A GAME THEORETIC APPROACH TO ADVERTISING STRATEGY

A Thesis

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Master of Science

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ABSTRACT

Developing an effective advertising strategy is a key component for a firm to be successfully competitive. This paper uses game theory to examine a duopolistic setting in which two firms have the same product. One firm has the higher price and larger market share and both want to compete through advertising. Assuming subgame perfection, we use backward induction to solve the game. We consider how changes in pricing, advertising expenditures and advertising effectiveness affect profit and market shares. We also investigate how changes in the size of the market affect the behavior of all factors. We find that as either firm becomes more effective, the other must respond through pricing and/or advertising. When both firms are highly effective in their advertising, it becomes more beneficial for neither firm to advertise. We also find that while the size of the market will affect the overall advertising expenditures, it will only minimally affect the price per unit and profit per unit to either firm.
To my advisers Dr. Clemons, Dr. Norfolk and Dr. Young, thank you for all your help and understanding through the long process of completing my thesis. To Mar, thank you for letting me call you at all hours of the night just to yell about how frustrated I was. I could not have asked for a better best friend. To my siblings Anna, Katie, Eddie and Andrew, thank you for being awesome and always being there to make me laugh and calm down when I let myself get overwhelmed. And especially to my Mom, thank you for all your love and encouragement. You taught me what it meant to work hard and whenever I got discouraged you were always there to keep me moving forward. I would not have made it this far without you. I love you. And to the rest of my family, extended family and friends, thank for all your support through college and the many years before.
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CHAPTER I
INTRODUCTION

In today’s economy, competition for consumers is constantly growing. In order to survive this competitive environment, firms must continually look for ways to set themselves apart from the competition. Current technology allows marketers to take advantage of easily accessible consumer and market information, enabling them to create a strong marketing mix strategy. A significant part of this strategy is developing an effective advertising plan to distinguish a firm’s brand from the competition.

There has been much work done on how different factors affect advertising expenditures and a firm’s overall profits. While the quality of a product impacts price and advertising decisions, the price tends to convey a quality level in the mind of consumers [1]. The perception of the consumer can dictate the advertising expenditures needed to reposition the product in the mind of consumers. When multiple objectives are sought, they are seen to affect the overall strategy of a brand. The brand seeks to maintain market share while increasing profits so its decision making can affect competitors within the market [2]. Venkatesh Shankar considers the case of a new market entrant and utilizes a game theoretic approach to model how a firm should alter its advertising strategy in response to the new entrant’s behavior. Shankar concludes that the firm’s reactions rely heavily on the firm’s own marketing
mix and competitive effectiveness.” If the firm can be competitive and continue to see a growth in sales, it will accommodate the new entrant into the market and retaliate using the strongest of its advertising tools [3].

Game theory has also played an important role in work done on the components of advertising and affecting factors. The quality of a product tends to affect firm profits and the overall market outcomes while quality equilibria become contingent on firm costs and the structure of the industry [4, 5]. Price and quality typically affect one another and there have been several ways to explore how these effects occur. In most cases, decisions about price and quality are made in two stage games [6, 7]. Sometimes Nash equilibria are found for price first and then quality is evaluated using a Stackelberg model [8]. Brand positioning tactics also come into consideration as a result of cooperation between price and quality decisions [6, 9].

Other than quality there are many elements to product design. For example, a cell phone can consist of several options that may be unique to the brand. In some cases a firm can choose to alter one or more of these elements in order to be more competitive. In certain markets it becomes advantageous for a firm to advertise in a manner to maximize the differentiation between itself and the competition in one dimension while minimizing the differentiation in another [10]. Pricing decisions also affect how advertising decisions are made while different advertising strategies can change how pricing is implemented. If advertising strategies increase demand for a firm’s product then prices can be raised and an increase in profit will be seen [11].
Along with developing an advertising strategy, firms must consider consumer preference and their brand loyalties. Firms compete not only for profits but also market shares. In this case, a firm should differentiate its product from other firms to avoid price competition but not too far as to alienate possible consumers [12]. Pricing of a product affects how a firm promotes to potential consumers, especially when using target pricing [13]. Quality also becomes a factor and changes how promotional strategies are implemented [14]. In Sandra Addo’s thesis, *A Game-Theoretic Framework to Competitive Individual Targeting*, a quality component is implemented to explore the affects on price, profit and promotion. Addo finds that consumers who are much more sensitive to quality should be targeted by the larger firm while the smaller firm should target the consumers that are less price sensitive [15].

This paper seeks to answer questions concerning when a firm should advertise and what expenditures should be made. We consider Shaffer and Zhang’s *Competitive One-to-One Promotions* model and exchange their promotional strategies with advertising strategies [13]. We introduce an advertising efficiency component which is dependent on advertising expenditures and the size of the market. As is common, we assume the cost of advertising to be quadratic with respect to advertising effectiveness [16]. Utilizing game theory, we model how advertising decisions can be made most effectively in a duopolistic setting. We explore how characteristics of a firm and the market affect the firm’s capability to advertise at an effective level. We consider two firms, *A* and *B*, where Firm *A* offers the product at a higher price and controls
the majority of the market. The production costs for both firms are fixed and the same.

To answer the advertising questions we must first examine the behavior of the market. We take into consideration how consumer loyalty plays a role in dictating the amount of market share a firm can gain through advertising. In this model, the loyalty to Firm $A$ is greater than the loyalty to Firm $B$, and we use the price differential to indicate the boundary of this loyalty. Firms use advertising to influence the consumers loyal to their competitors and to increase loyalty among their current consumers. A firm can only grow its market share so far since there is always a group of consumers that will remain loyal to a firm regardless of the amount of advertising from a competitive firm.

We must also consider the effectiveness of an advertising plan. When measuring effectiveness it is common to evaluate the reach and frequency of the advertisement [17]. As the number of consumers in the market grows, a firm must spend more in order to have the opportunity to reach each person at a certain frequency. For example, currently there are about 307 million people in the United States today and AT&T Inc. spent approximately $1.2 billion toward their national advertising campaign in the first six months of 2010 [18]. AT&T Inc. must spend more to reach a market the size of the United States than if it were only targeting the state of Ohio. However, as the market grows, it becomes less likely to obtain the reach and frequency goals. Even though AT&T Inc. has the second highest advertising expenditures of all companies advertising in the U.S., it still cannot reach every U.S.
consumer at the needed frequency [18]. The overall effectiveness of an advertising plan that is utilized in this paper takes into consideration both reach and frequency.

The next chapter outlines the model and game in greater detail. Chapters 3 and 4 describe the subgames and solution procedure. Chapter 5 illustrates and analyzes our findings, as well as determines the effects advertising has on market shares and profits. Finally, we summarize our efforts and give suggestions for further exploration.
CHAPTER II
THE MATHEMATICAL MODEL

Consider a duopolistic setting with firms $A$ and $B$. Using the index $i = A, B$ we have the following variables:

<table>
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<th>Variable</th>
<th>Definition</th>
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<tr>
<td>$c \geq 0$</td>
<td>Constant marginal cost for both firms</td>
<td>$\text{per item}$</td>
</tr>
<tr>
<td>$P_i$</td>
<td>Regular price per unit of firm $i$</td>
<td>$\text{per item}$</td>
</tr>
<tr>
<td>$l_i$</td>
<td>Brand loyalty to firm $i$</td>
<td>$\text{per item}$</td>
</tr>
<tr>
<td>$\tilde{l}_i$</td>
<td>Brand loyalty to firm $i$ after advertising</td>
<td>$\text{per item}$</td>
</tr>
<tr>
<td>$\hat{l}$</td>
<td>Location of marginal customer (price differential)</td>
<td>$\text{per item}$</td>
</tr>
<tr>
<td>$\mu_i$</td>
<td>Market share of firm $i$</td>
<td>$\text{people}$</td>
</tr>
<tr>
<td>$z_i \geq 0$</td>
<td>Total cost of advertising for firm $i$</td>
<td>$\text{dollars}$</td>
</tr>
<tr>
<td>$n$</td>
<td>Total number of people in the market</td>
<td>$\text{people}$</td>
</tr>
<tr>
<td>$\Pi_i$</td>
<td>Profit per unit sold for firm $i$</td>
<td>$\text{dollars per item}$</td>
</tr>
<tr>
<td>$\alpha_i(z_i)$</td>
<td>Advertising effectiveness of firm $i$</td>
<td>none</td>
</tr>
<tr>
<td>$\kappa_i$</td>
<td>Advertising effectiveness parameter for firm $i$</td>
<td>$\frac{1}{\sqrt{\text{people}^{\kappa_i}}}$</td>
</tr>
</tbody>
</table>
Given this duopolistic setting, we suppose that firms $A$ and $B$ are competing brands that sell a product produced at the same constant marginal cost, $c \geq 0$. We assume Firm $A$ sells its product at a higher price, $P_A > P_B$. Each person in the market buys exactly one unit of the product yielding a total of $n$ units sold. We consider each firm’s market share as their portion of the size of the market. If one firm increases their market share, the other firm’s market share will decrease proportionally.

