SIMULATIONS OF NANOFIBER ANTENNA AND ITS APPLICATIONS

A Thesis

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Master of Science

Suraj Adhikari

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Metal-coated nanofibers act as an antenna in the presence of electromagnetic fields. Due to the physical structure of nanofibers (i.e. radii in the range of nanometers, and length in the range of millimeters) they can be treated as excellent linear antenna. The application of the nano-antenna is in the early stages of development. A model for an infinitely long, finite conductivity carbon nanotube as an antenna can be found in current literature. We consider finite length metal coated nanofibers which are relatively inexpensive to produce.

This is a theoretical work to examine the feasibility and difficulties to implement metal-coated nano-antennas. The analysis presented here includes an exact formulation (well known as Hallen’s equation) which includes an exact kernel but is singular. Due to the presence of a singularity, use of approximate kernel, which is easy to implement but less accurate, is generally preferred. This thesis accurately calculates current distribution along a nanowire antenna by the use of the exact kernel. While doing so, the singularities present in it are analytically removed.

Furthermore, simulations for current due to closely spaced nanowire antennas are performed. A large number of such fibers are thought to construct a nanofiber mat, on which current distribution can be calculated. The result is important for
the calculation of the power profile over the mat due to the electromagnetic effect in general.
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CHAPTER I
INTRODUCTION

1.1 Nanowires

Nanowires are wires having diameters in the range of nanometer and having unconstrained longitudinal size. We are considering wires with a few millimeters of length (≈ 1 – 2) and few nanometers diameter (≈ 10 – 20). Nanowires can carry different physical properties according to the way they are obtained. We are assuming an ideal nanowire with all the other physical properties (e.g. conductivity, number of electrons on the surface etc.) being similar to those of normal wire.

1.2 Nanowire Antennas

A family of antennas made up of thin metal rod, wire or tubes are called linear antennas; nanowires fall into this antenna family. Due to their shape, i.e. length $l$ being very large relative to its radius $a$, they make up a perfect linear antenna.

In nanowire antennas, electric charges are systematically accelerated using generators on the transmitting side, these particles, exert force in a remote conductor on the receiving side due to the electromagnetic coupling, which, produces current in that conductor. Figure 1.1 shows transmitting and receiving linear antennas.
The nanowire antennas that we are analyzing are considered to be cylindrical structures, hence cylindrical coordinates are used while talking about the near field of the antennas, however, while talking about the far field and calculating the field due to the antennas we will be considering spherical coordinates.

1.2.1 Antenna Excitation

As can be seen in figure 1.1, depending upon antenna excitation, it can be thought of as a ‘receiving’ or ‘transmitting’ antenna. A transmitting antenna is excited using a delta-gap model or frill generator model [1]. For a receiving antenna, an arbitrary excitation is assumed. We are using delta-gap model for the transmitting antenna.

When using the delta gap model, a constant voltage is assumed to excite the gap of the antenna. The voltage is assumed to be zero everywhere else. Also, the
incident electric field will be constant throughout the gap. This method of excitation is accurate if the gap width is small.

The case of different source excitations is depicted in figure 1.2

![Figure 1.2: Different Source Modeling](image)

A Matlab plot of electric field for magnetic frill is shown in figure 1.3.

We are using delta-gap excitation. Current distribution along a linear antenna due to magnetic frill generator excitation is studied by [2].

1.2.2 Applications of Nanowire Antenna

Nanowire is increasingly becoming popular in a number of different applications.

Nanowires are being used as building blocks for nanoelectronics, an evolutionary path taken by microelectronic circuits while following Moore’s Law [3]. They are also potential candidate for making terahertz antennas. [3], [4].
On the other hand, the electromagnetic coupling, or so called "antenna effect", between nanofibers and MRI waves resulting in burns is studied by [5]. A similar use of nanowires in treating carcinogenic cells by copulation has been studied by [6].

The analysis undertaken in this paper to model and quantify current distribution in linear antennas can provide better understanding in the above mentioned applications of nanowires.

1.3 Current Distribution in Nanowire Antennas

Traditionally, the current distribution on a linear dipole antenna is considered to be sinusoidal. This approximation is assumed valid in order to derive an approximate antenna radiation pattern. Sinusoidal current representation is generally done using equation 1.1:
\[ I_z(z) = I_m \sin \kappa \left( \frac{l}{2} - |z| \right) \]  \hspace{1cm} (1.1)

Equation 1.1, however, is valid only for an infinitesimally thin, center driven dipole. A dipole with \( a << h \) and \( ka << 1 \), is considered an electrically thin dipole \([7]\) and other methods are employed to calculate current distribution of such an antenna.

\([8]\) and \([9]\) have a more accurate three term approximation, also known as King’s three term approximation.

Later Pocklington, deduced an integral equation for currents in thin wire. The work was further extended by L.V. King, E. Hallen and R.W.P King \([10]\).

The two most general equations used to solve for current distribution on linear antennas are Hallen’s and Pocklington’s equations, named after the inventors. The equations are usually solved numerically for a finite length antenna. Current distribution for an infinite length antenna is analytically solved using the Fourier Transform method in \([11]\)

1.4 Hallen’s Equation

As mentioned before, calculation of current distribution on a linear antenna is generally done by solving either Hallen’s or Pocklington’s equation \([1]\), \([9]\). These equations establish a relationship between current distribution and the electric field excitation of an antenna. The equations are then solved numerically to get current distribution on the linear antenna.
Hallen’s and Pocklington’s equations, both, are equally effective while calculating current distribution on a linear antenna. Pocklington’s equation is considered to be more straightforward in implementation. However, in a few cases Pocklington kernel becomes ill behaved [9](pg 899).

This paper undertakes the calculation of current distribution using Hallen’s equation.

The diagrammatical representation for the derivation of Hallen’s equation is depicted in figure 1.4. In figure, current distribution over the antenna is $I_z$, vector potential is given by $A_z$ and incident electric field is given by $E_{in}$. This relationship diagram depicts the following:

1. The relationship between vector potential $A_z$, and current distribution $I_z$, is established by the wave equation.

2. The relationship between vector potential $A_z$, and incident electric field $E_{in}$, is established by choice of a certain gauge, Lorenz Gauge is used in our case.

3. The relationship between two variables of interest namely $I_z$ and $E_{in}$ is derived.

Hallen’s equation is of the form of the first kind Fredholm integral equation. For a delta gap excitation, in general form it is written as:

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} G(z-z')I(z')dz' = C_1 \cos \kappa z + V_0 \sin \kappa |z|$$  \hspace{1cm} (1.2)

In 1.2, $l$ is the length of the antenna, $\kappa = 2\pi/\lambda$ is the free space wave number. $I(z)$ is the current distribution over the antenna length.
Kernel $G$ is dyadic Green’s function. Depending upon our choice of kernel, $G$ can take two forms: exact and approximate.

With $a$ as the radius of the wire, and $-l/2 < z < l/2$ the reduced form is written as:

$$G_{\text{red}}(z - z') = \frac{e^{-jkR}}{R}, R = \sqrt{(z - z')^2 + a^2} \quad (1.3)$$

And the exact form as:

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-jkR}}{R} d\phi', R = \sqrt{(z - z')^2 + 2a^2 - 2a^2\cos\phi'} \quad (1.4)$$

We have chosen both kernels and compared the performance.

The approximate kernel however, has no analytic solution [12],[13].
Looking closely at 1.2, it can be seen that the right hand side is not differentiable at $z = 0$ due to the presence of $|z|$. The left hand side, in contrast, is differentiable at all $z$ for a choice of reduced kernel in place of $G$.

The choice of exact kernel in place of $G$ on the other hand maintains the singularity at $z = 0$.

1.4.1 Solution of Hallen’s Equation

We have used Method of Moment (MoM) [14] to solve 1.2 numerically. 1.2 with reduced kernel is easy to solve numerically, but the solution will not converge. Exact kernels require careful considerations on singularity and are highly converging.

