PERFORMANCE EVALUATION OF MODAL AND LOCAL CONTROL METHODS FOR FLEXIBLE SYSTEMS

A Thesis
Presented to
The Graduate Faculty of The University of Akron

In Partial Fulfillment
of the Requirements for the Degree
Master of Science

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May, 2010
PERFORMANCE EVALUATION OF MODAL AND LOCAL CONTROL
METHODS FOR FLEXIBLE SYSTEMS

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Thesis

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ABSTRACT

Interest has increased in the active control of vibrations in mechanically flexible systems, e.g., attitude control of flexible spacecraft, ride quality improvement of air and surface transportation and adaptive optics. Because of their inherent flexibility, they are generally analyzed as distributed parameter systems which creates difficulties in the design and analysis of the appropriate controllers. To insure satisfactory performance of such systems, their distributed parameter nature must be taken into account in the design of the control system. The control task is normally thought of in terms of maintaining specified shape configurations, orientation, alignment, vibration suppression and pointing accuracy etc.

Modal control techniques have been developed to bypass problems associated with distributed parameter theory. Modal control is built upon the notion that certain specified system modes can be controlled by appropriate design of the associated closed-loop controls. This reduces the number of sensors and actuators needed to effect the control of the structure. An undesirable phenomenon, referred to as observation and control spillover can occur if the number of sensors and actuators used is small. Spillover refers to the phenomenon in which energy intended to go solely into the controlled modes leaks into the uncontrolled modes.
This thesis discusses the control of flexible systems described by a pinned-pinned beam equation, relating the structure displacement to the force distribution acting on the structure. Two different control approaches, local control and modal control methods, are evaluated based on cost in terms of coupling between sensors and actuators as well as error in the response. The main goal is to investigate how additional coupling between the sensors and actuators influence the cost and the performance of the modal control versus the local control methods.
ACKNOWLEDGEMENTS

First and foremost I offer my sincerest gratitude to my advisor Dr. Dane Quinn, who has supported me throughout my thesis with his patience and knowledge whilst allowing me the room to work on my thesis part-time after a two year gap. I attribute the level of my Masters degree to his encouragement and effort and without him this thesis, too, would not have been completed or written. One simply could not wish for a better or friendlier advisor.

I would like to thank the Department of Mechanical Engineering at the University of Akron for awarding me a teaching assistantship during my graduate study. I also would like to thank Dr. Graham Kelly and Dr. John Zhe for serving on my thesis committee. Finally, I thank my parents for supporting me throughout all my academic career at the university.
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CHAPTER I
INTRODUCTION

1.1 Overview

There is much interest in the control of large flexible structures. The control task is normally thought of in terms of maintaining specified shape configurations, orientation and alignment, vibration suppression and pointing accuracy, etc. Because of their inherent flexibility, they are generally analyzed as distributed parameter systems which creates difficulties in the design and analysis of controllers for them. Modal control techniques have been developed to bypass problems associated with distributed parameter theory. Modal control is built upon the notion that certain specified system modes can be controlled by appropriate design of the associated closed-loop controls. This reduces the number of sensors and actuators needed to effect the control of the structure. An undesirable phenomenon, referred to as observation and control spillover can occur if the number of sensors and actuators used is small. Spillover refers to the phenomenon in which energy intended to go solely into the controlled modes leaks into the uncontrolled modes.

The objective of this thesis is to evaluate two different control approaches, local control and modal control. They are evaluated based on cost in terms of coupling
between sensors and actuators as well as error in the response. The main goal is to investigate how additional coupling between the sensors and actuators, affect the cost and the performance of the modal control versus the local control as applied to the problem of a flexible pinned-pinned beam.

1.2 Motivation

For a communications satellite designer, providing precision surfaces for antenna reflectors has been a challenging problem. Surface errors are introduced by manufacturing errors, thermal distortion in orbit, moisture, loose joints, material degradation and creep. These reflectors are often made of graphite epoxy structures because of requirements for low thermal distortion. Significant time and cost are spent during fabrication, analysis and ground tests to minimize and determine the surface errors. Even with this effort, several current spacecraft antennas have experienced degraded performance due to higher than predicted surface errors [1].

Smart sensors and actuators with the ability to correct orbit surface errors have great potential for use in these microwave devices. Smart actuators can also provide a desired change in antenna beam shape due to change in coverage requirements. Therefore, smart structure technology has the potential of not only improving the performance of these structures, but also reduction in cost for analyses and ground tests. A number of smart materials are available which may be used as sensors or actuators. These materials include piezoelectric polymers and ceramics, shape memory alloys etc. While significant research effort has been devoted to the use of smart struc-
tures for active vibration suppression, considerably less attention has been focused on the use of smart structures for shape control.

1.3 Adaptive Optics

Adaptive Optics (AO) is a technology to improve the performance of optical systems by reducing the effects of rapidly changing optical distortion. It is commonly used on astronomical telescopes to remove the effects of atmospheric distortion, or astronomical seeing. Adaptive Optics works by measuring the distortion and rapidly compensating for it either using deformable mirrors or a material with variable refractive properties.

1.4 Deformable Mirrors

A deformable mirror is an advanced wave front control device consisting of a mirror membrane supported by an underlying actuator array. Each actuator in the array can be individually deflected by electrostatic actuation to achieve the desired pattern of deformation. Below are the list of Deformable Mirror (DM) types made.

1. Bulk micromachined model mirrors [2] and [3],

2. Surface micromachined deformable mirrors [4],

3. Piston motion segmented mirrors [4],

4. Deformable mirror with thermal actuators [5],
5. Electrostritive deformable mirrors.

1.5 Parameters of the Deformable Mirrors

Number of actuators. Number of actuators determines the number of degrees of freedom the mirror can correct. It is very common to compare an arbitrary DM to an ideal device that can perfectly reproduce wave front modes in the form of Zernike polynomials. Zernike polynomials are analyzed in detail in references [6], [7].

Actuator pitch and actuator stroke. Actuator pitch is the distance between actuators and actuator stroke is the maximum possible actuator displacement. Deformable mirrors with large actuator pitch and large number of actuators are bulky and expensive.

Influence function. The Influence function is the characteristic shape of the mirror corresponding to the response from the action of a single actuator. Different types of deformable mirrors have different influence functions. Moreover the influence functions can be different for different actuators of the same mirror. Influence functions that cover the whole mirror surface are called a “modal” function, while localized response are called “zonal”.

Actuator coupling. Actuator coupling shows how much the movement of one actuator will displace its neighbors. All “modal” mirrors have large cross-coupling, which
in fact is good as it secures the high quality of correction of smooth low-order optical aberrations that usually have the highest statistical weight.

Response time. Response time is the time a system or functional unit takes to react to a given input. Response time shows how quickly the mirror will react to the control signal. It can vary from microseconds for MEMS deformable mirrors to tens of seconds for thermally controlled deformable mirrors.

1.6 Overview of Thesis

The first chapter in the thesis introduce the reader to flexible system and a need for effective control systems. The second chapter discusses the previous work done on shape control of flexible systems and the optimization strategies developed. The main work done in this thesis is explained in Chapters 3, 4 and 5. In Chapter 3 the basic equations that form a differential model of the beam are developed using extended Hamilton’s principle. In Chapter 4, a Galerkin Reduction is used to reduce the differential equation of the continuum model to a finite degree of freedom model. Chapter 5 discusses the two control methods, local control and modal control. The performance evaluation of both methods is based on the steady state error and the cost. Also, the role of the coupling between the sensors and actuators is discussed, in particular how this coupling influence the performance of the control. Chapter 6 concludes the thesis with the performance evaluation of the controller for flexible
system and their effect on the shape control of the beam. Finally limitations and
recommendations for possible future work are stated.
2.1 Shape Control

The goal of shape control is to nullify the effects of certain external disturbances on the deformation of the structure by means of suitable control actuation. Shape control thus represents a branch of structural engineering that is closely related to control engineering. In the framework of a linear formulation, such a problem may be stated as seeking a control actuation such that a desired spatial distribution or shape of the corresponding displacement field is reached. In dynamic shape control, the desired shape has to be additionally prescribed as a function of time. It is the latter ambiguous field aspect that makes shape control situated on the very frontier of contemporary research in structural engineering as well as in automatic control.

2.2 Optimization Strategies

In shape control one intends to specify the spatial distribution, or the shape, of an actuating control, such that the displacement field of a structure distorted from its original shape or such that the structure follows some desired field of trajectories. Several papers have been published regarding shape control of smart structures. We...
classify the papers into five different areas of research which they are mainly focused on

1. Actuator placement,

2. Actuator placement and controller,

3. Electronics,

4. Structure,

5. Actuator-Structure.

2.2.1 Optimum position of actuators and sensors

The largest body of work related to optimization of smart structures can be categorized as optimal actuator placement, where the optimal locations of actuators are found for predetermined structures. The size of actuators may be optimized as well in actuator placement problems, but the passive structure is assumed to be of predetermined geometry and material. Almost all of this work is focused on optimal placement of piezoelectric actuator patches on simple structures such as on beams or plates. The papers dealing with optimal actuator placement are summarized.