Each firm possesses certain consumers that are loyal to their brand. A consumer’s brand loyalty is defined as the maximum price differential he or she is willing to pay in order to remain loyal to his or her preferred brand. If a consumer prefers brand $A$ by a loyalty of $l$, the consumer will continue to buy brand $A$ until the price of brand $A$ exceeds the price of brand $B$ by more than $l$, $P_A > P_B + l$ [13]. We assume Firm $A$ controls a larger share of the market with $l_A \geq l_B$ and that the representation of market shares is uniformly distributed for each firm. We use the price differential, $\hat{l} = P_A - P_B$, to define the marginal consumer and the boundary between brand loyalties. Consumers located at $l \geq \hat{l}$ will buy from firm $A$ while consumers located at $l < \hat{l}$ will buy from firm $B$. Figure 2.1 shows the initial distribution of the market. In our model, the number of people in the market is equal to the number of items sold since we assume that each person buys exactly one item.

The game is played in two stages with decisions being made about pricing in the first stage and advertising expenditures in the second. This structure demonstrates the general view that pricing decisions are made at high managerial levels and are slow to be adjusted [13]. Each firm looks to maximize their profits by choosing
The overall effectiveness of a firm’s advertising strategy is a complex component of the game. In general, there is no concrete way to define effectiveness since it differs with each firm and advertising strategy. Typically, certain goals of reach and frequency are established to determine the success of the delivery of an advertising campaign. The success of the delivery is dependent on the media channels utilized as well as the advertising expenditures of the firm. However, these performance indicators do not reflect much on the effectiveness of the advertisement itself. Predictors such as cognitive response measures are used to determine whether the advertisement achieves certain objectives. These objectives can range anywhere from seizing the attention of the viewer to solidifying a place in the viewer’s mind. In order to accurately determine the effectiveness of a campaign, both areas of analysis must be considered [17].
This paper defines the effectiveness of a firm’s advertising strategy as

$$\alpha_i(z_i) = \kappa_i \sqrt{z_i}. \quad (2.1)$$

The effectiveness depends directly on the cost of advertising, which is assumed to be quadratic with respect to advertising effectiveness, as is done in literature [16]. We assume that as advertising expenditures increase, factors such as reach and frequency will quickly increase and then eventually level out, as is typical of most advertising campaigns. The effectiveness parameter, $\kappa_i$, takes into consideration the effectiveness of the advertisement as well as any other variables which may alter the success of the campaign. We define this parameter as $\kappa_i = \tilde{\kappa}_i q$, where $\frac{1}{2} \leq q \leq 1$. The range of $q$ is critical to this model in order to achieve realistic results. The parameter $q$ is used as a measure of how the number of people in the market reflects the reality of the loss of effectiveness as more people are required to be reached. A more detailed explanation of the bounds of $q$ will be outlined further into the chapter. The parameter $\tilde{\kappa}_i$ corresponds to the potential effectiveness firm $i$ can achieve through its advertising campaign.

Figures 2.2 and 2.3 illustrate how components of the advertising effectiveness parameter are dependent on market size and advertising expenditures. In Figure 2.2 (a) we can see that as the market grows, the advertising effectiveness parameter, $\kappa_i = \tilde{\kappa}_i q$, decreases. As $q$ decreases, the decrease becomes slightly less drastic. In Figure 2.2 (b), we see that as $\tilde{\kappa}_i$ increases, advertising effectiveness increases. The
values of $\kappa_i$ and $q$ will vary for each advertising campaign. If the campaign is expected to be more effective, $q$ will be small and $\kappa_i$ will be large, while a less effective campaign will have a larger $q$ and a small $\kappa_i$. In Figure 2.3 we plot the advertising effectiveness equation, $\alpha = \kappa \sqrt{z}$, and see that as more is spent on advertising, the overall advertising effectiveness increases. Also note that as $q$ decreases, the advertising effectiveness parameter ($\kappa_i$), increases and the rate at which the advertising effectiveness is increasing becomes greater. In general, as the market grows it becomes harder to meet the reach and frequency goals, but as more is spent on advertising it becomes more likely to meet those goals. We incorporate this type of advertising effectiveness into the model to demonstrate how advertising affects the loyalty of a consumer. While some consumers will be loyal to a brand despite advertising from
any firm, others are strongly affected by how effectively a campaign reaches them. For example, if a consumer is loyal to Firm A but Firm B has an advertising plan that effectively reaches the consumer, then the consumer will now be loyal to Firm B. Firm B has decreased Firm A’s market share while simultaneously increasing its own through advertising.

A consumer’s loyalty after advertising occurs, and how advertising effects market shares, price and profit are the key contributions of this paper. We introduce a model for this by considering the starting loyalty and the effectiveness of both firms’ advertising plans. In the case of only Firm A advertises, we see that, as stated above, each firm is affected equally by Firm A’s effectiveness. If Firm B’s loyalty is originally located at $l_B$, its loyalty after Firm A advertises will be $\tilde{l}_B = l_B(1 - \alpha_A)$. Firm B’s loyalty is reduced by how many consumers Firm A was effectively able to gain. This is seen from multiplying the advertising effectiveness of Firm A with the original loyalty.
to Firm $B$ and subtracting that from the original loyalty to Firm $B$. The same effect is seen in the new loyalty to Firm $A$, except that Firm $A$ gains the consumers it was able to turn away from Firm $B$, $\hat{l}_A = l_A + \alpha_A l_B$. This differs from Shaffer and Zhang’s paper in that their model used promotional coupons to shift the location of the marginal consumer, $\hat{l}$, while this model shifts the loyalty spectrum through advertising. We use the marginal consumer as a bound for how far that loyalty can be shifted, $-l_B < \hat{l}$. This bound represents the consumers that will always be loyal to Firm $B$ regardless of how effective Firm $A$ advertises.

As stated earlier, $q$ and its bounds are key components of our model. Using Equation 4.7 (c), Figures 2.4 and 2.5 plot advertising expenditures when only Firm $A$ advertises and illustrate why the bounds $\frac{1}{2} \leq q \leq 1$ are necessary. In Figure 2.4 (a), when $q = \frac{1}{2}$, the cost of advertising grows linearly with the size of the market. If $q < \frac{1}{2}$, then advertising expenditures will begin to grow at an unrealistic rate. In Figure 2.4 (c), when $q = 1$, advertising expenditures are constant. In this case, the advertising effectiveness is so low that the firm will not aggressively advertise. This illustrates the real life occurrence of firms that use public relations, at minimal advertising cost, to still advertise its brand to consumers without actually paying to target consumers. It would be unrealistic for a firm to spend anything on advertising if $q > 1$. In Figure 2.5 (a), when $q = \frac{1}{2}$, advertising expenditures per person are constant. If $q < \frac{1}{2}$, the advertising expenditures per person would grow unrealistically as the market grows. In a real life situation we would not expect to pay more per person just because the market grows. In Figure 2.5 (c), when $q = 1$, advertising
expenditures per person are almost zero. This illustrates the same situation as when \( q = 1 \) in Figure 2.4 (c). The firm is so ineffective at advertising that it does not directly target its consumers but instead spends the bare minimum for PR. Again, if \( q > 1 \), it would be unrealistic to assume a firm would spend on advertising.

![Figure 2.4: Firm A’s advertising expenditures when only one firm advertises over varying \( q_A \) and \( n \), \( \hat{\kappa}_i = 1 \), \( l_A = 6 \), \( l_B = 4 \), \( c = 0.00001 \), \( \frac{l_A}{l_A + l_B} = 0.6 \), \( \frac{l_B}{l_A + l_B} = 0.4 \)](image)

(a): \( q_A = \frac{1}{2} \)  
(b): \( q_A = \frac{3}{4} \)  
(c): \( q_A = 1 \)

Figure 2.4: Firm A’s advertising expenditures when only one firm advertises over varying \( q_A \) and \( n \), \( \hat{\kappa}_i = 1 \), \( l_A = 6 \), \( l_B = 4 \), \( c = 0.00001 \), \( \frac{l_A}{l_A + l_B} = 0.6 \), \( \frac{l_B}{l_A + l_B} = 0.4 \)
Utilizing what we know about advertising expenditures and effectiveness, we implement a two-stage game in which the firms look to optimize their profits through pricing competition and advertising strategy. We assume perfect information for both firms at both stages. Pricing decisions are made in the first stage while the advertising strategy is developed in the second stage. We use subgame perfection to find the equilibrium of the game by first solving for the Nash equilibrium of the second stage and then solving the first stage. We break the main game into four subgames, each with both firms having the same strategy options of advertising and not advertising. Each subgame corresponds to the possible relationships of the regular price of a firm with its cost of production and the advertising expenditures per person in the market.