While solving for current, 1.2 is solved with following two constraints.

1. The currents, at the two antenna ends, $I(-l/2)$ and $I(l/2)$ are both equal to zero

2. The currents on either section of the wire are symmetric in nature

Reduced kernel is widely used in the literature; however, it is tricky to find an optimum number of divisions for which the solution converges. Number of subdivisions should be chosen based on length to radius ratio. If we have an exact kernel solution, that solution should be taken into account while determining number of subdivisions. A priori determination of this value is not possible [15],(pg 460). It can be noticed from the numerical solutions that while using the approximate kernel the imaginary value of current $I$ oscillates rapidly near the driving point, however, the real part of current matches closely with that of the exact kernel.
To use Method of Moment, the wire is first divided into a small number of segments. The size of each segment is chosen such that the change in physical parameters like conductivity inside that segment is negligible. Electric field and current distribution over that segment is also assumed to be constant.

![Figure 1.5: Numerical Analysis of a Single Strand of Nano-fiber](image)

We consider $N = 2M + 1$, where $M$ is number of divisions at each half of the antenna. The current is now represented as the sum of basis functions. Different basis functions such as pulse, triangular, delta, sinusoidal, etc., can be chosen. The rate of convergence varies with the choice of basis functions. A detailed discussion on solutions with different basis functions is done in [9] (pg 874-890). We have chosen delta function as our basis function.

All the numerical method calculations required to calculate current distribution are done using Matlab.
1.5 Antenna Array

In its most natural form a nanofiber exists in the form of a mat, rather than as a single strand. This thesis extends the study of a single antenna to an antenna array of two elements. This study is encouraged by the fact that similar extension performed to model a large number of elements can approximate the "antenna effect" on the mat and heating due to it. A detailed study of the antenna array is done in [1],[9] and [15].

When two elements are placed close to each other, current distribution on each of them contribute to a resultant field at a certain point in space. These two elements are then considered to form an antenna array. The antenna array is widely used to control the directivity of antenna. It should be noted that the current distribution of an antenna on each element is not only due to the length, radius and driving voltage, like in a single antenna, but also on the distribution of field on other antennas of the array [15]. Figure 1.6 shows a general configuration of a two element array.
Figure 1.6: Two-Element Array
2.1 Derivation of Hallen’s Equation

As seen in the figure 2.1, derivation of Hallen’s equation is done in following three steps:

1. Helmholtz equation is derived, which relates vector potential $A_z$, with current $I_z$, flowing over the antenna. The antenna is assumed to be in $z$ axis as shown in figure 2.1

We start with Maxwell’s equation:

$$\nabla \cdot B = 0 \quad \text{(2.1a)}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad \text{(2.1b)}$$

These equations hints presence of a vector potential $A$ such that,

$$E = -\nabla \phi - \frac{\partial A}{\partial t} \quad \text{(2.2a)}$$

$$B = \nabla \times A \quad \text{(2.2b)}$$

Maxwell’s equation relating $B$ and $E$ is:
\[ \nabla \times B = \mu J + \mu \epsilon \frac{\partial E}{\partial t} \]  

(2.3)

Replacing \( B \) and \( E \) from 2.2 in 2.3

\[ \nabla \times (\nabla \times A) - \mu \epsilon \frac{\partial}{\partial t} \left( -\nabla \phi - \frac{\partial A}{\partial t} \right) = \mu J \]  

(2.4a)

\[ \nabla (\nabla \cdot A + \mu \epsilon \frac{\partial \phi}{\partial t}) - \nabla^2 A + \mu \epsilon \frac{\partial^2 A}{\partial t^2} = \mu J \]  

(2.4b)

We try to impose Lorenz gauge in order to simplify the equation and later to get uniformity in the representation: Lorenz gauge is:

\[ \nabla \cdot A + \mu \epsilon \frac{\partial \phi}{\partial t} = 0 \]  

(2.5)
Imposing Lorenz Gauge to 2.4b we get vector potential $\mathbf{A}$ in a wave equation as follows:

$$\mu \epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \mu \mathbf{J}$$  \hspace{1cm} (2.6)

This is in time domain, this can be represented by Helmholtz equation in frequency domain.

Let us assume sinusoidal time dependence of $\mathbf{A}$ and $\mathbf{J}$ such that,

$$\mathbf{A}(\mathbf{r},t) = \mathbf{A}(\mathbf{r})e^{j\omega t}$$ \hspace{1cm} (2.7a)

$$\mathbf{J}(\mathbf{r},t) = \mathbf{J}(\mathbf{r})e^{j\omega t}$$ \hspace{1cm} (2.7b)

Replacing 2.7 in 2.6

$$\nabla^2 \mathbf{A} + \omega^2 \mu \epsilon \mathbf{A} = -\mu \mathbf{J}$$ \hspace{1cm} (2.8)

Since $I$ is $\hat{z}$ directed $\mathbf{A}$ is also $\hat{z}$ directed. Similarly for a thin antenna all other components of $\mathbf{J}$ and $\mathbf{A}$ goes to 0.

2.8 can thus be written as:

$$\nabla^2 A_z + \omega^2 \mu \epsilon A_z = -\mu J_z$$ \hspace{1cm} (2.9)
In cylindrical co-ordinates 2.9 can be represented as:

\[
\frac{\partial^2 A_z}{\partial r^2} + \frac{1}{r} \frac{\partial A_z}{\partial r} + \frac{\partial^2 A_z}{\partial z^2} + \frac{1}{r^2} \frac{\partial^2 A_z}{\partial \phi^2} + \omega^2 \mu \varepsilon A_z = -\mu J_z
\]

(2.10)

Wave number \( \kappa \) is defined:

\[
\kappa^2 = \omega^2 \mu \varepsilon
\]

(2.11)

Putting 2.11 in 2.10 we get:

\[
\frac{\partial^2 A_z}{\partial r^2} + \frac{1}{r} \frac{\partial A_z}{\partial r} + \frac{\partial^2 A_z}{\partial z^2} + \frac{1}{r^2} \frac{\partial^2 A_z}{\partial \phi^2} + \kappa^2 A_z = -\mu J_z
\]

(2.12)

We can use Green’s function method to solve 2.11

Where \( G(r) \) is Green’s function for the Helmholtz equation:

\[
\nabla^2 G + k^2 G = -\delta^{(3)}(\mathbf{r})
\]

(2.13a)

\[
G(r) = \frac{e^{-jkr}}{4\pi r}
\]

(2.13b)

The solution of 2.12 is convolved form of 2.13b so vector potential \( A_z \) can be written as:

\[
A_z(r) = \int_V \frac{\mu J_z(r')e^{-jkr}}{4\pi R} dv'
\]

(2.14)
Where $R$ is:

$$R = |\mathbf{r} - \mathbf{r}'|$$  \hspace{1cm} (2.15)

Now we represent $J_z(r')$ in terms of $I(z')$

For a thin wire antenna the following must be true:

$$J_z(r') = \hat{z}I(z')\delta(x')\delta(y')$$  \hspace{1cm} (2.16)

For a metallic cylindrical antenna with very small radius $a$ we can assume that all the charge distribution are distributed around the surface. So following must be true:

$$J_z(r') = \hat{z}I(z')\delta(r' - a)\frac{1}{2\pi}$$  \hspace{1cm} (2.17)

Putting 2.17 in 2.16 we get:

$$A_z = \frac{\mu}{4\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^a \int_0^{2\pi} I_z(z')\delta(r' - a)\frac{e^{-j\kappa R}}{R} dz' dr' d\phi'$$  \hspace{1cm} (2.18)

The wire is symmetrical over $\phi$, the integration of $\delta$ function results in $a$ so we are left with the following:
\[ A_z = \frac{\mu}{4\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{0}^{2\pi} e^{-j\kappa R} I(z')dz'd\phi' \]  

(2.19)

R is given by:

\[ R = |r - r'| \]  

(2.20a)

\[ R = \sqrt{r^2 + a^2 - 2ra\cos(\phi' - \phi) + (z - z')^2} \]  

(2.20b)

Looking closely at fig 2.1, we can see that \( r' \) is actually the radius \( a \), now replacing this for \( r' \) and looking at the difference \( |r - r'|^2 \), we see:

\[ |r - r'|^2 = |r^2 + a^2 - 2\cdot r \cdot r'| = |r^2 + a^2 - 2ra\cos(\phi' - \phi)| \]

The equation for vector potential is generally written in following form:

\[ A_z(z, r) = \frac{\mu}{4\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} I(z')G(z - z', r)dz' \]  

(2.21)

However \( G \) and \( R \) take different forms for approximate and exact kernel.

