the effect of the standard population on the Hispanic population is examined through various examples. The standard population may have a great impact on the Hispanic population if the rate at which Hispanics come into contact with individuals from the standard population is large enough.
I would like to acknowledge and extend my sincere gratitude to the following people who have made the completion of this thesis possible:

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“I can do all this through Him who gives me strength” -Philippians 4:13
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CHAPTER I
INTRODUCTION

The study of epidemiology dates back to the late 1800’s when scientists attempted to analyze disease in general and define its causes. Researchers such as Louis Pasteur and Robert Koch contributed to the definition and discovery of the Germ Theory of Disease, which states that microorganisms may be the cause of some or all disease. Scientists originally believed that spontaneous generation, or organisms arising from non-living objects, was the source of these microorganisms. Therefore, scientists had no reason to study the spread of disease, because there was no way to predict the spontaneous generation. Spontaneous generation, however, violated Basic Cell Theory, which maintains three basic concepts regarding cells. The concept violated by the theory of spontaneous generation is that all cells are produced by the division of pre-existing cells. Thus, spontaneous generation could not possibly explain the existence of microorganisms, let alone the cause of disease. Pasteur produced experiments that disproved spontaneous combustion and later came to theorize that contamination by microorganisms was due to airborne particles that were present in the air, but not created by the air. As a result of Pasteur’s work, scientists came to realize that they could find some pattern in the spread of disease [3].

Once scientists had more knowledge regarding the nature and causes of dis-
cases, they then wanted to apply a mathematical model to predict the outbreak of any given epidemic. The epidemic model describes how an epidemic evolves over time. As the mathematical models for epidemiology were created, they were applied to the knowledge of typical medical epidemics. More recent research, however, applies the epidemiological models to social issues and interactions. All of the models commonly used today assume that “disease” is spread through human interaction. By modeling epidemics, it may be possible to understand past events, predict what will happen in the future with respect to some event, disease, or interaction, and test control strategies for the different epidemic circumstances [3].

Many view alcohol abuse, or problem drinking, in today’s society as a true disease and epidemic. Regardless, alcoholism affects societies and cultures throughout the world, especially young adults. Alcohol abuse can be considered a disease due to the fact that there are specific and common short term and long-term effects on an individual who drinks heavily for some given period of time. Long-term effects include, but are not limited to: permanent damage to vital organs, cancer, gastrointestinal irritation, malnutrition, and high blood pressure. In fact, more than 100,000 U.S. deaths are caused by excessive alcohol consumption each year resulting from “drunk driving, cirrhosis of the liver, falls, cancer, and stroke” [4]. It is also estimated that more than 18% of Americans experience alcohol abuse or alcohol dependence at some time in their lives [4]. Alcohol abuse is prevalent in the U.S., as well as in many other developed countries. Because of the negative effects that alcohol abuse establishes, it is desirable to predict its growth or decay throughout a particular population [4].
Research examining the alcohol abuse rates among racial minorities has been of recent interest among various sociologists. In particular, research has focused on the alcohol abuse rates of immigrants entering the United States and their subsequent generations. “Actual patterns of changes in drinking vary for each ethnic group, since traditional patterns in each nation of origin differ as do circumstances, opportunities and constraints impacting on the various immigrant groups upon entry to the U.S.” [5]. This tendency to mimic the larger, more influential population is known as acculturation. According to Caetano’s [6] study on Hispanic acculturation and alcohol abuse, “those in the high acculturation group have higher rates of abuse and dependence than the lower acculturation groups.” For more information regarding sociological research on racial minorities, refer to Peralta and Steele [7].

Typically, as the first generation of immigrants moves to the United States, the problem drinking abuse rates model almost identically the abuse rates of their native countries. These abuse rates are almost always lower than the corresponding rates in the United States. With consecutive generations thereafter, alcohol abuse rates tend to increase from generation to generation until the rates are identical to those of the U.S. population. In some cases, the abuse rates of the latter immigrant generations even surpass those of the U.S. population.

Gilbert’s [5] study on Mexican-American women found that among these women, 68% of first generation individuals abstained from drinking, 37% of the second generation abstained from drinking, and only 33% of the third generation abstained from drinking. While any study is not without its limitations, the differences in the
percentages are drastic [5]. Another study of Mexican-American women confirms that behavior began to mimic that of the larger society, which has higher drinking rates than the women’s native country, especially among the third generation of women. The women confirmed the acculturation model hypothesis [8]. Another Study by Caetano et al. [6] claimed that “Mexican-Americans born in the U.S. had higher rates of abuse (16% versus 19.1%)” when compared to Mexican-Americans actually born in Mexico. Although both of these studies are representative of individuals of Mexican descent, there are similar findings regarding other Hispanic cultures.

The basic epidemiological models are deterministic when used with large populations. The models consist of differential equations, which are nonlinear. The initial epidemiological model, called the SIR (Susceptible-Infected-Recovered) model, was developed by Kermack and McKendrick in the 1920’s. Their purpose in creating such a model was to explain the rapid rise and fall in the number of infected patients observed in epidemics such as the plague and cholera, which both occurred during the 1800’s [9]. The SIR model maintains assumptions including:

1. The population size is fixed.

2. The incubation period of the infectious agent is instantaneous, and the duration of infectivity is the same as the length of the disease.

3. The population is completely homogeneous with no age or social structure.

4. All individuals have the same probability of coming in contact with an infected individual and, as a result, contracting the disease.
5. Those individuals who have recovered from the disease are assumed to have developed immunity to the disease.

6. The rate of infection and removal is much faster than the birth and death rates, therefore these factors are not considered in the original version of the model.

The system consists of three nonlinear differential equations that model epidemiology with respect to time, which takes into account the birth and death rates. Each equation models a different subgroup in the given population. In other words, there are three dependent variables in the system. For this manuscript, we refer to the “susceptible”, or mild to moderate drinker, group as $S$, the “infected”, or abusive drinking, group as $D$, and the “recovered” group as $R$.

While there are definite differences between the generation of addictive behaviors and the transmission of true infectious diseases, both can be modeled as the result of contacts between individuals in given environments. Social processes, such as drinking alcohol, can be modeled using an SIR model, because the situations in which people generally drink and/or become drinkers depends on a combination of social influence and access to alcohol. This implies that it may be possible to gain more understanding of the dynamics of drinking behaviors by using the SIR model of epidemiology to model drinking as a result of contacts between susceptible individuals with drinking individuals.

The goal of this manuscript is to review the model proposed by Sanchez et al [1] and to identify the quantitative mechanisms that either facilitate or limit the
conversion of the Hispanic population in an environment with or without contact with the U.S. (Standard) population from a population of non-drinkers to a population of drinkers. Quantifying these mechanisms will aid in understanding the role of social forces on the time evolution of drinking [10]. By solving the model, it may be possible to develop more accurate and effective treatments for alcoholics and legal drinking policies based on the given environment [1]. The model is used to examine the impact of acculturation among immigrants now living in the United States. Should this model be an accurate portrayal of this sub-population, we may then be able to predict the spread of alcohol abuse among such communities before it even occurs, thus giving the community more time to respond to the impending problem. The proposed model, which is slightly altered to accommodate the specific parameters of the social environment, maintains certain assumptions, including:

1. The total population is constant.

2. There is no influence to drink other than the influence of other drinkers.

3. The legal age to drink is not accounted for.

4. New people enter the population as occasional/moderate drinkers and mix at random with the rest of the population.

5. Everyone who has a problem with alcohol, regardless of consumption level, is placed in the problem drinkers ("infected") group.

6. The population is completely homogeneous with no age or social structure.
7. All individuals have the same probability of coming in contact with an infected individual and, as a result, becoming a problem drinker.

8. Those individuals who have temporarily recovered from problem drinking are assumed to have developed immunity to the disease.

9. All groups share the same environment.

10. Should a recovered alcoholic relapse, he or she will revert only to the infected problem drinking group and never to the susceptible moderate/occasional drinking group.

Using the SIR model without the birth and death rates as a foundation for a model, researchers have found that modifying the nonlinear equations provides an accurate system for modeling the spread of alcoholism. In this case, $S(t)$ represents occasional and moderate drinkers. $D(t)$ represents problem drinkers, which are considered the “infected” group. Finally, $R(t)$ represents temporarily recovered individuals [1]. Table 1.1 lists the variables and parameters used in this application of the SIR model.