Schulz and Heimbold [8] presented a method of maximizing dissipation energy by an optimal set of actuator/sensor and the feedback gains. Kondoh et al [9] used the linear quadratic-optimal control framework to perform actuator/sensor placement and feedback gains. Although both methods and cost functions in [8] [9] are distinct in terms of their control strategy, similar optimal solutions are obtained for the case
that the actuator/sensor pair is collocated. The solutions in both of them are initial condition dependent.

Coffignal et al [10] proposed a new approach to find the optimal location of piezoelectric actuators and sensors on beam structures by proposing to find the optimal actuators location by minimizing the mechanical energy integral of the system and the optimal sensors location by maximizing the energy of the state output. This method is used to find the optimal location of one actuator and one sensor on a cantilever beam and on a three-beam structure. Halim and Moheimani [11] suggest a criterion for the optimal placement of collocated piezoelectric actuator-sensor pairs on a thin flexible plate using modal and spatial controllability measures.

2.2.2 Actuator placement controller optimization

In this section, papers are discussed which deal with combined optimization of actuator locations and controller parameters. Many of these papers assume that sensors are collocated with actuators. Controller parameters such as feedback gain or actuation voltages are optimized either simultaneously or in sequence with actuator locations. Several actuator placement-controller papers are discussed below.

Yong Li et al [12] formulated a new optimization method is based on the maximization of dissipation energy due to control action, as well as the restriction on the sizing of collocated piezoelectric actuator/sensor pairs and the feedback gains. Onoda and Haftka [13] developed an approach to structure/control simultaneous optimization in which the optimal collocated sensor/actuator location and control gains
was obtained simultaneously by minimizing the total cost of structure and control system subject to constraints on the magnitude of response to a given disturbance involving both rigid-body and elastic modes.

Heng-Kwung Lee et al [14] propose a new controller design method for optimal control of the vibration in flexible systems, thereby allowing integrated determination of dislocated sensor/actuator locations and feedback gain. The optimization criterion is based on minimizing the sum of the integral flexible system energy. Yang and Lee [15] developed an analytical model for structural control optimization in which both non-collocated sensor/actuator placement and feedback control gain are considered as independent design variables. Two different control algorithms-active damping control and optimal output feedback control are applied to derive the control input.

2.2.3 Control of flexible structures

Meriovitch et al [16] developed a technique called Independent Modal Space Control (IMSC), based on idea of specific coordinate transformations to decoupled the equation of motion into set of independent coordinates. As a result, the control system design can be carried out for every second order system independently, so that the controller acts on each mode independently. This procedure not only guarantees controllability but also guarantees that no control spillover into modal modes occur provided number of actuators is equal to the order of discretized system.

Fanson and Caughey [17] introduced a technique for vibration suppression in large space structures, called Positive Position Feedback (PPF), making use of
generalized displacement measurements to accomplish vibration suppression. Poh and Baz [18] demonstrated the feasibility of a new Modal Positive Position Feedback (MPPF) strategy in controlling the vibration of a complex flexible structure using a single piezo-electric active structural member. The control strategy generates its control forces by manipulating only the modal position signals of the structure to provide a damping action to undamped modes. The performance of the new strategy is enhanced by augmenting it with a “time sharing” strategy to share a small number of actuators between larger number of modes.

Ambrosino et al [19] presented an control based on LQR and on spline reconstructor for active vibration control of flexible beam. The reconstructor introduce spatial filtering on the high frequency modes which allows to screen out un-modelled dynamics without introducing any phase lag in the control loop. In Lee et al [20] proposed a Adaptive Positive Position Feedback (APPF) for the multi-modal vibration control of frequency varying structures. Spillover phenomena and real-time system identification have been obviously difficult obstacles for the multi-modal adaptive vibration control. To overcome these problems, a algorithm is proposed to identify the frequencies of time-varying structures. Variable PPF controllers are adjusted with estimated natural frequencies at every time step, however the current Adaptive Positive Position Feedback (APPF) has some limitations. The structures shall be sufficiently linear so that the linear system identification model is valid.

Tzou and Hollkamp [21] proposed a distributed self-sensing piezoelectric actuators to achieve independent control of various natural modes. Provided, perfect
collocations of sensors and actuators in closed-loop structural controls. Mahmoodi and Ahmadian [22] developed a novel active vibration control technique based on positive position feedback method. This method, which is a modified version of positive position feedback, employs a first-order compensator that provides damping control and a second-order compensator for vibration suppression.

2.3 Shape Memory Alloys

Shape memory alloys (SMAs) are materials that “remember” their original shapes. SMAs are useful for such things as actuators which are materials that change shape, stiffness, position, natural frequency and other mechanical characteristics in response to temperature or electromagnetic fields. The potential uses for SMA’s especially as actuators, broadened the spectrum of many scientific fields.

Chen and Levy [23] presented a model of flexible beam covered with SMA layers and an active control method for a flexible beam by means of SMA layers and showed that changes in the Young’s modulus ratio and temperature of SMA layer will affect the natural frequency of the beam and that in the case of active vibration control. The response amplitude and vibration time will depend upon many factors such as the SMA material and Young’s modulus ratio of the material. Da Silva [24] presented an experimental investigation of an adaptive flexible beam actuated by means of an actuator made of SMA. The deflection of a flexible beam is controlled by means of resistive heating of a shape memory actuator and cooling in the surrounding
air, a feedback control system was implemented to control the deflection, taking the Joule heat as the control variable.

Dickinson and Wen [25] investigated the factors that hamper the usefulness of SMAs as actuators and presented a closed loop feedback control for SMA without the explicit identification of the hysteresis SMA behavior and used a thermodynamic constitutive model of SMA combined with a single modal model of the flexible beam to qualitatively analysis the closed loop stability of the system. Elahinia et al [26] presented a model for an SMA actuated manipulator which includes nonlinear dynamics of the manipulator and electrical and heat transfer behavior of SMA and designed a sliding model control to calculate the desired stress of the wire based on the desired angular position of the arm. The feedback controller maintains the desired stress by regulating the input voltage to the SMA wire.

2.4 Application of Shape Control in Adaptive Optics

Several analytical and numerical models have been developed on deformable mirrors with actuators. Vogel and Yang [27] developed a mathematical model for continuous face sheet MEMS deformable mirror device produced by Boston Micro Machines Corporation which consists of electrostatically driven actuators coupled to a continuous facesheet mirror. They modeled the deformations of the facesheet mirror under known actuator loading as the solution to a linear fourth order partial differential equation (PDE) known as a plate equation and the actuators are also modeled using the plate equation. To carry out the modeling effort they extended the Bifano
actuator model to account for the loading due to the rigid post and then combined the plate equation model of the continuous facesheet. Therefore they have coupled system of $n_a + 1$ fourth order PDEs, where $n_a$ represents the number of actuators. The facesheet PDEs are linear while the actuator PDEs are nonlinear. They solve the nonlinear PDEs by replacing the actuator PDEs with nonlinear algebraic equations. Therefore the end result is single a fourth order linear PDE coupled to $n_a$ algebraic equations. They solved the system of equations using a finite element discretization combined with Newton iteration to handle the nonlinearity.

Miller et al [28] developed a simple model for predicting static and dynamic behaviors for a fixed-fixed parallel plate electrostatic actuators. For static analysis, the deflection equations are derived based on minimization of total potential energy of the beam and for dynamics analysis, Lagrange’s method is used to derive the nonlinear equation of motion. A wide spectrum of analytical and numerical models have been developed for describing the motion of parallel plate electrostatic actuators. Beam bending models have been modified to include the nonlinear electrostatic force, stiffening due to beam stretch and residual stress from the fabrication process. The models are commonly used to predict static deflection, pull-in voltage and resonant frequency.

Tilmans et al [29] studied the response of a micro beam to a generalized transverse excitation and subject to an axial force. Rayleigh’s energy method was used to approximate the fundamental natural frequency of a straight, undeflected beam. An additional term was added to the expression to account for mid-plane
stretching. The model does not apply to electric actuation because it assumes that the transverse load is independent of the micro beam deflection. Assuming both beam and plate models and using the finite element method (FEM), Zook and Burns [30] calculated the natural frequencies of a micro beam subject to an axial load. They also determined experimentally the natural frequencies of the micro beam under electrostatic actuation. The FEM models accounted for shear deformation and rotary inertia, but neglected the mid-plane stretching of the elements. Their calculations showed no significant differences between the natural frequencies produced by the beam and plate models. However, the calculated natural frequencies were consistently higher than those measured experimentally. That was probably due to the fact that their model does not include the electrostatic force.

Ijntema and Tilmans [31] considered the static and dynamic responses of a micro beam under electric actuation but did not account for the mid-plane stretching. They calculated the static deflection due to an electrostatic force applied by the DC polarization voltage. The fundamental frequency was then approximated using Rayleigh’s energy method where the micro beam motion was linearized around the deflected shape obtained as a solution of the static problem. Tilmans and Legtenberg [32] solved the same static problem using the Rayleigh Ritz method assuming a single admissible trial function. They used this formulation to generate an analytical expression for the pull-in voltage. The calculated values of the pull-in voltage were in good agreement with the results of experiments they conducted on resonators of various lengths. They also compared an approximate analytical expression of the
fundamental natural frequency obtained from Rayleigh’s energy method to values of the fundamental natural frequency they obtained experimentally. They found out that the results obtained from the expression were only valid for small DC polarization voltages away from the pull-in voltage.

