VIBRATION-BASED ENERGY HARVESTING

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ABSTRACT

Advances in electronic and consumer technology are increasing the need for smaller, more efficient energy sources. Thus vibration-based energy harvesting, the scavenging of energy from existing ambient vibration sources and its conversion to useful electrical power, is becoming an increasingly attractive alternative to traditional power sources such as batteries. Energy harvesting devices have been developed based on a number of electro-mechanical coupling mechanisms and their design must be optimized to produce the maximum output for given environmental conditions. While the role of nonlinearities in the components has been shown to be significant in terms of the overall device efficiency, few studies have systematically investigated their influence on the system performance. Crawley and Anderson (1990) provided experimental evidence that a linear model for the piezoelectric coupling coefficient was not valid when large strains were applied. This was again seen in the work of du Toit (2005), who modeled and designed a MEMS piezoelectric vibration energy harvester. He noticed that his model, based on linear constitutive relations for the piezoelectric material, consistently under-predicted the experimental voltage produced from his device. Therefore, this thesis concentrates on the effects of nonlinear piezoelectric coupling models on the power output of an electromechanical energy harvester. An analytical study of
a one-dimensional mechanical attachment was conducted using a Poincaré-Lindstedt
perturbation analysis, including the effects of nonlinearities in both the stiffness and
the electro-mechanical coupling. The response was then compared against numerical
simulations of the original system, focusing on the relationship between the power
generated by the device, the ambient vibration characteristics, and the nonlinearities
in the system.
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CHAPTER I

INTRODUCTION

The increased need for smaller, more efficient energy sources has made the study of energy harvesting more imperative in scientific research and development. Many devices have been developed using a number of electro-mechanical coupling mechanisms, but the power produced by these devices was a concern. The most recent studies have concentrated on improving the power output by optimizing the electrical components of the harvesting device or by implementing an actuator into the mechanical component to increase the mechanical input. While these studies have moderately improved the power output for harvesting devices, the difficulties in correctly predicting the power output discourage many researchers. Taking into consideration nonlinearity in the mechanical input or the possibility of nonlinearity in the electro-mechanical coupling could alleviate the much of the inconsistencies in predicting the power output of the designed energy harvester. This chapter introduces the concept of energy harvesting, the different techniques used to develop a harvesting device and the difficulties that arise in the development.
1.1 Energy Harvesting

Energy harvesting is the process in which the energy created by some mechanical vibration or other mechanism is converted into electrical energy. Converting ambient vibrations into energy requires a mechanical oscillator, therefore a one-dimensional spring-mass damper system is often used to model the energy harvesting system. Figure 1.1 displays a one-dimensional spring-mass system, where $u(t)$ is the input force (for vibrations $u(t) = U_0 \sin(\omega_b t)$) and $c_{\text{mech}}$ and $c_e$ are the mechanical and electrical damping of the system. The equation of motion for such a system as Fig. 1.1 is

$$m\ddot{x} + c(\dot{x} - \dot{u}) + k(x - u) = 0,$$

where $c$ is the total damping ($c = c_{\text{mech}} + c_e$), $k$ is the spring stiffness, and $x$ is the absolute displacement. Equation 1.1 is in terms of the absolute position, but can be in terms of the relative displacement by defining $x(t) = u(t) + z(t)$ and substituting

Figure 1.1: A one-dimensional spring-mass damper system modeling energy harvesting is illustrated above.
in for \( x(t) \). The resulting equation of motion is

\[
m\ddot{z} + c\dot{z} + kz = -m\ddot{u}.
\] (1.2)

Then by substituting in for \( u(t) \) and dividing by the mass the following equation is obtained

\[
\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2z = \omega_b^2U \sin(\omega_b t),
\] (1.3)

where \( \omega_n^2 = k/m, \ c/\sqrt{km} = 2\zeta, \) and \( U \) is the non-dimensional applied force. By assuming a harmonic response of \( z(t) = Z_0 \sin(\omega_b t - \phi) \) and substituting into Eqn. 1.3, a solvable equation for \( Z_0 \) is obtained,

\[
[\omega_n^2 - \omega_b^2] Z_0 \sin(\omega_b t - \phi) + 2\zeta\omega_n\omega_b Z_0 \cos(\omega_b t - \phi) = \omega_b^2 U \sin(\omega_b t).
\] (1.4)