In this model, $\mu$ is considered the per person departure rate from drinking environments. This is the equivalent of “death” in the SIR model. The assumption is that the birth rate, which is equivalent to individuals entering the population, and the death rate, which is equivalent to individuals leaving the population, occur at the same rate. Thus, $\mu$ is representative of both the “birth” and “death” rates [1].
Table 1.1: *Variables and Parameters Used in the Sanchez [1] Model*

<table>
<thead>
<tr>
<th>Variables</th>
<th>Units</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S(t)$</td>
<td>People</td>
<td>Number of Occasional Drinkers</td>
</tr>
<tr>
<td>$R(t)$</td>
<td>People</td>
<td>Number of Recovered Drinkers</td>
</tr>
<tr>
<td>$D(t)$</td>
<td>People</td>
<td>Number of Problem Drinkers</td>
</tr>
<tr>
<td>$\beta$</td>
<td>1/Time</td>
<td>Transmission Rate</td>
</tr>
<tr>
<td>$\mu$</td>
<td>1/Time</td>
<td>Per-Person Departure Rate to $S$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>1/Time</td>
<td>Per-Person Recovery Rate</td>
</tr>
<tr>
<td>$\rho$</td>
<td>1/Time</td>
<td>Per-Person Relapse Rate</td>
</tr>
<tr>
<td>$t$</td>
<td>Time</td>
<td>Time</td>
</tr>
<tr>
<td>$N$</td>
<td>People</td>
<td>Total Population</td>
</tr>
</tbody>
</table>

A diagram of the various rates affecting the three groups of individuals is represented in Figure 1.1 below. In the diagram the arrows represent the direction of movement into or out of each drinking group. The parameters next to each of the arrows represent the rates that govern each transition. These rates are defined above in Table 1.1. People move from group to group as their drinking habits change [1].

Assumption number four refers to homogeneous mixing. This means that the likelihood of coming into contact with members of each drinking group is:

- $\frac{S}{N}$ for the probability of coming into contact with an occasional/moderate drinker,
Figure 1.1: Transition Between Subgroups in a Problem Drinking Population

- \( \frac{D}{N} \) for the probability of coming into contact with a problem drinker,

- \( \frac{R}{N} \) for the probability of coming into contact with a temporarily recovered drinker.

In the classic SIR model, the total population is equal to the sum of each of the drinking groups. In other words,

\[
N = S(t) + D(t) + R(t). \tag{1.1}
\]

The study considers a non-dimensionalized version of the model by using the above proportions,

\[
s = \frac{S}{N}, \quad d = \frac{D}{N}, \quad \text{and} \quad r = \frac{R}{N}. \tag{1.2}
\]
The governing equations after non-dimensionalization are

\[
\frac{ds}{dt} = \mu - \beta ds - \mu s,
\]

\[
\frac{dd}{dt} = \beta sd + \rho rd - (\mu + \phi)d,
\]

\[
\frac{dr}{dt} = \phi d - \frac{d}{r} r - \mu r,
\]

\[s + d + r = 1.\]

According to Sanchez et al. [1] “The rate of conversion from the susceptible state (occasional drinker) to the regular drinking state is assumed to be proportional to the size of the susceptible population, the likelihood of interacting with a randomly selected drinking partner, and the magnitude of intensity of contacts. The rate of relapse is the result of similar forces that involve contacts between \( R \) and \( D \) individuals.” As noted in assumption nine, individuals from all groups share the same environment, and since this is true, there are two cases. The first case occurs when drinking is not a part of the culture. In this case treatment is not available, thus the \( R \) group does not exist. The second case occurs when drinking is part of the culture, and there is some form of treatment available for problem drinkers. Relapse also affects the spread of alcohol abuse. If a recovered alcohol abuser reverts back to being a problem drinker, then he or she is considered to be “infected,” and can again instigate more cases of alcohol abuse.

To determine the realistic effect of the reproductive numbers, which are merely groups of parameters, Sanchez and his colleagues [1] computed the equilibrium states of the model. They did so by reducing the three nonlinear differential equa-
tions given in the model to a quadratic equation in $d$, where the coefficients depend on the reproductive numbers of the model. The solutions to the quadratic equation are then the equilibrium states. They found that different equilibrium states might arise depending on the values of the reproductive numbers with treatment. Overall, the results indicate that ineffective treatment programs with high relapse rates may increase the spread of alcoholism, rather than help to eliminate it. In such a case, the population may be better off without the treatment programs. A better solution may be to simply limit the amount of time a recovered drinker spends in environments where drinking, especially heavy drinking, occurs. Sanchez and his colleagues [1] studied current drinking literature pertaining to recovery, relapse, and the social interpersonal influences on drinking behavior in order to estimate the parameters of the model. From the literature, they found the minimum, maximum, median, and standard deviation values for $\beta$, $\mu$, $\phi$, and $\rho$. By taking sample values from within the minimum and maximum ranges, it is possible to calculate the critical value representation of the point at which a drinking population can be controlled. Below is an example of the parameter values of the SIR model relating to alcoholism as defined by Sanchez and his colleagues [1]. The values are taken from the values that the study found through literature research. Each given value represents the median that Sanchez and his colleagues determined from their research,

$$\beta = 0.6434, \quad \mu = 0.2553, \quad \phi = 0.1033, \quad \rho = 3.83e^{-5}. \quad (1.7)$$

In order to determine the sensitivity of the system to parameter variations,
Sanchez and his colleagues used the partial rank correlation coefficient. They found that the recovery rate, $\phi$, was the most sensitive parameter. Essentially, they concluded that if the initial recovery rate from treatment and the subsequent relapse rate, $\rho$, were high, then there would be a critical number of vulnerable people that can re-enter the abusive drinking group.

An important finding of this study is that once a drinking population is established, the effectiveness of the treatment becomes possibly the most critical factor in the attempt to eliminate the “infected” portion of the population. For example, if a population has a high rate of recovery but ineffective treatment programs and high relapse rates, the alcohol abuse will not diminish. The problem may actually worsen because there will be a higher number of individuals from the recovered group that will revert to the problem drinking group. As a result, there will be many more problem drinkers that have the ability to recruit others from either the recovered or susceptible groups to also become problem drinkers. In conclusion, the effectiveness of treatment is likely the most important variable in the recovery rate, because short-term effectiveness will eventually lead to a larger population of problem drinkers, while a truly effective treatment program will help to reduce the number of problem drinkers in the given population [1].

The model proposed by Sanchez and his colleagues [1] is not the only model for social epidemics and there has been other research in modeling alcohol abuse. Some research uses models different from that of the previously defined model; however, other research refers to extensions and variations of the Sanchez model. One
such model was studied by Patrick Newman [2], a student at Durham University, who was supervised by Professor B. Straughan and Dr. J.F. Blowey. The model consists of similar parameters and a similar SIR structure to Sanchez’ model, however, it also accounts for recovered drinkers who relapse to moderate/occasional drinking as well as those recovered drinkers who relapse to problem drinking. Newman’s model also considers two death rates, rather than just one. The model extension also accounts for legal drinking age. Newman assumes that the birth rate is equivalent to the rate at which individuals turn the legal drinking age, which is 21 in the United States [2].

The model proposed by Sanchez did not account for this parameter. Sanchez’ model did not account for any age variation, and “since contact rates between age groups vary greatly, it is often important to consider models with age structure” [11].

The Newman model makes many of the same assumptions as the Sanchez model, however, there are slight differences. These include:

1. The total population, $N$, is not constant.

2. The legal age to drink is accounted for.

3. An individual who relapses may revert to either the problem drinking group or the moderate/occasional drinking group. Both relapse rates are accounted for.

    Newman [2] chose to use parameter values different from those of Sanchez. Newman’s results concluded, “If the parameters used were reliable, then a sharp increase in the number of alcoholics should be expected” [2]. His research was based on numbers from the UK, but his findings matched the current research on alcohol
abuse in that population. He concluded that the UK is likely facing an epidemic of alcohol abuse.

