USE OF A DIFFUSIVE APPROXIMATION OF RADIATIVE TRANSFER FOR
MODELING THERMOPHOTOVOLTAIC SYSTEMS

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USE OF A DIFFUSIVE APPROXIMATION OF RADIATIVE TRANSFER FOR
MODELING THERMOPHOTOVOLTAIC SYSTEMS

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ABSTRACT

Thermophotovoltaic (TPV) energy conversion is the transfer of heat energy into electrical energy via light. A TPV material is heated and as a result radiates light. The infra-red light is collected by a photovoltaic diode and converted to an electric current. This process could be designed to re-use wasted heat energy. In almost every mechanical process, whether it is an active factory, a running engine, or an electronic device, energy escapes in the form of heat. These TPV materials can be designed and used in a smoke stack, in a car exhaust, or even as coated nanofibers, making energy recovery possible in almost every size of mechanical system. We model a TPV system using techniques of homogenization applied to a diffusion approximation of the radiative transfer equation. The aim is to optimize the geometry of the TPV material with respect to the medium of transfer to maximize emission with minimal TPV material.
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>LIST OF TABLES</th>
<th>vii</th>
</tr>
</thead>
<tbody>
<tr>
<td>LIST OF FIGURES</td>
<td>viii</td>
</tr>
</tbody>
</table>

## CHAPTER

### I. INTRODUCTION
- 1.1 Thermophotovoltaic energy conversion 1
- 1.2 Applications 1
- 1.3 Modeling the TPV System 3
- 1.4 Incorporation of wave length and temperature 7
- 1.5 Diffusion Equation 8
- 1.6 Solving the diffusion equation 10
- 1.7 Use of this Model 10

### II. HOMOGENIZATION
- 2.1 Homogenization Process 13
- 2.2 Homogenization with other systems 17

### III. RESULTS
- 3.1 Artificial Boundary Condition 21
### Contents

3.2 Irradiance Due to Point Source ........................................... 22  
3.3 Hollow Cylinder with Quartz Wall ................................. 25  
3.4 Two Dimensional Point Source in Polar Coordinates ............. 31  
3.5 Emission from Entire Interior Medium .............................. 39  
3.6 Boundary condition for optically transparent material .......... 47  

IV. CONCLUSIONS ................................................................. 52  
4.1 Summary and Results ...................................................... 52  
4.2 Effective Diffusion Coefficients ...................................... 53  
4.3 Effects of Temperature and Wavelength ............................ 54  
4.4 Applying conclusions .................................................... 54  

APPENDIX ................................................................. 55  

BIBLIOGRAPHY ............................................................... 66
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1 Parameter values used for solutions to emitting hollow sphere.</td>
<td>27</td>
</tr>
<tr>
<td>3.2 Parameter values used for solutions to emitting hollow cylindrical shell.</td>
<td>29</td>
</tr>
<tr>
<td>3.3 Parameter values used for solutions to point source emitter in polar coordinates</td>
<td>42</td>
</tr>
<tr>
<td>3.4 Parameter values used for solutions to solid disk emitter</td>
<td>43</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>3</td>
</tr>
<tr>
<td>1.2</td>
<td>4</td>
</tr>
<tr>
<td>1.3</td>
<td>5</td>
</tr>
<tr>
<td>1.4</td>
<td>6</td>
</tr>
<tr>
<td>1.5</td>
<td>11</td>
</tr>
<tr>
<td>1.6</td>
<td>11</td>
</tr>
<tr>
<td>2.1</td>
<td>13</td>
</tr>
<tr>
<td>2.2</td>
<td>18</td>
</tr>
<tr>
<td>3.1</td>
<td>22</td>
</tr>
<tr>
<td>3.2</td>
<td>25</td>
</tr>
<tr>
<td>3.3</td>
<td>26</td>
</tr>
</tbody>
</table>
3.21 Spectral irradiance in \( r \) with \( \theta \) fixed at \( \frac{\pi}{4} \) and a point source located at \((r_0 = 0.3 \text{ m}, \theta_0 = \frac{\pi}{4})\).
CHAPTER I

INTRODUCTION

1.1 Thermophotovoltaic energy conversion

Thermophotovoltaic (TPV) energy conversion is a process that converts heat energy to light and the light into an electric current. Certain materials, such as erbia, when heated, emit a light of a limited frequency range. A major issue in the conversion of light to electric current is a heat build up in the photovoltaic diode. Only certain wavelengths generate electricity, while the remaining wavelengths generate heat. Since good emitters are also good absorbers, this work studies the geometric placement of erbia in the TPV device to optimize the amount of emitted light using the least amount of erbia.

1.2 Applications

In almost every mechanical process, whether it is an active factory, a running engine, or an electronic device, energy escapes in the form of heat. TPV devices can be designed and used in a smoke stack, in a car exhaust, or even as coated nanofibers [1], making energy recovery possible in almost every size of mechanical system.
On micro scales the TPV energy conversion process is especially appealing. There are other sources of energy on a very small scale, besides TPV energy conversion, such as the combustion of hydrocarbons, but most of these involve moving parts [2]. The lack of friction due to moving parts is one reason why micro scale TVP systems are appealing [2].

TPV energy conversion can make a small, yet significant, contribution to hybrid automobiles, factory efficiency and any other system where energy is lost in the form of heat. We seek an economically designed process to convert wasted heat energy into a usable form.

TPV energy conversion was first proposed more than 30 years ago and has recently seen an increase in interest due to the advanced production of the photovoltaic equipment necessary for constructing the diodes and the development of microfabrication technology [3, 4]. Although, to date, no commercialization of this energy recovery scheme has been attempted [5]. The high cost and relatively low efficiency make a proper balance of materials and effectiveness a challenge [5, 6, 7]. The particular emitter used in our TPV model is called erbium oxide, or erbia, made from a rare earth metal called erbium. To make TPV devices economically feasible we seek geometric designs for efficient use of erbia. For this reason, this study focuses on only the emission part of the process.

The difference between TPV energy conversion and solar energy conversion is that TPV can be used where there is a source of wasted heat, where as solar energy conversion requires sunlight. [3, 8].
1.3 Modeling the TPV System

We model the emission of a variety of sources in different media, with a focus on the geometric configuration of both the source and the composition of the media. We consider a cylindrical tube with a source of heat, as shown in Figures 1.1 and 1.2. Along the edge of the tube is a fixed cylindrical light collector, which lies on the outside of a quartz enclosure inside of which is the emitting material. Quartz has a low index of refraction and a low absorption coefficient allowing much of the light emitted from the TPV source to escape and be collected. This model could easily be applied to an exhaust tube or smoke stack.

Figure 1.1: TPV source surrounded by quartz with light collectors (Top View)
Our investigation focuses on the geometrical configuration of the distribution of the emitting material within the quartz enclosure, as well as the structural composition of the medium that is enclosed by the quartz. We will consider cases of emitting material, namely pure quartz, pure erbia, and a homogenized mixture of quartz and erbia.

We choose practical geometries to investigate the effect of a medium being nearly homogeneous yet periodic in composition [9, 10, 11]. A similar concept has been considered by R. Maffione, et al. in [10, 11], with the medium of the ocean being uniform along surfaces of equal depth, but nonuniform relative to its depth. Periodic structures such as layered cylinders, flat sheets, or spokes of alternating materials are the types of media that are to be investigated, as seen in Figures 1.3
and 1.4. We assume that these periodic structures are very small; much smaller than the dimensions of the system. We use a homogenization technique involving multiple scale analysis in both Cartesian and cylindrical coordinate systems. Multiple length scales arise when one considers the system as being periodic on very small length scales, yet seemingly homogeneous when viewed at the scale of the system as a whole. This gives us the ability to model a composite medium of quartz and erbia or any emitting material. We can adjust the volume fractions of each of these materials and for each geometrical configuration calculate the net optical properties of the medium along each direction via homogenization theory.

Figure 1.3: The rectangular shaded region represents erbia, while the white region represents quartz.

Like blackbody emitters, TPV emitters radiate light when they are heated. Therefore, locations of high temperature will emit and act as a source of light. We
choose simple configurations of sources to compare with experiments done by Aljarrah, et al. [6]. Through such comparisons we find effective values for physical parameters, namely the absorption and scattering coefficients for the TPV emitter, erbia.

There have been investigations that show the thickness of the emitter plays an important role in the efficiency of the system [12, 13]. We verify that at a certain thickness the emission reaches a plateau. Beyond a certain thickness, increasing the thickness will not increase emission. Studies by Aljarrah [6] have been done to investigate the optical properties of both quartz and erbia. Through experiment, it was found that quartz has a scattering coefficient of 1cm$^{-1}$ and an absorption coefficient of 2cm$^{-1}$, while erbia has a scattering coefficient of 1cm$^{-1}$ and an absorption coefficient of 17cm$^{-1}$. These results will be used in our calculations.
1.4 Incorporation of wavelength and temperature

The spectral irradiance of a blackbody source depends on both the temperature of the system and the wavelength of the emitted light. The spectral irradiance as a function of temperature, $T$, and wavelength, $\lambda$, follows Planck’s law

$$P = \frac{2\pi h c^2}{\lambda^5 \left[e^{\frac{h}{\lambda k T}} - 1\right]},$$

(1.1)

where $k$ is Boltzmann’s constant, $h$ is Planck’s constant, and $c$ is the speed of light.

We will use this expression to model the strength of the emission from the TPV materials at the effective wavelengths \cite{14, 15, 16}.

In equation (1.1), the spectral irradiance decreases rapidly with the temperature. This will enable us to investigate high temperature regions with the corresponding spectral irradiance of the wavelength of interest, and to consider the emission elsewhere to be zero. Through experiments conducted at 773 K, the wavelength of highest emission from the TPV material erbia is approximately 1.5 $\mu$m \cite{6}.

The dependence on the wavelength and temperature is important. The strength of the source will vary greatly with temperature, and we obtain the frequency from the wavelength. From basic optics, we have the relationship that wavelength, $\lambda$, and frequency, $f$, are related by $\lambda = \frac{1}{f}$. A photoelectric diode will be chosen to have a frequency range for which it is most efficient.
1.5 Diffusion Equation

In the one dimensional case, we calculate how spectral radiance, \( L \), which is a watt per steradian per square meter per wavelength, changes along a spatial dimension \( x \) \cite{17, 18}

\[
\frac{dL}{dx} + (a + \sigma) L = aP. \tag{1.2}
\]

Here, \( a \) is the absorption coefficient and \( \sigma \) is the scattering coefficient.

The spectral radiance can be used to calculate an important measurable quantity for our model, called the spectral irradiance. The spectral irradiance, \( I \), of the light is a measure of the resulting power incident on a surface, in units of watts per square meter per wavelength. To arrive at the diffusion equation for spectral irradiance, Lopez integrates equation (1.2) over all solid angles, \( I = \int L d\omega \). Therefore, spectral irradiance is the net effect from the spectral radiance integrated over all possible directions. From Lopez et al. \cite{19}, we have

\[
-\nabla_r [D(\vec{r}) \nabla_r I(\vec{r})] + aI(\vec{r}) = aP, \tag{1.3}
\]

where the diffusion coefficient is

\[
D(\vec{r}) = \left\{a + \sigma [1 - g(\vec{r})] + 2/r\right\}^{-1}/3. \tag{1.4}
\]

Here \( P \) is as specified in equation (1.1), \( \vec{r} \) is the position vector, and \( r = |\vec{r}| \).

