FINITE ELEMENT ANALYSIS OF DROP DEFORMATION
IN THE ENTRANCE REGION OF A CYLINDRICAL TUBE

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Dissertation

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ABSTRACT

The finite element method was employed to investigate the large deformation of a viscoelastic drop, which was suspended in a viscoelastic medium, moving along the central axis of a cylindrical tube in the entrance region. For the investigation, finite element equations were developed, with the aid of the penalty function method, based on the system equations consisting of the continuity and momentum equations for both the drop and suspending medium, and auto-remeshing technique was employed to describe the moving interface between the drop and suspending medium. The system equations were formulated using the integral-type KBKZ constitutive equation to deal with a moving boundary problem. The effects of (1) the ratio of the relaxation times of the drop and the suspending medium, (2) the viscosity ratio of the drop and suspending medium, (3) the initial drop size, and (4) an apparent shear rate on the extent of the drop deformation were investigated by considering five combinations of viscoelastic drop and viscoelastic suspending medium. The computed shapes of a viscoelastic drop, suspended in another viscoelastic medium, moving along the central axis of a cylindrical tube in the entrance region elongated continuously as it moved towards the tube entrance and recoiled slightly after it passed the tube inlet. Further, the extent of drop deformation became greater as the ratio of the relaxation times of the drop and the suspending medium decreased, as the initial drop radius was increased, and as the apparent shear rate was increased. For comparison, also computed were the shapes of a Newtonian drop, suspended in another Newtonian medium, moving along the central axis of a cylindrical
tube in the entrance region. A maximum deformation of a Newtonian drop occurred at an axial position very close to the end of the converging section of the straight section of the cylindrical tube, while a maximum deformation of a viscoelastic drop occurred at an axial position slightly inside the straight section of the cylindrical tube. It has been found that a Newtonian drop deformed faster than a viscoelastic drop as the respective drops moved along the central axis of a conveying cylindrical tube. In order to explain the above observations, the extensional stress distributions along the centerline of the converging cylindrical tube were computed for both a Newtonian suspending medium and a viscoelastic suspending medium. The difference in the rate of drop deformation between Newtonian and viscoelastic drops, each moving along the central axis of a converging cylindrical tube, was attributable to the elasticity of a viscoelastic drop. This is because the large elastic modulus of a viscoelastic drop would tend to resist deformation, as compared to the deformation of a Newtonian drop. Interestingly enough, the rate of recoil of a viscoelastic drop was found to be much slower than that of a Newtonian drop, which was also attributable to the elasticity of a drop. It should be noted that the recoil of a viscoelastic drop is dictated predominantly by its relaxation modulus whereas the recoil of a Newtonian drop is determined strictly by its interfacial tension.
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4.7 Snap shots of the computed shapes of a viscoelastic drop (Case I) with the initial radius ($r_0$) of 0.45 mm and the numerical values of $\lambda_k$ given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at $\dot{\gamma}_{\text{app}} = 35.4$ s$^{-1}$, with the values of $\lambda_k$ and $a_k$ given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 1.148$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube…………………………..74

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4.9 Snap shots of the computed shapes of a viscoelastic drop (Case III) with the initial radius (r_0) of 0.45 mm and the numerical values of \( \lambda_k \) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \( \dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1} \), with the values of \( \lambda_k \) and \( a_k \) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions (z/R_0) along the centerline of the cylindrical tube in the entrance region: the position A at z/R_0 = -10, the position B at z/R_0 = -5, the position C at z/R_0 = 1.238, and the position D at z/R_0 = 5 with R_0 being the radius (3 mm) of the cylindrical tube.

4.10 Snap shots of the computed shapes of a viscoelastic drop (Case IV) with the initial radius (r_0) of 0.45 mm and the numerical values of \( \lambda_k \) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \( \dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1} \), with the values of \( \lambda_k \) and \( a_k \) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions (z/R_0) along the centerline of the cylindrical tube in the entrance region: the position A at z/R_0 = -10, the position B at z/R_0 = -5, the position C at z/R_0 = 1.286, and the position D at z/R_0 = 5 with R_0 being the radius (3 mm) of the cylindrical tube.

4.11 Snap shots of the computed shapes of a viscoelastic drop (Case V) with the initial radius (r_0) of 0.45 mm and the numerical values of \( \lambda_k \) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \( \dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1} \), with the values of \( \lambda_k \) and \( a_k \) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions (z/R_0) along the centerline of the cylindrical tube in the entrance region: the position A at z/R_0 = -10, the position B at z/R_0 = -5, the position C at z/R_0 = 1.337, and the position D at z/R_0 = 5 with R_0 being the radius (3 mm) of the cylindrical tube.

4.12 Snap shots of the computed shapes of five different viscoelastic drops (Case I through Case V) with the identical initial radius (r_0) of 0.45 mm and the numerical values of \( \lambda_k \) given in Table 4.2 when each of the drops
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4.13 Snap shots of the computed shapes of a Newtonian drop (Case VI) with the initial radius ($r_0$) of 0.45 mm when the drop was suspended in the Newtonian medium, moving at $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$, with the viscosity of the drop $\eta_d$ and the medium $\eta_m$ given in Table 4.6. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 0.146$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.

4.14 Snap shots of the computed shapes of a Newtonian drop (Case VII) with the initial radius ($r_0$) of 0.45 mm when the drop was suspended in the Newtonian medium, moving at $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$, with the viscosity of the drop $\eta_d$ and the medium $\eta_m$ given in Table 4.6. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 0.148$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.

4.15 Snap shots of the computed shapes of a Newtonian drop (Case VIII) with the initial radius ($r_0$) of 0.45 mm when the drop was suspended in the Newtonian medium, moving at $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$, with the viscosity of the drop $\eta_d$ and the medium $\eta_m$ given in Table 4.6. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 0.150$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.

4.16 Snap shots of the computed shapes of a Newtonian drop (Case IX) with the initial radius ($r_0$) of 0.45 mm when the drop was suspended in the
Newtonian medium, moving at $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$, with the viscosity of the drop $\eta_0d$ and the medium $\eta_0m$ given in Table 4.6. The snapshot of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 0.153$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.

4.17 Snap shots of the computed shapes of a Newtonian drop (Case X) with the initial radius ($r_0$) of 0.45 mm when the drop was suspended in the Newtonian medium, moving at $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$, with the viscosity of the drop $\eta_0d$ and the medium $\eta_0m$ given in Table 4.6. The snapshot of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 0.153$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.

4.18 Snap shots of the computed shapes of five different Newtonian drops (Case VI through Case X) with the identical initial radius ($r_0$) of 0.45 mm and the viscosity $\eta_0d$ given in Table 4.6 when each of the drops was suspended in the same Newtonian medium, moving at $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$, with the viscosity $\eta_0m$ given in Table 4.6. The snapshot of the computed shape of the drops are at the same dimensionless axial position of $z/R_0 = 5$ in the entrance region of a cylindrical tube for different drops: (a) Case VI, (b) Case VII, (c) Case VIII, (d) Case IX, and (e) Case X.

4.19 Extensional stress distribution along the centerline of the cylindrical tube: (1) Newtonian system, (2) viscoelastic system) when a Newtonian drop with initial radius ($r_0$) of 0.45 mm is suspended in the Newtonian medium, moving at $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$, (corresponding Case VI in Table 4.6), and a viscoelastic drop of 0.45 mm is suspended in the viscoelastic medium moving at $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$, (corresponding Case I in Table 4.1).

4.20 Extensional stress distribution along the centerline of the cylindrical tube when a viscoelastic drop with initial radius ($r_0$) of 0.45 mm is suspended in the viscoelastic medium (corresponding Case I in Table 4.6), moving at various shear rates: (1) $\dot{\gamma}_{\text{app}} = 15.4 \text{ s}^{-1}$; (2) $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$;
4.21 Extensional stress distribution along the centerline of the cylindrical tube when a Newtonian drop with initial radius \( r_0 \) of 0.45 mm is suspended in the Newtonian medium (corresponding Case VI in Table 4.6), moving at various shear rates: (1) \( \dot{\gamma}_{app} = 15.4 \text{ s}^{-1} \); (2) \( \dot{\gamma}_{app} = 35.4 \text{ s}^{-1} \); (3) \( \dot{\gamma}_{app} = 75.4 \text{ s}^{-1} \).

B.1 Snap shots of the computed shapes of a viscoelastic drop (Case I) with the initial radius \( r_0 \) of 0.45 mm and the numerical values of \( \lambda_k \) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \( \dot{\gamma}_{app} = 15.4 \text{ s}^{-1} \), with the values of \( \lambda_k \) and \( a_k \) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \( z/R_0 \) along the centerline of the cylindrical tube in the entrance region: the position A at \( z/R_0 = -10 \), the position B at \( z/R_0 = -5 \), the position C at \( z/R_0 = 1.136 \), and the position D at \( z/R_0 = 5 \) with \( R_0 \) being the radius (3 mm) of the cylindrical tube.

B.2 Snap shots of the computed shapes of a viscoelastic drop (Case II) with the initial radius \( r_0 \) of 0.45 mm and the numerical values of \( \lambda_k \) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \( \dot{\gamma}_{app} = 15.4 \text{ s}^{-1} \), with the values of \( \lambda_k \) and \( a_k \) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \( z/R_0 \) along the centerline of the cylindrical tube in the entrance region: the position A at \( z/R_0 = -10 \), the position B at \( z/R_0 = -5 \), the position C at \( z/R_0 = 1.182 \), and the position D at \( z/R_0 = 5 \) with \( R_0 \) being the radius (3 mm) of the cylindrical tube.

B.3 Snap shots of the computed shapes of a viscoelastic drop (Case III) with the initial radius \( r_0 \) of 0.45 mm and the numerical values of \( \lambda_k \) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \( \dot{\gamma}_{app} = 15.4 \text{ s}^{-1} \), with the values of \( \lambda_k \) and \( a_k \) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \( z/R_0 \) along the centerline of the cylindrical tube in the entrance region: the position A at \( z/R_0 = -10 \), the position B at \( z/R_0 = -5 \), the position C at \( z/R_0 = 1.215 \), and the position D at \( z/R_0 = 5 \) with \( R_0 \) being the radius (3 mm) of the cylindrical tube.
B.4 Snap shots of the computed shapes of a viscoelastic drop (Case IV) with the initial radius \( r_0 \) of 0.45 mm and the numerical values of \( \lambda_k \) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \( \dot{\gamma}_{\text{app}} = 15.4 \text{ s}^{-1} \), with the values of \( \lambda_k \) and \( a_k \) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \( (z/R_0) \) along the centerline of the cylindrical tube in the entrance region: the position A at \( z/R_0 = -10 \), the position B at \( z/R_0 = -5 \), the position C at \( z/R_0 = 1.270 \), and the position D at \( z/R_0 = 5 \) with \( R_0 \) being the radius (3 mm) of the cylindrical tube.

B.5 Snap shots of the computed shapes of a viscoelastic drop (Case V) with the initial radius \( r_0 \) of 0.45 mm and the numerical values of \( \lambda_k \) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \( \dot{\gamma}_{\text{app}} = 15.4 \text{ s}^{-1} \), with the values of \( \lambda_k \) and \( a_k \) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \( (z/R_0) \) along the centerline of the cylindrical tube in the entrance region: the position A at \( z/R_0 = -10 \), the position B at \( z/R_0 = -5 \), the position C at \( z/R_0 = 1.319 \), and the position D at \( z/R_0 = 5 \) with \( R_0 \) being the radius (3 mm) of the cylindrical tube.

B.6 Snap shots of the computed shapes of five different viscoelastic drops (Case I through Case V) with the identical initial radius \( r_0 \) of 0.45 mm and the numerical values of \( \lambda_k \) given in Table 4.2 when each of the drops was suspended in the same viscoelastic medium, moving at \( \dot{\gamma}_{\text{app}} = 15.4 \text{ s}^{-1} \), with the values of \( \lambda_k \) and \( a_k \) given in Table 4.1. The snap shots of the computed shape of the drops are at the same dimensionless axial position of \( z/R_0 = 5 \) in the entrance region of a cylindrical tube for different drops: (a) Case I, (b) Case II, (c) Case III, (d) Case IV, and (e) Case V.

B.7 Snap shots of the computed shapes of a viscoelastic drop (Case I) with the initial radius \( r_0 \) of 0.45 mm and the numerical values of \( \lambda_k \) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \( \dot{\gamma}_{\text{app}} = 75.4 \text{ s}^{-1} \), with the values of \( \lambda_k \) and \( a_k \) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \( (z/R_0) \) along the centerline of the cylindrical tube in the entrance region: the position A at \( z/R_0 = -10 \), the position B at \( z/R_0 = -5 \), the position C at \( z/R_0 = 1.160 \),
and the position D at \(z/R_0 = 5\) with \(R_0\) being the radius (3 mm) of the cylindrical tube.

B.8 Snap shots of the computed shapes of a viscoelastic drop (Case II) with the initial radius \((r_0)\) of 0.45 mm and the numerical values of \(\lambda_k\) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \(\dot{\gamma}_{\text{app}} = 75.4\ \text{s}^{-1}\), with the values of \(\lambda_k\) and \(a_k\) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \((z/R_0)\) along the centerline of the cylindrical tube in the entrance region: the position A at \(z/R_0 = -10\), the position B at \(z/R_0 = -5\), the position C at \(z/R_0 = 1.205\), and the position D at \(z/R_0 = 5\) with \(R_0\) being the radius (3 mm) of the cylindrical tube.

B.9 Snap shots of the computed shapes of a viscoelastic drop (Case III) with the initial radius \((r_0)\) of 0.45 mm and the numerical values of \(\lambda_k\) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \(\dot{\gamma}_{\text{app}} = 75.4\ \text{s}^{-1}\), with the values of \(\lambda_k\) and \(a_k\) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \((z/R_0)\) along the centerline of the cylindrical tube in the entrance region: the position A at \(z/R_0 = -10\), the position B at \(z/R_0 = -5\), the position C at \(z/R_0 = 1.253\), and the position D at \(z/R_0 = 5\) with \(R_0\) being the radius (3 mm) of the cylindrical tube.

B.10 Snap shots of the computed shapes of a viscoelastic drop (Case IV) with the initial radius \((r_0)\) of 0.45 mm and the numerical values of \(\lambda_k\) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \(\dot{\gamma}_{\text{app}} = 75.4\ \text{s}^{-1}\), with the values of \(\lambda_k\) and \(a_k\) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \((z/R_0)\) along the centerline of the cylindrical tube in the entrance region: the position A at \(z/R_0 = -10\), the position B at \(z/R_0 = -5\), the position C at \(z/R_0 = 1.303\), and the position D at \(z/R_0 = 5\) with \(R_0\) being the radius (3 mm) of the cylindrical tube.

B.11 Snap shots of the computed shapes of four different viscoelastic drops (Case I through Case IV) with the identical initial radius \((r_0)\) of 0.45 mm and the numerical values of \(\lambda_k\) given in Table 4.2 when each of the drops was suspended in the same viscoelastic medium, moving at
\[ \dot{\gamma}_{\text{app}} = 75.4 \text{ s}^{-1}, \] with the values of \( \lambda_k \) and \( a_k \) given in Table 4.1. The snap shots of the computed shape of the drops are at the same dimensionless axial position of \( z/R_0 = 5 \) in the entrance region of a cylindrical tube for different drops: (a) Case I, (b) Case II, (c) Case III, (d) Case IV, and (e) Case V.

B.12 Snap shots of the computed shapes of a viscoelastic drop (Case I) with the initial radius \( (r_0) \) of 0.30 mm and the numerical values of \( \lambda_k \) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \( \dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1} \), with the values of \( \lambda_k \) and \( a_k \) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \( (z/R_0) \) along the centerline of the cylindrical tube in the entrance region: the position A at \( z/R_0 = -10 \), the position B at \( z/R_0 = -5 \), the position C at \( z/R_0 = 1.133 \), and the position D at \( z/R_0 = 5 \) with \( R_0 \) being the radius (3 mm) of the cylindrical tube.

B.13 Snap shots of the computed shapes of a viscoelastic drop (Case II) with the initial radius \( (r_0) \) of 0.30 mm and the numerical values of \( \lambda_k \) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \( \dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1} \), with the values of \( \lambda_k \) and \( a_k \) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \( (z/R_0) \) along the centerline of the cylindrical tube in the entrance region: the position A at \( z/R_0 = -10 \), the position B at \( z/R_0 = -5 \), the position C at \( z/R_0 = 1.178 \), and the position D at \( z/R_0 = 5 \) with \( R_0 \) being the radius (3 mm) of the cylindrical tube.

B.14 Snap shots of the computed shapes of a viscoelastic drop (Case III) with the initial radius \( (r_0) \) of 0.30 mm and the numerical values of \( \lambda_k \) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \( \dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1} \), with the values of \( \lambda_k \) and \( a_k \) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \( (z/R_0) \) along the centerline of the cylindrical tube in the entrance region: the position A at \( z/R_0 = -10 \), the position B at \( z/R_0 = -5 \), the position C at \( z/R_0 = 1.211 \), and the position D at \( z/R_0 = 5 \) with \( R_0 \) being the radius (3 mm) of the cylindrical tube.

B.15 Snap shots of the computed shapes of a viscoelastic drop (Case IV) with the initial radius \( (r_0) \) of 0.30 mm and the numerical values of \( \lambda_k \)
given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$, with the values of $\lambda_k$ and $a_k$ given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 1.266$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.

B.16 Snap shots of the computed shapes of a viscoelastic drop (Case V) with the initial radius ($r_0$) of 0.30 mm and the numerical values of $\lambda_k$ given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$, with the values of $\lambda_k$ and $a_k$ given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 1.313$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.

B.17 Snap shots of the computed shapes of a viscoelastic drop (Case I) with the initial radius ($r_0$) of 0.60 mm and the numerical values of $\lambda_k$ given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$, with the values of $\lambda_k$ and $a_k$ given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 1.166$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.

B.18 Snap shots of the computed shapes of a viscoelastic drop (Case II) with the initial radius ($r_0$) of 0.60 mm and the numerical values of $\lambda_k$ given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$, with the values of $\lambda_k$ and $a_k$ given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 1.209$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.
B.19 Snap shots of the computed shapes of a viscoelastic drop (*Case III*) with the initial radius ($r_0$) of 0.60 mm and the numerical values of $\lambda_k$ given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at $\dot{\gamma}_{\text{app}} = 35.4$ s$^{-1}$, with the values of $\lambda_k$ and $a_k$ given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 1.258$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.

B.20 Snap shots of the computed shapes of a viscoelastic drop (*Case IV*) with the initial radius ($r_0$) of 0.60 mm and the numerical values of $\lambda_k$ given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at $\dot{\gamma}_{\text{app}} = 35.4$ s$^{-1}$, with the values of $\lambda_k$ and $a_k$ given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 1.309$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.

B.21 Snap shots of the computed shapes of a Newtonian drop (*Case VI*) with the initial radius ($r_0$) of 0.45 mm when the drop was suspended in the Newtonian medium, moving at $\dot{\gamma}_{\text{app}} = 15.4$ s$^{-1}$, with the viscosity of the drop $\eta_{0d}$ and the medium $\eta_{0m}$ given in Table 4.6. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 0.135$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.

B.22 Snap shots of the computed shapes of a Newtonian drop (*Case VII*) with the initial radius ($r_0$) of 0.45 mm when the drop was suspended in the Newtonian medium, moving at $\dot{\gamma}_{\text{app}} = 15.4$ s$^{-1}$, with the viscosity of the drop $\eta_{0d}$ and the medium $\eta_{0m}$ given in Table 4.6. The snap shots of the computed shape of the drop are at four different
dimensionless axial positions \(z/R_0\) along the centerline of the cylindrical tube in the entrance region: the position A at \(z/R_0 = -10\), the position B at \(z/R_0 = -5\), the position C at \(z/R_0 = 0.137\), and the position D at \(z/R_0 = 5\) with \(R_0\) being the radius (3 mm) of the cylindrical tube.

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CHAPTER I
INTRODUCTION

The capability of predicting the rheological behavior of immiscible polymer blends is very much in need in order to be able to help the processing of such polymer blends for the polymer fabrication industry. Here immiscible polymer blends refer to the situation where a very sharp interface exists between the two polymers. Unfortunately, at present there exists litter theoretical study reporting on the prediction of the rheological behavior of immiscible polymer blends. In view of the fact that the rheological behavior of a dispersed two-phase polymer blend is intimately related to its state of dispersion (i.e., its morphology), any attempts to predict the rheological behavior of a dispersed polymer blend must first be able to predict its morphology. In this regard, some attempts reported in the literature, which suggested empirical expressions for describing the rheology of immiscible polymer blends by disregarding its morphology, are of little rheological and practical significance. Thus, the first step for a better understanding of the rheological behavior of a dispersed immiscible polymer blend is to predict the shape of drops dispersed in the suspending medium. In this regard, the present research will pave a way upon which a better understanding of the rheological behavior of dispersed immiscible polymer blends can be achieved by conducting finite element computations for the deformation of drops in a dispersed two-phase polymer system.
The objectives of the proposed research were (1) to formulate finite element system equations, combined with the penalty function method, for predicting the drop deformation in the entrance region of a cylindrical tube, (2) to apply the Bowyer–Watson and Newton–Raphson methods to solve the finite element system equations using unstructured mesh as well as an auto-remeshing technique, and (3) to investigate the factors that affect the extent of drop deformation. In the present study, the following situations were considered: (i) the deformation of a viscoelastic drop suspended in a viscoelastic medium, and (ii) the deformation of a Newtonian drop suspended in a Newtonian medium. We considered the axisymmetric creeping motion of a single drop suspended in another liquid flowing through the entrance region of a cylindrical tube.

In the first part of this dissertation, we present the formulation of finite element system equations with the integral-type Kaye-Bernstein-Kearsley-Zapas (KBKZ) constitutive equation as a viscoelastic model. The penalty function method is employed to eliminate the pressure term appearing in the momentum equation, thus reducing the variables in the FEM system equations. The interface between the drop and suspending medium is tracked throughout the FEM computations, for which a computer code for auto-remeshing procedure has been developed, and unstructured meshes have been generated based on the Delaunay triangulation using the Bowyer–Watson algorithm. The method for calculating the strain tensor was also introduced, so that the stress tensor could be computed over the integration of time.

In the second part of this dissertation, we present the results of the computed shape of a viscoelastic drop suspend in a viscoelastic medium and a Newtonian drop suspended in a Newtonian medium while moving along the centerline of a cylindrical tube in the
entrance region, respectively. Five combinations of viscoelastic drop and viscoelastic suspending medium and also five combinations of Newtonian drop and Newtonian medium were investigated. The computed drop shapes showed that the drop elongates continuously as it moves towards the tube entrance and recoils slightly after it passes the tube inlet. For the viscoelastic system we found that the extent of drop deformation increased as the ratio of the relaxation times ($\lambda_k$) of the drop and the suspending medium was decreased, as the initial drop radius was increased, or as the apparent shear rate was increased. For the Newtonian system the extent of drop deformation increased as the viscosity ratio of the drop and the suspending medium was decreased, or as the initial drop radius was increased. We compared the differences in the extent of drop deformation between the viscoelastic and Newtonian systems and explained the differences in terms of the extensional stress distribution along the centerline of the cylindrical tube in the entrance region. We have correlated the deformation rate of a drop, the position where a maximum deformation occurs, and the rate of drop recoil with the distributions of the extensional stress along the centerline of a cylindrical tube in the entrance region.
A better understanding of the rheological behavior of immiscible polymer blends is of fundamental importance from both theoretical and practical points of view. However, the subject is very complicated in that immiscible polymer blends form a dispersed morphology in the equilibrium state, i.e., one of the constituent components forms the discrete phase (e.g., non-spherical drops) suspended in the other component forming the matrix phase. Therefore, the rheological behavior of a dispersed two-phase polymer blend is intimately related to its state of dispersion (i.e., its morphology). For this reason, any discussion of the rheological behavior of a dispersed immiscible polymer blend is of little physical significance unless the morphological state of the blend is considered. Since numerous, irregularly-shaped drops are dispersed in the matrix of an immiscible polymer blend, it is virtually impossible to describe precisely the state of dispersion of many drops in such a polymer blend. For this reason, during the past half century many research groups have investigated the deformation of a single drop suspended in the matrix subjected to hyperbolic and simple shear flow, extensional flow, steady-state shear flow, etc.

Taylor conducted a seminal experimental study on the deformation of a single drop in a hyperbolic plane and simple shear flows with the four-roll mill apparatus.
Subsequently, over the past five decades Taylor’s study has been extended by many research groups. An extensive experimental and theoretical investigation of the deformation of a single drop in non-uniform shear flow and extensional flow has been reviewed by Han and coworkers. There are numerous other experimental studies that report on the deformation of a single drop in the steady-state and/or extensional flow.

Equally important is the phenomenon of the breakup of drops in the flow of dispersed two-phase flow, which can be observed in emulsions as well. It is amply demonstrated in the literature that drop breakup is strongly associated with the capillary number, $\text{Ca} = \frac{r_0 \dot{\gamma} \eta_m}{\gamma}$ with $r_0$ being the radius of a drop, $\dot{\gamma}$ being the shear rate, $\eta_m$ being the viscosity of the suspending medium, and $\gamma$ being the interfacial tension. Such studies deal with a wide range of theoretical studies on the stability of a single drop in steady-state and time-dependent flow field, the dynamics of two-phase system in shear flow and in extensional flow, or other types of complex flow in mixing devices. Also, some experimental studies employ shear flow, channel flow, and microfluidic devices.

The modeling of the flow of two-phase polymeric fluids is a challenge in that it is associated with a non-linear moving-boundary problem with the boundary conditions prescribed on the surface that is yet to be determined as part of the solution of the system equations. Further, the complexity of the problem associated with the modeling of dispersed two-phase flow lies in that small deformations of a drop (say deviating only slightly from spherical shape) are of no practical importance although they have their
own merits. Thus, one must deal with large deformations of a drop whose shapes during flow are unknown a priori and thus they should be obtained as part of the solution of the system equations with appropriate boundary conditions. It can easily be surmised that under such circumstances analytical solutions would not be amenable, calling for a sophisticated numerical approach: finite element method (FEM).

However, an application of FEM to describe large deformations of a drop in a dispersed two-phase flow is not a trivial task, especially when dealing with the flow in a complicated flow geometry, such as flow in the entrance region of a cylindrical tube in which recoil of a deformed drop is expected to occur upon passing the tube inlet. The complexity of the problem will be compounded when dealing with viscoelastic, dispersed two-phase fluids, which are very often encountered when processing polymer blends of industrial importance.

In the past, some investigators$^{5,31,32,36,42,51–54}$ have reported on large deformations of a Newtonian drop suspended in another Newtonian medium flowing through a cylindrical tube, and only one study$^{55}$ has reported on large deformation of non-Newtonian drop suspended in another non-Newtonian medium in the entrance region of a cylindrical tube. However, to the best of our knowledge, very few studies have ever reported on the large deformations, via FEM, of a viscoelastic drop suspended in another viscoelastic medium flowing through the entrance region of a cylindrical tube. This observation has prompted us to undertake the dissertation presented herein. Specifically, we have developed FEM codes to solve the system equations based on the integral-type KBKZ constitutive equation and computed the shapes of large deformation of a single drop moving along the central axis of a cylindrical tube in the entrance region.
CHAPTER III

NUMERICAL METHOD FOR CALCULATING DROP DEFORMATION IN THE ENTRANCE REGION OF A CYLINDRICAL TUBE

In this chapter we describe the finite element formation for the simulation of large deformation of a drop suspended in another liquid. In doing so, the method of particle tracking is discussed and the time-dependent stress is calculated using an appropriate finite element algorithm. Also, the procedure for mesh generation method and refinement are explained. And the flow diagram for finite element analysis is illustrated.

Modeling and simulation of dispersed two-phase polymeric fluids is a challenging task in that the mathematical description is a nonlinear moving-boundary problem with boundary conditions prescribed on the surface that is determined only as part of the solution of system equations. Drop deformation is controlled by interface stress balance including the stress inside and outside a drop and the interfacial tension. In the past, the majority of the previous studies focused on steady-state homogeneous flow, while few investigations have been carried out on drop deformation in inhomogeneous flow, such as flow in channels or pipes with varying cross section. Chin and Han\textsuperscript{15,16} investigated the creeping motion of a drop in the conical section of a converging channel. They found that the viscoelastic drops were less deformable than the Newtonian drop and highly viscoelastic media gave rise to large deformation of drop. The results are consistent with
the behavior in homogeneous elongation. Later, computational results\textsuperscript{55} proved to be consistent with the experimental observations.

Bourry\textsuperscript{56} studied experimentally the drop deformation in slit flow and found that a viscoelastic drop initially deformed readily but then the deformation slowed down, whereas a Newtonian drop was initially slow to deform but then the deformation was rapid. Khayat\textsuperscript{57–63} carried out a series of numerical studies on the creep motion of a drop in different extensional flow channels, such as a hyperbolic convergent-divergent flow channel, and drew a conclusion consistent with Bourry’s experimental results. Their study was based on the linear Oldroyd-B fluid. All these investigations observed slight deformation of small-size drop in the relative large size channel and showed strong strain-rate dependent rheological properties of the system. However, it is still uncertain whether fluid elasticity or viscosity dominates the drop deformation. Mighri et al.\textsuperscript{28} conducted experiments on elastic fluids with a constant viscosity to investigate the elasticity effect on the drop deformation, and found that the drop deformation increased with increasing matrix elasticity. This observation seems to be consistent with the argument advanced in homogeneous elongational flows: polymer matrix suspending a drop hinders its deformation while the matrix enhances its deformation which, however, contradicts Khayat’s results.\textsuperscript{59,63} Using a nonlinear Giesekus model, Zhou et al.\textsuperscript{36} tried to clear up this issue by carrying out a numerical study on the drop deformation in the flow similar to the geometry employed by Chin and Han.\textsuperscript{15} They found that viscoelasticity in either component may promote or suppress drop deformation, depending on the capillary number $Ca$ and the drop-to-matrix viscosity ratio $\beta$. At present, the issue of large deformation of a drop as affected by the elasticity of a drop suspended in another
viscoelastic liquid remains unresolved. This observation has motivated us to investigate, via FEM, the extent of large deformation of a viscoelastic drop suspended in a viscoelastic medium in the entrance region of a cylindrical tube.

3.1 Formulation of Finite Element System Equations

Utilizing the penalty function method, the finite element system equation can be formulated for simulating the creep motion of a deformable drop in the entrance region of a cylindrical tube.

3.1.1 FEM Formulation for 2D Analysis

The governing equations for polymer flow can be written by

(i) continuity equation:

\[ \nabla \cdot \mathbf{v} = 0 \]  \hspace{1cm} (3.1)

(ii) momentum balance equation:

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \nabla \cdot \mathbf{\sigma} \]  \hspace{1cm} (3.2)

where \( \mathbf{v} \) is the velocity vector, \( \rho \) is the density, \( t \) is time, \( p \) is the hydrostatic pressure and \( \mathbf{\sigma} \) is the stress tensor.