Newman also suggests another model, which eliminates the assumption that the “birth and death” rates, or the rates at which individuals enter and leave the population, are equal. He also incorporates a new term to allow individuals who recover to return to the “susceptible” group, which includes individuals who are occasional or moderate drinkers, but not abusive drinkers. Newman [12] claims that this “reflects the fact that some treatment programmes actually encourage alcoholics to return to moderate drinking rather than attempting to abstain completely.” The equations in this version of Newman’s model are

\[
\begin{align*}
\frac{dS}{dt} &= \mu N - \frac{\beta DS}{N} + \epsilon R - \gamma S, \\
\frac{dD}{dt} &= \frac{\beta DS}{N} + \frac{\rho RD}{N} - \phi D - (\gamma + \delta)D, \\
\frac{dR}{dt} &= \phi D - \frac{\rho DR}{N} - \epsilon R - (\gamma + \delta)R, \\
N &= S + D + R.
\end{align*}
\]  

(1.8)  

(1.9)  

(1.10)  

(1.11)

In this version of Newman’s model, the variables are the same as in the previous model proposed by Sanchez and colleagues [1], but the variables, which are slightly altered, are listed in Table 1.2. Note that “natural” death refers to any death not caused by alcohol [12].

At Kent State University, a group of students studied the dynamics of alcohol consumption on the Kent State campus. They used an SIR model similar to those previously mentioned; however, they chose to eliminate one of the ordinary differential
Table 1.2: *Variables and Parameters in the Newman [2] Model*

<table>
<thead>
<tr>
<th>Variables</th>
<th>Units</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>1/Time</td>
<td>Transmission Rate</td>
</tr>
<tr>
<td>$\mu$</td>
<td>1/Time</td>
<td>Birth Rate</td>
</tr>
<tr>
<td>$\phi$</td>
<td>1/Time</td>
<td>Recovery Rate</td>
</tr>
<tr>
<td>$\rho$</td>
<td>1/Time</td>
<td>Relapse Rate to Abusive Drinking</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>1/Time</td>
<td>Relapse Rate to Moderate Drinking</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>1/Time</td>
<td>Rate of &quot;Natural&quot; Death</td>
</tr>
<tr>
<td>$\delta$</td>
<td>1/Time</td>
<td>Rate of Alcohol Related Deaths</td>
</tr>
<tr>
<td>$t$</td>
<td>Time</td>
<td>Time</td>
</tr>
<tr>
<td>$N$</td>
<td>People</td>
<td>Total Population</td>
</tr>
</tbody>
</table>

equations from the system of nonlinear equations. They removed the recovered group from the model, by claiming that recovery and relapse rarely occur during the average time in college. Their system consists of only two nonlinear differential equations [13].

Although the models presented are accurate under their assumptions, there are some major limitations:

1. The model presented by Sanchez assumes that the population is constant. In reality, this is not the case.

2. Both models assume that there is no influence to drink other than existing drinkers. Realistically, there may be a variety of reasons that people become
alcoholics, such as depression or some other abnormal mental state.

3. Sanchez’ model does not account for the legal drinking age. This limits the results because there are some individuals who begin to drink simply because they become of legal age to do so.

4. Both models assume that as people enter the population, they mix randomly. In reality, there are social connections, which makes the probability of coming in contact with a problem drinker vary for different individuals.

5. Sanchez’ model assumes that an individual who enters the recovered drinking group maintains immunity indefinitely. Realistically, a recovered individual can relapse into alcohol abuse at any point. Should this happen, the individual can theoretically become a problem drinker and able to “infect” others. Newman’s model accounts for such circumstances and even allows for recovered individuals to re-enter the drinking population as either a moderate/occasional drinker or a problem drinker.

6. Finally, both models assume that all individuals in the population share the same environment. In reality, no two people share the exact same environment. Additionally, a single person may experience several different environments through different aspects of his or her life (ex. school, home, friends, work, etc.)

In this manuscript, we modify the Sanchez model to consider the Hispanic situation, in which the rate at which individuals convert to abusive drinking is depen-
dent on the number of drinkers in the U.S. population. The original three equations will model the alcohol abuse rates of the standard U.S. population, while a slightly modified set of three nonlinear equations, with different parameters, will be implemented to model the alcohol abuse rates of the Hispanic Immigrant population. The parameter governing the transition of Hispanics from the susceptible group to the problem drinking group will incorporate the rate at which the standard population is converting to problem drinking.

Once the new systems of equations are determined, we will find the steady state solutions and determine their stability. Finally, we will implement the system of equations on sample populations. We expect that this system of equations will be accurate in predicting the spread of abusive drinking patterns in consecutive Hispanic-American generations, because the model incorporates parameters whose values are chosen to mimic the population and also accounts for the interaction between the Hispanic population and the U.S. Population.

The remainder of this thesis is organized as follows: In Chapter II, the set of nonlinear differential equations for the Hispanic Immigrant population will be developed. In Chapter III, the equations will be non-dimensionalized and analyzed, and the steady state solutions will be defined. In Chapter IV, we determine the stability of the steady state solutions. In Chapter V, we analyze how the standard population affects the Hispanic population. Finally, Chapter VI contains a summary and possible future work related to this model.
CHAPTER II
GOVERNING DIFFERENTIAL EQUATIONS

2.1 Standard Population

The standard population is governed by the set of differential equations explored by Sanchez and his colleagues [1]. In this model, the population is divided into three subgroups: individuals who are susceptible to becoming abusive drinkers, individuals who are alcohol abusers, and individuals who are recovered alcohol abusers. The variables and parameters for this system of equations are listed in Table 2.1.

The governing system of equations is

\[
\frac{dS}{dt} = \mu N - \beta \frac{D}{N} S - \mu S, \tag{2.1}
\]

\[
\frac{dD}{dt} = \beta \frac{S}{N} D + \rho \frac{R}{N} D - (\mu + \phi) D, \tag{2.2}
\]

\[
\frac{dR}{dt} = \phi D - \rho \frac{D}{N} R - \mu R, \tag{2.3}
\]

\[S + D + R = N. \tag{2.4}\]

2.1.1 Non-dimensionalization

For the purpose of later finding the steady state solutions of the system, it is necessary that these equations be non-dimensionalized. After dividing equations (2.1) through (2.4) by \(N\), the total number of individuals in the system, the equations are non-dimensionalized as follows:
Table 2.1: Variables and Parameters for the Standard Population

<table>
<thead>
<tr>
<th>Variables</th>
<th>Units</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S(t)$</td>
<td>People</td>
<td>Number of Occasional Drinkers</td>
</tr>
<tr>
<td>$R(t)$</td>
<td>People</td>
<td>Number of Recovered Drinkers</td>
</tr>
<tr>
<td>$D(t)$</td>
<td>People</td>
<td>Number of Problem Drinkers</td>
</tr>
<tr>
<td>$\beta$</td>
<td>1/Time</td>
<td>Transmission Rate</td>
</tr>
<tr>
<td>$\mu$</td>
<td>1/Time</td>
<td>Per-Person Departure Rate to S</td>
</tr>
<tr>
<td>$\phi$</td>
<td>1/Time</td>
<td>Per-Person Recovery Rate</td>
</tr>
<tr>
<td>$\rho$</td>
<td>1/Time</td>
<td>Per-Person Relapse Rate</td>
</tr>
<tr>
<td>$t$</td>
<td>Time</td>
<td>Time</td>
</tr>
<tr>
<td>$N$</td>
<td>People</td>
<td>Total Population</td>
</tr>
</tbody>
</table>

\[
\frac{ds}{dt} = \mu - \beta ds - \mu s, \quad \quad \quad (2.5)
\]
\[
\frac{dd}{dt} = \beta sd + \rho rd - (\mu + \phi)d, \quad \quad \quad (2.6)
\]
\[
\frac{dr}{dt} = \phi d - \frac{\rho d}{r} - \mu r, \quad \quad \quad (2.7)
\]
\[
s + d + r = 1, \quad \quad \quad (2.8)
\]

where $s = \frac{S}{N}, d = \frac{D}{N}$, and $r = \frac{R}{N}$. 