Equation 1.4 is solved by using the trig identity of

\[
a \sin(\omega t - \phi) + b \cos(\omega t - \phi) = \sqrt{a^2 + b^2} \sin(\omega t),
\] (1.5)

where \( \phi = \tan^{-1}(a^2/b^2) \). The solution to Eqn. 1.4 gives

\[
Z_0 = \frac{\omega_b^2 U}{\sqrt{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}}
\] (1.6)

or by multiplying the numerator and the denominator by \( 1/\omega_n^2 \) the non-dimensional form of the amplitude is derived,

\[
Z = \frac{r^2 U}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}.
\] (1.7)

Therefore, the forced steady-state response solution is

\[
z(t) = \frac{r^2 U}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \sin(\omega_b t - \phi),
\] (1.8)
where $\phi = \tan^{-1}(2\zeta r/[1 - r^2])$, $r = \omega_b/\omega_n$ and $\zeta$ is defined as the damping ratio.

Note, the transient response is not derived in this section because energy harvesting optimal performance is in the steady-state response. A more detailed derivation of the transient and steady-state response solutions from a spring-mass damper system can be found in Appendix A.

The instantaneous power absorbed by the damper ($c_e$) is

$$P_{inst} = c_e(\dot{z})^2,$$

where $\dot{z} = \omega Z \cos(\omega t - \phi)$ and for simplicity reasons $\omega = \omega_b$. Therefore, the average power can then be calculated by integrating over the whole period (T) to achieve Eqn. 1.10.

$$P_{avg} = \frac{1}{T} \int_0^T c_e \omega^2 Z^2 \cos^2(\omega t - \phi) \, dt$$

(1.10)

Knowing that the period is $2\pi/\omega$, Eqn. 1.10 can be reduced to

$$P_{avg} = \frac{c_e \omega^2 Z^2}{2\pi} \left[ \frac{1}{2} t + \frac{1}{4} \sin(2\omega t + \phi) \right]_0^{\frac{2\pi}{\omega}}$$

$$= \frac{c_e \omega^2 Z^2}{2\pi} \left( \frac{\pi}{\omega} \right)$$

$$= \frac{c_e \omega^2 Z^2}{2},$$

(1.11)

which can then be non-dimensionalized by dividing by $Z^2/(\omega_n m)$ to give

$$\hat{P}_{avg} = \frac{\zeta r^2}{(1 - r^2)^2 + 4\zeta^2 r^2}. $$

(1.12)

The power produced by Eqns. 1.9 and 1.11 depends on the displacement amplitude. From Eqn. 1.7, the resulting amplitude ($Z$) of the force displacement
magnitude is dependent on the damping ratio ($\zeta$) and $r$. Figures 1.2 and 1.3 display this influence on the amplitude response,

$$
\frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}},
$$

and the average power for different values of $\zeta$. Notice, as the damping ratio, $\zeta$, becomes larger the amplitude decreases dramatically. This makes sense because the more damping in the system the smaller the mechanical amplitude, which directly corresponds to the power output. Therefore, the damping ratio is a very important consideration when trying to obtain optimal power output from an energy harvester.

Figure 1.2: The above graph displays how the damping ratio, $\zeta$, influences the amplitude response.
Figure 1.3: Plotted above is the average power dependence on the frequency ratio, $r$.

Research conducted by [15] revealed the existence of a singularity in the optimal power output at low damping ratios (below a bifurcation damping ratio). Certain electrical circuit elements were thus needed to regulate the resistor and inductor to achieve constant optimal power. This theory was not valid for all frequency ratios when the damping ratio was below the bifurcation damping ratio. Since the damping ratio consists of the mechanical and electrical damping terms ($c_{\text{mech}}$ and $c_e$), an investigation on how the electrical damping effects the power output was needed. Note, the mechanical damping was a property of the materials forming the harvester and can only be changed by modifying the dimensions of the harvester. Therefore only the electrical damping contributions to the power output were investigated. Equation 1.11 was used in a MatLab program named FrequencyRatio.m (Appendix B),
where $\omega = \omega_n = 1$, $m = 0.1$, $k = 0.1$, $c_{mech} = 0.1$, $Z = 1$, and $c_e$ was varied.