It can be noted that the diffusion approximation assumes non-anisotropic light propagation \cite{20}. The term \( g(\vec{r}) \) is a result of anisotropic scattering which
is neglected, hence $g(\vec{r}) = 0$. We also neglect the $2/r$ term, which represents the divergence coefficient and is zero for media of constant refractive index [19]. Thus, we take

$$D(\vec{r}) = (a + \sigma)^{-1}/3.$$ (1.5)

The diffusion equation (1.3) is an approximation of the actual radiative transfer equation through a scattering and absorbing media [9, 13, 18, 21, 22, 23]. The first term approximates the radiative transfer of light as a diffusive term, the second term accounts for the absorption of light that is emitted. The right hand side represents spectral irradiance from a source. Note that the absorption and scattering coefficients can both depend on $\vec{r}$.

The diffusion approximation is appropriate to apply for the media considered, as the quantity $l_m/H < 1$, where $l_m = 1/(a + \sigma)$ and $H$ is the distance over which radiant energy density changes significantly [24]. In the media considered, we have $l_m << 1$ since the absorption coefficient is large.

Others have arrived at an approximate diffusion coefficient independent of the absorption coefficient [22]. The approximation in equation (1.4) is accurate and consistent with other approximations provided $I$ is much larger than the radiant flux density vector component along any direction [22, 25].
1.6 Solving the diffusion equation

In order to solve the diffusion equation, boundary conditions are required. We must account for the change in the indices of refraction at the boundary of the quartz enclosure. In particular, light absorption and scattering at the quartz boundary must be considered. Since the light is collected somewhere near the edge of the enclosure we impose an artificial far field boundary condition [26, 27]. That is, we set the spectral irradiance to zero beyond a certain point where in actuality it is arbitrarily small and nearing zero. We then measure the light collected in the collector by the solution to this slightly modified problem, which yields a slight under estimate of the results [27, 28]. The calculation of this distance will be discussed in Chapter 3. This saves us from solving the diffusion equation in a third domain outside the quartz wall, as in Figure 1.5. Instead we simply artificially extend the quartz wall to achieve a conservative approximation for what the spectral irradiance will actually be at the location of the light collector, as in Figure 1.6.

1.7 Use of this Model

We aim to calculate the spectral irradiance due to different sources in different media. Using homogenization we will develop effective coefficients of diffusion, absorption, and scattering. This will simplify the development of analytical solutions and allow us to calculate the spectral irradiance resulting from many sources in different media. We intend to find the optimal geometric distribution of the emitting medium. This
Figure 1.5: System without artificial boundary condition, in which we must solve the diffusion equation in three regions of space.

Figure 1.6: System with artificial boundary condition, in which we must solve the diffusion equation in only two regions of space.
will contribute towards the more effective use of TPV materials, as well as improve
the efficiency of the devices.

The configuration of the medium in which the source is emitting, as well
as the geometric configuration of the source, play an important roll in the spectral
irradiance that escapes far from the source. Since the light is collected outside the
quartz enclosure, we are most interested in the light that is away from the location
of the source.

In the remainder of the text, we will calculate the parameters for our model
and use them to calculate the spectral irradiance. We will use homogenization theory
as well as boundary conditions from Contini [27]. This will allow us to directly
compare each medium. We will also study the effects of the thickness of the emitting
material, as well as the effects of varying the amount of quartz and erbia in the
emitting medium.

In Chapter 2, we will go through the homogenization process to calculate the
effective optical properties of the homogenized media. In Chapter 3, we will use these
results to calculate the spectral irradiance. We aim to find the composition of the
interior medium that will yield the highest spectral irradiance away from the source.
CHAPTER II

HOMOGENIZATION

2.1 Homogenization Process

Consider emission from a source in a medium that is periodic in $\theta$, as in Figure 2.1.

![Figure 2.1: Medium containing small periodicities in $\theta$ that do not vary in $r$. $\varepsilon$ represents the angle subtended by each periodic section, and $\phi$ represents the volume fraction of quartz.](image)

Each periodic section subtends an angle of $\varepsilon$ radians, where $\varepsilon \ll 1$. The volume fraction of quartz is denoted by $\phi$, where $\phi \in [0, 1]$. The volume fraction of erbia is therefore $1 - \phi$. 

13
Lopez [19] defines the steady-state diffusion equation for the spectral irradiance, $I$ from a source, $S$ as

$$-\nabla \cdot [D \nabla I] + aI = S,$$  \hspace{1cm} (2.1)

where the diffusion coefficient $D$ is

$$D = \frac{1}{3[a + \sigma]}.$$

Here, $a$ is the absorption coefficient and $\sigma$ is the scattering coefficient. In the definition of $D$ we have neglected the contribution from the phase function for anisotropic scattering. In the configuration shown in Figure 2.1, it makes sense that $D$ depends on $\theta$, since the system does not vary in the radial direction.

We define the multiple length scales $r$, $\theta$, and $s = \frac{\theta}{\varepsilon}$, where $r$, $\theta$, and $s$ are $O(1)$. By the chain rule we have that

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta} + \frac{1}{\varepsilon} \frac{\partial}{\partial s},$$

$$\frac{\partial^2}{\partial \theta^2} = \frac{\partial^2}{\partial \theta^2} + \frac{2}{\varepsilon} \frac{\partial^2}{\partial \theta \partial s} + \frac{1}{\varepsilon^2} \frac{\partial^2}{\partial s^2}.$$

Hence, equation (2.1) becomes

$$-D \left( I_{rr} + \frac{1}{r} I_r \right) - \frac{1}{r^2} \left( D_\theta + \frac{1}{\varepsilon} D_s \right) \left( I_\theta + \frac{1}{\varepsilon} I_s \right) - \frac{D}{r^2} \left( I_{\theta \theta} + \frac{2}{\varepsilon} I_{\theta s} + \frac{1}{r^2} I_{ss} \right) + aI = aP.$$  \hspace{1cm} (2.2)

Here we have defined the source term, $S = aP$, based upon the blackbody emission as in Chubb [14, 15, 16], with $P$ defined in equation (1.1). For our purposes we restrict $P$ to be a constant. This means the temperature, $T$, is constant. We assume that $\varepsilon << 1$ and expand $I(r, \theta, s) = I_0(r, \theta, s) + \varepsilon I_1(r, \theta, s) + \varepsilon^2 I_2(r, \theta, s) + ....$
The $O(\frac{1}{\varepsilon^2})$ equation is

$$-[DI_0]_s = 0,$$

whose solution is

$$I_0 = A(r, \theta) + \int_0^s \frac{B(r, \theta)}{D} ds. \quad (2.3)$$

Note $B(r, \theta)$ and $A(r, \theta)$ are arbitrary functions of $r$ and $\theta$. We require that the integral is zero to ensure periodicity in $s$, for $s \in [0, 1]$. Thus $I_0 = A(r, \theta)$.

At $O(\frac{1}{\varepsilon})$ we find

$$-[DI_1]_s = D_s A_\theta(r, \theta).$$

We integrate with respect to $s$ twice to obtain

$$I_1 = -A_\theta s - \int_0^s \frac{C'(r, \theta)}{D} ds + E(r, \theta),$$

where $C(r, \theta)$ and $E(r, \theta)$ are arbitrary functions. To enforce periodicity in $s \in [0, 1]$ we choose

$$C = -\frac{A_\theta}{\int_0^1 \frac{1}{D} ds},$$

so that

$$I_1 = A_\theta \left[ -s + \int_0^s \frac{1}{D} ds \right] + E(r, \theta). \quad (2.4)$$

At $O(1)$ we find

$$-[DI_2]_s = DI_{0\theta\theta} + 2DI_{1\theta} + r^2 DI_{0rr} + Dr I_{0r} - r^2 a I_0 + D_\theta I_{0\theta} + D_\theta I_{1s} + D_s I_{1\theta} + r^2 a P.$$

We integrate with respect to $s$ to obtain

$$-DI_2 = \left[ r^2 A_{rr} + rA_r \right] + A_{\theta\theta} \int_0^1 D ds \quad (2.5)$$
\[ +\left[-r^2A - r^2P\right]\int_0^s \text{ads} + \int_0^s DA_{\theta\theta} \left[-1 + \frac{1}{\int_0^1 \frac{1}{D} ds}\right] ds \]  
\[ + D \left[A_{\theta\theta} \left(-s + \frac{\int_0^s \frac{1}{D} ds}{\int_0^1 \frac{1}{D} ds} + E_\theta\right) + F(r, \theta), \right] \]

where \( F(r, \theta) \) is arbitrary. If we force periodicity in \( s \), then

\[-\frac{1}{r^2} \left(\frac{1}{\int_0^1 \frac{1}{D} ds}\right) A_{\theta\theta} - \left(\int_0^1 \text{Dds}\right) \left(A_{rr} + \frac{1}{r} A_r\right) + \left(\int_0^1 \text{ads}\right) A = \left(\int_0^1 \text{ads}\right) P. \]  

(2.6)

From Figure 2.1 we see that

\[ \int_0^1 \text{ads} = \int_0^\phi a_q ds + \int_\phi^1 a_e ds = a_q \phi + a_e (1 - \phi) = a_{eff}, \]

(2.7)

where \( a_q \) is the absorption coefficient of quartz, \( a_e \) is the absorption coefficient of erbia, and \( a_{eff} \) is the effective absorption coefficient. Note that \( a_q \) and \( a_e \) are assumed to be constants over the spatial coordinate \( s \). If \( \kappa_q = a_q + \sigma_q \) and \( \kappa_e = a_e + \sigma_e \) are the extinction coefficients of quartz and erbia, respectively, then it can be shown that

\[ \int_0^1 \text{Dds} = \frac{\phi}{3[a_q + \sigma_q]} + \frac{1 - \phi}{3[a_e + \sigma_e]} = \frac{1}{3} \left[ \frac{\phi \kappa_e + (1 - \phi) \kappa_q}{\kappa_e \kappa_q} \right] = D_{reff}, \]

(2.8)

\[ \frac{1}{\int_0^1 \frac{1}{D} ds} = \frac{1}{3} \left[ \frac{\phi \kappa_q + (1 - \phi) \kappa_e}{\phi \kappa_q + (1 - \phi) \kappa_e} \right] = D_{\theta eff}. \]

(2.9)

Hence, equation (2.1), defined in the domain of Figure 2.1, is approximated by the following equation for \( I_0 \)

\[ -D_{\theta eff} I_{0\theta\theta} - D_{reff} \left(I_{0rr} + \frac{1}{r} I_{0r}\right) + a_{eff} I_0 = a_{eff} P. \]

(2.10)

If we consider that \( D_{eff} \) has the general form \( D_{eff} = \frac{1}{3 \kappa_{eff}} \), then from equations (2.8) and (2.9) we have that

\[ \kappa_{reff} = \frac{\kappa_e \kappa_q}{\phi \kappa_q + (1 - \phi) \kappa_e}, \]

(2.11)
\[ \kappa_{\theta eff} = \phi \kappa_q + (1 - \phi) \kappa_e. \] (2.12)

From here we conclude that

\[ \sigma_{\text{reff}} = \frac{\sigma_e \sigma_q}{\phi \sigma_e + (1 - \phi) \sigma_q}, \] (2.13)

\[ \sigma_{\theta eff} = \phi \sigma_q + (1 - \phi) \sigma_e. \] (2.14)

Hence, for the configuration shown in Figure 2.1, light is absorbed equally in the \( r \) and \( \theta \) directions per the effective absorption coefficient defined in equation (2.7). However, light is scattered differently in different directions, as described in equations (2.13) and (2.14). For example, if \( \kappa_q \) is much smaller than \( \kappa_e \), then \( D_{\text{reff}} \) will be much larger than \( D_{\theta eff} \). Thus, the spectral irradiance in the radial direction will be larger than in the \( \theta \)-direction.