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2.2 Hispanic Population

The Hispanic population has the same three subgroups as the standard population. Hence, the system of equations governing the Hispanic Population is of the same form as those equations governing the standard population. The only differences are the variables and parameters used to represent the individuals and their transition rates from group to group. The variables differ since they represent a different population. The parameters are also different, since the Hispanic population’s transition rates may vary from those of the standard population prior to initial contact with the standard population. Another difference between the standard and Hispanic populations is the transition rate from the susceptible group to the alcohol abuser group. In the standard population, there is a constant parameter that governs the transition. In the Hispanic population, we have a parameter $\delta(d)$, which is the sum of a constant and a constant multiplied by the number of alcohol abusers, $d$, in the standard population. This is expressed in equation (2.9) below:

$$\delta(d) = c_1 + c_2d.$$  \hspace{1cm} (2.9)

This conversion rate from susceptible to problem drinker in the Hispanic population is defined such that the first constant $c_1$ represents the conversion rate of the Hispanic population regardless of the population's interaction with any other population. This conversion rate can be increased if the Hispanics have any contact with the U.S., which is assumed to have an equal or greater conversion rate. The
$c_2 d$ represents the increased contact with the U.S. population, which results in an increased overall conversion from susceptible to problem drinker in the Hispanic population. An important assumption we make here is that the system for the standard population has reached its steady-state, and the steady-state value of $d$ is substituted into the Hispanic model in the parameter $\delta(d)$.

The interaction between the systems of the two populations is expressed in Figure 2.1 below.

![Figure 2.1: System Interaction between the Hispanic and standard populations](image)

The parameters and variables for the whole Hispanic population system of equations is listed in Table 2.2.
Table 2.2: Variables and Parameters for the Hispanic Population

<table>
<thead>
<tr>
<th>Variables</th>
<th>Units</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H(t)$</td>
<td>People</td>
<td>Number of Occasional Drinkers</td>
</tr>
<tr>
<td>$B(t)$</td>
<td>People</td>
<td>Number of Recovered Drinkers</td>
</tr>
<tr>
<td>$Q(t)$</td>
<td>People</td>
<td>Number of Problem Drinkers</td>
</tr>
<tr>
<td>$\delta(d)$</td>
<td>1/Time</td>
<td>Transmission Rate</td>
</tr>
<tr>
<td>$\eta$</td>
<td>1/Time</td>
<td>Per-Person Departure Rate to S</td>
</tr>
<tr>
<td>$\xi$</td>
<td>1/Time</td>
<td>Per-Person Recovery Rate</td>
</tr>
<tr>
<td>$\psi$</td>
<td>1/Time</td>
<td>Per-Person Relapse Rate</td>
</tr>
<tr>
<td>$t$</td>
<td>Time</td>
<td>Time</td>
</tr>
<tr>
<td>$M$</td>
<td>People</td>
<td>Total Population</td>
</tr>
</tbody>
</table>

The governing system of equations for the Hispanic population is

\[
\frac{dH}{dt} = \eta M - \delta(d) \frac{B}{M} H - \eta H, \tag{2.10}
\]

\[
\frac{dB}{dt} = \delta(d) \frac{H}{M} B + \xi \frac{Q}{M} B - (\eta + \psi) B, \tag{2.11}
\]

\[
\frac{dQ}{dt} = \psi B - \xi \frac{B}{M} Q - \eta Q, \tag{2.12}
\]

\[
H + B + Q = M. \tag{2.13}
\]

2.2.1 Non-dimensionalization

It is necessary to non-dimensionalize equations (2.10) through (2.13) for later analysis.

After dividing equations (2.10) through (2.13) by $M$, the total number of individuals
in the Hispanic population, the equations are non-dimensionalized as follows:

\[
\frac{dh}{dt} = \eta - \delta(d)bh - \eta h, \quad (2.14)
\]
\[
\frac{db}{dt} = \delta(d)hb + \xi q - (\eta + \psi)b, \quad (2.15)
\]
\[
\frac{dq}{dt} = \psi b - \xi bq - \eta q, \quad (2.16)
\]
\[
h + b + q = 1, \quad (2.17)
\]

where \( h = \frac{H}{M}, \ b = \frac{B}{M}, \) and \( q = \frac{Q}{M}. \)

2.3 Discussion of Parameters

In the standard population, we assume that the easiest parameter to control is the rate at which individuals convert from mild or moderate drinking to abusive drinking, \( \beta. \) This is due to the idea that the rate at which individuals convert is influenced by social factors that we, as a population, can control. These social factors include, but are not limited to: preventative programs, social support, and the transition from one peer group to another. All of these factors would ideally be easier to control than trying to help abusive drinkers recover or preventing individuals from relapsing back into abusive drinking from a recovery state.

In the Hispanic population, we choose to control the parameter \( \delta(d), \) which is the rate at which individuals from the Hispanic population convert to abusive drinking. We consider this the controllable parameter, because if we can control the standard population by focusing on minimizing the number of drinkers, we can then decrease the social interaction that Hispanics have with abusive drinkers from
the standard population. If we can decrease the number of abusive drinkers in the standard population, the Hispanic population will have less contact with alcohol abusers. So, the parameters $\beta$ and $\delta(d)$ are the parameters we will use in our analyses of these systems of equations.
CHAPTER III

BIFURCATION ANALYSIS

3.1 Standard Population Steady State Solutions

We first examine the steady state solutions of the standard population. In order to find the steady state solutions, where time is held constant, we set equations (2.5) through (2.7) equal to zero,

\[ s' = \mu - \beta ds - \mu s = 0, \quad (3.1) \]
\[ d' = \beta sd + \rho rd - (\mu + \phi)d = 0, \quad (3.2) \]
\[ r' = \phi d - \frac{d}{r} - \mu r = 0. \quad (3.3) \]

One solution to this system is the case where \( d = 0 \), or there are no abusive drinkers in the population. Hence, all of the people in the population remain in the “susceptible” group. Since there is not a single abusive drinker available to “infect” any other individuals, the system always remains with zero abusive drinkers. This is the base state solution of the system. Next, it is desirable to find the bifurcating state of this system. Since the abusive drinkers are the subjects of interest, we solve equation (3.1) first for \( s \). The solution is

\[ s = \frac{\mu}{\mu + \beta d}. \quad (3.4) \]
Now, we solve Equation (2.8) for $r$, and substitute in the value of $s$ from Equation (3.4), so that $s$ and $r$ are both rewritten in terms of $d$. The result is

$$r = 1 - \frac{\mu}{\mu + \beta d} - d. \quad (3.5)$$

Now that we have values for $s$ and $r$ in terms of $d$, we place these new values for $s$ and $r$ into equation (3.2). We simplify the resulting equation, so that it appears as a function of $d$. The result is

$$d^2 - (1 - \frac{\mu}{\beta} - \frac{\mu + \phi}{\rho})d + \frac{\mu}{\beta} (\frac{\mu + \phi}{\rho} - \frac{\beta}{\rho}) = 0. \quad (3.6)$$

This equation implies that

$$d = \frac{1 - \frac{\mu}{\beta} - \frac{\mu + \phi}{\rho} \pm \sqrt{(1 - \frac{\mu}{\beta} - \frac{\mu + \phi}{\rho})^2 - 4(\frac{\mu}{\beta} (\frac{\mu + \phi}{\rho} - \frac{\beta}{\rho}))}}{2}. \quad (3.7)$$

From Chapter II, recall that the parameter of interest associated with the standard population is $\beta$, the rate that individuals convert to abusive drinking. In order to analyze the system in terms of $\beta$, we rewrite Equation (3.6) by multiplying through by $\beta$ and then solving the equation for $\beta$. The result is

$$\beta = -\frac{\mu(d + \frac{\mu + \phi}{\rho})}{d^2 - (1 - \frac{\mu + \phi}{\rho})d - \frac{\mu}{\rho}}. \quad (3.8)$$

As an example, we define sample values for the standard population as $\phi = 0.25$, $\mu = 0.15$, and $\rho = 0.95$. The graph of Equation (3.8) is displayed in Figure 3.1.