Let us consider a two-dimensional (2D) flow depicted in Figure 3.1. In Figure 3.1, \( H \) is the height of the channel, \( W \) is the width of the channel, \( n_1 - n_4 \) show the normal direction of the boundaries \( \Gamma_1 - \Gamma_4 \).
If the geometry of flow region does not change rapidly, lubrication approximation can be used to analyze the flow in slowly varying narrow gaps.\textsuperscript{64} The following equations can be easily written.

\begin{equation}
\frac{\partial p}{\partial s} = \mu \frac{\partial^2 v_s}{\partial h^2} \tag{3.3}
\end{equation}

where \( s \) the velocity direction, \( h \) denotes the channel height direction and \( \mu \) is the viscosity. Integrating Eq. (3.3) we obtain

\begin{equation}
q_s = \frac{H^3}{12\mu} \frac{\partial p}{\partial s} \tag{3.4}
\end{equation}
or

\[ \frac{\partial p}{\partial s} = \frac{\pi_p \overline{\mu} q_s}{H^3} \]  

(3.5)

where \( q_s \) is the volumetric flow rate per unit width and, \( \pi_p \) is the dimensionless pressure gradient, which is a constant and only depends on the geometry of the cavity, and \( \overline{\mu} \) is the mean viscosity under the local average shear rate \( q_s/H^2 \).

For 2D flow, integrating the continuity Eq. (3.1) on the channel height we obtain

\[ \frac{\partial}{\partial x} \int_{-H/2}^{H/2} v_x \, dy + \frac{\partial}{\partial z} \int_{-H/2}^{H/2} v_z \, dy = 0 \]  

(3.6)

or

\[ \frac{\partial q_x}{\partial x} + \frac{\partial q_z}{\partial z} = 0 \]  

(3.7)

Obviously there is a stream function \( \psi (x, z) \) such that the following equations are satisfied:

\[ \frac{\partial \psi}{\partial z} = q_x \]  

(3.8)

\[ \frac{\partial \psi}{\partial x} = -q_z \]  

(3.9)

According to Eq. (3.4), \( q_x \) and \( q_z \) can be expressed as

\[ \frac{\partial p}{\partial x} = \frac{\pi_p \overline{\mu} q_x}{H^3} \]  

(3.10)

\[ \frac{\partial p}{\partial z} = \frac{\pi_p \overline{\mu} q_z}{H^3} \]  

(3.11)

By substituting Eqs. (3.10) and (3.11) into the flowing equation
\[
\frac{\partial}{\partial x} \left( \frac{\partial p}{\partial z} \right) - \frac{\partial}{\partial z} \left( \frac{\partial p}{\partial x} \right) = 0
\]  
(3.12)

we obtain

\[
\frac{\partial}{\partial x} \left( \frac{\pi \mu \bar{H}}{H^3} \frac{\partial \psi}{\partial z} \right) + \frac{\partial}{\partial z} \left( \frac{\pi \mu \bar{H}}{H^3} \frac{\partial \psi}{\partial x} \right) = 0
\]  
(3.13)

The relation between the stream function \( \psi(x, z) \) and the volumetric flow rate per unit width \( q_s \) is expanded by:

\[
q_s = q_x^2 + q_z^2 = \left( \frac{\partial \psi}{\partial z} \right)^2 + \left( \frac{\partial \psi}{\partial x} \right)^2
\]  
(3.14)

The boundary conditions for Eq. (3.13) are

\[
\frac{\partial \psi}{\partial x} = 0 \quad \text{on} \quad \Gamma_1
\]  
(3.15)

\[
\psi = 0 \quad \text{on} \quad \Gamma_2
\]  
(3.16)

\[
\frac{\partial \psi}{\partial x} = 0 \quad \text{on} \quad \Gamma_3
\]  
(3.17)

\[
\psi = -\frac{Q}{2} \quad \text{on} \quad \Gamma_4
\]  
(3.18)

Where \( \Gamma_1 - \Gamma_4 \) are the boundaries shown in Figure 3.1.

According to the Weighted Residual Method, the stream function \( \psi(x, z) \) in Eq. (3.13) can be regarded as a variable, and thus the Galerkin finite element equation can be obtained from Eq. (3.13) as follows (see Appendix A):

\[
\iint_{\Gamma} \mu H^3 W_n \frac{\partial \psi}{\partial n} \, d\Gamma - \iint_{\Omega} \mu H^3 \left( \frac{\partial \psi}{\partial x} \frac{\partial W_n}{\partial x} + \frac{\partial \psi}{\partial z} \frac{\partial W_n}{\partial z} \right) \, d\Omega = 0
\]  
(3.19)
where $\Gamma$ is the boundary contour of the flow region, $\Omega$ is the flow region, and $W_n$ is the weight functions. The flow domain can be divided by six nodes of triangular elements in the x-z coordinate. For each element $e$, the stream function $\psi$ is assumed to be

$$\psi (x, z) = \sum_{i=1}^{6} Q_i(x, z) \Psi_i$$  \hspace{1cm} (3.20)

where $\Psi_i$ represents the flow rate at the $i$th node, and $Q_i$ is the quadratic shape function. In the context of the Galerkin finite element method, we regard the weight function $W_n$ to be the quadratic shape function $Q_i$, i.e., $W_n = Q_i$. Substitution of Eq. (3.20) into Eq. (3.19) gives the finite element equation at local specific element $e$ as:

$$\int_{\Omega^e} \frac{\mu}{H^3} \left[ \frac{\partial Q^e_i}{\partial x} \frac{\partial Q^e_j}{\partial x} + \frac{\partial Q^e_i}{\partial z} \frac{\partial Q^e_j}{\partial z} \right] \psi_j^e \, dx \, dz = 0$$  \hspace{1cm} (3.21)

Applying the boundary conditions given by Eqs. (3.15) – (3.18) to Eq. (3.21) in the flow domain, we can determine the flow rate.

In the method described above, we only integrate the continuity equation, Eq. (3.1). Integrating Eqs. (3.1) and (3.2), from Eq. (3.7) we obtain the following expression

$$\frac{\partial \bar{p}}{\partial x} = \frac{\partial^2 q_x}{\partial x^2} + \frac{\partial^2 q_x}{\partial z^2}$$  \hspace{1cm} (3.22)

$$\frac{\partial \bar{p}}{\partial z} = \frac{\partial^2 q_z}{\partial x^2} + \frac{\partial^2 q_z}{\partial z^2}$$  \hspace{1cm} (3.23)

where $\bar{p}$ is the total pressure over the entire channel height per unit width defined by:

$$\bar{p} = \int_{-h/2}^{h/2} p \, dy$$  \hspace{1cm} (3.24)
The boundary conditions for the above equation are given by:

\[
\frac{\partial q_x}{\partial n} = \frac{\partial q_z}{\partial n} \quad \text{on} \quad \Gamma_1, \Gamma_2 \quad \text{and} \quad \Gamma_4 \quad \text{(3.25)}
\]

\[ q_x = q_z = 0 \quad \text{on} \quad \Gamma_2 \quad \text{(3.26)}
\]

\[ q_x = 0 \quad \text{on} \quad \Gamma_4 \quad \text{(3.27)}
\]

where \( \Gamma_1 - \Gamma_4 \) are the boundaries shown in Figure 3.1.

Using the Galerkin method, \( p, q_x \) and \( q_z \) can be regarded as variables and the finite element equations can be written as:

\[
\int_{\Omega^e} H_j \left( \frac{\partial q_x}{\partial x} + \frac{\partial q_z}{\partial z} \right) \, dx \, dz = 0 \quad \text{(3.28)}
\]

\[
\int_{\Omega^e} W_i \left[ \mu \left( \frac{\partial^2 q_x}{\partial x^2} + \frac{\partial^2 q_z}{\partial z^2} \right) - \frac{\partial p^e}{\partial x} \right] \, dx \, dz = 0 \quad \text{(3.29)}
\]

\[
\int_{\Omega^e} W_i \left[ \mu \left( \frac{\partial^2 q_z}{\partial z^2} + \frac{\partial^2 q_x}{\partial z^2} \right) - \frac{\partial p^e}{\partial z} \right] \, dx \, dz = 0 \quad \text{(3.30)}
\]

where \( H_j \) and \( W_i \) are the weight functions. Generally, they are determined by

\[ H_j = N_j^p \quad \text{(3.31)} \]

\[ W_j = N_i^q \quad \text{(3.32)} \]

in which \( N_j^p \) and \( N_i^q \) are the shape functions for \( p \) and \( q \), respectively. When the six-node isoparametric triangular elements are employed, the quadratic interpolation for \( p \) and \( q \) can be expressed as
The global finite element equation can be solved by an iterative method. Based on
the result of the kth iteration, the velocity and viscosity can be updated, and the
coefficients in the matrix equation can be obtained so that we can obtain the (h+1)th
solution for p and q.65,66

### 3.1.2 Mixed Finite Element Analysis versus Penalty Function Method

The finite element system equation can be formulated by various methods. The
mixed finite element analysis and the penalty function method are widely used in
formulating the finite element system equations.

#### 3.1.2.1 Mixed Finite Element Analysis

The above method can be used to solve a 2D flow problem. However, in most cases
3D analysis is necessary in flow through a channel having a complex geometry. When
the flow is in steady state and the body force can be neglected, the governing equations
can be written as:

\[
\nabla \cdot \mathbf{V} = 0 \tag{3.35}
\]

\[
\nabla \mathbf{p} + \nabla \cdot \mathbf{\sigma} = 0 \tag{3.36}
\]

where \( j = 1, 2, 3 \) indicates the three axes in the orthogonal coordinate system.
When the entire flow domain is divided into a series of elements, we can choose several nodes as integration points for the velocity and pressure, i.e., \( r \) nodes for velocity and \( s \) nodes for pressure. Because the derivatives of velocity are always larger than those of pressure in Eqs. (3.35) and (3.36), we choose \( r > s \), so that similar precision of velocity and pressure can be achieved.

For a given element \( e \), we have

\[
V^e_{x_j} = \sum_{i=1}^{r} N^v_i(x, y, z) \left(V_{x_j}\right)_i = \left[N^v\right] \{V_{x_j}\} \quad (j = 1, 2, 3) \tag{3.37}
\]

\[
p^e = \sum_{k=1}^{s} N^p_k(x, y, z) p_k = \left[N^p\right] \{p\} \tag{3.38}
\]

where the \( N^v_i \) and \( N^p_k \) are the quadratic interpolation function of velocity and pressure on the \( i \)th and \( k \)th nodes respectively, so called shape function. \( (V_{x_j})_i \) is the value of \( V_{x_j} \) at the \( i \)th node and \( p_k \) is the value of pressure at the \( k \)th node.

Selecting \( N^v_i \) and \( N^p_k \) as the weight functions in weak form of Eqs. (3.35) and (3.36), finite element formulation based on the Galerkin method can be expressed as:

\[
\int_{\Omega^e} N^p_k \nabla \cdot V_{x_j} \, d\Omega = 0 \quad (j = 1, 2, 3; \ k = 1, 2, \ldots, s) \tag{3.39}
\]

\[
\int_{\Omega^e} \eta \nabla N^v_i \cdot \nabla \left[N^v\right] \{V_{x_j}\} - \int_{\Omega^e} \frac{\partial N^v_i}{\partial x_j} \left[N^p\right] \, d\Omega \{p\} = -\int_{S^e} N^v_i X^*_j \, d\Gamma 
\tag{3.40}
\quad (j = 1, 2, 3; \ i = 1, 2, 3, \ldots, r)
\]

where the \( \Omega^e \) and \( S^e \) are the volume and boundary, respectively, of the element \( e \). \( X^*_j \) is the component of force on the \( j \)th \((x, y, z)\) direction along the boundary.
where $\mathbf{n}$ is the normal vector on the boundary and $n_x$, $n_y$ and $n_z$ are its components in the orthogonal coordinate $(x, y, z)$.

Combing Eq. (3.39) and Eq. (3.40), we obtain the local finite element equations.

$$
\begin{bmatrix}
K^e
\end{bmatrix}_{(3r+s)\times(3r+s)} \begin{bmatrix}
\phi^e
\end{bmatrix}_{(3r+s)\times1} = \begin{bmatrix}
F^e
\end{bmatrix}_{(3r+s)\times1} \quad (4.44)
$$

where

$$
\begin{bmatrix}
\phi^e
\end{bmatrix} = \left(\begin{array}{c}
V_{x1}, V_{x2}, \cdots, V_{xr}, V_{y1}, V_{y2}, \cdots, V_{yr}, V_{z1}, V_{z2}, \cdots, V_{zr}; -p_1, -p_2, \cdots, -p_s
\end{array}\right)^T \quad (4.45)
$$

$$
\begin{bmatrix}
F^e
\end{bmatrix} = (F_1, F_2, \cdots, F_{3r+s})^T \quad (4.46)
$$

$$
F_i = \begin{cases}
\int_{\Gamma^e} N_i^Y X^* \, d\Gamma & \text{i = 1, 2, \cdots, r} \\
\int_{\Gamma^e} N_i^Y Y^* \, d\Gamma & \text{i = r + 1, r + 2, \cdots, 2r} \\
\int_{\Gamma^e} N_i^Y Z^* \, d\Gamma & \text{i = 2r + 1, 2r + 2, \cdots, 3r} \\
0 & \text{i = 3r + 1, 3r + 2, \cdots, 3r + s}
\end{cases} \quad (4.47)
$$

$$
\begin{bmatrix}
K^e
\end{bmatrix} = \begin{bmatrix}
[K_1]_{r\times r} & [0]_{r\times r} & [0]_{r\times r} & [K_2]_{r\times r} \\
[0]_{r\times r} & [K_1]_{r\times r} & [0]_{r\times r} & [K_3]_{r\times r} \\
[0]_{r\times r} & [0]_{r\times r} & [K_1]_{r\times r} & [K_4]_{r\times r} \\
[K_2]^T_{r\times r} & [K_3]^T_{r\times r} & [K_4]^T_{r\times r} & [0]_{r\times r}
\end{bmatrix}_{(3r+s)\times(3r+s)} \quad (4.48)
$$
\[
\left( K_1 \right)_{ij} = \int_{\Omega} \eta \nabla N_i^\gamma \cdot \nabla N_j^\gamma \, d\Omega = 0 \quad (i = 1, 2, \ldots, r; \ j = 1, 2, \ldots, r) \quad (3.49)
\]

\[
\left( K_{m+1} \right)_{ij} = \int_{\Omega} \frac{\partial N_i^\gamma}{\partial x_m} N_j^\beta \, d\Omega = 0 \quad (m = 1, 2, 3; \ i = 1, 2, \ldots, r; \ j = 1, 2, \ldots, s) \quad (3.50)
\]

Combining the local finite element equations on each element, the global finite element equation can be obtained. Obviously, after we solve the global finite element equation, we can obtain the values of pressure and velocity simultaneously. However, due to the very large number of unknown variables in the matrix, the size of global matrix becomes large, requiring large memories for storage and computation, even though the 0 elements in the diagonal of the global matrix decrease the efficiency of programming. And it is difficult to have fine mesh size to obtain better computing precision. Both the accuracy and computing time need to be taken into account in FEM formulation and programming. If we decrease the number of variables in system equations, we may save CPU time considerably. If we refine the mesh size, we will have better accuracy, but the refined mesh size is more desirable than the choice of numerical method.

3.1.2.2 Penalty Function Method

The penalty function method is one way of decreasing the number of variables in the system equation. It is an approximation technique of including constraints into the variational formulation, which transforms a given constrained variational problem into an unconstrained variational problem by the introduction of a penalty on the infringement of
constraints. Equation (3.35) can be considered as a constraint to Eq. (3.36). By introducing a small variation of pressure into Eq. (3.35), we obtain

\[ e_p p + \nabla \cdot \mathbf{V}_{x_j} = 0; \quad j = 1, 2, 3 \]  

(3.51)

or

\[ p = -\frac{1}{e_p} \nabla \cdot \mathbf{V}_{x_j} = -\alpha \nabla \cdot \mathbf{V}_{x_j} \quad j = 1, 2, 3 \]  

(3.52)

where \( e_p \) is a very small number, \( \alpha \) is a penalty constant, which is a very large number that can be chosen depending on the viscosity and Reynolds number.

Substituting Eq. (3.52) into Eq. (3.38), we obtain

\[ \left[ N^p \right] \{ p \} = -\alpha \left( \frac{\partial N^v_i}{\partial x} \{ V_x \} + \frac{\partial N^v_i}{\partial y} \{ V_y \} + \frac{\partial N^v_i}{\partial z} \{ V_z \} \right) \]  

(3.53)

Introducing Eq. (3.53) into Eq. (3.40), we obtain the x-component of the momentum equation as

\[ \int_{\Omega^e} \left( \eta \nabla N^v_i \cdot \nabla \left[ N^v_j \right] + \alpha \frac{\partial N^v_i}{\partial x} \frac{\partial \left[ N^v_j \right]}{\partial x} \right) \ d\Omega \{ V_x \} + \int_{\Omega^e} \alpha \frac{\partial N^v_i}{\partial x} \frac{\partial \left[ N^v_j \right]}{\partial x} \ d\Omega \{ V_y \} \]  

(3.54)

\[ + \int_{\Omega^e} \alpha \frac{\partial N^v_i}{\partial x} \frac{\partial \left[ N^v_j \right]}{\partial x} \ d\Omega \{ V_z \} = -\int_{\Gamma^e} N^v_i X^i \ d\Gamma \]

where

\[ \nabla N^v_i \cdot \nabla \left[ N^v_j \right] = \frac{\partial N^v_i}{\partial x} \frac{\partial \left[ N^v_j \right]}{\partial x} + \frac{\partial N^v_i}{\partial y} \frac{\partial \left[ N^v_j \right]}{\partial y} + \frac{\partial N^v_i}{\partial z} \frac{\partial \left[ N^v_j \right]}{\partial z} \]  

(3.55)
and \( i = 1, 2, \ldots, r, \ j = 1, 2, \ldots, r \). We can also obtain the y-component and z-component
by iteratively substituting \( x \rightarrow y \rightarrow z \rightarrow x \). The finite element equation is in the form of
Eq. (3.44), with only selecting the 1st to \((3r)\)th component of matrix \( \{F^e\} \) and \( \{\varphi^e\} \).

\[
\begin{bmatrix}
K^e & \{\varphi^e\}
\end{bmatrix}_{3r \times 3r} = \begin{bmatrix}
F^e
\end{bmatrix}_{3r \times 1}
\tag{3.56}
\]

where

\[
\{\varphi^e\} = (V_{x1}, V_{x2}, \ldots, V_{xr}, V_{y1}, V_{y2}, \ldots, V_{yr}, V_{z1}, V_{z2}, \ldots, V_{zr})^T
\tag{3.57}
\]

\[
\{F^e\} = (F_1, F_2, \ldots, F_{3r})^T
\tag{3.58}
\]

\[
F_i = \begin{cases}
\int_{\Gamma^e} N_i^Y X^* d\Gamma & i = 1, 2, \ldots, r \\
\int_{\Gamma^e} N_i^Y Y^* d\Gamma & i = r+1, r+2, \ldots, 2r \\
\int_{\Gamma^e} N_i^Y Z^* d\Gamma & i = 2r+1, 2r+2, \ldots, 3r
\end{cases}
\tag{3.59}
\]

\[
K^e = \begin{bmatrix}
K_{11} & K_{12} & K_{13} \\
K_{12}^T & K_{22} & K_{23} \\
K_{13}^T & K_{23}^T & K_{33}
\end{bmatrix}_{3r \times 3r}
\tag{3.60}
\]

\[
(K_{km})_{ij} = \int_{\Omega^e} \left( \delta_{km} \eta \nabla N_i^Y \cdot [\nabla N_j^Y] + \alpha \frac{\partial N_i^Y}{\partial x_k} \frac{\partial [\nabla N_j^Y]}{\partial x_m} \right) d\Omega
\tag{3.61}
\]

where
Combining Eq. (3.56) with the boundary condition, the global finite element matrix can be solved. Then, the velocity field will be determined and the pressure can be calculated using Eq. (3.53).

It should be mentioned that the choice of a penalty constant $\alpha$ will directly affect the result. If $\alpha$ is too small, then it is not easy to satisfy the zero-divergence condition from Eq. (3.51), even more a faulty solution of pressure may be obtained (e.g., $p$ tends to be 0 in the flow domain). If $\alpha$ is too large, then the elements that contain $\alpha$ in the global stiffness matrix can be much larger than the others, thus the round-off error (depending on the precision of the computer) will result in the faulty solution of velocity; e.g., $v$ tends to be 0 in the flow domain even though the zero-divergence condition is satisfied automatically. Liu\textsuperscript{72} used the penalty method to study the effect of $\alpha$ on the finite element solution in studying the flow between two parallel plates. When the penalty constant $\alpha$ is fixed, refined mesh will result in a better numerical solution. In the case of a fixed mesh, when increasing $\alpha$ in sequence (from 10 to $10^{17}$), the numerical solution will become close to the real analytical solution, and then will deviate from it. This implies that the penalty constant $\alpha$ should be chosen within a certain range.

Comparing the mixed finite element method with the penalty method, the following conclusion can be drawn. The mixed finite element method can solve for the velocity and pressure simultaneously, so the nodes of the element contain both velocity and pressure components. There are $(3r + s)$ unknown variables in each element. On the other hand, there are only $3r$ unknown variables in each element when the penalty
method is employed. In this case, the computational speed will be improved considerably and meshes can be refined during computation resulting in improvement in precision. For the mixed finite element method, the degree of freedom on each node of the element is not the same; some nodes containing velocity or pressure while the others contain both of them. When integrating the element stiffness matrix into the global stiffness matrix, it may cause a problem while compiling the program code. Using the penalty method, there is only the velocity quadratic integration on each node. Pressure can only be calculated using Eq. (3.53). Comparing Eq. (3.45) with Eq. (3.57), we assume that there are 0 diagonal elements of the global matrix in Eq. (3.45), while no 0 diagonal elements of the global matrix in Eq. (3.57). This affects the solution of the velocity matrix. When the mixed finite element is employed, the solution of pressure is sensitive to the initial guess and the density of the meshes. According to the literature, the convergence of iteration may not be achieved.

3.1.3 Drop Deformation in the Entrance Region of a Cylindrical Tube

In the previous section we have shown how to formulate 3D finite element equations based on the Galerkin weak form. The penalty function method was also introduced to decrease the number of variables. Here we will formulate the finite element system equation for large deformation in the axisymmetric creep motion of a drop in the entrance region of a cylindrical tube using a truncated power law model and an integral-type viscoelastic mode, KBKZ model, respectively.

Consider the flow geometry depicted in Figure 3.2.
In Figure 3.2, $\eta_m$ and $\eta_d$ are the viscosity of matrix and drop, respectively; $\rho_m$ and $\rho_d$ are the density of matrix and drop, respectively; $r_0$ is the radius of drop and $R_0$ is the radius of the tube.

In the present investigation, we consider the axisymmetric creeping motion of a single drop suspended in another liquid flowing through the entrance region of a cylindrical tube with the radius $R_0$ (see Figure 3.2). Interface force balance is subjected to normal and tangential boundary conditions. For isothermal flow, according to the normal stress balance condition, the difference between the stress tensors on the normal direction is equal to the surface tension on the curvature $H$ ($H = 1/R$):

$$
(T_m - T_d) \cdot n = \frac{\gamma}{R} n
$$

(3.63)
where $\mathbf{T}$ is the total stress tensor, subscript $m$ denotes the medium, subscript $d$ denotes the drop, $\mathbf{n}$ is the local unit outward vector normal to the interface, $R$ is the local radius and, $\gamma$ is the interfacial tension.

The shear stress balance on the interface can be written as

$$\left(\mathbf{T}_m - \mathbf{T}_d\right) : \mathbf{nt} = 0$$  \hspace{1cm} (3.64)

where the $\mathbf{t}$ is the local unit tangential vector to the interface;

Bozzi et al.\textsuperscript{73} presented a method to calculate the local mean curvature of the interface between two phases. Following Bozzi et al., the normal force balance can be expressed as

$$\frac{1}{\eta_0 v_0} \left(\mathbf{T}_m - \mathbf{T}_d\right) \cdot \mathbf{n} = \frac{1}{Ca} \left(\frac{dt}{ds} + \frac{n dz}{r ds}\right) - n St \left(\frac{R_0}{r_0}\right)^2 z$$  \hspace{1cm} (3.65)

where $\eta_0$ and $v_0$ are the reference viscosity and characteristic velocity, respectively, $s$ is the length along the interface measured from the forward stagnation point, and $r$ is the local radius, $R_0$ is the radius of capillary, $r_0$ is the initial radius of the drop, respectively, and $St$ is the dimensionless buoyancy force which is defined by

$$St = \frac{(\rho_d - \rho_m)gr_0}{\eta_0 v_0}$$  \hspace{1cm} (3.66)

and $Ca$ is the capillary number defined by

$$Ca = \frac{\eta_0 v_0}{\gamma}$$  \hspace{1cm} (3.67)

The second term on the right-hand side of Eq. (3.65) can be neglected for practical purpose,\textsuperscript{73} and the term \( \left(\frac{dt}{ds} + \frac{n dz}{r ds}\right) \) in Eq. (3.65) can be calculated from
\[
\frac{1}{R} = \frac{dt}{ds} n_i + \frac{1}{r \, ds}
\]

where \( \bar{R} \) is the mean local radius. Therefore Eq. (3.65) can be rewritten as

\[
(T_m - T_d) \cdot n \approx \frac{\eta_0 v_0}{\bar{R}} \frac{1}{Ca} n
\]

or

\[
n_i n_j (T_{m,ij} - T_{d,ij}) = \frac{\gamma}{R}
\]

Equation (3.64) can also be rewritten as:

\[
t_i n_j (T_{m,ij} - T_{d,ij}) = 0
\]

in which \( T_{m,ij} \) and \( T_{d,ij} \) are the total stress for the medium and the drop, respectively.

3.1.3.1 Drop Deformation of Power-Law Fluid

Below we formulate finite element system equations for the truncated power-law model. The truncated power-law model can be expressed as

\[
\eta_i(\dot{\gamma}) = \begin{cases} 
\eta_{0,i} & \text{for } \dot{\gamma} < \dot{\gamma}_{c,i} \\
K_i \dot{\gamma}^{n_i - 1} & \text{for } \dot{\gamma} > \dot{\gamma}_{c,i}
\end{cases}
\]

for \( i = m \) and \( d \), respectively, where \( \eta_i(\dot{\gamma}) \) is the shear-rate dependent viscosity, \( \eta_{0,i} \) is the zero-shear (Newtonian) viscosity, \( K_i \) is the power-law consistency, \( n_i \) is the power-law index, \( \dot{\gamma}_{c,i} \) is the critical shear rate at which the viscosity begins to deviate from the Newtonian behavior, \( \dot{\gamma} = \sqrt{\sum d_{ij}^2} \), and \( d_{ij} \) is the \( ij \)th component of the rate-of-deformation tensor \( d \).
For the dispersed two-phase polymer blends, the continuity and momentum equations are given by:

\[ \nabla \cdot \mathbf{v}_m = 0 \quad (3.73) \]
\[ \nabla \cdot \mathbf{v}_d = 0 \quad (3.74) \]
\[ \rho_m \left( \frac{\partial \mathbf{v}_m}{\partial t} + \mathbf{v}_m \cdot \nabla \mathbf{v}_m \right) = -\nabla p_m + \nabla \cdot \mathbf{\sigma}_m \quad (3.75) \]
\[ \rho_d \left( \frac{\partial \mathbf{v}_d}{\partial t} + \mathbf{v}_d \cdot \nabla \mathbf{v}_d \right) = -\nabla p_d + \nabla \cdot \mathbf{\sigma}_d \quad (3.76) \]

where \( \mathbf{\sigma} \) is the deviatoric stress tensor which is defined by:

\[ \mathbf{T} = -p \mathbf{I} + \mathbf{\sigma} \quad (3.77) \]

where \( \mathbf{I} \) is the identity tensor.

There are two different ways of formulating the finite element equations; namely, one method is to treat both the velocity and pressure as unknown variables, which is known as the primitive variable formulation, and another method is to remove the incompressibility constraint using the penalty function method. As shown above, the interface balances shown by Eqs. (3.63) and (3.64) can be dealt more easily when the penalty function method is employed. Below the penalty function method will be used in the formulation of FEM system equations.

Using the procedures described above, we obtain the following equation

\[ p = -\frac{1}{\epsilon_p} \nabla \cdot \mathbf{v} = -\alpha \nabla \cdot \mathbf{v} \quad (3.78) \]

For the non-Newtonian creeping flow, the penalty constant \( \alpha \) can take the following form\(^{67,71} \).
\[ \alpha = \frac{\beta \bar{n}(\dot{\gamma})}{\eta_0} \]  
(3.79)

where the \( \beta \) is a large dimensionless constant, and \( \eta_0 \) is the reference viscosity.