For the approximate kernel we let \( a \rightarrow 0 \) so \( R \) of equation 2.20b becomes

\[ R = \sqrt{r^2 + (z - z')^2}. \] \( \phi \) dependence of kernel \( G \) vanishes so \( G_a \) for approximate kernel becomes

\[ G_a(z - z', r) = \frac{e^{j\kappa R}}{R} \]  

(2.22)
For exact kernel $G$ takes the form:

$$
G(z - z', r) = \frac{1}{2\pi} \int_0^{2\pi} e^{-j\kappa R} R d\phi' 
$$

(2.23)

2. Now we start with Lorenz gauge to establish the relationship between vector potential $A_z$ and incident electric field $E_{in}$

Lorenz gauge in frequency domain:

$$
\nabla \cdot A + j\omega \mu \varepsilon \Phi = 0 
$$

(2.24a)

From this we get electric potential $\Phi$ as:

$$
\Phi = -\frac{\nabla \cdot A}{\omega \mu \varepsilon} 
$$

(2.24b)

Rewriting $E$ from 2.2a in frequency domain we will get:

$$
E = -\nabla \Phi - j\omega A 
$$

(2.25)

Using value of $\Phi$ from 2.24b in 2.25 we get:

$$
E = \frac{1}{j\omega \mu \varepsilon} \nabla (\nabla \cdot A) - j\omega A 
$$

(2.26a)

This gives the final $E$ in terms of $A$ as:
Replacing $\omega^2 \mu \epsilon = \kappa^2$

The $z$ component of (2.26b) can be written as:

$$j \omega \mu \epsilon E_z = \partial_z^2 A_z + \kappa^2 A_z$$

(2.27)

At $r = a$ i.e. at the surface we have electric field inside the antenna is equal and opposite to the incident electric field $E_{in}$ so following should be true:

$$E_z(z) = -E_{in}(z)$$

(2.28)

2.27 can be written now as:

$$\partial_z^2 A_z + \kappa^2 A_z = -j \omega \mu \epsilon E_{in}$$

(2.29)

3. Finally From 2.29 and 2.19 we get Hallen’s integral equation as:

$$\frac{\mu}{4\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{2\pi} \frac{e^{-jnR}}{2\pi R} I(z') dz' d\phi' = -j \omega \mu \epsilon (\partial_z^2 + \kappa^2)^{-1} E_{in}(z)$$

(2.30)

Pocklington’s Equation is the equivalent expression for Hallen and is written as:
\[
\frac{\mu}{4\pi} \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_{0}^{2\pi} e^{-j\kappa R} \frac{I(z')}{2\pi R} dz' d\phi' (\partial_z^2 + \kappa^2) = -j\omega \mu \epsilon E_{in}(z) \tag{2.31}
\]

It should be noted that this \( R \) is different than \( R \) of 2.20b. That is because now we are only focused on the surface of wire and hence \( r \) of 2.20b is replaced by \( a \). Here \( R = \sqrt{(z - z')^2 + 2a^2 - 2a^2 \cos \phi'} \)

For approximate kernel the same can be written as:

Hallen’s Equation:

\[
\frac{\mu}{4\pi} \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{e^{-j\kappa R}}{R} I(z') dz' = -j\omega \mu \epsilon (\partial_z^2 + \kappa^2)^{-1} E_{in}(z) \tag{2.32}
\]

Pocklington’s Equation:

\[
\frac{\mu}{4\pi} \int_{-\frac{l}{2}}^{\frac{l}{2}} (\partial_z^2 + \kappa^2) \frac{e^{-j\kappa R}}{R} I(z') dz' = -j\omega \mu \epsilon E_{in}(z) \tag{2.33}
\]

\( R \) for approximate kernel is \( R = \sqrt{(z - z')^2 + a^2} \)

[1] talks mostly about 2.33. The above derivation is found in almost all introductory chapters in current distribution. [9] has an exhaustive derivation.

2.2 Closer Look at Hallen’s Equation

A closer inspection of Hallen’s equation will show us the components that make up the current distribution.

Lets go back to 2.29:
\[
\frac{\partial^2 A_z}{z^2} + \kappa^2 A_z = -j\omega\mu\epsilon E_{in} \tag{2.34}
\]

For delta gap case $E_{in}$ can be represented as:

\[
E_{in} = V_0\delta(z) \tag{2.35}
\]

Where $V_0$ is the voltage difference in the gap. From 2.35 and 2.34, and replacing $\kappa^2 = \omega^2\mu\epsilon$ the differential equation can be written as:

\[
\frac{\partial^2 A_z}{z^2} + \kappa^2 A_z = -\frac{j\kappa^2}{\omega} V_0\delta(z) \tag{2.36}
\]

This differential equation will have solution composed of a homogeneous solution and a particular solution.

Homogeneous solution of this is $A_{zh} = C_1 \cos \kappa z + C_2 \sin \kappa z$ where $C_1$ and $C_2$ are constants. Similarly the particular solution for this can be verified to be $A_{zp} = -jV/2c \sin \kappa|z|$. Calculation of homogeneous equation is done in Appendix B.

$A_z(z)$ is even function so coefficient $C_2 = 0$, hence the final solution for $A_z(z)$ at the surface of the antenna can be written as:

\[
A_z(z) = -\frac{jV_0}{2c} \sin \kappa|z| + C_1 \cos \kappa z \tag{2.37}
\]

Now equating $A_z$ from 2.37 with $A_z$ of 2.19, we get modified form of Hallen’s equation as:
\[ \frac{\mu}{4\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{0}^{2\pi} e^{-j\kappa R} \frac{I(z')dz'd\phi'}{2\pi R} = -jV_0 \frac{\eta}{2c} \sin \kappa |z| + C_1 \cos \kappa z \] (2.38)

An equivalent form of this is in [9] as:

\[ \frac{j\eta}{2\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{0}^{2\pi} e^{-j\kappa R} \frac{I(z')dz'd\phi'}{2\pi R} = V_0 \sin \kappa |z| + C \cos \kappa z \] (2.39)

In [15] another form of same equation is written as:

\[ \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\pi}^{\pi} e^{-j\kappa \sqrt{z^2+4a^2 \sin^2(\phi'/2)}} \frac{I(z')dz'd\phi'}{2\pi \sqrt{z^2+4a^2 \sin^2(\phi'/2)}} = -j \frac{2\pi V_0}{\eta} \sin \kappa |z| + C \cos \kappa z \] (2.40)

The equivalence of 2.39 and 2.40 can be proven.

2.2.1 Crude Approximation of Current from Hallen’s Equation

As mentioned before, most of the analysis on a linear dipole antenna is done by assuming sinusoidal current approximation. This section will show how Hallen’s equation can result in that type of solution.

