INTEGRATING HEALTH EDUCATION AND LEISURE TIME INTO ECONOMIC GROWTH MODELING

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INTEGRATING HEALTH EDUCATION AND LEISURE TIME INTO ECONOMIC GROWTH MODELING

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ABSTRACT

This paper models the economic growth of a developing country. An integrated approach incorporating, capital, technology, and human capital is used. The model is a Hamiltonian problem that maximizes utility by determining the amount of time that should be spent each day in production, investment in human capital, investment in health education, and leisure. Adding leisure time into the model is used to determine if this will allow economic growth of the country. Numerical solutions of the model are found and show that adding leisure increases productivity and technology, but results in a decrease to life expectancy. Stability analysis of the model shows that leisure provides multiple unstable paths for the country, although a trap stable fixed point can still occur.
ACKNOWLEDGEMENTS

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I want to say a special thanks to my family and friends who have prayed for me throughout working on my thesis, without which I wouldn't have finished.
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1.1 Overview

We often hear that “The rich get richer and the poor get poorer,” but is it really true? In an article in the New York Times, Conley suggests that this may be true because the more money you make and the richer you are, the more you want to work to keep the money you have [1]. In summary of the above-mentioned article, the rich like to compete against each other, and if one sees that another has money, that drives him/her to work more and make more money. On the other end of the spectrum, the poor stay in a stable state because they have no one with whom to compete. Therefore, the poor just stay poor, while the rich really do get richer [1]. Moving to a larger scale, the same distinction holds true for developed versus developing countries. Can this be helped? What can be done for these developing countries to get them out of their economic state, which is causing them to stay poor?

To try and answer these questions, economic models are examined and modified to fit the economy of a developing country. Different terms are integrated into an existing model [2] to form a new model.
After adding such terms, the new model is solved and the governing equations are interpreted, according to the economy of a developing country.

This Masters Thesis is an extension of Tucker (2008) [2]. Tucker incorporates life expectancy into a model that integrates technology and human capital. Tucker’s approach is an attempt to validate whether international health aid and education is an economically viable and profitable endeavor [2]. Tucker discovered that it is worthwhile to include health education into a struggling economy because it does increase life expectancy of that country. In addition to incorporating life expectancy, this proposed Masters Thesis model will extend Tucker’s approach by adding leisure into the model. This is included to determine whether having a break in the work day would make the workers more productive in the long run. By incorporating leisure, looking at fixed points and varying the parameter that measures leisure, we find that a country produces more but has a shorter life expectancy as leisure time increases. After finding the fixed points and doing a linear stability analysis on the model proposed in this thesis, we see that it is necessary to perform a change of variables. We also find the fixed points of the newly transformed model and a linear stability analysis is performed using Maple. The stability analysis shows that there are multiple unstable paths in which an economy is improving in some way, whether it be through capital increasing or technology improving. Despite the opportunity for continued improvement, there is still opportunity for the economy to be trapped on a stagnant path, although it may be at higher state of living. Future research for this project would be to break up a person’s life into different stages and see when he/she
is more productive and when he/she starts becoming less productive or completely unproductive for the economy.

The remainder of this chapter presents a literature survey on existing models and presents the assumptions. These models are evaluated and explained in later sections and are used as stepping stones for this proposed model. The model proposed in this thesis will be presented in Chapter 2 and so will the governing equations and interpretations. Chapter 3 presents the steady state solutions for this model and Chapter 4 introduces a change of variables and a transformed model is presented with its steady state solutions. The results and conclusions are provided in Chapter 5. In the Appendix, the derivations of the model are explained.

1.2 Literature Survey

In [3], Charles Jones reviews the Solow Model, which uses a Cobb-Douglas form production function. This model treats technology as an exogenous force, which is growth from an outside source, and treats socio-political stability as a coefficient that can either help or hinder economic growth for an economy [3].

Jones focuses on how to invest excess money: should it be invested in human or physical capital? This question leads to the Romer model [4]. Romer focuses on distributing investments across different areas, such as human capital, research and development, or physical capital, with a goal of maximizing growth and utility [4]. These models are good for developed countries, but the assumptions for developed countries are different from those of developing countries. For instance, life
expectancies are different for developed and developing countries, which would cause
the economy to invest differently. Therefore, we seek to find a model that would be
a better fit for a developing country, rather than a developed country.

Two models that more accurately relate to a developing country are those by
Fabrizio Zilibotti [5] and Michal Kejak [6]. Both models are derived from Robert
Lucas [7]. Zilibotti uses the Romer and Solow models in his paper. Zilibotti’s
model separates a region where growth is Solow-type, with convergence to a sta-
tionary steady-state, from a region where growth is Romer-type, with endogenous
self-sustained growth [5]. The logistic equation and its solution are used in Zilibotti’s
model to relate technology to capital accumulation [5]. The logistic equation and its
solution are:

\[
\frac{\dot{A}}{A} = \phi \frac{a - A}{a} \dot{K},
\]

\[
A = \frac{a}{1 + \left( \frac{a}{A_0} - 1 \right)e^{-\phi K}},
\]

where \( \cdot = \frac{d}{dt} \). In these equations \( A \) is technology, \( K \) is aggregate capital, which is
the total capital available throughout the economy. The variable \( t \) is time and \( \phi \)
is a diffusion parameter that is positive. Many variables and parameters are used
throughout the model. For a concise definition of each variable and parameter used,
refer to Tables 1.1 and 1.2. Notice that a change is made in Zilibotti’s notation
in order to remain consistent with the notation of the proposed model. What is
important to note here is that as \( K \) approaches 0, \( A \) approaches \( A_0 \), which means that
technology is approaching its lowest level. On the other hand, when \( K \) approaches
∞, $A$ approaches $a$, which is the highest feasible level of technology in the economy. We have that $K = 0$ is the initial condition and, as mentioned, we get $A_0$, the lowest level of technology. Due to the endogenous growth, as capital increases, technology grows also. This proposed model uses (1.1) and (1.2).

We use Zilibotti’s non-linear production function, $y$,

$$y = DA(K)k + Z[A(K)]^\theta.$$  \hfill (1.3)

Here $D$ and $Z$ are both parameters, $k$ is capital and $\theta$ is a parameter for the non-linear term associated with production growth. Aggregate capital, $K$, is independent of the economy’s capital $k$. Therefore, when taking derivatives with respect to $k$, any term with $K$ is not affected.

Zilibotti and Daron Acemoglu [8] consider productivity differences among countries with the same level of technology. It is argued that even if a developed and a less developed country have the same level of technology, the productivity of a less developed country will be lacking because they have unskilled workers using technology designed for skilled workers [8]. Learning how to use the new technology is essential for having high productivity levels.
Michal Kejak [6] extends Lucas’ model [7] by incorporating human capital and productivity endogenously. The following equations from Kejak are also important to this paper’s model:

\[
\frac{\dot{B}}{B} = \psi \frac{B_H - B}{B_H} \dot{H},
\]

\[
B(H) = \frac{B_H}{1 + \left( \frac{B_H}{B_0} - 1 \right) e^{-\psi H}},
\]

\[
\dot{h} = B(H)(1 - u)h.
\]

In these equations, \(B_0\) is the lowest level of productivity, \(B_H\) is the highest level of productivity and \(B\) is productivity at time \(t\). According to Kejak, \(\psi\) is a measure of the speed of diffusion and \(H\) is the average level of human capital in the economy [6]. As \(H\) increases, productivity approaches the highest level of production and as \(H\) decreases, productivity decreases to the lowest level of productivity. Equations (1.5) and (1.6) will be used in this model. The argument used for capital to delineate \(k\) and \(K\) holds true for human skill level \(h\) and \(H\). It should also be noted that at equilibrium, \(k = K\) and \(h = H\).

In equation (1.6), \((1 - u)\) is the amount of non-leisure time spent investing in human capital. Kejak’s model maximizes economic growth if \((1 - u)\) is optimized. This proposed model allots time for production, \(u\), investment in productivity, \(e_p\), for health education, \(e_q\) and for leisure time in the work day, \(l\). Therefore, we have

\[
u = 1 - e_p - e_q - l.
\]

To incorporate the leisure term, \(l\), we follow [9]. These authors also take their model from Lucas, but incorporate leisure into the utility function. The utility
function for this paper’s model is taken from [9], but is modified with a time-discounting factor. Incorporating leisure into the model introduces a potential source of non-convexities in the optimization problem and leads to the possible existence of multiple growth paths [9]. The utility function is defined as

$$U[c(t), l(t)] = \frac{[c(t)\beta l(t)^{1-\beta}]^{1-\sigma}}{1-\sigma}.$$  \hspace{1cm} (1.8)

In this utility function, $l$ is leisure, $t$ is time, $c$ is consumption at time $t$, $\beta$ is a parameter that measures the value of leisure with $0 < \beta < 1$, and $\sigma$ is a parameter which measures risk aversion, $0 < \sigma < 1$. Ladron, Ortigueira and Santos mention that there is no previous study on the role of leisure in the process of growth and they incorporate leisure to determine how time can be allocated between production of goods, education and leisure throughout the work week [9]. It is found that in those balanced growth paths, having higher physical capital, individuals will devote more time to leisure and less time to education. The proposed model will help determine if adding leisure to the work day will aid with improving the economic state of a country. We maximize the above utility function.

In [10], Paul A. de Hek also incorporates leisure and consumption into his utility function, where he looks at the relationship between consumption and leisure. As found in [11], as the price of one product increases/decreases the demand of the substitute product will increase/decrease. If Coca-cola and Pepsi are substitutes, the increase in the price of Pepsi will result in an increased demand for Coca-cola. On the otherhand, as the price increases/decreases, the demand for a complement product
decreases/increases. Considering hot dogs and hot dog buns, which are complements, if the price of hot dogs increase, the demand for hot dog buns will decrease. Looking at consumption and leisure as goods, Hek proposes that if the two are substitutes, the model can generate multiple steady states and if they are complements, the optimal path may turn out to be cyclical [10]. If they are substitutes and an individual has low-income in one period, he/she is likely to still have low-income in the next period and same for those with high-income. If consumption and leisure are complements then an individual who has low-income in one period may work harder in the next period to have a higher income and an individual with high-income in one period may have low-income in the next, which results in a cyclical behavior [10]. In [12], Salvador Ortigueira defines leisure as the amount of time devoted to activities which are factored by the level of human capital. Individuals derive utility from consumption and leisure and those having a greater amount of human capital can obtain a higher level of utility [12].

Using foreign investment as an aid to promote growth is an issue of dispute and in [13] it is expressed that having capital mobility helps a stagnant country change course and gives support for foreign investment and growth. Tucker introduces life expectancy into the model to determine if having a longer life will help improve the state of the economy, such as improving upon and increasing capital. This issue is also studied in [14]. A model is built where increasing life expectancy is endogenously determined by public investment in health and it is determined that a longer life expectancy results in an increase in savings and a larger workforce [14]. The conclusion
drawn is the more effective the health expenditure, the bigger the increase in growth because it reduces mortality, which is a key element in growth [14]. This conclusion is also supported in [15], where it is shown that health is a complement to growth. These articles support this model also, which shows that effective health education does increase life expectancy, which helps economic growth. In [16], the engine of growth is given by the change in human capital. Human capital depends on the person’s decisions on schooling and age of retirement, which affect productivity. A model is built in which individuals choose education time and retirement age and pensions depend on the contributions made by workers during their active lives [16], which is a good start for looking into further research to determine the most productive part in an individual’s life. This model found that individuals retire sooner when they expect to receive social security and that social security also produces a negative effect on education [16].

Tucker’s model [2] adjusts previous models by incorporating health education and life expectancy to show the positive economic returns of a healthy society. In addition to Tucker, we are adding leisure time to determine if a healthy society, that is also well-rested throughout the work day, will further increase both the economic growth and the economic returns of a developing country. In other articles regarding the health of developing countries, it was discovered that AIDS is among the top causes of economic stagnation [17]. The AIDS epidemic is so real and severe that it prohibits workers from being as productive as they can be. With health education, it is hoped that individuals in these economies will not contract AIDS as easily and will
grow economically. Tucker discovered that teaching about AIDS and other diseases through health education increases the life expectancy of a developing country.

1.3 Recovering Tucker

This proposed model is an extension of [2]. In order to get the model from [2], certain limits are taken. If leisure approaches zero and $\beta = 1$ we get Tucker’s model. From his model, additional limits are taken to retrieve Zilibotti and Kejak. Tucker’s model reduces to Zilibotti, by setting $e_p, e_q$ and $h$ to zero and $q$ is set as a constant. To recover Kejak’s model from Tucker’s model set $e_q = 0$, and $\theta = 1$. Also one must set $q$ and $A$ as fixed constants.

1.4 Assumptions

This paper seeks to maximize the utility, in the model proposed in thesis but not the transformed model, with respect to consumption, leisure, human skill level, capital, health education and production education in developing countries, so that their economy is able to leave the stagnant state it is in, and prosper. Production, health education and leisure make up the three-sector market and we want to determine how much time should be spent within each sector. The idea is to see if adding leisure time in the production sector will create more productive workers, thus increasing overall funding for the economy. A second sector is health education, which creates more time learning about health and will increase life expectancy. The final sector is
time spent in human capital, which will yield higher productivity in the production sector.

This model focuses on the effects of health education and leisure time in the economy. The model is assumed to be for a developing country, but can also be used for developed countries so long as the following assumptions are met. There is a highest life expectancy $Q$, that is attainable by an individual in a country. The life expectancy, $q$ is assumed to be initially low for a developing country, because the life expectancies are not high in such a country. It is also assumed that a person is productive throughout his/her entire life. Another option for future work would be to look at the different stages in one’s life to see when he/she is more productive and will be more beneficial to economic growth. This model neglects the factor of stress throughout a person’s life, and adding such a factor to the life expectancy equation would also be helpful for future work.