The dashed lines represent the asymptotes, which are located at

$$d = \frac{(1 - \frac{\mu + \phi}{\rho}) \pm \sqrt{(1 - \frac{\mu + \phi}{\rho})^2 + \frac{4\mu}{\rho}}}{2}. \quad (3.9)$$
Because we are only interested in the applicable parts of this graph, the standard population's bifurcation diagram is focused on the restricted relationship depicted in Figure 3.2 and also in Figure 3.3 using the data from Figure 3.1. This represents the portion of Figure 3.1 graphed by a solid line. Note that Figure 3.2 is an exaggerated version of Figure 3.3. This is the only applicable portion of Figure 3.1, because we assume that $\beta$ must be positive and the population of abusive drinkers, $d$, is greater than or equal to zero people. The dotted portions of the graph correspond to the irrelevant solutions of the system that do not make sense.
Figure 3.2: Bifurcation Diagram of the Standard Population

Note that the intercept on the $\beta$ axis is located at the value $\beta = \mu + \phi$. Recall that $\beta$ is the rate that individuals enter the drinking state, while the sum of $\phi$ and $\mu$ is the rate that individuals leave the drinking state. This means that at the $\beta$ intercept, we have $\beta = \mu + \phi$, or the rate that individuals enter the drinking state is equal to the rate at which individuals leave the drinking state. If $\beta > \mu + \phi$, meaning we move to the right of the intercept, the population will have a finite number of abusive drinkers. This is because individuals are entering the drinking population at a greater rate than individuals are leaving the drinking population. Conversely, if $\beta < \mu + \phi$, and we move to the left of the intercept, the population will have zero abusive drinkers. This is because individuals are leaving the drinking population at a
greater rate than individuals are entering the drinking population. The critical points
in the standard population’s system of equations are located at
\[
d = -\frac{\mu + \phi}{\rho} \pm \sqrt{\frac{\phi}{\rho}}.
\] (3.10)

Equation (3.10) implies that there are two critical points in the system. Referring
back to Figure 3.1, we see that this is the case. The vertex of the applicable portion
of the graph is located at
\[
d = -\frac{\mu + \phi}{\rho} + \sqrt{\frac{\phi}{\rho}}.
\] (3.11)

In Figures 3.1 and 3.2, the vertex is in the first quadrant, but this is not
always the case. The vertex may also fall in the second quadrant. The location of

Figure 3.3: Restricted and Rotated Portion of Figure 3.1
the vertex is the determining factor for whether or not a finite population of abusive drinkers becomes established. If we have a transcritical bifurcation, as depicted in Figures 3.1 and 3.3, then there will be a finite population of abusive drinkers, which corresponds to the upper branch of the transcritical bifurcation diagram. If we have a supercritical bifurcation, a finite population of abusive drinkers still exists, but the size of the drinking sub-population is less than in the transcritical case. Also, the drinking sub-population only occurs above the bifurcation point. From Equation (3.10), we know that the vertex will be located in the first quadrant if

\[ d = -\frac{\mu + \phi}{\rho} + \sqrt{\frac{\phi}{\rho}} > 0. \] (3.12)

If \( \rho \) is big and \( \mu \) and \( \phi \) are small, or the rate at which individuals enter the drinking sub-population is greater than the rate at which individuals leave the drinking sub-population, then more people enter the drinking sub-population than leave, which causes the establishment of a finite drinking population. Conversely, if \( \rho \) is small and \( \mu \) and \( \phi \) are big, then

\[ d = -\frac{\mu + \phi}{\rho} + \sqrt{\frac{\phi}{\rho}} < 0. \] (3.13)

is true. This means that the rate at which individuals enter the drinking sub-population is less than the rate at which individuals leave the drinking sub-population. In this case, there is no established drinking population. To determine the effect of the different parameters on the establishment of a drinking population, we consider some examples.
First, we fix $\mu$ and $\phi$ and let $\rho$ vary. Define $\mu = 0.15$, $\phi = 0.25$, and $\rho = 0.95$. The graph of this situation is the exact graph depicted in Figure 3.1. Note that the vertex is located at $d = 0.092$. Now, let $\rho = 0.4$. This corresponds to a decrease in the rate that individuals relapse back to abusive drinking and results in a population of no abusive drinkers, as displayed in Figure 3.4 (a). The vertex in this situation is located at $d = -0.209$. If we let $\rho = 2.0$, which corresponds to an increase in the rate at which individuals relapse back to abusive drinking, there exists a finite population of abusive drinkers, like in Figure 3.1. This relationship is depicted in Figure 3.4 (b). Note that the vertex is located at $d = 0.154$, which is greater than in the case of Figure 3.1.

![Graphs](image)

Figure 3.4: (a) Decreased $\rho$: $\mu = 0.15$, $\phi = 0.25$, $\rho = 0.4$ (b) Increased $\rho$: $\mu = 0.15$, $\phi = 0.25$, $\rho = 2.0$

Next, we fix $\mu$ and $\rho$ and let $\phi$ vary. Define $\mu = 0.15$, $\rho = 0.95$, and $\phi = 0.25$. The graph depicting this situation is, again, the graph in Figure 3.1. The vertex is
located at $d = 0.092$. Now, let $\phi = 0.1$, which corresponds to a decrease in the rate that individuals recover from problem drinking. This results in an increased finite population of abusive drinkers, as displayed in Figure 3.5 (a). The vertex in this situation is located at $d = 0.061$. If we let $\phi = 0.6$, which corresponds to an increase in the rate at which individuals recover from problem drinking, then the population eventually has no abusive drinking population. This relationship is depicted in Figure 3.5 (b). The vertex in this case is located at $d = -0.015$.

Finally, we fix $\rho$ and $\phi$ and vary $\mu$. Consider our standard parameter values, $\mu = 0.15$, $\rho = 0.95$, and $\phi = 0.25$. These parameter values correspond to Figure 3.1. The vertex is located at $d = 0.092$. Now, let $\mu = 0.05$, which corresponds to another decrease in the rate at which individuals leave the abusive drinking population. This results in an increased finite population of abusive drinkers, as displayed in Figure
3.6 (a). The vertex in this situation is located at \( d = 0.197 \), which is greater than in the case of Figure 3.1. If we let \( \mu = 0.5 \), which corresponds to an increase in the rate at which individuals leave the drinking population, then the population eventually has no abusive drinking population. This relationship is depicted in Figure 3.6 (b). The vertex in this case is located at \( d = -0.276 \).

![Figure 3.6: (a) Decreased \( \mu \): \( \mu = 0.05, \phi = 0.25, \rho = 0.95 \) (b) Increased \( \mu \): \( \mu = 0.5, \phi = 0.25, \rho = 0.95 \)](image)

3.2 Hispanic Population Steady State Solutions

The steady state solutions of the Hispanic population are identical in form to those of the standard population, since the systems are nearly identical. A major assumption that we make in order to analyze the Hispanic population is that the standard population has already reached its steady state. The results, whether or not the standard population has reached a state of finite abusive drinkers or zero abusive drinkers, are
then input into the Hispanic population’s system of equations. Recall that Equations (2.14) through (2.17) are the non-dimensionalized equations representing the Hispanic population. Similar to the standard population, we set Equations (2.14) through (2.16) equal to zero

\[ h' = \eta - \delta(d)bh - \eta h = 0, \]  
\[ b' = \delta(d)hb + \xi qb - (\eta + \psi)b = 0, \]  
\[ q' = \psi b - \xi bq - \eta q = 0. \]

The base state solution for this system of equations is the case where the number of abusive drinkers, \( b \), is equal to zero. This case is identical to the base state solution of the standard population.

The bifurcating state for the Hispanic population is found the same way as the bifurcating state of the standard population. The subjects of interest are the abusive drinkers in the Hispanic population, so we aim to get an equation in \( b \) for our bifurcating state. To do so, we first solve Equation (3.14) for \( h \). The solution is

\[ h = \frac{\eta}{\eta + \delta(d)b}. \]  

Now we solve Equation (2.17) for \( q \) and substitute in the value of \( h \) from Equation (3.17), so that \( h \) and \( q \) are both rewritten in terms of \( b \). The result is

\[ q = 1 - \frac{\eta}{\eta + \delta(d)b} - b. \]  

We now have values for \( h \) and \( q \) in terms of \( b \), and we place these new values into Equation (3.15) and simplify the resulting equation, so that it appears as a
function of \( b \). The result is

\[
\begin{align*}
\frac{b^2}{2} & - \left(1 - \frac{\eta}{\delta(d)} - \frac{\eta + \psi}{\xi}\right)b + \frac{\eta}{\delta(d)} \left( \frac{\eta + \psi}{\xi} - \frac{\delta(d)}{\xi}\right) = 0. \\
\end{align*}
\]  

(3.19)

After applying the quadratic formula, we find that Equation (3.19) becomes

\[
b = \frac{1 - \frac{\eta}{\delta(d)} - \frac{\eta + \psi}{\xi}}{2} \pm \sqrt{\left(1 - \frac{\eta}{\delta(d)} - \frac{\eta + \psi}{\xi}\right)^2 - 4\left(\frac{\eta}{\delta(d)} \left( \frac{\eta + \psi}{\xi} - \frac{\delta(d)}{\xi}\right)\right)}.
\]  