### 2.2 Homogenization with other systems

An identical process could be considered for different homogenized media. If we let our structure be uniform in \( \theta \), and have a small periodicity in \( r \), we would see a medium like Figure 2.2.

Each periodic section has a radial thickness of \( \varepsilon \), where \( \varepsilon << 1 \). The volume fraction of quartz is denoted by \( \phi \), where \( \phi \in [0, 1] \). The volume fraction of erbia is therefore \( 1 - \phi \).

If we perform the analogous calculations, we see that the effective diffusion coefficients along the \( r \) and \( \theta \) directions simply switch. That is,

\[
D_{\text{reff}} = \frac{1}{3} \left[ \frac{1}{\phi \kappa_q + (1 - \phi) \kappa_e} \right].
\]
Figure 2.2: Medium containing small periodicities in $r$ that do not vary in $\theta$. $\varepsilon$ represents the radial thickness of each periodic section.

and,

$$D_{\theta\text{eff}} = \frac{1}{3} \left[ \frac{\phi \kappa_e + (1 - \phi) \kappa_q}{\kappa_e \kappa_q} \right].$$

We see a similar result for the scattering coefficients. In the configuration in Figure 2.2, we have

$$\sigma_{\theta\text{eff}} = \phi \sigma_q + (1 - \phi) \sigma_e, \quad (2.15)$$

$$\sigma_{\text{ref}} = \frac{\sigma_e \sigma_q}{\phi \sigma_e + (1 - \phi) \sigma_q}. \quad (2.16)$$

Hence, for the configuration shown in Figure 2.2, light is absorbed equally in the $r$ and $\theta$ directions per the effective absorption coefficient defined in equation (2.7). However, how light is scattered depends upon direction, as described in equations (2.15) and (2.16). For example if $\kappa_q$ is much smaller than $\kappa_e$, then $D_{\theta\text{eff}}$ will be much larger than $D_{\text{ref}}$. Thus, the spectral irradiance in the angular direction will be larger than in the radial direction.
These effective coefficients will be investigated as we calculate the spectral irradiance of sources in a variety of different media.
CHAPTER III

RESULTS

In this chapter we calculate the spectral irradiance due to a variety of sources. These sources model locations of high temperature. As we see in our blackbody term in equation (1.1) we have high spectral irradiance from regions of higher temperature and as our temperature decreases the spectral irradiance will quickly go to zero. We will use delta functions as point sources to model single hot spots in the emitting medium. Since our equations are linear, we invoke the principle of superposition to model more complicated sources as the sum of the simple sources considered here. This allows us to consider many geometrical configurations of the source of emission.

It should be noted that this idealization of the source to a delta function will be acceptable, so long as our interest lies away from the source [26, 29].

We evaluated $P$ for $\lambda = 1.5\mu m$, which is the highest wavelength emitted by the erbium doped nanofibers used in experiment [6]. We will let $T = 773 K$, which is the temperature at which experimental results from Aljarrah, et. al [6] were recorded. The resulting magnitude is $6.7908 \cdot 10^{-10}$. We use this value in the calculations that follow.

We consider four types of media within the quartz wall. We have two different uniform media, one of pure quartz and one pure erbium, and two homogenized media of
pure erbia and pure quartz. One medium is the spokes illustrated in Figure 2.1 and discussed in Chapter 2. The second is the bands, illustrated in Figure 2.2, and discussed in Chapter 2. For the general comparison between uniform and homogenized media, we will let $\phi = 0.5$. This means that the banded and spoke configurations will be half erbia and half quartz. We will then investigate the effects of varying $\phi$, and the resulting spectral irradiance.

3.1 Artificial Boundary Condition

We now calculate the artificial boundary condition as prescribed in Contini [27]. The artificial boundary condition sets the spectral irradiance to zero at some location beyond the actual thickness of the quartz wall. For the models considered here, we let the quartz wall begin at $r = R_g m$ and be 0.1 m thick, as in Figure 3.1. We calculate the distance, which we denote by $z$, beyond the quartz wall where we impose our artificial boundary condition. Contini found that $z = 2AD$, where $D$ is the diffusion coefficient, and $A$ is calculated in equation (3.1). We therefore set the spectral irradiance equal to zero at $R_a = R_g + 0.1 + z$, where

$$A = 504.332889 - 2641.00214n + 5923.699064n^2 - 7376.355814n^3 + 5507.53041n^4$$

$$- 2463.357945n^5 + 610.956547n^6 - 64.8047n^7. \quad (3.1)$$
Here $n = \frac{n_q}{n_{\text{outside}}}$. $n_{\text{outside}}$ is the index of refraction of the outside medium, in this case we will use air ($n_{\text{outside}} = 1.00029$ for air), and $n_q = 1.5$ is the index of refraction of quartz [27].

Figure 3.1: The interior medium extends to $R_g$. The quartz wall has thickness 0.1m, and it is being artificially extended by a distance $z$.

3.2 Irradiance Due to Point Source

To approximate a point source in three dimensions, we will calculate the spectral irradiance due to a small hollow spherical shell. We do not expect our spectral irradiance to change in any direction except radially, since our source is symmetric. Our spectral irradiance can be approximated by the following partial differential
equation,
\[-D_{\text{ref}} \left[ I_{rr} + \frac{2}{r} I_r \right] + a_{\text{eff}} I = \frac{a_{\text{eff}} P \delta(r - r_0)}{4\pi r^2}.\]  \hspace{1cm} (3.2)

Here, $P$ is the power of the source distributed on the spherical shell of radius $r_0$, the magnitude of $P$ is defined by equation (1.1). Recall that $a_{\text{eff}}$ is the effective absorption coefficient and $D_{\text{ref}}$ is the effective diffusion coefficient. The first two terms are representative of the diffusion of the radiation, while the third term accounts for absorption of the light back into the media. The right hand side accounts for spectral irradiance from the source, in this case a hollow radiating sphere of radius $r_0$. To solve this equation we split the domain of $r$ into the two regions where the delta function is zero ($0 < r < r_0$ and $r > r_0$). We denote $I = I^-$ for $0 < r < r_0$, and $I = I^+$ for $r > r_0$. This leads to the following governing equations:

\[-D_{\text{ref}} \left[ I_{rr} + \frac{2}{r} I_r \right] + a_{\text{eff}} I^- = 0 \hspace{1cm} (3.3)\]

and

\[-D_{\text{ref}} \left[ I_{rr} + \frac{2}{r} I_r \right] + a_{\text{eff}} I^+ = 0.\]  \hspace{1cm} (3.4)

We enforce the boundary conditions:

\[I^+(r_0) = I^-(r_0)\]

to ensure continuity in $I$ at the location of the source,

\[
\left. \frac{d}{dr} I^+ \right|_{r=r_0} - \left. \frac{d}{dr} I^- \right|_{r=r_0} = -\frac{aP}{D_{\text{ref}} 4\pi r_0^2}.\]  \hspace{1cm} (3.5)
to account for the change in our flux at the source, also known as the jump condition,

\[ \lim_{r \to 0} I^- < \infty \]

and

\[ \lim_{r \to \infty} I^+ < \infty, \]

to ensure that our solution is bounded everywhere. After solving equations (3.3) and (3.4), and using the parameters specified in Table 3.1, we obtain the solutions:

\[ I^+ = \frac{c_1}{r} \sinh(ggr) \] (3.6)

for \( r < r_0 \), and

\[ I^- = \frac{c_2}{r} e^{-ggr} \] (3.7)

for \( r > r_0 \). Here \( gg = \sqrt{\frac{\alpha_{eff}}{D_{refj}}} \), \( \alpha = -\frac{aP}{D_{refj}4\pi r_0^4} \) and the constants \( c_1 \) and \( c_2 \) are defined in the Appendix.

Figure 3.2 shows the spectral irradiance due to a spherical source in a uniform medium of pure quartz and of pure erbia. The emission within the erbia is much greater than that within the quartz close to the source. This can be explained by the high absorption coefficient which multiplies the source term in equation (3.2). Therefore, high absorbing media are also high emitting media. To the contrary, in the medium of pure quartz, less absorption leads to less emission. We also see that the spectral irradiance is higher in pure quartz than in pure erbia far from the source. This is also because of the high absorption coefficient in erbia not allowing the light to travel far.
Figure 3.2: Emission from a hollow sphere with radius $r_0 = 0.3$ m.

3.3 Hollow Cylinder with Quartz Wall

Next, we consider the emission from a single emitting cylinder of radius $r_0$. The system is axisymmetric if we center the cylinder on the $z$-axis. In this case, we have the diffusion equation independent of $\theta$ and $z$. We consider effects from the cylindrical quartz wall on the outside of the system. We treat the system as if the quartz were running from $R_g$ to $R_a$, see Figure 3.3. We denote the spectral irradiance and the effective coefficients inside the interior medium with a subscript $g$, and those inside the quartz with a subscript $q$. It should be noted that the medium inside the quartz enclosure could be uniform or homogenized, as discussed in Chapter 2.
Figure 3.3: System enclosed with quartz containing an emitting hollow cylindrical shell of radius $r_0$. 
Table 3.1: Parameter values used for solutions to emitting hollow sphere.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Pure Erbia</th>
<th>Pure Quartz</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_0$</td>
<td>0.3m</td>
<td>0.3m</td>
</tr>
<tr>
<td>$D_{reff}$</td>
<td>$\frac{1}{34}$cm</td>
<td>$\frac{1}{9}$cm</td>
</tr>
<tr>
<td>$a_{eff}$</td>
<td>17cm$^{-1}$</td>
<td>2cm$^{-1}$</td>
</tr>
<tr>
<td>$P$</td>
<td>$6.7908 \cdot 10^{-10} W$</td>
<td>$6.7908 \cdot 10^{-10} W$</td>
</tr>
</tbody>
</table>

Now we shall calculate the spectral irradiance by solving

$$-D_{qreff} \left[ I_{gr} + \frac{1}{r} I_{gr} \right] + a_{eff} I_g = \frac{a_{eff} P \delta(r - r_0)}{2\pi r}$$

(3.8)

for $0 \leq r \leq R_g$, and

$$-D_{qreff} \left[ I_{qrr} + \frac{1}{r} I_{qr} \right] + a_{eff} I_q = 0$$

(3.9)

for $R_g \leq r \leq R_a$.