Closely looking at the integral \[ \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\pi}^{\pi} e^{-j\kappa \sqrt{(z-z')^2+a^2}} \frac{I(z')dz'd\phi'}{\sqrt{(z-z')^2+a^2}}, \] we can see that its value is dominated by those values of \( z' \) which are very close to \( z \), that is because, the denominator \( \sqrt{(z-z')^2+a^2} \) will have very small value at \( z \approx z' \). Table 2.1 shows that for smaller values of \( z \), value of \( e^z/z \) is large.
Table 2.1: Value of $e^z/z$ for different value of $z$

<table>
<thead>
<tr>
<th>$z$</th>
<th>$e^z/z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>1000.0</td>
</tr>
<tr>
<td>0.005</td>
<td>200.0</td>
</tr>
<tr>
<td>0.01</td>
<td>100.0</td>
</tr>
<tr>
<td>0.05</td>
<td>20.0</td>
</tr>
<tr>
<td>0.1</td>
<td>10.0</td>
</tr>
<tr>
<td>0.5</td>
<td>2.0</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

It is a relatively good approximation if the integral $\int_{\frac{1}{2}}^{\frac{1}{2}} \frac{e^{-j\kappa \sqrt{(z-z')^2 + a^2}}}{\sqrt{(z-z')^2 + a^2}}$ is replaced by the average value of $\frac{e^{-j\kappa \sqrt{(z-z')^2 + a^2}}}{\sqrt{(z-z')^2 + a^2}}$ for values of $z' \approx z$. Let us call that value $\hat{Z}$.

The LHS of 2.39, can be replaced as:

$$\hat{Z}I(z) = V_0 \sin \kappa |z| + C \cos \kappa z$$ (2.41)

To find the constant value $C$ of 2.41, we use the end condition that $I(h) = 0$

With that replacement we will get:

$$C = -\frac{V_0 \sin \kappa h}{\cos \kappa h}$$ (2.42)
Replacing the value of $C$ from 2.42, to 2.41, we will get:

\[ \hat{Z}I(z) = V_0 \sin \kappa |z| - \frac{V_0 \sin \kappa h}{\cos \kappa h} \cos \kappa z \]  
\[ I(z) = \frac{V_0 \sin \kappa |z| \cos \kappa h - V_0 \sin \kappa h \cos \kappa |z|}{\hat{Z} \cos \kappa h} \]  
\[ (2.43a) \]
\[ (2.43b) \]

Finally $I(z)$, in compact form can be written sinusoidal form as:

\[ I(z) = \frac{V_0 \sin \left( \kappa (h - |z|) \right)}{\hat{Z} \cos \kappa h} \]  
\[ (2.44) \]

Plots of current with this equation is done in results section.

2.3 A Deeper Analysis on Exact Kernel

A closer look at the exact kernel will enable us to find numerically friendly expression for the same. This kernel has been widely studied. Few of the popular ones can be found in [16], [12] and [17].

This thesis compares two approaches taken in order to solve the exact kernel. A more straightforward and simple approach is found in [17].

In this approach the singular part of the exact kernel is made explicit by isolation. The non singular part is treated assuming the $R$ of the approximate kernel and the singular part is approximated by an elliptic integral of first kind.

The exact kernel is given by:

\[ G(z - z') = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-j\kappa \sqrt{(z-z')^2 + 2a^2 - 2a^2 \cos \phi'}}}{\sqrt{(z-z')^2 + 2a^2 - 2a^2 \cos \phi'}} d\phi' \]  
\[ (2.45) \]
\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-j\kappa R}}{R} d\phi' \text{ can be re-written as:}
\]

\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-j\kappa R}}{R} d\phi' = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{R} d\phi' + \frac{1}{2\pi} \int_0^{2\pi} \frac{(e^{-j\kappa R} - 1)}{R} d\phi' \quad (2.46)
\]

First part of 2.46 is singular whereas the second part is not. In this approach the integral of singular part, \(\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{R} \), is calculated using an elliptic integral of the first kind whereas the second part is treated like the approximate kernel.

We have:

\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{R} d\phi' = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\sqrt{(z - z')^2 + 2a^2 - 2a^2 \cos \phi'}} d\phi' \\
= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\sqrt{(z - z')^2 + 4a^2 \sin^2 \phi' / 2}} d\phi' \quad (2.47)
\]

Making change of variable with \(\phi' = \pi + 2\theta\) in 2.47 we will get:

\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{R} d\phi' = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{\sqrt{(z - z')^2 + 4a^2 \cos^2 \theta}} d\theta \\
= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{\sqrt{(z - z')^2 + 4a^2(1 - \sin^2 \theta)}} d\theta \\
= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{\sqrt{(z - z')^2 + 4a^2 - 4a^2 \sin^2 \theta}}} d\theta \\
= \frac{1}{\pi((z - z')^2 + 4a^2)} \int_{-\pi/2}^{\pi/2} \frac{1}{\sqrt{1 - \frac{4a^2}{(z - z')^2 + 4a^2} \sin^2 \theta}}} d\theta \\
\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{R} d\phi' = \frac{2}{\pi((z - z')^2 + 4a^2)} \int_{0}^{\pi/2} \frac{1}{\sqrt{1 - \frac{4a^2}{(z - z')^2 + 4a^2} \sin^2 \theta}}} d\theta \quad (2.48)
\]

An elliptic integral \(K(m)\), is defined such that:
\[ K(m) = \int_{0}^{\pi/2} \frac{d\phi}{\sqrt{1 - m \sin^2 \phi}} \]  \tag{2.49}

Considering \( m = \frac{4a^2}{(z-z')^2 + 4a^2} \), 2.48 can be written in the form of 2.49 as:

\[
\frac{1}{2\pi} \int_{0}^{2\pi} \frac{1}{R} d\phi' = \frac{2}{\pi((z - z')^2 + 4a^2)} K\left[ \frac{4a^2}{(z - z')^2 + 4a^2} \right] \tag{2.50}
\]

This elliptic integral can be represented by using the following series\[18]\:

\[ K(m) = [a_0 + a_1(1 - m) + \cdots + a_4(1 - m)^4] - [b_0 + b_1(1 - m) + \cdots + b_4(1 - m)^4]ln(1 - m) \tag{2.51} \]

The value of co-efficients for 2.51 are listed in 2.2, for error factor \( \epsilon \leq 2 \times 10^{-8} \)

Table 2.2: Coefficients for \( K(m) \)

\[
a_0 = 1.38629436112 \quad b_0 = 0.5 \\
a_1 = 0.09666344259 \quad b_1 = 0.12498593597 \\
a_2 = 0.03590092383 \quad b_2 = 0.06880248576 \\
a_3 = 0.03742563713 \quad b_3 = 0.03328355346 \\
a_4 = 0.01451196212 \quad b_4 = 0.00441787012 
\]

When the value of \( m \) is close to 1 the term \( log(1 - m) \) will go to infinity.

For those values \[9\](Pg 871) and \[19\](pg 338), suggest following approximation of the exact kernel:

26
\[ G(z) = \frac{1}{\pi a} ln\left( \frac{8a}{|z - z'|} \right) \] (2.52)

A more rigorous approach for the expansion of the exact kernel is found in [9].

The numerical section that follows has the numerical implementation of exact and approximate kernels, which in turn is used to calculate the current distributions in a linear nanowire antenna.
2.4 Two Element Antenna Array

Discussion for two element array is followed that of [9], equations for two element array given in pg(929-937). If \( I_0 \) be the current in element 1 and \( I_1 \) be the current in the element 2, then \( I_0 \) and \( I_1 \) will be given by solving the following Hallen’s equation.

\[
\frac{j\eta}{2\pi} \sum_{q=0}^{1} \sum_{m=-M}^{M} I_q(z_m) \int_{-\Delta_q/2}^{\Delta_q/2} G_{pq}(z_n - z_m - z)dz = C_p \cos kz_n + V_p \sin k|z_n| \quad (2.53)
\]
CHAPTER III
NUMERICAL SECTION

Calculation of current distribution of a linear antenna is mostly performed using Method of Moments (MoM), which was first introduced by Harrington in [14]. A detailed discussion can also be found at [20].