The augment function approach is based on variational principles.\(^74,75\) It requires that among the admissible velocities \( \mathbf{v} \) that meet the condition of compatibility and incompressibility, as well as the velocity boundary conditions, the actual solution gives the following augment function:

\[ J = \int_{\Omega} \rho \mathbf{b} \cdot \mathbf{v} \, d\Omega + \int_{\Omega} \mathbf{T} : \mathbf{d} \, d\Omega - \int_{\Gamma} \mathbf{f} \cdot \mathbf{v} \, d\Gamma \]  
(3.80)

where \( \Gamma \) is the boundary contour of the flow region, \( \Omega \) is the flow region, \( \mathbf{f} \) is the boundary force vector, and \( \mathbf{b} \) is the pseudo body force. In this study, the unsteady-state term \( \frac{\partial \mathbf{v}}{\partial t} \) and the convective term \( \mathbf{v} \cdot \nabla \mathbf{v} \) appearing in Eq. (3.75) and (3.76) can be considered as additional body force.\(^76\) \( \mathbf{v} \cdot \nabla \mathbf{v} \) can be calculated as \( \mathbf{v}^{\text{old}} \cdot \nabla \mathbf{v}^{\text{old}} \), hence \( \mathbf{b} \) in Eq. (3.80) can be written as

\[ \mathbf{b} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}^{\text{old}} \cdot \nabla \mathbf{v}^{\text{old}} \]  
(3.81)

Obviously, the augment function works in the entire flow region, including the continuous phase and dispersed phase. The solution of the original boundary value problem is then obtained from the solution of variational form of the augment problem.

\[ \delta J = \int_{\Omega} \rho \delta \mathbf{b} \cdot \delta \mathbf{v} \, d\Omega + \int_{\Omega} \mathbf{T} : \delta \mathbf{d} \, d\Omega - \int_{\Gamma} \mathbf{f} \cdot \delta \mathbf{v} \, d\Gamma = 0 \]  
(3.82)

Note that Eqs. (3.73) and (3.74) can be considered as an incompressibility constraint for Eqs. (3.75) and (3.76), so that the pressure \( p \) can be removed with the aid of Eq. (3.78).
Now, the variational form of Eq. (3.80) is given by

\[
0 = \delta J = \int_{\Omega} \rho \left( \frac{\partial v_i}{\partial t} \delta v_i + (v \cdot \nabla v) \cdot \delta v \right) \, d\Omega + \int_{\Omega} \sigma : \nabla (\delta v) \, d\Omega
+ \int_{\Omega} \alpha (\nabla \cdot v) (\nabla \cdot (\delta v)) \, d\Omega - \int_{\Gamma} f \cdot \delta v \, d\Gamma
\]  

(3.83)

or

\[
0 = \int_{\Omega} \rho \left( \frac{\Delta v_i}{\Delta t} + v_j^{\text{old}} \frac{\partial v_i^{\text{old}}}{\partial x_j} \right) \delta v_i \, d\Omega + \int_{\Omega} \alpha d_{ii} \delta d_{ij} \, d\Omega
+ \int_{\Omega} \sigma_{ij} \delta d_{ij} \, d\Omega - \int_{\Gamma} f_i \delta v_i \, d\Gamma
\]  

(3.84)

where the last term on the right-hand side of Eq. (3.84) corresponds to the virtual work due to the prescribed surface traction. It should be noted that Eq. (3.84) is valid for the entire flow domain. Along the interface of the two phases, neither the velocity nor the pressure is known a priori. The surface traction relates to the total stress tensor as:

\[
f_i = T_{ij} n_j
\]  

(3.85)

Using Eq. (3.85), the boundary term in Eq. (3.84) can be split into two regions giving rise to:

\[
0 = \int_{\Omega} \rho \frac{\partial v_i}{\partial t} \delta v_i \, d\Omega + \int_{\Omega} \rho v_j^{\text{old}} \frac{\partial v_i^{\text{old}}}{\partial x_j} \delta v_i \, d\Omega + \int_{\Omega} \alpha d_{ii} \delta d_{ij} \, d\Omega + \int_{\Omega} \sigma_{ij} \delta d_{ij} \, d\Omega
- \int_{\Gamma_1} T_{m,ij} n_j \delta v_i \, d\Gamma - \int_{\Gamma_2} T_{m,ij} n_j \delta v_i \, d\Gamma - \int_{\Gamma_2} T_{d,ij} n_j \delta v_i \, d\Gamma
\]  

(3.86)

where the \( \Gamma_1 \) is the outside boundary of continuous phase flow and the \( \Gamma_2 \) is the interface between the drop and matrix. It is convenient to introduce a new orthogonal coordinate \((t, n)\) instead of cylindrical coordinate \((r, z)\), in treating the stress balance given by Eqs. (3.70) and (3.71). The last two term in Eq. (3.86) can be written as \( T_{\Gamma_2} \).
\[ T_{r2} = -\int_{\Gamma_2} T_{m,ij} n_j \delta v_i \text{ d}\Gamma - \int_{\Gamma_2} T_{d,ij} n_j \delta v_i \text{ d}\Gamma \quad (3.87) \]

Using \( \delta v_i = t_i \delta v_t + n_i \delta v_n \) in Eq. (87), we have

\[
T_{r2} = -\int_{\Gamma_2} T_{m,ij} n_j \delta v_i \text{ d}\Gamma - \int_{\Gamma_2} T_{d,ij} n_j \delta v_i \text{ d}\Gamma \\
= -\int_{\Gamma_2} (T_{m,ij} - T_{d,ij}) n_j \delta v_i \text{ d}\Gamma \\
= -\int_{\Gamma_2} (T_{m,ij} - T_{d,ij}) n_j (t_i \delta v_t + n_i \delta v_n) \text{ d}\Gamma \\
= -\int_{\Gamma_2} (T_{m,ij} - T_{d,ij}) n_j t_i \delta v_t \text{ d}\Gamma - \int_{\Gamma_2} (T_{m,ij} - T_{d,ij}) n_j n_i \delta v_n \text{ d}\Gamma 
\quad (3.88)

According to Eqs. (3.71), the first term on the right-hand side of Eq. (3.88) is equal to zero. Using Eq. (3.70), Eq. (3.88) can be rewritten as:

\[ T_{r2} = -\int_{\Gamma_2} \gamma \frac{1}{R} \delta v_n \text{ d}\Gamma \quad (3.89) \]

Hence Eq. (3.84) can be rewritten as:

\[
\int_{\Omega} \rho \frac{\partial \delta v_i}{\partial t} \delta v_i \text{ d}\Omega + \int_{\Omega} \rho v_j^{old} \frac{\partial \delta v_i^{old}}{\partial x_j} \delta v_i \text{ d}\Omega + \int_{\Omega} \alpha_{ij} \delta d_{ij} \text{ d}\Omega + \int_{\Omega} \sigma_{ij} \delta d_{ij} \text{ d}\Omega \\
= \int_{\Gamma_1} T_{m,ij} n_j \delta v_n \text{ d}\Gamma + \int_{\Gamma_2} \gamma \frac{1}{R} \delta v_n \text{ d}\Gamma 
\quad (3.90)
\]

Rewriting Eq. (3.90) in term of r and z component, the left-hand side of the r-component of Eq. (3.90) is given by
and the left-hand side of z-component of Eq. (3.90) is given by

\[
\int_\Omega \left( \rho \frac{\partial V_z}{\partial t} \frac{\partial V_z}{\partial z} + \rho V_r \frac{\partial V_z}{\partial r} \frac{\partial V_z}{\partial r} + \rho V_z \frac{\partial V_z}{\partial z} \frac{\partial V_z}{\partial z} \right) \, d\Omega \tag{3.92}
\]

Six-node isoparametric triangular elements are employed to obtain finite element matrix equations from Eq. (3.90). Quadratic interpolation for the velocity can be expressed as:

\[
V_r = N(\zeta, \eta)_k V_{rk} \quad k = 1, 2, 3, 4, 5, 6
\]

\[
V_z = N(\zeta, \eta)_k V_{zk} \quad k = 1, 2, 3, 4, 5, 6
\]

and the velocity gradient is:

\[
\frac{\partial V_r}{\partial x_i} = \frac{\partial N(\zeta, \eta)_k}{\partial x_i} V_{rk} \quad x_i = r, z \quad k = 1, 2, 3, 4, 5, 6
\]

\[
\frac{\partial V_z}{\partial x_i} = \frac{\partial N(\zeta, \eta)_k}{\partial x_i} V_{zk} \quad x_i = r, z \quad k = 1, 2, 3, 4, 5, 6
\]

where \(N(\zeta, \eta)\) is the shape function for the velocity in the natural coordinate \((\zeta, \eta)\).
It is convenient to employ the local coordinate system along the interface of the two fluids. The velocity components in the local coordinate system \((t, n)\), defined along the interface, can be transformed in the global coordinate \((r, z)\), using a transformation matrix \([Q]\) as:

\[
\{V\} = [Q] \{\overline{V}\}
\]  

(3.97)

where \(\{V\}\) is the vector of the variables in the global coordinate, \(\{\overline{V}\}\) is the vector of the variables in the local coordinate system, and \([Q]\) is a coordinate transformation matrix.

Using Eqs. (3.93) – (3.97), we can rewrite each term in expression (3.91) as follows.

\[
\rho \frac{\partial V_r}{\partial t} \delta V_r \approx \frac{\rho}{\Delta t} \overline{\delta V}_{rk}^T N_k^T \left( V_r - V_{old}^r \right)
\]

\[
= \frac{\rho}{\Delta t} \overline{\delta V}_{rk}^T N_k^T \left( N_j Q \overline{v}_{rj} - V_{r}^r \right) = \overline{\delta V}_{rk}^T \left( Q^T A_{r1} Q \overline{v}_{rj} + Q^T A_{r2} \right)
\]

(3.98)

\[
\rho V_r^old \frac{\partial V_r^old}{\partial t} \delta V_r + \rho V_z^old \frac{\partial V_z^old}{\partial t} \delta V_r
\]

\[
= \rho \overline{\delta V}_{rk}^T N_k^T \left( V_r^old \frac{\partial V_r^old}{\partial t} + V_z^old \frac{\partial V_z^old}{\partial t} \right) = \overline{\delta V}_{rk}^T \left( B_{r1} + B_{r2} \right)
\]

(3.99)

\[
2\alpha \frac{V_r}{r} \frac{\partial V_r}{\partial t} + \alpha \frac{\partial V_r}{\partial t} \frac{\partial V_r}{\partial t} + \alpha \frac{\partial V_z}{\partial t} \frac{\partial V_r}{\partial t} + \frac{V_r}{r^2} \frac{\partial V_r}{\partial r} + \frac{\partial V_z}{\partial t} \frac{\partial V_r}{\partial r}
\]

\[
= 2\alpha \overline{\delta V}_{rk}^T N_k^T \left( \frac{\partial N_j}{\partial t} \frac{\partial V_r}{\partial r} + \frac{\partial N_j}{\partial t} \frac{\partial V_r}{\partial r} \right) + \alpha \overline{\delta V}_{rk}^T N_k^T \frac{\partial N_j}{\partial t} Q \overline{v}_{rj} + \alpha \overline{\delta V}_{rk}^T N_k^T \frac{\partial N_j}{\partial t} \frac{\partial N_j}{\partial t} Q \overline{v}_{rj}
\]

\[
+ \alpha \overline{\delta V}_{rk}^T N_k^T \frac{\partial N_j}{\partial t} \frac{\partial N_j}{\partial t} Q \overline{v}_{rj} + \alpha \overline{\delta V}_{rk}^T N_k^T \frac{\partial N_j}{\partial t} \frac{\partial N_j}{\partial t} Q \overline{v}_{rj}
\]

\[
= \overline{\delta V}_{rk}^T \left[ Q^T \left( C_{r1} + C_{r2} + C_{r3} \right) Q \overline{v}_{rj} + Q^T \left( C_{r4} + C_{r5} \right) Q \overline{v}_{rj} \right]
\]

(3.100)
\[2\eta \frac{\partial V_i}{\partial r} \frac{\partial V_r}{\partial r} + \eta \frac{\partial V_i}{\partial z} \frac{\partial V_r}{\partial z} + \eta \frac{\partial V_i}{\partial r} \frac{\partial V_r}{\partial z} + 2\eta \frac{1}{r^2} V_r \delta V_i = \\
2\eta \overline{\delta V}_{rk} T \frac{\partial N_{k}^T}{\partial r} \frac{\partial N_{j}^T}{\partial r} QV_{ij} + \eta \overline{\delta V}_{rk} T \frac{\partial N_{k}^T}{\partial z} \frac{\partial N_{j}^T}{\partial z} QV_{ij} = \\
2\eta \overline{\delta V}_{rk} T \frac{\partial N_{k}^T}{\partial r} \frac{\partial N_{j}^T}{\partial r} QV_{ij} + \eta \overline{\delta V}_{rk} T \frac{\partial N_{k}^T}{\partial z} \frac{\partial N_{j}^T}{\partial z} QV_{ij} + \eta \overline{\delta V}_{rk} T \frac{\partial N_{k}^T}{\partial z} \frac{\partial N_{j}^T}{\partial z} QV_{ij} = \\
\delta V_{rk} T \left[ Q^T \left( D_{r1} + D_{r2} + D_{r3} \right) QV_{ij} + Q^T D_{r4} QV_{ij} \right]
\]

where

\[A_{r1} = \frac{\rho}{\Delta t} N_{k}^T N_{j} \]  

\[A_{r2} = -\frac{\rho}{\Delta t} N_{k}^T V_{r}^{old} \]  

\[B_{r1} = \rho N_{k}^T V_{r}^{old} \frac{\partial V_{r}^{old}}{\partial r} \]  

\[B_{r2} = \rho N_{k}^T V_{z}^{old} \frac{\partial V_{z}^{old}}{\partial z} \]  

\[C_{r1} = 2\alpha \frac{\partial N_{k}^T}{\partial r} \frac{N_{j}}{r} \]  

\[C_{r2} = \alpha \frac{\partial N_{k}^T}{\partial r} \frac{\partial N_{j}}{\partial r} \]  

\[C_{r3} = \alpha \frac{N_{k}^T}{r} \frac{N_{j}}{r} \]  

\[C_{r4} = \alpha \frac{\partial N_{k}^T}{\partial z} \frac{\partial N_{j}}{\partial z} \]  

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\[ C_{r5} = \alpha \frac{N_k^T}{r} \frac{\partial N_j}{\partial r} \] (3.110)

\[ D_{r1} = 2\eta \frac{\partial N_k^T}{\partial r} \frac{\partial N_j}{\partial r} \] (3.111)

\[ D_{r2} = \eta \frac{\partial N_k^T}{\partial z} \frac{\partial N_j}{\partial r} \] (3.112)

\[ D_{r3} = 2\eta \frac{N_k^T N_j}{r^2} \] (3.113)

\[ D_{r4} = \eta \frac{\partial N_k^T}{\partial z} \frac{\partial N_j}{\partial r} \] (3.114)

where the notations with superscript prime donate the terms related to the force.

Similarly, we can rewrite each term in expression (3.92) as

\[ \rho \frac{\partial V_z}{\partial t} \delta V_z \approx \rho \frac{\partial V_z^T}{\partial t} N_k^T \left( N_j Q V_{zj} - V^\text{old}_z \right) \] (3.115)

\[ = \delta V_{zk}^T \left( Q^T A_{z1} Q V_{zj} - Q^T A_{z2} \right) \]

\[ \rho V_z^\text{old} \frac{\partial V_z^\text{old}}{\partial r} \delta V_z + \rho V_z \frac{\partial V_z}{\partial z} \delta V_z \]

\[ = \rho \delta V_{zk}^T N_k^T \left( V_z^\text{old} \frac{\partial V_z}{\partial z} + V_z \frac{\partial V_z}{\partial z} \right) = \delta V_{zk}^T Q^T \left( B_{z1} + B_{z2} \right) \] (3.116)

\[ \alpha \frac{\partial V_z}{\partial r} \delta V_z + \alpha \frac{\partial V_z}{\partial z} \delta V_z + \alpha \frac{\partial V_z}{\partial z} \delta V_z \]

\[ = \alpha \delta V_{zk}^T Q^T \frac{\partial N_k^T N_j}{\partial z} Q V_{zj} + \alpha \delta V_{zk}^T Q^T \frac{\partial N_k^T}{\partial r} \frac{\partial N_j}{\partial z} Q V_{zj} \]

\[ + \alpha \delta V_{zk}^T Q^T \frac{\partial N_k^T}{\partial z} \frac{\partial N_j}{\partial z} Q V_{zj} \]

\[ = \delta V_{zk}^T \left[ Q^T (C_{z1} + C_{z2}) Q V_{zj} + Q^T C_{z3} Q V_{zj} \right] \]
\[
\eta \frac{\partial v_x}{\partial z} \frac{\partial \delta v_z}{\partial r} + \eta \frac{\partial v_z}{\partial r} \frac{\partial \delta v_z}{\partial z} + 2\eta \frac{\partial v_z}{\partial z} \frac{\partial \delta v_z}{\partial r} \\
= \eta \delta v_z^T Q^T \frac{\partial N_k}{\partial r} \frac{\partial N_j}{\partial z} Q\delta v_j + \eta \delta v_z^T Q^T \frac{\partial N_k}{\partial r} \frac{\partial N_j}{\partial z} Q\delta v_j \\
+ 2\eta \delta v_z^T Q^T \frac{\partial N_k}{\partial z} \frac{\partial N_j}{\partial z} Q\delta v_j \\
= \delta v_z^T \left[ Q^T D_z zl Q\delta v_j + Q^T (D_z z2 + D_z z3) Q\delta v_j \right]
\]

(3.118)

where

\[
A_{z1} = \frac{\rho}{\Delta t} N_k^T N_j
\]

(3.119)

\[
A_{z2}' = -\frac{\rho}{\Delta t} N_k^T V_z^{\text{old}}
\]

(3.120)

\[
B_{z1}' = \rho N_k \frac{\partial V_z^{\text{old}}}{\partial r}
\]

(3.121)

\[
B_{z2}' = \rho N_k \frac{\partial V_z^{\text{old}}}{\partial z}
\]

(3.122)

\[
C_{z1} = \alpha \frac{\partial N_k^T N_j}{\partial r}
\]

(3.123)

\[
C_{z2} = \alpha \frac{\partial N_k^T \partial N_j}{\partial r}
\]

(3.124)

\[
C_{z3} = \alpha \frac{\partial N_k^T \partial N_j}{\partial z}
\]

(3.125)

\[
D_{z1} = \eta \frac{\partial N_k^T \partial N_j}{\partial r}
\]

(3.126)

\[
D_{z2} = \eta \frac{\partial N_k^T \partial N_j}{\partial r}
\]

(3.127)
Using Eqs. (3.98) – (3.101) and Eqs. (3.115) – (3.118), Eq. (3.90) can be expressed in the finite element matrix equation as:

\[
\begin{bmatrix}
\{Q\}^T \left[ \begin{bmatrix} K_{rr} & K_{rz}^p \\ K_{rz} & K_{zz}^p \end{bmatrix} \right] \{Q\} \\
\{Q\}^T \left[ \begin{bmatrix} K_{rr} & K_{rz}^p \\ K_{rz} & K_{zz}^p \end{bmatrix} \right] \{Q\} \\
\end{bmatrix}
\begin{bmatrix}
\{v_{rj}\} \\
\{v_{zj}\} \\
\end{bmatrix}
\] = \begin{bmatrix}
\{Q\}^T \{F_r\} \\
\{Q\}^T \{F_z\} \\
\{F_N\} \\
\end{bmatrix}

(3.129)

In Eq. (3.129), the terms with superscript p came from eliminating the pressure terms appearing in Eqs. (3.75) and (3.76), with the aid of Eq. (3.78). The force vector \(\{F_N\}\), indicated by the subscript N is due to the normal stress balance at the interface. \(v_{rj}\) and \(v_{zj}\) are the jth component of local velocities corresponding to the shape function \(N(\zeta, \eta)_j\) for \((j = 1, 2, 3, 4, 5, 6)\).

The element of the stiffness matrix and force vector in Eq. (3.129) can be summarized as:

\[
K_{rr} = \int_{\Omega^e} \left( A_{r1} + D_{r1} + D_{r2} + D_{r3} \right) d\Omega 
\]

(3.130)

\[
K_{rr}^p = \int_{\Omega^e} \left( C_{r1} + C_{r2} + C_{r3} \right) d\Omega 
\]

(3.131)

\[
K_{rz} = \int_{\Omega^e} \left( D_{r4} \right) d\Omega 
\]

(3.132)

\[
K_{rz}^p = \int_{\Omega^e} \left( C_{r4} + C_{r5} \right) d\Omega 
\]

(3.133)
\[ K_{zz} = \int_{\Omega^e} (D_{z_1}) d\Omega \]  
(3.134)

\[ K_{zz}^P = \int_{\Omega^e} (C_{z_1} + C_{z_2}) d\Omega \]  
(3.135)

\[ K_{zz} = \int_{\Omega^e} (A_{z_1} + D_{z_2} + D_{z_3}) d\Omega \]  
(3.136)

\[ K_{zz}^P = \int_{\Omega^e} (C_{z_3}) d\Omega \]  
(3.137)

\[ \{F_N\} = \left\{ \begin{array}{c} 0 \\ \int_{\Gamma_2 \cap \partial\Omega^e} S_i^T \frac{\gamma}{R} d\Gamma \end{array} \right\} \]  
(3.138)

\[ F_r = \int_{\Omega^e} \left( -A_{r_2} - B_{r_1} - B_{r_2} \right) d\Omega \]  
(3.139)

\[ F_z = \int_{\Omega^e} \left( -A_{z_2} - B_{z_1} - B_{z_1} \right) d\Omega \]  
(3.140)

where \( \Omega^e \) is the volume of an element and \( \partial \Omega^e \) is its boundary. \( S_i \) (i= 1, 2, 3) in Eq. (3.138) is the quadratic shape function. And \( \{F_N\} \) can be calculated with the aid of \( S_i \) along the element boundary on the surface \( \Gamma_2 \) using the known values of the surface tension and curvature.

3.1.3.2 Drop Deformation of Viscoelastic Fluid

**Integral-Type Constitutive Equation of State, KBKZ Model**

Two types of constitutive equations have been developed in an effort to describe the viscoelasticity of polymeric fluid: the differential-type constitutive equation and the integral-type constitutive equation. Generally, much more effort has been spent on
solving polymer flow problem with the differential-type constitutive model due to its compatibility with the differential representation of momentum and continuity equations.\textsuperscript{2, 79, 80}

A major challenge of employing the integral-type constitutive equation for FEM lies in that the solution of the integral constitutive equation requires an accurate particle tracking and backward integration along the path-line due to the fading memory of viscoelastic materials. However an advantage of using the integral-type model lies in that it is explicit in the stress tensor, and therefore it does not introduce the stress as primary unknowns in the numerical computation. The integral form of the Upper Convected Maxwell (UCM) model can be written as

$$\sigma(t) = \frac{\eta_0}{\lambda} \int_{-\infty}^{t} \left( C^{-1}(t, t') - I \right) e^{ \frac{t - t'}{\lambda} } \, dt'$$

where $\sigma(t)$ is the deviatoric stress tensor at current time $t$, $\lambda$ is the relaxation time, and $C^{-1}(t, t')$ is the Finger strain tensor which is related to past time $t'$. When using the integral form of the UCM model, the stress tensor can be calculated from the velocity (gradient) explicitly and no iteration is necessary. Therefore, the stress can be treated as a pseudo body force in solving the global finite element equations. Only the velocity has to be calculated by iteration.

Consider the following form of an integral-type constitutive equation, known as the KBKZ model, that relates $\sigma$ to the deformation history which has been used by Papanastasiou et al.\textsuperscript{81}

$$\sigma(t) = \int_{-\infty}^{t} M(t, t') h\left( I_{C^{-1}}, II_{C^{-1}} \right) C^{-1}(t, t') \, dt'$$

(3.142)
where $M(t, t')$ is a memory function, $I_{C^{-1}}$ and $II_{C^{-1}}$ are the first and second invariants of the Finger strain tensor $C^{-1}(t, t')$, and $h(I_{C^{-1}}, II_{C^{-1}})$ is the strain damping function.

Generally, the memory function $M(t, t')$ can be expressed as a series of exponential functions:

$$M(t - t') = \sum_{k=1}^{N} \frac{a_k}{\lambda_k} \exp\left(-\frac{t - t'}{\lambda_k}\right)$$

(3.143)

where $\lambda_k$ and $a_k$ are the relaxation time and relaxation modulus coefficients, respectively, at a reference temperature $T_0$.

The strain damping function $h(I_{C^{-1}}, II_{C^{-1}})$ has been presented in many different forms; one form introduced by Papanastasiou et al.\textsuperscript{82} is given by

$$h(I_{C^{-1}}, II_{C^{-1}}) = \frac{\alpha}{(\alpha - 3) + \beta I_{C^{-1}} + (1 - \beta) II_{C^{-1}}}$$

(3.144)

where $\alpha$ and $\beta$ are material constants.

Because of the inherent complexity of using an integral-type model, relatively little effort has been devoted on integral model in computation of polymeric flow. Obviously, the calculation of the time-dependant memory function and strain function in Eq. (3.142) gives rise to the problem of strain history calculation and particle tracking which are the main challenges associated with the application of the integral-type model. In the next section, we will discuss the tracking algorithm in detail.

It is difficult to calculate the stress defined by Eq. (3.142) which is related to the time integration. One approximation is to evaluate the integral function at a finite number of points (positions) along the upward streamline and use proper numerical
integration algorithm to calculate the infinite integral approximately. Bernstein et al.\textsuperscript{83} developed a series of special elements, and assumed constant velocity gradients in the element for the purpose of calculating strain history. With the assumption of a constant velocity gradient, the material tracking equation becomes a homogeneous linear ordinary differential system with constant coefficients which can be solved by finding the characteristic root from the characteristic equation of the system. The streamline can be obtained and the integration of the memory function $M(t, t')$ is calculated using the Gauss–Laguerre numerical quadrature formula.\textsuperscript{84} Based on a special stream element scheme, Luo and coworkers\textsuperscript{85,86} set the trajectories of material points identical to those of two element boundaries of the streamline elements which make the tracking of past position of material points a relatively easy task. The deformation gradient tensor is calculated by numerically integrating a simple expression for the material derivative along the existing streamlines. Also the infinite memory integral is approximated by Laguerre or Gaussian integration.

In this study (see Figure 3.2), the interface between the drop and continuous matrix will change step by step, while the drop moves along the flow direction. This results in the challenge of constructing the grids for the finite element analysis. Refined mesh is necessary for high resolution computation. However the mesh generation cannot be easily connected to the particle path-line, meaning that the range of integration over time and position may be located as part of the discretized elements. A new method other than locating a finite element node on the streamline has been employed in this study.

Let us confine our attention to the specific problem depicted in Figure 3.2. The governing equation can still be written as Eqs. (3.73) – (3.76).
Employing the penalty function method represented by Eq. (3.78), we take Eqs. (3.73) and (3.74) as constraints of Eqs. (3.75) and (3.76), and formulate the finite element equations through variational method similar to the one presented above. On a typical element, the variational form of Eqs. (3.73) – (3.76) is given by

$$\delta J = 0 = \int_\Omega \rho \left( \frac{\partial \mathbf{v}}{\partial t} \cdot \delta \mathbf{v} + (\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \delta \mathbf{v} \right) d\Omega + \int_\Omega \sigma(t, t') : \nabla(\delta \mathbf{v}) d\Omega$$

$$+ \int_\Omega \alpha(\nabla \cdot \mathbf{v})(\nabla \cdot (\delta \mathbf{v})) d\Omega - \int_\Gamma \mathbf{f} \cdot \delta \mathbf{v} d\Gamma$$

(3.145)

or

$$0 = \int_\Omega \rho \left( \frac{\Delta \mathbf{v}_i}{\Delta t} + \mathbf{v}_i^{\text{old}} \frac{\partial \mathbf{v}_i^{\text{old}}}{\partial x_j} \right) \delta \mathbf{v}_i d\Omega + \int_\Omega \alpha d_{ii} \delta d_{ij} d\Omega$$

$$+ \int_\Omega \sigma(t, t')_{ij} \delta d_{ij} d\Omega - \int_\Gamma f_i \delta \mathbf{v}_i d\Gamma$$

(3.146)

in which the third term on the right-hand side is given by

$$\int_\Omega \sigma(t, t')_{ij} \delta d_{ij} d\Omega = \int_\Omega \int_{-\infty}^t \mathbf{M}(t - t') h\left(I_{\mathbf{C}^{-1}}, II_{\mathbf{C}^{-1}}\right) (t', t')_{ij} dt' \delta d_{ij} d\Omega$$

(3.147)

Note that the memory function $\mathbf{M}(t, t')$, the strain damping function $h(I_{\mathbf{C}^{-1}}, II_{\mathbf{C}^{-1}})$, and the Finger strain tensor $\mathbf{C}^{-1}(t, t')$ can be calculated once the deformation gradient tensor $\mathbf{E}$ is determined, and the integral form of stress over the volume $\Omega$ can be evaluated numerically. Introducing the time discretization form of the strain tensor into Eq. (3.146) while keeping the present time $t$ and splitting the time integral in Eq. (3.147), the finite element equation may be expressed by
\[
\int_{\Omega} \rho \frac{\partial V_i}{\partial t} \delta V_i \, d\Omega + \int_{\Omega} \rho V_j \frac{\partial \delta V_i}{\partial t} \delta V_i \, d\Omega + \int_{\Omega} \alpha \delta d_i \delta \delta_{ij} \, d\Omega \\
+ \int_{\Omega} \sigma(t, t') \delta \delta_{ij} \, d\Omega = \int_{\Gamma_1} T_m, ij n_j v_n \, d\Gamma + \int_{\Gamma_1} \frac{\gamma}{R} \delta v_n \, d\Gamma
\]

(3.148)

which can be rewritten into \( r \) and \( z \) components, respectively. In writing Eq. (3.148), we employed the same procedure as given in Eqs. (3.87) – (3.89).

The left-hand side of the \( r \)-component of Eq. (3.148) is given by

\[
\int_{\Omega} \left\{ \rho \frac{\partial V_r}{\partial t} \delta V_r + \rho V_r \frac{\partial V_r}{\partial r} \delta V_r + \rho V_z \frac{\partial V_r}{\partial z} \delta V_r + \rho V_r \frac{\partial V_r}{\partial r} \delta V_r + 2\alpha \frac{V_r}{r} \frac{\partial V_r}{\partial r} \delta V_r \right. \\
+ \frac{\partial V_r}{\partial r} \frac{\partial \delta V_r}{\partial r} + \frac{\partial V_r}{\partial z} \frac{\partial \delta V_r}{\partial z} + \frac{\partial V_z}{\partial r} \frac{\partial \delta V_r}{\partial z} + \frac{\partial V_z}{\partial z} \frac{\partial \delta V_r}{\partial z} + \frac{\partial V_z}{r} \frac{\partial \delta V_r}{\partial r} \\
+ \left. \tau r = \frac{\partial \delta V_r}{\partial r} + \tau \delta V_r + \tau \delta V_r \right\} \, d\Omega
\]

(3.149)

and the left-hand side of the \( z \)-component of Eq. (3.148) is given by

\[
\int_{\Omega} \left\{ \rho \frac{\partial V_z}{\partial t} \delta V_z + \rho V_r \frac{\partial V_z}{\partial t} \delta V_z + \rho V_z \frac{\partial V_z}{\partial z} \delta V_z + \rho V_z \frac{\partial V_z}{\partial z} \delta V_z \\
+ \frac{\partial V_r}{\partial z} \frac{\partial \delta V_z}{\partial z} + \frac{\partial V_z}{\partial z} \frac{\partial \delta V_z}{\partial z} + \frac{\partial V_z}{\partial z} \frac{\partial \delta V_z}{\partial z} + \frac{\partial V_z}{r} \frac{\partial \delta V_z}{\partial z} \\
+ \left. \tau z = \frac{\partial \delta V_z}{\partial z} + \tau \delta V_z \right\} \, d\Omega
\]

(3.150)

Six-node isoparametric triangular elements are employed to obtain a finite element matrix equation, by taking similar procedure described above for Eqs. (3.93) – (3.96).