Choi and Lovell [33] use a shooting method to solve the nonlinear force equilibrium equation. They noted the significant effects of beam stretch and residual stress on static displacement. Najar et al [34] use the differential quadrature method to discretized the beam equation of motion. Abdel-Rahman et al [35] present a detailed parametric analysis of the beam stretch effect. Several strategies have been adopted to reduce computation time. Choi and Lovell [33] derived a closed form solution for the actuator displacement based on one and two-term linear approximations of the electrostatic force. Chowdhury et al [36] describe a semi-empirical closed form model for the pull-in voltage. Anantha suresh et al [37] investigated the relationship between number of mode shapes and accuracy of reduced-order macro models. Mehner et al [38] describe a process for generating macro models based on modal methods and polynomial fits of finite-element results. Younis et al [39] generated a reduced-order model by using up to five linear undamped dynamic mode shapes to represent the beam.
CHAPTER III
CONTINUUM MODEL

This chapter describes a mathematical model for a deformable structure with distributed actuators. The structure considered is modeled as pinned-pinned beam, and the actuators are modeled as discrete control forces acting on the beam, as shown in Figure 3.1. In this model $u(x,t)$ denotes the transverse displacement and $f(x,t)$ represents the actuator force, while $m(x)$ is the mass per unit length and $EI(x)$ represents the flexural rigidity. We develop the resulting boundary value problem using an extended Hamilton’s principle.

3.1 Extended Hamilton’s Principle

The extended Hamilton’s principle is variational principle of mechanics derived from the generalized D’Alembert principle. We can state the generalized D’Alembert principle as

The virtual work performed by the effective forces through infinitesimal virtual displacements compatible with the system constraints is zero.

In terms of a system of $N$ particles, this can be written as

$$
\sum_{i=1}^{N} (F_i - m_i \ddot{r}_i) \cdot \delta r_j = 0,
$$

(3.1)
where $m_i$ is mass of particle $i$ acted upon by applied force $F_i$ and $-m_i \ddot{r}_i$ is inertia force which is negative of the time rate change of momentum $m_i \dot{r}_i$. We begin with the case in which the position vectors $r_i$ are independent. The virtual work of the applied forces is defined as

$$\sum_{i=1}^{N} F_i \cdot \delta r_j = \delta W; \quad (3.2)$$

and includes conservative and non-conservative forces. To reduce the second term in Eq. (3.1) we consider

$$\frac{\partial}{\partial t} (m_i \dot{r}_i \cdot \delta r_j) = m_i \ddot{r}_i \cdot \delta r_j + m_i \dot{r}_i \cdot \delta \dot{r}_j,$$

$$= m_i \ddot{r}_i \cdot \delta r_j + \delta \left( \frac{m_i}{2} \dot{r}_i \cdot \dot{r}_j \right),$$

$$= m_i \ddot{r}_i \cdot \delta r_j + \delta T_i. \quad (3.3)$$

Where $T_i$ is the kinetic energy of particle $m_i$, rearranging and integrating with respect to time over interval $t_1 \leq t \leq t_2$, we have

$$- \int_{t_1}^{t_2} m_i \ddot{r}_i \cdot \delta r_j \, dt = \int_{t_1}^{t_2} \delta T_i \, dt - m_i \dot{r}_i \cdot \delta r_j |_{t_1}^{t_2}, \quad (3.4)$$
but the virtual displacements are arbitrary. Hence we choose them as to satisfy

\[ \delta r_j = 0 \text{ at } t = t_1 \text{ and } t = t_2 \text{ therefore it reduces to} \]

\[ - \int_{t_1}^{t_2} m_i \ddot{r}_i \cdot \delta r_j \, dt = \int_{t_1}^{t_2} \delta T_i \, dt. \]  \hspace{1cm} (3.5)

Summing up over \( i \) and integrating with respect to \( t \) over interval \( t_1 \leq t \leq t_2 \) we have

\[ - \int_{t_1}^{t_2} \sum_{i=1}^{N} m_i \ddot{r}_i \cdot \delta r_j \, dt = \int_{t_1}^{t_2} \delta T \, dt, \]  \hspace{1cm} (3.6)

in which \( T \) is the system’s kinetic energy. Finally integrating Eq. (3.1) with respect to time over interval \( t_1 \leq t \leq t_2 \) using Eq. (3.2) and Eq. (3.6) we obtain

\[ \int_{t_1}^{t_2} (\delta T + \delta W) \, dt = 0, \]  \hspace{1cm} (3.7)

which represents the mathematical statement of extended Hamilton’s principle. It is often convenient to divide the virtual work into two parts, one due to conservative forces and other due to non conservative forces. Hence we can write

\[ \delta W = \delta W_c + \delta W_{nc} = -\delta V + \delta W_{nc}, \]  \hspace{1cm} (3.8)

where \( \delta W_c \) is work done by conservative force, \( \delta W_{nc} \) is work done by non conservative force and \( V \) is the potential energy. We can rewrite the extended Hamilton’s principle in the form

\[ \int_{t_1}^{t_2} (\delta T - \delta V + \delta W_{nc}) \, dt = 0. \]  \hspace{1cm} (3.9)
3.2 Derivation of Continuum Model

We derive the boundary-value problem of the continuum model by means of extended

Hamilton’s principle. The kinetic energy of the continuum model is

\[ T(t) = \frac{1}{2} \int_0^1 \rho(x) \left[ \frac{\partial u(x,t)}{\partial t} \right]^2 dx = 0, \quad (3.10) \]

and the potential energy is

\[ V(t) = \frac{1}{2} \int_0^1 EI(x) \left[ \frac{\partial^2 u(x,t)}{\partial x^2} \right]^2 dx = 0, \quad (3.11) \]

while the virtual work of the distributed force \( f(x,t) \) is

\[ \delta W_{nc}(t) = \int_0^1 f(x,t) \delta u(x,t) \, dx. \quad (3.12) \]

Next we write the variation in kinetic energy

\[ \delta T = \int_0^1 \rho(x) \frac{\partial u}{\partial t} \delta \left( \frac{\partial u}{\partial t} \right) dx = \int_0^1 \rho(x) \frac{\partial u}{\partial t} \frac{\partial (\delta u)}{\partial t} dx, \quad (3.13) \]

and carry out the following integration by part with respect to time

\[ \int_{t_1}^{t_2} \delta T \, dt = - \int_{t_1}^{t_2} \left( \int_0^1 \rho(x) \frac{\partial^2 u}{\partial t^2} (\delta u) \, dx \right) dt, \quad (3.14) \]

where we consider \( \delta u = 0 \) at \( t = t_1 \) and \( t = t_2 \). Similarly we write the variation of the potential energy

\[ \delta V = \int_0^1 EI(x) \frac{\partial^2 u}{\partial x^2} \delta \left( \frac{\partial^2 u}{\partial x^2} \right) dx = \int_0^1 EI(x) \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 (\delta u)}{\partial x^2} dx, \quad (3.15) \]

integrating by parts and obtain

\[ \delta V = \left( EI(x) \frac{\partial^2 u}{\partial x^2} \right) \delta \left( \frac{\partial u}{\partial x} \right) \bigg|_{x=0}^{x=1} - \left[ \frac{\partial}{\partial x} \left( EI(x) \frac{\partial^2 u}{\partial x^2} \right) \right] \delta u \bigg|_{x=0}^{x=1}
+ \int_0^1 \frac{\partial^2}{\partial x^2} \left( EI(x) \frac{\partial^2 u}{\partial x^2} \right) \delta u \, dx. \quad (3.16) \]
Now inserting Eq. (3.12), Eq. (3.14) and Eq. (3.16) into Eq. (3.9) and collecting terms we have

\[
\int_{t_1}^{t_2} \left\{ - \int_0^1 \left[ \rho(x) \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left( EI(x) \frac{\partial^2 u}{\partial x^2} \right) - f(x, t) \right] \delta u \, dx 
- \left( EI(x) \frac{\partial^2 u}{\partial x^2} \right) \delta \left. \left( \frac{\partial u}{\partial x} \right) \right|_{x=0}^{x=1} + \left. \frac{\partial}{\partial x} \left( EI(x) \frac{\partial^2 u}{\partial x^2} \right) \delta u \right|_{x=0}^{x=1} \right\} \, dt = 0. \quad (3.17)
\]

Then, invoking the arbitrariness of the virtual displacement \( \delta u \) over the interval \( 0 \leq x \leq 1 \), subject to the boundary conditions \( \delta u = 0 \) at \( x = 0 \) and \( x = 1 \), Eq. (3.17) can be satisfied only if the coefficient of \( \delta u \) is zero over the same interval. Hence we obtain the fourth order partial differential equation for the bending vibration of the beam in the form

\[
\rho(x) \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2}{\partial x^2} \left( EI(x) \frac{\partial^2 u}{\partial x^2} \right) - f(x, t) = 0. \quad (3.18)
\]

Rearranging, this becomes

\[
\rho(x) \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left( EI(x) \frac{\partial^2 u}{\partial x^2} \right) = f(x, t). \quad (3.19)
\]

To complete the boundary-value problem we must specify two boundary condition at each end of the beam, with pinned-pinned conditions at both ends. At pinned ends, the deflection and curvature are zero, so the boundary conditions are

\[
u(0, t) = 0, \quad \frac{\partial^2 u(0, t)}{\partial x^2} = 0 \quad (3.20)\
u(1, t) = 0, \quad \frac{\partial^2 u(1, t)}{\partial x^2} = 0. \quad (3.21)\]