Since we are assuming perfect information, the resulting strategy is limited to the unique situation of both firms having perfect information about themselves, the competition and the market. In reality, a firm will have some certainty of these factors but will have to rely on a few predictive assumptions concerning their advertising effectiveness, their competitors’ advertising effectiveness and expenditures, and the reaction of the market. A proper analysis of the industry and competition combined with a realistic understanding of a company’s own abilities give this model the potential to be effectively implemented in real-life, duopolistic settings.
Figure 2.5: Firm A’s advertising expenditures per person when only one firm advertises over varying $q_A$ and $n$, $\hat{\kappa}_i = 1$, $l_A = 6$, $l_B = 4$, $c = 0.00001$, $\frac{l_A}{l_A + l_B} = 0.6$, $\frac{l_B}{l_A + l_B} = 0.4$.
CHAPTER III

SUBGAMES

Subgame 1: \( P_A \leq c + \frac{2A}{n} \) and \( P_B \leq c + \frac{2B}{n} \)

In this case, the cost of advertising to each firm is significantly high enough that neither can advertise and return a positive profit. For each firm, the price per unit is less than the cost per unit to produce and advertise. We assume only advertising will alter the loyalty of consumers. Since the loyalty of a consumer is subject to the price differential, the next chapter will evaluate pricing levels which affect the market shares due to pricing competition only. The market share equations of each firm remain the same as the initial setup of the model and are

\[
\mu_A^1 = \frac{l_A - \hat{l}}{l_A + l_B} n , \quad \mu_B^1 = \frac{l_B + \hat{l}}{l_A + l_B} n .
\]

Table 3.1: No Firm Advertises

<table>
<thead>
<tr>
<th>Firm B</th>
<th>Advertise</th>
<th>No Advertise</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm A Advertise</td>
<td>(0, 0)</td>
<td>(0, ( \Pi_B^1 ))</td>
</tr>
<tr>
<td>No Advertise</td>
<td>(( \Pi_A^1 ), 0)</td>
<td>(( \Pi_A^1 ), ( \Pi_B^1 ))</td>
</tr>
</tbody>
</table>
In the payoff matrix shown in Table 3.1, we can see that if either firm chooses to promote, their payoff will be 0. This corresponds to a loss in profits or breaking even. The only way either firm can earn a profit is by not advertising. Hence, the Nash equilibrium occurs when neither firm advertises. The equilibrium payoffs for each firm are

\[
\Pi^1_A = \frac{l_A - \hat{l}}{l_A + l_B}(P_A - c), \quad \Pi^1_B = \frac{l_B + \hat{l}}{l_A + l_B}(P_B - c). \tag{3.1}
\]

**Subgame 2:** \(P_A > c + \frac{c_A}{n} \text{ and } P_B \leq c + \frac{c_B}{n}\)

In this case, the cost of advertising to Firm A is at a level where it can advertise and return a positive profit. The unit cost of producing and advertising to Firm B is still too high to make advertising beneficial. In this case, the advertising costs make it more profitable for Firm A to advertise than to not advertise. This allows Firm A the opportunity to increase its market share while simultaneously decreasing the market share of Firm B. Recall that advertising effectiveness directly affects the loyalty of a consumer that is loyal to another firm. Firm A was able to profitably advertise with a certain amount of advertising effectiveness and change the loyalty of a number of consumers that were loyal to Firm B. Given this, the new loyalty to Firm A is \(\tilde{l}_A = \kappa_A l_B \sqrt{z_A} + l_A\) and the new loyalty to Firm B is \(-\tilde{l}_B = \kappa_A l_B \sqrt{z_A} - l_B\).

These new loyalties shift the current loyalty spectrum and were derived from our earlier discussion of how Firm B’s loyalty changes by the number of people Firm A was effectively able to turn away from Firm B, or \(\tilde{l}_B = l_B(1 - \alpha_A)\). The new market
shares are
\[ \mu_A^2 = \frac{\kappa_A l_B \sqrt{z_A} + l_A - \hat{l}}{l_A + l_B} n , \quad \mu_B^2 = \frac{-\kappa_A l_B \sqrt{z_A} + l_B + \hat{l}}{l_A + l_B} n . \]

The market shares of Firm A increase exactly proportional to the decrease of the market shares of Firm B. This reflects the idea of market shares as a portion of a whole. Even if the actual size of the market changes, the firms can only exchange market shares, they cannot create nor eliminate them.

Table 3.2: Only Firm A Advertises

<table>
<thead>
<tr>
<th>Firm A</th>
<th>Firm B</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Advertise</td>
<td>(\Pi_A^2, 0)</td>
<td>(\Pi_A^2, \Pi_B^2)</td>
</tr>
<tr>
<td>No Advertise</td>
<td>(\Pi_A^1, 0)</td>
<td>(\Pi_A^1, \Pi_B^1)</td>
</tr>
</tbody>
</table>

In Table 3.2, we see the payoff matrix for this case. While Firm A will make a profit if it does not advertise, its payoff will be greater when it does advertise. Firm B will still have a payoff of 0 if it chooses to advertise. Therefore, the Nash equilibrium will occur when only Firm A advertises and the payoffs are

\[ \Pi_A^2 = \frac{\kappa_A l_B \sqrt{z_A} + l_A - \hat{l}}{l_A + l_B} (P_A - c) - \frac{z_A}{n} , \quad \Pi_B^2 = \frac{-\kappa_A l_B \sqrt{z_A} + l_B + \hat{l}}{l_A + l_B} (P_B - c) . \] (3.2)
Subgame 3: $P_A \leq c + \frac{z_A}{n}$ and $P_B > c + \frac{z_B}{n}$

In this case, the cost of advertising to Firm $B$ is at a level to make advertising preferred. The cost to Firm $A$ makes it unprofitable to advertise. This case is analogous to Subgame 2. The market shares to each firm are

$$
\mu_A^3 = \frac{-\kappa_B l_A \sqrt{z_B} + l_A - \hat{l}}{l_A + l_B} n , \quad \mu_B^3 = \frac{\kappa_B l_A \sqrt{z_B} + l_B + \hat{l}}{l_A + l_B} n .
$$

The payoff matrix is shown in Table 3.3.

<table>
<thead>
<tr>
<th>Firm B</th>
<th>Advertise</th>
<th>No Advertise</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm A</td>
<td>$\Pi_A^3$</td>
<td>$\Pi_A^1$</td>
</tr>
<tr>
<td>Advertise</td>
<td>$(0, \Pi_B^3)$</td>
<td>$(0, \Pi_B^1)$</td>
</tr>
<tr>
<td>No Advertise</td>
<td>$(\Pi_A^3, \Pi_B^3)$</td>
<td>$(\Pi_A^1, \Pi_B^1)$</td>
</tr>
</tbody>
</table>

The Nash equilibrium will occur when only Firm $B$ advertises and the payoffs are

$$
\Pi_A^3 = \frac{-\kappa_B l_A \sqrt{z_B} + l_A - \hat{l}}{l_A + l_B} (P_A - c) , \quad \Pi_B^3 = \frac{\kappa_B l_A \sqrt{z_B} + l_B + \hat{l}}{l_A + l_B} (P_B - c) - \frac{z_B}{n} . \quad (3.3)
$$
Subgame 4: $P_A > c + \frac{z_A}{n}$ and $P_B > c + \frac{z_B}{n}$

In this case, the cost to advertise for both firms is at a level in which both can advertise and return a positive profit. Since both firms have the opportunity to advertise, both will affect the redistribution of market shares. The new loyalties to each firm are

$$\tilde{l}_A = \kappa_A l_B \sqrt{z_A} - \kappa_B l_A \sqrt{z_B} + l_A$$ and $$-\tilde{l}_B = \kappa_A l_B \sqrt{z_A} - \kappa_B l_A \sqrt{z_B} - l_B.$$ The market shares to each firm are

$$\mu^A = \frac{\kappa_A l_B \sqrt{z_A} - \kappa_B l_A \sqrt{z_B} + l_A - \hat{l}}{l_A + l_B}, \quad \mu^B = \frac{-\kappa_A l_B \sqrt{z_A} + \kappa_B l_A \sqrt{z_B} + l_B + \hat{l}}{l_A + l_B}.$$

Table 3.4 shows the payoff matrix for this case.