We apply the boundary conditions:

$$\lim_{{r \to r_0^-}} I_g = \lim_{{r \to r_0^+}} I_g$$

to ensure continuity at the location of the source,

$$\lim_{{r \to r_0^-}} \frac{dI_g}{dr} - \lim_{{r \to r_0^+}} \frac{dI_g}{dr} = -\frac{a_{eff} P}{D_{qreff} 2\pi r_0}$$

to account for the jump in flux at the source,

$$n^2_q I_g(R_g) = n^2_q I_g(R_g)$$

27
to account for the change in index of refraction at the location of the quartz wall,

\[ D_{qreff} \frac{dI_g}{dr} \bigg|_{r=Ra} = D_{qreff} \frac{dI_q}{dr} \bigg|_{r=Ra} \]

to ensure continuity of flux,

\[ \lim_{r \to 0} I_g(r) < \infty \]

to ensure boundedness everywhere, and

\[ I_q(R_a) = 0 \]

per our artificial boundary condition.

Solving these equations, we find the solution as a linear combination of modified Bessel functions of order zero, \( I_0 \left( \sqrt{\frac{a_{eff}}{D_{reff}}} r \right) \) and \( K_0 \left( \sqrt{\frac{a_{eff}}{D_{reff}}} r \right) \). Note that \( I_0 \) is the modified Bessel function of the first kind of order zero, and is not the leading order term in the asymptotic expansion of spectral irradiance discussed in Chapter 2. Imposing the boundary conditions leads to the solutions

\[ I_g^- = c_1 I_0(ggr) + c_2 K_0(ggr) \quad (3.10) \]

for \( r < r_0 \),

\[ I_g^+ = c_3 I_0(ggr) + c_4 K_0(ggr) \quad (3.11) \]

for \( r_0 < r < Ra \), and

\[ I_q = c_5 I_0(gqr) + c_6 K_0(gqr) \quad (3.12) \]

28
for $R_g < r < R_a$, where $g_g = \sqrt{\frac{a_{geff}}{D_{geff}}}$, $g_q = \sqrt{\frac{a_{qeff}}{D_{qeff}}}$, $\alpha = -\frac{a_{qeff}P}{D_{qeff}2\pi r_0}$, and the constants $c_1$ through $c_6$ are as specified in the Appendix.

These solutions are shown in Figure 3.4 and Figure 3.5, using the parameters specified in Table 3.2.

Table 3.2: Parameter values used for solutions to emitting hollow cylindrical shell.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Pure Quartz</th>
<th>Spokes</th>
<th>Bands</th>
<th>Pure Erbia</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_0$</td>
<td>0.3m</td>
<td>0.3m</td>
<td>0.3m</td>
<td>0.3m</td>
</tr>
<tr>
<td>$R_g$</td>
<td>0.4m</td>
<td>0.4m</td>
<td>0.4m</td>
<td>0.4m</td>
</tr>
<tr>
<td>$R_a$</td>
<td>1.3055m</td>
<td>1.3055m</td>
<td>1.3055m</td>
<td>1.3055m</td>
</tr>
<tr>
<td>$D_{geff}$</td>
<td>$\frac{1}{5}$cm</td>
<td>0.0648cm</td>
<td>0.0317cm</td>
<td>$\frac{1}{5}$cm</td>
</tr>
<tr>
<td>$D_{qeff}$</td>
<td>$\frac{1}{5}$cm</td>
<td>$\frac{1}{5}$cm</td>
<td>$\frac{1}{5}$cm</td>
<td>$\frac{1}{5}$cm</td>
</tr>
<tr>
<td>$a_{geff}$</td>
<td>2cm$^{-1}$</td>
<td>9.5cm$^{-1}$</td>
<td>9.5cm$^{-1}$</td>
<td>17cm$^{-1}$</td>
</tr>
<tr>
<td>$a_{qeff}$</td>
<td>2cm$^{-1}$</td>
<td>2cm$^{-1}$</td>
<td>2cm$^{-1}$</td>
<td>2cm$^{-1}$</td>
</tr>
<tr>
<td>$P$</td>
<td>$6.7908 \cdot 10^{-10}W$</td>
<td>$6.7908 \cdot 10^{-10}W$</td>
<td>$6.7908 \cdot 10^{-10}W$</td>
<td>$6.7908 \cdot 10^{-10}W$</td>
</tr>
<tr>
<td>$n_g$</td>
<td>1.5</td>
<td>1.9</td>
<td>1.9</td>
<td>2.3</td>
</tr>
<tr>
<td>$n_q$</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
</tr>
</tbody>
</table>

We observe that the spectral irradiance due to the cylindrical source, in Figure 3.4, behaves as we expect. It decays as $r \to 0$. We see that there is considerable spectral irradiance at $r = 0$ in each medium, except for that of pure erbia. This, however, is not of practical importance to us. We are concerned with the spectral
irradiance on the outside of the quartz wall. There is a drop in spectral irradiance at the quartz wall, $r = R_g = 0.4$m, in all cases, except for that of pure quartz. This is because light is not reflected back into the interior when both media are pure quartz, since there is no change in refractive index. We see that close to the source, erbia produces light of the highest spectral irradiance, but it decays the quickest. Pure erbia does not yield high spectral irradiance far from the source, because the high absorption coefficient results in more light being absorbed over a smaller distance. For this reason, we would not want our medium to be comprised of pure erbia. The spoke configuration produces more spectral irradiance far from the source than pure quartz and pure erbia. This is a result of the high emission for the erbia combined with the low absorption coefficient of the quartz. The periodic structure of the quartz
allows the light emitted from the erbia to travel in the radial direction through the quartz. In the case of bands, we see that light emitted from the interior bands is re-absorbed in each layer of erbia as it travels radially. It is clear from Figure 3.5 that for the purposes of collecting the light beyond the quartz wall, we should use the homogenized medium of spokes.

![Figure 3.5: View of spectral irradiance beyond the quartz wall from a hollow cylindrical shell source with radius $r_0 = 0.3$ m.](image)

3.4 Two Dimensional Point Source in Polar Coordinates

Now, let us consider the emission from a point source on the $r-\theta$ plane. See Figure 3.6. It should also be noted that the interior medium could be uniform or homogenized.
Our diffusion equation will be independent of \( z \). If we place the point source at some location, \((r_0, \theta_0)\), inside the quartz enclosure, we solve the following equation for spectral irradiance:

\[
-D_{g_{eff}} \left[ I_{grr} + \frac{1}{r} I_{gr} \right] - D_{g_{\theta eff}} \left[ \frac{1}{r^2} I_{g\theta} \right] + a_{g_{eff}} I_g = \frac{a_{g_{eff}} P \delta(r - r_0)\delta(\theta - \theta_0)}{r} \tag{3.13}
\]

for \( 0 < r < R_g \), and

\[
-D_{q_{eff}} \left[ I_{qrr} + \frac{1}{r} I_{qr} \right] - D_{q_{\theta eff}} \left[ \frac{1}{r^2} I_{q\theta} \right] + a_{q_{eff}} I_q = 0 \tag{3.14}
\]

for \( R_g < r < R_a \). These equations are subject to the boundary conditions:

\[
I_g(r, 0) = I_g(r, 2\pi),
\]
\[ I_{g\theta}(r, 0) = I_{g\theta}(r, 2\pi), \]

\[ I_q(r, 0) = I_q(r, 2\pi), \]

and

\[ I_{q\theta}(r, 0) = I_{q\theta}(r, 2\pi) \]

to ensure 2\pi periodicity in \( \theta \),

\[ \lim_{r \to 0} I_g(r, \theta) < \infty \]

to ensure spectral irradiance is bounded everywhere,

\[ I_q(R_a, \theta) = 0 \]

as implementation of the artificial boundary condition,

\[ n_q^2 I_g(R_g, \theta) = n_g^2 I_q(R_g, \theta) \]

to account for the change in index of refraction, and lastly

\[ D_{\text{qreff}} \frac{dI_g(R_g, \theta)}{dr} = D_{\text{qreff}} \frac{dI_q(R_g, \theta)}{dr} \]

to ensure continuity of flux.

For the interest of the reader, we present the solution procedure of this case in detail. This solution procedure is very similar to that of the other cases. Equations (3.13) and (3.14) are solvable using separation of variables, therefore \( I(r, \theta) = A(r)T(\theta) \). If we assume that we are away from the location of the source, then \((r, \theta) \neq (r_0, \theta_0)\). We solve the resulting ordinary differential equation in \( \theta \) to find
eigen-functions $\sin(n\theta)$ and $\cos(n\theta)$. We expand the solution as

$$I = \sum_{n=0}^{\infty} (A_n(r) \sin(n\theta) + B_n(r) \cos(n\theta)).$$

We find

$$A_n(r) = \frac{1}{\pi} \int_{0}^{2\pi} I(r, \theta) \sin(n\theta) d\theta,$$

and

$$B_n(r) = \frac{1}{\pi} \int_{0}^{2\pi} I(r, \theta) \cos(n\theta) d\theta.$$

We multiply equation (3.13) by $\frac{\sin(n\theta)}{\pi}$ and integrate from 0 to $2\pi$. After some manipulation, we find

$$r^2 \frac{d^2 A_{gn}}{dr^2} + r \frac{dA_{gn}}{dr} - \left( \frac{D_{qeff} n^2}{D_{qreff}} + a_{qeff} r^2 \right) A_{gn} = \frac{a_{qeff} Pr \delta(r - r_0)}{\pi} \sin(n\theta_0)$$

(3.15)

for $0 < r < R_g$, and

$$r^2 \frac{d^2 A_{qn}}{dr^2} + r \frac{dA_{qn}}{dr} - \left( \frac{D_{qeff} n^2}{D_{qreff}} + a_{qeff} r^2 \right) A_{qn} = 0$$

(3.16)

for $R_g < r < R_a$.

These equations lead to modified Bessel functions as solutions, whose order depends on $n$. We must now derive boundary conditions for $A_{gn}(r)$ and $A_{qn}(r)$. After consideration of the boundary conditions prescribed, we see the following conditions for each value of $n$:

$$\lim_{r \to 0} A_{gn}(r) < \infty$$

to ensure that $I$ is bounded as $r \to 0$,

$$n_q^2 A_{gn}(R_g) = n_q^2 A_{qn}(R_g)$$
as a result of the change in index of refraction at \( r = R_g \),
\[
D_{\text{ref}} \frac{dA_{gn}}{dr} \bigg|_{r=R_g} = D_{\text{ref}} \frac{dA_{gn}}{dr} \bigg|_{r=R_g}
\]
as a result of continuity of flux, and
\[
A_{gn}(R_a) = 0
\]
as a result of the artificial boundary condition that \( I = 0 \) at \( r = R_a \).

We now consider three regions of space in the \( r \) direction. If we are away from the source, then \( r \neq r_0 \) and the right hand side of equation (3.15) is zero. We require the spectral irradiance to be continuous at \( r = r_0 \); therefore,
\[
\lim_{r \to r_0^+} A_{gn}(r) = \lim_{r \to r_0^-} A_{gn}(r).
\]

By integrating equation (3.15) over an infinitesimal length containing \( r_0 \), we arrive at the boundary condition, known as the jump condition,
\[
\lim_{r \to r_0^+} \frac{dA_{gn}(r)}{dr} - \lim_{r \to r_0^-} \frac{dA_{gn}(r)}{dr} = -\frac{a_{geff}P \sin(n\theta_0)}{\pi r_0}.
\]

Satisfying these boundary conditions we obtain the solutions
\[
A_{gn}^{-}(r) = c_1 I_{vg}(ggr) + c_2 K_{vg}(ggr) \quad (3.17)
\]
for \( r < r_0 \),
\[
A_{gn}^{+}(r) = c_3 I_{vg}(ggr) + c_4 K_{vg}(ggr) \quad (3.18)
\]
for \( r_0 < r < R_g \), and
\[ A_{qn}(r) = c_5 I_{vq}(gqr) + c_6 K_{vq}(gqr) \] (3.19)

for \( R_g < r < R_a \). Here \( gg = \sqrt{\frac{\alpha_{qeff}}{D_{qreff}}} \), \( gq = \sqrt{\frac{\alpha_{qeff}}{D_{qreff}}} \), \( vg = \sqrt{\frac{D_{qθeff}}{D_{qreff}}} \), \( vq = \sqrt{\frac{D_{qθeff}}{D_{qreff}}} \), \( \alpha = \frac{a_{qeff} P \sin(nθ_0)}{\pi r_0} \), and the constants \( c_1 \) through \( c_6 \) are as specified in the Appendix.