Also, the velocity component in the local coordinate system \((t, n)\) defined along the interface can be transformed into the global coordinate \((r, z)\), by a transformation matrix \([Q]\), with the relationship shown in Eq. (3.97)
Using Eqs. (3.93) – (3.96), we can rewrite each term in expression (3.149), and express them similar to Eqs. (3.98) – (3.100), except the following term

\[ \tau_{rr} \frac{\partial \delta V_r}{\partial r} + \tau_{rz} \frac{\partial \delta V_z}{\partial z} + \tau_{\theta\theta} \frac{1}{r} \delta V_r \]

\[ = \delta V_{rk}^T Q^T \frac{\partial N_k^T}{\partial r} \tau_{rr} + \delta V_{rk}^T Q^T \frac{\partial N_k^T}{\partial z} \tau_{rz} + \delta V_{rk}^T Q^T \frac{N_k^T}{r} \tau_{\theta\theta} \]  

\[ = \delta V_{rk}^T Q^T \left[ (D_{r1}' + D_{r2}' + D_{r3}') \right] \]  

where

\[ D_{r1}' = \frac{\partial N_k^T}{\partial r} \tau_{rr} \]  

\[ D_{r2}' = \frac{\partial N_k^T}{\partial z} \tau_{rz} \]  

\[ D_{r3}' = \frac{N_k^T}{r} \tau_{\theta\theta} \]  

We can also rewrite each term in expression (3.150) by using Eqs. (3.93) – (3.96), and express them as similar to Eqs. (3.115) – (3.117), except the following term

\[ \tau_{zz} \frac{\partial \delta V_z}{\partial r} + \tau_{zz} \frac{\partial \delta V_z}{\partial z} \]

\[ = \delta V_{zk}^T Q^T \frac{\partial N_k^T}{\partial r} \tau_{zx} + \delta V_{zk}^T Q^T \frac{\partial N_k^T}{\partial z} \tau_{zz} \]  

\[ = \delta V_{zk}^T Q^T \left[ (D_{z1}' + D_{z2}') \right] \]  

where

\[ D_{z1}' = \frac{\partial N_k^T}{\partial r} \tau_{zx} \]  

\[ D_{z2}' = \frac{\partial N_k^T}{\partial z} \tau_{zz} \]
Using Eqs. (3.98) – (3.100), (3.115) – (3.117), (3.151) and (3.155), Eq. (3.148) can be expressed in the finite element matrix equation as:

$$
\begin{align*}
\begin{bmatrix}
Q^T 
& \begin{bmatrix}
K_{rr}^p 
& K_{rz}^p 
& K_{zz}^p
\end{bmatrix}

\end{bmatrix}
\begin{bmatrix}
Q \\
\begin{bmatrix}
K_{rr} 
& K_{rz} 
& K_{zz}
\end{bmatrix}

\end{bmatrix}

\begin{bmatrix}
\{v_j\} \\
\{v_{zj}\}
\end{bmatrix}

= 
\begin{bmatrix}
Q^T 
& \begin{bmatrix}
F_r 
\{F_z\}
\end{bmatrix}

\end{bmatrix}
\begin{bmatrix}
Q^T 
& \begin{bmatrix}
F_N 
\end{bmatrix}

\end{bmatrix}

\end{align*}
$$

(3.158)

The element in stiffness matrix on the left-hand side of Eq. (3.158) is given by,

$$
K_{rr} = \int_{\Omega_r} \left( A_{r1} \right) d\Omega
$$

(3.159)

$$
K_{zz} = \int_{\Omega_z} \left( A_{z1} \right) d\Omega
$$

(3.160)

and $K_{rr}^p$, $K_{rz}^p$, $K_{zz}^p$ are defined by Eqs. (3.131), (3.133), (3.135), and (3.137), respectively. The element force vector $\{F_N\}$ on the right-hand side of Eq. (3.158) is given by Eq. (3.138). $\{F_r\}$ and $\{F_z\}$ are given by

$$
F_r = \int_{\Omega_r} \left( -A_{r2}' - B_{r1}' - B_{r2}' - D_{r1}' - D_{r2}' - D_{r3}' \right) d\Omega
$$

(3.161)

$$
F_z = \int_{\Omega_z} \left( -A_{z2}' - B_{z1}' - B_{z2}' - D_{z1}' - D_{z2}' \right) d\Omega
$$

(3.162)

### 3.2 Moving Boundary and Particle Tracking

A moving boundary problem is always associated with the location of the discontinuity of stress, velocity, and/or pressure which can be calculated by adopting a different governing equation for each phase. It has to be located accurately during
computation, so that high resolution can be achieved. Such a technique is called interface tracking. The methods for interface tracking can be classified, depending on the nature of the grid used in the bulk of the phases: fixed and moving grids. For fixed-grid method, there is a set of predefined meshes that do not move with the interface. The interface somehow cuts across the fixed grids. These kinds of fixed grids may be either structured or unstructured, as shown in Figure 3.3 and Figure 3.4. As for the moving-grid methods, the interface is always located on the boundary between two grids which identify the two

![Figure 3.3 An interface cutting across a fixed, structured grid.](image)

![Figure 3.4 An interface cutting across a fixed, unstructured grid.](image)
phases of sub-domains. In this case, while some grids can be structured which may simplify the analysis near the interface, most of the grids are unstructured, especially when the interface undergoes abrupt deformation. Sometimes it is necessary to refine the grid close to the interface for high accuracy of computation.

There are many ways of computing the moving boundary problem in the multiphase flow. One way is to use the Lagrangian method, which employs Lagrangian coordinates that follow the individual particles as they move along the flow region. The grid also moves along with the coordinate. Using this method, extensive research has been reported, including the examination of the initial deformation of a buoyant bubble by Shopov et al., simulations of the unsteady-state two-dimensional motion of several particles by Feng, et al., and Hu et al., and axisymmetric simulation of the collision of a single drop with a wall by Fukai et al.

Another method is to capture the front (interface) directly on a regular, stationary grid, including the Volume of Fluid (VOF) method where a marker is used to identify each fluid, and the Marker-and-Cell (MAC) method where marker particles are used. These methods feature a sharp boundary between different fluids, and the major challenge of using this method lies in the management of deletion or addition of markers while the interface is stretched or compressed. And the computational speed becomes considerably slow as the number of markers increases. Recently, many improvements have been developed to increase the accuracy and applicability of this method, including Constrained Interpolation Profile (CIP) method by Yabe, the phase-field method by Jacqmin, the Level Sets (LS) method which is based on a smooth distance function whose zero value is located on the interface, and the modified level sets.
Brackbill \cite{109} developed a continuous surface tension function by defining continuous ‘color’ (marker) function at the transition regions of finite thickness. Bonometti \cite{110} took a numerical approach combining the features of the volume of fluid method and the level set method.

Further, the front tracking method has been employed, \cite{111-113} in which a set of fixed grids are used. This method uses an additional separate front marker on the interface which helps track the interface explicitly. The fixed grids are modified near the interface to enable the grid line to follow the moving interface. Therefore the information regarding location and curvature of the interface is explicitly available during the computation. Some researchers have developed hybrid methods with the advantage of both front tracking and front-capturing techniques. \cite{88,101,114} They employed a set of fixed grids for the entire flow region and track the interface by modifying the grids that contain the interface. Unlike the front tracking methods, which treat each phase separately, this method treats all phases together by solving a set of governing equations for the entire flow regime.

The fourth category of methods is to employ a set of meshes which have boundary-fitted grid points for each phase. The meshes deform along with the particles on both sides of the boundary and offer potentially the highest accuracy with the accurate explicit information of location and curvature of interface which entails direct and accurate application of boundary conditions. This method has been implemented in the finite element method, \cite{51,52,55,91,115,116} finite difference method, \cite{117,118} boundary element method \cite{57-60,62} and boundary integral method. \cite{5,53,54,119,120} Tracking of the moving particle entails a considerable logical and computational effort. Furthermore, a large
displacement of internal domain may cause mesh entanglement, especially when there is contraction flow or recirculation flow. Hence, the meshes close to the interface must be refined at each step of the computation, and an auto-remeshing scheme has to be employed, which may introduce interpolation error as well as additional cost. However it is very difficult to handle a merging and folding interface with this method. Generally, this method cannot handle some sophisticated problem like drop breakup, coalescence, and reconnection and thus is limited mostly to a single drop undergoing relatively mild deformation. Limited research has been presented on fully three-dimensional computations.\textsuperscript{53,60}

In this study, the integral constitutive equation, KBKZ model defined by Eq. (3.142), was employed, with which the time-dependent strain was calculated by integration over the particle path-line. In the use of this method, not only the particles on the interface have to be tracked downstream so that the future interface can be predicted, but also the entire particles inside the flow regime have to be tracked for the time integration purpose. Papanastasiou et al.\textsuperscript{81} developed a finite element formulation in solving the integral-type model by employing the Streamline Element Scheme (SES). The simultaneous construction of the streamline finite element is difficult to carry out directly when the conventional finite element technique is employed, and the evaluation of the primary unknowns and the streamline may randomly cross the element boundary, making the tracking of a particle path-line more complicated. With the SES method, the nodes of the finite elements are selectively located on a set of streamlines. This will simplify the integration of strain and make the integration over the unknown variables related to the path-line, i.e., time integration much easier. However the elements are no longer the
conventional, and this method cannot be applied to the complicated flow, like flow with closed streamlines (recirculation regions). When dealing with the conventional elements like triangular element, a special scheme has to been developed.

For a quasi-steady flow, there is the following relationship between the particle acceleration and the velocity field.

\[
\frac{Dv}{Dt} = v \cdot \nabla v \tag{3.163}
\]

Now we can calculate the shape of a deformed drop at time \( t + \Delta t \) by using the velocity on the interface as a reference at the current time \( t \):

\[
r_{t+\Delta t} \approx r_t + v_r \Delta t + \frac{\Delta t^2}{2} \left( v_r \frac{\partial v_r}{\partial t} + v_z \frac{\partial v_z}{\partial z} \right) \tag{3.164}
\]

\[
z_{t+\Delta t} \approx z_t + v_z \Delta t + \frac{\Delta t^2}{2} \left( v_r \frac{\partial v_r}{\partial t} + v_z \frac{\partial v_z}{\partial z} \right) \tag{3.165}
\]

where the \((r_t, z_t)\) and \((r_{t+\Delta t}, z_{t+\Delta t})\) are the nodal positions at the current time \( t \) and the following time step \( t + \Delta t \) on the interface \( \Gamma_2 \), respectively.

After the interface is updated, it is unavoidable that the volume of the drop may change due to the truncated error in computations, which will result in significant accumulated error in the following calculation step if the change of the drop volume is not corrected. Therefore the volume should be checked at each calculation step. Once the deviation of drop volume exceeds 1% of the original volume, a correction will be made by adjusting the shape of the deformed drop via zooming with a proper scaling factor with respect to the center of drop. Thus the conservation of drop volume is maintained within a 1% deviation from the original volume.
By using a similar scheme as the interface tracking continues, the following equation may be used for tracking the particles in the upstream of the entire flow regime.

\[ r_{t'} = r_t - v_r ds + \frac{ds^2}{2} \left( v_r \frac{\partial v_r}{\partial t} + v_z \frac{\partial v_z}{\partial z} \right) + 0(ds^3) \] (3.166)

\[ z_{t'} = z_t - v_z ds + \frac{ds^2}{2} \left( v_r \frac{\partial v_r}{\partial t} + v_z \frac{\partial v_z}{\partial z} \right) + 0(ds^3) \] (3.167)

where \( s = t - t' \) is the residence time of the particle, \((r_{t'}, z_{t'})\) is the particle position at a past time \( t' \) and \((r_t, z_t)\) is the particle position at current time \( t \).

### 3.3 Auto-Remeshing and Mesh Refinement

As a drop moves along the flow direction, the shape of the drop will change. After the interface position is located using a tracking algorithm, a new set of meshes have to be constructed and it becomes necessary to determine the positions of the node in the old meshes of the new integration points. With the given boundaries of the inlet, outlet, and wall, together with the updated shape of the drop, an auto-remeshing procedure will be employed, during which the meshes will be remapped so that no element can cross the interface.

The unique features of the proposed method lie in that node positions and connections are computed simultaneously. This method will be carried out by the Bowyer–Watson algorithm to compute the Delaunay triangulation. A computer program for the auto-remeshing procedure will be developed, and an unstructured grid will be generated based on the Delaunay triangulation (see Figure 3.5 and Figure 3.6).
Figure 3.5 A six-node triangular finite element that will be used in finite element analysis.

Figure 3.6 Delaunay triangulation (continuous) of the convex hull of a small set of points.

Using the Bowyer–Watson algorithm, appropriate distribution of interior points will be calculated and connected to form triangles simultaneously. A mesh relaxation technique proposed by Frey and Field\textsuperscript{123} is incorporated into the program code by adjusting the connection among points. This scheme consists of a procedure for iteratively making the mesh topology more regular by edge swapping. The triangular mesh density will be controlled by the selective refinement method.\textsuperscript{124} Therefore the node density on the
interface and close to the interface will be controlled as a function of local curvature, and the irregular element will be refined.

After a new set of the meshes are constructed, the position of all newly generated nodes on the new meshes will be changed. The position of the new interface as well as the integration points on the old mesh must be determined, so that the unknowns on the new node, i.e., velocities, can be calculated. Using the Newton–Raphson method, this procedure can be carried out by finding the element number and element coordinate in the old meshes of the positions corresponding to the coordinate for the points in the new meshes. The following mapping equations show a relation between the natural coordinates \((\zeta, \eta)\) corresponding to a given position \((r, z)\) in a six-node isoparametric triangular element.55

\[
\begin{align*}
F(\xi, \eta) &= r - N(\xi, \eta) r_j = 0 \\
G(\xi, \eta) &= z - N(\xi, \eta) r_j = 0
\end{align*}
\]

where \(j = 1, 2, 3, 4, 5\) and 6. Using this method, the old velocity and its gradient on the nodes of newly elements can be interpolated with the shape function using Eqs. (3.93) – (3.96).

### 3.4 Calculation of Strain Tensor

Once the path-line of the particle has been located, we can calculate the extra stress \(\sigma\) in Eq. (3.142) over time \(t\) if the Finger strain tensor \(C^{-1}(t, t')\) can be determined. Basically there are two methods to compute the time-dependent Finger strain tensor.
$C^{-1}(t, t')$. One method is to directly integrate the material derivatives in the Finger strain tensor $C^{-1}(t, t')$ or Cauchy–Green tensor $C(t, t')$. We may calculate $C^{-1}(t, t')$ from the premise of kinematic identity such that the upper convected derivative of the Finger strain tensor becomes zero\textsuperscript{86}, however it is very complicated and the solution is not accurate while long computational time is required.

An alternative method to compute the Finger strain tensor $C^{-1}(t, t')$ is to first obtain the deformation gradient tensor $E$ and then $C^{-1}(t, t')$ through the following relationship

$$C = E^T \cdot E$$  \hspace{1cm} (3.170)

With $C$ known, one can obtain $C^{-1}$ by means of Cayley–Hamilton theorem as\textsuperscript{125}:

$$C^{-1} = \frac{(C^2 - I_cC + II_c I)}{III_c}$$  \hspace{1cm} (3.171)

where the $I_c$, $II_c$ and $III_c$ are the first, second, and third invariants of $C$, respectively.

$$I_c = \text{tr}(C)$$  \hspace{1cm} (3.172)

$$II_c = \frac{I_c^2 - \text{tr}(C^2)}{2}$$  \hspace{1cm} (3.173)

$$III_c = \det(C) = (\det(E))^2$$  \hspace{1cm} (3.174)

The deformation gradient tensor $E$ can be calculated from:

$$\frac{DE(t - t')}{Dt'} = -L(t - t')E_t(t - t')$$  \hspace{1cm} (3.175)

where the $L(t - t')$ is velocity gradient tensor. For the present time $t' = t$, the deformation gradient tensor is identity tensor $I$.\textsuperscript{86}

$$E_t(t - t')|_{t=t'} = I$$  \hspace{1cm} (3.176)
The predictor-corrector formula of the Euler method will be used to calculate

\[
E_{t}(s) = I - \frac{ds}{2} (L + L' ) + \frac{ds^2}{2} L'L + 0(ds^3 )
\]  

(3.177)

where \( s = t - t' \).

In the case of our axisymmetric problem, the components of \( E \) are given by

\[
E_{rr} \approx 1 - \frac{ds}{2} \left( \frac{\partial v_r}{\partial r} + \frac{\partial v_r'}{\partial r} \right) + \frac{ds^2}{2} \left( \frac{\partial v_r'}{\partial r} \frac{\partial v_r'}{\partial r} + \frac{\partial v_r'}{\partial z} \right)
\]  

(3.178)

\[
E_{rz} \approx -\frac{ds}{2} \left( \frac{\partial v_z}{\partial z} + \frac{\partial v_z'}{\partial z} \right) + \frac{ds^2}{2} \left( \frac{\partial v_z'}{\partial r} \frac{\partial v_z'}{\partial r} + \frac{\partial v_z'}{\partial z} \right)
\]  

(3.179)

\[
E_{zz} \approx 1 - \frac{ds}{2} \left( \frac{\partial v_z}{\partial z} + \frac{\partial v_z'}{\partial z} \right) + \frac{ds^2}{2} \left( \frac{\partial v_z'}{\partial r} \frac{\partial v_z'}{\partial r} + \frac{\partial v_z'}{\partial z} \right)
\]  

(3.180)

\[
E_{00} = \frac{r'}{r}
\]  

(3.182)

Note that the \( E \) is not symmetric necessarily, \( E_{zz} \neq E_{rz} \). In strictly isochoric deformation, the determinant \( \det(E) = 1 \), and one can see that the above scheme has second-order accuracy.

Now, the Cauchy–Green strain tensor is obtained from Eq. (3.170) as

\[
C = \begin{bmatrix}
E_{rr}^2 + E_{zz}^2 & 0 & E_{zz}E_{rr} + E_{rr}E_{rz} \\
0 & E_{00}^2 & 0 \\
E_{zz}E_{rr} + E_{rr}E_{rz} & 0 & E_{rr}^2 + E_{zz}^2
\end{bmatrix}
\]  

(3.183)
Therefore, the $I_c$, $II_c$, and $III_c$ are the invariants of $C$, which can be calculated using Eqs. (3.172)–(3.174). Using Eq. (3.171), the components of $C^{-1}$ in term of the components of $E$ take the form:

$$
C^{-1} = \begin{bmatrix}
(E_{zx}^2 + E_{zy}^2)E_{00}^2 & 0 & -\left(E_{yz}E_{zz} + E_{zy}E_{zx}\right)E_{00}^2 \\
0 & (E_{xz}E_{zz} - E_{zx}E_{yz})^2 & 0 \\
-\left(E_{yx}E_{zz} + E_{yx}E_{yz}\right)E_{00}^2 & 0 & \left(E_{xy}^2 + E_{yz}^2\right)E_{00}^2
\end{bmatrix}
$$

(3.184)

With the known component of $C^{-1}$, the strain damping function $h(I_{C^{-1}}, II_{C^{-1}})$ can be calculated using Eq. (3.144).

Equations (3.178) – (3.182) give the relative deformation tensor concerning two points on the increment time $ds$ with second-order accuracy for each step, and only the velocity gradient is involved. Note that we can track the particle path-line and integrate the stress all the way upstream. However, the error generated by truncating Eq. (3.177) will accumulate due to the truncated errors in Eqs. (3.178) – (3.182) and summation of the integration. It is necessary to minimize the errors generated by $E$. Since the memory function $M(t, t')$ in Eq. (3.143) is in the form of exponential function, the Gauss–Laguerre scheme will be employed to numerically approximate the integration over time $t$ in Eq. (3.142). The following rule will help improve the numerical accuracy and the efficiency in computing the deformation tensor.

$$
E_{t_1'}(t_3') = E_{t_2'}(t_3') E_{t_2'}(t_2') \quad t_3' < t_2' < t_1'
$$

(3.185)

where the $t_1'$, $t_2'$ and $t_3'$ are the Gauss–Laguerre integration points. The global deformation tensor $E_{t}(t')$ can be obtained by connecting all the local $E$ on the Gauss–Laguerre points. The above rule is very important. While taking the current
configuration as reference configuration, the relationship \( E_t(t - t') \bigg|_{t=t'} = I \) is always satisfied. Furthermore, the determinant of the \( E \) always keeps second order accurate\(^8\).

The integral-type constitutive equation characterizes memory effect that fades with time. To account for the memory effects, the integration limit of Eq. (3.147) is from minus infinity to the present time. The minus infinity represents a position in the flow history before which any deformations that may occur do not affect the current stress; however such a limit is unrealistic and unnecessary. Therefore a cutoff time will help approximate the integration and needs to be determined.

Let us take the memory function with a single relaxation time as an example. The memory function is given by

\[
M(s) = \frac{\eta_0}{\lambda} \exp \left( -\frac{s}{\lambda} \right)
\]  

(3.186)

The cutoff time \( s_c \) can be found via criterion \( M(s_c) = \varepsilon \), which leads to

\[
s_c = -\lambda \ln \left( \frac{\lambda \varepsilon}{\eta_0} \right)
\]  

(3.187)

The above procedure can be extended to the case of the memory function with multiple relaxation time \( \lambda_k \) (see Eq. (3.143)), which has the form:

\[
M(s) = \frac{a_k}{\lambda_k} \exp \left( -\frac{s}{\lambda_k} \right)
\]  

(3.188)

The cutoff time is determined for each relaxation time by

\[
s_{ck} = -\lambda_k \ln \left( \frac{\lambda_k \varepsilon}{a_k} \right)
\]  

(3.189)
Many algorithms have been developed to deal with this kind of integration problem. Composite Simpson’s one-third rule\textsuperscript{82} and Gauss–Laguerre quadrature\textsuperscript{85,127} are the ones that have been extensively used. Compared with the Composite Simpson’s rule, the Gauss–Laguerre quadrature can solve the memory function more efficiently. Substitution of Eq. (3.143) into Eq. (3.142) gives

\[ \sigma(t) = \sum_{k} \frac{a_k}{\lambda_k} \int_{-\infty}^{t} \exp \left( -\frac{t-t'}{\lambda_k} \right) h \left( I_{C^{-1}}, \Pi_{C^{-1}} \right) C^{-1} \left( t, t' \right) dt' \] (3.190)

For each \( k \), a transformation \( \tau = (t - t')/\lambda_k \) can be used to write the \( k \)th term on the right-hand side of Eq. (3.190) as

\[ \frac{a_k}{\lambda_k} \int_{0}^{\infty} \exp \left( -\tau \right) h \left( I_{C^{-1}}, \Pi_{C^{-1}} \right) C^{-1} \left( -\lambda_k \tau \right) d\tau \] (3.191)

which can be approximated via Gauss–Laguerre quadrature formula.

\[ \frac{a_k}{\lambda_k} \int_{0}^{\infty} \exp \left( -\tau \right) h \left( I_{C^{-1}}, \Pi_{C^{-1}} \right) C^{-1} \left( -\lambda_k \tau \right) d\tau \approx \sum_{i=0}^{n_k} A_i^k h \left( I_{C^{-1}}, \Pi_{C^{-1}} \right) C^{-1} (T_i^k) \] (3.192)

where

\[ T_i^k = -\lambda_k \tau_i^k \] (3.193)

\[ A_i^k = a_k w_i^k \] (3.194)

\( \tau_i^k \) and \( w_i^k \) \((i = 1, 2, \ldots, n_k)\) are the Laguerre quadrature points and weight function, respectively.
Given the Finger strain tensor $C^{-1}$ which can be calculated from Eqs. (3.184), the extra stress can be calculated via the following form

$$\sigma = \sum_k \sum_{i=0}^{n_k} A^k_i \ h \left( I_{C^{-1}}, I_{C^{-1}} \right) C^{-1}(T^k_i)$$  \hspace{1cm} (3.195)

### 3.5 Flow Diagram and Numerical Algorithm

The entire flow regime can be discretized into a set of triangular finite elements, and the local finite element equation given by Eq. (3.129) for truncated power-law model and Eq. (3.158) for the integral-type KBKZ model can be assembled, yielding a global finite element equation in the form:

$$K\ V = F$$  \hspace{1cm} (3.196)

where $K$ is the stiffness matrix of $12\times12$, $V$ is the velocity vector of $12\times1$, and $F$ is the total body force vector of $12\times1$. Equation (3.196) can be solved by iteration and its solution gives the velocity field.

There are many different schemes to solve the global finite element equation. For integral-type KBKZ model, the stress is explicit and thus one can solve the stress separately after the velocity has been solved. The convergence of velocity and stress has to be satisfied simultaneously. For the power-law model, the stress term is expressed in the form of viscosity and shear rate.. There is no stress iteration procedure involved. Only velocity convergence is required for solving the global finite element equation.

For the power-law model, the general computation scheme can be represented by:

$$K_n \ V_n = F_n \quad \text{for } n = 0, 1, 2 \ldots$$
The solution for the velocity can be obtained after the global finite element equation is solved.

For the integral-type KBKZ model, the general computation scheme is as follows:

For \( n = 0 \),

\[
\sigma_0 = 2\eta_0 d
\]

\[
K_0 V_0 = F_0
\]

For \( n = 1, 2, \ldots \),

(a) \[
\sigma_n = \sum_k \sum_{i=0}^{n_i} A_i^k h \left( I_{C^{-1}}, II_{C^{-1}} \right) C^{-1} (T_i^k)
\]

(b) \[
K_n V_n = F_n
\]

This algorithm starts with the initial Newtonian stress \( \sigma_0 \) (called initial guess), and solves the governing equation (3.158) for Newtonian flow problem to obtain an initial value of velocity field \( V_0 \). Using the information on the preliminary velocity, the extra stress force \( \sigma \) can be calculated from Eq. (3.195), and subsequently the total body force vector \( F \) in the global finite element Eq. (3.196) can be constructed, enabling the calculation of velocity using Eq. (3.158). Once the new velocity field is found, the above procedures will be repeated until the convergence of both velocity and stress is reached. The flow diagram for the power-law model is given in Figure 3.7 and the flow diagram for the KBKZ model is given in Figure 3.8.
Figure 3.7 Finite element analysis flow diagram for drop deformation for the power-law model.
Figure 3.8  Finite element analysis flow diagram for drop deformation for the KBKZ model.
CHAPTER IV  
COMPUTATIONAL RESULTS AND DISCUSSION

4.1 Introduction

In this chapter we present the computational results describing the extent of deformation of a drop moving along the centerline of a cylindrical tube in the entrance region as affected by the rheological parameters and flow conditions. As described in the previous chapter, for the computations we employed the finite element codes developed in this study, which is based on the integral-type KBKZ constitutive equation. In order to facilitate the presentation of our computational results, below we will first summarize very briefly the main feature of the KBKZ model and then the numerical values of the rheological parameters in the KBKZ model, which were chosen for our numerical computations. Then we will present the rationale behind the choices of the numerical values of the rheological parameters.

Specifically, we employed the following form of the KBKZ model (see Chapter 3 for the details)

\[
\sigma(t) = \int_{-\infty}^{t} M(t, t') h(I_{c^{-1}}, I_{c^{-1}}) C^{-1}(t, t') \, dt' \quad (4.1)
\]
where $M(t, t')$ is a memory function, $I_{C^{-1}}$ and $II_{C^{-1}}$ are the first and second invariants of the Finger strain tensor $C^{-1}(t, t')$, and $h(I_{C^{-1}}, II_{C^{-1}})$ is the strain damping function. In general, the memory function $M(t, t')$ is expressed by a series of exponential functions:

$$M(t-t') = \sum_{k=1}^{N} \frac{a_k}{\lambda_k} \exp\left(-\frac{t-t'}{\lambda_k}\right)$$  \hspace{1cm} (4.2)

where $\lambda_k$ and $a_k$ are the relaxation time and relaxation modulus coefficient, respectively, at a reference temperature $T_0$. The strain damping function $h(I_{C^{-1}}, II_{C^{-1}})$ has been presented in many different forms; one form introduced by Papanastasiou et al.\footnote{Papanastasiou et al.} is given by

$$h(I_{C^{-1}}, II_{C^{-1}}) = \frac{\alpha}{(\alpha - 3) + \beta I_{C^{-1}} + (1-\beta) II_{C^{-1}}}$$  \hspace{1cm} (4.3)

where $\alpha$ and $\beta$ are material parameters. According to the literature,\footnote{Papanastasiou et al.} values of $\alpha$ may be determined from stress relaxation experiments and values of $\beta$ may be determined from uniaxial and biaxial extensional flow experiments.

In view of the fact that there is a scarcity of experimental data available in the literature over a wide range of the parameters ($\lambda_k$, $a_k$, $\alpha$, and $\beta$) appearing in Eqs. (4.2) and (4.3), in this study we chose eight numerical values of $\lambda_k$ and $a_k$ (with $k$ ranging from 1 to 8) for a low-density polyethylene (LDPE) at 150 °C as given in Table 4.1, which are taken from the literature,\footnote{Papanastasiou et al.} and we chose $\alpha = 14.38$ and $\beta = 0.018$ which are also taken from the literature.\footnote{Papanastasiou et al.} Then our strategy in carrying out numerical computations to investigate the extent of deformation of a viscoelastic drop suspended in a viscoelastic medium, while moving along the centerline of a cylindrical tube in the
Table 4.1  Numerical values of $\lambda_k$ and $a_k$ appealing in the KBKZ model with $\alpha = 14.38$ and $\beta = 0.018$ for a low-density polyethylene (LDPE) at 150 °C,\(^\text{86,129}\) representing the suspending medium in the present study.

<table>
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<th>$k$</th>
<th>$\lambda_k$ (s)</th>
<th>$a_k$ (Pa)</th>
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<td>129000</td>
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</tbody>
</table>

entrance region, was to regard the KBKZ model having the numerical values of $\lambda_k$ and $a_k$ given in Table 4.1 as representing the suspending medium. Figure 4.1 gives logarithmic plots of shear viscosity ($\eta$) versus shear rate ($\dot{\gamma}$) and first normal stress difference ($N_1$) versus $\dot{\gamma}$ for the suspending medium in steady-state shear flow, which were predicted from Eq. (4.1) using the numerical values of $\lambda_k$ and $a_k$ given in Table 4.1. As mentioned above, throughout this study the values of $\alpha$ and $\beta$ appearing in Eq. (4.3) were fixed as $\alpha = 14.38$ and $\beta = 0.018$.

For the reason that there are too many parameters associated with Eq. (4.1), we varied only the values of $\lambda_k$ appearing in Eq. (4.2) to represent the drop which is suspended in a viscoelastic medium whose rheological parameters are given in Table 4.1.
Figure 4.1 Logarithmic plots of $\eta$ versus $\dot{\gamma}$ (▲) and $N_1$ versus $\dot{\gamma}$ (▼) for a viscoelastic fluid which were predicted from the KBKZ model having the values of $\lambda_k$ and $a_k$ given in Table 4.1 for the memory function $M(t-t')$ and $\alpha = 14.38$ and $\beta = 0.018$. In this study this fluid is regarded as being the suspending medium.

With this strategy we considered five different cases (Case I through Case V) from the view point of the rheological parameter $\lambda_k$. The rationale behind the choice of the above strategy lies in that $\lambda_k$ represents the relaxation time of a viscoelastic fluid. Further our preliminary calculations indicated that among the parameters associated with Eqs. (4.1)–(4.3) the parameter $\lambda_k$ was found to be most sensitive to the prediction of
viscoelastic properties. Table 4.2 gives a summary of the numerical values of $\lambda_k$ for five different cases (Case I through Case V) considered in the numerical calculations conducted in this study.

For comparison, in this study we also conducted numerical computations for the extent of deformation of a Newtonian drop suspended in a Newtonian medium moving along the centerline of a cylindrical tube in the entrance region. For such purposes we calculated the zero-shear viscosity $\eta_0$ of both the drop and the suspending medium using the expression:

$$\eta_0 = \sum_{k=1}^{8} a_k \lambda_k$$  \hspace{1cm} (4.4)

Table 4.2 Numerical values of $\lambda_k$ appearing in the memory function $M(t-t')$ defined by Eq. (4.2) for five different cases, each representing a drop deforming in the same suspending medium.

<table>
<thead>
<tr>
<th>k</th>
<th>$\lambda_k$ (s) for Case I</th>
<th>$\lambda_k$ (s) for Case II</th>
<th>$\lambda_k$ (s) for Case III</th>
<th>$\lambda_k$ (s) for Case IV</th>
<th>$\lambda_k$ (s) for Case V</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0005</td>
<td>0.00025</td>
<td>0.00011</td>
<td>0.00005</td>
<td>0.000025</td>
</tr>
<tr>
<td>2</td>
<td>0.005</td>
<td>0.0025</td>
<td>0.0011</td>
<td>0.0005</td>
<td>0.00025</td>
</tr>
<tr>
<td>3</td>
<td>0.05</td>
<td>0.025</td>
<td>0.011</td>
<td>0.005</td>
<td>0.0025</td>
</tr>
<tr>
<td>4</td>
<td>0.5</td>
<td>0.25</td>
<td>0.11</td>
<td>0.05</td>
<td>0.025</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>2.5</td>
<td>1.1</td>
<td>0.5</td>
<td>0.25</td>
</tr>
<tr>
<td>6</td>
<td>50</td>
<td>25</td>
<td>11</td>
<td>5</td>
<td>2.5</td>
</tr>
<tr>
<td>7</td>
<td>500</td>
<td>250</td>
<td>110</td>
<td>50</td>
<td>25</td>
</tr>
<tr>
<td>8</td>
<td>5000</td>
<td>2500</td>
<td>1100</td>
<td>500</td>
<td>250</td>
</tr>
</tbody>
</table>
Later, the values of $\eta_0$ thus calculated were used to investigate the deformation of a Newtonian drop suspended in a Newtonian suspending medium, the results of which will be presented later in this chapter.