(3.20)

Since the parameter of interest for the Hispanic population is \( \delta(d) \), the rate at which individuals in the Hispanic population convert to alcohol abusers, we rewrite Equation (3.19) in terms of \( \delta(d) \). \( \delta(d) \) is now a function of \( b \), the percentage of alcohol abusers in the Hispanic population. To reach our desired equation, multiply through by \( \delta(d) \), and solve the equation for \( \delta(d) \). The result is

\[
\delta(d) = -\frac{\eta(b + \frac{\eta + \psi}{\xi})}{b^2 - \left(1 - \frac{\eta + \psi}{\xi}\right)b - \frac{\eta}{\xi}}.
\]  

(3.21)

As an example of how this equation appears graphically, we define sample values for the parameters. Let \( \eta = 0.15 \), \( \psi = 0.25 \), and \( \xi = 0.95 \). Note that this is the same case as the standard population. In other words, \( \rho = \xi \), \( \phi = \psi \), \( \beta = c_1 \), \( \mu = \eta \), and \( c_2 = 0 \). In other words, the standard and Hispanic populations have the same parameter values, and the Hispanic population has no interaction with the standard population. The graph of Equation (3.21) is displayed in Figure 3.7. Since we are only interested in the applicable parts of this graph, the Hispanic population’s bifurcation diagram is focused on the restricted relationship depicted in Figure 3.8.

Note that the intercept on the \( \delta(d) \) axis is located at the value \( \delta(d) = \eta + \psi \), which is the rate of conversion from abusive drinking to either a recovered state or a
Figure 3.7: $\delta(d)$ as a function of $b$

The susceptible state. Also, the vertices of the bifurcation diagram are located at

$$b = -\frac{\eta + \psi}{\xi} \pm \sqrt{\frac{\psi}{\xi}}. \quad (3.22)$$

Equation (3.23) implies that there are two critical points in the system, similar to the standard population. Referring to Figure 3.7, we see that this is the case. The vertex of the applicable portion of the graph is located at

$$b = -\frac{\eta + \psi}{\xi} + \sqrt{\frac{\psi}{\xi}}. \quad (3.23)$$

In Figures 3.7 and 3.8, the vertex is located in the first quadrant, but like the standard population, this is not always the case. The vertex may also be located in the second
quadrant. As before, the location of the vertex is the determining factor for whether or not a finite population of abusive drinkers becomes established in the Hispanic population. From Equation 3.23, we know that the vertex will be located in the first quadrant if

\[ b = -\frac{\eta + \psi}{\xi} + \sqrt{\frac{\psi}{\xi}} > 0. \] (3.24)

If \( \xi \) is big and \( \eta \) and \( \psi \) are small, or the rate at which individuals leave the drinking population is less than the rate at which individuals enter the drinking sub-population, then more people enter the drinking sub-population than leave, which causes the establishment of a finite drinking population. As a result, the vertex is
located in the first quadrant. Conversely, if $\xi$ is small and $\eta$ and $\psi$ are big, then

$$b = -\frac{\eta + \psi}{\xi} + \sqrt{\frac{\psi}{\xi}} < 0.$$  \hspace{1cm} (3.25)

This means that the rate at which individuals leave the drinking sub-population is greater than the rate at which individuals enter the drinking sub-population. In this case, there is an established drinking sub-population. Similar to the standard population, the size of this sub-population is less than the case where individuals enter the drinking population at a greater rate than individuals leave the drinking population.

Equations (3.8) and (3.21), as well as the base state solutions of both systems, are explored further in a linear stability analysis in Chapter IV.
CHAPTER IV
LINEAR STABILITY ANALYSIS

In order to determine the stability of the given systems of equations, we will perform a linear stability analysis. The base states will be evaluated analytically, but the bifurcating states will be evaluated numerically.

4.1 Standard Population

Recall from Chapter III, the base state of the standard population is located at \( d = 0 \), where \( d \) is the portion of the population that is classified as abusive drinkers. First, we analyze the base state solution and determine its stability. As previously discussed, the original system of equations can be simplified from three ordinary differential equations to two ordinary differential equations. So, the system that we work with for the purpose of the linear stability analysis is

\[
s' = \mu - \beta ds - \mu s, \tag{4.1}
\]

\[
d' = (\beta - \rho)ds - (\mu + \phi - \rho)d - \rho d^2. \tag{4.2}
\]
In order to determine the stability of the base state, we seek solutions to Equations (4.1) and (4.2) of the form:

\[
\begin{align*}
  s(t) &= \bar{s} + \gamma u(t), \\
  d(t) &= \bar{d} + \gamma v(t),
\end{align*}
\]

where \( \gamma \ll 1 \).

Note that if \( u, v \to \pm \infty \) as \( t \to \infty \), then the solution is unstable, but if \( u \) and \( v \) are bounded or \( u, v \to 0 \) as \( t \to \infty \), then the solution is stable. Next, we linearize the system about \( \bar{s} \) and \( \bar{d} \) to find

\[
\begin{align*}
  \bar{s}' + \gamma u' &= \mu - \beta(\bar{s} + \gamma u)(\bar{d} + \gamma v) - \mu(\bar{s} + \gamma u), \\
  \bar{d}' + \gamma v' &= (\beta - \rho)(\bar{s} + \gamma u)(\bar{d} + \gamma v) - (\mu + \phi - \rho)(\bar{d} + \gamma v) - \rho(\bar{d} + \gamma v)^2.
\end{align*}
\]

The leading order system of equations is the original system from Equations (4.1) and (4.2). So, we examine the order \( \gamma \) system of equations, which is represented below in Equations (4.7) and (4.8). Note that we are considering the case where \( d = 0 \) for the base state solution, hence it must be the case that \( \bar{d} = 0 \) and \( \bar{s} = 1 \). Thus,

\[
\begin{align*}
  u' &= -\beta v - \mu u, \\
  v' &= (\beta - \mu - \phi)v.
\end{align*}
\]

Since we are interested in the stability of the abusive drinking population, we only use Equation (4.8). The solution to this differential equation is

\[
v(t) = ce^{(\beta - \mu - \phi)t},
\]
where \( c \) is some constant dependent on the initial condition. Regardless of the initial condition, we can determine the stability. As \( t \to \infty \), if \( \beta - \mu - \phi < 0 \), then the solution is stable. Thus for \( \beta < \mu + \phi \), the solution is stable. This implies that if the rate that individuals enter the drinking population is less than the rate at which individuals leave the drinking population, the population will have zero abusive drinkers. On the other hand, if \( \beta - \mu - \phi > 0 \), then the solution is unstable. This implies that if \( \beta > \mu + \phi \), then the solution is unstable. So, if the rate that individuals enter the drinking population is greater than the rate at which individuals leave the drinking population, the population will have some finite number of abusive drinkers.

Graphically, the stability of the base state appears as in Figure 4.1.

Next, it is desirable to determine the stability of the bifurcating solution. Recall from Chapter III, that the bifurcating state is represented by

\[
d^2 - \left(1 - \frac{\mu}{\beta} - \frac{\mu + \phi}{\rho}\right)d + \frac{\mu}{\beta} \left(\frac{\mu + \phi}{\rho} - \frac{\beta}{\rho}\right) = 0.
\]

We determine the stability of this system numerically, using Equation (3.8). The Maple code can be found in the Appendix. The first case to consider is when there is an established drinking population. Recall that this occurs when the rate that people enter the drinking sub-population, \( \rho \), is greater than the rate that people leave the drinking sub-population, \( \mu + \phi \). In this case, the vertex lies in the first quadrant, so we consider the stability of both the upper and lower bifurcating branches. In the Maple code, we define values for the parameters and rewrite Equations (2.5) through (2.8) as a system of two equations with two dependent variables, \( s \) and \( r \). The result
is

\[ s' = \mu - \beta ds - \mu s, \quad (4.11) \]

\[ d' = (\beta - \rho)ds - (\mu + \phi - \rho)d - \rho d^2. \quad (4.12) \]

Next, we define the bifurcating branch and the value of \( s \) given the value of the bifurcating solution from Equation (3.8). We find the Jacobian of the system and substitute in the bifurcating solution, and finally we find the eigenvalues of the system. These eigenvalues only correspond to one point on the bifurcating branch, so multiple runs are necessary to determine the stability of the bifurcating solution. Once the
stability of one branch is found, we find the stability of the other branch. The results show that the lower branch of the bifurcating solution beginning at the vertex is unstable, while the upper branch beginning at the vertex is stable. Figure 4.2 illustrates the stability of both the base state and the bifurcating solutions of the standard population, given that a finite drinking population is established.