1.5 Parameter List

The definitions and explanations of each parameter and variable are given when they are introduced in an equation; however, for conciseness, a complete list of parameters and variables are provided in the following tables.
<table>
<thead>
<tr>
<th>Variable</th>
<th>Definition</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>Time</td>
<td>Years</td>
</tr>
<tr>
<td>$A$</td>
<td>Technology as a function of time</td>
<td>Unitless</td>
</tr>
<tr>
<td>$K$</td>
<td>Aggregate capital</td>
<td>Output</td>
</tr>
<tr>
<td>$k$</td>
<td>Capital</td>
<td>Output</td>
</tr>
<tr>
<td>$q$</td>
<td>Life expectancy</td>
<td>Years</td>
</tr>
<tr>
<td>$e_q$</td>
<td>Time spent in health education</td>
<td>Unitless</td>
</tr>
<tr>
<td>$B$</td>
<td>Productivity level</td>
<td>1/Years</td>
</tr>
<tr>
<td>$H$</td>
<td>Aggregate skill level</td>
<td>Unitless</td>
</tr>
<tr>
<td>$h$</td>
<td>Skill level</td>
<td>Unitless</td>
</tr>
<tr>
<td>$e_p$</td>
<td>Time spent in production education</td>
<td>Unitless</td>
</tr>
<tr>
<td>$l$</td>
<td>Time spent in leisure</td>
<td>Unitless</td>
</tr>
<tr>
<td>$u$</td>
<td>Time spent in production</td>
<td>Unitless</td>
</tr>
<tr>
<td>$c$</td>
<td>Consumption</td>
<td>Output/Years</td>
</tr>
</tbody>
</table>
Table 1.1: Definition of Variables (cont.)

<table>
<thead>
<tr>
<th>Variable</th>
<th>Definition</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w$</td>
<td>Wage rate</td>
<td>Output/Years</td>
</tr>
<tr>
<td>$F$</td>
<td>Cobb-Douglas Production function</td>
<td>Output/Years</td>
</tr>
<tr>
<td>$L$</td>
<td>Labor</td>
<td>Unitless</td>
</tr>
<tr>
<td>$\Pi$</td>
<td>Profit function</td>
<td>Output</td>
</tr>
<tr>
<td>$y$</td>
<td>Zilibotti Production function, GDP</td>
<td>Output/Years</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Shadow variable</td>
<td>Unitless</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Costate variable</td>
<td>Unitless</td>
</tr>
<tr>
<td>$x$</td>
<td>Capital to skill level ratio</td>
<td>Output</td>
</tr>
<tr>
<td>$v$</td>
<td>Consumption to capital ratio</td>
<td>1/Years</td>
</tr>
</tbody>
</table>

Table 1.2: Definition of Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_0$</td>
<td>Lowest life expectancy</td>
<td>Years</td>
</tr>
<tr>
<td>$Q$</td>
<td>Highest life expectancy</td>
<td>Years</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Health education diffusion</td>
<td>Unitless</td>
</tr>
<tr>
<td>$B_F$</td>
<td>Highest level of productivity</td>
<td>1/Years</td>
</tr>
<tr>
<td>$B_0$</td>
<td>Lowest level of productivity</td>
<td>1/Years</td>
</tr>
<tr>
<td>Parameter</td>
<td>Definition</td>
<td>Unit</td>
</tr>
<tr>
<td>-----------</td>
<td>----------------------------------</td>
<td>-----------------------</td>
</tr>
<tr>
<td>( \psi )</td>
<td>Productivity diffusion</td>
<td>Unitless</td>
</tr>
<tr>
<td>( \eta )</td>
<td>Capital diffusion</td>
<td>1/Output</td>
</tr>
<tr>
<td>( \rho )</td>
<td>Depreciation of capital</td>
<td>1/Years</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>Measure of risk aversion</td>
<td>Unitless</td>
</tr>
<tr>
<td>( a )</td>
<td>Highest level of technology</td>
<td>Unitless</td>
</tr>
<tr>
<td>( A_0 )</td>
<td>Lowest level of technology</td>
<td>Unitless</td>
</tr>
<tr>
<td>( \phi )</td>
<td>Technology diffusion</td>
<td>1/(Years*Output)</td>
</tr>
<tr>
<td>( \delta )</td>
<td>Time discount factor</td>
<td>1/Years</td>
</tr>
<tr>
<td>( r )</td>
<td>Interest rate</td>
<td>1/Years</td>
</tr>
<tr>
<td>( Z )</td>
<td>Non-linear output coefficient</td>
<td>Output^{1-\theta}/Years</td>
</tr>
<tr>
<td>( D )</td>
<td>Linear output coefficient</td>
<td>1/Years</td>
</tr>
<tr>
<td>( \theta )</td>
<td>Non-linear output coefficient</td>
<td>Unitless</td>
</tr>
<tr>
<td>( \beta )</td>
<td>Parameter on leisure</td>
<td>Unitless</td>
</tr>
<tr>
<td>( G )</td>
<td>Cobb-Douglas multi-factor productivity constant</td>
<td>Output^\alpha/Years</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>Capital share of production</td>
<td>Unitless</td>
</tr>
</tbody>
</table>
CHAPTER II
CONSTRUCTING THE ORIGINAL MODEL

2.1 The Original Model

This model is similar to Tucker’s model, with most of the equations being identical. Two of the equations in this model are modified versions of Tucker’s equations to accommodate the allotment of leisure time throughout the work day. The equation for technological growth and its differential equation, the equation for life expectancy and its differential equation are taken directly from Tucker’s model and will be presented with their respective explanations first. The technological growth equation is

$$\frac{dA}{dt} = \left(\frac{a - A}{a}\right) A \phi \frac{d}{dt}[qK].$$

(2.1)

In this equation, $A$ is the level of technology at time $t$, and $a$ is the highest level of technology the world can reach. Also, in this equation, $K$ represents aggregate capital, which is the total capital available throughout the economy, and $q$ stands for life expectancy measured in years. Finally, $\phi > 0$ is the diffusion of change in the technological level into the economic system. The solution to equation (2.1) is

$$A(t) = \frac{a}{1 + \left(\frac{a}{A_0} - 1\right) e^{-\phi[q(t)K(t)]}}.$$ 

(2.2)

Here we have $A_0$, which is the lowest level of technology. According to these equations, if a person lives a longer life, represented by $q$, he/she has more time to create and
use technology that is beneficial to the economy [2]. Using the same logic, as capital increases, technology approaches the highest point of production. Note that as \( qK \) tends to 0, then \( A \) tends to \( A_0 \), the lowest level of technology. The rate at which an economy or country reaches the highest level of technology, \( a \), depends on the value of \( \phi \), the rate of diffusion of technology. The parameter \( \phi \) also shows how effective individuals of a country are at using their resources, or capital, and using their wisdom to increase life expectancy, which, in turn, can improve and increase technology. With a small \( q \) and \( K \), there is not a long enough time to gain enough experience, nor are there enough resources to increase technology. This is illustrated in Figure 2.1. As a point of comparison, Kejak does not integrate technology into his model; therefore, \( \phi = 0 \) and \( A \) is constant.

It is also important to delve into the life expectancy function, as the changes in \( q \) affect \( A \). The differential equation for life expectancy is

\[
\frac{dq}{dt} = \frac{Q - q}{Q} \frac{d}{dt} \left[ \gamma e_q + \eta K \right],
\]

where \( Q \) is the highest possible life expectancy. In this equation, \( e_q \) represents non-leisure, non-production time spent in health education. The diffusion affect of the time spent in health education is represented by \( \gamma > 0 \). The parameter \( \eta \) is also positive and measures the influence of capital on life expectancy. Neither Zilibotti nor Kejak integrate life expectancy into their models, and hence \( q \) would be constant in each of their models. When this equation is solved, we find that

\[
q(t) = \frac{Q}{1 + \left( \frac{Q}{Q_0} - 1 \right) e^{-[\gamma(e_q) + \eta(K(t))]}}, \quad \text{(2.4)}
\]
Following the previous logistic functions, $Q_0$ is the lowest feasible life expectancy in the country. As $K$ and $e_q$ increase, $q$ approaches $Q$, which is the highest possible life expectancy. As $e_q$ tends to 0 and as $K$ tends to 0, $q$ tends to $Q_0$. A small $e_q$ implies there is not much time spent in health education, whereas a large $e_q$ implies there is much time. A small $K$ implies the country is poor and a large $K$ represents a rich country. The constraints $\gamma$ and $\eta$ determine how quickly the economy or country is able to reach the highest possible life expectancy. The size of $\gamma$, the diffusion effect of health education, can be thought of as the effectiveness of the teaching. A small $\gamma$ implies non-effective teaching, whereas a large $\gamma$ implies very effective teaching. The same logic applies for $\eta$, the diffusion effect of capital. A small $\eta$ implies that a country is non-effective in using resources to improve health. In contrast, a large $\eta$ implies a country is very effective in using resources to improve health. This is illustrated through Figures 2.2, 2.3 and 2.4.

Using Equation (2.5) from Tucker’s model, the following logistic equation is found

$$B(H) = \frac{B_F}{1 + \left(\frac{B_F}{B_0} - 1\right)e^{-\psi H}}. \tag{2.5}$$

This equation is based on assumptions from Kejak and Lucas regarding human capital and productivity. The idea is that spending time to improve human capital promotes technical progress [2]. In this equation, $B_F$ is the highest level of productivity throughout the world market. Also, $\psi > 0$, the speed of diffusion for productivity determines the ease with which a country can reach its highest feasible level of productivity. The lowest level of productivity of a country, which is attained only at the
lowest level of human capital, is denoted by $B_0$. It should be pointed out that for all time $t$, $B_F \geq B \geq B_0$. As human capital increases, $B$ approaches $B_F$ and as human capital decreases toward zero, $B$ approaches $B_0$.

The next equation follows the form of Kejak and assumes that the level of productivity in the education sector depends on the developmental level of a society, which is expressed by the average level of human capital [6],

$$\frac{dh}{dt} = B(H)e_p h.$$  \hfill (2.6)

In this equation, which is the constraint on human skill level, $B(H)$ is the level of productivity in the education sector and $h$ is the level of skill held by the population. Here, $e_p$ is the proportion of non-leisure time spent in production education, as opposed to health education. Zilibotti does not consider skill level, hence $h = 0$ for his model. Recall that the time allotment scheme of our country or economy, namely equation (1.7) is

$$1 = u + e_p + e_q + l.$$  \hfill (2.7)

As mentioned previously, $l$ represents the time allotted for leisure throughout the work day, $u$ is the time allotted for production, $e_p$ is the amount of non-leisure time spent in production education, or the human capital sector, and $e_q$ is the amount of non-leisure, non-production time spent in health education. The summation of these proportions is equal to one work day and $0 \leq u, e_q, e_p, l \leq 1$. To recover Kejak or Zilibotti, set $l = e_q = 0$. Additionally, to fully recover Zilibotti, one would set $e_p = 0$ as well.
The equations above are important in answering economic questions and determining how much time should be spent in each section of the economy. In order to optimally allot time between the three sectors, we can assume that an individual wants to maximize the combination of leisure and consumption over his/her lifetime. This assumption is analyzed through examination of the maximization problem and the differential equation for capital. The maximization problem uses the utility function from Ladron-De-Guevara et al. [9] and integrates over time

$$\max_{c,k,h,u,e,q,e_p,l} \left( \lim_{Q \to \infty} \int_0^Q \frac{[c^{\beta}l^{1-\beta}]^{1-\sigma} - \delta t}{1 - \sigma} e^{-\delta t} dt \right).$$

(2.8)

In order to maximize the combination of leisure and consumption over a person’s lifetime, it is necessary to choose the optimal time allotments for $u, e_q, e_p, l$ and also for variables $c, k$ and $h$. In this problem, $c$ is consumption at time $t$, and $\sigma$ is a measure of risk aversion. Utility is the risk of gaining and losing money or resources. For $\sigma$ close to 0, a person/country is increasingly risk averse, avoiding risk whenever possible, and current consumption is more important than future consumption. For $\sigma$ close to 1, a person/country is increasingly willing to take risk and considers current consumption less important than future consumption and thus will save money to use at a later time. This maximization problem is a trade-off between consumption and leisure, and $\beta$ is the relative importance of consumption over leisure. For $\beta$ close to 1, consumption becomes more important than leisure, and for $\beta$ close to 0, leisure becomes more important than consumption. The time discounting factor is represented by $e^{-\delta t}$, where $\delta$ is a factor that shows a consumer’s likelihood to invest.
at some point. A large $\delta$ implies people would rather spend their money now, and consume all they earn when they earn it. A small $\delta$, however, implies people are willing to save their earnings and invest in capital or something else with future value or benefit.

The equation for capital, which is a constraint on the maximization problem, is

$$\frac{dk}{dt} = kr + wuh - c,$$

where $r$ is the investment return rate, $w$ is the working wage rate and $k$ is capital. This equation states that the derivative of capital with respect to time equals the accumulating capital minus the capital spent. If a country does not consume all that is earned, it will be available for capital. The $w$ in this equation is found using the profit maximization equation, which uses the Cobb-Douglas production function, $F(k, L) = Gk^\alpha L^{1-\alpha}$, which models the production center. Here $k$ is capital, $L$ is labor, $G$ is a multi-productivity constant parameter and $\alpha$ is a parameter that denotes the capital share of production with $0 < \alpha < 1$. Consequently, $1 - \alpha$ corresponds to the labor share of production. By setting profit equal to production minus wage, $w$, plus Gross Domestic Product (GDP), we obtain the profit equation

$$\Pi = F(k, L) - wL - [DAk + Z(Ak)^\theta],$$

which uses the production equation. Equation (2.10) will be optimized and the result used to solve the associated Hamiltonian system of equation (2.8). The derivatives with respect to capital and labor are taken to implement additional constraints on
the parameters of production. The derivative with respect to labor lets us find $w$, and the derivative with respect to capital obtains both $D$ and $Z$. The derivations for each are found in the Appendix. It is found that parameters $D$, $Z$ and $G$ are not independent and maximizing profit yields the codependent relationship.