3.1 Method of Moments

In this method, moment is calculated by multiplying unknown variable with a chosen basis (weighing) function and integrating. This method can solve both differential and integral equations.

[19] has summarized the steps in implementation of Method of Moments in a number of steps. Those steps tailored for our purpose are listed as follows:

1. Derivation of Hallen’s integral equation.

2. Discretization of Hallen’s integral equation into matrix form. We are assuming pulse function as basis function.

3. Evaluation of each element of the given matrix. We will find an integral equation for each element of the matrix.
4. Expression of matrices in matrix equation form. This equation can then be solved to find out the current distribution over the antenna and constant $C$ which can be seen with cosine term in RHS of the Hallen’s equation.

3.1.1 Discretizing Integral Equation

While implementing this method on a linear nano-antenna, the antenna is divided into a number of smaller segments. We assume current over each segment to be constant. This is a reasonable approximation if the antenna is divided into a large number of segments. A trade off has to be made on accuracy vs speed/memory limitation of the machine.

Let each half of the antenna is divided into $N$, number of divisions. Considering gap to be one of the divisions we will have $2N + 1$ total divisions. If $\Delta$ be the size of each division we get:

$$\Delta = \frac{l}{2N + 1} \quad (3.1)$$

The divisions of the antenna is shown in the figure 3.1. Each box represents one division whose width is equal to $\Delta$. In this case $N = 4$, so the total number of divisions is given by $2N + 1 = 9$. Figure also shows that $z_n = n\Delta$

Now for a basis function a pulse function is defined such that each function is constant(1) over a segment $\Delta$ and zero everywhere else. If $P$ represents a pulse function it can be mathematically written as:
Figure 3.1: Segmentation of Linear Antenna For Numerical Approximation

\[ u(z' - z_n) = \begin{cases} 
1 & \text{for } |z' - z_n| \leq \Delta/2 \\
0 & \text{otherwise} 
\end{cases} \]  

(3.2)

The pictorial representation of the same is given in 3.2. The figure shows pulse functions for \( P(z' + 3\Delta) \) and \( P(z' - 2\Delta) \).
The current $I$, is first assumed to be sum of the pulse basis function $u$, which can be represented as:

$$I(z') = \sum_{n=-N}^{N} I_n u_n(z' - n\Delta)$$

(3.3)

Figure 3.3 shows the approximation of the current using pulse functions $u$.

![Figure 3.3: Current in Approximated by Step Functions](image)

Rewriting the Hallen’s equation from 2.39,

$$\frac{j\eta}{2\pi} \int_{\frac{l}{2}}^{\frac{l}{2}} G(z - z')I(z')dz' = V_0 \sin \kappa |z| + C \cos \kappa z$$

(3.4)

Putting value of $I$ from 3.3 to 3.4,

$$\sum_{n=-N}^{N} I_n \frac{j\eta}{2\pi} \int_{\frac{l}{2}}^{\frac{l}{2}} G(z - z'\Delta)u_n(z' - z_n)dz' = V_0 \sin \kappa |z| + C \cos \kappa z$$

(3.5)
Equation 3.5 cannot be satisfied for all $z$, a local weighted average is formed. Another weighting function is used to do so.

There are two popular choices of weighting function $W$. One is the use of a delta function, called point matching case and the other, called Galerkin method, uses weighting function same as the basis function. [15] implements the calculation for Galerkin method. We will use point matching in our analysis.

Let us calculate average at $p$ points. Evaluating equation 3.5 for $W(z - z_p) = \delta(z - z_p)$ we will get:

$$\sum_{n=-N}^{N} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} G(z - z')u_n(z' - z_n)\delta(z - z_p)dz' = V_0 \sin \kappa |z_p| + C \cos \kappa z_p$$  \hspace{1cm} (3.6)

The integration range can also be written in the following form:

$$\sum_{n,p=-N}^{N} \int_{z_n - \Delta/2}^{z_n + \Delta/2} G(z - z_p)u_n(z' - z_n)\delta(z - z_p)dz' = V_0 \sin \kappa |z_p| + C \cos \kappa z_p$$ \hspace{1cm} (3.7)

Let us consider digitized form of Green’s function to be represented by $K_{pn}$.

Performing change in variable $z = z' - z_n$ we get:

$$K_{pn} = \int_{z_n - \Delta/2}^{z_n + \Delta/2} G(z_p - z_n)u_n(z' - z_n)dz'$$ \hspace{1cm} (3.8a)

$$= \int_{-\Delta/2}^{+\Delta/2} G(z_p - z_n - z)dz$$ \hspace{1cm} (3.8b)

$n$, and $p$ both can range from $-N$ to $N$. 33
Table 3.1: Subintervals and Weights for Five Point Gaussian Quadrature

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$w_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{128}{225}$</td>
</tr>
<tr>
<td>$+\frac{1}{3}\sqrt{5 - 2\sqrt{10/7}} \cdot \frac{322 + 13\sqrt{70}}{900}$</td>
<td></td>
</tr>
<tr>
<td>$-\frac{1}{3}\sqrt{5 - 2\sqrt{10/7}} \cdot \frac{322 + 13\sqrt{70}}{900}$</td>
<td></td>
</tr>
<tr>
<td>$+\frac{1}{3}\sqrt{5 + 2\sqrt{10/7}} \cdot \frac{322 - 13\sqrt{70}}{900}$</td>
<td></td>
</tr>
<tr>
<td>$-\frac{1}{3}\sqrt{5 + 2\sqrt{10/7}} \cdot \frac{322 - 13\sqrt{70}}{900}$</td>
<td></td>
</tr>
</tbody>
</table>

$K_{pn}$ can further be folded into half which minimizes the integration range but essentially give same result. A final discretized kernel $K_{pn}$ can be written as:

$$K_{pn} = \int_{0}^{\Delta/2} [G(z_p - z_n - z) + G(z_p - z_n + z)]dz$$ (3.9)

3.1.2 Evaluation of Each Element of the Kernel Matrix

The integration in equation 3.9 is performed using five point Gaussian quadrature rule. In this rule the domain of integration in divided into five subpoints $x_i$. A particular weight $w_i$ is assigned for each subpoint. For five point Gaussian quadrature, table 3.1 lists subpoints and weights (based on Abramowitz and Stegun 1972).

A close look at $z_p - z_n = (m - n)\Delta$ of equation 3.9 shows that it is actually a distance vector. So the matrix $K_{pn}$ is Toeplitz in nature. Rather than calculating
each element of matrix, $K_{pn}$ is easily implemented as follows.

1. A new matrix $K_q$ is declared such that $K_q = \int_{-\Delta/2}^{+\Delta/2} G(q\Delta - z)dz$, $q$ range from 0 to $2N + 1$.

2. Now matrix $K_{pn}$ is defined by interchanging elements of $K_q$ in manner shown in Table 3.2. This operation in done by function `toeplitz` in matlab. This function will take lot less time than determining each element of the matrix because except for the first column the rest is calculated by shifting and adding the columns.