The distributed forces \( f(x, t) \) is used to represent discrete actuators acting at distinct locations along the length of the beam, so to represent the individual actuator forces
at their locations \( x = x_j \), this force is rewritten as

\[
f(x, t) = \sum_{j=0}^{m} F_j(t) \delta(x - x_j),
\]

where \( F_j(t) \) represents control force of the actuator at location \( x_j \) and \( \delta \) is a Dirac delta function. Substituting this into the PDE we have

\[
\rho \frac{\partial^2 u}{\partial t^2} + EI \frac{\partial^4 u}{\partial x^4} = \sum_{j=0}^{m} F_j(t) \delta(x - x_j),
\]

where \( m \) represents the number of actuators, and the density per unit length \( \rho \) and flexural rigidity \( EI \) have been assumed to be constant over the length of the beam. Therefore we have a fourth order PDE boundary value problem that describes the continuum model with multiple actuators. In Chapter 4 we apply a Galerkin reduction technique to reduce this continuum problem to a set of discrete equations.
CHAPTER IV
GALERKIN APPROACH

The Galerkin method is a means for converting a partial differential equation to a
system of ordinary differential equations. It relies on the weak formulation of an
equation and works in principle by restricting the possible solutions as well as the test
functions to a smaller space than for the original solution. These small systems are
easier to solve than the original problem, but their solution is only an approximation
to the original solution. The approach was invented by the Russian mathematician
Boris Galerkin.

The Galerkin method approximates the solution to the boundary value prob-
lem by a linear combination of basis functions determined by requiring that the
residual be orthogonal to each of the homogeneous basis functions, i.e., those that
vanish on the boundary, and that the boundary conditions be satisfied.

4.1 Galerkin Reduction

Let $\phi_r(x)$ be a set of basis functions that satisfy the boundary conditions. We will
approximate the solution to the boundary value problem by a linear combination
$u(x, t) = \phi_1(x) A_1(t) + \ldots + \phi_n(x) A_n(t)$ of these basis functions, where the vector
$A = [A_r]$ of coefficients is to be determined. The coefficients of the homogeneous
basis functions are determined by an orthogonality condition, as described below. The result is a system of equations that can be solved for the coefficients $A_r$ of the approximate solution for $u(x, t)$. This system of equations will be nonlinear if the original PDE is nonlinear. The necessary integrals can be evaluated by numerical quadrature, and the accuracy of the solution will be affected by the accuracy of this numerical integration.

Assume the approximate solution to the continuum model

$$u^{(n)}(x, t) = \sum_{r=1}^{n} A_r(t) \phi_r(x),$$

where the trial functions $\phi_r(x)$ are the independent comparison functions and $A_r(t)$ are undetermined coefficients. Comparison functions are trial functions which are differentiable as many times as the order of the system and satisfy all the boundary conditions. The solution given in Eq. (4.1) does not satisfy exact differential equation defining the eigenvalue problem, so that some error denoted by $R(u^{(n)}, x)$, known as the residual is incurred. Because $u^{(n)}$ is a linear combination of comparison functions, the boundary conditions are satisfied exactly.

To determine the coefficients $A_r$ we multiply the residual $R(u^{(n)}, x)$ by $\phi_i(x)$ in sequence, integrate over the domain of the system, and set the result equal to zero

$$\int_{0}^{1} \phi_i(x) R(u^{(n)}, x) \, dx = 0, \quad i = 1, 2, \ldots, n, \quad (4.2)$$

or

$$\int_{0}^{1} \phi_i(x) \left\{ \frac{\rho}{dt^2} \frac{d^2 u(x, t)}{dt^2} + EI \frac{d^4 u(x, t)}{dx^4} - \sum_{j=1}^{m} F_j(t) \delta(x - x_j) \right\} \, dx = 0. \quad (4.3)$$
Substituting Eq. (4.1) into this expression yields

\[
\int_0^1 \phi_i(x) \left\{ \rho \frac{d^2}{dt^2} \left( \sum_{r=1}^n \phi_r(x) A_r(t) \right) + EI \frac{d^4}{dx^4} \left( \sum_{r=1}^n \phi_r(x) A_r(t) \right) - \sum_{j=1}^m (F_j(t) \delta(x - x_j)) \right\} \, dx = 0. \tag{4.4}
\]

Rearranging

\[
\int_0^1 \left\{ \sum_{r=1}^n \left( \rho \phi_i(x) \phi_r(x) \frac{d^2(A_r(t))}{dt^2} \right) \right\} \, dx + \int_0^1 \left\{ \sum_{r=1}^n \phi_i(x) EI \frac{d^4 \phi_r(x)}{dx^4} A_r(t) \right\} \, dx
- \int_0^1 \left\{ \sum_{j=1}^m \phi_i(x) (F_j(t) \delta(x - x_j)) \right\} \, dx = 0. \tag{4.5}
\]

Rewriting the above equation in general form

\[
\sum_{r=1}^n \left( m_{ir} \frac{d^2(A_r(t))}{dt^2} \right) + \sum_{r=1}^n (k_{ir} A_r(t)) = f_i(t), \tag{4.6}
\]

where \( m_{ir} \) are the elements of the mass matrix

\[
m_{ir} = m_{ri} = \int_0^1 \rho \phi_i(x) \phi_r(x) \, dx, \quad i, r = 1, 2, \ldots, n, \tag{4.7}
\]

and \( k_{ir} \) are the elements of the stiffness matrix

\[
k_{ir} = k_{ri} = \int_0^1 \phi_i(x) EI \frac{d^4 \phi_r(x)}{dx^4} \, dx, \quad i, r = 1, 2, \ldots, n, \tag{4.8}
\]

and \( f_i(t) \) is modal control force, given as

\[
f_i(t) = \sum_{j=1}^m F_j(t) \phi_i(x_j). \tag{4.9}
\]

\( F_j(t) \) represents the individual actuator force at location \( x = x_j \). Substituting the modal control force back into Eq. (4.6) we have

\[
m_{ir} \left[ \sum_{r=1}^n \frac{d^2(A_r(t))}{dt^2} \right] + k_{ir} \left[ \sum_{r=1}^n A_r(t) \right] = \sum_{j=1}^m F_j(t) \phi_i(x_j). \tag{4.10}
\]
Rewriting the modal equations in matrix form

\[
\begin{bmatrix}
  m_{1,1} & m_{1,2} & \cdots & m_{1,n} \\
  m_{2,1} & m_{2,2} & \cdots & m_{2,n} \\
  \vdots & \vdots & \ddots & \vdots \\
  m_{n,1} & m_{n,2} & \cdots & m_{n,n}
\end{bmatrix}
\begin{bmatrix}
  \ddot{A}_1(t) \\
  \ddot{A}_2(t) \\
  \vdots \\
  \ddot{A}_n(t)
\end{bmatrix}
=\begin{bmatrix}
  k_{1,1} & k_{1,2} & \cdots & k_{1,n} \\
  k_{2,1} & k_{2,2} & \cdots & k_{2,n} \\
  \vdots & \vdots & \ddots & \vdots \\
  k_{n,1} & k_{n,2} & \cdots & k_{n,n}
\end{bmatrix}
\begin{bmatrix}
  A_1(t) \\
  A_2(t) \\
  \vdots \\
  A_n(t)
\end{bmatrix}
+\begin{bmatrix}
  \sum_{j=1}^{m} F_j(t) \phi_1(x_j) \\
  \sum_{j=1}^{m} F_j(t) \phi_2(x_j) \\
  \vdots \\
  \sum_{j=1}^{m} F_j(t) \phi_n(x_j)
\end{bmatrix}
\]  
(4.11)

where \( j \) represents the actuators and \( i \) represents the modes, with spatial modes of form

\[
\phi_r(x) = \sin(r\pi x).
\]
(4.12)

Following an \( n \)-mode Galerkin reduction, for \( i = 1, \ldots, n \), the equations of motion can be written as

\[
M \ddot{A} + K A = f(t) = T F(t),
\]
(4.13)

with

\[
[M]_{ir} = \int_0^1 \rho \phi_i(x) \phi_r(x) \, dx, \quad i, r = 1, 2, \ldots, n,
\]
(4.14)

\[
[K]_{ir} = \int_0^1 EI \frac{d^2\phi_i}{dx^2}(x) \frac{d^2\phi_r}{dx^2}(x) \, dx, \quad i, r = 1, 2, \ldots, n,
\]
(4.15)

and

\[
[T]_{ij} = \phi_i(x_j).
\]
(4.16)
Even if the spatial modes are chosen to decouple the mass and stiffness matrix, these equations of motion can once again be coupled through a general feedback control $F(t)$. Finally the response of the structure $u(x,t)$ at the actuator locations $\mathbf{x} = [x_j]$ is given by $\mathbf{U} = \mathbf{T}^T \mathbf{A}$. The above differential equation is numerically solved by using the MATLAB function `ode45`, all the numeric results and conclusion are discussed in the following chapters.
5.1 Introduction

In this chapter we discuss two different controller approaches to control the modes of the continuum model, in order to approximate desired shape and characterize the performance of the controller. The performance is characterized in terms of steady state error and also the coupling between sensors and actuators. This work focuses on how the performance of the control system is influenced by the coupling between sensors and actuators.