We repeat this process to obtain \( B_n(r) \), which will have the same functional form as \( A_n(r) \), with slightly modified constants. This will give us all the information we need to build the solution for the spectral irradiance. We see the solutions in Figures 3.7 and 3.8 using the parameters specified in Table 3.3.

![Spectral Irradiance Graph](image)

Figure 3.7: Spectral irradiance in \( r \) direction with \( θ \) fixed at \( \frac{π}{4} \) and a point source located at \( (r_0 = 0.3 \text{m}, θ_0 = \frac{π}{4}) \).

Notice in Figure 3.7, that spokes have the most spectral irradiance far from
the source. Since we are concerned with the spectral irradiance that makes it into the quartz wall, we see that the most ideal structure considered here is spokes. It should be noted that the spectral irradiance due to a point source in two dimensions, which would be a line source in three dimensions, gives off more spectral irradiance far from the source than the cylinder source only along the ray on which the source is located. This is because we have a higher density of power in the single line as compared to over the surface of a cylinder, therefore the spectral irradiance is higher. However, the spectral irradiance decays very quickly, and therefore there will not be light collected around the entire cylindrical collector.

Since the pure erbia is the best emitter, the spectral irradiance is highest close to the source. However, due to the high absorption coefficient, less light gets out far from the source. The most ideal situation is when there is an amount of erbia to emit a high spectral irradiance, yet with the quartz structured in the medium to allow light to escape. With bands we see light does escape, however, the microstructure of the quartz and erbia allows less light in the $r$-direction and more light in the $\theta$-direction. We are more interested in light traveling far in the radial direction and less concerned with light moving in the $\theta$-direction. When we have nothing but quartz as the emitter we see that the spectral irradiance is not very high. The spectral irradiance does however travel without decaying very much compared to the other media. This is not an ideal situation. With erbia we get much more spectral irradiance both inside the quartz enclosure and within the quartz itself.

We see a drop in spectral irradiance at the quartz wall, $r = 0.4$ m, in all cases.
except for the pure quartz case. We take a closer look at the spectral irradiance within the quartz in Figure 3.8. Since the light collector will be placed just outside the quartz wall, we consider the spectral irradiance at the quartz air boundary. We see that the spoke configuration has considerably higher spectral irradiance than any of the other configuration.

![Figure 3.8: Spectral irradiance in $r$ with $\theta$ fixed at $\frac{\pi}{4}$ and a point source located at $(r_0 = 0.3m, \theta_0 = \frac{\pi}{4})$.](image)

We now examine the contour lines of spectral irradiance in the $x - y$ plane. Figure 3.9 shows the level curves of spectral irradiance resulting from the medium of spokes, while Figure 3.10 shows the level curves of spectral irradiance resulting from the medium of bands. Since the effective diffusion coefficient is larger in the radial
direction in the spoke configuration, the spectral irradiance travels further radially. In the medium of bands the spectral irradiance travels more along the $\theta$ direction and less in the radial direction.

![Spectral Irradiance Level Curves](image)

Figure 3.9: Level curves of spectral irradiance in the $x - y$ plane due to a source located at $(r_0 = 0.3, \theta_0 = \frac{\pi}{4})$ in the spoke configuration.

3.5 Emission from Entire Interior Medium

Consider emission from a solid source having power equal to the blackbody emission. If the source in two dimensions is a disk of radius $r_0$, as in Figure 3.11, we can write the equation for spectral irradiance independent of $\theta$. 

39
Figure 3.10: Level curves of spectral irradiance in the $x−y$ plane due to a source located at $(r_0 = 0.3, \theta_0 = \frac{\pi}{4})$ in the banded configuration.

\[-D_{g\text{eff}} \left[ I_{grr} + \frac{1}{r} I_{gr} \right] + a_{g\text{eff}} I_g = a_{g\text{eff}} P \]

(3.20)

for $r < R_g$, and

\[-D_{q\text{eff}} \left[ I_{qrr} + \frac{1}{r} I_{qr} \right] + a_{q\text{eff}} I_q = 0 \]

(3.21)

for $R_g < r < R_a$. We see the solution to the homogeneous problem is a linear combination of modified Bessel functions of order zero, $I_0\left(\frac{a_{\text{eff}}}{D_{\text{eff}}} r\right)$ and $K_0\left(\frac{a_{\text{eff}}}{D_{\text{eff}}} r\right)$. Again, it should be noted that this $I_0$ is the modified Bessel function of the first kind of order zero, not the leading order term in the asymptotic expansion of spectral irradiance. We must now find the particular solution within the region of space where
Figure 3.11: Solid emitting disk of radius $r_0$ in polar coordinates.
Table 3.3: Parameter values used for solutions to point source emitter in polar coordinates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Pure Quartz</th>
<th>Spokes</th>
<th>Bands</th>
<th>Pure Erbia</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_0$</td>
<td>0.3m</td>
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<td>0.3m</td>
<td>0.3m</td>
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<td>$\pi/4$</td>
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<tr>
<td>$R_g$</td>
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<td>0.4m</td>
</tr>
<tr>
<td>$R_a$</td>
<td>1.3055m</td>
<td>1.3055m</td>
<td>1.3055m</td>
<td>1.3055m</td>
</tr>
<tr>
<td>$D_{g\text{eff}}$</td>
<td>$\frac{1}{5}$cm</td>
<td>0.0648cm</td>
<td>0.0317cm</td>
<td>$\frac{1}{51}$cm</td>
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<tr>
<td>$D_{q\text{eff}}$</td>
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<td>$\frac{1}{5}$cm</td>
<td>$\frac{1}{5}$cm</td>
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<td>9.5cm$^{-1}$</td>
<td>9.5cm$^{-1}$</td>
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<tr>
<td>$P$</td>
<td>$6.7908 \cdot 10^{-10}$W</td>
<td>$6.7908 \cdot 10^{-10}$W</td>
<td>$6.7908 \cdot 10^{-10}$W</td>
<td>$6.7908 \cdot 10^{-10}$W</td>
</tr>
</tbody>
</table>

there is an emitter present. If we add the blackbody spectral irradiance, $P$, whose magnitude is determined by equation (1.1), to our homogeneous solution, we have found our general solution. We must now satisfy the following boundary conditions:

$$\lim_{r \to 0} I < \infty$$

to ensure boundedness everywhere,

$$\lim_{r \to r_0^+} n_g^2 I(r) = \lim_{r \to r_0^-} n_q^2 I(r)$$
to account for the change of index of refraction at the quartz wall,

\[
D_{qref} \frac{dI_g}{dr} = D_{qref} \frac{dI_q}{dr}
\]

to ensure continuity of flux, and

\[
I(R_a) = 0
\]

per the artificial boundary condition.

Table 3.4: Parameter values used for solutions to solid disk emitter

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Pure Quartz</th>
<th>Spokes</th>
<th>Bands</th>
<th>Pure Erbia</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R_g)</td>
<td>0.4m</td>
<td>0.4m</td>
<td>0.4m</td>
<td>0.4m</td>
</tr>
<tr>
<td>(R_a)</td>
<td>1.3055m</td>
<td>1.3055m</td>
<td>1.3055m</td>
<td>1.3055m</td>
</tr>
<tr>
<td>(D_{qref})</td>
<td>(\frac{1}{5})cm</td>
<td>0.0648cm</td>
<td>0.0317cm</td>
<td>(\frac{1}{54})cm</td>
</tr>
<tr>
<td>(D_{qref})</td>
<td>(\frac{1}{5})cm</td>
<td>(\frac{1}{5})cm</td>
<td>(\frac{1}{5})cm</td>
<td>(\frac{1}{5})cm</td>
</tr>
<tr>
<td>(a_{qref})</td>
<td>2cm(^{-1})</td>
<td>9.5cm(^{-1})</td>
<td>9.5cm(^{-1})</td>
<td>17cm(^{-1})</td>
</tr>
<tr>
<td>(a_{qref})</td>
<td>2cm(^{-1})</td>
<td>2cm(^{-1})</td>
<td>2cm(^{-1})</td>
<td>2cm(^{-1})</td>
</tr>
<tr>
<td>(n_g)</td>
<td>1.5</td>
<td>1.9</td>
<td>1.9</td>
<td>2.3</td>
</tr>
<tr>
<td>(n_q)</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
</tr>
<tr>
<td>(P)</td>
<td>(6.7908 \cdot 10^{-10} W)</td>
<td>(6.7908 \cdot 10^{-10} W)</td>
<td>(6.7908 \cdot 10^{-10} W)</td>
<td>(6.7908 \cdot 10^{-10} W)</td>
</tr>
</tbody>
</table>

After satisfying these boundary conditions, we find the following solutions:

\[
I_g = c_1 I_0(ggr) + P
\]

(3.22)

for \(r < R_g\), and
Figure 3.12: Resulting spectral irradiance from a solid disk source of radius $r_0 = 0.4m$.

\[ I_q = c_3 I_0(gqr) + c_4 K_0(gqr) \]  
(3.23)

for $R_g < r < R_a$. Here $g_g = \sqrt{\frac{a_{geff}}{D_{geff}}}$, $g_q = \sqrt{\frac{a_{qeff}}{D_{qeff}}}$, $P$ is the blackbody spectral irradiance, and constants $c_1$ through $c_4$ are specified in the Appendix.

Using the parameters specified in Table 3.4, we see the solutions in Figures 3.12 and 3.13. In Figure 3.12 we most clearly see the effect of the entire interior medium emitting. In all the cases we see the spectral irradiance is highest toward the center of the system. In each case, most extremely in the case of pure erbia, we see the spectral irradiance is relatively flat within the interior medium. The spectral irradiance does not begin to decay rapidly until we reach the edge of the emitting medium. We also see a drop in spectral irradiance at the quartz wall in all cases,
Figure 3.13: Resulting spectral irradiance from a solid disk source of radius $r_0 = 0.4\,\text{m}$ beyond the quartz wall.

except that of pure quartz.

In Figure 3.13 we investigate the effects inside the quartz interior. We see that the spoke configuration yields the highest spectral irradiance. Again, light emitted from the erbia will be able to travel radially through the quartz with little absorption.

We now let the thickness of the solid emitter vary. We will consider the emitter to be solid erbia using the results previously calculated, and allow $R_g$ to vary. We see the result that beyond a certain thickness, increasing the thickness will not increase the spectral irradiance, as seen in Figure 3.14, using the parameters in Table 3.4.