Table 3.4: Both Firms Advertise

<table>
<thead>
<tr>
<th>Firm A</th>
<th>Firm B</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Advertise</td>
<td>No Advertise</td>
<td></td>
</tr>
<tr>
<td>Advertise</td>
<td>$(\Pi_A^4, \Pi_B^4)$</td>
<td>$(\Pi_A^2, \Pi_B^2)$</td>
<td></td>
</tr>
<tr>
<td>No Advertise</td>
<td>$(\Pi_A^3, \Pi_B^3)$</td>
<td>$(\Pi_A^1, \Pi_B^1)$</td>
<td></td>
</tr>
</tbody>
</table>

Each firm has the opportunity to earn a positive profit regardless of whether or not it advertises, however it is optimal for each firm to advertise given the conditions. Therefore, the Nash equilibrium occurs when both firms advertise and the payoffs are
\[ \Pi_A^4 = \frac{\kappa_A l_B \sqrt{z_A} - \kappa_B l_A \sqrt{z_B} + l_A \hat{t}}{l_A + l_B} (P_A - c) - \frac{z_A}{n}, \]

\[ \Pi_B^4 = \frac{-\kappa_A l_B \sqrt{z_A} + \kappa_B l_A \sqrt{z_B} + l_B + \hat{t}}{l_A + l_B} (P_B - c) - \frac{z_B}{n}. \]

In the next chapter we outline the solution procedure for each subgame.
CHAPTER IV
SOLUTION PROCEDURE

In this chapter we outline the solution procedure for solving the game. This two-stage game seeks to optimize the payoffs to firms $A$ and $B$ through price competition and advertising strategy. In the first stage of the game, pricing decisions are made while advertising decisions are established in the second stage. We assume perfect information is available to both firms at both stages of the game. This allows us to use a solution procedure of subgame perfection where backward induction is used to first solve for the Nash equilibria of the second stage [13]. Our game is subdivided into four subgames which represent all the possible relationships of the main game. Solving these subgames yields the Nash equilibria of the second stage. We then optimize the equilibria of the second stage and analyze them in the first stage.

In the previous chapter we found the Nash equilibria of the second stage, which will be differentiated with respect to price and advertising expenditure to find the optimal payoffs. These optimal payoffs will then be evaluated in the first stage of the game. In the first stage, both firms still have the same strategy options, advertise and do not advertise. The variation of this stage occurs as a result of the pricing decisions being made. By analyzing each subgame we determine the optimal pricing for all possible pricing and advertising cost relationships and ultimately find the op-
timal pricing strategy to make advertising beneficial for both firms. We continue the solution procedure by optimizing the Nash equilibria found in the second stage.

**Subgame 1: Neither Firm Advertise**

In this case, the cost of advertising to each firm was at a high enough level to prevent both firms from being able to advertise. The equilibrium payoffs are

\[
\Pi_A^1 = \frac{l_A - \hat{l}}{l_A + l_B} (P_A - c),
\]

\[
\Pi_B^1 = \frac{l_B + \hat{l}}{l_A + l_B} (P_B - c).
\]

Since neither firm advertises, each only has control over its own pricing level. This initiates a pricing competition to change the price differential or the brand loyalty limit. To find the pricing level that will yield the best payoffs, we differentiate Equations (4.1) with respect to their regular prices,

\[
\frac{\partial \Pi_A^1}{\partial P_A} = \frac{l_A - 2P_A + P_B + c}{l_A + l_B},
\]

\[
\frac{\partial \Pi_B^1}{\partial P_B} = \frac{l_B + P_A - 2P_B + c}{l_A + l_B}.
\]

We set Equations (4.2) equal to zero and solve the system of equations to find the optimizing equilibrium price levels.
\[ P_{A}^{1*} = \frac{2l_{A} + l_{B}}{3} + c \, , \quad (a) \]  
\[ P_{B}^{1*} = \frac{l_{A} + 2l_{B}}{3} + c \, . \quad (b) \]

From these optimized equilibrium prices we obtain the optimal payoffs, the resulting market shares and the loyalty of the marginal consumer

\[ \Pi_{A}^{1*} = \frac{(2l_{A} + l_{B})^2}{9(l_{A} + l_{B})} \, , \quad (a) \]
\[ \Pi_{B}^{1*} = \frac{(l_{A} + 2l_{B})^2}{9(l_{A} + l_{B})} \, , \quad (b) \]

\[ \mu_{A}^{1*} = \frac{2l_{A} + l_{B}}{3(l_{A} + l_{B})} n \, , \quad (c) \]
\[ \mu_{B}^{1*} = \frac{l_{A} + 2l_{B}}{3(l_{A} + l_{B})} n \, , \quad (d) \]
\[ \hat{l} = \frac{l_{A} - l_{B}}{3} \, . \quad (e) \]

The case of no firm promotes will serve as a benchmark for the other three subgames.

Letting \( l_{A} = 6, l_{B} = 4, \) and \( c = 0.00001 \) we find that, under these conditions, Firm A’s optimal price per unit and profit per unit are \( P_{A}^{*} = 5.3333 \) and \( \Pi_{A}^{*} = 2.8444. \)

Under these same conditions, Firm B’s optimal price per unit and profit per unit are
\( P_B^* = 4.6667 \) and \( \Pi_B^* = 2.1778 \). We use the same conditions throughout the paper to illustrate the changes between each subgame.

**Subgame 2: Only Firm A Advertises**

In this case, advertising costs make it only beneficial for Firm A to advertise. The equilibrium payoffs are

\[
\Pi_A = \frac{\kappa_A l_B \sqrt{z_A} + l_A - \hat{\lambda}(P_A - c) - \frac{z_A}{n}}{l_A + l_B},
\]

(4.5)

\[
\Pi_B = \frac{-\kappa_A l_B \sqrt{z_A} + l_B + \hat{\lambda}(P_B - c)}{l_A + l_B}.
\]

Since Firm A is advertising it now has control over its advertising expenditures. To find the optimal pricing levels and advertising expenditures, we differentiate Equations (4.5) with respect to their regular prices and the advertising expenditures of Firm A,

\[
\frac{\partial \Pi_A^2}{\partial P_A} = \frac{\kappa_A l_B \sqrt{z_A} + l_A - 2P_A + P_B + c}{l_A + l_B},
\]

\[
\frac{\partial \Pi_A^2}{\partial z_A} = \frac{\kappa_A l_B \sqrt{z_A}}{l_A + l_B} \frac{1}{(P_A - c) - \frac{1}{n}},
\]

(4.6)

\[
\frac{\partial \Pi_B^2}{\partial P_B} = \frac{-\kappa_A l_B \sqrt{z_A} + l_B + P_A - 2P_B + c}{l_A + l_B}.
\]

We set Equations (4.6) equal to zero and solve the system of equations to find the optimizing equilibrium pricing levels and advertising expenditures:
\[ P_A^* = \frac{\kappa_A l_B \sqrt{z_A^*}}{3} + \frac{2l_A + l_B}{3} + c \quad (a) \]

\[ P_B^* = -\frac{\kappa_A l_B \sqrt{z_A^*}}{3} + \frac{l_A + 2l_B}{3} + c \quad (b) \]

\[ z_A^* = \left[ \frac{\kappa_A (2l_A + l_B)n}{6(l_A + l_B) - \kappa_A^2 l_B n} \right]^2 \quad (c) \]

In this case, Firm A’s price increased by \( \left( \frac{\kappa_A l_B \sqrt{z_A^*}}{3} \right) \) and Firm B’s price decreased by the same amount from Subgame 1. The amount Firm A must spend on advertising depends strongly on the effectiveness of the advertising campaign, the number of people in the market, and how strongly loyal consumers are to Firm B. In this case, \( z_A \) illustrates how as the market size increases, advertising expenditures will increase but also as advertising effectiveness increases, advertising expenditures will decrease. The trade off between market size and advertising effectiveness make it possible for Firm A to advertise, even in a growing market.

From Equations (4.7) we obtain the optimal equilibrium payoffs and the resulting market shares.
\[ \Pi_A^{2*} = \frac{[\kappa_A l_B \sqrt{z_A^*} + (2l_A + l_B)]^2}{9(l_A + l_B)} - \frac{z_A^*}{n}, \quad (a) \]

\[ \Pi_B^{2*} = \frac{[-\kappa_A l_B \sqrt{z_A^*} + (l_A + 2l_B)]^2}{9(l_A + l_B)}, \quad (b) \]

\[ \mu_A^{2*} = \frac{\kappa_A l_B \sqrt{z_A^*} + (2l_A + l_B)}{3(l_A + l_B)} n, \quad (c) \]

\[ \mu_B^{2*} = \frac{-\kappa_A l_B \sqrt{z_A^*} + (l_A + 2l_B)}{3(l_A + l_B)} n, \quad (d) \]

\[ \hat{l} = \frac{2\kappa_A l_B \sqrt{z_A^*}}{3} + \frac{l_A - l_B}{3}. \quad (e) \]

In this case, \( \Pi_A \) increases due to the loyalty gained from Firm B, \( \left( \frac{\kappa_A l_B \sqrt{z_A^*}}{9(l_A + l_B)} \right) \), and decreases by the amount spent on advertising per person, \( \frac{z_A^*}{n} \). Firm B’s profits only decrease in terms of the loyalty lost, \( \left( \frac{\kappa_A l_B \sqrt{z_A^*}}{9(l_A + l_B)} \right) \).