Basically, there are three fundamental physical parameters which affect the extent of deformation of a viscoelastic drop suspended in a viscoelastic medium moving along the centerline of a cylindrical tube in the entrance region; they are (1) the ratio of relaxation time of the drop and the suspending medium, (2) the initial size of drop, and (3) the apparent shear rate. Since the entrance region flow accompanies an acceleration of a fluid, we anticipate that the drop in the entrance region will keep elongated along the centerline of a cylindrical tube. Therefore the prediction, via finite element method (FEM), of the shape of a drop moving along the centerline of a cylindrical tube in the entrance region is not a trivial matter because it is a moving boundary problem, thus requiring a continuous remeshing of the finite elements chosen during the numerical computation. In spite of some difficulties encountered initially, in this study we have successfully completed our numerical computations via FEM. In this chapter we present the highlights of our numerical computations and the details of the computational results are presented in the Appendix B of this dissertation.

4.2 Comparison of the Steady-State Shear Flow Properties of the Drop with Those of the Suspending Medium as Predicted from the KBKZ Model.

Figure 4.2 gives logarithmic plots of $\eta$ versus $\dot{\gamma}$ (▲, △) and $N_1$ versus $\dot{\gamma}$ (▼, ▽) in steady-state shear flow predicted from Eq. (4.1) for the suspending medium (filled
Figure 4.2 Logarithmic plots of $\eta$ versus $\dot{\gamma}$ ($\blacktriangle$, $\triangle$) and $N_1$ versus $\dot{\gamma}$ ($\blacklozenge$, $\blacktriangledown$) for the suspending medium (the filled symbols) and the drop (the open symbols), Case I, which were predicted from the KBKZ model using the numerical values of $\lambda_k$ and $a_k$ given in Table 4.1 for the suspending medium and the numerical values of $\lambda_k$ given in Table 4.2 for the drop Case I.

Symbols) using the numerical value of the parameters given in Table 4.1 and for the drop Case I (open symbols), using the numerical value of $\lambda_k$ given in Table 4.2. It should be mentioned that the numerical values of $\lambda_k$, $a_k$, $\alpha$, and $\beta$ employed for the suspending medium are the same as those given in Table 4.1. It is seen in Figure 4.2 that the drop
(open symbols) has higher values of \( \eta \) and \( N_1 \) than the suspending medium (filled symbols) over the entire range of \( \dot{\gamma} \) considered, which is attributable to the fact that the relaxation times \( \lambda_k \) for the drop (Case I in Table 4.2) are larger than those for the suspending medium (see Tables 4.1).

Figure 4.3 gives logarithmic plots of \( \eta \) versus \( \dot{\gamma} \) (▲, △) and \( N_1 \) versus \( \dot{\gamma} \) (▼, ▽) in steady-state shear flow predicted from Eq. (4.1) for the suspending medium (filled

![Figure 4.3 Logarithmic plots of \( \eta \) versus \( \dot{\gamma} \) (▲, △) and \( N_1 \) versus \( \dot{\gamma} \) (▼, ▽) for the suspending medium (the filled symbols) and the drop Case II (the open symbols), which were predicted from the KBKZ model using the numerical values of \( \lambda_k \) and \( a_k \) given in Table 4.1 for the suspending medium and the numerical values of \( \lambda_k \) given in Table 4.2 for the drop Case II.](image.png)
symbols) using the numerical value of the parameters given in Table 4.1 and for the drop
Case II (open symbols), using the numerical value of \( \lambda_k \) given in Table 4.2. It is seen in
Figure 4.3 that the drop (open symbols) has higher values of \( \eta \) and \( N_1 \) than the
suspending medium over the entire range of \( \dot{\gamma} \) considered, which is attributable to the
fact that the relaxation times \( \lambda_k \) for the drop (Case II in Table 4.2) are larger than those
for the suspending medium (see Table 4.1).

Figure 4.4 gives logarithmic plots of \( \eta \) versus \( \dot{\gamma} \) (▲, △) and \( N_1 \) versus \( \dot{\gamma} \) (▼, ▽) in
steady-state shear flow predicted from Eq. (4.1) for the suspending medium (filled
symbols) using the numerical value of the parameters given in Table 4.1 and the drop
Case III (open symbols), using the numerical value of \( \lambda_k \) given in Table 4.2. It is seen in
Figure 4.4 that the drop (open symbols) and the suspending medium (filled symbols)
have virtually the values of \( \eta \) and \( N_1 \) over the entire range of \( \dot{\gamma} \) considered, which is
attributable to the fact that the drop and suspending medium have virtually the same
value of relaxation times \( \lambda_k \) (see Tables 4.1 and 4.2).

Figure 4.5 gives logarithmic plots of \( \eta \) versus \( \dot{\gamma} \) (▲, △) and \( N_1 \) versus \( \dot{\gamma} \) (▼, ▽) in
steady-state shear flow predicted from Eq. (4.1) for the suspending medium (filled
symbols) using the numerical value of the parameters given in Table 4.1 and the drop
Case IV (open symbols), using the numerical value of \( \lambda_k \) given in Table 4.2. It is seen in
Figure 4.5 that the drop has lower values of \( \eta \) and \( N_1 \) than the suspending medium over
the entire range of \( \dot{\gamma} \) tested, which is attributable to the fact that the relaxation times \( \lambda_k \)
for the drop (Case IV in Table 4.2) are smaller than those for the suspending medium (see
Tables 4.1).
Figure 4.4 Logarithmic plots of $\eta$ versus $\dot{\gamma}$ ($\triangle, \Delta$) and $N_1$ versus $\dot{\gamma}$ ($\blacktriangle$, $\blacktriangledown$) for the suspending medium (the filled symbols) and the drop Case III (the open symbols), which were predicted from the KBKZ model using the numerical values of $\lambda_k$ and $a_k$ given in Table 4.1 for the suspending medium and the numerical values of $\lambda_k$ given in Table 4.2 for the drop Case III.

Figure 4.6 gives logarithmic plots of $\eta$ versus $\dot{\gamma}$ ($\blacktriangle, \blacktriangledown$) and $N_1$ versus $\dot{\gamma}$ ($\blacktriangle$, $\blacktriangledown$) in steady-state shear flow for the suspending medium (filled symbols) predicted from Eq. (4.1) using the numerical value of the parameters given in Table 4.1 and the drop Case V (open symbols), using the numerical value of $\lambda_k$ given in Table 4.2. It is seen in
Figure 4.5 Logarithmic plots of $\eta$ versus $\dot{\gamma}$ (▲, △) and N$_1$ versus $\dot{\gamma}$ (▼, ▽) for the suspending medium (the filled symbols) and the drop Case IV (the open symbols), which were predicted from the KBKZ model using the numerical values of $\lambda_k$ and $a_k$ given in Table 4.1 for the suspending medium and the numerical values of $\lambda_k$ given in Table 4.2 for the drop Case IV.

Figure 4.6 the drop has lower values of $\eta$ and N$_1$ than the suspending medium over the entire range of $\dot{\gamma}$ considered, which is attributable to the fact that the relaxation times $\lambda_k$ for the drop (Case V in Table 4.2) are smaller than those for the suspending medium (see Table 4.1).
Figure 4.6 Logarithmic plots of $\eta$ versus $\dot{\gamma}$ ($\blacktriangle$, $\blacktriangledown$) and $N_1$ versus $\dot{\gamma}$ ($\blacktriangleleft$, $\blacktriangleright$) for the suspending medium (the filled symbols) and the drop Case V (the open symbols), which were predicted from the KBKZ model using the numerical values of $\lambda_k$ and $a_k$ given in Table 4.1 for the suspending medium and the numerical values of $\lambda_k$ given in Table 4.2 for the drop Case V.

4.3 The Deformation of a Viscoelastic Drop Suspended in a Viscoelastic Medium Moving along the Centerline of a Cylindrical Tube in the Entrance Region

In the numerical computations we considered (i) the radius ($R_0$) of a cylindrical tube having 3.0 mm, (ii) three different values (0.3, 0.45, and 0.6 mm) of the radius ($r_0$) of the
initial drop, and (iii) three different values (15.4, 35.4, and 75.4 s\(^{-1}\)) of apparent shear rate (\(\dot{\gamma}_{\text{app}}\)) based on the radius (R_0) of the cylindrical tube.

### 4.3.1 Effects of the Relaxation Times of the Drop and Suspending Medium on the Extent of Deformation of a Viscoelastic Drop Suspended in a Viscoelastic Medium Moving along the Centerline of a Cylindrical Tube in the Entrance Region

Figure 4.7 gives snapshots of the computed shapes of a viscoelastic drop (designated as *Case I* in Table 4.2) with the radius (r_0) of 0.45 mm at four different positions (A, B, C, and D) along the centerline of a cylindrical tube in the entrance region traveling at \(\dot{\gamma}_{\text{app}} = 35.4\) s\(^{-1}\). Note that the drop was suspended in the viscoelastic medium whose relaxation times are given in Table 4.1. Since the size of drop was so small that the enlarged shapes of the drop at four different position in the entrance region of the cylindrical tube are given at the bottom of the flow channel. It is clearly seen in Figure 4.7 that the drop elongates continuously as it moves towards the tube entrance and recoils slightly after it passed the tube inlet. Figure 4.8 gives snapshots of the computed shapes of a viscoelastic drop (designated as *Case II* in Table 4.2), Figure 4.9 gives snapshots of the computed shapes of a viscoelastic drop (designated as *Case III* in Table 4.2), Figure 4.10 gives snapshots of the computed shapes of a viscoelastic drop (designated as *Case IV* in Table 4.2), and Figure 4.11 gives snapshots of the computed shapes of a viscoelastic drop (designated as *Case V* in Table 4.2), under the otherwise identical initial drop size (r_0 = 0.45 mm) and flow conditions (at \(\dot{\gamma}_{\text{app}} = 35.4\) s\(^{-1}\)) as those in Figure 4.7.
It is seen in Figures 4.7–4.11 that the extent of drop deformation increases as the drop travels from the upstream side of the cylindrical tube to its entrance and further inside the tube. This observation is in good qualitative agreement with the experimental results of Chin and Han.\textsuperscript{15}

Figure 4.7 Snap shots of the computed shapes of a viscoelastic drop (Case I) with the initial radius ($r_0$) of 0.45 mm and the numerical values of $\lambda_k$ given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$, with the values of $\lambda_k$ and $a_k$ given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 1.148$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.
Figure 4.8  Snap shots of the computed shapes of a viscoelastic drop (*Case II*) with the initial radius \((r_0)\) of 0.45 mm and the numerical values of \(\lambda_k\) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \(\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}\), with the values of \(\lambda_k\) and \(a_k\) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \((z/R_0)\) along the centerline of the cylindrical tube in the entrance region: the position A at \(z/R_0 = -10\), the position B at \(z/R_0 = -5\), the position C at \(z/R_0 = 1.193\), and the position D at \(z/R_0 = 5\) with \(R_0\) being the radius (3 mm) of the cylindrical tube.
Figure 4.9 Snap shots of the computed shapes of a viscoelastic drop (Case III) with the initial radius \( r_0 \) of 0.45 mm and the numerical values of \( \lambda_k \) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \( \dot{\gamma}_{\text{app}} = 35.4 \, \text{s}^{-1} \), with the values of \( \lambda_k \) and \( a_k \) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \( (z/R_0) \) along the centerline of the cylindrical tube in the entrance region: the position A at \( z/R_0 = -10 \), the position B at \( z/R_0 = -5 \), the position C at \( z/R_0 = 1.238 \), and the position D at \( z/R_0 = 5 \) with \( R_0 \) being the radius (3 mm) of the cylindrical tube.
Figure 4.10 Snap shots of the computed shapes of a viscoelastic drop (Case IV) with the initial radius \( r_0 \) of 0.45 mm and the numerical values of \( \lambda_k \) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \( \dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1} \), with the values of \( \lambda_k \) and \( a_k \) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \( z/R_0 \) along the centerline of the cylindrical tube in the entrance region: the position A at \( z/R_0 = -10 \), the position B at \( z/R_0 = -5 \), the position C at \( z/R_0 = 1.286 \), and the position D at \( z/R_0 = 5 \) with \( R_0 \) being the radius (3 mm) of the cylindrical tube.
Figure 4.11  Snap shots of the computed shapes of a viscoelastic drop (Case V) with the initial radius ($r_0$) of 0.45 mm and the numerical values of $\lambda_k$ given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$, with the values of $\lambda_k$ and $a_k$ given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 1.337$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.
Figure 4.12 compares the extent of deformation of five drops (designated as Case I, Case II, Case III, Case IV, and Case V) having different relaxation times $\lambda_k$, which are given in Table 4.2, at the same axial position $z/R_0 = 5$ which is slightly inside the cylindrical tube. Note that all five drops in Figure 4.12 have the same initial drop radius ($r_0 = 0.45$ mm), and each drop is deformed in the same suspending medium, the numerical values of its physical parameters being given in Table 4.1, and the suspending medium flows through the entrance region of a cylindrical tube at an apparent shear rate of 35.4 s$^{-1}$. It is clearly seen in Figure 4.12 that the extent of drop deformation becomes greater as the ratio of the relaxation times ($\lambda_k$) of the drop and the suspending medium decreases from 5 to 0.25 (compare the values of $\lambda_k$ given for each drop given in Table 4.2 with those for the suspending medium given in Table 4.1). It should be remembered that in Figures 4.2–4.6 the steady-state shear viscosity ($\eta$) and first normal stress difference ($N_1$) of each drop are compared with those of the suspending medium. Referring to Figures 4.2–4.6, at present it is difficult to conclude which of the two rheological properties, $\eta$ or $N_1$, of the drop plays a predominant role in controlling the extent of drop deformation.

In order to have quantitative comparison of the extent of drop deformation among the five drops investigated, we calculated the axial ratio ($a/b$) of the long side ($a$) and short side ($b$) of the deformed drop, as depicted schematically below. Table 4.3 gives a summary of the calculated values of the axial ratio ($a/b$) for the five drops at four different positions (A, B, C, and D) in the entrance region of the cylindrical tube, showing a very clear trend for the extent of drop deformation as affected by the ratio of
Figure 4.12 Snap shots of the computed shapes of five different viscoelastic drops (Case I through Case V) with the identical initial radius \( r_0 \) of 0.45 mm and the numerical values of \( \lambda_k \) given in Table 4.2 when each of the drops was suspended in the same viscoelastic medium, moving at \( \dot{\gamma}_{app} = 35.4 \) s\(^{-1} \), with the values of \( \lambda_k \) and \( a_k \) given in Table 4.1. The snap shots of the computed shape of the drops are at the same dimensionless axial position of \( z/R_0 = 5 \) in the entrance region of a cylindrical tube for different drops: (a) Case I, (b) Case II, (c) Case III, (d) Case IV, and (e) Case V.
the relaxation times of the drop and the suspending medium. The computed shapes of
the drops presented in Figures 4.7–4.11 and summarized in Table 4.3 are in good
qualitative agreement with the experimental results of Chin and Han\textsuperscript{15}, which were
published three decades ago.

![Diagram of an ellipse with axes labeled a and b]

Table 4.3 The axial ratio $a/b$ of five different viscoelastic drops (designated as \textit{Case I},
\textit{Case II}, \textit{Case III}, \textit{Case IV}, and \textit{Case V} in Table 4.2) with the initial radius ($r_0$) of 0.45
mm, each suspended in a viscoelastic medium (see Table 4.1 for the physical parameters
appearing in Eqs. (4.1)–(4.3)), moving along the centerline of a cylindrical tube in the
entrance region at $\dot{\gamma}_{\text{app.}} = 35.4$ s$^{-1}$. The values of $a/b$ ratio listed here are determined
from the calculated drop shapes, given in Figures 4.7–4.11, at different positions ($z/R_0$)
along the centerline a cylindrical tube in the entrance region.

<table>
<thead>
<tr>
<th>Drop</th>
<th>The $a/b$ ratio of a drop at the axial position A, $z/R_0 = -10$</th>
<th>The $a/b$ ratio of a drop at the axial position B, $z/R_0 = -5$</th>
<th>The $a/b$ ratio of a drop at the axial position C</th>
<th>The $a/b$ ratio of a drop at the axial position D, $z/R_0 = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I</td>
<td>1.572</td>
<td>1.924</td>
<td>4.341 ($z/R_0 = 1.148$)</td>
<td>4.209</td>
</tr>
<tr>
<td>Case II</td>
<td>1.693</td>
<td>2.302</td>
<td>7.907 ($z/R_0 = 1.193$)</td>
<td>7.633</td>
</tr>
<tr>
<td>Case III</td>
<td>1.838</td>
<td>2.741</td>
<td>11.989 ($z/R_0 = 1.238$)</td>
<td>11.543</td>
</tr>
<tr>
<td>Case IV</td>
<td>2.263</td>
<td>3.613</td>
<td>16.924 ($z/R_0 = 1.286$)</td>
<td>16.153</td>
</tr>
<tr>
<td>Case V</td>
<td>2.997</td>
<td>5.517</td>
<td>21.241 ($z/R_0 = 1.337$)</td>
<td>20.139</td>
</tr>
</tbody>
</table>
4.3.2 Effects of Initial Drop Size on the Extent of Deformation of a Viscoelastic Drop Suspended in a Viscoelastic Medium Moving along the Centerline of a Cylindrical Tube in the Entrance Region

We also computed the shapes of a viscoelastic drop at various positions along the centerline of a cylindrical tube in the entrance region by varying the initial radius of a drop, each suspended in the viscoelastic medium whose relaxation times are given in Table 4.1. Table 4.4 gives a summary of the effect of the initial radius of a drop on the axial ratio \(a/b\) of the drop (designated as Case III in Table 4.2) at three different positions (B, C, and D) in the entrance region of the cylindrical tube. The details of the computed shapes of the drops are presented in the Appendix B of this dissertation. It is clearly seen in Table 4.4 that the extent of drop deformation increases as the initial drop radius is increased. Again, this observation is in good qualitative agreement with the

Table 4.4 The axial ratio \(a/b\) of viscoelastic drops (designated as Case III in Table 4.2) with three different values of initial radius (0.3, 0.45, and 0.6 mm), each suspended in a viscoelastic medium (see Table 4.1 for the physical parameters appearing in Eqs. (4.1)–(4.3)), moving along the centerline of a cylindrical tube in the entrance region at \(\dot{\gamma}_{\text{app.}} = 35.4 \text{ s}^{-1}\). The values of \(a/b\) ratio listed here are determined from the calculated drop shapes, which are given in the Appendix B of this dissertation, at different positions along the centerline of a cylindrical tube in the entrance region.

<table>
<thead>
<tr>
<th>Initial drop radius (mm)</th>
<th>The (a/b) ratio of a drop at the axial position B, (z/R_0 = -5)</th>
<th>The (a/b) ratio of a drop at the axial position C</th>
<th>The (a/b) ratio of a drop at the axial position D, (z/R_0 = 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.30</td>
<td>2.420</td>
<td>9.365 ((z/R_0 = 1.211))</td>
<td>9.009</td>
</tr>
<tr>
<td>0.45</td>
<td>2.741</td>
<td>11.989 ((z/R_0 = 1.238))</td>
<td>11.543</td>
</tr>
<tr>
<td>0.60</td>
<td>2.993</td>
<td>16.448 ((z/R_0 = 1.258))</td>
<td>15.823</td>
</tr>
</tbody>
</table>
experimental results of Chin and Han.15

4.3.3 Effects of Apparent Shear Rate on the Extent of Deformation of a Viscoelastic Drop Suspended in a Viscoelastic Medium Moving along the Centerline of a Cylindrical Tube in the Entrance Region

Further we computed the shapes of a viscoelastic drop at various positions along the centerline of a cylindrical tube in the entrance region as affected by apparent shear rate for the five drops whose relaxation times are given in Table 4.2. Table 4.5 gives a summary of the effect of apparent shear rate on the axial ratio (a/b) of the drop (designated as Case II in Table 4.2) at three different positions (B, C, and D) in the entrance region of the cylindrical tube. The details of the computed shapes of the drops are presented in the Appendix B of this dissertation. It is clearly seen in Table 4.5 that the extent of drop deformation increases as the apparent shear rate is increased. Again, this observation is in good qualitative agreement with the experimental results of Chin and Han 15.

4.4 The Deformation of a Newtonian Drop Suspended in a Newtonian Medium Moving along the Centerline of a Cylindrical Tube in the Entrance Region

For comparison, we also conducted numerical computations for the extent of deformation of a Newtonian drop suspended in a Newtonian medium moving along the centerline of a cylindrical tube in the entrance region. In the numerical computations we replaced the parameters appearing in the KBKZ fluid with zero-shear viscosity (η₀) (see
Table 4.5  The axial ratio $a/b$ of viscoelastic drops (designated as Case II in Table 4.2) with the initial radius of 0.45 mm, suspended in a viscoelastic medium (see Table 4.1 for the physical parameters appearing in Eqs. (4.1)–(4.3)), moving along the centerline of a cylindrical tube in the entrance region at three different shear rates, $\dot{\gamma}_{app} = 15.4$, 35.4, and 75.4 s$^{-1}$. The values of $a/b$ ratio listed here are determined from the calculated shapes of drops, which are given in the Appendix B of this dissertation, at different positions along the centerline of a cylindrical tube in the entrance region.

| Apparent shear rate ($s^{-1}$) | The $a/b$ ratio of a drop at the axial position B, $z/R_0 = -5$ | The $a/b$ ratio of a drop at the axial position C | The $a/b$ ratio of a drop at the axial position D, $z/R_0 = 5$ |
|--------------------------------|--------------------------------------------------|--------------------------------------------------|
| 15.4                          | 1.680                                            | 5.397 ($z/R_0 = 1.182$)                          | 5.193                                            |
| 35.4                          | 2.302                                            | 7.907 ($z/R_0 = 1.193$)                          | 7.633                                            |
| 75.4                          | 3.180                                            | 11.458 ($z/R_0 = 1.205$)                         | 11.060                                           |

Eq. (4.4)) with other parameters fixed, the radius ($R_0$) of a cylindrical tube being 3.0 mm and the radius ($r_0$) of the initial drop being 0.45 mm, and we considered three different values (15.4 s$^{-1}$, 35.4 s$^{-1}$, and 75.4 s$^{-1}$) of apparent shear rate ($\dot{\gamma}_{app}$).

4.4.1 Effects of the Viscosity Ratio of the Drop and Suspending Medium on the Extent of Deformation of a Newtonian Drop Suspended in a Newtonian Medium Moving along the Centerline of a Cylindrical Tube in the Entrance Region

For the computation, we considered five values of zero viscosity ($\eta_{0m}$) for the suspending medium, thus giving rise to five pairs of drop and medium. Table 4.6 gives
Table 4.6  Numerical values of zero-shear viscosity of the drop and suspending medium for Case VI through Case X.

<table>
<thead>
<tr>
<th>Drop</th>
<th>Zero-shear viscosity of the drop, $\eta_{0d}$ (Pa s)</th>
<th>Zero-shear viscosity of the suspending medium, $\eta_{0m}$ (Pa s)</th>
<th>$\eta_{0d}/\eta_{0m}$ ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case VI</td>
<td>255315</td>
<td>51063</td>
<td>5.0</td>
</tr>
<tr>
<td>Case VII</td>
<td>127657</td>
<td>51063</td>
<td>2.5</td>
</tr>
<tr>
<td>Case VIII</td>
<td>56169</td>
<td>51063</td>
<td>1.1</td>
</tr>
<tr>
<td>Case IX</td>
<td>25532</td>
<td>51063</td>
<td>0.5</td>
</tr>
<tr>
<td>Case X</td>
<td>12766</td>
<td>51063</td>
<td>0.25</td>
</tr>
</tbody>
</table>

the numerical values of $\eta_{0d}$ and $\eta_{0m}$ for Case VI, Case VII, Case VIII, Case IX, and Case X, respectively. It should be mentioned that the values of $\eta_{0d}$ and $\eta_{0d}$ given in Table 4.6 were calculated using Eq. (4.4). Thus, below we will compare the extent of drop deformation between Case VI and Case I, between Case VII and Case II, between Case VIII and Case III, between Case IX and Case IV, and between Case X and Case V.

Figure 4.13 gives snap shots of the computed shapes of a Newtonian drop corresponding to Case VI (see Table 4.6) with the radius ($r_0$) of 0.45 mm at four different positions (A, B, C, and D) along the centerline of a cylindrical tube in the entrance region traveling at $\dot{\gamma}_{\text{app}} = 35.4$ s$^{-1}$. Note that the drop was suspended in the Newtonian medium whose viscosity is given in Table 4.6. It is clearly seen in Figure 4.13 that the drop elongates continuously as it moves towards the tube entrance and recoils after it passed the tube inlet. Figure 4.14 gives snap shots of the computed shapes of a Newtonian drop corresponding to Case VII (see Table 4.6), Figure 4.15 gives snap shots of the computed
Figure 4.13 Snap shots of the computed shapes of a Newtonian drop (Case VI) with the initial radius \( r_0 \) of 0.45 mm when the drop was suspended in the Newtonian medium, moving at \( \dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1} \), with the viscosity of the drop \( \eta_0d \) and the medium \( \eta_0m \) given in Table 4.6. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \( z/R_0 \) along the centerline of the cylindrical tube in the entrance region: the position A at \( z/R_0 = -10 \), the position B at \( z/R_0 = -5 \), the position C at \( z/R_0 = 0.146 \), and the position D at \( z/R_0 = 5 \) with \( R_0 \) being the radius (3 mm) of the cylindrical tube.
Figure 4.14 Snap shots of the computed shapes of a Newtonian drop (Case VII) with the initial radius \(r_0\) of 0.45 mm when the drop was suspended in the Newtonian medium, moving at \(\dot{\gamma}_{\text{app}} = 35.4\) s\(^{-1}\), with the viscosity of the drop \(\eta_0d\) and the medium \(\eta_0m\) given in Table 4.6. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \(z/R_0\) along the centerline of the cylindrical tube in the entrance region: the position A at \(z/R_0 = -10\), the position B at \(z/R_0 = -5\), the position C at \(z/R_0 = 0.148\), and the position D at \(z/R_0 = 5\) with \(R_0\) being the radius (3 mm) of the cylindrical tube.
Figure 4.15 Snap shots of the computed shapes of a Newtonian drop (Case VIII) with the initial radius ($r_0$) of 0.45 mm when the drop was suspended in the Newtonian medium, moving at $\dot{\gamma}_{\text{app}} = 35.4$ s$^{-1}$, with the viscosity of the drop $\eta_{0d}$ and the medium $\eta_{0m}$ given in Table 4.6. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 0.150$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.
shapes of a Newtonian drop corresponding to Case VIII (see Table 4.6), Figure 4.16 gives
snap shots of the computed shapes of a Newtonian drop corresponding to Case IX (see
Table 4.6), and Figure 4.17 gives snap shots of the computed shapes of a Newtonian drop
 corresponding to Case X (see Table 4.6), each drop having the same initial size of 0.45
mm and being suspended in the same medium flowing at $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$. It is seen in
Figures 4.13–4.17 that the extent of drop deformation increases as the drop travels from
the upstream side of the cylindrical tube to its entrance and then decreases after it passes
the tube inlet.

Figure 4.18 compares the extent of deformation of five separate drops
(corresponding to Case VI, Case VII, Case VIII, Case IX, and Case X) having different
values of $\eta_0d$ (see Table 4.6), at the same axial position $z/R_0 = 5$ which is slightly inside
the cylindrical tube. Note that all five drops in Figure 4.18 have the same initial drop
radius ($r_0 = 0.45 \text{ mm}$), and each drop is deformed in the same suspending medium, whose
viscosity is given in Table 4.6, and the suspending medium flows through the entrance
region of a cylindrical tube at an apparent shear rate of $35.4 \text{ s}^{-1}$. It is clearly seen in
Figure 4.18 that the extent of drop deformation becomes greater as the viscosity ratio of
the drop and the suspending medium ($\eta_{0d}/\eta_{0m}$) decreases from 5 to 0.25 (see Table 4.6).

In order to have quantitative comparison in the extent of drop deformation among
the five drops investigated, we calculated the axial ratio ($a/b$) of the long side ($a$) and
short side ($b$) of the deformed drop. Table 4.7 gives a summary of the calculated values
of the axial ratio ($a/b$) for the five drops at four different positions (A, B, C, and D) in the
entrance region of the cylindrical tube, showing a very clear trend for the extent of drop
Figure 4.16 Snap shots of the computed shapes of a Newtonian drop (Case IX) with the initial radius ($r_0$) of 0.45 mm when the drop was suspended in the Newtonian medium, moving at $\dot{\gamma}_{\text{app}} = 35.4$ s$^{-1}$, with the viscosity of the drop $\eta_{\text{od}}$ and the medium $\eta_{\text{om}}$ given in Table 4.6. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 0.152$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.
Figure 4.17 Snap shots of the computed shapes of a Newtonian drop (Case X) with the initial radius ($r_0$) of 0.45 mm when the drop was suspended in the Newtonian medium, moving at $\dot{\gamma}_{\text{app}} = 35.4$ s$^{-1}$, with the viscosity of the drop $\eta_{0d}$ and the medium $\eta_{0m}$ given in Table 4.6. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 0.153$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.
Figure 4.18 Snap shots of the computed shapes of five different Newtonian drops (*Case VI* through *Case X*) with the identical initial radius ($r_0$) of 0.45 mm and the viscosity $\eta_0$ given in Table 4.6 when each of the drops was suspended in the same Newtonian medium, moving at $\dot{\gamma}_{app} = 35.4$ s$^{-1}$, with the viscosity $\eta_{0m}$ given in Table 4.6. The snap shots of the computed shape of the drops are at the same dimensionless axial position of $z/R_0 = 5$ in the entrance region of a cylindrical tube for different drops: (a) *Case VI*, (b) *Case VII*, (c) *Case VIII*, (d) *Case IX*, and (e) *Case X*. 
Table 4.7 The axial ratio $a/b$ of five different Newtonian drops (designated as Case VI, Case VII, Case VIII, Case IX, and Case X) with the initial radius ($r_0$) of 0.45 mm, each suspended in a Newtonian medium (see Table 4.6), moving along the centerline of a cylindrical tube in the entrance region at $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$. The values of $a/b$ ratio listed here are determined from the calculated drop shapes, given in Figures 4.13–4.17, at different positions along the centerline of a cylindrical tube in the entrance region.

<table>
<thead>
<tr>
<th>Drop</th>
<th>The $a/b$ ratio of a drop at the axial position A, $z/R_0 = -10$</th>
<th>The $a/b$ ratio of a drop at the axial position B, $z/R_0 = -5$</th>
<th>The $a/b$ ratio of a drop at the axial position C</th>
<th>The $a/b$ ratio of a drop at the axial position D, $z/R_0 = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case VI</td>
<td>1.628</td>
<td>1.990</td>
<td>4.188 ($z/R_0 = 0.146$)</td>
<td>3.272</td>
</tr>
<tr>
<td>Case VII</td>
<td>1.789</td>
<td>2.483</td>
<td>7.687 ($z/R_0 = 0.148$)</td>
<td>6.096</td>
</tr>
<tr>
<td>Case VIII</td>
<td>1.958</td>
<td>2.871</td>
<td>11.681 ($z/R_0 = 0.150$)</td>
<td>9.211</td>
</tr>
<tr>
<td>Case IX</td>
<td>2.443</td>
<td>3.901</td>
<td>16.523 ($z/R_0 = 0.152$)</td>
<td>13.080</td>
</tr>
<tr>
<td>Case X</td>
<td>3.268</td>
<td>6.044</td>
<td>20.725 ($z/R_0 = 0.153$)</td>
<td>16.510</td>
</tr>
</tbody>
</table>

decomposition as affected by the $\eta_{0d}/\eta_{0m}$ ratio. Namely, the extent of drop deformation increases dramatically as the $\eta_{0d}/\eta_{0m}$ is decreased from 5 to 0.25.