Another important assumption to note is

$$kr - k\rho \approx y(t) = DAk + Z(Ak)^\theta,$$

which states that capital, adjusted by its investment rate minus the depreciation of capital, shown as $k\rho$, is approximately equal to the Gross Domestic Product of the economy [2]. This is used in (2.10). The first term, $kr$ corresponds to earning by investing and $k\rho$ corresponds to loss by depreciation. When they are subtracted, it is the net rate of growth or decay of capital. This equation uses Zilibotti’s productivity function, which has a linear and a non-linear term. The linear term is used in Kejak also, where $A$ is constant. The non-linear term is homogeneous and is added so that a country is able to escape the trap of underdeveloped countries and provide an opportunity for the fixed point to be unstable, so that it can initiate and maintain growth. With the linear term alone, the fixed point is stable and countries are not able to grow. Using Maple and, setting $Z = 0$, to eliminate the non-linear term, we were able to verify that the fixed point was stable and there is little opportunity for growth. Here, $D$ and $Z$ are parameters for an individual country and $0 \leq \theta \leq 1$ is a parameter associated with the non-linear factor of output. If we set $\rho = 0$, then we recover Zilibotti and similarly, setting $\theta = 1$ recovers Kejak.
2.2 Interpretation of Technology and Life Expectancy Equations

Recall that the interpretation of equation (2.2) in Section 2.1 is that as $qK$ tends to 0, $A$ tends to $A_0$, which is the lowest level of technology. Also as $qK$ increases, $A$ approaches $a$, the highest level of production technology. Figure 2.1 plots $A$ versus $qK$ for different values of $\phi$, the rate of diffusion of technology. As $\phi$ increases, the economy more effectively utilizes resources to gain life expectancy and $A$ approaches $a$ more quickly. However, if $\phi$ is too small, the economy is not effectively utilizing resources, so it will not reach the highest level of technology very quickly, if at all.

![Figure 2.1: Technology Equation (2.2): Varying $\phi$, the rate of diffusion of technology. Here $a = 0.075$ and $A_0 = 0.015$.](image)

Equation (2.2) can represent any country, developed or developing, depending on the range of $qK$. Each country would start at a different point on the curve, on its level of capital, life expectancy and on the value of the diffusion parameter, the
effectiveness of using resources to gain life expectancy. A parametric study of (2.2) with respect to diffusion, $\phi$, of technology is important for determining how easily a country or economy may reach $a$.

Figure 2.2 shows the approach to the highest life expectancy while varying $\eta$, the measure of influence of capital. Here $Q = 85$ and $Q_0 = 35$.

Figure 2.2 shows the approach to the highest life expectancy while varying $\eta$, the measure of influence or effectiveness in using capital to improve health, for a 'rich' country and for a 'poor' country. The rich country is represented by $K = 200$ and the poor country is represented by $K = 100$. We also show a country with no aggregate capital, namely $K = 0$. These levels for capital, $K$ are also used in Figures 2.3 and 2.4. Similar to (2.2), equation (2.4) is studied with respect to $e_q$, rather than time. This gives opportunity for a country, developed or developing, to
have a similar initial condition, or starting point, on a curve. As Figure 2.2 shows, the increase in \( \eta \) drastically increases the approach to \( Q \), the highest life expectancy. When \( \eta = 0.03 \), thus more effectively utilizing capital or resources, the country’s life expectancy starts at or near the highest life expectancy, \( Q \). If individuals of a country effectively utilize resources, the life expectancy of that country increases. However, if individuals do not effectively use resources, life expectancy is lower, and the country may not reach the highest life expectancy. Also, as \( e_q \) increases, that is time spent in health education increases, an economy reaches the highest life expectancy more quickly.

![Figure 2.3: Life Expectancy Equation (2.4): Varying \( \gamma \), the measure of influence of time spent in health education. Here \( Q = 85 \) and \( Q_0 = 35 \).](image)

Figure 2.3 demonstrates life expectancy influenced by the effectiveness of time spent in education, \( \gamma \). As with the previous case, the approach to the highest life
expectancy is more rapid with larger $\gamma$. Increasing $\gamma$ from two to three, implying that the effectiveness of teachers teaching about the importance of health and teaching about the risks of disease improves, adds several years to the life expectancy of that country. When $e_q$ tends to 0, the average life expectancy approaches $Q_0$, the feasible minimum. As shown in Figure 2.3, even when $K = 0$, with an increase in the amount of time spent in health education, $e_q$, the country can approach the highest life expectancy. Again, with the larger $K$, the approach to $Q$ is faster for the two values of $\gamma$.

Figure 2.4: Life Expectancy Equation (2.4): Comparing developed versus developing countries with varied $\gamma$, $\eta$ and $K$. Here $Q = 85$ and $Q_0 = 35$.

Figure 2.4 compares rich and poor countries. A small $\gamma$ or $\eta$ implies that the country, rich or poor, is not very effective or mature with respect to using resources
to improve health or health education. Countries that are not effective in using the capital at hand or at teaching may not reach the highest life expectancy, regardless of it being a rich or a poor country. The figure shows that, with small $\gamma$ and small $\eta$, the country, whether rich or poor, does not reach the highest life expectancy, although the life expectancy does increase. Rich countries have the advantage, however, regardless of the values of $\gamma$ or $\eta$, because they start with a higher life expectancy than do poor countries. Examining Figure 2.4, we notice that a country with $K = 200$ is very effective in both using resources and teaching. Therefore, it begins at the highest life expectancy and maintains that level. A country with $K = 100$ which is effective in both using resources and teaching does not start at the highest life expectancy, but does reach it. Therefore, it is not as important to begin with a lot of resources or capital, it is more important to effectively utilize what you have.

2.3 Solving the Original Model

The Hamiltonian operator is used to maximize consumption over the variables $c$, $k$, $e_p$, $e_q$, $l$, and $h$. We seek six governing equations for the six unknown variables. The Hamiltonian is constructed by taking the utility function and adding to it two constraints, one for human skill level and one for capital. Each contraint is multiplied by a Lagrange multiplier, $\lambda$ or $\mu$. The Hamiltonian operator is

$$HAM(c, e_q, e_p, l, k, h; \lambda, \mu) = \left[ \frac{(c^\beta)^{1-\beta)}{1-\sigma} e^{-\delta t} \right] + \lambda [kr + wh - c]$$

$$+\mu [B(H)e_p,h] . \quad (2.12)$$

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The Hamiltonian is rewritten by using equation (1.7) for \( u \) and adding a zero in the form of "+\( k\rho - k\rho \)" which yields

\[
H \text{AM} = \left[ \frac{(e^\beta l^{1-\beta})^{1-\sigma}}{1-\sigma} e^{-\delta t} \right] + \lambda [(kr + k\rho) - k\rho + w(1 - e_p - e_q - l)h - c] + \mu [B(H)e_p h].
\] (2.13)

To continue solving the Hamiltonian, partial derivatives are taken with respect to the eight variables and parameters, \( c, k, e_p, e_q, l, h, \lambda \) and \( \mu \), giving us eight equations and eight unknowns, as follows:

\[
H \text{AM}_c = 0 = [c^\beta l^{1-\beta}]^{1-\sigma} \beta l^{1-\beta} c^{\beta-1} e^{-\delta t} - \lambda,
\] (2.14)

\[
H \text{AM}_\lambda = \frac{dk}{dt} = (kr + k\rho) - k\rho + w(1 - e_p - e_q - l)h - c,
\] (2.15)

\[
H \text{AM}_\mu = \frac{dh}{dt} = B(H)e_p h,
\] (2.16)

\[
H \text{AM}_k = -\frac{d\lambda}{dt} = \lambda[DA + ZA^\theta k^{\theta-1} \theta - \rho],
\] (2.17)

\[
H \text{AM}_h = \frac{d\mu}{dt} = \lambda w(1 - e_p - e_q - l) + \mu B(H)e_p,
\] (2.18)

\[
H \text{AM}_e_p = 0 = \mu B(H)h - \lambda wh,
\] (2.19)

\[
H \text{AM}_e_q = 0 = \lambda \left[ Dk \frac{dA}{dq} \frac{dq}{de_q} + Zk^\theta \theta A^\theta -1 \frac{dA}{dq} \frac{dq}{de_q} \right] - \lambda wh,
\] (2.20)

\[
H \text{AM}_l = 0 = e^{\beta(1-\sigma)}(1 - \beta)l^{(1-\beta)(1-\sigma)-1} e^{-\delta t} - \lambda wh,
\] (2.21)

where \( \frac{dA}{dq} \) and \( \frac{dq}{de_q} \), which come from (2.2) and (2.4), are:

\[
\frac{dA}{dq} = \frac{a\phi K(t)(\frac{Q}{Q_0} - 1)e^{-\phi[QK]}}{[1 + (\frac{Q}{Q_0} - 1)e^{-\phi[QK]}]^2},
\] (2.22)

\[
\frac{dq}{de_q} = \frac{Q\gamma(\frac{Q}{Q_0} - 1)e^{-[\gamma(e_q) + \eta(K(t))]}\right]}{[1 + (\frac{Q}{Q_0} - 1)e^{-[\gamma(e_q) + \eta(K(t))]}]^2},
\] (2.23)
Now the above system of equations (2.14)-(2.21) can be solved by eliminating \( \lambda \) and \( \mu \) and reducing it to a system of six equations. We first, solve for \( \lambda \) using (2.14) and use the Cobb-Douglas production function and the profit function to find \( w \). After such values are defined, we make the proper substitutions then continue solving to find the governing equations. Appendix B has complete derivations to find the following:

\[
\frac{\dot{c}}{c} = \frac{DA + Z A^\theta k^{\theta-1} \theta - \rho - \delta}{\sigma},
\]

(2.24)

\[
\frac{\dot{h}}{h} = B(H)e_p,
\]

(2.25)

\[
\frac{\dot{k}}{k} = DA + Z A^\theta k^{\theta-1} + G k^{\alpha-1} L^{1-\alpha} (1 - \alpha) - \rho - \frac{c}{k},
\]

(2.26)

\[
\frac{d(DAk + Z(Ak)^\theta)}{de_q} = wh,
\]

(2.27)

\[
\frac{de_p + de_q + dl}{1 - e_p - e_q - l} = \frac{\dot{h}}{h} - \frac{\dot{k}}{k} + \frac{1}{\alpha} \left[ DA + Z A^\theta k^{\theta-1} \theta - \rho \right] - \frac{1}{\alpha} (1 - e_q - l) B(H) + \frac{1}{\alpha H} dB e_p h,
\]

(2.28)

\[
l = \frac{1 - \beta}{\beta} \frac{c}{wh}.
\]

(2.29)

Note when \( \beta = 1 \), equation (2.29) yields \( l = 0 \) and leisure is not considered as important as consumption. Maximization of (2.8) is equivalent to solving these equations to determine the unknowns \( c, k, e_p, e_q, l \) and \( h \).

2.4 Interpretations of the Original Model

Equation (2.24) means that the average rate of change of consumption is equal to the marginal effect of capital on gross output, minus both the depreciation of capital
and a time discounting factor, which is scaled by a relative risk-aversion coefficient, \( \sigma \). This equation shows that an increase in national output with respect to capital results in an increase in average rate of consumption.

Equation (2.25) implies that the average rate of change of human skill level is equal to the productivity scaled by production education. Interpreting this equation suggests that more time spent in production education will produce workers with higher skill levels, which will result in higher output [2]. Similarly, equation (2.27) shows that the marginal effect of health education on production is equal to wage times human skill level.

Equation (2.26), which is the capital governing equation, shows that capital is equal to output and scaled production, less depreciation of capital and consumption. As depreciation increases, the average rate of capital decreases, and this also decreases as consumption increases. The average rate of change of capital increases as capital, labor or technology increase.

Equation (2.28) shows that the average rate of change of time spent in production is equal to the average rate of change of skill less the average rate of change of capital plus scaled marginal effect of capital on output less depreciation of capital and scaled productivity plus the rate of change of productivity scaled by time spent in production. The implication is that time spent in education affects the entire economic system.
Equation (2.29) shows that leisure is equal to a scaled consumption divided by a factor of a worker’s wage and the skill level of that worker. This shows that when the wage is increased, leisure time decreases. Also, leisure time decreases when the human skill level $h$, increases. This agrees with [1].
CHAPTER III

STEADY STATE OF THE ORIGINAL MODEL

3.1 Steady State Solutions

The steady state solutions are found by setting the time derivatives of equations (2.24) through (2.29) equal to zero. The equations and their solutions will be given in this section, but the complete derivation process will be shown in Appendix C.