The second case to consider is when the vertex falls in the second quadrant. This occurs when $\rho$ is small or when $\phi$ and $\mu$ are large, which implies that more people leave the drinking sub-population than enter, so a drinking population does
not become established. In this case, there is only one bifurcating branch located in the first quadrant, so it is the only applicable bifurcating solution. To test the stability of this branch, we follow the same procedure as in the first case, but only with one branch. The values in the Appendix are sample values for this case. Note that on the bifurcating branch, at the sample point, the eigenvalues are both negative, which indicates stability. This is the stability of only one point on the bifurcating branch, but we run multiple cases and find the same outcome on the entire bifurcating branch in this case. In the case that the vertex is not incorporated in the bifurcation diagram, because it lies in the negative $d$ region, or the second quadrant, the bifurcating branch is always stable. See Figure 4.3.

4.2 Hispanic Population

The Hispanic Population’s system of equations is identical in form to that of the Standard Population, therefore, the stability of the base state and the bifurcating states are also identical in form. The only difference is that we choose a different parameter to manipulate, $\delta(d)$. Hence, the base state is stable from $\delta(d) = 0$ to the intercept, $\delta(d) = \eta + \psi$, and is unstable from the intercept on. Approaching the linear stability analysis in the exact manner as with the standard population, we find that the linearized equation representing the rate of change of the portion of abusive drinkers in the population, call it $w(t)$, is defined as

$$w(t) = ae^{(\delta(d)-\eta-\psi)t}.$$ \hspace{1cm} (4.13)
Figure 4.3: *Standard Population System Stability without Vertex*

Note that $a$ is an arbitrary constant dependent on the initial condition chosen for the system. If $\delta(d) > \eta + \psi$, then the base state solution is unstable. In other words, the base state solution to the right of the intercept is unstable. Conversely, if $\delta(d) < \eta + \psi$, then the base state solution is stable. So if $\delta(d)$ is between 0 and $\eta + \psi$, then the solution is stable. This analysis is expressed in Figure 4.4.

Now, consider the bifurcating state of the system of equations. Since the system is identical in form to that of the standard population, the stability of the bifurcating state will be the same. The Hispanic population’s stability graphically
Figure 4.4: Hispanic Population Base State Stability

appears exactly the same as the standard population’s, except the parameters will differ. Figure 4.5 represents the stability of the Hispanic population’s base state and bifurcating state solutions.
Figure 4.5: *Hispanic Population System Stability*
CHAPTER V
SAMPLE POPULATIONS

Now that we have analyzed both the standard population’s and the Hispanic population’s governing equations, we want to analyze the effect that the standard population has on the Hispanic population. We consider three examples in this chapter.

5.1 A Standard Population of Finite Abusive Drinkers

For the first example, we consider a standard population that has some finite number of abusive drinkers. The population falls somewhere on the stable portion of the bifurcating solution. Let $\beta = 0.6$, $\phi = 0.2$, $\mu = 0.1$, and $\rho = 0.95$. These parameter values are chosen so that the rate at which individuals leave the drinking sub-population is less than the rate at which individuals enter the population. Then based on Equation (3.7), the portion of the standard population that becomes alcohol abusers is $d = 0.605$, which is approximately 60.5% of the population. Now, we consider the Hispanic population and define $\delta(d) = 0.5 + 0.8d$. If we substitute $d$ into the equation for $\delta(d)$, we have $\delta(d) = 0.984$. Next, we define the remaining parameters for the Hispanic population as: $\eta = 0.1$, $\psi = 0.2$, and $\xi = 0.95$. Then from Equation (3.20), the portion of the Hispanic population that becomes alcohol abusers is $b = 0.689$. Approximately 68.9% of the Hispanic population is driven to alcohol abuse after the
interaction with the standard population. Figure 5.1 compares the standard and Hispanic populations, assuming the standard population reaches its steady state and then influences the Hispanic population. The “x” denotes where the populations fall on their respective bifurcation diagrams with respect to the defined parameter values. We compare this to the case in which the Hispanic population has no interaction with the standard population. Then, $\delta(d) = 0.5$, and let the other parameters retain the same values. In this case, the portion of alcohol abusers in the Hispanic population is $b = 0.559$. The difference between these two cases in the Hispanic population is approximately 13%. This implies that the standard population is causing an extra 13% of the Hispanic population to become alcohol abusers. Figure 5.2 compares the Hispanic population with the influence of the standard population to the Hispanic population without the influence of the standard population.

5.2 A Standard Population with No Established Drinking Sub-Population

For the second example, we consider a standard population that has no established drinking sub-population. The population falls somewhere on the stable portion of the base state solution. Then, the only stipulation that the parameters must follow is that $\beta$ must be less than $\mu + \phi$. In application, this means that the rate that individuals relapse to alcohol abuse has to be less than the rate at which alcohol abusers recover or leave the sub-population. In this case, we know that there are absolutely no abusive drinkers in the population, or $d = 0$. Now, we consider the Hispanic population. Define the parameters exactly as in the last example. Hence,
Figure 5.1: Standard Population Denoted by “x”: $\mu = 0.1$, $\phi = 0.2$, $\rho = 0.95$; Hispanic Population Denoted by “+”: $\eta = 0.1$, $\psi = 0.2$, $\xi = 0.95$, $c_1 = 0.5$, $c_2 = 0.8$.

$\eta = 0.1$, $\psi = 0.2$, and $\xi = 0.95$. Also, $\delta(d) = 0.5 + 0.8d$, but because $d = 0$, we know that $\delta(d) = 0.5$. This becomes the same case as if the Hispanic population were to have absolutely no contact with the standard population. Just as in the last example, the portion of the Hispanic population that become alcohol abusers is $b = 0.559$. The “x” in Figure 5.2 (b) signifies where the Hispanic population falls on the bifurcation diagram with respect to these sample parameters.

Until now we have assumed that $c_1 = 0.5$. To see the impact that $c_1$ has on the Hispanic population, redefine $c_1 = 0.3$. This means that the Hispanic popu-
Figure 5.2: “+” Denotes the Hispanic Population with Standard Population Contact: \( \eta = 0.1, \, \psi = 0.2, \, \xi = 0.95, \, c_1 = 0.5, \, c_2 = 0.8; \) “x” Denotes the Hispanic Population with no Standard Population Contact: \( \eta = 0.1, \, \psi = 0.2, \, \xi = 0.95, \, c_1 = 0.5, \, c_2 = 0. \\

lation has a lower rate of conversion from susceptible to abusive drinking than the previous examples, before any contact with the standard population. Consider the same parameter values and percentage of drinkers in the standard population as in the previous example. So, \( \eta = 0.1, \, \psi = 0.2, \, \xi = 0.95, \) and \( d = 0. \) Now, \( \delta(d) = 0.3 + 0.8d. \) Since \( d = 0, \) we have \( \delta(d) = 0.3. \) The portion of abusive drinkers in the Hispanic population, then is \( b = 0.351, \) or about 35.1\%. Compared to the last example, where \( c_1 = 0.5, \) there is a 20.8\% decrease in the percent of alcohol abusers in the Hispanic population. Figure 5.3 illustrates this difference. This shows that \( c_1 \)
Figure 5.3: “+” Denotes the Hispanic Population with Larger $c_1$: $\eta = 0.1$, $\psi = 0.2$, $\xi = 0.95$, $c_1 = 0.5$, $c_2 = 0$; “x” Denotes the Hispanic Population with Smaller $c_1$: $\eta = 0.1$, $\psi = 0.2$, $\xi = 0.95$, $c_1 = 0.3$, $c_2 = 0$.

has a significant impact on the Hispanic population. Note that Figure 5.3 illustrates a case where $\delta(d) = \eta + \psi$, or the rate that individuals enter the drinking state is equal to the rate that individuals leave the drinking state. If we define $\delta(d) = 0.2$, $\eta = 0.1$, and $\psi = 0.2$, then we have $\delta(d) < \eta + \psi$. In this case, we have $d = 0$, or there are no abusive drinkers in the population. The population limits to the stable base state solution. Figure 5.4 illustrates where such a population would lie on the base state.
Figure 5.4: “x” Denotes the Hispanic Population with No Abusive Drinkers: $\eta = 0.1$, $\psi = 0.2$, $\xi = 0.95$, $c_1 = 0.2$, $c_2 = 0$.