4.4.2 Effects of Apparent Shear Rate on the Extent of Deformation of a Newtonian Drop Suspended in a Newtonian Medium Moving along the Centerline of a Cylindrical Tube in the Entrance Region

We also computed the shapes of a Newtonian drop at various positions along the centerline of a cylindrical tube in the entrance region as affected by apparent shear rate for the five drops whose viscosities are given in Table 4.6. Table 4.8 gives a summary of
Table 4.8  The axial ratio \(a/b\) of Newtonian drops (designated as \textit{Case VII} in Table 4.6) with the initial radius of 0.45 mm, suspended in a Newtonian medium (see Table 4.6), moving along the centerline of a cylindrical tube in the entrance region at three different shear rates, \(\dot{\gamma}_{\text{app}} = 15.4, 35.4, \text{ and } 75.4 \text{ s}^{-1}\). The values of \(a/b\) ratio listed here are determined from the calculated shapes of drops, which are given in the Appendix B of this dissertation, at different positions along the centerline of a cylindrical tube in the entrance region.

<table>
<thead>
<tr>
<th>Apparent shear rate (s(^{-1}))</th>
<th>The (a/b) ratio of a drop at the axial position B, (z/R_0 = 5)</th>
<th>The (a/b) ratio of a drop at the axial position C</th>
<th>The (a/b) ratio of a drop at the axial position D, (z/R_0 = 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>15.4</td>
<td>1.736</td>
<td>5.302 ((z/R_0 = 0.137))</td>
<td>4.182</td>
</tr>
<tr>
<td>35.4</td>
<td>2.483</td>
<td>7.687 ((z/R_0 = 0.148))</td>
<td>6.096</td>
</tr>
<tr>
<td>75.4</td>
<td>3.287</td>
<td>11.228 ((z/R_0 = 0.162))</td>
<td>8.849</td>
</tr>
</tbody>
</table>

The effect of apparent shear rate on the axial ratio \((a/b)\) of the drop (designated as \textit{Case VII} in Table 4.6) at three different positions (B, C, and D) in the entrance region of the cylindrical tube. The details of the computed shapes of the drops are presented in the Appendix B of this dissertation. It is clearly seen in Table 4.8 that the extent of drop deformation increases as the apparent shear rate is increased.

4.5  The Effect of Fluid Elasticity on the Deformation of a Viscoelastic Drop and Subsequent Recoil in the Entrance Region of a Cylindrical Tube

In reference to the computed extent of drop deformation presented in Figures 4.7–4.11 for viscoelastic drops and in Figures 4.13–4.17 for Newtonian drops, here we
wish to elaborate on the significance of the results presented above, namely the effect of fluid elasticity on the extent of drop deformation followed by subsequent drop recoil in the entrance region of a cylindrical tube. Specifically, the following computational results are worth elaborating on: (1) the position at which a maximum drop deformation (i.e., the largest aspect ratio of a drop) occurs along the axis of a cylindrical tube, (2) the rate of increase in the extent of drop deformation until reaching the largest aspect ratio, and (3) the rate of drop recoil upon passing the position at which a maximum aspect ratio of drop occurs.

To help facilitate our discussion below, we wish to make the following observations on the deformation of viscoelastic drops given in Figures 4.7–4.11 and on the deformation of Newtonian drops given in Figures 4.13–4.17. Specifically we observe that the extent of a maximum deformation of a Newtonian drop (i.e., the largest value of aspect ratio) occurs at an axial position very close to the end of the converging section or very close to the entrance of the straight section (i.e., at $z/R_0 \approx 0$) of the cylindrical tube, while the extent of a maximum deformation of a viscoelastic drop occurs at an axial position slightly inside the straight section (i.e., at $z/R_0 > 1.0$) of the cylindrical tube; namely (i) compare Figure 4.13 for a Newtonian drop (Case VI) with Figure 4.7 for a viscoelastic drop (Case I), (ii) compare Figure 4.14 for a Newtonian drop (Case VII) with Figure 4.8 for a viscoelastic drop (Case II), (iii) compare Figure 4.15 for a Newtonian drop (Case VIII) with Figure 4.9 for a viscoelastic drop (Case III), (iv) compare Figure 4.16 for a Newtonian drop (Case IX) with Figure 4.10 for a viscoelastic drop (Case IV),
and (iv) compare Figure 4.17 for a Newtonian drop (Case X) with Figure 4.11 for a viscoelastic drop (Case V).

For quantitative comparison, Table 4.9 gives a summary of the calculated values of aspect ratio \( a/b \) of two deformed drops, a Newtonian drop (Case VI) suspended in a Newtonian medium and a viscoelastic drop (Case I) suspended in a viscoelastic medium, both having the same initial radius \( r_0 \) of 0.45 mm, moving along the centerline of a cylindrical tube in the entrance region at \( \dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1} \). The values of \( a/b \) ratio listed in

<table>
<thead>
<tr>
<th>Axial position ((z/R_0))</th>
<th>The (a/b) ratio in Case I</th>
<th>The (a/b) ratio in Case VI</th>
</tr>
</thead>
<tbody>
<tr>
<td>–5</td>
<td>1.922</td>
<td>1.989</td>
</tr>
<tr>
<td>–4</td>
<td>2.030</td>
<td>2.206</td>
</tr>
<tr>
<td>–3</td>
<td>2.149</td>
<td>2.483</td>
</tr>
<tr>
<td>–2</td>
<td>2.278</td>
<td>2.848</td>
</tr>
<tr>
<td>–1</td>
<td>2.488</td>
<td>3.399</td>
</tr>
<tr>
<td>0</td>
<td>2.876</td>
<td>4.160</td>
</tr>
<tr>
<td>0.146 (F)</td>
<td></td>
<td>4.188</td>
</tr>
<tr>
<td>1</td>
<td>4.280</td>
<td>3.686</td>
</tr>
<tr>
<td>1.148 (E)</td>
<td>4.341</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4.286</td>
<td>3.380</td>
</tr>
<tr>
<td>3</td>
<td>4.258</td>
<td>3.311</td>
</tr>
<tr>
<td>4</td>
<td>4.233</td>
<td>3.291</td>
</tr>
<tr>
<td>5</td>
<td>4.209</td>
<td>3.271</td>
</tr>
</tbody>
</table>

Table 4.9 The \(a/b\) ratio of two drops with the initial radius \(r_0\) of 0.45 mm, each suspended in a medium (corresponding to Case I and Case VI), moving along the centerline of a cylindrical tube in the entrance region at \( \dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1} \). The values of \(a/b\) ratio listed here are determined from the calculated drop shapes, at different positions along the centerline of a cylindrical tube in the entrance region. Note that a maximum of \(a/b\) ratio occurs at position (E) \(z/R_0 = 1.148\) for Case I and at position (F) \(z/R_0 = 0.146\) for Case VI.
Table 4.9 are determined from the calculated drop shapes at different positions along the centerline of a cylindrical tube in the entrance region. Notice in Table 4.9 that a maximum of \( a/b \) ratio occurs at position (E) \( z/R_0 = 1.148 \) for a viscoelastic drop (Case I) and at position (F) \( z/R_0 = 0.146 \) for a Newtonian drop (Case VI). We have made similar observations in Figures 4.7–4.11 for other viscoelastic drops and in Figures 4.13–4.17 for other Newtonian drops. The detailed analyses of the computational results for those situations are presented in the Appendix B of this dissertation.

In order to explain the above observation, in this study we computed the extensional stress distribution along the centerline, \( \sigma_{zz}(z/R_0) \), of the converging cylindrical tube for a Newtonian suspending medium and a viscoelastic suspending medium, respectively, and the results are summarized in Figure 4.19 in which a Newtonian drop (Case VI) and a viscoelastic drop (Case I), respectively, are suspended in a respective suspending medium at an apparent shear rate of 35.4 s\(^{-1}\). It is of great interest to observe in Figure 4.19 that a maximum of extensional stress \( \sigma_{zz}(z/R_0) \) of a Newtonian suspending medium occurs before reaching the entrance \( (z/R_0 = 0) \) of the straight section of the converging cylindrical tube, while a maximum of \( \sigma_{zz}(z/R_0) \) of a viscoelastic suspending medium occurs after passing the entrance of the straight section \( (at z/R_0 > 0) \) of the converging cylindrical tube. Since a drop (viscoelastic or Newtonian) moves along the central axis of a cylindrical tube by extensional flow, it is very reasonable to speculate that the deformation of a drop is subjected to extensional stress along the central axis of the cylindrical tube. Thus we can conclude that the position at which a maximum of aspect ratio \( a/b \) of a drop is observed in the converging section of a cylindrical tube is closely
related to the position at which a maximum of extensional stress $\sigma_{zz}(z/R_0)$ is observed along the axis of a cylindrical tube.

The above observation suggests that a Newtonian drop would deform faster than a viscoelastic drop as the respective drops move along the central axis of a converging cylindrical tube. This indeed is borne out to be the case from our computational results when the extent of deformation of a Newtonian drop (Case VI) given in Figure 4.13 is
compared with that of a viscoelastic drop (Case I) given in Figure 4.7. Similar observations can be made (i) when the extent of deformation given in Figure 4.14 for a Newtonian drop (Case VII) is compared with that given Figure 4.8 for a viscoelastic drop (Case II), (ii) when the extent of deformation given in Figure 4.15 for a Newtonian drop (Case VIII) is compared with that given Figure 4.9 for a viscoelastic drop (Case III), (iii) when the extent of deformation given in Figure 4.16 for a Newtonian drop (Case IX) is compared with that given in Figure 4.10 for a viscoelastic drop (Case IV), and (iv) when the extent of deformation given in Figure 4.17 for a Newtonian drop (Case X) is compared with that given in Figure 4.11 for a viscoelastic drop (Case V). The difference in the rate of the extent of drop deformation between Newtonian and viscoelastic drops, each moving along the central axis of a converging cylindrical tube, is attributable to the elasticity of a viscoelastic drop. This is because the large elastic modulus of a viscoelastic drop would tend to resist deformation, as compared to the deformation of a Newtonian drop.

Another significant observation of our computed shape of drops is the rate at which an elongated drop recoils after passing the position at which a maximum aspect ratio occurs. Notice in Table 4.9 that a viscoelastic drop recoils at much a slower rate as compared to a Newtonian drop after the respective drop passed the position at which a maximum aspect ratio is attained. It is clearly seen in Figure 4.19 that after passing the position at which a maximum aspect ratio of drop occurs, values of extensional stress $\sigma_{zz}(z/R_0)$ for a Newtonian medium decrease very quickly to very small values while values of $\sigma_{zz}(z/R_0)$ for a viscoelastic medium decrease very slowly. Similar observations
were also made for other pairs of viscoelastic and Newtonian drops, which are presented in the Appendix B of this dissertation. The above observation can be explained as follows. The recoil of a viscoelastic drop is dictated predominantly by its relaxation modulus whereas the recoil of a Newtonian drop is determined strictly by its interfacial tension. It should be mentioned that the magnitude of the elastic force of a viscoelastic fluid is much greater than that of interfacial force. Consequently, the relaxation time of a viscoelastic drop is much greater (on the orders of magnitude) than that of a Newtonian drop. Hence the relaxation modulus of a viscoelastic drop is expected to slow down its recoil more than the interfacial tension of a Newtonian drop. Thus we conclude once again that the elasticity of a viscoelastic drop plays a predominant role in determining its rate of recoil, whereas only the interfacial tension determines the rate of recoil of a Newtonian drop. This conclusion now explains why the rate of recoil of a viscoelastic drop is much slower than that of a Newtonian drop in the entrance region of a cylindrical tube.

Figure 4.20 gives computed extensional stress profiles $\sigma_{zz}(z/R_0)$ of a viscoelastic suspending medium at three different values of apparent shear rates ($\dot{\gamma}_{app}$), 15.4, 35.4, and 75.4 s$^{-1}$. It is seen in Figure 4.20 that values of $\sigma_{zz}(z/R_0)$ increase with increasing $\dot{\gamma}_{app}$, indicating that the extent of drop deformation would increase with increasing $\dot{\gamma}_{app}$. This is because as pointed out above, the larger the values of $\sigma_{zz}(z/R_0)$ the greater the extent of drop deformation would be. From the numerical computations conducted in our study we have found that this indeed was the situation. The computed shapes of
Extensional stress distribution along the centerline of the cylindrical tube when a viscoelastic drop with initial radius \( r_0 \) of 0.45 mm is suspended in the viscoelastic medium (corresponding Case I in Table 4.6), moving at various shear rates:

1. \( \dot{\gamma}_{\text{app}} = 15.4 \text{ s}^{-1} \);
2. \( \dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1} \);
3. \( \dot{\gamma}_{\text{app}} = 75.4 \text{ s}^{-1} \).

Figure 4.20 gives computed extensional stress profiles \( \sigma_{zz}(z/R_0) \) of a Newtonian suspending medium at three different values of \( \dot{\gamma}_{\text{app}} \), 15.4, 35.5, and 76.4 s\(^{-1}\). It is seen in Figure 4.21 that values of \( \sigma_{zz}(z/R_0) \) increase with increasing \( \dot{\gamma}_{\text{app}} \), indicating that the extent of drop deformation would increase with increasing \( \dot{\gamma}_{\text{app}} \). From the numerical computations conducted in our study we have found that this indeed was the situation.
The computed shapes of Newtonian drops at three different values of $\dot{\gamma}_{\text{app}}$ for selected Cases (VI, VIII, and IX) are presented in the Appendix B.

Figure 4.21  Extensional stress distribution along the centerline of the cylindrical tube when a Newtonian drop with initial radius ($r_0$) of 0.45 mm is suspended in the Newtonian medium (corresponding Case VI in Table 4.6), moving at various shear rates: (1) $\dot{\gamma}_{\text{app}} = 15.4$ s$^{-1}$; (2) $\dot{\gamma}_{\text{app}} = 35.4$ s$^{-1}$; (3) $\dot{\gamma}_{\text{app}} = 75.4$ s$^{-1}$.
5.1 Conclusions

In this dissertation we have formulated the finite element system equations for solving the system equation of the drop deformation in the entrance region of a cylindrical tube. The penalty function method was employed to eliminate the pressure term in the momentum equation, thereby reducing the variables in the finite system equations. An adaptive unstructured grid was generated based on the Bowyer–Watson algorithm and an auto-remeshing technique was employed, so that the grid could be refined to fit the moving interface between the drop and suspending medium. The positions of the new interface as well as the integration points on the old meshes had to be computed again, so that the unknown interfaces on the new node, i.e., velocity, were calculated. At each time step, the Newton–Raphson method was used to obtain the numerical solution for the finite element system equations.

We conducted numerical computations for the extent of deformation of a drop suspended in another fluid moving along the centerline of a cylindrical tube in the entrance region. Viscoelastic and Newtonian fluids were applied to both the drop and suspending medium, respectively. The KBKZ integral-type constitutive equation was
employed as a viscoelastic model and the method to calculate the strain tensor was
developed, so that the stress tensor could be computed via integration over time using the
Gauss–Laguerre quadrature.

For the case of a viscoelastic drop suspended in another viscoelastic medium
moving along the centerline of a cylindrical tube in the entrance region, we considered (i)
the radius \( R_0 \) of a cylindrical tube having 3.0 mm, (ii) five cases (Case I through Case V)
using the KBKZ integral-type constitutive equation with various relaxation time \( \lambda_k \), (iii)
three different values (0.3, 0.45, and 0.6 mm) of the radius \( r_0 \) of the initial drop, and (iv)
three different values (15.4, 35.4, and 75.4 s\(^{-1}\)) of apparent shear rate \( \dot{\gamma}_{app} \) based on the
fixed radius of the cylindrical tube. The computed result of a viscoelastic drop suspended
in another viscoelastic medium showed that the drop elongated continuously as it moved
towards the tube entrance and recoiled slightly after it passed the tube inlet. Further
observations showed that the extent of drop deformation became greater as the ratio of
the relaxation times \( \lambda_k \) of the drop and the suspending medium decreases from 5 to 0.25,
with the otherwise identical initial drop size and flow condition. For each pair of drop
and suspending medium (Case I through Case V), by varying the initial radius of a drop
at fixed flow condition, we found that the extent of drop deformation increased as the
initial drop radius was increased from 0.10 to 0.20 mm, and by varying the flow
condition with fixed initial radius of drop, we found that the extent of drop deformation
increased as the apparent shear rate was increased from 15.4 s\(^{-1}\) to 75.4 s\(^{-1}\).

We observed that the computational results of the drop deformation are in
qualitative agreement with the experimental result in the literature\(^{15}\).
For the case of a Newtonian drop suspended in another Newtonian medium moving along the centerline of a cylindrical tube in the entrance region, we considered (i) the radius ($R_0$) of a cylindrical tube having 3.0 mm, (ii) five cases (Case VI through Case X) for Newtonian systems with various viscosities which were calculated based on the corresponding parameters of the KBKZ integral model, (iii) the radius of the drop having 0.45 mm, and (iv) three different values (15.4, 35.4, and 75.4 s$^{-1}$) of apparent shear rate ($\dot{\gamma}_{app}$) based on the fixed radius of the cylindrical tube. The computed result of a Newtonian drop suspended in another Newtonian medium showed that the drop elongated continuously as it moved towards the tube entrance and recoiled after it passed the tube inlet. Further observation showed that the extent of drop deformation became greater as the viscosity ratio of the drop and the suspending medium ($\eta_{0d}/\eta_{0m}$) was decreased from 5 to 0.25, with the otherwise identical initial drop size and flow conditions. For each pair of drop and suspending medium (Case VI through Case X), we found that the extent of drop deformation increased as the apparent shear rate was increased from 15.4 s$^{-1}$ to 75.4 s$^{-1}$.

By comparing the computed results of the viscoelastic system and the Newtonian system, we observed that the extent of a maximum deformation of a Newtonian drop occurs at an axial position very close to the end of the converging section or very close to the entrance of the straight section (i.e., at $z/R_0 \approx 0$) of the cylindrical tube, while the extent of a maximum deformation of a viscoelastic drop occurs at an axial position slightly inside the straight section (i.e., at $z/R_0 > 1.0$) of the cylindrical tube. We also observed that a Newtonian drop deformed faster than a viscoelastic drop as the respective
drops moved along the central axis of a conveying cylindrical tube. In order to explain the above observations, we computed the extensional stress distributions along the centerline, $\sigma_{zz}(z/R_0)$, of the converging cylindrical tube for a Newtonian suspending medium and a viscoelastic suspending medium, respectively, and found that the position at which a maximum of aspect ratio $a/b$ of a drop in the converging section of a cylindrical tube was closely related to the position at which a maximum of extensional stress $\sigma_{zz}(z/R_0)$ along the axis of a cylindrical tube. We also found that the values of $\sigma_{zz}(z/R_0)$ until attaining a maximum are larger for a Newtonian suspending medium than for a viscoelastic medium which suggested that a Newtonian drop would deform faster than a viscoelastic drop as the respective drops move along the central axis of a converging cylindrical tube. The difference in the rate of the extent of drop deformation between Newtonian and viscoelastic drops, each moving along the central axis of a converging cylindrical tube, is attributable to the elasticity of a viscoelastic drop. This is because the large elastic modulus of a viscoelastic drop would tend to resist deformation, as compared to the deformation of a Newtonian drop.

We also have learned that the rate of recoil of a viscoelastic drop is much slower than that of a Newtonian drop after passing the position at which a maximum aspect ratio occurs, which was attributed to the distribution of the extensional stress distribution along the centerline, $\sigma_{zz}(z/R_0)$. The recoil of a viscoelastic drop is dictated predominantly by its relaxation modulus whereas the recoil of a Newtonian drop is determined strictly by its interfacial tension. The magnitude of the elastic force of a viscoelastic fluid is much greater than that of interfacial force. Consequently, the relaxation time of a viscoelastic
drop is much greater (on the orders of magnitude) than that of a Newtonian drop. Hence the relaxation modulus of a viscoelastic drop is expected to slow down its recoil more than the interfacial tension of a Newtonian drop. Thus we conclude that the elasticity of a viscoelastic drop played a predominant role in determining its rate of recoil, whereas only the interfacial tension determines the rate of recoil of a Newtonian drop. This conclusion now explains why the rate of recoil of a viscoelastic drop is much slower than that of a Newtonian drop in the entrance region of a cylindrical tube. Also we conclude that the larger the values of $\sigma_{zz}(z/R_0)$ the greater the extent of drop deformation would be, which is consistent with the fact that the values of $\sigma_{zz}(z/R_0)$ increases with increasing $\dot{\gamma}_{app}$, indicating that the extent of drop deformation would increase with increasing $\dot{\gamma}_{app}$.

5.2 Recommendations

This dissertation has investigated large deformation of a viscoelastic drop suspended in a viscoelastic medium and a Newtonian drop suspended in a Newtonian medium. However, owing to the practical computational difficulty, we dealt with only a situation where a single drop travels along the central axis of a cylindrical tube in the entrance region. We recognize the fact that such situation is far removed from the industrial situations where hundreds or thousands of drops are suspended in another liquid. Nevertheless, in this study we have revealed the important roles which the elasticity of a fluid plays in the large deformation of a drop in the entrance region of a cylindrical tube. The following investigations are worth pursuing in the future. (1) A development of computer codes for three-dimensional (3D) auto-remeshing technique is highly
recommended. (2) An investigation of large deformation of multiple drops (by increasing the number of drops from say 3, 5, 7, 9, etc. progressively) is recommended. It should be mentioned that the development of computer codes for multiple drops will require a 3D auto-remeshing technique. (3) An investigation of drop breakup in the entrance region of a cylindrical tube is recommended. This investigation will require a criterion for determining a critical maximum aspect ratio of a drop at which breakup might occur. (4) Other integral-type constitutive equations might be worth considering for the development of finite element system equations. Granted that the recommendations made above would require great patience and perseverance before obtaining meaningful results.
REFERENCES


APPENDIX A
DERIVATION OF EQUATION (3.19)

According to Weighted Residual Method (WRM), for Eq. (3.13) there is a weight function $W$ so that the following equation is satisfied for all the points inside the flow region.

$$
0 = \int_{\Omega} W \left[ \frac{\partial}{\partial x} \left( \frac{\pi_p}{H^3} \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial z} \left( \frac{\pi_p}{H^3} \frac{\partial \psi}{\partial z} \right) \right] d\Omega
$$

$$
= \int_{\Omega} \left[ W \frac{\pi_p}{H^3} \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial x} \right) + W \frac{\pi_p}{H^3} \frac{\partial}{\partial z} \left( \frac{\partial \psi}{\partial z} \right) \right] dx \, dz
$$

(A.1)

The first term on the right-hand side of Eq. (A.1) can be written as:

$$
\int_{\Omega} W \frac{\pi_p}{H^3} \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial x} \right) dx \, dz = \int_{\Omega} W \frac{\pi_p}{H^3} d \left( \frac{\partial \psi}{\partial x} \right) dz
$$

$$
= \frac{\pi_p}{H^3} \int_{\Omega} W \left( \frac{\partial \psi}{\partial x} \right) dx \, dz - \frac{\pi_p}{H^3} \int_{\Omega} \frac{\partial \psi}{\partial x} dW
$$

(A.2)

Similarly the second term on the right-hand side of Eq. (A.1) can be written as
\[
\iint_{\Omega} W \frac{\pi_{p\mu}}{H^3} \frac{\partial}{\partial z} \left( \frac{\partial \psi}{\partial z} \right) \, dx \, dz = \frac{\pi_{p\mu}}{H^3} \iint_{\Omega} \left[ W \left( \frac{\partial \psi}{\partial x} \right) - \frac{\partial \psi}{\partial x} \frac{dW}{dz} \right] \, dz
\]

\[
= \frac{\pi_{p\mu}}{H^3} \int_{\Gamma} W \left( \frac{\partial \psi}{\partial n} \right) n_x \, dz - \frac{\pi_{p\mu}}{H^3} \iint_{\Omega} \frac{\partial \psi}{\partial x} \frac{\partial W}{\partial x} \, d\Omega
\]

Since \( \frac{\partial \psi}{\partial n} = \frac{\partial \psi}{\partial x} n_x + \frac{\partial \psi}{\partial z} n_z \), Eq. (A.1) can be written as

\[
\frac{\pi_{p\mu}}{H^3} \int_{\Gamma} W \left( \frac{\partial \psi}{\partial n} \right) \, d\Gamma - \frac{\pi_{p\mu}}{H^3} \iint_{\Omega} \left( \frac{\partial \psi}{\partial x} \frac{\partial W}{\partial x} + \frac{\partial \psi}{\partial z} \frac{\partial W}{\partial z} \right) \, d\Omega = 0
\]  

(3.19)
APPENDIX B

ADDITIONAL RESULTS OF DROP SHAPES COMPUTED

B.1 The Deformation of a Viscoelastic Drop Suspended in a Viscoelastic Medium
Moving along the Centerline of a Cylindrical Tube in the Entrance Region

B.1.1 Effects of the Relaxation Times of the Drop and Suspending Medium on the
Extent of Deformation of a Viscoelastic Drop Suspended in a Viscoelastic
Medium Moving along the Centerline of a Cylindrical Tube in the Entrance
Region

In Chapter 4 we presented the shapes of a viscoelastic drop (designated as Case I
through Case V in Table 4.2) with the radius \( r_0 \) of 0.45 mm at four different positions
(A, B, C, and D) along the centerline of a cylindrical tube in the entrance region traveling
at \( \dot{\gamma}_{app} = 35.4 \text{ s}^{-1} \). Here we present snap shots of a viscoelastic drop (designated as Case
I through Case V in Table 4.2) with the radius \( r_0 \) of 0.45 mm at four different positions
(A, B, C, and D) along the centerline of a cylindrical tube in the entrance region traveling
at \( \dot{\gamma}_{app} = 15.4 \text{ s}^{-1} \) and \( \dot{\gamma}_{app} = 75.4 \text{ s}^{-1} \), respectively.
Figure B.1 gives snap shots of the computed shapes of a viscoelastic drop (designated as Case I in Table 4.2) with the radius ($r_0$) of 0.45 mm at four different positions (A, B, C, and D) along the centerline of a cylindrical tube in the entrance region traveling at $\dot{\gamma}_{\text{app}} = 15.4$ s$^{-1}$. Note that the drop was suspended in the viscoelastic medium whose relaxation times are given in Table 4.1. It is clearly seen in Figure B.1 that the drop elongates continuously as it moves towards the tube entrance and recoils slightly after it passes the tube inlet. Figure B.2 gives snap shots of the computed shapes of a viscoelastic drop (designated as Case II in Table 4.2), Figure B.3 gives snap shots of the computed shapes of a viscoelastic drop (designated as Case III in Table 4.2), Figure B.4 gives snap shots of the computed shapes of a viscoelastic drop (designated as Case IV in Table 4.2), and Figure B.5 gives snap shots of the computed shapes of a viscoelastic drop (designated as Case V in Table 4.2), under the otherwise identical initial drop size ($r_0 = 0.45$ mm) and flow conditions (at $\dot{\gamma}_{\text{app}} = 15.4$ s$^{-1}$) as those in Figure B.1. It is seen in Figures B.1–B.5 that the extent of drop deformation increases as the drop travels from the upstream side of the cylindrical tube to its entrance and further inside the tube.

Figure B.6 compares the extent of deformation of five drops (designated as Case I, Case II, Case III, Case IV, and Case V) having different relaxation times $\lambda_k$, which are given in Table 4.2, at the same axial position $z/R_0 = 5$ which is slightly inside the cylindrical tube. Note that all five drops in Figure B.6 have the same initial drop radius ($r_0 = 0.45$ mm), and each drop is deformed in the same viscoelastic suspending medium, the numerical values of its physical parameters being given in Table 4.1, and the
Figure B.1 Snap shots of the computed shapes of a viscoelastic drop (Case I) with the initial radius \( r_0 \) of 0.45 mm and the numerical values of \( \lambda_k \) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \( \dot{\gamma}_{\text{app}} = 15.4 \text{ s}^{-1} \), with the values of \( \lambda_k \) and \( a_k \) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \( z/R_0 \) along the centerline of the cylindrical tube in the entrance region: the position A at \( z/R_0 = -10 \), the position B at \( z/R_0 = -5 \), the position C at \( z/R_0 = 1.136 \), and the position D at \( z/R_0 = 5 \) with \( R_0 \) being the radius (3 mm) of the cylindrical tube.
Figure B.2 Snap shots of the computed shapes of a viscoelastic drop (Case II) with the initial radius ($r_0$) of 0.45 mm and the numerical values of $\lambda_k$ given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at $\dot{\gamma}_{app} = 15.4$ s$^{-1}$, with the values of $\lambda_k$ and $a_k$ given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 1.182$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.
Figure B.3 Snap shots of the computed shapes of a viscoelastic drop (Case III) with the initial radius \((r_0)\) of 0.45 mm and the numerical values of \(\lambda_k\) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \(\dot{\gamma}_{\text{app}} = 15.4 \text{ s}^{-1}\), with the values of \(\lambda_k\) and \(a_k\) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \((z/R_0)\) along the centerline of the cylindrical tube in the entrance region: the position A at \(z/R_0 = -10\), the position B at \(z/R_0 = -5\), the position C at \(z/R_0 = 1.215\), and the position D at \(z/R_0 = 5\) with \(R_0\) being the radius (3 mm) of the cylindrical tube.
Figure B.4 Snap shots of the computed shapes of a viscoelastic drop (Case IV) with the initial radius \(r_0\) of 0.45 mm and the numerical values of \(\lambda_k\) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \(\dot{\gamma}_{\text{app}} = 15.4\ \text{s}^{-1}\), with the values of \(\lambda_k\) and \(a_k\) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \(z/R_0\) along the centerline of the cylindrical tube in the entrance region: the position A at \(z/R_0 = -10\), the position B at \(z/R_0 = -5\), the position C at \(z/R_0 = 1.270\), and the position D at \(z/R_0 = 5\) with \(R_0\) being the radius (3 mm) of the cylindrical tube.
Figure B.5 Snap shots of the computed shapes of a viscoelastic drop (Case V) with the initial radius ($r_0$) of 0.45 mm and the numerical values of $\lambda_k$ given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at $\dot{\gamma}_{\text{app}} = 15.4$ s$^{-1}$, with the values of $\lambda_k$ and $a_k$ given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 1.319$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.
Figure B.6 Snap shots of the computed shapes of five different viscoelastic drops (Case I through Case V) with the identical initial radius (r₀) of 0.45 mm and the numerical values of λₖ given in Table 4.2 when each of the drops was suspended in the same viscoelastic medium, moving at $\dot{\gamma}_{\text{app}} = 15.4$ s⁻¹, with the values of λₖ and aₖ given in Table 4.1. The snap shots of the computed shape of the drops are at the same dimensionless axial position of z/R₀ = 5 in the entrance region of a cylindrical tube for different drops: (a) Case I, (b) Case II, (c) Case III, (d) Case IV, and (e) Case V.
suspending medium flows through the entrance region of a cylindrical tube at an apparent shear rate of \(15.4 \text{ s}^{-1}\). It is clearly seen in Figure B.6 that the extent of drop deformation becomes greater as the ratio of the relaxation times (\(\lambda_k\)) of the drop and the suspending medium decreases from 5 to 0.25 (compare the values of \(\lambda_k\) given for each drop given in Table 4.2 with those for the suspending medium given in Table 4.1).