3.1.3 Writing Matrix Equation and Finding Constant C

If $s_p = \sin |k|z_p$ and $c_p = \cos k z_p$, the equation 3.9 can be written in the matrix form as shown in 3.10

$$\frac{j \eta}{4\pi} \begin{pmatrix}
K_{1,1} & K_{1,2} & \cdots & K_{1,N} \\
K_{2,1} & K_{2,2} & \cdots & K_{2,N} \\
\vdots & \vdots & \ddots & \vdots \\
K_{P,1} & K_{P,2} & \cdots & K_{P,N}
\end{pmatrix} \begin{pmatrix}
I_1 \\
I_2 \\
\vdots \\
I_P
\end{pmatrix} = \begin{pmatrix}
s_1 \\
s_2 \\
\vdots \\
s_P
\end{pmatrix} + C \begin{pmatrix}
c_1 \\
c_2 \\
\vdots \\
c_P
\end{pmatrix}$$
Table 3.2: Nature of Discretized Kernel

\[
\begin{array}{cccccc}
\text{pn} = 1 & \text{n} = 2 & n = 3 & n = 4 & n = 5 \\
1 q = 0 & q = 1 & q = 2 & q = 3 & q = 4 \\
K_0 & K_1 & K_2 & K_3 & K_4 \\
2 q = 1 & q = 0 & q = 1 & q = 2 & q = 3 \\
K_1 & K_0 & K_1 & K_2 & K_3 \\
3 q = 2 & q = 1 & q = 0 & q = 1 & q = 2 \\
K_2 & K_1 & K_0 & K_1 & K_2 \\
4 q = 3 & q = 2 & q = 1 & q = 0 & q = 1 \\
K_3 & K_2 & K_1 & K_0 & K_1 \\
5 q = 4 & q = 3 & q = 2 & q = 1 & q = 0 \\
K_4 & K_3 & K_2 & K_1 & K_0
\end{array}
\]
\[
\begin{pmatrix}
I_1 \\
I_2 \\
\vdots \\
I_P
\end{pmatrix} = \frac{4\pi}{j\eta}
\begin{pmatrix}
K_{1,1} & K_{1,2} & \cdots & K_{1,N} \\
K_{2,1} & K_{2,2} & \cdots & K_{2,N} \\
\vdots & \vdots & \ddots & \vdots \\
K_{P,1} & K_{P,2} & \cdots & K_{P,N}
\end{pmatrix}\begin{pmatrix}
s_1 \\
s_2 \\
\vdots \\
s_P
\end{pmatrix}

+ \frac{4\pi}{j\eta}
\begin{pmatrix}
K_{1,1} & K_{1,2} & \cdots & K_{1,N} \\
K_{2,1} & K_{2,2} & \cdots & K_{2,N} \\
\vdots & \vdots & \ddots & \vdots \\
K_{P,1} & K_{P,2} & \cdots & K_{P,N}
\end{pmatrix}\begin{pmatrix}
c_1 \\
c_2 \\
\vdots \\
c_P
\end{pmatrix}
\]

Matrices $\overline{Z}, I, \overline{s}$ and $\overline{c}$ are symmetric. Constant $C$ is obtained in literature by two popular methods. They are listed as follows:

1. First one is done in [9]. This method utilizes the fact that current $I$ is zero at end points, i.e. $I(\pm h) = 0$

2. Second one is done in [19] and [17]. This method considers $C$ to be another unknown variable. To calculate this an additional row is added in all matrices.

We have implemented both the methods. It is seen that both the methods give comparable results.

Considering $I(\pm h) = 0$

This method is implemented in [9]. In this method the matrices are wrapped and then multiplied by the step function which is then equated to 0.
Matrix $K$ can be written as:

$$K_{p,n} = \begin{pmatrix} K_{1,1} & K_{1,2} & \cdots & K_{1,n} \\ K_{2,1} & K_{2,2} & \cdots & K_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ K_{m,1} & K_{m,2} & \cdots & K_{m,n} \end{pmatrix}$$

(3.11a)

$$= \begin{pmatrix} A & a^R & B \\ a^R & a_0 & \bar{a} \\ \bar{B} & \bar{a} & \bar{A} \end{pmatrix}$$

(3.11b)

The matrix equation 3.10 can now be written as:

$$\begin{pmatrix} A & a^R & B \\ a^R & a_0 & \bar{a} \\ \bar{B} & \bar{a} & \bar{A} \end{pmatrix} \begin{pmatrix} I_1^R \\ I_0 \\ I_1 \end{pmatrix} = \begin{pmatrix} s_1 + Cc_1^R \\ s_0 + Cc_0 \\ s_1 + Cc_1 \end{pmatrix}$$

(3.12)

This can be written in equation form as shown in equation 3.13

$$\overline{AI_1^R} + \overline{a^RI_0} + \overline{BI_1} = V_0 s_1 + \overline{Cc_1^R}$$

$$\overline{a^TRI_1} + a_0 I_0 + \overline{a^TI_1} = V_0 s_0 + \overline{Cc_0}$$

$$\overline{BI_1^R} + \overline{aI_0} + \overline{AI_1} = V_0 s_1 + \overline{Cc_1}$$

(3.13)

First and last equations of equation 3.13 are equivalent to each other and hence redundant. So a reduced system can be written as:
$$\begin{pmatrix} a_0 & 2\bar{a}^T \\ \bar{a}I_0 & \bar{A} + \bar{B}J \end{pmatrix} \begin{pmatrix} I_0 \\ I_1 \end{pmatrix} = \begin{pmatrix} Cc_0 + V_0s_0 \\ C\bar{c}_1 + V_0\bar{s}_1 \end{pmatrix} \quad (3.14)$$

Where $J$ is reversal matrix.

This can be written in matrix form as equation 3.15

$$ZI = C\bar{c} + V_0\bar{s} \quad (3.15)$$

Now to get $C$ using end point conditions we define a unit step function such that $\bar{u} = [0, \ldots, 0, 1]$ Multiplying both sides of 3.15 by transpose of $u$

$$I = C\bar{Z}^{-1}\bar{c} + V_0\bar{Z}^{-1}\bar{s}$$

$$\bar{u}^T I = \bar{u}^T C\bar{Z}^{-1}\bar{c} + \bar{u}^T V_0\bar{Z}^{-1}\bar{s}$$

We thus get final $C$ as:

$$C = -V_0 \frac{\bar{u}^T \bar{Z}^{-1}\bar{s}}{\bar{u}^T \bar{Z}^{-1}\bar{c}} \quad (3.17)$$

Considering $C$ as yet another constant

This method is fairly straightforward. In this method the cosine term with constant $C$ is also brought to left side of the matrix equation of 3.10 and one row is added to all the matrices. The final form of the equation will look like the following.
\[
\begin{pmatrix}
K_{1,1} & K_{1,2} & \cdots & K_{1,N} \\
K_{2,1} & K_{2,2} & \cdots & K_{2,N} \\
\vdots & \vdots & \ddots & \vdots \\
K_{P,1} & K_{P,2} & \cdots & K_{P,N} \\
K_{P+1,1} & K_{P+1,2} & \cdots & K_{P+1,N}
\end{pmatrix}
\begin{pmatrix}
I_1 \\
I_2 \\
\vdots \\
I_P \\
I_{P+1}
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
\vdots \\
c_P \\
c_{P+1}
\end{pmatrix}
= 
\begin{pmatrix}
I_1 \\
I_2 \\
\vdots \\
I_P \\
C
\end{pmatrix}
\begin{pmatrix}
s_1 \\
s_2 \\
\vdots \\
s_P \\
s_{P+1}
\end{pmatrix}
\]

(3.18a)

\[
\begin{pmatrix}
j\eta & 0 & \cdots & 0 \\
0 & j\eta & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & j\eta \\
j\eta & 0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
I_1 \\
I_2 \\
\vdots \\
I_P \\
I_{P+1}
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
\vdots \\
c_P \\
c_{P+1}
\end{pmatrix}
= 
\begin{pmatrix}
I_1 \\
I_2 \\
\vdots \\
I_P \\
C
\end{pmatrix}
\begin{pmatrix}
s_1 \\
s_2 \\
\vdots \\
s_P \\
s_{P+1}
\end{pmatrix}
\]

(3.18b)

All of the elements of matrix \( I \), are the currents for elements of wire except the last one which is constant \( C \).
CHAPTER IV
RESULTS AND DISCUSSION

Our main aim is to observe a current distribution over a linear nano-antenna and a frequency range at which a useful result can be observed. Due to its constraint on size a linear nano-antenna can’t be very much longer or shorter than its radius to give a proper radiation pattern.