With these assumptions, the equations become

\[ 0 = \frac{DA + ZA^\theta k^{\theta-1} - \rho - \delta}{\sigma}, \]  \hspace{1cm} (3.1)

\[ 0 = B(H)e_ph, \] \hspace{1cm} (3.2)

\[ 0 = DA + ZA^\theta k^{\theta-1} + Gk^{\alpha-1}L^{1-\alpha}(1 - \alpha) - \rho - \frac{c}{k}, \] \hspace{1cm} (3.3)

\[ \frac{d(DAk + Z(Ak)^\theta)}{de_q} = wh, \] \hspace{1cm} (3.4)

\[ 0 = \frac{0}{h} - \frac{0}{k} + \frac{1}{\alpha} \left[ DA + ZA^\theta k^{\theta-1} - \rho \right] \]
\[ - \frac{1}{\alpha} (1 - e_q - l)B(H) + \frac{1}{\alpha} dB \frac{dH}{H} e_ph, \] \hspace{1cm} (3.5)

\[ l = \frac{1 - \beta}{\beta} \frac{c}{wh}. \] \hspace{1cm} (3.6)

It is easiest to start with equation (3.2) to find a solution for \( e_p \). First, using the productivity equation (2.5), this equation can be rewritten as follows:

\[ \frac{dh}{dt} = B(H)e_ph = \frac{B_F}{1 + \left( \frac{B_F}{B_0} - 1 \right) e^{-\psi H}} e_ph. \] \hspace{1cm} (3.7)
However, from above, the steady state solutions require time derivatives to be set to zero. The only way \( \frac{dh}{dt} \) can equal zero is if \( h = 0 \), \( e_p = 0 \) or \( B_0 = 0 \). It is assumed that individuals of a country have at least some skill level, so it wouldn’t make sense for the skill level \( h \) to be equal to zero. Setting \( B_0 \) equal to zero would imply that productivity never grows and that \( B(H) = \frac{dB}{dH} \), which is not reasonable, so that is eliminated as well. The only option remaining is \( e_p = 0 \), which would imply that there is no time spent is production education; therefore, the skill level \( h \) will not increase. Now that we have \( e_p \) set to zero and the time derivatives are equal to zero, we look at the steady state solution for equation (3.5)

\[
0 = \frac{1}{\alpha} \left[ DA + Z A^\theta k^{\theta-1} \theta - \rho \right] - \frac{1}{\alpha} (1 - e_q - l) B(H). \tag{3.8}
\]

Using equation (3.1), \( \delta \) can be found, which will help solve for \( e_q \). We multiply both sides by \( \sigma \) and add the \( \delta \) to the other side to yield

\[
DA + Z A^\theta k^{\theta-1} \theta - \rho = \delta. \tag{3.9}
\]

Now, equation (3.8) can be simplified as follows,

\[
0 = \frac{1}{\alpha} \delta - \frac{1}{\alpha} (1 - e_q - l) B(H). \tag{3.10}
\]

Following Kejak, it is assumed that \( B(H) \) is a constant and is a measure of productivity that is dependent upon \( h \).
We can finish solving for $e_q$, holding $B(H)$ constant. We next factor out $\frac{1}{\alpha}$ in equation (3.10) to obtain

$$0 = 1\frac{1}{\alpha}(\delta - (1 - e_q - l)B(H)),$$

(3.11)

then multiply both sides by $\alpha$ to get

$$0 = \delta - (1 - e_q - l)B(H).$$

(3.12)

Next, we add the righthand side of the minus sign and divide by $B(H)$ to obtain

$$1 - e_q - l = \frac{\delta}{B(H)}.$$

(3.13)

Now, we finish solving for $e_q$, which yields

$$e_q = 1 - l - \frac{\delta}{B(H)}.$$

(3.14)

Recall that $0 < e_q < 1$. Therefore, we must ensure that $0 \leq (1 - l - \frac{\delta}{B(H)}) \leq 1$. This implies that $1 > l + \frac{\delta}{B(H)} > 0$ and that $\delta \leq B(H)$. This assumption can be made because $\delta$ is a time discounting factor that is small and positive and $B(H)$ is a positive value but is larger than a discount factor. Since we have all positive values, we are ensured that it is positive. It is important that $\delta$ is less than $B(H)$, because if $\delta$ is greater than $B(H)$, the value would be too large and when added to $l$, would be greater than one. Furthermore, $l$ would have to be a value small enough that when added to $\frac{\delta}{B(H)}$, the value would not be greater than one.

We have solutions for $e_p, e_q$, and $l$, which is equation (3.6). Now we have to solve for $k, c$, and $h$, where $h$ is solved for numerically. We can get $c$ and $k$ in terms
of $h$ using the Cobb-Douglas production function and the definition of labor, $L = uh$. Using these equations and simplifying, as shown in Appendix C, yields the solutions for $k$ and $c$:

$$k = \frac{\delta}{B(H)h} \left( \frac{G\alpha}{\rho + \delta} \right)^{\frac{1}{1-\alpha}},$$  \hspace{1cm} (3.15)

$$c = \frac{\delta}{B(H)h} \left[ \left( \frac{G\alpha}{\rho + \delta} \right)^{\frac{1}{1-\alpha}} (DA - \rho) + G \left( \frac{G\alpha}{\rho + \delta} \right)^{\frac{\alpha}{1-\alpha}} (1 - \alpha) \right]$$  \hspace{1cm} (3.16)

Using the derivatives of the profit function and algebraically manipulating equation (2.20), yields the algebraic equation for $h$

$$\frac{dA}{dq} \frac{dq}{de_q} = A \frac{1 - \alpha}{\alpha} \frac{B(H)}{\delta},$$  \hspace{1cm} (3.17)

which is solved numerically, as shown in Appendix C.

3.2 Results of the Original Model

Using both Maple and equations (3.6) and (3.17), we find $e_q$ and $h$. We are able to use these values and equations (3.14)-(3.16) to find $l, k$ and $c$. Technology $A$, and life expectancy $q$, are found using equations (2.2) and (2.4). All of these values, found in Table 3.1, are derived using specific values of parameters, which most are taken from Zilibotti [5] and Kejak [6]. We have that $\delta = 0.08, \rho = 0.05, G = 5, Z = 10, D = 1, \theta = 0.72, B(H) = 0.0876, a = 0.075, A_0 = 0.746 \times 10^{-4}, Q = 85, Q_0 = 35$ and $\sigma = 0.4$, where $\gamma = 1.5$ in the original model. We also have that $\phi = 0.00877/5.5$ and $\eta = 0.01/5.5$. These parameters are adjusted and scaled by 5.5 so that the values
in the original model and transformed model match and a country will be at an equal state on the curve for each model. For the steady state of the original model, we assume that \( e_p = 0 \). Looking at Table 3.1, we can see the trend as \( \beta \) increases. Varying \( \beta \) is important, because it measures the value of leisure in a society. As expected, \( e_q \), the time spent in health education, increases as \( \beta \) increases. As \( \beta \) increases, the time spent in leisure decreases, therefore time spent in health education would increase. Capital, \( k \) and consumption, \( c \), decrease as \( \beta \) increases. Consumption, therefore, moves in the same direction as leisure, which validates equation (2.29) from Chapter 2. The human skill level \( h \) also decreases as \( \beta \) increases, indicating the more leisure time one has, the better his/her skill level becomes. Life expectancy \( q \), however, decreases with more leisure time. For \( \beta \) close to 1, we get Tucker’s results.

After we find the fixed point, we linearized the equations and solve the linear equations using Maple. After solving these equations, which is a linear stability analysis, we find the eigenvalues and eigenvectors of the fixed point. These are listed in Table 3.2. Three eigenvalues corresponding to the fixed points are positive, which indicate an unstable manifold. However, there is a negative eigenvalue, which is a stable path leading to an economic trap. Should a developing country’s economy be on a stable path, its economy will be trapped. Looking at the positive, real eigenvalue, the eigenvector is pointing in the \( c \) direction.
### Table 3.1: Equilibrium Values for the Original Model

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$e_p$</th>
<th>$e_q$</th>
<th>$h$</th>
<th>$l$</th>
<th>$k$</th>
<th>$q$</th>
<th>$A$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta = 0.95$</td>
<td>0</td>
<td>0.0164</td>
<td>5.3853</td>
<td>0.0700</td>
<td>100.5585</td>
<td>39.3453</td>
<td>0.0265</td>
<td>37.1287</td>
</tr>
<tr>
<td>$\beta = 0.97$</td>
<td>0</td>
<td>0.0462</td>
<td>5.1970</td>
<td>0.0405</td>
<td>97.0415</td>
<td>40.1465</td>
<td>0.0249</td>
<td>54.2846</td>
</tr>
<tr>
<td>$\beta = 0.99$</td>
<td>0</td>
<td>0.0738</td>
<td>5.0154</td>
<td>0.0130</td>
<td>93.6516</td>
<td>40.8908</td>
<td>0.0232</td>
<td>51.4799</td>
</tr>
<tr>
<td>$\beta = 0.999999$</td>
<td>0</td>
<td>0.0868</td>
<td>4.9261</td>
<td>0.0000013</td>
<td>91.9833</td>
<td>41.2403</td>
<td>0.0223</td>
<td>50.0750</td>
</tr>
</tbody>
</table>

### Table 3.2: Eigenvalues and Eigenvectors for the Original Model

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>Eigenvalues</th>
<th>Eigenvectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta = 0.95$</td>
<td>0.8 ± 0.5i, 0.8 ± 0.5i</td>
<td>$c, h, k, and e_q$</td>
</tr>
<tr>
<td></td>
<td>(-29.4 ± 68.5i, 0.4 ± 0.1i, 30.1 ± 66.1i, -0.2 ± 0.1i)</td>
<td></td>
</tr>
<tr>
<td>$\beta = 0.95$</td>
<td>-0.1</td>
<td>(-35.1, -9.4, 40.9, -0.4)</td>
</tr>
<tr>
<td>$\beta = 0.999999$</td>
<td>0.8 ± 0.5i</td>
<td>$c, h, k, and e_q$</td>
</tr>
<tr>
<td></td>
<td>(-21.2 ± 69.9i, 0.3 ± 0.6i, 31.2 ± 58.5i, -0.1 ± 0.4i)</td>
<td></td>
</tr>
<tr>
<td>$\beta = 0.999999$</td>
<td>-0.03</td>
<td>(-42.8, -9.2, 31.3, -0.4)</td>
</tr>
<tr>
<td></td>
<td>0.04</td>
<td>(57.7, 11.1, 44.5, -0.7)</td>
</tr>
</tbody>
</table>
CHAPTER IV
CONSTRUCTING THE TRANSFORMED MODEL

4.1 The Transformed Model

A fixed point in the original model has a stable path which indicates a possible economic trap, as was shown in Chapter 3. Since there is opportunity for an economic trap, it is beneficial to perform a change of variables of the original model, creating a transformed model. The transformed model will give a new set of fixed points which are not fixed points of the original model. Note that $e_p = 0$ in the original model, but $e_p \neq 0$ in the transformed model. We are considering a completely different model whose fixed points describe dynamic paths in the original variables. We find in [9] a change of variables to the model. We create variables for $x$ and $v$:

$$ x = \frac{k}{h}, \quad (4.1) $$

$$ v = \frac{c}{k}. \quad (4.2) $$

These correspond to $x$ being capital per human skill level and $v$ the consumption per unit of capital. Using these equations for $x$ and $v$, we will seek a system of 5 equations with 5 unknowns: $x, v, e_q, e_p$ and $l$. Our equations for technology (2.1) and life expectancy (2.3) are similarly transformed.

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The transformation of equations (2.1) and (2.3) will set up the new 5-variable system. Looking at the original technology equation (2.1), we see that

\[ \frac{dA}{dt} = \left( \frac{a - A}{a} \right) A \phi \left[ q \frac{dK}{dt} + \frac{dq}{dt} K \right]. \]  

(4.3)

The technology equation becomes

\[ A(t) = \frac{a}{1 + \left( \frac{q}{A_0} - 1 \right) e^{-\phi[q(t)x(t)]}}. \]  

(4.4)

Assuming \( k \) is constant as \( h \) increases, the level of technology would not approach \( a \) very quickly. This would imply that as human skill level increases, there isn’t as much of a need for new technology. It is assumed the individuals have mastered the technology they currently have. However, \( k \) may not stay constant. Looking at (2.15), \( k \) increases as \( h \) increases; therefore, increasing \( h \) could lead to an increase in the level of technology, rather than a decrease. This increase in technology could be a result of individuals requiring new technology to complement their enhanced skill level. The parameter, \( \phi \), is the effectiveness of utilizing capital per human skill level to gain life expectancy as technology increases. Equation (2.3), the life expectancy equation, is also modified with a transformation of \( k \) to \( x \) as

\[ \frac{dq}{dt} = \frac{Q - q}{Q} q \frac{d}{dt} [\gamma e_q + \eta x]. \]  

(4.5)

The solution to this logistic equation is

\[ q(t) = \frac{Q}{1 + \left( \frac{Q}{q_0} - 1 \right) e^{-[\gamma e_q + \eta x(t)]}}. \]  

(4.6)
With these transformations, we are now able to reduce our 6-variable system of equations to a 5-variable system of equations. Considering the dependency of $A$ upon $q$, and $q$ upon $e_q$, equation (2.20) and its components for $\frac{dA}{dq}$ and $\frac{dq}{de_q}$ become

$$HAME_q = 0 = \lambda \left[ Dk \frac{dA}{dq} \frac{dq}{de_q} + Zk^{\theta} A^{\theta-1} \frac{dA}{dq} \frac{dq}{de_q} \right] - \lambda wh,$$  \hspace{1cm} (4.7)

$$\frac{dA}{dq} = \frac{\alpha \phi x(t)(\frac{\alpha}{A_0} - 1)e^{-\phi[qx(t)]}}{[1 + (\frac{\alpha}{A_0} - 1)e^{-\phi[qx]}]^2},$$  \hspace{1cm} (4.8)

$$\frac{dq}{de_q} = \frac{Q \gamma (\frac{Q}{Q_0} - 1)e^{-[\gamma e_q + \eta x(t)]}}{[1 + (\frac{Q}{Q_0} - 1)e^{-[\gamma e_q + \eta x]}]^2}.$$  \hspace{1cm} (4.9)

Using equation (2.10) and taking the derivative with respect to capital, then setting that equal to zero, $\frac{d\Pi}{dk} = 0$, we are able to solve for $k$. This $k$ is used in equation (4.7), as shown in Appendix C. Now, to find the first of our five equations used to solve for our five unknowns, we use equation (3.17) and equation (3.13), which yields

$$\frac{dA}{dq} \frac{dq}{de_q} = A_{1} - \alpha \frac{1}{\alpha} \frac{1}{1 - e_p - e_q - l}.$$  \hspace{1cm} (4.10)