Table B.1 gives a summary of the calculated values of the axial ratio (\(a/b\)) for the five drops at four different positions (A, B, C, and D) in the entrance region of the cylindrical tube, showing a very clear trend for the extent of drop deformation as affected by the ratio of the relaxation times of the drop and the suspending medium.

<table>
<thead>
<tr>
<th>Drop</th>
<th>The a/b ratio of a drop at the axial position A, (z/R_0 = -10)</th>
<th>The a/b ratio of a drop at the axial position B, (z/R_0 = -5)</th>
<th>The a/b ratio of a drop at the axial position C, (z/R_0 = 1.136)</th>
<th>The a/b ratio of a drop at the axial position D, (z/R_0 = 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I</td>
<td>1.331</td>
<td>1.823</td>
<td>3.071 ((z/R_0 = 1.136))</td>
<td>2.981</td>
</tr>
<tr>
<td>Case II</td>
<td>1.680</td>
<td>2.682</td>
<td>5.397 ((z/R_0 = 1.182))</td>
<td>5.193</td>
</tr>
<tr>
<td>Case III</td>
<td>2.028</td>
<td>3.784</td>
<td>7.677 ((z/R_0 = 1.215))</td>
<td>7.385</td>
</tr>
<tr>
<td>Case IV</td>
<td>2.562</td>
<td>4.888</td>
<td>10.556 ((z/R_0 = 1.270))</td>
<td>10.106</td>
</tr>
<tr>
<td>Case V</td>
<td>3.682</td>
<td>6.883</td>
<td>15.011 ((z/R_0 = 1.319))</td>
<td>14.253</td>
</tr>
</tbody>
</table>
Figure B.7 gives snap shots of the computed shapes of a viscoelastic drop (designated as Case I in Table 4.2) with the radius \( r_0 \) of 0.45 mm at four different positions (A, B, C, and D) along the centerline of a cylindrical tube in the entrance region traveling at \( \dot{\gamma}_{app} = 75.4 \, \text{s}^{-1} \). Note that the drop was suspended in the viscoelastic medium whose relaxation times are given in Table 4.1. It is clearly seen in Figure B.7 that the drop elongates continuously as it moves towards the tube entrance and recoils slightly after it passed the tube inlet. Figure B.8 gives snap shots of the computed shapes of a viscoelastic drop (designated as Case II in Table 4.2), Figure B.9 gives snap shots of the computed shapes of a viscoelastic drop (designated as Case III in Table 4.2), and Figure B.10 gives snap shots of the computed shapes of a viscoelastic drop (designated as Case IV in Table 4.2), under the otherwise identical initial drop size \( r_0 = 0.45 \, \text{mm} \) and flow conditions (at \( \dot{\gamma}_{app} = 75.4 \, \text{s}^{-1} \)) as those in Figure B.7. It is seen in Figures B.7–B.10 that the extent of drop deformation increases as the drop travels from the upstream side of the cylindrical tube to its entrance and further inside the tube.

Figure B.11 compares the extent of deformation of four drops (designated as Case I, Case II, Case III, and Case IV) having different relaxation times \( \lambda_k \) which are given in Table 4.2, at the same axial position \( z/R_0 = 5 \) which is slightly inside the cylindrical tube. Note that all four drops in Figure B.11 have the same initial drop radius \( r_0 = 0.45 \, \text{mm} \), and each drop is deformed in the same suspending medium, the numerical values of its physical parameters being given in Table 4.1, and the suspending medium flows through the entrance region of a cylindrical tube at an apparent shear rate of 75.4 \, \text{s}^{-1}. \) It is clearly seen in Figure B.11 that the extent of drop deformation becomes greater as the ratio of
Figure B.7 Snap shots of the computed shapes of a viscoelastic drop (Case I) with the initial radius ($r_0$) of 0.45 mm and the numerical values of $\lambda_k$ given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at $\dot{\gamma}_{\text{app}} = 75.4$ s$^{-1}$, with the values of $\lambda_k$ and $a_k$ given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 1.160$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.
Figure B.8 Snap shots of the computed shapes of a viscoelastic drop (Case II) with the initial radius \( r_0 \) of 0.45 mm and the numerical values of \( \lambda_k \) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \( \dot{\gamma}_{\text{app}} = 75.4 \text{ s}^{-1} \), with the values of \( \lambda_k \) and \( a_k \) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \( (z/R_0) \) along the centerline of the cylindrical tube in the entrance region: the position A at \( z/R_0 = -10 \), the position B at \( z/R_0 = -5 \), the position C at \( z/R_0 = 1.205 \), and the position D at \( z/R_0 = 5 \) with \( R_0 \) being the radius (3 mm) of the cylindrical tube.
Figure B.9 Snap shots of the computed shapes of a viscoelastic drop (Case III) with the initial radius ($r_0$) of 0.45 mm and the numerical values of $\lambda_k$ given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at $\dot{\gamma}_{app} = 75.4 \text{ s}^{-1}$, with the values of $\lambda_k$ and $a_k$ given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 1.253$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.
Figure B.10 Snap shots of the computed shapes of a viscoelastic drop (Case IV) with the initial radius ($r_0$) of 0.45 mm and the numerical values of $\lambda_k$ given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at $\dot{\gamma}_{app} = 75.4$ s$^{-1}$, with the values of $\lambda_k$ and $a_k$ given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 1.303$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.
Figure B.11 Snap shots of the computed shapes of four different viscoelastic drops (Case I through Case IV) with the identical initial radius \( r_0 \) of 0.45 mm and the numerical values of \( \lambda_k \) given in Table 4.2 when each of the drops was suspended in the same viscoelastic medium, moving at \( \dot{\gamma}_{\text{app}} = 75.4 \text{ s}^{-1} \), with the values of \( \lambda_k \) and \( a_k \) given in Table 4.1. The snap shots of the computed shape of the drops are at the same dimensionless axial position of \( z/R_0 = 5 \) in the entrance region of a cylindrical tube for different drops: (a) Case I, (b) Case II, (c) Case III, (d) Case IV, and (e) Case V.
the relaxation times ($\lambda_k$) of the drop and the suspending medium decreases from 5 to 0.5 (compare the values of $\lambda_k$ given for each drop given in Table 4.2 with those for the suspending medium given in Table 4.1).

Table B.2 gives a summary of the calculated values of the axial ratio ($a/b$) for the five drops at four different positions (A, B, C, and D) in the entrance region of the cylindrical tube, showing a very clear trend for the extent of drop deformation as affected by the ratio of the relaxation times of the drop and the suspending medium. Again the computed shapes of the drops presented in Figures B.1–B.11 and summarized in Tables B.1 and B.2 are similar to the those presented in Chapter 4 (see Figure 4.7–4.12 and Table 4.3) and in good qualitative agreement with the experimental results of Chin and Han.$^{15}$

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**Table B.2** The axial ratio $a/b$ of four different viscoelastic drops (designated as *Case I*, *Case II*, *Case III*, *Case IV*, and *Case V* in Table 4.2) with the initial radius ($r_0$) of 0.45 mm, each suspended in a viscoelastic medium (see Table 4.1 for the physical parameters appearing in Eqs. (4.1)–(4.3)), moving along the centerline of a cylindrical tube in the entrance region at $\dot{\gamma}_{ap} = 75.4$ s$^{-1}$. The values of $a/b$ ratio listed here are determined from the calculated drop shapes, given in Figures B.7–B.10, at different positions ($z/R_0$) along the centerline a cylindrical tube in the entrance region.

<table>
<thead>
<tr>
<th>Drop</th>
<th>The a/b ratio of a drop at the axial position A, $z/R_0 = -10$</th>
<th>The a/b ratio of a drop at the axial position B, $z/R_0 = -5$</th>
<th>The a/b ratio of a drop at the axial position C, $z/R_0 = 1.160$</th>
<th>The a/b ratio of a drop at the axial position D, $z/R_0 = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>Case I</em></td>
<td>1.980</td>
<td>2.681</td>
<td>6.577 ($z/R_0 = 1.160$)</td>
<td>6.376</td>
</tr>
<tr>
<td><em>Case II</em></td>
<td>2.182</td>
<td>3.180</td>
<td>11.458 ($z/R_0 = 1.205$)</td>
<td>11.060</td>
</tr>
<tr>
<td><em>Case III</em></td>
<td>2.511</td>
<td>3.989</td>
<td>17.684 ($z/R_0 = 1.253$)</td>
<td>17.012</td>
</tr>
<tr>
<td><em>Case IV</em></td>
<td>3.114</td>
<td>5.072</td>
<td>24.629 ($z/R_0 = 1.270$)</td>
<td>23.617</td>
</tr>
</tbody>
</table>
B.1.2 Effects of Initial Drop Size on the Extent of Deformation of a Viscoelastic Drop Suspended in a Viscoelastic Medium Moving along the Centerline of a Cylindrical Tube in the Entrance Region

We also computed the shapes of a viscoelastic drop (designated as Case I through Case V in Table 4.2) with the radii \(r_0\) of 0.30 mm and 0.60 mm, respectively, at four different positions (A, B, C, and D) along the centerline of a cylindrical tube in the entrance region traveling at \(\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}\).

Figure B.12 gives snap shots of the computed shapes of a viscoelastic drop (designated as Case I in Table 4.2) with the radius \(r_0\) of 0.30 mm at four different positions (A, B, C, and D) along the centerline of a cylindrical tube in the entrance region traveling at \(\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}\). Note that the drop was suspended in the viscoelastic medium whose relaxation times are given in Table 4.1. It is clearly seen in Figure B.12 that the drop elongates continuously as it moves towards the tube entrance and recoils slightly after it passed the tube inlet. Figure B.13 gives snap shots of the computed shapes of a viscoelastic drop (designated as Case II in Table 4.2), Figure B.14 gives snap shots of the computed shapes of a viscoelastic drop (designated as Case III in Table 4.2), Figure B.15 gives snap shots of the computed shapes of a viscoelastic drop (designated as Case IV in Table 4.2), and Figure B.16 gives snap shots of the computed shapes of a viscoelastic drop (designated as Case V in Table 4.2), under the otherwise identical initial drop size \(r_0 = 0.30 \text{ mm}\) and flow conditions (at \(\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}\)) as those in Figure B.12. It is seen in Figures B.12–B.16 that the extent of drop deformation increases as the drop
Figure B.12 Snap shots of the computed shapes of a viscoelastic drop (Case I) with the initial radius \( r_0 \) of 0.30 mm and the numerical values of \( \lambda_k \) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \( \dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1} \), with the values of \( \lambda_k \) and \( a_k \) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \( (z/R_0) \) along the centerline of the cylindrical tube in the entrance region: the position A at \( z/R_0 = -10 \), the position B at \( z/R_0 = -5 \), the position C at \( z/R_0 = 1.133 \), and the position D at \( z/R_0 = 5 \) with \( R_0 \) being the radius (3 mm) of the cylindrical tube.
Figure B.13 Snap shots of the computed shapes of a viscoelastic drop (*Case II*) with the initial radius \((r_0)\) of 0.30 mm and the numerical values of \(\lambda_k\) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \(\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}\), with the values of \(\lambda_k\) and \(a_k\) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \((z/R_0)\) along the centerline of the cylindrical tube in the entrance region: the position A at \(z/R_0 = -10\), the position B at \(z/R_0 = -5\), the position C at \(z/R_0 = 1.178\), and the position D at \(z/R_0 = 5\) with \(R_0\) being the radius (3 mm) of the cylindrical tube.
Figure B.14  Snap shots of the computed shapes of a viscoelastic drop (Case III) with the initial radius \( r_0 \) of 0.30 mm and the numerical values of \( \lambda_k \) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \( \dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1} \), with the values of \( \lambda_k \) and \( a_k \) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \( z/R_0 \) along the centerline of the cylindrical tube in the entrance region: the position A at \( z/R_0 = -10 \), the position B at \( z/R_0 = -5 \), the position C at \( z/R_0 = 1.211 \), and the position D at \( z/R_0 = 5 \) with \( R_0 \) being the radius (3 mm) of the cylindrical tube.
Figure B.15 Snap shots of the computed shapes of a viscoelastic drop (Case IV) with the initial radius ($r_0$) of 0.30 mm and the numerical values of $\lambda_k$ given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at $\dot{\gamma}_{\text{app}} = 35.4$ s$^{-1}$, with the values of $\lambda_k$ and $a_k$ given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 1.266$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.
Figure B.16  Snap shots of the computed shapes of a viscoelastic drop (Case V) with the initial radius ($r_0$) of 0.30 mm and the numerical values of $\lambda_k$ given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at $\dot{\gamma}_{\text{app}} = 35.4$ s$^{-1}$, with the values of $\lambda_k$ and $\alpha_k$ given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 1.313$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.
travels from the upstream side of the cylindrical tube to its entrance and further inside the tube.

Figure B.17 gives snap shots of the computed shapes of a viscoelastic drop (designated as \textit{Case I} in Table 4.2) with the radius ($r_0$) of 0.60 mm at four different positions (A, B, C, and D) along the centerline of a cylindrical tube in the entrance region traveling at $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$. Note that the drop was suspended in the viscoelastic medium whose relaxation times are given in Table 4.1. It is clearly seen in Figure B.17 that the drop elongates continuously as it moves towards the tube entrance and recoils slightly after it passed the tube inlet. Figure B.18 gives snap shots of the computed shapes of a viscoelastic drop (designated as \textit{Case II} in Table 4.2), Figure B.19 gives snap shots of the computed shapes of a viscoelastic drop (designated as \textit{Case III} in Table 4.2), and Figure B.20 gives snap shots of the computed shapes of a viscoelastic drop (designated as \textit{Case IV} in Table 4.2), under the otherwise identical initial drop size ($r_0 = 0.60 \text{ mm}$) and flow conditions (at $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$) as those in Figure B.17. It is seen in Figures B.17–B.20 that the extent of drop deformation increases as the drop travels from the upstream side of the cylindrical tube to its entrance and further inside the tube.

In Table 4.4 of Chapter 4, we have summarized the axial ratio $a/b$ of a viscoelastic drop at various positions (designated as \textit{Case III} in Table 4.2) suspended in the viscoelastic medium whose relaxation times are given in Table 4.1 $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$.

Table B.3 gives a summary of the effect of the initial radius of a drop on the axial ratio ($a/b$) of the drop (designated as \textit{Case I} in Table 4.2) at three different positions (B, C, and D) in the entrance region of the cylindrical tube at $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$, Table B.4 gives a
Figure B.17 Snap shots of the computed shapes of a viscoelastic drop (Case I) with the initial radius (r₀) of 0.60 mm and the numerical values of $\lambda_k$ given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$, with the values of $\lambda_k$ and $a_k$ given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 1.166$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.
Figure B.18 Snap shots of the computed shapes of a viscoelastic drop (Case II) with the initial radius ($r_0$) of 0.60 mm and the numerical values of $\lambda_k$ given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at $\dot{\gamma}_{\text{app}} = 35.4$ s$^{-1}$, with the values of $\lambda_k$ and $a_k$ given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 1.209$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.
Figure B.19 Snap shots of the computed shapes of a viscoelastic drop (Case III) with the initial radius \( r_0 \) of 0.60 mm and the numerical values of \( \lambda_k \) given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at \( \dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1} \), with the values of \( \lambda_k \) and \( a_k \) given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \( z/R_0 \) along the centerline of the cylindrical tube in the entrance region: the position A at \( z/R_0 = -10 \), the position B at \( z/R_0 = -5 \), the position C at \( z/R_0 = 1.258 \), and the position D at \( z/R_0 = 5 \) with \( R_0 \) being the radius (3 mm) of the cylindrical tube.
Figure B.20 Snap shots of the computed shapes of a viscoelastic drop (*Case IV*) with the initial radius ($r_0$) of 0.60 mm and the numerical values of $\lambda_k$ given in Table 4.2 when the drop was suspended in the viscoelastic medium, moving at $\dot{\gamma}_{\text{app}} = 35.4$ s$^{-1}$, with the values of $\lambda_k$ and $a_k$ given in Table 4.1. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 1.309$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.
Table B.3  The axial ratio a/b of viscoelastic drops (designated as Case I in Table 4.2) with three different values of initial radius (0.3, 0.45, and 0.6 mm), each suspended in a viscoelastic medium (see Table 4.1 for the physical parameters appearing in Eqs. (4.1)–(4.3)), moving along the centerline of a cylindrical tube in the entrance region at $\dot{\gamma}_{app} = 35.4$ s$^{-1}$. The values of a/b ratio listed here are determined from the calculated drop shapes, which are given in the Appendix B of this dissertation, at different positions along the centerline of a cylindrical tube in the entrance region.

<table>
<thead>
<tr>
<th>Initial drop radius (mm)</th>
<th>The a/b ratio of a drop at the axial position B, $z/R_0 = -5$</th>
<th>The a/b ratio of a drop at the axial position C</th>
<th>The a/b ratio of a drop at the axial position D, $z/R_0 = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.30</td>
<td>1.832</td>
<td>3.868 ($z/R_0 = 1.133$)</td>
<td>3.751</td>
</tr>
<tr>
<td>0.45</td>
<td>1.924</td>
<td>4.341 ($z/R_0 = 1.148$)</td>
<td>4.209</td>
</tr>
<tr>
<td>0.60</td>
<td>2.018</td>
<td>5.367 ($z/R_0 = 1.166$)</td>
<td>5.206</td>
</tr>
</tbody>
</table>

Table B.4  The axial ratio a/b of viscoelastic drops (designated as Case II in Table 4.2) with three different values of initial radius (0.3, 0.45, and 0.6 mm), each suspended in a viscoelastic medium (see Table 4.1 for the physical parameters appearing in Eqs. (4.1)–(4.3)), moving along the centerline of a cylindrical tube in the entrance region at $\dot{\gamma}_{app} = 35.4$ s$^{-1}$. The values of a/b ratio listed here are determined from the calculated drop shapes, which are given in the Appendix B of this dissertation, at different positions along the centerline of a cylindrical tube in the entrance region.

<table>
<thead>
<tr>
<th>Initial drop radius (mm)</th>
<th>The a/b ratio of a drop at the axial position B, $z/R_0 = -5$</th>
<th>The a/b ratio of a drop at the axial position C</th>
<th>The a/b ratio of a drop at the axial position D, $z/R_0 = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.30</td>
<td>2.156</td>
<td>6.849 ($z/R_0 = 1.178$)</td>
<td>6.611</td>
</tr>
<tr>
<td>0.45</td>
<td>2.302</td>
<td>7.907 ($z/R_0 = 1.193$)</td>
<td>7.633</td>
</tr>
<tr>
<td>0.60</td>
<td>2.508</td>
<td>9.540 ($z/R_0 = 1.209$)</td>
<td>9.208</td>
</tr>
</tbody>
</table>

summary of the effect of the initial radius of a drop on the axial ratio (a/b) of the drop (designated as Case II in Table 4.2) at three different positions (B, C, and D) in the
entrance region of entrance region of the cylindrical tube at $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$, Table B.4 gives a summary of the effect of the initial radius of a drop on the axial ratio ($a/b$) of the drop (designated as Case II in Table 4.2) at three different positions (B, C, and D) in the entrance region of the cylindrical tube at $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$, Table B.5 gives a summary of the effect of the initial radius of a drop on the axial ratio ($a/b$) of the drop (designated as Case IV in Table 4.2) at three different positions (B, C, and D) in the entrance region of the cylindrical tube at $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$. It is clearly seen in Tables B.3–B.5 that the extent of drop deformation increases as the initial drop radius is increased.

Table B.5 The axial ratio $a/b$ of viscoelastic drops (designated as Case IV in Table 4.2) with three different values of initial radius (0.3, 0.45, and 0.6 mm), each suspended in a viscoelastic medium (see Table 4.1 for the physical parameters appearing in Eqs. (4.1)–(4.3)), moving along the centerline of a cylindrical tube in the entrance region at $\dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1}$. The values of $a/b$ ratio listed here are determined from the calculated drop shapes, which are given in the Appendix B of this dissertation, at different positions along the centerline of a cylindrical tube in the entrance region.

<table>
<thead>
<tr>
<th>Initial drop radius (mm)</th>
<th>The $a/b$ ratio of a drop at the axial position B, $z/R_0 = -5$</th>
<th>The $a/b$ ratio of a drop at the axial position C</th>
<th>The $a/b$ ratio of a drop at the axial position D, $z/R_0 = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.30</td>
<td>3.190</td>
<td>13.177 ($z/R_0 = 1.266$)</td>
<td>12.615</td>
</tr>
<tr>
<td>0.45</td>
<td>3.613</td>
<td>16.924 ($z/R_0 = 1.286$)</td>
<td>16.153</td>
</tr>
<tr>
<td>0.60</td>
<td>4.301</td>
<td>23.797 ($z/R_0 = 1.166$)</td>
<td>22.784</td>
</tr>
</tbody>
</table>
B.1.3 Effects of Apparent Shear Rate on the Extent of Deformation of a Viscoelastic Drop Suspended in a Viscoelastic Medium Moving along the Centerline of a Cylindrical Tube in the Entrance Region

In Table 4.5 of Chapter 4 we have summarized the effect of apparent shear rate on the axial ratio \(a/b\) of the drop (designated as Case II in Table 4.2) at three different positions (B, C, and D) in the entrance region of the cylindrical tube. Table B.6 gives a summary of the effect of apparent shear rate on the axial ratio \(a/b\) of the drop (designated as Case I in Table 4.2) at three different positions (B, C, and D) in the entrance region of the cylindrical tube. Table B.7 gives a summary of the effect of the apparent shear rate on the axial ratio \(a/b\) of the drop (designated as Case III in Table 4.2) at three different positions (B, C, and D) in the entrance region of the cylindrical tube.

Table B.6 The axial ratio \(a/b\) of viscoelastic drops (designated as Case I in Table 4.2) with the initial radius of 0.45 mm, suspended in a viscoelastic medium (see Table 4.1 for the physical parameters appearing in Eqs. (4.1)–(4.3)), moving along the centerline of a cylindrical tube in the entrance region at three different shear rates, \(\dot{\gamma}_{app} = 15.4, 35.4,\) and 75.4 s\(^{-1}\). The values of \(a/b\) ratio listed here are determined from the calculated shapes of drops, which are given in the Appendix B of this dissertation, at different positions along the centerline of a cylindrical tube in the entrance region.

<table>
<thead>
<tr>
<th>Apparent shear rate (s(^{-1}))</th>
<th>The a/b ratio of a drop at the axial position B, (z/R_0 = -5)</th>
<th>The a/b ratio of a drop at the axial position C</th>
<th>The a/b ratio of a drop at the axial position D, (z/R_0 = 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>15.4</td>
<td>1.331</td>
<td>3.071 ((z/R_0 = 1.136))</td>
<td>2.981</td>
</tr>
<tr>
<td>35.4</td>
<td>1.924</td>
<td>4.341 ((z/R_0 = 1.148))</td>
<td>4.209</td>
</tr>
<tr>
<td>75.4</td>
<td>2.681</td>
<td>6.577 ((z/R_0 = 1.160))</td>
<td>6.376</td>
</tr>
</tbody>
</table>
The axial ratio $a/b$ of viscoelastic drops (designated as *Case III* in Table 4.2) with the initial radius of 0.45 mm, suspended in a viscoelastic medium (see Table 4.1 for the physical parameters appearing in Eqs. (4.1)–(4.3)), moving along the centerline of a cylindrical tube in the entrance region at three different shear rates, $\dot{\gamma}_{\text{app}} = 15.4, 35.4,$ and $75.4$ s$^{-1}$. The values of $a/b$ ratio listed here are determined from the calculated shapes of drops, which are given in the Appendix B of this dissertation, at different positions along the centerline of a cylindrical tube in the entrance region.

<table>
<thead>
<tr>
<th>Apparent shear rate (s$^{-1}$)</th>
<th>The $a/b$ ratio of a drop at the axial position B, $z/R_0 = -5$</th>
<th>The $a/b$ ratio of a drop at the axial position C</th>
<th>The $a/b$ ratio of a drop at the axial position, $z/R_0 = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15.4</td>
<td>2.028</td>
<td>7.677 ($z/R_0 = 1.215$)</td>
<td>7.385</td>
</tr>
<tr>
<td>35.4</td>
<td>2.741</td>
<td>22.989 ($z/R_0 = 1.238$)</td>
<td>11.543</td>
</tr>
<tr>
<td>75.4</td>
<td>3.989</td>
<td>17.684 ($z/R_0 = 1.253$)</td>
<td>17.102</td>
</tr>
</tbody>
</table>

Table B.8 gives a summary of the effect of apparent shear rate on the axial ratio ($a/b$) of the drop (designated as *Case IV* in Table 4.2) at three different positions (B, C, and D) in the entrance region of the cylindrical tube. It is clearly seen in Tables B.6–B.8 that the extent of drop deformation increases as the apparent shear rate is increased. Again, this observation is in good qualitative agreement with the experimental results of Chin and Han.$^{15}$

### B.2 The Deformation of a Newtonian Drop Suspended in a Newtonian Medium Moving along the Centerline of a Cylindrical Tube in the Entrance Region

In Chapter 4 we have presented the computed results of a Newtonian drop suspended in a Newtonian medium moving along the centerline of a cylindrical tube in the entrance region at the apparent shear rate ($\dot{\gamma}_{\text{app}}$) based on the radius ($R_0$) of the
The axial ratio $a/b$ of viscoelastic drops (designated as Case I in Table 4.2) with the initial radius of 0.45 mm, suspended in a viscoelastic medium (see Table 4.1 for the physical parameters appearing in Eqs. (4.1)–(4.3)), moving along the centerline of a cylindrical tube in the entrance region at three different shear rates, $\dot{\gamma}_{\text{app}} = 15.4, 35.4,$ and $75.4 \text{ s}^{-1}$. The values of $a/b$ ratio listed here are determined from the calculated shapes of drops, which are given in the Appendix B of this dissertation, at different positions along the centerline of a cylindrical tube in the entrance region.

<table>
<thead>
<tr>
<th>Apparent shear rate ($s^{-1}$)</th>
<th>The $a/b$ ratio of a drop at the axial position B, $z/R_0 = -5$</th>
<th>The $a/b$ ratio of a drop at the axial position C</th>
<th>The $a/b$ ratio of a drop at the axial position D, $z/R_0 = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15.4</td>
<td>2.562</td>
<td>10.556 ($z/R_0 = 1.270$)</td>
<td>10.106</td>
</tr>
<tr>
<td>35.4</td>
<td>3.613</td>
<td>16.294 ($z/R_0 = 1.286$)</td>
<td>16.153</td>
</tr>
<tr>
<td>75.4</td>
<td>5.072</td>
<td>24.629 ($z/R_0 = 1.303$)</td>
<td>23.617</td>
</tr>
</tbody>
</table>

cylindrical tube being $35.4 \text{ s}^{-1}$. Below we present more computed results of a Newtonian drop suspended in a Newtonian medium moving along the centerline of a cylindrical tube in the entrance region at the apparent shear rate ($\dot{\gamma}_{\text{app}}$) of 4.4 $\text{s}^{-1}$ and 75.4 $\text{s}^{-1}$, respectively, based on the tube radius ($R_0$). We also summarize the effect of apparent shear rate on the axial ratio ($a/b$) of the drop at different positions in the entrance region of a cylindrical tube.

**B.2.1 Effects of the Viscosity Ratio of the Drop and Suspending Medium on the Extent of Deformation of a Newtonian Drop Suspended in a Newtonian Medium Moving along the Centerline of a Cylindrical Tube in the Entrance Region**
Figure B.21 gives snap shots of the computed shapes of a Newtonian drop corresponding to Case VI (see Table 4.6) with the radius \( r_0 \) of 0.45 mm at four different positions (A, B, C, and D) along the centerline of a cylindrical tube in the entrance region traveling at \( \dot{\gamma}_{\text{app}} = 15.4 \text{ s}^{-1} \). Note that the drop was suspended in the Newtonian medium whose viscosity is given in Table 4.6. It is clearly seen in Figure B.21 that the drop elongates continuously as it moves towards the tube entrance and recoils after it passes the tube inlet. Figure B.22 gives snap shots of the computed shapes of a Newtonian drop corresponding to Case VII (see Table 4.6), Figure B.23 gives snap shots of the computed shapes of a Newtonian drop corresponding to Case VIII (see Table 4.6), Figure B.24 gives snap shots of the computed shapes of a Newtonian drop corresponding to Case IX (see Table 4.6), and Figure B.25 gives snap shots of the computed shapes of a Newtonian drop corresponding to Case X (see Table 4.6), each drop having the same initial size of 0.45 mm and being suspended in the same medium flowing at \( \dot{\gamma}_{\text{app}} = 15.4 \text{ s}^{-1} \).