Figure 1.4 displays how the non-dimensionalized average power was maximized with an electrical damping equivalent to the mechanical damping. Therefore, to obtain an optimal power output the electrical and mechanical damping should be the same order of magnitude.

Figure 1.4: The optimal electrical damping for the largest power output is displayed in the above graph, (a) log-log scale (b) cartesian scale.

The power produced also greatly depends on the technique used to harvest the mechanical energy. There are four basic techniques that are used to convert mechanical energy to electrical energy: electromagnetic, piezoelectric, magnetostrictive, and electrostatic. Each of these technique’s power output depends on a variety of factors such as the input vibration, the system coupling coefficient, the quality factor
of the device, the mass density of the generator, and the degree of which the electrical load maximizes power transmission [17]. This power is rated by the efficiency of the system. The standard definition of efficiency is
\[
\eta = \frac{P_{\text{out}}}{P_{\text{in}}},
\]
(1.14)

One factor that has an enormous amount of effect on the efficiency and the power output is the coupling coefficient. The coupling coefficient relates the total energy put into the system to the amount of energy converted by the transducer (energy stored at output divided by the total energy input). The following sections discuss these factors as they are used in the different energy harvesting techniques.

1.1.1 Electromagnetic Energy Harvesting

An electromagnetic energy harvester consists of an oscillating spring-mass system attached to a conductor (coil) or a magnet. When this system vibrates the coil passes through a magnetic field created by a magnetic core. Not all of the energy from the mechanical system can be transmitted to the load (electrical system), because some energy is lost within the coil of the conductor. Therefore, the resistance in the coil must be taken into consideration as seen in Eqn. 1.15 [3], where \( P_{L\text{max}} \) is the power in the load, \( m \) is the mass of the system, \( \omega_n \) is the natural frequency, \( Y \) is the amplitude of the harmonic base excitation, \( \xi_p \) is the mechanical losses, \( R_{\text{load}} \) is the load resistance and \( R_{\text{coil}} \) is the coil resistance. A significant amount of energy is also lost from the parasitic loss mechanisms (air damping) and the transduction mechanism.

\[
P_{L\text{max}} = \frac{m\omega_n^3Y^2}{16\xi_p} \times \frac{R_{\text{load}}}{R_{\text{load}} + R_{\text{coil}}}
\]
(1.15)
Modifying the resistances affect the damping factor and thus the electromagnetic transduction \( (c_E) \), which can be estimated as

\[
c_E = \frac{(NlB)^2}{R_{\text{load}} + R_{\text{coil}} + j\omega L_{\text{coil}}} \quad (1.16)
\]

is also affected. In Eqn. 1.16, \( B \) is the flux density to which the coil is subjected, \( l \) is the side length of the coil, \( N \) is the number of turns in the coil, and \( L_{\text{coil}} \) is the coil inductance.

One advantage of electromagnetic energy harvesting is that it does not require a separate voltage source, but a permanent magnetic field is required and that could consist of bulky magnets. The output voltage is normally small as well, which makes it necessary to transform the voltage into useable levels.

### 1.1.2 Piezoelectric Energy Harvesting

Piezoelectric Energy Harvesting is based on the piezoelectric effect which exists in two domains. The first domain is the ability to convert an applied electrical potential to a mechanical strain and the second is the converse effect, the ability to convert mechanical strain into electrical charge. The conversion of mechanical strain to electrical charge process is utilized in energy harvesting. A piezoelectric energy harvester depends greatly on the elastic and dielectric properties of the piezoelectric material. As the loads change on the system the electric properties of the system also change, but the stiffness of the material will always be larger than the stiffness of the
structure. The constitutive equations for a piezoelectric material are

\[
\begin{bmatrix}
  S \\
  D
\end{bmatrix} = \begin{bmatrix}
  s & d \\
  d & \epsilon
\end{bmatrix} \begin{bmatrix}
  T \\
  E
\end{bmatrix},
\]  
(1.17)