Recall that the advertising parameter is a key component of the model and will affect the price and profit of each firm. Figures 4.1 and 4.2 plot Equations 4.7 (a) and 4.8 (a), respectively, and illustrate how as the advertising parameter changes for Firm A, its price per person and profit per person change. In Figure 4.1 (a), Firm A is so effective at advertising that it can charge a larger price and still maintain the needed loyalty to grow market share and profits. In Figure 4.1 (b), Firm A’s effectiveness...
begins to decrease so its price must also decrease in order to gain consumers from Firm B. Initially, when $q_A = \frac{3}{4}$, Firm A can set price to be slightly higher but it must decrease as the market grows. Note that as the market grows, Firm A’s price closely approaches its subgame 1 price of neither firm advertises. Figure 4.2 illustrates the same concepts. When $q_A = \frac{1}{2}$, Firm A is extremely effective and has a higher profit than when it becomes less effective. It is also important to note that the difference in price per unit from $q_A = \frac{1}{2}$ to $q_A = \frac{3}{4}$ is much larger than the difference in profit per unit for these two situations. This is a result of advertising expenditures being higher in order to achieve the higher effectiveness. Likewise, when $q_A = \frac{1}{2}$, Firm A has to set a higher price to offset its advertising expenditures.

\[ \text{(a): } q_A = \frac{1}{2} \]

\[ \text{(b): } q_A = \frac{3}{4} \text{ and } q_A = 1 \]

Figure 4.1: Firm A’s price per item over varying $q_A$ and $n$, $\hat{\kappa}_i = 1$, $l_A = 6$, $l_B = 4$, $c = 0.00001$, $\frac{l_A}{l_A+l_B} = 0.6$, $\frac{l_B}{l_A+l_B} = 0.4$
(a): $q_A = \frac{1}{2}$

(b): $q_A = \frac{3}{4}$ and $q_A = 1$

Figure 4.2: Firm $A$’s profit per person over varying $q_A$ and $n$, $\hat{\kappa}_i = 1$, $l_A = 6$, $l_A = 4$, $c = 0.00001$, $\frac{l_A}{l_A + l_B} = 0.6$, $\frac{l_B}{l_A + l_B} = 0.4$

Subgame 3: Only Firm $B$ Advertises

In this case, advertising costs make it only beneficial for Firm $B$ to advertise. This case is analogous to Subgame 2 and is optimized similarly. The optimal equilibrium pricing levels, advertising strategies, payoffs and resulting market shares are
\[ P_A^* = \frac{-\kappa_B l_A \sqrt{z_B^*}}{3} + \frac{2l_A + l_B}{3} + c , \]

\[ P_B^* = \frac{\kappa_B l_A \sqrt{z_B^*}}{3} + \frac{l_A + 2l_B}{3} + c , \]

\[ z_B^* = \left[ \frac{\kappa_B (l_A + 2l_B)n}{6(l_A + l_B) - \kappa_B^2 l_A n} \right]^2 , \]

\[ \Pi_A^3 = \frac{[-\kappa_B l_A \sqrt{z_A^*} + (2l_A + l_B)]^2}{9(l_A + l_B)} , \]

\[ \Pi_B^3 = \frac{[-\kappa_B l_A \sqrt{z_A^*} + (l_A + 2l_B)]^2}{9(l_A + l_B)} - \frac{z_B^*}{n} , \]

\[ \mu_A^3 = \frac{\kappa_A l_B \sqrt{z_A^*} + (2l_A + l_B)}{3(l_A + l_B)} , \]

\[ \mu_B^3 = \frac{-\kappa_A l_B \sqrt{z_A^*} + (l_A + 2l_B)}{3(l_A + l_B)} , \]

\[ \hat{l} = \frac{-2\kappa_B l_A \sqrt{z_B^*}}{3} + \frac{l_A - l_B}{3} . \]
Subgame 4: Both Firms Advertise

In this case, the advertising costs for both firms make it beneficial for each to advertise.

The equilibrium payoffs are

$$\Pi_A^4 = \frac{\kappa_A l_B \sqrt{z_A} - \kappa_B l_A \sqrt{z_B} + l_A - \hat{l}}{l_A + l_B} (P_A - c) - \frac{z_A}{n} ,$$

$$\Pi_B^4 = \frac{-\kappa_A l_B \sqrt{z_A} + \kappa_B l_A \sqrt{z_B} + l_B + \hat{l}}{l_A + l_B} (P_B - c) - \frac{z_B}{n} .$$

Since both firms have the opportunity to advertise, they each have control over their own pricing levels and advertising expenditures. To find the optimal levels of these variables we differentiate Equations (4.10) with respect to their regular prices and advertising expenditures

$$\frac{\partial \Pi_A^4}{\partial P_A} = \frac{\kappa_A l_B \sqrt{z_A} - \kappa_B l_A \sqrt{z_B} + l_A - 2P_A + P_B + c}{l_A + l_B} ,$$

$$\frac{\partial \Pi_A^4}{\partial z_A} = \frac{\kappa_A l_B}{2 \sqrt{z_A}} (P_A - c) - \frac{1}{n} ,$$

$$\frac{\partial \Pi_B^4}{\partial P_B} = \frac{-\kappa_A l_B \sqrt{z_A} + \kappa_B l_A \sqrt{z_B} + l_B + P_A - 2P_B + c}{l_A + l_B} ,$$

$$\frac{\partial \Pi_B^4}{\partial z_B} = \frac{\kappa_B l_A}{2 \sqrt{z_B}} (P_B - c) - \frac{1}{n} .$$
Setting Equations (4.11) equal to zero and solving the resulting system gives the optimal equilibrium pricing levels and advertising expenditures

\[
P^*_A = \frac{\kappa_A l_B \sqrt{z^*_A}}{3} - \frac{\kappa_B l_A \sqrt{z^*_B}}{3} + \frac{2l_A + l_B}{3} + c, \quad (a)
\]

\[
P^*_B = \frac{-\kappa_A l_B \sqrt{z^*_A}}{3} + \frac{\kappa_B l_A \sqrt{z^*_B}}{3} + \frac{l_A + 2l_B}{3} + c, \quad (b)
\]

\[
z^*_A = \left[ \frac{\kappa_A [-\kappa_B l_A \sqrt{z^*_B} + (2l_A + l_B)]}{6(l_B/l_A) - \kappa_B^2 l_A n} \right]^2, \quad (c)
\]

\[
z^*_B = \left[ \frac{\kappa_B [-\kappa_A l_B \sqrt{z^*_A} + (l_A + 2l_B)]}{6(l_A/l_B) - \kappa_A^2 l_B n} \right]^2. \quad (d)
\]

From Equations (4.12) we obtain the following optimal equilibrium payoffs and the resulting market shares.
\[ \Pi_A^{i*} = \frac{[\kappa_A l_B \sqrt{z_A^*} - \kappa_B l_A \sqrt{z_B^*} + (2l_A + l_B)]^2}{9(l_A + l_B)} - \frac{z_A^*}{n}, \quad (a) \]

\[ \Pi_B^{i*} = \frac{[-\kappa_A l_B \sqrt{z_A^*} + \kappa_B l_A \sqrt{z_B^*} + (l_A + 2l_B)]^2}{9(l_A + l_B)} - \frac{z_B^*}{n}, \quad (b) \]

\[ \mu_A^{i*} = \frac{\kappa_A l_B \sqrt{z_A^*} - \kappa_B l_A \sqrt{z_B^*} + (2l_A + l_B)}{3(l_A + l_B)} n, \quad (c) \]

\[ \mu_B^{i*} = \frac{-\kappa_A l_B \sqrt{z_A^*} + \kappa_B l_A \sqrt{z_B^*} + (2l_A + l_B)}{3(l_A + l_B)} n, \quad (d) \]

\[ \hat{l} = \frac{2\kappa_A l_B \sqrt{z_A^*} - 2\kappa_B l_A \sqrt{z_B^*}}{3} - \frac{l_A - l_B}{3}. \quad (e) \]

In this situation, advertising expenditures of each firm depend on each other and the advertising effectiveness parameter. Figure 4.3 plots Equation 4.12 (c) with the advertising effectiveness parameter, \( q \), varying for both firms. In all cases of varying \( q_A \), Firm A’s advertising expenditures are basically the same when \( q_B = \frac{3}{4} \) or \( q_B = 1 \). In these cases, Firm A takes advantage of Firm B being ineffective and strongly advertises to gain market share. There is little change between the two cases because Firm A’s effectiveness is bounded by how far it can move Firm B’s loyalty, \( l_B < \hat{l} \). Firm B will always have consumers that will be loyal regardless of advertising from Firm A. Firm B also had a smaller loyalty to begin with and Firm A cannot take more consumers away from Firm B than it initially had. When \( q_B = \frac{3}{4} \), Firm A spends more than when \( q_B = \frac{1}{2} \) to maintain its consumer base and take as many
consumers as possible from Firm B. When $q_B = 1$, it makes almost no impact on Firm A’s loyalty and Firm A only has to spend slightly more than when $q_B = \frac{3}{4}$ to obtain the remaining consumers that can be taken from Firm B. When Firm B’s advertising effectiveness parameter is $q_B = \frac{1}{2}$, Firm A finds it more beneficial to use a less intense advertising strategy. This is because Firm B is so effective that Firm A will be spending with no return on its investment. By using a less intense strategy, Firm A loses less on advertising but is able to maintain some profit and market share. When Firm A also has $q_A = \frac{1}{2}$, it still advertises but can spend less than before since it is more effective. When $q_A = \frac{3}{4}$, Firm A spends the minimal to maintain a few of its consumers and when $q_A = 1$, Firm A finds it most beneficial to essentially not advertise.