The analysis of a current distribution of a linear antenna are mostly done numerically. Literature lacks the proper experimental measurements of a current distribution of a linear antenna. So we are starting our discussion comparing different numerical models. When a convincing numerical model is found, we will implement that to see the range of frequencies for which a good current distribution and hence a radiation pattern is obtained for a normal nanowire.

A brief analysis on a current distribution for two element nanowire is also undertaken.

Let’s start with the kernel comparisons. Table 4.1 shows the different values of the approximate and the exact kernel.

Two things can be concluded from this table.

1. Values generated from real and exact kernels do not deviate from each other by large values except at zero, where,
Table 4.1: Comparison between approximate and exact kernel

\[ a = 30e^{-9}, \ l = 0.6\text{mm}, \ f = 1\text{THZ} \]

<table>
<thead>
<tr>
<th>( z )</th>
<th>( K(z,a,’ex’) )</th>
<th>( K(z,a,’ap’) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-6.0e-4</td>
<td>500.0000e-003 + 7.8540e-009i</td>
<td>500.0000e-003 + 7.8540e-009i</td>
</tr>
<tr>
<td>-4.8e-4</td>
<td>-505.6356e-003 -367.3658e-003i</td>
<td>-505.6356e-003 -367.3658e-003i</td>
</tr>
<tr>
<td>-3.6e-4</td>
<td>257.5141e-003 +792.5471e-003i</td>
<td>257.5141e-003 +792.5471e-003i</td>
</tr>
<tr>
<td>-2.4e-4</td>
<td>386.2713e-003-1.1888e+000i</td>
<td>386.2713e-003 - 1.1888e+000i</td>
</tr>
<tr>
<td>-1.2e-4</td>
<td>-2.0225e+000 + 1.4695e+000i</td>
<td>-2.0225e+000 + 1.4695e+000i</td>
</tr>
<tr>
<td>+0.0e-4</td>
<td>Inf + 6.2832e+000i</td>
<td>10.0000e+003 + 6.2832e+000i</td>
</tr>
<tr>
<td>+1.2e-4</td>
<td>-2.0225e+000 + 1.4695e+000i</td>
<td>-2.0225e+000 + 1.4695e+000i</td>
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<td>+6.0e-4</td>
<td>500.0000e-003 + 7.8540e-009i</td>
<td>500.0000e-003 + 7.8540e-009i</td>
</tr>
</tbody>
</table>
2. the value of the kernel is infinity for the exact kernel, shown in gray, and a finite
value for approximate kernel

A nonzero value of the approximate kernel may look like a good thing but in
fact this causes the non convergence of the current values with the increasing number
of divisions. This is discussed in detail in the analytical section.

4.1 Current Distributions

First of all we plot the one term expansion of Hallen’s equation done in 2.44. This is
the sinusoidal approximation of the Hallen’s equation.

We have analyzed for the same antenna for which table 4.1 is generated. The
approximate values of $Z$ is obtained by averaging middle five largest values of the
kernel, which are the values of kernel for the values of $z$ close to 0.

Larger number of values for the kernel $Z$ is required for the approximate sinu-
soidal current distribution. Figure 4.1 shows the plot for one term Hallen’s expansion.
This is done for number of subdivisions $N = 100$ on each half of the antenna.

Similarly the value of $C$ is obtained as $C = \frac{-\sin\kappa(l/2)}{\cos\kappa(l/2)}$. Thus for $V_0 = 1$ we
will get $I(z) = \frac{C}{Z} + \frac{1}{Z} \sin\kappa|z|$.

To avoid getting very small values of distance we have non-dimensionalized
all the lengths with the wavelength of $\lambda = 0.3 \times 10^{-3}$ which is the wavelength at one
terahertz frequency.
The approximate kernel is assumed while obtaining average value of $Z$ for figure 4.1.

Now let's move on with a current distribution with the approximate kernel and compare it with 4.1. Figure 4.2 shows the current distribution using the approximate kernel.

Another graph 4.3 shows the comparison between the approximate and the exact kernel; also between the exact kernel and the sinusoidal approximation.

Figure 4.3 shows that the current distribution due to the approximate and the exact kernel are matching. Looking at figure 4.3 one might wonder why so much of hard work for the exact kernel. For this [15] has an excellent comparison case at page 458. We will try to replicate the test case using our program.
Figure 4.2: Sinusoidal Approximation for Hallen’s Equation

The test case assumes $L/\lambda = 0.5$ and $a/\lambda = 0.006$. Using these values, the real and the imaginary parts of the current obtained for the approximate and the exact kernel are obtained. [15] uses a different numerical scheme for the calculation of $I$, and this comparison, if come out to be the same will make us more confident on simulation result obtained by our program.

Since our $\lambda$ is fixed at $0.3mm$, $L = 0.15 \times 10^{-3}$ and $a = 1800 \times 10^{-9}$. For this value we will compare the real and the imaginary parts of the current for the approximate and the exact kernel respectively. To be consistent with the comparison we are also looking at only one half of the antenna.

Figure 4.4 shows the comparison.

The comparison matches exactly with the current calculations done in [15]. The magnitudes are also matching.
The deviation of the values are seen only at the end points. An oscillation of the real part and the imaginary part can be observed at the end points. The oscillations are very large for the imaginary part which forced us to take 30 less points near the driving point for the imaginary current.

The choice of number of divisions over the antenna should be based on the ratio $L/a$, [15]. Oscillations occur when $N \gg L/a$. Figure 4.5 shows the distortion due to the increasing $N = 40, 200$ for a fixed $L/a = 100$.

For the exact kernel however, this kind of oscillations is not seen.

4.6 replicates the analysis done for figure 4.5 for exact kernel.
4.2 Nanometer Terahertz Antenna

Till this point we were looking at the nature of kernels and attempting to prove the veracity of the analysis. Now we delve into high frequency nano-antennas.

A linear antenna of the given length has a window of frequency ranges for which it can operate properly. If the antenna length is very small or very large as compared to the wavelength, the interaction will not be efficient. This is due to a parameter called radiation resistance. For a linear antenna when the ratio, $L/\lambda$, is very small, radiation resistance will become very large and when the antenna length is very large, radiation resistance becomes very small resulting in inefficient antennas.
Figure 4.5: Changing Current due to Increasing $N$ in Approximate Kernel

The analysis is done for a full wavelength dipole. The radiation pattern and the current distribution for different nanowire antennas is given in figure 4.7

Similarly the radiation pattern for the given radius range is given in the figure 4.8

We are looking at the antenna length of 0.3$mm$, which is $1\lambda$ for 1$THz$ frequency.

Same simulation is performed for the antenna length 0.6$mm$, twice the wavelength. The figures are shown in 4.9 and 4.10

Similarly comparison for current distributions of micro antennas with multiple radii is done in 4.11. The current is more distorted near the excitation. This
distortion is more clear when we look at the radiation pattern on 4.12

A similar distortion can be observed in figures 4.13 and 4.14 for the antenna length of 0.6mm and terahertz frequency.
Figure 4.7: Current due to different $a$ for $L = 0.3\,mm, f = 1THz$
Figure 4.8: Radiation Pattern due to different $a$ for $L = 0.3\text{mm}, f = 1\text{THz}$
Figure 4.9: Current due to different $a$ for $L = 0.6\, mm, f = 1\, THz$
Figure 4.10: Radiation Pattern due to different $a$ for $L = 0.6\,mm$, $f = 1\,THz$
Figure 4.11: Current due to different $a$ for $L = 0.3\,mm$, $f = 1THz$
Figure 4.12: Radiation Pattern due to different $a$ for $L = 0.3\,mm$, $f = 1\,THz$
Figure 4.13: Current due to different $a$ for $L = 0.3\, mm$, $f = 1\, THz$
Figure 4.14: Radiation Pattern due to different $a$ for $L = 0.3\text{mm}, f = 1\text{THz}$
4.3 Two Element Array

When two identical antennas are placed closed to each other, the current pattern on each antenna can be obtained as shown in figure 4.15.