where \( S \) is the strain, \( D \) is the electric displacement, \( s \) is the compliance, \( d \) is the electro-mechanical or piezoelectric coupling coefficient, \( \epsilon \) is the dielectric constant, \( T \) is the stress and \( E \) is the electric field. The material typically exhibits anisotropic characteristics, and therefore its properties differ depending on the direction of the forces and the orientation of polarization and electrodes. The power output produced by a piezoelectric energy harvester can be modified by changing the thickness of the piezoelectric material or by creating multiple layers of the piezoelectric material. As the thickness or the number of layers is increased the power output improves. Other aspects of the harvester are also affected by the dimensions of the piezoelectric material. Piezoelectric energy harvesters produce high voltages and low currents and as the size of the device decreases the current also decreases. This can have a very large effect on the usage of the power output. The largest advantage of using piezoelectrics as energy harvesters is that no external voltage source is needed. Therefore, the energy harvester can be manifested into smaller portable devices easier.

The equations of motion for a piezoelectric energy harvester are

\[
m\ddot{x} + c\dot{x} + kx - \frac{d(x)}{C}Q = A\sin(\omega t) \quad \text{and} \quad (1.18)
\]

\[
V = \frac{1}{C}Q - \frac{d(x)}{C}x, \quad (1.19)
\]
where $C$ is the capacitance of the piezoelectric material, $q$ is the charge, $A\sin(\omega t)$ is the input force, and $V$ is the voltage. Note that $d(x)$ represents the piezoelectric constant but its linearity/nonlinearity is dependent on the strain ($x$). Piezoelectric energy harvesters and their nonlinear behavior will be discussed in great detail in the later chapters.

1.1.3 Magnetostrictive Energy Harvesting

The magnetostrictive technique is very similar to piezoelectric energy harvesting. Its output power is dependent on the design parameters such as a cylindrical coil. A magnetostrictive material’s constitutive equations are given by

$$
\begin{bmatrix}
S \\
B
\end{bmatrix} =
\begin{bmatrix}
s & d \\
d & \mu
\end{bmatrix}
\begin{bmatrix}
T \\
H
\end{bmatrix},
$$

where $B$ is the magnetic flux, $\mu$ is the permeability constant and $H$ is the magnetic field strength. Note, that these constitutive equations are vary similar to the piezoelectric material but with the replacement of $D$, $E$, and $\epsilon$ with $B$, $H$, and $\mu$. The equations of motion are therefore equivalent to Eqns. 1.18 and 1.19.

1.1.4 Electrostatic Energy Harvesting

The basis of any electrostatic energy harvester consists of two conductors separated by a capacitor and the two components move relative to one another. When the conductors move the energy stored in the capacitor changes. This energy can then be converted into usable electrical energy. Electrostatic energy harvesting depends heavily on the geometry and operating conditions, such as the position and a constant
voltage or charge of the harvester. An example of a MEMS electrostatic transducer is two parallel capacitors. The range of motion of the energy harvester must be one hundred times greater than the minimum distance between the two plates. As this distance decreases, the power and efficiency of the energy harvester will decrease dramatically. This technique is also very easy to implement into micro-systems, but has a disadvantage of needing a separate voltage source.

The equation of motion for an electrostatic transducer energy harvester as seen in Fig. 1.5(a) is

\[ m\ddot{z} + (b_m + b_e)\dot{z} + kz = -m\dot{y}, \]  

(1.21)

where \( b_m \) and \( b_e \) represent the mechanical and electrical damping and both differ depending on the type of electrostatic converter is used (in-plane overlap, in-plane gap closing, etc), [18]. A simple charge pump circuit representation (Fig. 1.5(b)) can be used to calculate the output energy. The resulting output power is represented by Eqns. 1.22 and 1.23, [18], where \( C_{\text{max}} \) and \( C_{\text{min}} \) are the maximum and minimum
capacitance of the variable capacitor \((C_v)\), \(C_{\text{par}}\) is the parasitic capacitance, \(V_{\text{max}}\) is the maximum voltage on \(C_v\), and \(V_{\text{in}}\) is the input voltage.