Consider a structure subjected to varying thermal conditions. Unless carefully designed it will distort as a result of the thermal gradients. One way to prevent this is to build the structure from a thermally stable material. This is the passive approach. An alternative way is to use a set of actuators and sensors connected by a feedback loop, such a structure is active.

An active structure consists of a mechanical component structure with a set of actuators and sensors coupled by a controller. If the bandwidth of the controller includes some vibration modes of the structure its dynamic response must be considered when developing the controller design.
There are two different types of control strategies, feedback and feedforward. The principle of feedback control is represented in Figure 5.1, with $G(s)$ representing the Laplace transformation model of the system and $H(s)$ describing the controller. The output $y$ of the system is compared to the reference input $r$ and the error signal $e = r - y$ is the input to the controller [40]. The design problem consists of finding the appropriate controller $H(s)$ such that the closed-loop system is stable and behaves in the appropriate manner.

Feedforward is a term describing an element within a control system which passes a controlling signal from a source in the control system’s external environment, often a command signal from an external operator, to a load elsewhere in its external environment. A control system which has only feedforward behavior applies its control signal in a pre-defined way without responding to how the load reacts. This is in contrast with a system that also has feedback, which adjusts the output to take account of the response of the system to the load. In addition the load itself may
vary unpredictably, considered to belong to the external environment of the system.

5.2 Control of Continuum Model

Two different types of control approaches have been applied to the continuum model. The first is defined as local control method and second is the modal control method.

1. Local control method: Sensor and actuator are placed in the same position or collocated, and the action of each actuator depends only on the measured response of the system at that location.

2. Modal control method: Independently control each vibrational mode of the system to obtain a desired response.

5.2.1 Control Performance

The objective of this work is to examine how the underlying coupling between sensors and actuators influences the ability of the controller to implement a desired shape. The performance characterization of the controller is evaluated based on the error and cost and is given by the equation

\[ B = \text{Cost}^2 + \text{Error}^2. \]  

(5.1)

The error \( E \) is defined between actual response of the beam and the desired response of the beam, given by

\[ E = \sqrt{\int_0^1 (U(x,t) - V(x,t))^2 \, dx}, \]  

(5.2)
where $U(x, t)$ is the actual shape or response and $V(x, t)$ is desired shape.

The actual response of the beam in terms of modal amplitudes is given below

$$U(x, t) = \sum_{i=1}^{n} A_i(t)\phi_i(x),$$  \hspace{1cm} (5.3)

and the desired response

$$V(x, t) = \sum_{i=1}^{n} v_i(t)\phi_i(x),$$  \hspace{1cm} (5.4)

where $A_i$ and $v_i$ are the modal amplitudes of actual and desired response.

5.2.2 Component of Cost

Component of the cost is given by the equation below

$$\text{Cost} = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}.$$  \hspace{1cm} (5.5)

For $i \neq j$ $C_{ij}$ is the cost required to transmit information between sensor $i$ and $j$ is given by

$$C_{ij} = W_t |x_i - x_j|,$$  \hspace{1cm} (5.6)

where $W_t$ is the transmission weight. For $i = j$ the cost depends on the force produced by actuator $j$, so that

$$C_{jj} = W_f |f_j|,$$  \hspace{1cm} (5.7)

where $W_f$ is force weight and $f_j$ is the force at actuator $j$. Finding the total cost and error values, we can characterize the performance of the local and modal control. This enables us to compare the two control methods under certain conditions to minimize the performance measure $B$ in the response.
Figure 5.2: Actual and Desired Shape of Beam

The performance of the two control methods, local and modal control is evaluated based on two desired profiles, the first profile is \( v(x, t) = 0.50 \sin(\pi x) \) where \( x \) is the displacement along the beam. The desired profile is symmetric as shown in Figure 5.3. The second profile is of the form \( v(x, t) = 0.30 \sin(\pi x) + 0.50 \sin(\pi x) \), where the profile shown in Figure 5.6. The performance of the two controls is also evaluated as the number of actuators vary along the length of the beam and also the location of the actuators. In the two actuator model, the performance is evaluated with one actuator fixed and varying the second actuator and finally the performance is evaluated as the number of modes in the system vary.
5.3 Local Control Method

We restate the equation of motion in general form as

\[ M \ddot{A} + K A = f(t), \quad (5.8) \]

while the modal force \( f(t) \) is represented as

\[ [f(t)]_i = \sum_{j=1}^{m} F_j(t) \phi_i(x_j). \quad (5.9) \]

Here \( j \) indexes the actuators, \( m \) represents the number of actuators, and \( i \) indexes the modes. Rewriting \([f(t)]_i\) in general form we have

\[ f = TF, \quad (5.10) \]

where \( T \) is transformation matrix and \( F \) is control force. Rewriting the above equation in matrix form

\[
\begin{bmatrix}
  f_1(t) \\
  f_2(t) \\
  \vdots \\
  f_n(t)
\end{bmatrix} =
\begin{bmatrix}
  \phi_1(x_1) & \phi_1(x_2) & \cdots & \phi_1(x_m) \\
  \phi_2(x_1) & \phi_2(x_2) & \cdots & \phi_2(x_m) \\
  \vdots & \vdots & \ddots & \vdots \\
  \phi_n(x_1) & \phi_n(x_2) & \cdots & \phi_n(x_m)
\end{bmatrix}
\begin{bmatrix}
  F_1(t) \\
  F_2(t) \\
  \vdots \\
  F_m(t)
\end{bmatrix}
\]

where \( F \) is local control forcing is given by the equation

\[ F = -KU, \quad (5.11) \]

for \( V = 0 \) and

\[ F = -K(U - V), \quad (5.12) \]
for $V \neq 0$. Where $K$ is the controller gain. $U$ is the approximate solution to the equation of motion from Eq. (5.3) and $V$ is the desired displacement at that actuator location. Substituting terms and rewriting the equation for control force

$$F_j(t) = -KU(x_j,t),$$

$$= -K \sum_{r=1}^{n} A_r \phi_r(x_j), \quad (5.13)$$

where $A_r$ is of mode $r$. Rewriting $F_j(t)$ in matrix form.

$$
\begin{bmatrix}
F_1(t) \\
F_2(t) \\
\vdots \\
F_m(t)
\end{bmatrix} = -K
\begin{bmatrix}
\phi_1(x_1) & \phi_2(x_1) & \cdots & \phi_n(x_1) \\
\phi_1(x_2) & \phi_2(x_2) & \cdots & \phi_n(x_2) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_1(x_m) & \phi_2(x_m) & \cdots & \phi_n(x_m)
\end{bmatrix}
\begin{bmatrix}
A_1(t) \\
A_2(t) \\
\vdots \\
A_n(t)
\end{bmatrix}
$$

Rewriting $U(x_j,t)$ in general form

$$U = S A, \quad (5.14)$$

where

$$[S]_{jl} = \phi_l(x_j). \quad (5.15)$$

Therefore

$$T = S^T, \quad (5.16)$$

and substituting for $U(x_j,t)$ in Eq. (5.11) we have

$$F = -K S A. \quad (5.17)$$

Substituting the control force $F$ into Eq. (5.10) leads to

$$f = T (-K S A). \quad (5.18)$$
Rearranging terms we have the modal forcing equation for local control approach as

\[ f = -KT T^T A, \]  
(5.19)

where \( T \) is \( n \times m \) transformation matrix and \( T^T \) is \( m \times n \) matrix. Recall that \( n \) is number of modes and \( m \) is number of actuators. Rewriting, the above equation in matrix form we have modal forcing for \( n \) number of modes.

\[
\begin{bmatrix}
  f_1(t) \\
  f_2(t) \\
  \vdots \\
  f_n(t)
\end{bmatrix}
= -K
\begin{bmatrix}
  \phi_1(x_1) & \phi_1(x_2) & \cdots & \phi_1(x_m) \\
  \phi_2(x_1) & \phi_2(x_2) & \cdots & \phi_2(x_m) \\
  \vdots & \vdots & \ddots & \vdots \\
  \phi_n(x_1) & \phi_n(x_2) & \cdots & \phi_n(x_m)
\end{bmatrix}
\begin{bmatrix}
  A_1(t) \\
  A_2(t) \\
  \vdots \\
  A_n(t)
\end{bmatrix},
\]  
(5.20)

5.3.1 Performance Evaluation of Local Control

The goal of this work is to investigate how the control gain \( K \) and the position of the actuators along the length of the beam influences the performance of the controlled system, the performance of the controlled system is evaluated based on cost and error as described in Eq. (5.1).

Consider a single actuator model with a one degree of freedom in which sensors and actuators are collocated so the response of the system is at the actuators, described by \( U \). That is sensors and actuators are located at identical spatial points
Figure 5.3: Single Mode-Single Actuator Model
Figure 5.4: Gain $K$ vs. Error $E$, Cost $C$ and Performance $B$
Figure 5.5: Position of Actuator $x$ vs. Error $E$, Cost $C$ and Performance $B$
$x = x_j$ here we also assume that actuators are equally spaced along the length of the beam. Figure 5.3 shows the response of the single actuator model with a varying gain $K$.