Equation (3.13) is the steady state equation which assumes $e_p = 0$, the above equation does not make this assumption. Finding three of the next four equations is not quite as straightforward and involves finding the derivatives and growth rates of $x$ and $v$, which are

$$\frac{dx}{dt} = \frac{dk}{dt} h - \frac{dh}{dt} \frac{k}{h} = \frac{dk}{dt} h - \frac{dh}{dt} k,$$  \hspace{1cm} (4.11)

$$\frac{dv}{dt} = \frac{dc}{dt} k - \frac{dk}{dt} \frac{c}{k} = \frac{dc}{dt} c - \frac{dk}{dt} k.$$  \hspace{1cm} (4.12)
Now, substitute equations (2.24), (2.25) and (2.26) into equations (4.11) and (4.12) and transform the appropriate variables, when needed, to obtain three of the next four equations:

\[
\dot{x} = DA \left( 1 - \frac{1}{\theta} \right) + G \left( \frac{1 - e_p - e_q - l}{x} \right)^{1-\alpha} \left( 1 - \alpha + \frac{\alpha}{\theta} \right)
- \rho - v - B(H)e_p, \quad (4.13)
\]

\[
\dot{v} = G \left( \frac{1 - e_p - e_q - l}{x} \right)^{1-\alpha} \left( \frac{\alpha}{\sigma} - 1 + \alpha - \frac{\alpha}{\theta} \right) + v + \rho \left( 1 - \frac{1}{\sigma} \right)
- \frac{\delta}{\sigma} - DA \left( 1 - \frac{1}{\theta} \right), \quad (4.14)
\]

\[
\frac{dx}{dt} + \frac{de_p}{dt} + \frac{dl}{dt} = -\frac{dx}{x} + \frac{1}{\alpha} \left[ G\alpha \left( \frac{1 - e_p - e_q - l}{x} \right)^{1-\alpha} - \rho \right]
- \frac{1}{\alpha} (1 - e_q - l) B(H). \quad (4.15)
\]

The last equation that will be used to solve for the five unknowns is found by using equation (2.29), performing a multiplication and division by \( k \) and transforming the appropriate variables, to yield

\[
l = \frac{\left( 1 - \beta \right)x v \left( \frac{1 - e_p - e_q - l}{x} \right)^{\alpha}}{\beta G(1 - \alpha)}. \quad (4.16)
\]

We can now use (4.10), (4.13), (4.14), (4.15) and (4.16) to solve for our fixed points \( x, v, e_p, e_q \) and \( l \).

### 4.2 Finding the Fixed Points

In order to solve for \( x, v, e_p, e_q \) and \( l \), it is necessary to manipulate (4.10) and (4.13)-(4.16) for use in Maple. First, we use equation (4.10) and equations (4.8) and (4.9)
to find \( e_p \) and use a simple division in equation (4.16) to find \( v \). We also, isolate \( \frac{de_p}{dt} \) in equation (4.15) to find equation (4.22). The complete derivations are found in Appendix C. The following equations are used in Maple:

\[
e_p = 1 - e_q - l
\]

\[
\frac{1 - \alpha}{\alpha} \left[ 1 + \left( \frac{\alpha}{A_0} - 1 \right) e^{-\phi(Qx)} \right] \left[ 1 + \left( \frac{Q}{Q_0} - 1 \right) e^{-\gamma|e_q| - \eta|x|} \right]^2
\]

\[
\phi x \left( \frac{\alpha}{A_0} - 1 \right) e^{-\phi(Qx)} Q \gamma \left( \frac{Q}{Q_0} - 1 \right) e^{-\gamma|e_q| - \eta|x|}
\]

(4.17)

\[
v = \frac{l \beta G (1 - \alpha) x^\alpha}{(1 - \beta) x (1 - e_p - e_q - l)^\alpha^2}
\]

(4.18)

\[
\frac{dx}{dt} = DA \left( 1 - \frac{1}{\theta} \right) + G \left( \frac{1 - e_p - e_q - l}{x} \right)^{1 - \alpha} \left( 1 - \alpha + \frac{\alpha}{\theta} \right)
\]

\[-\rho - v - B(H) e_p,
\]

(4.19)

\[
\frac{dv}{dt} = G \left( \frac{1 - e_p - e_q - l}{x} \right)^{1 - \alpha} \left( \frac{\alpha}{\theta} - 1 + \alpha - \frac{\alpha}{\theta} \right) + v
\]

\[+ \rho \left( 1 - \frac{1}{\sigma} \right) - \frac{\delta}{\sigma} - DA \left( 1 - \frac{1}{\theta} \right),
\]

(4.20)

\[
\frac{de_p}{dt} + \frac{de_q}{dt} + \frac{dl}{dt} = -\frac{dx}{dt} x + G \left( \frac{1 - e_p - e_q - l}{x} \right)^{1 - \alpha}
\]

\[-\rho - \frac{B(H)}{\alpha} \left( 1 - e_q - l \right),
\]

(4.21)
\[
\frac{de_q}{dt} = \frac{(1 - e_p - eq - l) \left[ -\frac{de_p}{dx} + G \left( \frac{1-e_p-e_q-l}{x} \right)^{1-\alpha} - \frac{\rho}{\alpha} \right]}{1 + \frac{de_p}{de_q}} \]
\[
- \frac{(1 - e_p - eq - l) \left[ \frac{B(H)}{\alpha}(1 - e_q - l) - \frac{de_p}{dx} \frac{dx}{dt} \right]}{1 + \frac{de_p}{de_q}},
\]
(4.22)

Since \( e_q, x \) and \( l \) depend on time, we use the chain rule to calculate \( \frac{de_p}{dt} \).
\[
\frac{de_p}{dt} = \frac{de_p}{de_q} \frac{de_q}{dt} + \frac{de_p}{dx} \frac{dx}{dt} + \frac{de_p}{dl} \frac{dl}{dt},
\]
(4.23)

where \( \frac{de_p}{dt} = -1 \) and \( \frac{de_p}{de_q} \) and \( \frac{de_p}{dx} \) are as follows:
\[
\frac{de_p}{de_q} = -1 + \frac{\alpha - 1}{\alpha x \phi \left( \frac{a}{A_0} - 1 \right) Q \gamma \left( \frac{Q}{Q_0} - 1 \right)} \left[ \frac{dT}{de_q} B - \frac{dB}{de_q} T \right],
\]
(4.24)

\[
T = \left[ 1 + \left( \frac{a}{A_0} - 1 \right) e^{-\phi[x]} \right] \left[ 1 + \left( \frac{Q}{Q_0} - 1 \right) e^{-\gamma[e_q] - \eta[x]} \right]^2,
\]

\[
\frac{dT}{de_q} = - \left( \frac{a}{A_0} - 1 \right) e^{-\phi[x]} \phi x \frac{dq}{de_q} \left[ 1 + \left( \frac{Q}{Q_0} - 1 \right) e^{-\gamma[e_q] - \eta[x]} \right]^2
\]
\[
- 2\gamma \left[ 1 + \left( \frac{Q}{Q_0} - 1 \right) e^{-\gamma[e_q] - \eta[x]} \right] \left( \frac{Q}{Q_0} - 1 \right) e^{-\gamma[e_q] - \eta[x]}
\]
\[
* \left[ 1 + \left( \frac{a}{A_0} - 1 \right) e^{-\phi[x]} \right],
\]

\[
B = e^{-\phi[x]} e^{-\gamma[e_q] - \eta[x]},
\]

where \( B \) is with respect to \( e_q \) and is replaced in equation (4.24) and
\[
\frac{dB}{de_q} = -\gamma e^{-\phi[x]} - \phi x \frac{dq}{de_q} e^{-\gamma[e_q] - \eta[x]},
\]
\[
\frac{de_p}{dx} = \frac{\alpha - 1}{\alpha \phi \left( \frac{a}{A_0} - 1 \right) Q \gamma \left( \frac{Q}{Q_0} - 1 \right)} \left[ \frac{dT}{dx} B - \frac{dB}{dx} T \right],
\]
(4.25)
\[
\frac{dT}{dx} = -\left(\frac{a}{A_0} - 1\right) e^{-\phi[x]} \left( x \frac{dq}{dx} + q \right) \left[ 1 + \left( \frac{Q}{Q_0} - 1 \right) e^{-\gamma[e_q]-\eta[x]} \right]^2 \\
- 2\eta \left[ 1 + \left( \frac{Q}{Q_0} - 1 \right) e^{-\gamma[e_q]-\eta[x]} \right] \left( \frac{Q}{Q_0} - 1 \right) e^{-\gamma[e_q]-\eta[x]} \\
* \left[ 1 + \left( \frac{a}{A_0} - 1 \right) e^{-\phi[x]} \right],
\]

\[B = xe^{-\phi[x]-\gamma[e_q]-\eta[x]},\]

where \(B\) is with respect to \(x\) and is replaced in equation (4.25) and finally

\[
\frac{dB}{dx} = e^{-\phi[qx]-\gamma[e_q]-\eta[x]} \left[ 1 - x \left( \eta + \phi \left( \frac{dq}{dx}x + q \right) \right) \right].
\]

We still need \(\frac{dl}{dt}\), which requires the use of the chain rule on (4.18) because of the time dependent nature of \(l, x, e_p\) and \(e_q\) as follows:

\[
\frac{dv}{dt} = \frac{dv}{dl} \frac{dl}{dt} + \frac{dv}{dx} \frac{dx}{dt} + \frac{dv}{de_p} \frac{de_p}{dt} + \frac{dv}{de_q} \frac{de_q}{dt}.
\]

Rearranging and making the substitution for \(\frac{de_p}{dt}\) to solve for \(\frac{dl}{dt}\) we obtain

\[
\frac{dl}{dt} = \frac{dv}{dt} - \frac{dv}{dx} \frac{dx}{dt} - \frac{dv}{de_q} \frac{de_q}{dt} - \frac{dv}{de_p} \frac{de_q}{dt} - \frac{dv}{de_p} \frac{dx}{dt},
\]

where

\[
\frac{dv}{dl} = \frac{\beta G(1 - \alpha)x^{\alpha - 1}}{1 - \beta} \left[ 1 + l\alpha(1 - e_p - e_q - l)^{-1} \right],
\]

\[
\frac{dv}{dx} = \frac{\beta G(1 - \alpha)(\alpha - 1)x^{\alpha}}{(1 - \beta)(1 - e_p - e_q - l)^{\alpha+1}},
\]

\[
\frac{dv}{de_p} = \frac{dv}{de_q} = \frac{\beta G(1 - \alpha)l^{\alpha-1}x^\alpha}{(1 - \beta)(1 - e_p - e_q - l)^{\alpha+1}}.
\]

Notice that \(\frac{dv}{de_p}\) and \(\frac{dv}{de_q}\) are equivalent.
4.3 Results of the Transformed Model

In Chapter 2 we assumed a starting technology and life expectancy level when interpreting equations (2.2) and (2.4). After changing variables, it is appropriate to assume that $Q_0 = 35$, however $A_0$ needs to be much smaller, because we want $A$ to have a similar curve and initial position on the curve for the transformed model. We compare Figure 4.1 with Figure 2.1 to see that adjustments were made with our values of $A_0$ and $\phi$ so that the two figures will have curves that start at the same initial position. Our goal is to use the fixed points of $c, k$ and $h$ to find similar fixed points in $x$ and $v$.

![Figure 4.1: $a = 0.075$ and $A_0 = 0.1404 \times 10^{-3}$.](image)

Figure 4.1 shows the new value of $A_0 = 0.1404 \times 10^{-3}$, and the approach to the highest level of technology as $q$ increases. Figure 4.2 shows the approach to the
highest life expectancy as $e_q$ increases. For a country to reach a life expectancy of $Q$, the time spent in health education, $e_q$, has to approach 1. It is unlikely that a country would spend its entire work day learning health education. Therefore, it is not likely that a country’s life expectancy will reach 85, according to this model.

![Figure 4.2: $Q = 85$ and $Q_0 = 35$.](image)

Again, using both Maple and equations (4.19), (4.22) and (4.27), we find $x, e_q$ and $l$. Using these values and equations (4.17) and (4.18), we are able to find $e_p$ and $v$. These fixed points are dynamic in $h, k$ and $c$. We use equations (4.4) and (4.6) to find $A$ and $q$. The parameters used to find the fixed points in the transformed model are: $\delta = 0.08, \rho = 0.05, G = 5, Z = 10, D = 1, \theta = 0.72, B(H) = 0.0876, a = 0.075, A_0 = 0.1404 \times 10^{-3}, Q = 85, Q_0 = 35, \sigma = 0.4, \gamma = 4, \phi = 0.00932$ and $\eta = 0.02$. Since the size of $\beta$ determines the amount of time spent in leisure, simulations were
run varying $\beta$ to determine how the fixed points change. Each of the fixed points and the additional variables $u$, time spent in production, $q$, life expectancy, and $A$, technology, are plotted with respect to $\beta$, so the results can be seen more clearly.

![Graph showing fixed points and sums versus $\beta$]

Figure 4.3: The fixed points $e_p, e_q$ and $l$ and the sum of the three versus $\beta$ with two values of $\gamma$ where $Q_0 = 35, \phi = 0.00932$ and $\eta = 0.02$.