The advertising parameter also affects how price, profit and market share have changed from when no one advertises. We first consider the cases when Firm B is not extremely effective and then look at the case of when $q_B = \frac{1}{2}$. Figure 4.4 shows the difference between Firm A’s price when both firms advertise, Equation 4.12 (a), and its price when no firm advertises, Equation 4.3 (a), as Firm B’s advertising effectiveness parameter is $q_B = \frac{3}{4}$ and $q_B = 1$. When Firm A has $q_A = \frac{1}{2}$, it can increase its price for both cases of Firm B. As $q$ gets larger for Firm A its price difference becomes much smaller. When $q_B = \frac{3}{4}$, Firm A only slightly lowers its price compared to when no one advertises and raises it when $q_B = 1$. When Firm B has $q_B = \frac{3}{4}$, there is still advertising expenditures that must be made up in price. When both $q_A = 1$ and $q_B = 1$ they are both so ineffective that neither want to directly
target consumers with advertising and will lower price to try and maintain market share.

Similarly, profit change from the case of no firm advertises is affected. Figure 4.5 shows the difference between Firm A’s profit per person when both firms advertise, Equation 4.13 (a), and its profit per person when no firm advertises, Equation 4.4 (a), as Firm B’s advertising effectiveness parameter is $q_B = \frac{3}{4}$ and $q_B = 1$. When Firm A has $q_A = \frac{1}{2}$ its profit increases quickly when the market is small and then increases at a slower rate as the market grows. This is very realistic since we would not expect a firm to gain more per person just because the market increases. As $q_A$ increases, Firm A’s profit decreases from when no firm advertises. Its profit is still positive but will be less than if no one had advertised. As $q_B$ increases for Firm B, Firm A sees more of a profit increase, in the case of $q_A = \frac{1}{2}$ for Firm A, and decreases less for the other cases.

Market share change from when no firm advertises is also affected by $q_A$ and $q_B$. Figure 4.6, illustrates the difference between Firm A’s market share when both firms advertise, Equation 4.13 (c), and its market share when no firm advertises, Equation 4.4 (c), as Firm B’s advertising effectiveness parameter is $q_B = \frac{3}{4}$ and $q_B = 1$. When $q_A = \frac{1}{2}$, Firm A is very effective while Firm B is relatively ineffective. In this case it seems that Firm A is able to gain a significant amount of market share compared to when no firm advertises. However, recall that Firm A cannot take all of those consumers loyal to Firm B and it definitely cannot take more consumers than are actually in the market. Figure 4.6 (a) must be bounded by the limits of
$z_A$. In reality, the upper bound on Firm $A$’s spending is dependent on the loyalty to Firm $B$, Firm $B$’s advertising expenditures and the number of people in that market that can actually be obtained. When $q_A = \frac{3}{4}$, The market is only slightly altered for either of Firm $B$’s cases. When the firms have equal $q$, Firm $A$ loses market share. This is a realistic occurrence since Firm $A$ has more market share to defend while Firm $B$ has the opportunity to attack with both a lower price and an equally effective advertising strategy. When Firm $B$ is very ineffective, Firm $A$ is able to gain market share. When Firm $A$ is ineffective $q_B = \frac{3}{4}$, it loses market share to Firm $B$. When both firms are equally ineffective, Firm $A$ maintains the same market share as when no firm advertises.

A unique situation arises when Firm $B$ has $q_B = \frac{1}{2}$. Figure 4.7 shows how the price, profit and market share difference from the case of no firm advertises as defined above are affected by this. When Firm $A$ has $q_A = \frac{3}{4}$ and $q_A = 1$ the price and profit decrease similarly for both cases. Firm $A$ makes less profit than if neither firm would have advertised. There is a very interesting situation that is illustrated by these figures. When Firm $A$ has $q_A = \frac{1}{2}$ it is just as effective as Firm $B$ and must lower its price significantly as another way to be competitive. Its profit per person decreases much more than in other cases from when no firm promotes. It cannot make up the profit loss in market share because Firm $B$ is just as effective at taking loyalty away from Firm $A$. In this situation, both firms are so effective they cancel each other out and their profits are affected. Market shares for Firm $A$ decrease rapidly. Just as with Firm $A$, Firm $B$’s advertising expenditures are bounded by the
loyalty to Firm A. The only way to compete is to spend more on advertising than is actually beneficial. There is no return on investment for either firm. In this case it may be in their best interest for both firms to not advertise and compete solely on price.
(a): $q_A = \frac{1}{2}$, $q_B$ varies

(b): $q_A = \frac{3}{4}$, $q_B$ varies

(c): $q_A = 1$, $q_B$ varies

Figure 4.3: Firm $A$’s advertising expenditures over varying $q_A$, $q_B$ and $n$, $\hat{\kappa}_i = 1$ for both Firms, $l_A = 6$, $l_A = 4$, $c = 0.00001$, $\frac{l_A}{l_A + l_B} = 0.6$, $\frac{l_B}{l_A + l_B} = 0.4$
Figure 4.4: Firm A’s price change from when no firm advertises to both firms advertise over varying $q_A$, $q_B$ and $n$, $\hat{\kappa}_i = 1$ for both Firms, $l_A = 6$, $l_A = 4$, $c = 0.00001$, $rac{l_A}{l_A + l_B} = 0.6$, $\frac{l_B}{l_A + l_B} = 0.4$.
(a): $q_A = \frac{1}{2}$, $q_B = \frac{3}{4}$ and $q_B = 1$ 

(b): $q_A = \frac{3}{4}$, $q_B = \frac{3}{4}$ and $q_B = 1$

(c): $q_A = 1$, $q_B = \frac{3}{4}$ and $q_B = 1$

Figure 4.5: Firm A’s profit change from when no firm advertises to both firms advertise over varying $q_A$, $q_B$ and $n$, $\hat{\kappa}_i = 1$ for both Firms, $l_A = 6$, $l_B = 4$, $c = 0.00001$, $\frac{l_A}{l_A + l_B} = 0.6$, $\frac{l_B}{l_A + l_B} = 0.4$
(a): $q_A = \frac{1}{2}$, $q_B = \frac{3}{4}$ and $q_B = 1$

(b): $q_A = \frac{3}{4}$, $q_B = \frac{3}{4}$ and $q_B = 1$

(c): $q_A = 1$, $q_B = \frac{3}{4}$ and $q_B = 1$

Figure 4.6: Firm A’s market share change from when no firm advertises to both firms advertise over varying $q_A$, $q_B$ and $n$, $\hat{\kappa}_i = 1$ for both Firms, $l_A = 6$, $l_B = 4$, $c = 0.00001$, $\frac{l_A}{l_A + l_B} = 0.6$, $\frac{l_B}{l_A + l_B} = 0.4$
(a): Firm A’s price change and dependency on market size,

\[ P_A^{1*} = 5.333, \text{ } q_A \text{ varies, } q_B = \frac{1}{2} \]

(b): Firm A’s profit change and dependency on market size,

\[ \Pi_A^{1*} = 2.8444, \text{ } q_A \text{ varies, } q_B = \frac{1}{2} \]

(c): Firm A’s market share change and dependency on market size, \( q_A \) varies, \( q_B = \frac{1}{2} \)

Figure 4.7: Firm A’s price, profit and market share change from when no firm advertises to both firms advertise over varying \( q_A \) and \( n \), \( q = \frac{1}{2} \) for Firm B, \( \hat{k}_i = 1 \) for both Firms, \( l_A = 6, l_A = 4, c = 0.00001, \frac{l_A}{l_A+l_B} = 0.6, \frac{l_B}{l_A+l_B} = 0.4 \)
CHAPTER V

ANALYSIS AND RESULTS

In this section we look to answer the key questions of this paper: When is it beneficial for each firm to advertise and how much should each firm spend on advertising as a function of $q$ and $n$? We also seek to analyze how different advertising strategies affect market shares and profits. We first analyze the solutions for each subgame and then determine the optimal strategy for the overall game.

5.1 Subgame Analysis

Subgame 1: No firm advertises

Consider the solution

$P_A^1 = \frac{2l_A + l_B}{3} + c$, $P_B^1 = \frac{l_A + 2l_B}{3} + c$,

$\Pi_A^1 = \frac{(2l_A + l_B)^2}{9(l_A + l_B)}$, $\Pi_B^1 = \frac{(l_A + 2l_B)^2}{9(l_A + l_B)}$,

$\mu_A^1 = \frac{2l_A + l_B}{3(l_A + l_B)}n$, $\mu_B^1 = \frac{l_A + 2l_B}{3(l_A + l_B)}n$.