As the gain $K$ increases the error $E$ between the actual response and desired response is reduced as shown in Figure 5.4 and the cost $C$ increases as the gain $K$ increases, but the performance $B$ decreases up to a gain $K = 2.7$ with additional increase in gain the performance increases. The error continues to decrease but the increase in the cost compensates so that at gain $K = 2.7$ the optimum performance exists for local control. With fixed gain $K = 2.7$, the position of the actuator is varied along the length of the beam and the error $E$, cost $C$ and performance $B$ is evaluated as shown in the Figure 5.5. The optimum location of the sensor/actuator for this single actuator model occur at $x = 0.5$ where performance $B$ is minimum.

For the second example a model with two modes and two actuators is considered. The actuators are equally spaced along the length of the beam. Figure 5.6 shows the response of the two actuator model with a varying gain $K$. From Figure 5.6 the error is significantly large from the actual response to the desired response as the gain $K$ increases. As the gain $K$ increases the error $E$ is decreased but the cost $C$ increases because more control action needed. The cost $C$, error $E$ and the performance $B$ are plotted with the gain $K$ in Figure 5.7. As the gain $K$ increases the error $E$ between the actual response and desired response is reduced as shown in Figure 5.7 and the cost $C$ increases as the gain $K$ increases but the performance $B$ decreases up to a gain $K = 1.8$ and increases back again, so at gain $K = 1.8$ we get
Figure 5.6: Two Mode-Two Actuator Model
Figure 5.7: Gain $K$ vs. Error $E$, Cost $C$ and Performance $B$
Figure 5.8: Position of Actuator $x$ vs. Error $E$, Cost $C$ and Performance $B$
the optimum values for the performance of the two actuator model with local control.

With fixed gain at $K = 1.8$, the position of the first actuator is varied along the length of the beam while fixing the second actuator, and the error $E$, cost $C$ and performance $B$ is evaluated as shown in Figure 5.8, from Figure 5.8 the optimum location of the first sensor/actuator occurs at $x_1 = 0.26$ while the second actuator/sensor is fixed at $x_1 = 0.67$. In this way the optimum location of the actuators is selected to minimize the value of $B$, representing the performance of the system.

5.4 Modal Control Method

Modal control approaches attempt the control each mode of the structure independently. One such example, developed by Meirovitch [16], is known as Independent Modal Space Control (IMSC). In this, the controller drives each mode independently and has the effect of diagonalizing the above gain matrix $K TT^T$. Unfortunately, for underactuated systems it is not possible to control each mode independently. Instead, with $m$ sensor/actuator pairs only $m$ modes can be independently controlled and the system suffers from spillover, whereby control energy leaks into uncontrolled modes and then unavoidably influences the dynamic response of the controlled modes.

Without loss of generality assume these to be the first $m$ modes and identify these controlled modal amplitudes $A_c$. The equations of motion of this generalized system as represented in Eq. (4.13), is restated as

$$M\ddot{A} + KA = f(t).$$
To control the amplitudes $A_c$ of certain number of $m$ modes the forcing term $f$ is re-written as

$$f = \begin{bmatrix} [f_c]_m \\ [f_u]_p \end{bmatrix},$$

(5.21)

where $f_c$ is the generalized forcing for the amplitudes of $m$ number of modes to control, and $f_u$ is the generalize forcing for the amplitudes of $p$ number of uncontrolled modes, where $p = n - m$. Then, assume the modal control force to be a constant gain $K$ for every mode times amplitudes of the controlled modes

$$f_c = -K A_c.$$

(5.22)

Recall that the modal control force $f$ is the product of a transformation matrix $T$ times the physical control forcing $F$

$$f = T F.$$

(5.23)

Therefore the modal control vector can be written as

$$f = \begin{bmatrix} [f_c]_m \\ [f_u]_p \end{bmatrix} = \begin{bmatrix} T_c \end{bmatrix}_{m \times m} (F)^m,$$

(5.24)

where $T_c$ is the transformation matrix for controlled modes and $T_u$ is the transformation matrix for the uncontrolled modes. From Eq. (5.24) we have

$$f_c = T_c F,$$

(5.25)

and

$$f_u = T_u F.$$

(5.26)
Also from Eq. (5.22) we have \( f_c = -K A_c \), therefore

\[
f_c = T_c F = -K A_c.
\]  

(5.27)

Therefore solving for the physical actuator force

\[
F = -K (T_c)^{-1} A_c.
\]  

(5.28)

Substituting this into Eq. (5.26) we have

\[
f_u = -K T_u (T_c)^{-1} A_c,
\]  

(5.29)

substituting \( f_u \) and \( f_c \) back into Eq. (5.24), the modal control force \( f \) for modal control method becomes

\[
f = \begin{pmatrix} [f_c]_m \\ [f_u]_p \end{pmatrix} = -K \begin{pmatrix} I \\ T_u (T_c)^{-1} \end{pmatrix} A_c.
\]  

(5.30)

5.4.1 Performance Evaluation of Modal Control

Modal control approaches attempt the control each mode of the structure independently. Begin with single actuator model with two modes in which one sensor/actuator pair are coupled. Figure 5.9 shows the response of the single actuator model with a varying gain \( K \), as the gain \( K \) increases the error \( E \) between the actual response and desired response is reduced as shown in Figure 5.10 and also the cost \( C \) increases. The performance \( B \) decreases up to gain \( K = 2.4 \) and increases back again and stabilizes, so at gain \( K = 2.4 \) we get the optimum values for error and the cost for modal control.
Figure 5.9: Two Mode-Single Actuator Model
Figure 5.10: Gain $K$ vs. Error $E$, Cost $C$ and Performance $B$
Figure 5.11: Position of Actuator $x$ vs. Error $E$, Cost $C$ and Performance $B$
Figure 5.12: Three Mode-Two Actuator Model
Figure 5.13: Gain $K$ vs. Error $E$, Cost $C$ and Performance $B$
With fixed gain at $K = 2.4$, the position of the actuator is varied along the length of the beam and the error $E$, cost $C$ and performance $B$ is evaluated as shown in the Figure 5.11, form the Figure 5.11 the optimum location of the sensor/actuator for this single actuator two mode model is at $x = 0.5$ where performance $B$ is minimum.

For second example we assume a model with three modes and two actuators in which two actuator/sensor pairs are coupled and the actuators are equally spaced along the length of the beam. Figure 5.12 shows the response of the system with a varying gain $K$. From Figure 5.13 as the gain $K$ increases the error $E$ decreased and the cost $C$ increases and stabilizes. The cost $C$, error $E$ and the performance
$B$ are plotted with the gain $K$ in Figure 5.13. As the gain $K$ increases the error $E$ between the actual response and desired response is reduced as shown in Figure 5.13 and the cost $C$ increases as the gain $K$ increases but the performance $B$ decreases up to a gain $K = 1.6$ and increases back and stabilizes, so at gain $K = 1.6$ we get the optimum values for error and the cost for two actuator model approach.

With fixed gain at $K = 1.6$, the position of the first actuator is varied along the length of the beam while fixing the second actuator. The error $E$, cost $C$ and performance $B$ is evaluated as shown in the Figure 5.14 and the optimum location of the first actuator is at $x_1 = 0.36$ while the second actuator is fixed at $x_2 = 0.67$. In this way the optimum location of the actuators is selected to minimize the error $E$ between the actual and desired response and the cost $C$.

5.5 Comparison between Local and Modal Control

From the two control methods above, examples of the local control method has lower performance values compared to the examples of modal control method. The lower the performance value the better the control, but the error $E$ in local control examples is higher compared to examples of modal control method.

As in the case of the modal control method, controlling $m$ independent sensor/actuator pairs results in more cost, but controlling individually modes gives the less error. Therefore the actual response of the beam is close to the desired response. But the cost $C$ increases form transmission of information from one sensor to another as show in Figure 5.10 and 5.13. As the gain $K$ increases the error $E$ between the
Figure 5.15: Gain $K$ vs. Cost $C$ on 2-Actuators, 4-Mode Model
Figure 5.16: Gain $K$ vs. Performance $B$ on 2-Actuators, 4-Mode Model
Figure 5.17: Gain $K$ vs. Cost $C$ on 2-Actuators, 6-Mode Model
Figure 5.18: Gain $K$ vs. Performance $B$ on 2-Actuators, 6-Mode Model
Figure 5.19: Gain $K$ vs. Error $E$ on 3-Actuators, 6-Mode Model
Figure 5.20: Gain $K$ vs. Cost $C$ on 3-Actuators, 6-Mode Model
Figure 5.21: Gain $K$ vs. Performance $B$ on 3-Actuators, 6-Mode Model
actual and desired shape decreases considerably, but the cost \( C \) increases to certain value and stabilizes. This resulting in a higher performance values for modal control method.

When comparing modal control versus local control for two actuator four mode model in Figures 5.15 and 5.16, the cost \( C \) and performance \( B \) is higher in modal control compared to local control. Figure 5.15 shows gain \( K \) versus cost \( C \), the cost \( C \) is in terms of coupling and control force for both modal and local control, since there is coupling between the sensor and actuators in modal control, the initial cost is higher as the gain \( K \) varies the cost stabilizes. Also from Figure 5.16, the performance of the modal control is higher initially because of the cost \( C \) as the gain \( K \) increases the performance drops and stabilizes thereafter. However, in case of local control method the initial cost is lower, since there is no coupling involved between one sensor to another, so the only cost term in local control is from the control force at the actuators as the gain \( K \) varies the cost gradually increases.