As $\beta$ tends to 1, leisure approaches zero, which is seen in Figure 4.3. On the other hand, $e_p$ and $e_q$ increase as $\beta$ increases. As the amount of time in leisure decreases, the amount of time spent in the other areas of the work day increase. Varying $\gamma$ shows how the effectiveness of health education changes the fixed points. The trends of the fixed points are similar for $\gamma = 3$ and $\gamma = 4$. The time spent in production education, $e_p$, and the time spent in health education, $e_q$, still increase and the amount of time in leisure, $l$, still decreases. There is not much of a change in the graphs of leisure with the varying $\gamma$, indicating the effectiveness of health
education does not play a role in leisure time. However, the effectiveness of health education is important for $e_p$ and $e_q$. Notice that when $\gamma = 3$, the graph for $e_p$ is higher than when $\gamma = 4$. If a country is not effectively teaching health education, the individuals will seek education elsewhere. On the other hand, the values of $e_q$ for $\gamma = 4$ is greater than that for $\gamma = 3$, because a country would spend more time in health education if it is being taught properly. The sum of the three variables used in Figure 4.3 is plotted versus $\beta$ also to show that the increase in $e_p$ and $e_q$ is greater than the decrease in $l$, hence the graph increases as $\beta$ tends to 1.

![Graph showing time spent in production, $u$, versus $\beta$ with two values of $\gamma$.](image)

Figure 4.4: Time spent in production, $u$, versus $\beta$ with two values of $\gamma$ where $Q_0 = 35$, $\phi = 0.00932$ and $\eta = 0.02$.

Looking at Figure 4.4, as $\beta$ tends to 1, the time spent in production, $u$, decreases. Recall that, as $\beta$ increases, the amount of time spent in leisure decreases.
This indicates that as leisure time increases slightly, the production $u$, increases. The result is similar for both $\gamma = 3$ and $\gamma = 4$. In this case, a larger $\gamma$ increases $u$, indicating more effective teaching leaves more time for production.

![Graph showing life expectancy $q$ versus $\beta$ with two values of $\gamma$ where $Q_0 = 35$, $\phi = 0.00932$ and $\eta = 0.02$.](image)

Figure 4.5: Life expectancy, $q$, versus $\beta$ with two values of $\gamma$ where $Q_0 = 35$, $\phi = 0.00932$ and $\eta = 0.02$.

Although productivity increases as leisure time increases, the result is not the same for life expectancy $q$. Figure 4.5 shows that $q$ increases as $\beta$ increases. Therefore, the less time spent in leisure, the longer the life expectancy. One may be more productive throughout his/her life with having more time for leisure, but that person’s life expectancy decreases as a result. Notice that more effectively teaching about health ($\gamma$ increasing), increases life expectancy.
As shown in Figure 4.6, the technology level for a country decreases as $\beta$ tends to 1. This indicates that as leisure decreases, a country’s technology level decreases also. More effectively teaching health also increases the level of technology.

![Figure 4.6: Technology, $A$, versus $\beta$ with two values of $\gamma$ where $Q_0 = 35$, $\phi = 0.00932$ and $\eta = 0.02$.](image)

Recall that $v = \frac{c}{k}$, which is consumption per level of capital. In Figure 4.7, this ratio increases with the increase in $\beta$. With less time available for leisure, $v$ increases. On the other hand, Figure 4.8 shows that $x$ decreases as $\beta$ increases. Also recall that $x = \frac{k}{h}$, so that as leisure decreases so does the ratio of capital per human skill level.
Figure 4.7: Ratio of consumption per capital, $v$, versus $\beta$ with two values of $\gamma$ where $Q_0 = 35, \phi = 0.00932$ and $\eta = 0.02$.

A stability analysis for the fixed points is important to determine if a country will reach a better state of living. The parameters used for the stability analysis are the same as are used when finding the fixed points, which are: $\delta = 0.08, \rho = 0.05, G = 5, Z = 10, D = 1, \theta = 0.72, B(H) = 0.0876, a = 0.075, A_0 = 0.1404 \times 10^3, Q = 85, Q_0 = 35, \sigma = 0.4, \gamma = 4, \phi = 0.00932$ and $\eta = 0.02$. 
Table 4.1 shows there are two positive eigenvalues and one negative eigenvalue for each $\beta$ used. The two positive eigenvalues indicate instability and change in the state of an economy. This change is for the better and is improving the state of the economy. Negative eigenvalues, however, indicate a trap. The country’s economy is trapped at a certain state, although it is at a better place than before, perhaps with improved capital or technology. If a country happens to get trapped, changing $\beta$ will improve the economic state at which the country is trapped. Notice that as $\beta$ increases, there is not a significant change with the positive eigenvalues, but the negative eigenvalue decreases. The same analysis was performed with $\gamma = 3$ to see if there was significant change in the eigenvalues. For $\gamma = 3$, the positive eigenvalues did not have a significant change when varying $\beta$, but the negative eigenvalue
decreased as $\beta$ increased. The positive eigenvalues for $\gamma = 3$ and $\gamma = 4$ did not show significant change, while the negative eigenvalues were smaller for the smaller $\gamma$. The eigenvectors corresponding to each eigenvalue are also given showing that for the positive eigenvalues, the eigenvectors point in the $x$ direction in the $(e_q, x, l)$ coordinate system.

Table 4.1: Eigenvalues and Eigenvectors for the Transformed Model

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>Eigenvalues</th>
<th>Eigenvectors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-0.5288</td>
<td>$\langle 0.0948, -3.1736, -0.0044 \rangle$</td>
</tr>
<tr>
<td>$\beta = 0.95$</td>
<td>0.5164</td>
<td>$\langle -0.9432, 41.7519, 0.0303 \rangle$</td>
</tr>
<tr>
<td></td>
<td>0.1159</td>
<td>$\langle -0.7551, 34.0059, 0.1270 \rangle$</td>
</tr>
<tr>
<td></td>
<td>-0.5408</td>
<td>$\langle 0.9543, -29.8633, -0.0080 \rangle$</td>
</tr>
<tr>
<td>$\beta = 0.99$</td>
<td>0.5188</td>
<td>$\langle -8.4528, 358.7516, 0.0491 \rangle$</td>
</tr>
<tr>
<td></td>
<td>0.1171</td>
<td>$\langle -4.1906, 182.4620, 0.1345 \rangle$</td>
</tr>
</tbody>
</table>

Figures 4.9-4.11 are direct numerical solutions of equations (4.17)-(4.19), (4.22) and (4.27). Using the linear stability analysis, a small factor, of order $10^{-12}$, is added to the fixed point to determine an initial condition, which is as close as possible to stability. Adding such a small factor does not drastically alter the fixed point and looking at Figures 4.9-4.11, when time is equal to zero, we can see that the initial
condition is close to the value of the fixed point. These figures demonstrate an eco-
nomic trap and that the curve is in the direction of a stable path. This indicates that
the economy can stay stagnant for a long time, without any improvement. The path
stays close to the fixed point for at least 40 years before it shoots away. Also looking
at Figure 4.9 we can see that the time spent in health education goes to zero over
the course of life expectancy. This indicates that learning about health education is
helpful for only a certain amount of years, after which time it is no longer beneficial.

![Graph](image)

Figure 4.9: Time spent in health education, $e_q$, versus time, in years to see if the
stable path can move away from the fixed point.
Figure 4.10: Time spent in leisure, $l$, versus time, in years to see if the stable path can move away from the fixed point.

Figure 4.11: Ratio of capital per human skill level, $x$, versus time, in years to see if the stable path can move away from the fixed point.
CHAPTER V
CONCLUSIONS

What can be done to help developing countries grow? Developing countries are stagnant in a bad economic state, and the goal is to discover if there is anything that can help these countries reach a better one. Various models were studied to find the best fit one for a developing country. We started with the models of Jones and Romer, but their models reflect a developed country more than a developing country. These two led into Zilibotti and Kejak, whose models better reflect a developing country.

Tucker added life expectancy into his model, which incorporated health education in the work day. He found that adding health education does increase life expectancy and thus does improve the economic state of a developing country. A country can still be trapped, but will be trapped at a better place. In addition to health education, we incorporate leisure and try to determine if this modification helps an economy reach a higher economic state.

After solving the original variable model and doing a stability analysis, we deduced that the country can get trapped. Studying the transformed model was more promising. Although a country can still be trapped, the trap can occur at a better economic state. It is not determined whether or not a developing country actually grows. It would be necessary to study per-capita income of a country to determine
economic growth. For the purpose of this model, it has been shown that incorporating health education and leisure improves the economic state of a developing country.

Further research is necessary to get a clearer understanding of economic growth for a developing country. Including a stress factor into the life expectancy equation would be a good idea for continuing this research. Another good idea to continue this research would be to break ones life into stages to see when someone is most productive and how that affects the economic state of a country. Using a model that incorporates per-capita income would provide an answer to the question of what can help developing countries grow, which is also a good idea for future research.


APPENDICES
APPENDIX A

RECOVERING TUCKER

Fabrizio Zilibotti’s and Michal Kejak’s models played an important role in defining this model. Their models can be recovered by substituting a 1 or 0 for certain variables, which can be seen in the Appendix of Tucker [2]. For the purpose of this model, Tucker’s model will be recovered. Showing that his model can be recovered indicates that Zilibotti and Kejak will also be recovered when following the steps Tucker uses.

In order to recover Tucker’s equations, set $\beta = 1$ and $l = 0$. $l$ directly relates to leisure and $\beta$ comes from the utility function that incorporates leisure into the model. Therefore, a good place to start is with the maximization problem, which comes from the utility function. In (2.8), setting $\beta = 1$ and $l = 0$, will yield Tucker’s maximization problem, which is

$$\max_{c,k,h,u,e_p,e_q} \left( \lim_{Q \to \infty} \int_0^Q \frac{c^{1-\sigma} - \sigma e^{-\delta t}}{1 - \sigma} dt \right).$$

(A.1)

To recover Tucker’s formula for the time allotment scheme for a country or economy in a given work day, it is necessary to set $l$ equal to zero in (2.7),

$$1 = u + e_p + e_q.$$ 

(A.2)
Looking at the Hamiltonian operator, (2.13), with $\beta = 1$ and $l = 0$ we have:

$$HAM = \left[ \frac{c^{1-\sigma}}{1-\sigma} e^{-\delta t} \right] + \lambda [(kr + k\rho - k\rho + w(1 - e_p - e_q)h - c]$$

$$+ \mu [B(H)e_p h]. \quad (A.3)$$

Again, for completeness, the remaining equations will be manipulated to recover Tucker’s equations. Starting with (2.14), set the parameter, $\beta$, equal to one, to find

$$HAM_c = 0 = c^{-\sigma} e^{-\delta t} - \lambda. \quad (A.4)$$

Equations (2.16),(2.17),(2.19) and (2.20), match Tucker’s equations exactly. Also, (2.21), the derivative with respect to $l$, would not be in this model if setting $l = 0$, removing one equation from the system. Therefore, this model would not have an additional equation which Tucker does not have. To finish recovering Tucker’s model, in (2.15) and (2.18) we set $l = 0$ to find

$$HAM_\lambda = \frac{dk}{dt} = (kr + k\rho) - k\rho + w(1 - e_p - e_q)h - c, \quad (A.5)$$

$$HAM_h = \frac{-d\mu}{dt} = \lambda w(1 - e_p - e_q) + \mu B(H)e_p. \quad (A.6)$$

Making these adjustments recovers Tucker’s model completely.
The Hamiltonian and its governing equations are presented in Chapter 2, but the derivations of such equations were not provided. In this section, the derivations of the governing equations will be shown.

B.1 Consumption Governing Equation

Use (2.14) to solve for $\lambda$ and find

$$[c^\beta l^{1-\beta}]^{-\sigma} \beta l^{1-\beta} e^{-\delta t} = \lambda.$$  \hfill (B.1)

This can be substituted into (2.17) to obtain

$$-\beta [[(l^{1-\beta} c^{\beta-1} e^{-\delta t}) (-\sigma(c^\beta l^{1-\beta})^{-\sigma-1})(c^\beta (1 - \beta) l^{1-\beta} \frac{dl}{dt} + l^{1-\beta} \beta c^{\beta-1} \frac{dc}{dt})$$

$$+[(e^\beta l^{1-\beta})^{-\sigma} (c^{\beta-1} e^{-\delta t} (1 - \beta) l^{-\beta} \frac{dl}{dt} + l^{1-\beta} e^{-\delta t} (\beta - 1) c^\beta \frac{dc}{dt}$$

$$+l^{1-\beta} e^{-\delta t} - \delta)]] = [c^\beta l^{1-\beta}]^{-\sigma} \beta l^{1-\beta} c^{\beta-1} e^{-\delta t} [DA + Z A^\theta k^{\theta-1} \theta - \rho],$$  \hfill (B.2)

which simplifies to

$$\frac{l}{c} \sigma c^{-1} \dot{c} + \frac{l}{c} \delta = \frac{l}{c} [DA + Z A^\theta k^{\theta-1} \theta - \rho].$$  \hfill (B.3)

Finally, after dividing through by $\frac{l}{c}$, one can manipulate this to obtain (2.24):

$$\frac{\dot{c}}{c} = DA + Z A^\theta k^{\theta-1} \theta - \rho - \delta.$$  \hfill (B.4)
B.2 Capital Governing Equation

Before showing the derivation of the Capital Governing Equation (2.26), we explain the derivation of (2.25). Using (2.16), a simple division will yield (2.25). However, the derivation for (2.26) is not so straightforward. In order to solve for this, a one sector production firm in the economy will be assumed through a Cobb-Douglas production function. As mentioned earlier, the Cobb-Douglas production function is:

\[
F(k, L) = Gk^\alpha L^{1-\alpha},
\]

with profit,

\[
\Pi = F(k, L) - wL - [DAk + Z(Ak)^\theta].
\]

In order to maximize profits, the following condition must be met:

\[
\frac{d\Pi}{dL} = 0 = F_L - w.
\]

For completeness, the derivative with capital is also taken, which is

\[
\frac{d\Pi}{dk} = 0 = F_k - DA - ZA^\theta k^{\theta-1}\theta.
\]

This maximization adds another constraint on \(e_p, e_q, h, A, k, D\) and \(Z\).