These replicate the solutions of Shaffer and Zhang when no firm promotes [13]. Fig-

43
Figure 5.1 shows the details of *Subgame 1* where the shaded area illustrates the revenues to each firm assuming there is no marginal cost. Since we assume that \( l_A > l_B \) and \( P_A > P_B \), Firm A maintains the largest portion of market shares and the greatest revenue.

\[
\text{Profit } \Pi_A = \frac{2l_A + l_B}{3} + c
\]

\[
\text{Profit } \Pi_B = \frac{l_A + 2l_B}{3} + c
\]

\[
\text{Fixed Cost}
\]

\[
\frac{-l_B}{l_A + l_B}
\]

\[
\frac{l_A}{l_A + l_B}
\]

\[
\frac{l_B}{l_A + l_B}
\]

\[
\frac{2l_A + l_B}{3} + c
\]

\[
\frac{l_A + 2l_B}{3} + c
\]

\[
\frac{l_A}{l_A + l_B}
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\[
\frac{l_B}{l_A + l_B}
\]

\[
\frac{2l_A + l_B}{3} + c
\]

\[
\frac{l_A + 2l_B}{3} + c
\]

\[
\frac{-l_B}{l_A + l_B}
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\frac{l_A}{l_A + l_B}
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\frac{l_B}{l_A + l_B}
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\frac{2l_A + l_B}{3} + c
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\frac{2l_A + l_B}{3} + c
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\frac{2l_A + l_B}{3} + c
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\frac{l_A + 2l_B}{3} + c
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\frac{l_A}{l_A + l_B}
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\frac{l_B}{l_A + l_B}
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\[
\frac{2l_A + l_B}{3} + c
\]

\[
\frac{l_A + 2l_B}{3} + c
\]

\[
\frac{l_A}{l_A + l_B}
\]

\[
\frac{l_B}{l_A + l_B}
\]
**Subgame 2: Only Firm A advertises**

The solution for the case of only Firm A advertises is

\[ P^*_A = \frac{\kappa A l_B \sqrt{z^*_A}}{3} + \frac{2l_A + l_B}{3} + c, \quad P^*_B = \frac{-\kappa A l_B \sqrt{z^*_A}}{3} + \frac{l_A + 2l_B}{3} + c, \]

\[ z^*_A = \left[ \frac{\kappa A (2l_A + l_B)n}{6(l_A + l_B) - \kappa^2 A l_B n} \right]^2, \]

\[ \Pi^*_A = \frac{[\kappa A l_B \sqrt{z^*_A} + (2l_A + l_B)]^2}{9(l_A + l_B)} - \frac{z^*_A}{n}, \quad \Pi^*_B = \frac{[-\kappa A l_B \sqrt{z^*_A} + (l_A + 2l_B)]^2}{9(l_A + l_B)}, \]

\[ \mu^*_A = \frac{\kappa A l_B \sqrt{z^*_A} + (2l_A + l_B)}{3(l_A + l_B)} n, \quad \mu^*_B = \frac{-\kappa A l_B \sqrt{z^*_A} + (l_A + 2l_B)}{3(l_A + l_B)} n. \]

In this case, the cost of advertising to Firm A makes it beneficial for Firm A to advertise. This is Firm A’s ideal case since it can gain the largest market share and profit with the least amount spent on advertising. Both firms must still operate at the optimal pricing level in order for both firms to gain the largest market share and profit.

Figure 5.2 illustrates the profits for each firm. Since Firm A is the only firm advertising, it is the only one that can increase its profits. It is not profitable for Firm B to advertise so it cannot counter Firm A’s advertising campaign. Note that
the price differential does not change the same amount as the consumer loyalties. This is because the price differential only depends on the pricing strategies of each firm while the consumer loyalty is affected by both pricing and advertising.

Subgame 3: Only Firm B Advertises

The solution to the case when only Firm B advertises is

\[
 \begin{align*}
 P_A^* &= \frac{-\kappa_B l_A \sqrt{z_B^*} + 2l_A + l_B}{3} + c , \\
 P_B^* &= \frac{\kappa_B l_A \sqrt{z_B^*} + l_A + 2l_B}{3} + c ,
\end{align*}
\]

\[
 z_B^* = \left[ \frac{\kappa_B (l_A + 2l_B)n}{6(l_A + l_B) - \kappa_B^2 l_A n} \right]^2 ,
\]

(5.1)
In this situation only Firm $B$ finds it beneficial to advertise. The situation is very similar to Subgame 2 in that the firms still need to operate at the optimal pricing level to achieve the largest market shares and profits. This results in Firm $B$ increasing the size of its profit while Firm $A$’s decreases. Again, the price differential does not change proportionally to the consumer loyalties.

Figure 5.3: Subgame 3: Only Firm B Advertises - Shaded portions are the revenues to each firm over loyalty levels and optimal pricing. Dotted line represents the original market distribution.
Subgame 4: Both firms advertise

The solution to the case where both firms advertise is

\[
P_A^* = \frac{\kappa_A l_B \sqrt{z_A^*}}{3} - \frac{\kappa_B l_A \sqrt{z_B^*}}{3} + \frac{2l_A + l_B}{3} + c ,
\]

\[
P_B^* = \frac{-\kappa_A l_B \sqrt{z_A^*}}{3} + \frac{\kappa_B l_A \sqrt{z_B^*}}{3} + \frac{l_A + 2l_B}{3} + c ,
\]

\[
z_A^* = \left[ \frac{\kappa_A (-\kappa_B l_A \sqrt{z_B^*} + 2l_A + l_B) n}{6 \left( \frac{l_A + l_B}{l_A} \right) - \kappa_A^2 l_A n} \right]^2 ,
\]

\[
z_B^* = \left[ \frac{\kappa_B (-\kappa_A l_B \sqrt{z_A^*} + l_A + 2l_B) n}{6 \left( \frac{l_A + l_B}{l_A} \right) - \kappa_B^2 l_B n} \right]^2 ,
\]

\[
\Pi_A^{4*} = \frac{\left[ \kappa_A l_B \sqrt{z_A^*} - \kappa_B l_A \sqrt{z_B^*} + (2l_A + l_B) \right]^2}{9(l_A + l_B)} - \frac{z_A^*}{n} ,
\]

\[
\Pi_B^{4*} = \frac{-\left[ \kappa_A l_B \sqrt{z_A^*} + \kappa_B l_A \sqrt{z_B^*} + (l_A + 2l_B) \right]^2}{9(l_A + l_B)} - \frac{z_B^*}{n} ,
\]

\[
\mu_A^{4*} = \frac{\kappa_A l_B \sqrt{z_A^*} - \kappa_B l_A \sqrt{z_B^*} + (2l_A + l_B)}{3(l_A + l_B)} n ,
\]

\[
\mu_B^{4*} = \frac{-\kappa_A l_B \sqrt{z_A^*} + \kappa_B l_A \sqrt{z_B^*} + (2l_A + l_B)}{3(l_A + l_B)} n ,
\]

This case illustrates the instance when both firms are able to advertise. This
situation is not ideal for either firm since each would like to be the only one advertising. Each firm’s advertising expenditures rely on the spending and market share of the other. As one firm spends more, the other must become more effective or also spend more. Each would find it more beneficial to have no other spending to counter.

![Diagram of Figure 5.4](image)

Figure 5.4: Subgame 4: Both Firms Advertise - Shaded portions are the revenues to each firm over loyalty levels and optimal pricing assuming. Dotted line represents the original market distribution.

Figure 5.4 illustrates how each firm’s profits change when both firms advertise. Since both firms advertise, the change is smaller than in Subgames 2 or 3 because each is counter acting the efforts of the other. This figures assumes that Firm A is more effective than Firm B. As a result of these assumptions, Firm A is able to gain profit while Firm B loses profit. Firm A is not able to gain as much profit as in the case when it is the only one advertising since Firm B ”wins back” some of the customers Firm A could have previously won.
5.2 The Optimal Strategy

Consider the optimized equilibrium solutions described in the previous section and displayed in Table 5.1.

<table>
<thead>
<tr>
<th>Firm A</th>
<th>Firm B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Advertise</td>
<td>(\Pi^4_A, \Pi^4_B)</td>
</tr>
<tr>
<td>No Advertise</td>
<td>(\Pi^3_A, \Pi^3_B)</td>
</tr>
</tbody>
</table>

Both firms find it in their best interest to be the only person advertising. It is also in each firm’s best interest to always advertise when the other advertises in order to minimize its loss. As a result, each firm will always choose to advertise given the appropriate conditions outlined in the previous section.