Since the control is trying to lower the error from actual shape to the desired shape, the performance of the local control is shown the Figure 5.16, based on the plot the local control has a better performance than modal control up to gain \( K = 5.2 \) and as the gain \( K \) increases the performance increases. In the Figures 5.17 and 5.18, the number of modes are increased, adding more sensor/actuator connections in the modal control. As be seen from Figure 5.17, the cost \( C \) is increased when compared to a four mode model, as the number of connections increase between the sensors. The cost for transmitting data also increases, so the initial cost for modal control is
higher compared to the local control. In Figure 5.18 the performance is plotted for both modal and local control. From the figure, the modal control method has higher performance values than the local control, the performance of the modal control stabilizes after gain $K = 4$. However, in case of the local control the performance increases as the gain $K$ increases.

For a three actuator model with six mode model, the error, cost and performance curves are shown for both local and modal controls shown in Figures 5.19, 5.20 and 5.21. From the figures, comparing the performance plot of the both the controls to Figure 5.18, the performance values decreases this is because of the extra actuator. This gives less error between the actual shape and the desired shape, but the cost increases as expected. Adding an extra actuator to the model increases the amount of control force and also there are more connections between the actuators/sensors pairs.

When the two control methods are compared for a three actuator six mode model the initial performance values and the cost values are higher in modal control. The performance value become stable as the gain $K$ increases but in case of local control as the gain $K$ increases the performance values increases over the modal control values at gain $K = 9$, and stabilizes at higher values of the gain $K$.

Therefore form the examples above the initial cost and performance values of the modal control is higher compared to the local control. Since the coupling between the sensors/actuator increases the cost $C$ increases effecting the control performance. The error values are much lower in the modal control meaning the actual response
is much closer to the desired response in modal control. Therefore at a increased cost we get the actual response of the beam closer to the desired response.

In case of local control the initial cost and performance values are lower but the error between the actual shape to desired shape is higher in local control as the gain $K$ increases both the performance and cost and considerably increasing over the values of the modal control. Based on the results, models with more number of actuators and higher modes, using modal control approach gives less error $E$ and the actual beam response is closer to desired response at a higher controller performance index.
CHAPTER VI

CONCLUSION AND FUTURE WORK

6.1 Conclusion

This work illustrates the difference between local and modal control methods for flexible systems. The uncollocated sensors and actuators are uncoupled in local control method and coupled in modal control method. The effect on the control is characterized by minimizing the performance index. This work investigates how the coupling between sensors and actuators as well as the control force and the position of the actuators along the length of the pinned-pinned beam, influences the performance of the controlled system. The performance of the controlled system is evaluated based on cost elements such as control force, cost for coupling between the sensors and actuators and error between the actual response and the desired response. This work also investigates how the position of the actuators also influences the performance index of the control and optimum locations are specified when having single and two actuator model on a pinned-pinned beam.

Based on these results, modal control has better performance compared to local control. Coupling of the sensors and the cost to transmit information between them add extra cost to modal control. But the error between the actual and desired
shape is lower in modal control method. Therefore with lower control gains $K$ its easy to get the shape of the continuum model close to the desired shape, without having higher control force cost. Compared to modal control, local control has lower cost $C$, but the error $E$ between actual and desired shape is higher particularly as the number of actuator increases. Based on the results, for models with more number of actuators, using modal control approach gives less error $E$ and response is closer to desired response required.

6.2 Future Work

Modal control generally requires that the system be not only controllable, but allow for full coupling between all sensors and actuators. While recent advances in the area of smart and adaptive structures have allowed for substantial increases in the capability to include sensors and actuators, the coupling between two sensors and actuators is still an issue, as it requires $m^2$ connections between the $m$ sensor/actuator pairs. Thus, further work is needed in the area of the structure of the feedback control gain matrix and the coupling between sensors and actuators together with optimal placement of actuators for minimal physical inputs to control the system.
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APPENDICES
APPENDIX A
MATLAB CODE-LOCAL CONTROL METHOD

Deriv Function.m

function y = deriv6(t,A,par,x,v)

LOCAL APPROACH
E = Modulus of elasticity
I = Cross-sectional area moment of inertia
K = Controller Gain
M = Mass per Unit Length (m = rho*a)
P = Number of modes
Q = Number of Actuators
c = Damping coefficient
x = Location point of actuators on the beam
v = Desired Displacement.
Wa=par(1); %Wa = E*I/pho
Wb=par(2); %Wb = 2*K/pho
P= par(3);
Q=par(4);
n=length(x);
for i=1:P
z(i)=0;
for n=1:Q
u(n)=0;
for j=1:P
u(n)=(sin(j*pi*(x(n)))*A(2*j-1)+u(n));
end
z(i)= z(i)+ sin(i*pi*(x(n)))*(u(n)-v(n));
end
% LOCAL APPROACH - BEAM RESPONSE

K_Values = zeros(1,100);
x_Values = zeros(1,100);
L_Values = zeros(1,100);
Performance_Values = zeros(1,100);
Cost_Values = zeros(1,100);
Value_count = 1;

for K = 1:0.1:10;
    for x = 0.05:0.02:0.98
        P = 1; % No of modes
        Q = 1; % No of actuators
        x(1,:) = [0.33333 0.66667];
        x(1,:) = [0 0.66667];
        v(1,:) = [0.3 0.5];
        x(1,:) = [0.5];
        v(1,:) = [0.5];
        K = 2.7;
        rho = 1;
        E = 0.1;
        I = 0.1;
        Wa = (E*I)/rho;
        Wb = (2*K)/(rho);
        end
    end
end

Beam Response.m

clc
close all
clear all
format short

y(2*i-1,1) = A(2*i);
y(2*i,1) = -Wa*pi^4*(i)^4*A(2*i-1)-0.1*A(2*i)-Wb*z(i);
end

y;

Beam Response.m

clc
close all
clear all
format short

% LOCAL APPROACH - BEAM RESPONSE

K_Values = zeros(1,100);
x_Values = zeros(1,100);
L_Values = zeros(1,100);
Performance_Values = zeros(1,100);
Cost_Values = zeros(1,100);
Value_count = 1;

for K = 1:0.1:10;
    for x = 0.05:0.02:0.98
        P = 1; % No of modes
        Q = 1; % No of actuators
        x(1,:) = [0.33333 0.66667];
        x(1,:) = [0 0.66667];
        v(1,:) = [0.3 0.5];
        x(1,:) = [0.5];
        v(1,:) = [0.5];
        K = 2.7;
        rho = 1;
        E = 0.1;
        I = 0.1;
        Wa = (E*I)/rho;
        Wb = (2*K)/(rho);
% Wt = 0.1;
Wf = 2.6;
par = [Wa Wb P Q];
initial = zeros(1, 2*P);
% No_Of_Iterations = (0.98 - 0.05) / 0.02;
% Increment_of_x = 0.02;
% Iteration_Count = 0;
% for i = 1: No_Of_Iterations
% x(1,1) = x(1,1) + Increment_of_x;
% Iteration_Count = Iteration_Count + 1;
[t, A] = ode45(@deriv6, [0, 1000], initial, [], par, x, v);
% Actual shape plot.
figure(13)
q = 0:1/1000:1;
Zsum = 0;
Qsum = 0;
z = 0;
for i = 1:P
z = A(end, 2*i-1) * sin(i*pi*q);
% plot(q, z, 'b')
% xlabel('q'), ylabel('mode shape equation'), grid; hold on
Zsum = Zsum + z;
end
plot(q, Zsum, 'g')
xlabel('q (collocated)'), ylabel('mode shape equation'), grid; hold on
% desired shape plot.
G = 0;
Gsum = 0;
for i = 1:P
Bsum = 0;
B = 0;
for j = 1:Q
B = v(end, j) * sin(i*pi*x(j));
end
plot(q, Bsum, 'g')
xlabel('q (collocated)'), ylabel('mode shape equation'), grid; hold on
%
Bsum=Bsum+B;
end
G=Bsum*sin(i*pi*q);
Gsum=Gsum+G;
end
plot(q,Gsum,'r');hold on;
xlabel('q'),ylabel('Actual shape'),grid;hold on
% Q = Area between actual and desired shape.
Y=(Zsum-Gsum).^2;
L=100*trapz(q,Y);

%%%%% ACTUATOR FORCES%%%%%
for n=1:Q
u(n)=0;
for j=1:P
u(n)=sin(j*pi*(x(n)))*A(end,2*j-1)+u(n);
end
R(n)=Wb*(u(n)-v(n));
end

%%%%COMPONENT OF COST%%%%
Cost=0;
C=zeros(P,Q);
for i=1:P
for j=1:Q
if (i==j)
C(i,j)=Wf*abs(R(i));
else
C(i,j)=0;
end
C(i,j);
end
Cost=Cost+C(i,j);
end
end
Cost;
Performance = Cost^2 + L^2;

%K_Values(Value_count) = K;

%x_Values(Value_count) = x(1,1);

x_Values(Value_count) = x;

L_Values(Value_count) = L;

Performance_Values(Value_count) = Performance;

Cost_Values(Value_count) = Cost;

Value_count = Value_count + 1;

end

%K_Values;

x_Values;

L_Values;

Cost_Values;

Performance_Values;

Value_count = Value_count - 1;

%figure(15)