We now vary $\phi$, the volume fraction of quartz, in the spoke configuration. We have seen that of the configurations considered here, the spokes yield the highest
Figure 3.14: Irradiance at the edge of a solid emitter of pure erbia as a function of the thickness of the emitter. spectral irradiance with each source. The dependence of $D_{ref}$ on $\phi$ is as discussed in Chapter 2. We use a linear interpolation between the index of refraction and absorption coefficient of pure erbia and pure quartz. That is, $n_g = 1.5 + 0.8\phi$ and $a_{geff} = 2 + 15\phi$. Figure 3.15 illustrates the spectral irradiance due to spokes as $\phi$ varies from 0 to 1.

We see from this figure that there is a maximum close to $\phi = \frac{1}{2}$. For this reason, when manufacturing a composite medium of quartz and erbia, it will be most effective if equal amounts of materials are used. However, the graph is not strongly peaked. Economics would dictate that we should use less erbia if it is possible to yield the same emission. For example, instead of using 50% erbia one might use 25% erbia.
3.6 Boundary condition for optically transparent material

We notice a large loss in spectral irradiance at the location of the quartz wall. This is a result of light being reflected back into the interior medium. However, investigations by Siegel and Howell [24] have indicated that the boundary condition

\[ n_q^2 I_g = n_g^2 I_q \]

at the quartz wall will be better for translucent media. In the case of quartz and other optically transparent media, we will obtain a better approximation using the condition

\[ I_q = \left[ 1 - \frac{(n_q - n_g)^2}{(n_q + n_g)^2} \right] I_g \]
Figure 3.16: Resulting spectral irradiance from a solid disk source of radius $r_0 = 0.4$ m at the quartz wall [14, 15, 16, 24]. If we solve equations (3.20) and (3.21) with this replacement of boundary conditions, and using the parameters specified in Table 3.4, we see the solutions in Figures 3.16 and 3.17.

The solutions remain the same qualitatively, with spokes being the medium with highest emission. There is a much less significant loss of spectral irradiance at the quartz wall. This may be more physically realistic, given the low index of refraction of quartz. This boundary condition yields the result of the pure quartz medium having the lowest emission everywhere, which is expected physically.

We solve equations (3.8) and (3.9) with this replaced boundary condition. Using the parameters from Table 3.2, we see the solutions in Figures 3.18 and 3.19.

There is much less of a loss in spectral irradiance at the quartz wall. We
Figure 3.17: Resulting spectral irradiance from a solid disk source of radius $r_0 = 0.4$m beyond the quartz wall

notice that the medium of pure erbia has the least spectral irradiance far from the source. This to be expected, since light emitted by the cylinder will be absorbed.

We solve equations (3.13) and (3.14) with this replaced boundary condition. Using the parameters from Table 3.3, we see the solutions in Figures 3.20 and 3.21.

There is much less of a loss in spectral irradiance at the quartz wall. We notice that the medium of pure erbia has the least spectral irradiance far from the source. This to be expected, since light emitted by the point source will be absorbed.

Using this replaced boundary condition, our results agree more with physical expectations. However, qualitatively, the solutions remain the same. Spokes remains as the preferred configuration for collecting spectral irradiance far from the source.
Figure 3.18: Emission from a hollow cylindrical shell of radius $r_0 = 0.3$ m.

Figure 3.19: View of spectral irradiance beyond the quartz wall from a hollow cylindrical shell source with radius $r_0 = 0.3$ m.
Figure 3.20: Spectral irradiance in $r$ direction with $\theta$ fixed at $\frac{\pi}{4}$ and a point source located at $(r_0 = 0.3\,\text{m}, \theta_0 = \frac{\pi}{4})$.

Figure 3.21: Spectral irradiance in $r$ with $\theta$ fixed at $\frac{\pi}{4}$ and a point source located at $(r_0 = 0.3\,\text{m}, \theta_0 = \frac{\pi}{4})$. 

51
CHAPTER IV
CONCLUSIONS

4.1 Summary and Results

We have investigated the effects of the geometric configuration of both the emitter and the medium of radiation for thermophotovoltaic energy conversion. We have characterized the effects on the spectral irradiance as a result of changes in either the geometry of the emitter or the medium. We have homogenized periodic, non-uniform media to find effective coefficients of diffusion, as well as coefficients of absorption and scattering. These effects are then combined with a diffusion approximation of the radiative transfer equation to approximate the spectral irradiance due to different sources within different media.

In the polar configurations proposed with the cylindrical collector located at a fixed radial position, we see that the homogenized medium of spokes resulted in the highest spectral irradiance far from the source for every scenario considered. The high emission of the erbia and the high diffusion coefficient of the quartz allow light that is emitted from the erbia to travel through the quartz.

When we compare the bands and spokes, we see that the bands allow for light to travel more easily through the medium in the $\theta$ direction, but more is absorbed
in the radial direction. The homogenized geometry explains this, as the light will be absorbed by each band of erbia as the light travels radially. This is not the case in the spoke configuration; the channel-like wedges of quartz allow for more light to travel radially and less to travel in the $\theta$ direction within the medium itself.

We have been able to investigate the effects of the effective diffusion coefficients in the homogenized media. We adjust the volume fraction to calculate the value of these coefficients depending on the ratio of quartz and erbia.

4.2 Effective Diffusion Coefficients

Through homogenization, our model calculated the effective diffusion coefficients for a homogenization of materials with periodicities in the $r-\theta$ plane. A similar process can be done to calculate the effective diffusion coefficients in other coordinate systems. In any case, we see a similar result. Light will travel with highest spectral irradiance in the direction of the highest diffusion coefficient. In the context of thermophotovoltaic energy conversion, if we can control the location of the light collectors, we should place them where the most spectral irradiance will escape. If we can control the geometric configuration of the TPV materials, we make the material so that the light diffuses with most spectral irradiance in the direction of the light collectors.
4.3 Effects of Temperature and Wavelength

We have defined the strength of our source based on a uniform temperature distribution. In practice, the temperature within the system will be a function of space, so the blackbody emission term would also be a function of space. This would affect the spectral irradiance considerably, since it will affect the power of the sources. This is a limitation to this model that could be expanded upon.

We have only considered the spectral irradiance at the frequencies of highest emission from erbia. Future research would further incorporate wavelength dependence. Radiation that is incident on the photovoltaic diode that is not in the effective range of wavelengths will heat the diode and decrease its efficiency. For this reason, one could strive to find the configuration that emits the highest in the efficient range, while minimizing emission outside of that range of wavelengths.

4.4 Applying conclusions

In the construction of these TPV materials, one should take note of the structural composition, while using roughly equal parts erbia and quartz. Making layered bands of erbia and quartz will be less effective than combining the two like spokes. We aim to allow light that is emitted from the erbia to travel more easily through the quartz in the radial direction, resulting in the most light far from the source.
APPENDIX

COMPLETE ANALYTICAL SOLUTIONS

The spectral irradiance due to an emitting hollow sphere of radius $r_0$ is

$$I_− = \frac{c_1}{r} \sinh(ggr) \quad (A.1)$$

for $r < r_0$, and

$$I_+ = \frac{c_2}{r} e^{-ggr} \quad (A.2)$$

for $r > r_0$. Here $gg = \sqrt{g_{ref}/L_{ref}}$, $\alpha = -\frac{aP}{L_{ref}4\pi r_0^2}$ and

$$c_1 = 2\alpha r_0^2 e^{ggr_0} \cdot \left[ ggr_0(e^{ggr_0})^2 + ggr_0 - (e^{ggr_0})^2 + 1 + ggr_0^2(e^{ggr_0})^2 - ggr_0^2 \right]^{-1},$$

and

$$c_2 = -((e^{ggr_0})^2 - 1)\alpha r_0^2 e^{ggr_0} \cdot \left[ ggr_0(e^{ggr_0})^2 + ggr_0 - (e^{ggr_0})^2 + 1 + ggr_0^2(e^{ggr_0})^2 - ggr_0^2 \right]^{-1}.$$

The spectral irradiance due to an emitting hollow cylinder is

$$I_−_s = c_1 I_0(ggr) + c_2 K_0(ggr) \quad (A.3)$$

for $r < r_0$,
\[ I^+_g = c_3 I_0(ggr) + c_4 K_0(ggr) \]  
(A.4)

for \( r_0 < r < R_g \),

\[ I_q = c_5 I_0(gqr) + c_6 K_0(gqr) \]  
(A.5)

for \( R_g < r < R_a \), where \( gg = \sqrt{\frac{a_{\text{eff}}}{D_{\text{gref}}}}, gq = \sqrt{\frac{a_{\text{eff}}}{D_{\text{qref}}}}, \) and \( \alpha = -\frac{a_{\text{eff}} P}{D_{\text{gref}} 2\pi r_0} \) and

\[
c_1 = -\left( \alpha (I_0(ggr_0)) ggn_g^2 D_{\text{gref}} I_0(gqR_g) K_0(gqR_g) K_1(ggR_g) \right.
\]

\[-I_0(ggr_0) ggn_g^2 D_{\text{gref}} I_0(gqR_g) K_0(gqR_a) K_1(ggR_g) \]

\[-I_0(ggr_0) K_0(ggR_g) n_q^2 I_0(gqR_a) D_{\text{qref}} K_1(ggR_g) gq \]

\[-I_0(ggr_0) K_0(ggR_g) n_q^2 D_{\text{qref}} I_1(gqR_g) gq K_0(gqR_g) \]  

\[+K_0(ggr_0) n_q^2 I_0(ggR_g) I_0(gqR_a) D_{\text{qref}} K_1(ggR_g) gq \]

\[+K_0(ggr_0) n_q^2 I_0(ggR_g) D_{\text{qref}} I_1(gqR_g) gq K_0(gqR_a) \]

\[+K_0(ggr_0) D_{\text{gref}} I_1(ggR_g) gq I_0(gqR_a) n_q^2 K_0(ggR_g) \]

\[-K_0(ggr_0) D_{\text{gref}} I_1(ggR_g) ggn_g^2 I_0(gqR_g) K_0(gqR_a) \) \)

\[
\cdot (I_0(ggr_0) K_1(ggr_0) + I_1(ggr_0) K_0(ggr_0) )
\]

\[
\cdot gg (n_q^2 I_0(ggR_g) I_0(gqR_a) D_{\text{qref}} K_1(ggR_g) gq
\]

\[+n_q^2 I_0(ggR_g) D_{\text{qref}} I_1(gqR_g) gq K_0(gqR_a) \]

\[+D_{\text{gref}} I_1(ggR_g) gg I_0(gqR_a) n_q^2 K_0(ggR_g) \]