Figure 5.5 illustrates how the key factors of the model change between Subgames for Firm A. In Subgame 1, Firm A’s price and profit are not affected by the size of the market and it maintains the majority of the market. In Subgame 2, its price will be raised to cover advertising expenditures and its profit will increase as a result. In Subgame 4, price must be lowered to compete at a price level since Firm B is just as effective at advertising. Its profit per unit will decrease slightly as a result.
of this price change. In both Subgame 2 and 4, market share does not change from Subgame 1. This is because in Subgame 2, Firm A raised its price allowing Firm B to maintain customers based solely on price. In Subgame 4, both firms are equally effective at advertising so the market shares are unaltered.

In Figure 5.6 we let Firm A be more effective than Firm B and examine how the model changes between Subgames for Firm A. Subgame 1 remains unchanging since it is not affected by advertising effectiveness. In Subgame 2, Firm A’s price will be significantly raised from Subgame 1 since it is highly effective which allows for it to raise price while maintaining market. Firm A’s profit will increase similarly as a result. In Subgame 4, Firm A can still raise its price since it is so effective but cannot raise it as much as in Subgame 2 since it still must account for Firm B’s advertising campaign. Its profit per unit will increase slightly as a result of this price change but still not as much as in Subgame 2. In Subgame 2 market share increases from Subgame 1 as a result of Firm A being so effective. In Subgame 4, Firm A’s market share only slightly increases from Subgame 1 and is not nearly as high as Subgame 2. This is because in Subgame 2, Firm A did not have to account for Firm B’s advertising strategy. In Subgame 4, Firm B is not as effective as Firm A but still reduces the amount of market share Firm A can obtain.

In Figure 5.7 we let Firm B be more effective than Firm A and examine how the model changes between Subgames for Firm A. Again, Subgame 1 remains unchanging since it is not affected by advertising effectiveness. Subgame 2 is only affected by Firm A’s advertising effectiveness so Subgame 2 behaves the same as in
Figure 5.5. In Subgame 4, Firm A must lower its price from the other subgames since it is not as effective as Firm B. This is one of the ways Firm A can hold on to some of the market that Firm B is trying to acquire. Its profit per unit will decrease from the other subgames as a result of this price change. Firm A’s market share only slightly decreases from Subgame 1 despite Firm B being highly effective. This is because Firm A competes at a price level in response to Firm B’s effective advertising strategy and our initial assumption that Firm A has a greater initial loyalty than Firm B.

5.3 Analysis of Advertising Effectiveness

The advertising effectiveness parameter $\alpha_i(z_i) = \kappa_i \sqrt{z_i}$ introduces the question of what exactly is effectiveness? In general, the effectiveness parameters of a campaign are determined by the firm and can become numerous. As noted before, for the purpose of this paper we will consider one common method of measuring effectiveness through the reach and frequency of the advertisement. As the number of consumers in the market grows, a firm must spend more in order to have the opportunity to reach each person at a certain frequency. However, as the market grows, it becomes less likely to obtain those reach and frequency goals. The overall effectiveness of an advertising plan must take into consideration both of these aspects. As the size of the market increases it becomes nearly impossible to reach 100% of the intended audience at the needed level of frequency.
The size of the market also affects the amount needed to be spent on advertising in order to obtain the desired effectiveness. Figure 5.8 plots Equation 4.7 (c) and illustrates how as the market grows, advertising expenditures per person initially grow quickly and then begin to grow at a slower rate. This is typical of most advertising campaigns. AT&T would have spent about $4 a person in the first six months of 2010 if it had targeted every person in the United States [18]. Since its market is smaller than the entire U.S. population, we begin to see the numbers reflected in Figure 5.8. It becomes increasingly important to develop an effective campaign to offset the need for extreme advertising spending.

Advertising effectiveness also has a large effect on advertising expenditures. As one firm’s advertising campaign becomes more effective, its advertising spending decreases while the other firm must make up the difference. Figure 5.9 shows the situations where each firm’s advertising effectiveness varies for the case when both firms advertise. We let \( \hat{\kappa}_i = 1 \) which gives the bound \( 0 < \kappa_i \leq 1 \).

In Figure 5.9, Equation 2.1 is solved for \( z \) and plotted. The advertising effectiveness parameter of Firm A is varying and \( \hat{\kappa}_B = 10 \). Consider first when \( q_A = 0.5 \). In this case, \( \kappa_A = \kappa_B \) when \( \hat{\kappa}_A = 10 \). When \( \kappa_A \ll \kappa_B \), Firm A must spend more to make up for the efficiency of Firm B. As \( \kappa_A \) gets close to \( \kappa_B \), Firm A’s advertising costs begin to level out. This leveling out occurs after \( \kappa_A = \kappa_B \) as a result of the assumptions we made about Firm A having a higher price and a larger initial
market share. Once $\kappa_A > \kappa_B$ occurs, Firm A’s advertising costs continue to level out and Firm B’s advertising costs continue to increase at a faster rate to make up for the increased efficiency of Firm A.

When $q_A = 0.55$, Firm A’s advertising expenditures stay higher than Firm B’s even after $\hat{\kappa}_A > \hat{\kappa}_B$. It is not until $\hat{\kappa}_A$ is about 50% larger than $\hat{\kappa}_B$ that Firm A’s advertising expenditures begin to level out. This illustrates the large effect $q$ has on advertising expenditures. In order to be effective, $q$ must stay close to $\frac{1}{2}$.
Figure 5.5: Firm A’s price, profit and market share over varying \( n \), \( q_A = \frac{3}{4} \), \( q_B = \frac{3}{4} \), \( \hat{\kappa}_i = 1 \) for both Firms, \( l_A = 6 \), \( l_A = 4 \), \( c = 0.00001 \), \( \frac{l_A}{l_A+l_B} = 0.6 \), \( \frac{l_B}{l_A+l_B} = 0.4 \).
Figure 5.6: Firm $A$’s price, profit and market share over varying $n$, $q_A = \frac{1}{2}$, $q_B = \frac{3}{4}$, $\hat{\kappa}_i = 1$ for both Firms, $l_A = 6$, $l_A = 4$, $c = 0.00001$, $\frac{l_A}{l_A+l_B} = 0.6$, $\frac{l_B}{l_A+l_B} = 0.4$
(a): Firm A’s price and dependency on market size

(b): Firm A’s profit and dependency on market size

(c): Firm A’s market share and dependency on market size

Figure 5.7: Firm A’s price, profit and market share over varying $n$, $q_A = \frac{3}{4}$, $q_B = \frac{1}{2}$, $\hat{\kappa}_i = 1$ for both Firms, $l_A = 6$, $l_A = 4$, $c = 0.00001$, $l_A + l_B = 0.6$, $l_A + l_B = 0.4$
Figure 5.8: Advertising Expenditures as size of the market grows, $l_A = 0.6$, $l_B = 0.4$, $c = 0.00001$, $q_A = \frac{1}{2}$, $\kappa_A = \kappa_B$

Figure 5.9: Advertising expenditures dependency on advertising effectiveness parameter and $q$, $\kappa_B = 10$, $n = 1,000,000$
CHAPTER VI

CONCLUSION

In this paper we seek to answer these questions through game theory: When is it beneficial for each firm to advertise and how much should each firm spend on advertising? Following Shaffer and Zhang [13], we implement a two-stage game involving two firms, each with different market shares, pricing and advertising expenditures. We assume that of these two firms, Firm A controls the majority of the market and has higher pricing. Assuming perfect information is available to both firms at both stages of the game, we utilize a solution procedure of subgame perfection. In this procedure, backward induction is used to first solve for the Nash equilibria of the second stage and then solve for the Nash equilibria of the first stage. Once both stages of the game are solved for all subgames, we find the optimal solution for the game.

Given the appropriate pricing levels and advertising spending, we find that both firms will always choose to advertise. The choices on pricing and advertising spending are dependent on one another. A firm must be sure to set pricing at a level to make advertising beneficial and to maintain positive profits. As the size of the market grows, there becomes more opportunity for growth to both firms. However, this also impacts advertising spending and effectiveness. In this situation,
the campaign becomes less effective and advertising costs become more expensive. In order to compete at an advertising level given these factors, a firm must alter their pricing strategy to offset the advertising expenditures. We found that as the market grows, price per unit and profit per unit appear not to be dependent on $n$. This is the result of the advertising effectiveness parameter decreasing as advertising expenditures increase with the market. The two effects essentially cancel each other out leaving a minimal impact on price per unit and profit per unit.

A firm must take into consideration all factors that affect their spending and profits in order to develop a strong, competitive advertising campaign. Changes in the size of the market greatly alter the effectiveness of the campaign which alters the advertising expenditures needed to make advertising beneficial. Changes in the amount of spending on advertising shifts the zones in which advertising is an optimal choice. Each firm must set its advertising expenditures such that the unit difference in price and production cost is greater than the cost of advertising per person in order to make advertising beneficial. If advertising expenditures become too large, neither firm will find it beneficial to advertise.

This paper did not consider advertising expenditures over time where a firm may take a loss initially in hopes of growing its market share to positively affect profits in the future.
BIBLIOGRAPHY