Now that the background has been set, the solution for the Capital Governing Equation can be shown. First, we divide (2.15) through by \(k\), keeping in mind that \(k = K\), to yield

\[
\frac{\dot{k}}{k} = DA + ZA^\theta k^{\theta-1} - \rho + \frac{w(1 - e_q - e_p - l)h}{k} - \frac{c}{k}.
\]
At this time, we introduce the labor equality:

\[ L = uh = (1 - e_q - e_p - l)h. \]  \hspace{1cm} (B.10)

Therefore, (B.9) can be rewritten as

\[
\frac{\dot{k}}{k} = DA + Z A^\theta k^{\theta-1} - \rho + \frac{wL}{k} - \frac{c}{k}. \hspace{1cm} (B.11)
\]

We solve for \( w \) using (B.7) to yield

\[ w = F_L = Gk^\alpha L^{-\alpha} (1 - \alpha). \]  \hspace{1cm} (B.12)

Now, (B.11) above can be rewritten as

\[
\frac{\dot{k}}{k} = DA + Z A^\theta k^{\theta-1} - \rho + \frac{Gk^\alpha L^{-\alpha} (1 - \alpha)L}{k} - \frac{c}{k}. \hspace{1cm} (B.13)
\]

The final governing equation is

\[
\frac{\dot{k}}{k} = DA + Z A^\theta k^{\theta-1} + Gk^{\alpha-1} L^{1-\alpha} (1 - \alpha) - \rho - \frac{c}{k}. \hspace{1cm} (B.14)
\]

**B.3 Health Education Governing Equation**

Equation (2.20), with a simple division by \( \lambda \), can be written as

\[ wh = \left[ Dk \frac{dA}{dq} \frac{dq}{de_q} + Z k^\theta \theta A^{\theta - 1} \frac{dA}{dq} \frac{dq}{de_q} \right]. \hspace{1cm} (B.15) \]

From these two, we looked at models that include leisure. Notice each term on the right side has a common factor, so that

\[ wh = \frac{dA}{dq} \frac{dq}{de_q} \left[ Dk + Z k^\theta \theta A^{\theta - 1} \right]. \hspace{1cm} (B.16) \]
This equation can be reduced to

\[ wh = \frac{d(DA_k + Z(Ak)^6)}{de_q}. \]  

(B.17)

It is seen through this equation that one skilled work hour is equal to the marginal effect of health education on production.

B.4 Production Education Governing Equation

The solution for this equation is a little more involved and tedious than the previous equations, and starts with (2.19) by rewriting it as

\[ \mu B(H) = \lambda w. \]  

(B.18)

This is substituted into (2.18) to obtain

\[ \frac{-d\mu}{dt} = \mu B(H)(1 - e_p - e_q - l) + \mu B(H) e_p. \]  

(B.19)

This can be simplified to

\[ \frac{-d\mu}{dt} = \mu B(H)(1 - e_q - l). \]  

(B.20)

Starting with (2.19), dividing throughout by \( h \), and taking the derivative with respect to time yields

\[ \frac{d\lambda}{dt} w + \frac{dw}{dt} \lambda + \frac{-d\mu}{dt} B(H) - B'(H) \frac{dh}{dt} \mu = 0. \]  

(B.21)

Using (B.19) and (2.16) and solving for \( \mu \) we find that \( \mu = \frac{\lambda w}{B(H)} \), and

\[ \frac{d\lambda}{dt} w + \frac{dw}{dt} \lambda + \mu B(H)(1 - e_q - l) B(H) - B'(H) B(H) e_p h \frac{\lambda w}{B(H)} = 0. \]  

(B.22)
Using the above equation, we simplify and divide by $\lambda w$ to obtain

$$
\frac{d\lambda}{\lambda} + \frac{dw}{w} + (1 - e_q - l)B(H) - B'(H)e_p h = 0. \quad (B.23)
$$

We can use (2.17) to make further simplifications as follows,

$$
\frac{dw}{dt} = [DA + Z A^\theta k^{\theta - 1} - \rho] + (1 - e_q - l)B(H) - B'(H)e_p h = 0. \quad (B.24)
$$

We next solve for $\frac{dw}{dt}$ by taking a derivative with respect to time in (B.12), which yields

$$
\frac{dw}{dt} = G(1 - \alpha)(\alpha k^{-1} \dot{k} L^{-\alpha} - \alpha L^{-1} \dot{L} k^\alpha). \quad (B.25)
$$

Divide through by $w$,

$$
\frac{dw}{dt} = \frac{G(1 - \alpha)k^\alpha L^{-\alpha}(\alpha k^{-1} \dot{k} - \alpha L^{-1} \dot{L})}{G(1 - \alpha)k^\alpha L^{-\alpha}}. \quad (B.26)
$$

This can be simplified further as

$$
\frac{dw}{dt} = \alpha \frac{\dot{k}}{k} - \alpha \frac{\dot{L}}{L}, \quad (B.27)
$$

where $\cdot = \frac{d}{dt}$.

We know $\frac{\dot{k}}{k}$ which is given (2.26), but we have to solve for $\frac{\dot{L}}{L}$ using (B.9):

$$
\frac{\dot{L}}{L} = \frac{\frac{d}{dt}(1 - e_p - e_q - l)h}{(1 - e_p - e_q - l)h} = \frac{\frac{d}{dt}(1 - e_p - e_q - l) - (\frac{de_p}{dt} + \frac{de_q}{dt} + \frac{dl}{dt})h}{(1 - e_p - e_q - l)h}. \quad (B.28)
$$
This equation is substituted back into the above equations and simplifications are made to yield (2.28)

\[
\frac{de_p}{dt} + \frac{de_q}{dt} + \frac{dl}{dt} + \frac{1}{1 - e_p - e_q - l} \left( \hat{h} - \frac{\hat{k}}{k} + \frac{1}{\alpha} \left[ DA + ZA^\theta k^\theta \rho - \rho \right] - \frac{1}{\alpha} (1 - e_q - l) B(H) + \frac{1}{\alpha} \frac{dB}{dH} e_p h \right).
\]

(B.29)

B.5 Leisure Time Governing Equation

The final governing equation, dealing with leisure, is solved in a straightforward fashion. Using (2.21), we rewrite as follows:

\[
c^{\beta(1 - \sigma)}(1 - \beta) l^{(1 - \beta)(1 - \sigma) - 1} e^{-\delta t} = \lambda wh.
\]

(B.30)

Substituting the known value of \( \lambda \), using (B.1), we obtain

\[
[c^{\beta l_{1 - \beta} - \sigma} l^{1 - \beta} e^{-\delta t} wh] = c^{\beta(1 - \sigma)}(1 - \beta) l^{(1 - \beta)(1 - \sigma) - 1} e^{-\delta t}.
\]

(B.31)

This equation can be simplified to

\[
\frac{1}{c} l_{1 - \beta} wh = 1 - \beta.
\]

(B.32)

Solving for \( l \) yields (2.29),

\[
l = \frac{1 - \beta}{\beta} \frac{c}{wh}.
\]

(B.33)
B.6 Transforming the Governing Equations

The derivations for finding the governing equations from Chapter 4 are shown here in more detail. To continue the detail of the solution procedure, the time derivatives for $x$ and $v$ are found, and are substituted into the original governing equations

$$\frac{dx}{dt} = \frac{dk}{dt} h - \frac{dh}{dt} k h^2,$$  \hfill (B.34)

$$\frac{dv}{dt} = \frac{dc}{dt} k - \frac{dk}{dt} c k^2.$$

Now, the derivatives and growth rates, equations (4.11) and (4.12), are as follows:

$$\frac{dx}{dt} = \frac{dk}{dt} h - \frac{dh}{dt} k h^2 k = \frac{dk}{dt} - \frac{dh}{dt} h,$$  \hfill (B.36)

$$\frac{dv}{dt} = \frac{dc}{dt} k - \frac{dk}{dt} c k^2 c = \frac{dc}{dt} - \frac{dk}{dt} k.$$  \hfill (B.37)

We have expressions for $\frac{dk}{dt}, \frac{dh}{dt}$ and $\frac{dc}{dt}$, which are substituted into the equations above to solve for $x$ and $v$.

We have

$$\frac{dx}{dt} = DA + ZA^\theta k^{\theta-1} + G k^{\alpha-1} L^{1-\alpha}(1 - \alpha)$$

$$- \rho - \frac{c}{k} - B(H)e_p.$$  \hfill (B.38)

The $\frac{c}{k}$ term can be transformed into $v$ immediately, but the term with $L$ will require algebra to transform. Recall that $L = uh = (1 - e_p - e_q - l)h$. We replace this with the $L$ in the equation above and obtain

$$\frac{dx}{dt} = DA + ZA^\theta k^{\theta-1} + G k^{\alpha-1} ((1 - e_p - e_q - l)h)^{1-\alpha}(1 - \alpha)$$

$$- \rho - v - B(H)e_p.$$  \hfill (B.39)
After manipulating this equation, we obtain $x^{-(1-\alpha)}$, from which

$$\frac{dx}{dt} = DA + ZA^\theta k^{\theta-1} + G \left( \frac{1 - e_p - e_q - l}{x} \right)^{1-\alpha} (1 - \alpha)$$

$$-\rho - v - B(H) e_p.$$  \hfill (B.40)

Notice, that there is still a $k$ in this equation, which also needs a transformation. In order to transform $k$, recall equation (C.1), the profit maximization equation with respect to capital. Using this equation with other manipulations we solve for the $k$ in the $ZA^\theta k^{\theta-1}$ term as follows:

$$G\alpha k^{\alpha-1} L^{1-\alpha} - DA - ZA^\theta k^{\theta-1} \theta = 0,$$ \hfill (B.41)

and

$$G\alpha k^{\alpha-1} L^{1-\alpha} = DA + ZA^\theta k^{\theta-1} \theta.$$ \hfill (B.42)

Using the same substitution for $L$ as above we find:

$$G\alpha k^{\alpha-1} ((1 - e_p - e_q - l)h)^{1-\alpha} = DA + ZA^\theta k^{\theta-1} \theta,$$ \hfill (B.43)

and

$$\frac{1}{\theta} \left( G\alpha \left( \frac{1 - e_p - e_q - l}{x} \right)^{1-\alpha} - DA \right) = ZA^\theta k^{\theta-1}.$$ \hfill (B.44)

Now that we have an equivalent expression for our $ZA^\theta k^{\theta-1}$ term, we can plug this into our existing equation and simplify to get the differential equation for $x$, namely equation (4.13):

$$\frac{dx}{dt} = DA + \frac{1}{\theta} \left( G\alpha \left( \frac{1 - e_p - e_q - l}{x} \right)^{1-\alpha} - DA \right)$$

$$+ G \left( \frac{1 - e_p - e_q - l}{x} \right)^{1-\alpha} (1 - \alpha) - \rho - v - B(H) e_p.$$ \hfill (B.45)
\[
\frac{dx}{dt} = DA \left(1 - \frac{1}{\theta}\right) + G \left(1 - \frac{e_p - e_q - l}{x}\right)^{1-\alpha} \left(1 - \alpha + \frac{\alpha}{\theta}\right)
\]
\[-\rho - v - B(H)e_p. \quad (B.46)
\]

The solution procedure for \( v \) uses the same simplifications as we use for \( x \). Therefore, we will just show the simplified form yielding (4.14):

\[
\frac{dv}{dt} = \frac{DA + ZA^\theta k^{\theta-1} \theta - \rho - \delta}{\sigma}
\]
\[-DA - ZA^\theta k^{\theta-1} - Gk^{\alpha-1}L^{1-\alpha}(1 - \alpha) + \rho + \frac{c}{k}, \quad (B.47)
\]

thus

\[
\frac{dv}{dt} = G \left(1 - \frac{e_p - e_q - l}{x}\right)^{1-\alpha} \left(\frac{\alpha}{\sigma} - 1 + \alpha - \frac{\alpha}{\theta}\right)
\]
\[+v + \rho \left(1 - \frac{1}{\sigma}\right) - \frac{\delta}{\sigma} - DA \left(1 - \frac{1}{\theta}\right). \quad (B.48)
\]

Equation (2.28) is transformed in a similar fashion:

\[
\frac{de_p}{dt} + \frac{de_q}{dt} + \frac{dl}{dt} \frac{1}{1 - e_p - e_q - l} = \frac{\dot{h}}{h} - \frac{\dot{k}}{k} + \frac{1}{\alpha} \left[DA + ZA^\theta k^{\theta-1} \theta - \rho\right]
\]
\[-\frac{1}{\alpha}(1 - e_q - l)B(H) + \frac{1}{\alpha}\frac{dB}{dH} e_p h. \quad (B.49)
\]

To complete this transformation, use equation (B.43) and equation (4.11), and recall that \( B(H) \) is a constant, making its derivative equal to zero, which yields equation (4.15)

\[
\frac{de_p}{dt} + \frac{de_q}{dt} + \frac{dl}{dt} \frac{1}{1 - e_p - e_q - l} = -\frac{dx}{dt} \alpha + \frac{1}{\alpha} \left[G\alpha \left(1 - \frac{e_p - e_q - l}{x}\right)^{1-\alpha} - \rho\right]
\]
\[-\frac{1}{\alpha}(1 - e_q - l)B(H). \quad (B.50)
\]
The last equation, (4.16), is found by using the profit maximization equation with respect to $L$, $w = F_L = Gk^\alpha L^{-\alpha}(1 - \alpha)$, and making a multiplication and division by $k$. We use our equation for $l$, (2.29), as the starting point, as follows:

\[
 l = \frac{1 - \beta}{\beta} \frac{c}{wh},
\]  
\(\text{(B.51)}\)

\[
 l = \frac{1 - \beta}{\beta} \frac{ck}{khw},
\]  
\(\text{(B.52)}\)

and thus

\[
 l = \frac{1 - \beta}{\beta} \frac{1}{Gk^\alpha L^{-\alpha}(1 - \alpha)xv}.
\]  
\(\text{(B.53)}\)

Using $L = uh = (1 - e_p - e_q - l)h$ we get $x^{-\alpha}$ in the denominator. Bring this to the numerator and yield equation (4.16)

\[
 l = \frac{(1 - \beta)xv(\frac{1 - e_p - e_q - l}{x})^\alpha}{\beta G(1 - \alpha)}.
\]  
\(\text{(B.54)}\)
APPENDIX C

STEADY STATE DERIVATIONS

C.1 Original Model

From Chapter 3, we know that \( e_p = 0 \) and that \( e_q = 1 - l - \frac{\delta}{B(H)} \). The derivations for \( c \) and \( k \) are presented in this section and so is the derivation for the algebraic expression used to solve for \( h \).