Figure B.26 gives snap shots of the computed shapes of a Newtonian drop corresponding to Case VI (see Table 4.6) with the radius \( r_0 \) of 0.45 mm at four different positions (A, B, C, and D) along the centerline of a cylindrical tube in the entrance region traveling at \( \dot{\gamma}_{\text{app}} = 75.4 \text{ s}^{-1} \). Note that the drop was suspended in the Newtonian medium whose viscosity is given in Table 4.6. It is clearly seen in Figure B.26 that the drop elongates continuously as it moves towards the tube entrance and recoils after it passes the tube inlet. Figure B.27 gives snap shots of the computed shapes of a Newtonian drop corresponding to Case VII (see Table 4.6), Figure B.28 gives snap shots of the computed shapes of a Newtonian drop corresponding to Case VIII (see Table 4.6), Figure B.29
Figure B.21 Snapshots of the computed shapes of a Newtonian drop (Case VI) with the initial radius ($r_0$) of 0.45 mm when the drop was suspended in the Newtonian medium, moving at $\dot{\gamma}_{\text{app}} = 15.4$ s$^{-1}$, with the viscosity of the drop $\eta_{0d}$ and the medium $\eta_{0m}$ given in Table 4.6. The snapshots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 0.135$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.
Figure B.22 Snap shots of the computed shapes of a Newtonian drop (Case VII) with the initial radius ($r_0$) of 0.45 mm when the drop was suspended in the Newtonian medium, moving at $\dot{\gamma}_{\text{app}} = 15.4$ s$^{-1}$, with the viscosity of the drop $\eta_{\text{od}}$ and the medium $\eta_{\text{om}}$ given in Table 4.6. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 0.137$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.
Figure B.23 Snap shots of the computed shapes of a Newtonian drop (Case VIII) with the initial radius \( r_0 \) of 0.45 mm when the drop was suspended in the Newtonian medium, moving at \( \dot{\gamma}_{\text{app}} = 15.4 \text{ s}^{-1} \), with the viscosity of the drop \( \eta_{\text{ld}} \) and the medium \( \eta_{\text{lm}} \) given in Table 4.6. The snap shots of the computed shape of the drop are at four different dimensionless axial positions (\( z/R_0 \)) along the centerline of the cylindrical tube in the entrance region: the position A at \( z/R_0 = -10 \), the position B at \( z/R_0 = -5 \), the position C at \( z/R_0 = 0.140 \), and the position D at \( z/R_0 = 5 \) with \( R_0 \) being the radius (3 mm) of the cylindrical tube.
Figure B.24 Snap shots of the computed shapes of a Newtonian drop (*Case IX*) with the initial radius \((r_0)\) of 0.45 mm when the drop was suspended in the Newtonian medium, moving at \(\dot{\gamma}_{\text{app}} = 15.4\) s\(^{-1}\), with the viscosity of the drop \(\eta_0d\) and the medium \(\eta_0m\) given in Table 4.6. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \((z/R_0)\) along the centerline of the cylindrical tube in the entrance region: the position A at \(z/R_0 = -10\), the position B at \(z/R_0 = -5\), the position C at \(z/R_0 = 0.143\), and the position D at \(z/R_0 = 5\) with \(R_0\) being the radius (3 mm) of the cylindrical tube.
Figure B.25  Snap shots of the computed shapes of a Newtonian drop (Case X) with the initial radius ($r_0$) of 0.45 mm when the drop was suspended in the viscoelastic medium, moving at $\dot{\gamma}_{\text{app}} = 15.4$ s$^{-1}$, with the viscosity of the drop $\eta_{0d}$ and the medium $\eta_{0m}$ given in Table 4.6. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 0.145$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.
Figure B.26 Snap shots of the computed shapes of a Newtonian drop (*Case VI*) with the initial radius \((r_0)\) of 0.45 mm when the drop was suspended in the Newtonian medium, moving at \(\dot{\gamma}_{\text{app}} = 75.4\) s\(^{-1}\), with the viscosity of the drop \(\eta_d\) and the medium \(\eta_m\) given in Table 4.6. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \((z/R_0)\) along the centerline of the cylindrical tube in the entrance region: the position A at \(z/R_0 = -10\), the position B at \(z/R_0 = -5\), the position C at \(z/R_0 = 0.159\), and the position D at \(z/R_0 = 5\) with \(R_0\) being the radius (3 mm) of the cylindrical tube.
Figure B.27 Snap shots of the computed shapes of a Newtonian drop (Case VII) with the initial radius \( r_0 \) of 0.45 mm when the drop was suspended in the Newtonian medium, moving at \( \dot{\gamma}_{\text{app}} = 75.4 \text{ s}^{-1} \), with the viscosity of the drop \( \eta_d \) and the medium \( \eta_m \) given in Table 4.6. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \( z/R_0 \) along the centerline of the cylindrical tube in the entrance region: the position A at \( z/R_0 = -10 \), the position B at \( z/R_0 = -5 \), the position C at \( z/R_0 = 0.162 \), and the position D at \( z/R_0 = 5 \) with \( R_0 \) being the radius (3 mm) of the cylindrical tube.
Figure B.28 Snap shots of the computed shapes of a Newtonian drop (Case VIII) with the initial radius \( r_0 \) of 0.45 mm when the drop was suspended in the Newtonian medium, moving at \( \dot{\gamma}_{\text{app}} = 75.4 \, \text{s}^{-1} \), with the viscosity of the drop \( \eta_{\text{d}} \) and the medium \( \eta_{\text{m}} \) given in Table 4.6. The snap shots of the computed shape of the drop are at four different dimensionless axial positions \( z/R_0 \) along the centerline of the cylindrical tube in the entrance region: the position A at \( z/R_0 = -10 \), the position B at \( z/R_0 = -5 \), the position C at \( z/R_0 = 0.165 \), and the position D at \( z/R_0 = 5 \) with \( R_0 \) being the radius (3 mm) of the cylindrical tube.
Figure B.29 Snap shots of the computed shapes of a Newtonian drop (Case IX) with the initial radius ($r_0$) of 0.45 mm when the drop was suspended in the Newtonian medium, moving at $\dot{\gamma}_{\text{app}} = 75.4$ s$^{-1}$, with the viscosity of the drop $\eta_{\text{od}}$ and the medium $\eta_{\text{om}}$ given in Table 4.6. The snap shots of the computed shape of the drop are at four different dimensionless axial positions ($z/R_0$) along the centerline of the cylindrical tube in the entrance region: the position A at $z/R_0 = -10$, the position B at $z/R_0 = -5$, the position C at $z/R_0 = 0.167$, and the position D at $z/R_0 = 5$ with $R_0$ being the radius (3 mm) of the cylindrical tube.
gives snap shots of the computed shapes of a Newtonian drop corresponding to Case IX (see Table 4.6), each drop having the same initial size of 0.45 mm and being suspended in the same medium flowing at \( \dot{\gamma}_{app} = 75.4 \text{ s}^{-1} \). It is seen in Figures B.21–B.29 that the extent of drop deformation increases as the drop travels from the upstream side of the cylindrical tube to its entrance and then decreases after it passes the tube inlet.

Figure B.30 compares the extent of deformation of five separate drops (corresponding to Case VI, Case VII, Case VIII, Case IX, and Case X) having different values of \( \eta_{0d} \) (see Table 4.6), at the same axial position \( z/R_0 = 5 \) which is slightly inside the cylindrical tube. Note that all five drops in Figure B.30 have the same initial drop radius \( r_0 = 0.45 \text{ mm} \), and each drop is deformed in the same suspending medium, whose viscosity is given in Table 4.6, and the suspending medium flows through the entrance region of a cylindrical tube at an apparent shear rate of 15.4 \text{ s}^{-1}. Figure B.31 compares the extent of deformation of four separate drops (corresponding to Case VI, Case VII, Case VIII, and Case IX) having different values of \( \eta_{0d} \) (see Table 4.6), at the same axial position \( z/R_0 = 5 \) which is slightly inside the cylindrical tube. Note that all four drops in Figure B.31 have the same initial drop radius \( r_0 = 0.45 \text{ mm} \), and each drop is deformed in the same suspending medium, whose viscosity is given in Table 4.6, and the suspending medium flows through the entrance region of the cylindrical tube at an apparent shear rate of 75.4 \text{ s}^{-1}. It is clearly seen that the extent of drop deformation becomes greater as the viscosity ratio of the drop and the suspending medium \( (\eta_{0d}/\eta_{0m}) \) decreases from 5 to 0.25 in Figure B.30 or from 5 to 0.5 in Figure B.31 (see Table 4.6).
Figure B.30  Snap shots of the computed shapes of five different Newtonian drops (Case VI through Case X) with the identical initial radius \( r_0 \) of 0.45 mm and the viscosity \( \eta_0d \) given in Table 4.6 when each of the drops was suspended in the same Newtonian medium, moving at \( \dot{\gamma}_{app} = 15.4 \text{ s}^{-1} \), with the viscosity \( \eta_{0m} \) given in Table 4.6. The snap shots of the computed shape of the drops are at the same dimensionless axial position of \( z/R_0 = 5 \) in the entrance region of a cylindrical tube for different drops: (a) Case VI, (b) Case VII, (c) Case VIII, (d) Case IX, and (e) Case X.
Figure B.31 Snap shots of the computed shapes of five different Newtonian drops (Case VI through Case IX) with the identical initial radius ($r_0$) of 0.45 mm and the viscosity $\eta_{0d}$ given in Table 4.6 when each of the drops was suspended in the same Newtonian medium, moving at $\dot{\gamma}_{app} = 75.4$ s$^{-1}$, with the viscosity $\eta_{0m}$ given in Table 4.6. The snap shots of the computed shape of the drops are at the same dimensionless axial position of $z/R_0 = 5$ in the entrance region of a cylindrical tube for different drops: (a) Case VI, (b) Case VII, (c) Case VIII, and (d) Case IX.
In order to have quantitative comparison of the extent of drop deformation among the five drops investigated, we calculated the axial ratio \(a/b\) of the long side \(a\) and short side \(b\) of the deformed drop. Table B.9 gives a summary of the calculated values of the axial ratio \(a/b\) for the five drops at four different positions (A, B, C, and D) in the entrance region of the cylindrical tube at an apparent shear rate of 15.4 s\(^{-1}\), and Table B.10 gives a summary of the calculated values of the axial ratio \(a/b\) for the five drops at four different positions (A, B, C, and D) in the entrance region of the cylindrical tube at an apparent shear rate of 75.4 s\(^{-1}\), showing a very clear trend for the extent of drop deformation as affected by the \(\eta_{0d}/\eta_{0m}\) ratio. Namely, the extent of drop deformation increases dramatically as the \(\eta_{0d}/\eta_{0m}\) is decreased from 5 to 0.25 (Table B.9) or from 5 to 0.5 (Table B.10).

Table B.9 The axial ratio \(a/b\) of five different Newtonian drops (designated as Case VI, Case VII, Case VIII, Case IX, and Case X) with the initial radius \(r_0\) of 0.45 mm, each suspended in a Newtonian medium (see Table 4.6), moving along the centerline of a cylindrical tube in the entrance region at \(\dot{\gamma}_{\text{app}} = 15.4\) s\(^{-1}\). The values of \(a/b\) ratio listed here are determined from the calculated drop shapes, given in Figures B.21–B.25, at different positions along the centerline of a cylindrical tube in the entrance region.

<table>
<thead>
<tr>
<th>Drop</th>
<th>The (a/b) ratio of a drop at the axial position A, (z/R_0 = -10)</th>
<th>The (a/b) ratio of a drop at the axial position B, (z/R_0 = -5)</th>
<th>The (a/b) ratio of a drop at the axial position C</th>
<th>The (a/b) ratio of a drop at the axial position D, (z/R_0 = 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case VI</td>
<td>1.222</td>
<td>1.376</td>
<td>2.967 ((z/R_0 = 0.135))</td>
<td>2.318</td>
</tr>
<tr>
<td>Case VII</td>
<td>1.385</td>
<td>1.736</td>
<td>5.302 ((z/R_0 = 0.137))</td>
<td>4.182</td>
</tr>
<tr>
<td>Case VIII</td>
<td>1.510</td>
<td>2.182</td>
<td>7.484 ((z/R_0 = 0.140))</td>
<td>5.904</td>
</tr>
<tr>
<td>Case IX</td>
<td>1.825</td>
<td>2.760</td>
<td>10.324 ((z/R_0 = 0.143))</td>
<td>8.198</td>
</tr>
<tr>
<td>Case X</td>
<td>2.385</td>
<td>4.227</td>
<td>14.631 ((z/R_0 = 0.145))</td>
<td>11.691</td>
</tr>
</tbody>
</table>
Table B.10  The axial ratio \(a/b\) of four different Newtonian drops (designated as Case VI, Case VII, Case VIII, and Case IX) with the initial radius \((r_0)\) of 0.45 mm, each suspended in a Newtonian medium (see Table 4.6), moving along the centerline of a cylindrical tube in the entrance region at \(\dot{\gamma}_{app} = 75.4\) s\(^{-1}\). The values of \(a/b\) ratio listed here are determined from the calculated drop shapes, given in Figures B.26–B.29, at different positions along the centerline of a cylindrical tube in the entrance region.

<table>
<thead>
<tr>
<th>Drop</th>
<th>The (a/b) ratio of a drop at the axial position (z/R_0 = -10)</th>
<th>The (a/b) ratio of a drop at the axial position (z/R_0 = -5)</th>
<th>The (a/b) ratio of a drop at the axial position (z/R_0 = 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case VI</td>
<td>2.051</td>
<td>2.778</td>
<td>6.343 ((z/R_0 = 0.159))</td>
</tr>
<tr>
<td>Case VII</td>
<td>2.280</td>
<td>3.287</td>
<td>11.228 ((z/R_0 = 0.162))</td>
</tr>
<tr>
<td>Case VIII</td>
<td>2.659</td>
<td>4.153</td>
<td>17.328 ((z/R_0 = 0.165))</td>
</tr>
<tr>
<td>Case IX</td>
<td>3.362</td>
<td>5.485</td>
<td>23.415 ((z/R_0 = 0.167))</td>
</tr>
</tbody>
</table>

B.2.2  Effects of Apparent Shear Rate on the Extent of Deformation of a Newtonian Drop Suspended in a Newtonian Medium Moving along the Centerline of a Cylindrical Tube in the Entrance Region

In Table 4.8 of Chapter 4, we have presented a summary of the effect of apparent shear rate on the axial ratio \((a/b)\) of the drop (designated as Case VII in Table 4.6) at three different positions (B, C, and D) in the entrance region of the cylindrical tube.

Table B.11 gives a summary of the effect of apparent shear rate on the axial ratio \((a/b)\) of the drop (designated as Case VI in Table 4.6) at three different positions (B, C, and D) in the entrance region of the cylindrical tube. Table B.12 gives a summary of the effect of apparent shear rate on the axial ratio \((a/b)\) of the drop (designated as Case VIII
Table B.11  The axial ratio $a/b$ of Newtonian drops (designated as Case VI in Table 4.6) with the initial radius of 0.45 mm, suspended in a Newtonian medium (see Table 4.6), moving along the centerline of a cylindrical tube in the entrance region at three different shear rates, $\dot{\gamma}_{\text{app}} = 15.4, 35.4, \text{and } 75.4 \text{ s}^{-1}$. The values of $a/b$ ratio listed here are determined from the calculated shapes of drops, which are given in the Appendix B of this dissertation, at different positions along the centerline of a cylindrical tube in the entrance region.

<table>
<thead>
<tr>
<th>Apparent shear rate (s$^{-1}$)</th>
<th>The a/b ratio of a drop at the axial position B, $z/R_0 = -5$</th>
<th>The a/b ratio of a drop at the axial position C</th>
<th>The a/b ratio of a drop at the axial position D, $z/R_0 = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15.4</td>
<td>1.376</td>
<td>2.906 ($z/R_0 = 0.135$)</td>
<td>2.318</td>
</tr>
<tr>
<td>35.4</td>
<td>1.990</td>
<td>4.188 ($z/R_0 = 0.146$)</td>
<td>3.272</td>
</tr>
<tr>
<td>75.4</td>
<td>2.778</td>
<td>6.343 ($z/R_0 = 0.159$)</td>
<td>4.936</td>
</tr>
</tbody>
</table>

Table B.12  The axial ratio $a/b$ of Newtonian drops (designated as Case VIII in Table 4.6) with the initial radius of 0.45 mm, suspended in a Newtonian medium (see Table 4.6), moving along the centerline of a cylindrical tube in the entrance region at three different shear rates, $\dot{\gamma}_{\text{app}} = 15.4, 35.4, \text{and } 75.4 \text{ s}^{-1}$. The values of $a/b$ ratio listed here are determined from the calculated shapes of drops, which are given in the Appendix B of this dissertation, at different positions along the centerline of a cylindrical tube in the entrance region.

<table>
<thead>
<tr>
<th>Apparent shear rate (s$^{-1}$)</th>
<th>The a/b ratio of a drop at the axial position B, $z/R_0 = -5$</th>
<th>The a/b ratio of a drop at the axial position C</th>
<th>The a/b ratio of a drop at the axial position D, $z/R_0 = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15.4</td>
<td>2.182</td>
<td>7.484 ($z/R_0 = 0.140$)</td>
<td>5.904</td>
</tr>
<tr>
<td>35.4</td>
<td>2.871</td>
<td>11.681 ($z/R_0 = 0.150$)</td>
<td>9.211</td>
</tr>
<tr>
<td>75.4</td>
<td>4.153</td>
<td>17.328 ($z/R_0 = 0.165$)</td>
<td>13.566</td>
</tr>
</tbody>
</table>

in Table 4.6) at three different positions (B, C, and D) in the entrance region of the cylindrical tube. Table B.13 gives a summary of the effect of apparent shear rate on the
Table B.13  The axial ratio $a/b$ of Newtonian drops (designated as Case IX in Table 4.6) with the initial radius of 0.45 mm, suspended in a Newtonian medium (see Table 4.6), moving along the centerline of a cylindrical tube in the entrance region at three different shear rates, $\dot{\gamma}_{\text{app}} = 15.4, 35.4, \text{ and } 75.4 \text{ s}^{-1}$. The values of $a/b$ ratio listed here are determined from the calculated shapes of drops, which are given in the Appendix B of this dissertation, at different positions along the centerline of a cylindrical tube in the entrance region.

<table>
<thead>
<tr>
<th>Apparent shear rate ($s^{-1}$)</th>
<th>The $a/b$ ratio of a drop at the axial position B, $z/R_0 = -5$</th>
<th>The $a/b$ ratio of a drop at the axial position C</th>
<th>The $a/b$ ratio of a drop at the axial position D, $z/R_0 = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15.4</td>
<td>2.760</td>
<td>10.324 ($z/R_0 = 0.143$)</td>
<td>8.198</td>
</tr>
<tr>
<td>35.4</td>
<td>3.901</td>
<td>16.523 ($z/R_0 = 0.152$)</td>
<td>13.080</td>
</tr>
<tr>
<td>75.4</td>
<td>5.485</td>
<td>23.415 ($z/R_0 = 0.167$)</td>
<td>18.959</td>
</tr>
</tbody>
</table>

axial ratio ($a/b$) of the drop (designated as Case IX in Table 4.6) at three different positions (B, C, and D) in the entrance region of the cylindrical tube. It is clearly seen in Tables B.11–B.13 that the extent of drop deformation increases as the apparent shear rate is increased.

B.3 The Effect of Fluid Elasticity on the Deformation of a Viscoelastic Drop and Subsequent Recoil in the Entrance Region of a Cylindrical Tube

In Chapter 4 we have summarized the calculated values of aspect ratio $a/b$ of two deformed drops, a Newtonian drop (Case VI) suspended in a Newtonian medium and a viscoelastic drop (Case I) suspending in a viscoelastic medium, both having the same
initial radius \( r_0 \) of 0.45 mm, moving along the centerline of a cylindrical tube in the entrance region at \( \dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1} \) (see Table 4.9).

Table B.14 gives a summary of the calculated values of aspect ratio \( a/b \) of two deformed drops, a Newtonian drop (Case VII) suspended in a Newtonian medium and a viscoelastic drop (Case II) suspending in a viscoelastic medium, both having the same initial radius \( r_0 \) of 0.45 mm, moving along the centerline of a cylindrical tube in the entrance region at \( \dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1} \). Table B.15 gives a summary of the calculated values

Table B.14  The \( a/b \) ratio of two drops with the initial radius \( r_0 \) of 0.45 mm, each suspended in a medium (corresponding to Case II and Case VII), moving along the centerline of a cylindrical tube in the entrance region at \( \dot{\gamma}_{\text{app}} = 35.4 \text{ s}^{-1} \). The values of \( a/b \) ratio listed here are determined from the calculated drop shapes, at different positions along the centerline of a cylindrical tube in the entrance region. Note that a maximum of \( a/b \) ratio occurs at position (E) \( z/R_0 = 1.193 \) for Case II and at position (F) \( z/R_0 = 0.148 \) for Case VII.

<table>
<thead>
<tr>
<th>Axial position ((z/R_0))</th>
<th>The ( a/b ) ratio in Case II</th>
<th>The ( a/b ) ratio in Case VII</th>
</tr>
</thead>
<tbody>
<tr>
<td>–5</td>
<td>2.302</td>
<td>2.379</td>
</tr>
<tr>
<td>–4</td>
<td>2.510</td>
<td>2.801</td>
</tr>
<tr>
<td>–3</td>
<td>2.741</td>
<td>3.358</td>
</tr>
<tr>
<td>–2</td>
<td>3.019</td>
<td>4.217</td>
</tr>
<tr>
<td>–1</td>
<td>3.319</td>
<td>5.383</td>
</tr>
<tr>
<td>0</td>
<td>3.809</td>
<td>7.535</td>
</tr>
<tr>
<td>0.148 (F)</td>
<td></td>
<td>7.687</td>
</tr>
<tr>
<td>1</td>
<td>7.530</td>
<td>6.688</td>
</tr>
<tr>
<td>1.193 (E)</td>
<td>7.907</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>7.771</td>
<td>6.280</td>
</tr>
<tr>
<td>3</td>
<td>7.720</td>
<td>6.163</td>
</tr>
<tr>
<td>4</td>
<td>7.675</td>
<td>6.125</td>
</tr>
<tr>
<td>5</td>
<td>7.632</td>
<td>6.089</td>
</tr>
</tbody>
</table>
Table B.15 The a/b ratio of two drops with the initial radius ($r_0$) of 0.45 mm, each suspended in a medium (corresponding to Case III and Case VIII), moving along the centerline of a cylindrical tube in the entrance region at $\dot{\gamma}_{app} = 35.4 \text{ s}^{-1}$. The values of a/b ratio listed here are determined from the calculated drop shapes, at different positions along the centerline of a cylindrical tube in the entrance region. Note that a maximum of a/b ratio occurs at position (E) $z/R_0 = 1.238$ for Case III and at position (F) $z/R_0 = 0.150$ for Case VIII.

<table>
<thead>
<tr>
<th>Axial position ($z/R_0$)</th>
<th>The a/b ratio in Case III</th>
<th>The a/b ratio in Case VIII</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>2.741</td>
<td>2.823</td>
</tr>
<tr>
<td>-4</td>
<td>3.027</td>
<td>3.447</td>
</tr>
<tr>
<td>-3</td>
<td>3.481</td>
<td>4.363</td>
</tr>
<tr>
<td>-2</td>
<td>4.004</td>
<td>5.732</td>
</tr>
<tr>
<td>-1</td>
<td>4.601</td>
<td>7.998</td>
</tr>
<tr>
<td>0</td>
<td>5.597</td>
<td>11.552</td>
</tr>
<tr>
<td>0.150 (F)</td>
<td></td>
<td>11.681</td>
</tr>
<tr>
<td>1</td>
<td>11.419</td>
<td>10.102</td>
</tr>
<tr>
<td>1.238 (E)</td>
<td>11.989</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>11.750</td>
<td>9.501</td>
</tr>
<tr>
<td>3</td>
<td>11.664</td>
<td>9.323</td>
</tr>
<tr>
<td>4</td>
<td>11.595</td>
<td>9.267</td>
</tr>
<tr>
<td>5</td>
<td>11.528</td>
<td>9.213</td>
</tr>
</tbody>
</table>

of aspect ratio $a/b$ of two deformed drops, a Newtonian drop (Case VIII) suspended in a Newtonian medium and a viscoelastic drop (Case III) suspended in a viscoelastic medium, Table B.16 gives a summary of the calculated values of aspect ratio $a/b$ of two deformed drops, a Newtonian drop (Case IX) suspended in a Newtonian medium and a viscoelastic drop (Case VI) suspended in a viscoelastic medium, and Table B.17 gives a summary of the calculated values of aspect ratio $a/b$ of two deformed drops, a Newtonian drop (Case X) suspended in a Newtonian medium and a viscoelastic drop (Case V)
Table B.16 The a/b ratio of two drops with the initial radius ($r_0$) of 0.45 mm, each suspended in a medium (corresponding to Case IV and Case IX), moving along the centerline of a cylindrical tube in the entrance region at $\gamma_{app} = 35.4 \text{ s}^{-1}$. The values of a/b ratio listed here are determined from the calculated drop shapes, at different positions along the centerline of a cylindrical tube in the entrance region. Note that a maximum of a/b ratio occurs at position (E) $z/R_0 = 1.286$ for Case IV and at position (F) $z/R_0 = 0.152$ for Case IX.

<table>
<thead>
<tr>
<th>Axial position ($z/R_0$)</th>
<th>The a/b ratio in Case IV</th>
<th>The a/b ratio in Case IX</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>3.619</td>
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<td>16.161</td>
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<tr>
<td>0.152 (F)</td>
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<tr>
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<td>15.795</td>
<td>14.272</td>
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<tr>
<td>1.286 (E)</td>
<td>16.924</td>
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</tr>
<tr>
<td>2</td>
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<td>13.503</td>
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<td>13.177</td>
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<tr>
<td>5</td>
<td>16.202</td>
<td>13.099</td>
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suspended in a viscoelastic medium, under the otherwise identical initial drop size ($r_0 = 0.45 \text{ mm}$) and flow conditions (at $\gamma_{app} = 35.4 \text{ s}^{-1}$). Notice that a maximum of a/b ratio occurs at position (E) $z/R_0 = 1.193$ for a viscoelastic drop (Case II) and at position (F) $z/R_0 = 0.148$ for a Newtonian drop (Case VII) (see Table B.14), a maximum of a/b ratio occurs at position (E) $z/R_0 = 1.238$ for a viscoelastic drop (Case III) and at position (F) $z/R_0 = 0.150$ for a Newtonian drop (Case VIII) (see Table B.15), a maximum of a/b ratio occurs at position (E) $z/R_0 = 1.286$ for a viscoelastic drop (Case IV) and at position (F)
Table B.17  The a/b ratio of two drops with the initial radius ($r_0$) of 0.45 mm, each suspended in a medium (corresponding to Case V and Case X), moving along the centerline of a cylindrical tube in the entrance region at $\dot{\gamma}_{app} = 35.4$ s$^{-1}$. The values of a/b ratio listed here are determined from the calculated drop shapes, at different positions along the centerline of a cylindrical tube in the entrance region. Note that a maximum of a/b ratio occurs at position (E) $z/R_0 = 1.338$ for Case V and at position (F) $z/R_0 = 0.153$ for Case X.

<table>
<thead>
<tr>
<th>Axial position ($z/R_0$)</th>
<th>The a/b ratio in Case V</th>
<th>The a/b ratio in Case X</th>
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<td>0.153 (F)</td>
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</tr>
<tr>
<td>1.338 (E)</td>
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<td>5</td>
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<td>16.460</td>
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</table>

$z/R_0 = 0.152$ for a Newtonian drop (Case IX) (see Table B.16), a maximum of a/b ratio occurs at position (E) $z/R_0 = 1.338$ for a viscoelastic drop (Case II) and at position (F) $z/R_0 = 0.153$ for a Newtonian drop (Case VII) (see Table B.17).

In order to explain the above observation, in this study we computed the extensional stress distribution along the centerline, $\sigma_{zz}(z/R_0)$, of the converging cylindrical tube for a Newtonian suspending medium and a viscoelastic suspending medium, respectively, and the results are summarized in Figure 4.19 in which a Newtonian drop (Case VI) and a
viscoelastic drop (*Case I*), respectively, are suspended in a respective suspending medium at an apparent shear rate of 35.4 s$^{-1}$. Figures B.32 and B.33 give the extensional stress distribution along the centerline, $\sigma_{zz}(z/R_0)$, of the converging cylindrical tube for a Newtonian suspending medium (*Case VI*) and a viscoelastic suspending medium (*Case I*), respectively, an apparent shear rate of 15.4 s$^{-1}$ (Figure B.32) and 75.4 s$^{-1}$ (Figure B.33), respectively.

Figure B.32 Extensional stress distribution along the centerline of the cylindrical tube: (1) Newtonian system, (2) viscoelastic system) when a Newtonian drop with initial radius ($r_0$) of 0.45 mm is suspended in the Newtonian medium, moving at $\dot{\gamma}_{\text{app}} = 15.4$ s$^{-1}$, (corresponding *Case VI* in Table 4.6), and a viscoelastic drop of 0.45 mm is suspended in the viscoelastic medium moving at $\dot{\gamma}_{\text{app}} = 15.4$ s$^{-1}$, (corresponding *Case I* in Table 4.1).
Figure B.33  Extensional stress distribution along the centerline of the cylindrical tube: (1) Newtonian system, (2) viscoelastic system) when a Newtonian drop with initial radius \(r_0\) of 0.45 mm is suspended in the Newtonian medium, moving at \(\dot{\gamma}_{\text{app}} = 75.4 \text{ s}^{-1}\), (corresponding Case VI in Table 4.6), and a viscoelastic drop of 0.45 mm is suspended in the viscoelastic medium moving at \(\dot{\gamma}_{\text{app}} = 75.4 \text{ s}^{-1}\), (corresponding Case I in Table 4.1).

It is of great interest to observe in Figures B.32, B.33 and 4.19 that a maximum of extensional stress \(\sigma_{zz}(z/R_0)\) of a Newtonian suspending medium occurs before reaching the entrance \((z/R_0 = 0)\) of the straight section of the converging cylindrical tube, while a maximum of \(\sigma_{zz}(z/R_0)\) of a viscoelastic suspending medium occurs after passing the entrance of the straight section \((z/R_0 > 0)\) of the converging cylindrical tube. Since a
drop (viscoelastic or Newtonian) moves along the central axis of a cylindrical tube by extensional flow, it is very reasonable to speculate that the deformation of a drop is subjected to extensional stress along the central axis of the cylindrical tube. Thus we can conclude that the position at which a maximum of aspect ratio $a/b$ of a drop is observed in the converging section of a cylindrical tube is closely related to the position at which a maximum of extensional stress $\sigma_{zz}(z/R_0)$ is observed along the axis of a cylindrical tube.

The above observations suggest that a Newtonian drop would deform faster than a viscoelastic drop as the respective drops move along the central axis of a converging cylindrical tube. This indeed is borne out to be the case from our computational results when the extent of deformation of a Newtonian drop ($\textit{Case VI}$) given in Figures 4.13, B.21, and B.26 is compared with that of a viscoelastic drop ($\textit{Case I}$) given in Figures 4.7, B.1, B.7, B.12, and B.17. Similar observations can be made (i) when the extent of deformation given in Figures 4.14, B.22, and B.27 for a Newtonian drop ($\textit{Case VII}$) is compared with that given Figures 4.8, B.2, B.8, B.13, and B.18 for a viscoelastic drop ($\textit{Case II}$), (ii) when the extent of deformation given in Figures 4.15, B.23 and B.28 for a Newtonian drop ($\textit{Case VIII}$) is compared with that given Figures 4.9, B.3, B.9, B.14, and B.19 for a viscoelastic drop ($\textit{Case III}$), (iii) when the extent of deformation given in Figures 4.16, B.24 and B.29 for a Newtonian drop ($\textit{Case IX}$) is compared with that given in Figures 4.10, B.4, B.10, B.15 and B.20 for a viscoelastic drop ($\textit{Case IV}$), and (iv) when the extent of deformation given in Figures 4.17 and B.25 for a Newtonian drop ($\textit{Case X}$) is compared with that given in Figures 4.11, B.5 and B.16 for a viscoelastic drop ($\textit{Case V}$). The difference in the rate of the extent of drop deformation between Newtonian and
viscoelastic drops, each moving along the central axis of a converging cylindrical tube, is attributable to the elasticity of a viscoelastic drop. This is because the large elastic modulus of a viscoelastic drop would tend to resist deformation, as compared to the deformation of a Newtonian drop.

Another significant observation of our computed shape of drops is the rate at which an elongated drop recoils after passing the position at which a maximum aspect ratio occurs. Notice in Tables B.14–B.17 and Table 4.9 that a viscoelastic drop recoils at a much slower rate as compared to a Newtonian drop after the respective drop passes the position at which a maximum aspect ratio is attained. It is clearly seen in Figures B.32, B.33 and 4.19 that after passing the position at which a maximum aspect ratio of drop occurs, values of extensional stress $\sigma_{zz}(z/R_0)$ for a Newtonian medium decrease very quickly to very small values while values of $\sigma_{zz}(z/R_0)$ for a viscoelastic medium decrease very slowly. Similar observations were also made for other pairs of viscoelastic and Newtonian drops, which are presented above. The above observation can be explained as follows. The recoil of a viscoelastic drop is dictated predominantly by its relaxation modulus whereas the recoil of a Newtonian drop is determined strictly by its interfacial tension. It should be mentioned that the magnitude of the elastic force of a viscoelastic fluid is much greater than that of an interfacial force. Consequently, the relaxation time of a viscoelastic drop is much greater (on the orders of magnitude) than that of a Newtonian drop. Hence the relaxation modulus of a viscoelastic drop is expected to slow down its recoil more than the interfacial tension of a Newtonian drop. Thus we conclude once again that the elasticity of a viscoelastic drop plays a
predominant role in determining its rate of recoil, whereas only the interfacial tension
determines the rate of recoil of a Newtonian drop. This conclusion now explains why the
rate of recoil of a viscoelastic drop is much slower than that of a Newtonian drop in the
entrance region of a cylindrical tube.