56
\[ -D_{\text{eff}1}(ggR_g)gn^2 \theta_0(ggR_a)K_0(ggR_a) ]^{-1}, \]

\[ c_2 = 0, \]

\[ c_3 = - \left( gn^2 D_{\text{eff}1}(ggR_a)K_0(ggR_a)K_1(ggR_a) \right. \]
\[ -gn^2 D_{\text{eff}1}(ggR_a)K_0(ggR_a)K_1(ggR_a) \]
\[ -K_0(ggR_a)n^2 \theta_0(ggR_a)D_{\text{eff}}K_1(ggR_a)gq \]
\[ -K_0(ggR_a)n^2 D_{\text{eff}1}(ggR_a)gqK_0(ggR_a) \]
\[ \cdot \theta_0(gg) \cdot \left( [I_0(gg)K_1(gg)gq + I_1(gg)K_0(gg)]^{-1} \right. \]
\[ +I_1(gg)K_0(gg)gq \left. \right) \]
\[ +n^2 \theta_0(gg)D_{\text{eff}1}(ggR_a)gqK_0(ggR_a) \]
\[ +D_{\text{eff}1}(ggR_a)gqI_0(ggR_a)n^2 K_0(ggR_a) \]
\[ -D_{\text{eff}1}(ggR_a)gn^2 \theta_0(ggR_a)K_0(ggR_a) \]

\[ c_4 = \frac{-I_0(gg)\theta_0}{gg(I_0(gg)K_1(gg) + I_1(gg)K_0(gg))}, \]

\[ c_5 = n^2 D_{\text{eff}1}(ggR_a)\theta_0(I_0(ggR_a)K_1(ggR_a) \]
\[ +I_1(ggR_a)K_0(ggR_a))K_0(ggR_a) \]
\[ \cdot [n^2 \theta_0(ggR_a)I_0(ggR_a)K_1(ggR_a)I_0(ggR_a)D_{\text{eff}1}(ggR_a)gq \]

57
\[ + n_q^2 I_0(ggR_g)I_0(ggr_0)K_1(ggr_0)D_{\text{q}_{\text{eff}}} I_1(ggR_g)gqK_0(gqR_a) \]
\[ + n_q^2 I_0(ggR_g)I_1(ggr_0)K_0(ggr_0)I_0(gqR_a)D_{\text{q}_{\text{eff}}} K_1(gqR_g)gq \]
\[ + n_q^2 I_0(ggR_g)I_1(ggr_0)K_0(ggr_0)D_{\text{q}_{\text{eff}}} I_1(ggR_g)gqK_0(gqR_a) \]
\[ + D_{\text{q}_{\text{eff}}} I_1(ggR_g)I_1(ggr_0)gqK_0(ggr_0)I_0(gqR_a)n_g^2 K_0(gqR_g) \]
\[ + D_{\text{q}_{\text{eff}}} I_1(ggR_g)I_0(ggr_0)K_1(ggr_0)ggI_0(gqR_a)n_g^2 K_0(gqR_g) \]
\[ - D_{\text{q}_{\text{eff}}} I_1(ggR_g)I_0(ggr_0)K_1(ggr_0)ggn_g^2 I_0(gqR_g)K_0(gqR_a) \]
\[ - D_{\text{q}_{\text{eff}}} I_1(ggR_g)I_1(ggr_0)ggK_0(ggr_0)n_g^2 I_0(gqR_g)K_0(gqR_a)]^{-1}, \]
and

\[ c_o = - n_q^2 I_0(gqR_a)D_{\text{q}_{\text{eff}}} I_0(ggr_0)\alpha \left(I_0(ggR_g)K_1(ggR_g) \right) \]
\[ + I_1(ggR_g)K_0(ggR_g)) \]
\[ \cdot \left[ n_q^2 I_0(ggR_g)I_0(ggr_0)K_1(ggr_0)I_0(gqR_a)D_{\text{q}_{\text{eff}}} K_1(gqR_g)gq \right] \]
\[ + n_q^2 I_0(ggR_g)I_0(ggr_0)K_1(ggr_0)D_{\text{q}_{\text{eff}}} I_1(ggR_g)gqK_0(gqR_a) \]
\[ + n_q^2 I_0(ggR_g)I_1(ggr_0)K_0(ggr_0)I_0(gqR_a)D_{\text{q}_{\text{eff}}} K_1(gqR_g)gq \]
\[ + n_q^2 I_0(ggR_g)I_1(ggr_0)K_0(ggr_0)D_{\text{q}_{\text{eff}}} I_1(ggR_g)gqK_0(gqR_a) \]
\[ + D_{\text{q}_{\text{eff}}} I_1(ggR_g)I_1(ggr_0)ggK_0(ggr_0)I_0(gqR_a)n_g^2 K_0(gqR_g) \]
\[ + D_{\text{q}_{\text{eff}}} I_1(ggR_g)I_0(ggr_0)K_1(ggr_0)ggI_0(gqR_a)n_g^2 K_0(gqR_g) \]
\[ - D_{\text{q}_{\text{eff}}} I_1(ggR_g)I_0(ggr_0)K_1(ggr_0)ggn_g^2 I_0(gqR_g)K_0(gqR_a) \]
\[ - D_{\text{q}_{\text{eff}}} I_1(ggR_g)I_1(ggr_0)ggK_0(ggr_0)n_g^2 I_0(gqR_g)K_0(gqR_a)]^{-1}. \]
The solution to equations (3.15) and (3.16) is

\[ A_{gn}^{-}(r) = c_1 I_{vg}(ggr) + c_2 K_{vg}(ggr) \]  \hspace{1cm} (A.6)

for \( r < r_0 \),

\[ A_{gn}^{+}(r) = c_3 I_{vg}(ggr) + c_4 K_{vg}(ggr) \]  \hspace{1cm} (A.7)

for \( r_0 < r < R_g \),

\[ A_{qn}(r) = c_5 I_{vq}(gqr) + c_6 K_{vq}(gqr) \]  \hspace{1cm} (A.8)

for \( R_g < r < R_a \). Here \( gg = \sqrt{\frac{a_{eff}}{D_{gref}}} \), \( gq = \sqrt{\frac{a_{eff}}{D_{qref}}} \), \( vg = \sqrt{\frac{D_{gθeff}}{D_{gref}}} n \), \( vq = \sqrt{\frac{D_{qθeff}}{D_{qref}}} n \) and \( \alpha = \frac{a_{eff} P \sin(\eta_0)}{\pi r_o} \), and

\[ c_1 = (\alpha \left( -D_{gref}ggn_g^2 R_g I_{vg}(ggr_0) I_{vq}(gqR_g) K_{vg}(gqR_a) K_{vg+1}(ggR_g) \right. \]
\[ + D_{gref}ggn_g^2 R_g I_{vg}(ggr_0) I_{vq}(gqR_g) K_{vg}(gqR_a) K_{vg+1}(ggR_g) \]
\[ + K_{vg}(ggR_g)n_q^2 I_{vg}(ggr_0) I_{vq}(gqR_a) D_{qref} vqK_{vq}(gqR_g) \]
\[ - K_{vg}(ggR_g)n_q^2 I_{vg}(ggr_0) K_{vq}(gqR_a) D_{qref} I_{vq}(ggR_g)gqR_g \]
\[ - K_{vg}(ggR_g)n_q^2 I_{vg}(ggr_0) K_{vq}(gqR_a) D_{qref} vqI_{vq}(ggR_g) \]
\[ - K_{vg}(ggR_g)n_q^2 I_{vg}(ggr_0) I_{vq}(gqR_a) D_{qref} K_{vg+1}(ggR_g)gqR_g \]
\[ - K_{vg}(ggR_g)vqI_{vg}(ggr_0) D_{gref} I_{vq}(gqR_a)n_q^2 K_{vq}(gqR_g) \]
\[ +K_{vg}(ggR_g)vgI_{vg}(ggr_0)D_{qref f}n_g^2I_{vg}(gqR_g)K_{vg}(gqR_g) \]
\[ -ggI_{vg+1}(ggR_g)R_gD_{qref f}K_{vg}(ggr_0)n_g^2I_{vg}(gqR_g)K_{vg}(gqR_g) \]
\[ +ggI_{vg+1}(ggR_g)R_gD_{qref f}K_{vg}(ggr_0)I_{vg}(gqR_a)n_g^2K_{vg}(gqR_g) \]
\[ -n_g^2I_{vg}(ggR_g)K_{vg}(ggr_0)I_{vg}(gqR_a)D_{qref f}vqK_{vg}(gqR_g) \]
\[ +n_g^2I_{vg}(ggR_g)K_{vg}(ggr_0)K_{vg}(gqR_a)D_{qref f}I_{vg+1}(gqR_g)gqR_g \]
\[ +n_g^2I_{vg}(ggR_g)K_{vg}(ggr_0)K_{vg}(gqR_a)D_{qref f}vqI_{vg}(gqR_g) \]
\[ +n_g^2I_{vg}(ggR_g)K_{vg}(ggr_0)I_{vg}(gqR_a)D_{qref f}K_{vg+1}(gqR_g)gqR_g \]
\[ +vgI_{vg}(ggR_g)D_{qref f}K_{vg}(ggr_0)I_{vg}(gqR_a)n_g^2K_{vg}(gqR_g) \]
\[ -K_{vg}(gqR_a)I_{vg}(gqR_g)n_g^2K_{vg}(ggr_0)D_{qref f}I_{vg}(ggR_g)vq \]
\[ \cdot (I_{vg}(ggr_0)K_{vg+1}(ggr_0) + I_{vg+1}(ggr_0)K_{vg}(ggr_0)) \]
\[ \cdot D_{qref f}gg(-ggI_{vg+1}(ggR_g)R_gD_{qref f}n_g^2I_{vg}(gqR_g)K_{vg}(gqR_a) \]
\[ +ggI_{vg+1}(ggR_g)R_gD_{qref f}I_{vg}(gqR_a)n_g^2K_{vg}(gqR_g) \]
\[ -n_g^2I_{vg}(ggR_g)I_{vg}(gqR_a)D_{qref f}vqK_{vg}(gqR_g) \]
\[ +n_g^2I_{vg}(ggR_g)K_{vg}(gqR_a)D_{qref f}I_{vg+1}(gqR_g)gqR_g \]
\[ +n_g^2I_{vg}(ggR_g)K_{vg}(gqR_a)D_{qref f}vqI_{vg}(gqR_g) \]
\[ +n_g^2I_{vg}(ggR_g)I_{vg}(gqR_a)D_{qref f}K_{vg+1}(gqR_g)gqR_g \]
\[ +vgI_{vg}(ggR_g)D_{qref f}I_{vg}(gqR_a)n_g^2K_{vg}(gqR_g) \]
\[ -vgI_{vg}(ggR_g)D_{qref f}n_g^2I_{vg}(gqR_g)K_{vg}(gqR_a)) \]^{-1} \]