As mentioned in Chapter 3, the Cobb-Douglas production function is used in the solution process for \( k \) and \( c \). Recall, the Cobb-Douglas function is \( F = Gk^\alpha L^{1-\alpha} \) and the profit function is \( \Pi = F(k, L) - wL - [DAk + Z(Ak)^\theta] \). We optimize the profit function with respect to capital

\[
\frac{\partial \Pi}{\partial k} = 0 = \frac{\partial F}{\partial k} - DA - ZA^\theta k^{\theta-1} \theta. \tag{C.1}
\]

Since this yields \( \frac{\partial F}{\partial k} = G\alpha k^{\alpha-1} L^{1-\alpha} \), we substitute in the above equation and obtain

\[
0 = G\alpha k^{\alpha-1} L^{1-\alpha} - DA - ZA^\theta k^{\theta-1} \theta, \tag{C.2}
\]

which can be rewritten as follows

\[
G\alpha k^{\alpha-1} L^{1-\alpha} = DA + ZA^\theta k^{\theta-1} \theta. \tag{C.3}
\]

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Also, as mentioned in Chapter 3, we use the fact that \( L = uh = (1 - e_p - e_q - l)h \) and from equation (3.9), we use \( DA + Z A^\theta k^{\theta-1} \theta = \rho + \delta \) to solve for \( k \) as follows:

\[
G^\alpha k^{\alpha-1}((1 - e_p - e_q - l)h)^{1-\alpha} = \rho + \delta, \tag{C.4}
\]

\[
k^{\alpha-1} = ((1 - e_p - e_q - l)h)^{\alpha-1} \frac{\rho + \delta}{G^\alpha}. \tag{C.5}
\]

and thus

\[
k = (1 - e_p - e_q - l)h \left( \frac{\rho + \delta}{G^\alpha} \right)^{\frac{1}{\alpha-1}}. \tag{C.6}
\]

Here we substitute the values of the steady-state solutions of \( e_p \) and \( e_q \), resulting in

\[
k = \left( 1 - \left( 1 - l - \frac{\delta}{B(H)} \right) - l \right) h \left( \frac{G^\alpha}{\rho + \delta} \right)^{\frac{1}{\alpha-1}}, \tag{C.7}
\]

\[
k = \frac{\delta}{B(H)} h \left( \frac{G^\alpha}{\rho + \delta} \right)^{\frac{1}{\alpha-1}}. \tag{C.8}
\]

Using this equation for \( k \) and rearranging equation (3.3), we can make a substitution and solve for \( c \). Again, we use \( L = uh = (1 - e_p - e_q - l)h \). Before making the substitution for \( k \), we solve for \( c \) in equation (3.3)

\[
0 = DA + Z A^\theta k^{\theta-1} + G k^{\alpha-1} L^{1-\alpha} (1 - \alpha) - \rho - \frac{c}{k}, \tag{C.9}
\]

\[
0 = DA k + Z(Ak)^\theta + Gk^\alpha L^{1-\alpha} (1 - \alpha) - \rho k - c, \tag{C.10}
\]

and thus

\[
c = DA k + Z(Ak)^\theta + Gk^\alpha L^{1-\alpha} (1 - \alpha) - \rho k. \tag{C.11}
\]
Now, we make the replacement for \(k\) and \(L\) in equation (C.11) as follows

\[
c = DA \frac{\delta}{B(H)} h \left( \frac{G\alpha}{\rho + \delta} \right)^{\frac{1}{1-\alpha}} + Z(Ak)^\theta
\]

\[
+ G \left( \frac{\delta}{B(H)} h \left( \frac{G\alpha}{\rho + \delta} \right)^{\frac{1}{1-\alpha}} \right)^\alpha \left( (1 - e_p - e_q - l)h \right)^{1-\alpha} (1 - \alpha)
\]

\[
- \rho \frac{\delta}{B(H)} h \left( \frac{G\alpha}{\rho + \delta} \right)^{\frac{1}{1-\alpha}}
\].

(C.12)

After some simplification and substituting the known values of \(e_p\) and \(e_q\), we get

\[
c = \frac{\delta}{B(H)} h \left( \frac{G\alpha}{\rho + \delta} \right)^{\frac{1}{1-\alpha}} (DA - \rho)
\]

\[
+ G \left( \frac{\delta}{B(H)} h \left( \frac{G\alpha}{\rho + \delta} \right)^{\frac{1}{1-\alpha}} \right)^\alpha \left( \frac{\delta}{B(H)} h \right)^{1-\alpha} (1 - \alpha) + Z(Ak)^\theta.
\]

(C.13)

After further simplification, we get our solution for \(c\)

\[
c = \frac{\delta}{B(H)} h \left[ \left( \frac{G\alpha}{\rho + \delta} \right)^{\frac{1}{1-\alpha}} (DA - \rho) + G \left( \frac{G\alpha}{\rho + \delta} \right)^{\frac{\alpha}{1-\alpha}} (1 - \alpha) \right]
\]

\[
+ Z(Ak)^\theta.
\]

(C.14)

Now we have \(c\) and \(k\) in terms of \(h\); therefore, we have to go through the solution procedure for the algebraic equation used to solve for \(h\). We will use the partial derivative with respect to \(L\) of the profit function \(w = Gk^\alpha L^{-\alpha}(1 - \alpha) = Gk^\alpha(h(1 - e_p - e_q - l))^{-\alpha}(1 - \alpha)\). Using equation (2.27) in the following way we find our algebraic solution for \(h\):

\[
\frac{d(DAk + Z(Ak)^\theta)}{de_q} = wh,
\]

(C.15)

\[
Dk \frac{dA}{dq} \frac{dq}{de_q} + Z(k)^\theta A^{\theta-1} \frac{dA}{dq} \frac{dq}{de_q} = wh,
\]

(C.16)
\[
\frac{dA}{dq} \frac{dq}{de} [Dk + Zk^\theta A^{\theta-1} \theta] = Gk^\alpha (h(1 - e_p - e_q - l)^{-\alpha}(1 - \alpha)h. \quad (C.17)
\]

Using the profit function from equation (C.2) and factoring out an \(\frac{A}{k}\) with other rearrangements, we obtain:

\[
G\alpha k^{\alpha-1} L^{1-\alpha} \alpha A \left[ Dk + ZA^{\theta-1} k^{\theta} \theta \right] = 0, \quad (C.18)
\]

and multiply by \(\frac{k}{A}\) to yield

\[
Dk + ZA^{\theta-1} k^{\theta} \theta = \frac{k}{A} G\alpha k^{\alpha-1} L^{1-\alpha}. \quad (C.19)
\]

We substitute this into equation (C.17) to continue solving for the solution in terms of \(h\):

\[
\frac{dA}{dq} \frac{dq}{de} \left[ \frac{k}{A} G\alpha k^{\alpha-1} L^{1-\alpha} \right] = Gk^\alpha (h(1 - e_p - e_q - l)^{-\alpha}(1 - \alpha)h, \quad (C.20)
\]

\[
\frac{dA}{dq} \frac{dq}{de} \left[ \frac{k}{A} G\alpha k^{\alpha-1} (h(1 - e_p - e_q - l)^{1-\alpha} \right] = Gk^\alpha (h(1 - e_p - e_q - l)^{-\alpha}(1 - \alpha)h, \quad (C.21)
\]

so that

\[
\frac{dA}{dq} \frac{dq}{de} = A^{1-\alpha} \frac{1}{\alpha} \frac{1}{1 - e_p - e_q - l}. \quad (C.22)
\]

Making the proper substitutions for \(e_p\) and \(e_q\) we get

\[
\frac{dA}{dq} \frac{dq}{de} = A^{1-\alpha} \frac{B(H)}{\alpha} \frac{1}{\delta}, \quad (C.23)
\]

which must be solved numerically to find \(h\).
C.2 Transformed Model

Equations (4.17)-(4.25) in Chapter 4 are used to solve for the five unknowns of the transformed model. The derivations of those equations are given in this section. Equations (4.19), (4.20) and (4.21) are taken directly from (4.13), (4.14) and (4.15), respectively. Also, the derivations for (4.24) and (4.25) are entirely presented in Chapter 4.

The first equation uses (4.10), and making the proper substitutions yields equation (4.17)

\[
\frac{dA}{dq} \frac{dq}{d\epsilon_q} = A \frac{1 - \alpha}{\alpha} \frac{1}{1 - e_p - e_q - l}.
\]  

(C.24)

Making the substitutions for \(\frac{dA}{dq}\) and \(\frac{dq}{d\epsilon_q}\) from equations (4.8) and (4.9), dividing by \(A\), and simplifying, one obtains

\[
\frac{a \phi x (\frac{a}{A_0} - 1) e^{-\phi [qx]} Q \gamma (\frac{Q}{Q_0} - 1) e^{-\gamma [e_q]} - \eta [x]}{[1 + (\frac{a}{A_0} - 1) e^{-\phi [qx]}][1 + (\frac{Q}{Q_0} - 1) e^{-\gamma [e_q]} - \eta [x]]^2} = \frac{1 - \alpha}{\alpha} \frac{1}{1 - e_p - e_q - l}.
\]  

(C.25)

To finish solving for \(e_p\), multiply both sides by \((1 - e_p - e_q - l)\) and move the necessary terms around to obtain

\[
e_p = 1 - e_q - l
\]  

\[- \frac{1 - \alpha}{\alpha} \frac{1 + (\frac{a}{A_0} - 1) e^{-\phi [qx]} [1 + (\frac{Q}{Q_0} - 1) e^{-\gamma [e_q]} - \eta [x]]^2}{\phi x (\frac{a}{A_0} - 1) e^{-\phi [qx]} Q \gamma (\frac{Q}{Q_0} - 1) e^{-\gamma [e_q]} - \eta [x]}.
\]  

(C.26)
Finding the equation for $v$ is more straightforward than the previous derivation and begins with equation (4.16)

$$l = \frac{(1-\beta)xv(1-e_p-e_q-l)x}{\beta G(1-\alpha)}. \quad (C.27)$$

We divide both sides by $v$ and $l$:

$$\frac{1}{v} = \frac{(1-\beta)x(1-e_p-e_q-l)x}{l\beta G(1-\alpha)}, \quad (C.28)$$

$$v = \frac{l\beta G(1-\alpha)x^{\alpha}}{(1-\beta)x(1-e_p-e_q-l)^{\alpha}}. \quad (C.29)$$

To find (4.22), we begin with (4.15)

$$\frac{de_q}{dt} + \frac{de_p}{dt} + \frac{dl}{dt} = -\frac{dx}{x} + \frac{1}{\alpha} \left[ G\alpha \left( \frac{1-e_p-e_q-l}{x} \right)^{1-\alpha} - \rho \right]$$

$$- \frac{1}{\alpha} (1-e_q-l) B(H). \quad (C.30)$$

We multiply both sides by $(1-e_p-e_q-l)$, subtract $\frac{de_p}{dt}$ and $\frac{dl}{dt}$ and substitute equation (4.23) to obtain

$$\frac{de_q}{dt} = (1-e_p-e_q-l)$$

$$\left[ -\frac{dx}{x} G\left( \frac{1-e_p-e_q-l}{x} \right)^{1-\alpha} - \frac{\rho}{\alpha} - \frac{B(H)}{\alpha} (1-e_q-l) \right]$$

$$- \frac{de_p}{dt} \frac{de_q}{dt} - \frac{de_p}{dx} \frac{dx}{dt} - \frac{de_p}{dl} \frac{dl}{dt} - \frac{dl}{dt}. \quad (C.31)$$
We continue to solve for $\frac{de_q}{dt}$

$$\frac{de_q}{dt} + \frac{de_p}{de_q} \frac{de_q}{dt} = (1 - e_p - e_q - l)$$

$$\left[ -\frac{dx}{dt} G \left( \frac{1 - e_p - e_q - l}{x} \right)^{1-\alpha} - \frac{H}{\alpha} - \frac{B(H)}{\alpha} (1 - e_q - l) \right]$$

$$= \frac{d\varepsilon_p}{dx} \frac{dx}{dt} + \frac{d\varepsilon_p}{dl} \frac{dl}{dt} - \frac{dl}{dt}.$$  \hfill (C.32)

Recalling that $\frac{de_p}{dt} = -1$, we make the substitution, continue to solve and get the equation for $\frac{de_q}{dt}$

$$\frac{de_q}{dt} = \frac{(1 - e_p - e_q - l) \left[ -\frac{dx}{dt} + G \left( \frac{1 - e_p - e_q - l}{x} \right)^{1-\alpha} - \frac{H}{\alpha} \right]}{1 + \frac{d\varepsilon_p}{de_q}}$$

$$= \frac{(1 - e_p - e_q - l) \left[ \frac{B(H)}{\alpha} (1 - e_q - l) - \frac{d\varepsilon_p}{dx} \frac{dx}{dt} \right]}{1 + \frac{d\varepsilon_p}{de_q}}.$$ \hfill (C.33)