%plot(K_Values(1:Value_count),L_Values(1:Value_count),'r'),grid;hold on

%xlabel('Gain'),ylabel('Error'),grid;hold on

%plot(K_Values(1:Value_count),Cost_Values(1:Value_count),'g'),grid;hold on

%plot(K_Values(1:Value_count),Performance_Values(1:Value_count),'b'),grid;hold on

%xlabel('Gain'),ylabel('Error,Cost,Performance'),grid;hold on

figure(16)

plot(x_Values(1:Value_count),L_Values(1:Value_count),'r'),grid;hold on

%xlabel('Gain'),ylabel('Error'),grid;hold on

plot(x_Values(1:Value_count),Cost_Values(1:Value_count),'g'),grid;hold on

plot(x_Values(1:Value_count),Performance_Values(1:Value_count),'b'),grid;hold on

xlabel('Position of actuator'),ylabel('Error,Cost,Performance'),grid;hold on

%figure(16)

%plot(K_Values(1:Value_count),Cost_Values(1:Value_count),'g'),grid;hold on

%xlabel('Gain'),ylabel('Cost'),grid; hold on

%figure(17)

%plot(K_Values(1:Value_count),Performance_Values(1:Value_count),'b'),grid;hold on

%xlabel('Gain'),ylabel('Performance'),grid; hold on
figure(10)
plot(x_Values(1:Value_count),L_Values(1:Value_count),'r')
xlabel('position of actuators'), ylabel('Error'), grid; hold on

figure(11)
plot(x_Values(1:Value_count),Cost_Values(1:Value_count),'r')
xlabel('position of actuators'), ylabel('Cost'), grid; hold on

figure(12)
plot(x_Values(1:Value_count),Performance_Values(1:Value_count),'r')
xlabel('position of actuators'), ylabel('Performance'), grid; hold on

% Time plots.
for i=1:P
    figure(2*i)
    subplot(2,1,1), plot(t,A(:,2*i-1),'r')
xlabel('time'), ylabel('displacement'), grid; hold on
    subplot(2,1,2), plot(t,A(:,2*i),'g')
xlabel('time'), ylabel('velocity'), grid; hold on
end
APPENDIX B
MATLAB CODE-MODAL CONTROL METHOD

Deriv Function.m

function y = deriv6(t,A,par,x,v)

MODAL APPROACH
% E = Modulus of elasticity
% I = Cross-sectional area moment of inertia
% K = Stiffness
% M = Mass per Unit Length (m = rho*a)
% P = Number of modes
% Q = Number of Actuators
% c = Damping coefficient
% x = Location point of actuators on the beam
% v = Desired Displacement
Wa=par(1); %Wa = E*I/pho
Wb=par(2); %Wb = 2/pho
P= par(3);
Q=par(4);
n=length(x);
%Cm table
%X=(1:Q)/(Q+1);
T=zeros(P,Q);
for i=1:P
T(i,:)=sin(i*pi*x);
end
Cm =[eye(Q);T(Q+1:P,1:Q)*inv(T(1:Q,1:Q))];
Cm;
%C Table

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C = zeros(Q,1);
U = zeros(Q,1);
for i=1:Q
    for n=1:Q
        U(n) = (sin(i*pi*(x(n)))*v(n));
        C(i) = C(i) + U(n);
    end
end
for i=1:P
    S = zeros(P,1);
    for j=1:P
        E = zeros(Q,1);
        for n=1:Q
            E(n) = (A(2*n-1)-C(n));
        end
        S(j) = Cm(j,:) * E(:,1);
    end
    y(2*i-1,1) = A(2*i);
y(2*i,1) = -Wa*pi^4*(i)^4*A(2*i-1)-0.1*A(2*i)-Wb*S(i);
end
y;

Beam Response.m

clc
close all
clear all
format short
% MODAL APPROACH
% K_Values = zeros(1,100);
% x_Values = zeros(1,100);
% L_Values = zeros(1,100);
% Performance_Values = zeros(1,100);
Cost_Values = zeros(1,100);
Value_count=1;
%for K=1:.1:10;
for x=0.05:0.02:0.98
P=2; % modes
Q=1; % Actuators
%x(1,:)=[0.33333 0.66667];
%x(1,:)=[0 0.66667];
%xv(1,:)=[0.3 0.5];
%x(1,:)=[0.5];
v(1,:)= [0.5];
K=2.4;
rho=1;
E=0.1;
I=.1;
Wt= 1;
Wf= 2.4;
Wa = (E*I)/rho;
Wb = (2*K)/(rho);
par = [Wa Wb P Q ];
initial= zeros(1,2*P);
%No_Of_Iterations = (0.98 - 0.02) / 0.04
%Increment_of_x = 0.04
%Iteration_Count =0
%for i=1:No_Of_Iterations
%x(1,1)= x(1,1)+Increment_of_x
%Iteration_Count = Iteration_Count + 1;
[t,A]=ode45(@deriv6,[0,1000],initial,[],par,x,v);
% Actual shape plot.
figure(1)
q=0:1/1000:1;
Zsum=0;
z=0;
for i=1:P
    z=A(end,2*i-1)*sin(i*pi*q);
    %plot(q,z,'b')
    %xlabel('q'),ylabel('mode shape equation'),grid;hold on
    Zsum= Zsum+z;
end
plot(q,Zsum,'g')
xlabel('q'),ylabel('mode shape equation'),grid;hold on
%Desired shape plot.
G=0;
Gsum=0;
for i=1:P
    Bsum=0;
    B=0;
    for j=1:Q
        B=v(end,j)*sin(i*pi*x(j));
        Bsum=Bsum+B;
    end
    G=Bsum*sin(i*pi*q);
    Gsum=Gsum+G;
end
plot(q,Gsum,'r');hold on;
xlabel('q'),ylabel('Actual shape'),grid;hold on
% Q = Area between actual and desired shape.
Y=(Zsum-Gsum).^2;
L=100*trapz(q,Y);

% ACUTATORS FORCING
O=[A(end,1)];
%O=[A(end,1);A(end,3)];
T=zeros(P,Q);
for n=1:Q
    for i=1:P
        T(i,:)=sin(i*pi*x);
    end
end
D = inv(T(1:Q,1:Q));
Fc(n) = D(n,:) * O(:,1);

%%%COMPONENT OF COST%%%  
g(1,:) = [x 0.65];  
%g(1,:) = [x(1,1) 0.53 0.66667];  
%g(1,:) = [0.33333 0.43 .57 0.66667];  
Cost = 0;
c = zeros(P,Q);
for i = 1:P
    for j = 1:Q
        if (i == j)
            C(i,j) = Wf * abs(Fc(i));
        else
            C(i,j) = Wt * abs(g(i) - g(j));
        end
    end
    Cost = Cost + C(i,j);
end
Performance = Cost^2 + L^2;
%K_Values(Value_count) = K;
x_Values(Value_count) = x;
%x_Values(Value_count) = x(1,1);
L_Values(Value_count) = L;
Performance_Values(Value_count) = Performance;
Cost_Values(Value_count) = Cost;
Value_count = Value_count + 1;
end
%K_Values;
x_Values;
L_Values;
Cost_Values;
Performance_Values;
Value_count = Value_count - 1;

figure(14)
plot(K_Values(1:Value_count),L_Values(1:Value_count),'r'), grid; hold on
xlabel('Gain'), ylabel('Error'), grid; hold on
plot(K_Values(1:Value_count),Cost_Values(1:Value_count),'g'), grid; hold on
plot(K_Values(1:Value_count),Performance_Values(1:Value_count),'b'), grid; hold on
xlabel('Gain'), ylabel('Cost, Performance'), grid; hold on

figure(16)
plot(x_Values(1:Value_count),L_Values(1:Value_count),'r'), grid; hold on
xlabel('Gain'), ylabel('Error'), grid; hold on
plot(x_Values(1:Value_count),Cost_Values(1:Value_count),'g'), grid; hold on
plot(x_Values(1:Value_count),Performance_Values(1:Value_count),'b'), grid; hold on
xlabel('Position of actuator'), ylabel('Error, Cost, Performance'), grid; hold on

figure(10)
plot(K_Values(1:Value_count),L_Values(1:Value_count),'r')
xlabel('Gain'), ylabel('Error'), grid; hold on

figure(11)
plot(K_Values(1:Value_count),Cost_Values(1:Value_count),'g')
xlabel('Gain'), ylabel('Cost'), grid; hold on

figure(12)
plot(K_Values(1:Value_count),Performance_Values(1:Value_count),'g')
xlabel('Gain'), ylabel('Performance'), grid; hold on

figure(10)
plot(x_Values(1:Value_count),L_Values(1:Value_count),'r')
xlabel('position of actuators'), ylabel('Error'), grid; hold on

figure(11)
plot(x_Values(1:Value_count),Cost_Values(1:Value_count),'g')
xlabel('position of actuators'), ylabel('Cost'), grid; hold on

figure(12)
plot(x_Values(1:Value_count),Performance_Values(1:Value_count),'g')
xlabel('position of actuators'), ylabel('Performance'), grid; hold on

% Time plots.
% for i=1:P
% figure(2*i)
% subplot(2,1,1), plot(t,A(:,2*i-1),'r')
% xlabel('time'), ylabel('displacement'), grid; hold on
% subplot(2,1,2), plot(t,A(:,2*i),'g')
% xlabel('time'), ylabel('velocity'), grid; hold on
% end