60
\[ c_2 = 0, \]

\[ c_3 = \left( - D_{\text{gref}} gg n_g^2 R_g I_{vq}(gq R_g) K_{vq}(gq R_a) K_{vq+1}(gq R_g) \right. \]

\[ + D_{\text{gref}} gg n_g^2 R_g I_{vq}(gq R_a) K_{vq}(gq R_g) K_{vq+1}(gq R_g) \]

\[ + K_{vq}(gq R_g) n_q^2 I_{vq}(gq R_a) D_{\text{gref}} vq K_{vq}(gq R_g) \]

\[ - K_{vq}(gq R_g) n_q^2 K_{vq}(gq R_a) D_{\text{gref}} I_{vq+1}(gq R_g) gq R_g \]

\[ - K_{vq}(gq R_g) n_q^2 K_{vq}(gq R_a) D_{\text{gref}} I_{vq}(gq R_g) \]

\[ - K_{vq}(gq R_g) n_q I_{vq}(gq R_a) D_{\text{gref}} K_{vq+1}(gq R_g) gq R_g \]

\[ - K_{vq}(gq R_g) v q D_{\text{gref}} I_{vq}(gq R_a) n_g^2 K_{vq}(gq R_g) \]

\[ + K_{vq}(gq R_g) K_{vq}(gq R_a) I_{vq}(gq R_g) n_g^2 D_{g_{\text{ref}} v q} \alpha I_{vq}(g g r_0) \]

\[ \cdot \left( I_{vq}(g g r_0) K_{vq+1}(g g r_0) + I_{vq+1}(g g r_0) K_{vq}(g g r_0) \right) \]

\[ \cdot D_{\text{gref}} gg \left( - g g I_{vq+1}(g g R_g) R_g D_{g_{\text{ref}}} n_g^2 I_{vq}(g q R_g) K_{vq}(g q R_a) \right. \]

\[ + g g I_{vq+1}(g g R_g) R_g D_{g_{\text{ref}}} n_g^2 I_{vq}(g q R_a) n_g^2 K_{vq}(g q R_g) \]

\[ - n_q^2 I_{vq}(g g R_g) I_{vq}(g q R_a) D_{g_{\text{ref}}} v q K_{vq}(g q R_g) \]

\[ + n_q I_{vq}(g g R_g) K_{vq}(g q R_a) D_{g_{\text{ref}}} I_{vq+1}(g q R_g) g q R_g \]

\[ + n_q^2 I_{vq}(g g R_g) K_{vq}(g q R_a) D_{g_{\text{ref}}} I_{vq}(g q R_g) \]

\[ + n_q I_{vq}(g g R_g) I_{vq}(g q R_a) D_{g_{\text{ref}}} K_{vq+1}(g q R_g) g q R_g \]

\[ + v g I_{vq}(g g R_g) D_{g_{\text{ref}}} I_{vq}(g q R_a) n_g^2 K_{vq}(g q R_g) \]
\[-v_g I_{vg} (g g R_g) D_{\text{eff}} n_g^2 I_{vq} (g q R_q) K_{vq} (g q R_a)]^{-1},\]

c_4 = I_{vg} (g g r_0) \alpha \cdot [g g D_{\text{eff}} (I_{vg} (g g r_0) K_{vq+1} (g g r_0) + I_{vg+1} (g g r_0) K_{vq} (g g r_0))]^{-1},

c_5 = - (n_q^2 I_{vg} (g g r_0) \alpha R_g (I_{vg} (g g R_g) K_{vq+1} (g g R_g) K_{vq} (g q R_a))
+ I_{vg+1} (g g R_g) K_{vq} (g g R_g) K_{vq} (g q R_a))
\cdot [ - g g I_{vg+1} (g g R_g) R_g I_{vg} (g g r_0) D_{\text{eff}} K_{vq+1} (g g r_0) n_g^2 I_{vq} (g q R_q) K_{vq} (g q R_a)
- g g I_{vg+1} (g g R_g) R_g D_{\text{eff}} I_{vq+1} (g g r_0) K_{vq} (g g r_0) n_g^2 I_{vq} (g q R_q) K_{vq} (g q R_a)
+ g g I_{vg+1} (g g R_g) R_g D_{\text{eff}} I_{vq+1} (g g r_0) K_{vq} (g g r_0) I_{vq} (g q R_a) n_g^2 K_{vq} (g q R_g)
- n_q^2 I_{vg} (g g R_g) I_{vq} (g g r_0) K_{vq+1} (g g r_0) I_{vq} (g q R_a) D_{\text{eff}} I_{vq+1} (g q R_g) g g R_g
+ n_q^2 I_{vg} (g g R_g) I_{vq} (g g r_0) K_{vq+1} (g g r_0) K_{vq} (g q R_a) D_{\text{eff}} I_{vq+1} (g q R_g) g q K_{vq}
+ n_q^2 I_{vg} (g g R_g) I_{vq} (g g r_0) K_{vq+1} (g g r_0) K_{vq} (g q R_a) D_{\text{eff}} I_{vq+1} (g q R_g) I_{vq} (g q R_g) g g R_g
+ n_q^2 I_{vg} (g g R_g) I_{vq} (g g r_0) K_{vq+1} (g g r_0) K_{vq} (g q R_a) D_{\text{eff}} K_{vq+1} (g q R_g) g q K_{vq}
- n_q^2 I_{vg} (g g R_g) I_{vq} (g g r_0) K_{vq+1} (g g r_0) K_{vq} (g q R_a) D_{\text{eff}} I_{vq+1} (g q R_g) g q R_g
+ n_q^2 I_{vg} (g g R_g) I_{vq} (g g r_0) K_{vq+1} (g g r_0) K_{vq} (g q R_a) D_{\text{eff}} I_{vq+1} (g q R_g) g q R_g
+ n_q^2 I_{vg} (g g R_g) I_{vq} (g g r_0) K_{vq+1} (g g r_0) K_{vq} (g q R_a) D_{\text{eff}} I_{vq+1} (g q R_g) I_{vq} (g q R_g) g g R_g
+ n_q^2 I_{vg} (g g R_g) I_{vq} (g g r_0) K_{vq+1} (g g r_0) K_{vq} (g q R_a) D_{\text{eff}} I_{vq+1} (g q R_g) g q R_g
+ n_q^2 I_{vg} (g g R_g) I_{vq} (g g r_0) K_{vq+1} (g g r_0) K_{vq} (g q R_a) D_{\text{eff}} I_{vq+1} (g q R_g) g q R_g
+ g g I_{vg+1} (g g R_g) R_g I_{vg} (g g r_0) D_{\text{eff}} K_{vq+1} (g g r_0) I_{vq} (g q R_a) n_g^2 K_{vq} (g q R_g)\]
\[ \begin{align*}
+vgI_{vg}(ggR_g)D_{qreff}I_{vg+1}(ggr_0)K_{vg}(ggr_0)I_{vq}(gqR_a)n^2_gK_{vq}(gqR_g) \\
+vgI_{vg}(ggR_g)I_{vg}(ggr_0)D_{qreff}K_{vg+1}(ggr_0)I_{vq}(gqR_a)n^2_gK_{vq}(gqR_g) \\
-vgI_{vg}(ggR_g)I_{vg}(ggr_0)D_{qreff}K_{vg+1}(ggr_0)n^2_I_{vq}(gqR_g)K_{vq}(gqR_a) \\
-K_{vg}(gqR_a)I_{vq}(gqR_g)n^2_gK_{vg}(ggr_0)I_{vg+1}(ggr_0)D_{qreff}I_{vq}(gqR_g)vg)^{-1},
\end{align*} \]

and

\[ c_6 = n^2_gI_{vg}(ggr_0)\alpha I_{vq}(gqR_a)R_g(I_{vg}(ggR_g)K_{vg+1}(ggR_g) + I_{vg+1}(ggR_g)K_{vg}(ggR_g)) \]

\[ \cdot \left[ -ggI_{vg+1}(ggR_g)R_gI_{vg}(ggr_0)D_{qreff}K_{vg+1}(ggr_0)n^2_I_{vq}(gqR_g)K_{vq}(gqR_a) \\
-ggI_{vg+1}(ggR_g)R_gD_{qreff}I_{vg+1}(ggr_0)K_{vg}(ggr_0)n^2_I_{vq}(gqR_g)K_{vq}(gqR_a) \\
+ggI_{vg+1}(ggR_g)R_gD_{qreff}I_{vg+1}(ggr_0)K_{vg}(ggr_0)I_{vq}(gqR_a)n^2_gK_{vq}(gqR_g) \\
-n^2_I_{vg}(ggR_g)I_{vg}(ggr_0)K_{vg+1}(ggr_0)I_{vq}(gqR_a)D_{qreff}vqK_{vq}(gqR_g) \\
+n^2_gI_{vg}(ggR_g)I_{vg}(ggr_0)K_{vg+1}(ggr_0)K_{vq}(gqR_a)D_{qreff}I_{vq+1}(gqR_g)gqR_g \\
+n^2_gI_{vg}(ggR_g)I_{vg}(ggr_0)K_{vg+1}(ggr_0)K_{vq}(gqR_a)D_{qreff}vqI_{vq}(gqR_g) \\
+n^2_I_{vg}(ggR_g)I_{vg}(ggr_0)K_{vg+1}(ggr_0)I_{vq}(gqR_a)D_{qreff}K_{vq+1}(gqR_g)gqR_g \\
-n^2_I_{vg}(ggR_g)I_{vg+1}(ggr_0)K_{vg}(ggr_0)I_{vq}(gqR_a)D_{qreff}vqK_{vq}(gqR_g) \\
+n^2_gI_{vg}(ggR_g)I_{vg+1}(ggr_0)K_{vg}(ggr_0)I_{vq}(gqR_a)D_{qreff}I_{vq+1}(gqR_g)gqR_g \\
+n^2_I_{vg}(ggR_g)I_{vg+1}(ggr_0)K_{vg}(ggr_0)K_{vq}(gqR_a)D_{qreff}vqI_{vq}(gqR_g) \\
+n^2_I_{vg}(ggR_g)I_{vg+1}(ggr_0)K_{vg}(ggr_0)I_{vq}(gqR_a)D_{qreff}K_{vq+1}(gqR_g)gqR_g \right] \]
The spectral irradiance due to a solid emitter is

\[ I = c_1 I_0(ggr) + BB \]  \hspace{1cm} (A.9)

for \( r < R_g \), and

\[ I = c_3 I_0(gqr) + c_4 K_0(gqr) \]  \hspace{1cm} (A.10)

for \( R_g < r < R_a \). Here \( gg = \sqrt{\frac{a_{qref}}{D_{qreff}}} \), \( gq = \sqrt{\frac{a_{qref}}{D_{qreff}}} \), and \( BB \) is the blackbody power term.

\[
c_1 = - \left( D_{qreff} gq I_0(gq R_a) K_1(gq R_g) \right) + I_1(gq R_g) K_0(gq R_a) n_q^2 BB) \\
\cdot \left[ D_{qreff} I_1(gq R_g) gg I_0(gq R_a) n_q^2 K_0(gq R_g) \right] \\
- D_{qreff} I_1(gq R_g) ggn_q^2 I_0(gq R_a) K_0(gq R_a) \\
+ n_q^2 I_0(gq R_g) I_0(gq R_a) D_{qreff} K_1(gq R_g) gq
\]
\[+n_q^2 I_0(ggR_g)D_{qref f}I_1(gqR_g)gqK_0(gqR_a)]^{-1},\]

\[c_3 = -D_{qref f}I_1(ggR_g)ggn_q^2BBK_0(gqR_a)\]
\[\cdot[D_{qref f}I_1(ggR_g)ggI_0(gqR_a)n_q^2K_0(gqR_g)\]
\[+n_q^2 I_0(ggR_g)I_0(gqR_a)D_{qref f}K_1(gqR_g)gq\]
\[+n_q^2 I_0(ggR_g)D_{qref f}I_1(gqR_g)gqK_0(gqR_a)]^{-1},\]

and

\[c_4 = D_{qref f}I_1(ggR_g)ggI_0(gqR_a)n_q^2BB\]
\[\cdot[D_{qref f}I_1(ggR_g)ggI_0(gqR_a)n_q^2K_0(gqR_g)\]
\[+n_q^2 I_0(ggR_g)I_0(gqR_a)D_{qref f}K_1(gqR_g)gq\]
\[+n_q^2 I_0(ggR_g)D_{qref f}I_1(gqR_g)gqK_0(gqR_a)]^{-1}.\]
BIBLIOGRAPHY