5.3 A Hispanic Population with Less Interaction with the Standard Population

In both of the previous examples, 80% of abusive drinkers came into contact with the Hispanic population. This may not necessarily be a realistic scenario. For a final example, define the standard population in the same manner as example one. In other words, let $\beta = 0.6$, $\phi = 0.2$, $\mu = 0.1$, and $\rho = 0.95$. This implies that the portion of the standard population that becomes alcohol abusers is $d = 0.605$. Define $\delta(d) = 0.5 + 0.1d$ for the Hispanic population. In other words, only 10% of alcohol abusers from the standard population come into contact with the Hispanic population.
Since \( d = 0.605 \), \( \delta(d) = 0.561 \). We define the remaining parameters for the Hispanic Population as: \( \eta = 0.1 \), \( \psi = 0.2 \), and \( \xi = 0.95 \). Then the portion of the Hispanic population that becomes alcohol abusers is \( b = 0.589 \). Figure 5.3 exemplifies the comparison between the standard and Hispanic populations using an “x” to denote where the populations fall on their own respective bifurcation diagrams. We compare this to the case where the standard population has absolutely no contact with and thus no impact on the Hispanic population. We have \( \delta(d) = 0.5 \), \( \eta = 0.153 \), \( \psi = 0.2 \), and \( \xi = 0.95 \). The portion of the Hispanic population classified as alcohol abusers is \( b = 0.559 \). Figure 5.5 exemplifies the difference between the Hispanic population with less (a) or no (b) contact with the standard population. Here, we have approximately a 3.0% difference. We can say that the standard population causes approximately a 3.0% increase in alcohol abuse rates of the Hispanic population. Although this percentage is much smaller than in the first example, it is still a significant increase when referring to an entire sub-population of people.

From these examples, we see that the standard population can have a severe impact on the Hispanic population depending on how much contact individuals from both populations have with one another.
Figure 5.5: Standard Population Denoted by “x”: \( \mu = 0.1, \phi = 0.2, \rho = 0.95 \); Hispanic Population Denoted by “+”: \( \eta = 0.1, \psi = 0.2, \xi = 0.95, c_1 = 0.5, c_2 = 0.1 \).
Figure 5.6: “+” Denotes the Hispanic Population with Standard Population Contact: \( \eta = 0.1, \psi = 0.2, \xi = 0.95, c_1 = 0.5, c_2 = 0.1; \) “x” Denotes the Hispanic Population with no Standard Population Contact: \( \eta = 0.1, \psi = 0.2, \xi = 0.95, c_1 = 0.5, c_2 = 0.0. \)
Sanchez et al. [1] proposed a system of three differential equations to model the alcohol abuse behavior of any given population. Because research shows that the standard U.S. population influences the Hispanic Immigrant population, we have proposed a model for the Hispanic sub-population that is similar to that of the standard population. The only difference between the models is that the rate at which individuals in the Hispanic population become alcohol abusers is dependent on how many alcohol abusers there are in the standard population, as well as how frequently the two populations come into contact. The purpose of this research was to examine the effects of the standard population on the Hispanic population, using the proposed system of equations.

The Sanchez et al. [1] model was not without limitations, thus the proposed model also has various limitations due to the underlying assumptions made. First and foremost, we assume that the total population is constant. The rates at which individuals leave and enter the population are equal. Also, there is no influence to drink other than the influence of other drinkers. Individuals become alcohol abusers by merely coming into contact with other alcohol abusers. Individuals entering the system enter the “susceptible” sub-population and then mix at random with the rest
of the population. Another major assumption is that everyone who has a problem with alcohol, regardless of consumption level, is placed in the abusive drinker sub-population. The definition of alcohol abuser ought to be particularly defined. Also, all individuals have the same probability of coming into contact with an alcohol abuser and, as a result, becoming an abusive drinker themselves. This is a limitation in that individuals do not all surround themselves with the same crowd of people. It is definite that some individuals would realistically have a higher chance of coming into contact with an alcohol abuser than others in any given population. Finally, we also assumed that the standard population had already reached its steady state.

After defining the model, we found the steady state solutions of both the standard population and the Hispanic population and then classified the solutions as either base state or bifurcating state solutions. Then, the graphical difference between a population with an established finite abusive drinking sub-population and a population with no established abusive drinking sub-population was explored. The stability of both solutions was examined using a linear stability analysis. The base states were examined analytically, while the bifurcating states were analyzed numerically. We found that the base state is stable if the rate at which individuals enter the drinking sub-population is less than the rate at which individuals leave the drinking sub-population. In this case, the population has zero abusive drinkers. Conversely, if the rate at which individuals enter the drinking sub-population is greater than the rate at which individuals leave the drinking sub-population, the base state is unstable. For the bifurcating solution, we found that the upper branch is stable and the
lower branch is unstable. If the population lies on the upper branch, then there exists a finite population of abusive drinkers in the population. Finally, the effect of the standard population on the Hispanic population was discussed by comparing various examples of the models. We found that the standard population may have a great impact on the Hispanic population if the rate at which Hispanics come into contact with individuals from the standard population is large enough. We also determined that if there are no drinkers in the standard population, the Hispanic population will retain its’ original parameter values and not be influenced by the standard population.

For future works, one may consider other applicable uses for the proposed model. Numerous studies, including one such study by Caetano and colleagues [14], have found that most romantic partners share similar drinking habits. In fact, this study found that “69% of the couples [in their study] were in the same or adjacent drinking categories” [14]. One use of the proposed model would be to allow one gender to represent the standard population, and to let the opposite gender represent the dependent population.

Other enhancements to the model include defining in detail the meaning of “alcohol abuser”, so that there is a clearly defined difference between each of the three sub-populations in each model. Also, we could incorporate the legal drinking age into our model, since the majority of alcohol abusers are over the age of 21, and individuals over 21 tend to be the individuals most at risk for “coming into contact” with an alcohol abuser. Additionally, there are other reasons that some individuals convert to abusive drinking patterns other than the influence of their peers. Other
causes, such as emotional causes, could be accounted for in the model. Finally, the proposed model assumes that the U.S. population is approximately the same size as the Hispanic Population. In the United States, though, we know that the standard population is actually much larger than the Hispanic population. In order to make the model more accurate and representative of the actual populations, it is desirable that these limitations and weaknesses be minimized.
BIBLIOGRAPHY


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APPENDIX

LINEAR STABILITY ANALYSIS OF THE STANDARD POPULATION: CODE

The following code was entered into Maple to determine the stability of the standard population’s bifurcating solution to the system of differential equations.

> restart;
> Digits := 20;
> beta := .2;

> phi := .6033;

> mu := .2553;

> rho := .95;

> sdot := proc (nsr) options operator, arrow; mu*beta*d*s - mu*s end proc;

> ddot := proc (nsr) options operator, arrow; (beta-rho)*s*d end proc;

(s, d) -> (beta - rho) s d - (mu + phi - rho) d - rho d^2

\[\text{dstar} := \frac{(1-mu/beta-(mu+phi)/rho+sqrt((1-mu/beta-(mu+phi)/rho)^2-4*mu*((mu+phi)/rho-beta/rho)/beta))(1/2)}{0.47576102491897470522}\]

\[\text{sstar} := \frac{mu/(beta*dstar+mu)}{0.45475220589358783445}\]

\[\text{with(linalg);}\]
\[\text{J := jacobian([s, d]), [s, d]);}\]

\[\text{Jstar := map(proc (X) options operator, arrow; subs(s = sstar, d = dstar, X) end proc, J);}\]

\[\text{collect(charpoly(Jstar, lambda), lambda);}\]
\[\text{lambda}^2 + 1.0133776171058942953 \text{ lambda} + 0.21106046594747479748\]

\[\text{ev1 := eigenvalues(Jstar);}\]
\[-0.720401425787705385, -0.292976191318188750\]