The Function $h_{coupled2}$ is used from [9] for this calculation. Driving element is given the voltage excitation of 1V whereas the driven element acts as a parasitic element.

4.4 Conclusion and Future Work

From the results one can derive that Hallen’s equation holds true for the higher frequencies as long as we keep the length of wire within the range comparable to the wavelength of our concern. For a nanowire, with nanometer radius the assumption (that the radius is smaller than the length) holds true for the higher frequency ranges including the terahertz frequency range. Most important observation of this work is the finding that the nanowire antenna can work as a linear antenna for the terahertz frequency range. We saw how that is not true for the micro-radii wires.

Our works also shows that the numerical analysis of Hallen’s equation is possible for the dimensions in the range of nanometers. Hallen’s equation was traditionally tested only for the large antennas.

A more accurate model for the conductivity of the wire can be implemented in Hallen’s equation to take into account of change in the conductivity of the wire.
Figure 4.15: Current Distribution for Two Element Array
due to a lack of electrons while doing the calculations in nano scale. This will be a valuable extension for this work.

Similarly a nanofiber mat, the application of which is assumed to be in the range of medical industry to space explorations, can be treated as an array of large number of the linear nanoantennas. Calculation of a current distribution and a power profile for such a mat can be other extension of this work.
BIBLIOGRAPHY


APPENDICES
APPENDIX A

CALCULATION OF GREEN’S FUNCTION

Wave equation in frequency domain is written in form of Helmholtz equation. We got the following non dimensionalized Helmholtz equation.

\[ \nabla^2 A + \omega^2 \mu \epsilon A = -\mu J \]  \hspace{1cm} (A.1)

Corresponding Green’s function would be the solution for the following PDE.

\[ \nabla^2 G + \kappa^2 G = \delta(x - x', y - y', z - z') \]  \hspace{1cm} (A.2)

The Green’s function \( G \) should satisfy the following:

1. \( G \) has to satisfy \( \nabla^2 G + \kappa^2 G = 0 \)

2. \( G \) has to be continuous

3. \( G \) has to satisfy the following at point \( z' \)

From A.2

\[
\iiint_V (\nabla^2 G + \kappa^2 G) dV = \iiint_V \delta(x - x', y - y', z - z') \]  \hspace{1cm} (A.3a)

\[
\iiint_V div(\nabla G) + \iiint_V (\kappa^2 G) = 1 \]  \hspace{1cm} (A.3b)

\[
\int \frac{\partial G}{\partial n} dS + \kappa^2 \iiint_V (G) dV = 1 \]  \hspace{1cm} (A.3c)
In our case, we can consider Green’s function to be spherically symmetric so we may write:

\[ \nabla^2 G = \frac{\partial^2 G}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial G}{\partial \rho} \]  

(A.4a)

where \( \rho \) is given by:

\[ \rho = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} \]  

(A.4b)

Helmholtz will then becomes the following:

\[ \frac{\partial^2 G}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial G}{\partial \rho} + \kappa^2 G = 0 \]  

(A.5a)

\[ \rho \frac{\partial^2 G}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial G}{\partial \rho} + \kappa^2 \rho G = 0 \]  

(A.5b)

\[ \frac{\partial^2}{\partial \rho^2} (\rho G) + \kappa^2 (\rho G) = 0 \]  

(A.5c)

Assuming \( u = \rho G \)

A.5c can be rewritten in terms of \( u \) as:

\[ \frac{\partial^2 u}{\partial \rho^2} + \kappa^2 u = 0 \]  

(A.5d)

For this homogeneous equation the solution can be written as:

\[ u = c_1 e^{i\kappa \rho} + c_2 e^{-i\kappa \rho} \]  

(A.6)

Finally A.6 can be written in terms of \( G \) as:
\[ G = \frac{A}{\rho} e^{i\kappa \rho} + \frac{B}{\rho} e^{-i\kappa \rho} \quad (A.7) \]

For incoming waves we only need second part of \( G \)

\[ G = \frac{B}{\rho} e^{-i\kappa \rho} \quad (A.8) \]

To determine coefficient \( B \) we use the properties of Green’s function

\[ \frac{\partial G}{\partial \rho} = B \left( \frac{-i\kappa}{\rho} - \frac{1}{\rho^2} \right) e^{-i\kappa \rho} \quad (A.9) \]

For \( \epsilon \to 0 \)

\[ B \left( \frac{-i\kappa}{\rho} - \frac{1}{\rho^2} \right) e^{-i\kappa \rho} 4\pi \rho^2 + \kappa^2 \frac{4}{3} \pi \rho^3 = 1 \quad (A.10a) \]

\[ -B.4\pi = 1 \quad (A.10b) \]

\[ B = -\frac{1}{4\pi} \quad (A.10c) \]

This results in the following final value of \( G \) which can be used to get the solution for the Helmholtz equation.

\[ G = -\frac{e^{-i\kappa \rho}}{4\pi \rho} \quad (A.11) \]
APPENDIX B
SECOND APPENDIX: CALCULATION OF HOMOGENEOUS SOLUTION AND VERIFICATION OF PARTICULAR SOLUTION

The equation of our concern is:

\[ \frac{\partial^2 A_z}{z^2} + \kappa^2 A_z = -\frac{j\kappa^2}{\omega} V_0 \delta(z) \] (B.1)

For calculation of homogeneous equation we look at the homogeneous differential equation:

\[ \frac{\partial^2 A_z}{z^2} + \kappa^2 A_z = 0 \] (B.2)

Assuming solution to be \( A_z = e^{\lambda z} \) The characteristic equation is:

The roots for this are \( i\kappa \) and \( -i\kappa \), so we obtain two solutions for this, which are \( A_{z1} = e^{i\kappa z} \) and \( A_{z2} = e^{-i\kappa z} \). So the general solution for this is:
\[ A_z = c_1 e^{i\kappa z} + c_2 e^{-i\kappa z} \]  \hspace{1cm} (B.3a)

\[ A_z = c_1 (\cos \kappa z + i \sin \kappa z) + c_2 (\cos \kappa z - i \sin \kappa z) \]  \hspace{1cm} (B.3b)

\[ A_z = (c_1 + c_2) \cos \kappa z + i(c_1 - c_2) \sin \kappa z \]  \hspace{1cm} (B.3c)

\[ A_z = C_1 \cos \kappa z + C_2 \sin \kappa z \]  \hspace{1cm} (B.3d)
APPENDIX C

PROGRAM DESCRIPTION

Kernel Calculation: nano_ker.m does the kernel calculation. The input and output arguments are listed below:

- Input z The point at which value of kernel is to be calculated
- Input a Radius of the wire
- Input 'met' Method which can take form exact('ex'), or approximate ('ap') form.
- Output G Value of Kernel

Current Calculation: Two programs currenta.m and currente.m does the current calculation for approximate kernel method and exact kernel method respectively. Both of the functions have same input and output variables. The only difference lies in the internal of the program, which either calls exact kernel or approximate kernel.

- Input L Length of the nanowire in mm
- Input a Radius of the nanowire in nm
- Input N Number of divisions in one half of the antenna
• Output I Discrete current values

• Output z Points at which the values of currents are calculated

**Radiation Pattern Calculation**: `radpat.m` calculates radiation pattern for given antenna. It considers each element of observation as an element of dipole array and uses the formula of antenna array to calculate the total $E$ field of antenna. Its input and output parameters are listed below.

• Input I Discrete values of current at different points

• Input z The points at which the current is being calculated

• Input $R$ Magnitude of $E$ field

• Output theta Points at which $E$ field is being calculated