\[
E = \frac{1}{2} V_{\text{in}}^2 (C_{\text{max}} - C_{\text{min}}) \left( \frac{C_{\text{max}} + C_{\text{par}}}{C_{\text{min}} + C_{\text{par}}} \right) \tag{1.22}
\]

\[
E = \frac{1}{2} V_{\text{max}} V_{\text{in}} (C_{\text{max}} - C_{\text{min}}) \tag{1.23}
\]

Figure 1.5: The above diagram represents (a) a general vibration converter and (b) a simple circuit for an electrostatic converter, where \(C_{\text{stor}}\) is the storing capacitor [18].
Piezoelectric energy harvesting is favored among researchers due the fact an external voltage source is not needed. Research on piezoelectrics has appeared as early as the 1970’s, but only within the last two decades has its use with energy harvesting grown due the need of more efficient energy sources. The following sections give a brief background of the usage of piezoelectrics in energy harvesting and the difficulties predicting the power output that has lead to the reasoning behind the research completed in this thesis.

2.1 Previous Work

In 1997, [20] researched the theoretical and experimental considerations of the energy storage characteristics of a piezoelectric harvester. The harvester, developed previously by the authors, consisted of a steel ball and a piezoelectric vibrator. When the ball impacted the piezoelectric material, electrical energy was produced and a circuit containing a bridge-rectifier and a capacitor stored that energy. It was found that the output voltage depended upon the capacitance of capacitor connected to the rectifier and after numerous impacts the voltage decayed. The resulting harvester achieved a maximum efficiency of 35% under conditions of high initial voltage.
Building off of the previous research, [4] investigated the power requirements of a piezoelectric actuator operating in a damping augmentation loop. This was based on the knowledge that piezoelectric actuators are effective in suppressing vibrations in control systems. A clear understanding of the electro-mechanical power flow characteristics of a piezoelectric actuator was necessary though for the actuators to be implemented into the design of the amplifiers. While the piezoelectric actuator was driven by a current amplifier, the accelerated feedback augmented the damping of the structure without changing the stiffness. This became very useful in determining the real and reactive power flow through the structure and actuator into the amplifier. From their investigation, it was concluded that controlling the current through the actuator to be proportional to the acceleration actively dampened the mechanical structure. This allowed the mechanical energy from external sources to be absorbed by the piezoelectric actuator and transmit it to an electrical source. The control of piezoelectric actuators was a great concern in energy harvesting, but another concern was the actual energy conversion mechanisms.

In 2003, [18] investigated and evaluated different energy conversion mechanisms for both Micro Electro-mechanical Systems (MEMS) and piezoelectric converters. The mechanisms studied were piezoelectric, electrostatic and electromagnetic. In electrostatic energy conversion, the basis was a variable capacitor in which the capacitance was inversely proportional to the voltage. These converters were very easy to integrate into a microsystem, but needed a separate voltage source. Piezoelectric converters did not require a separate voltage source. It was concluded that piezoelectric
converters were capable of converting more power per unit volume than electrostatic and electromagnetic converters, but an in-plane gap closing capacitor was the best design when the electrostatic and electromagnetic converters were compared.

Also in 2003, [5] demonstrated a robust strain energy harvesting system for powering wireless devices. The system consisted of a composite material laminated with unidirectional aligned piezoelectric fibers. Then the material was subject to three point cyclic bending using an electrodynamic actuator. The strain energy was stored by rectifying the piezoelectric fiber output into a capacitor until a preset threshold was met. Then the charge was transferred to a wireless sensor node. It was found that the time required to charge the storage capacitor from 2.5 to 9.5V was inversely proportional to the rate at which the energy was harvested. When a higher strain was applied, the required amount of time decreased (rate of energy harvesting increased). It was also found that the amount power output was proportional to the applied frequency. The devise designed in this paper produced enough energy to power a transmitter for approximately 250ms.

Not only are wireless devices requiring more efficient energy sources, implementing energy harvesting into MEMS devices has also become a necessity. A study of microgenerators was completed by [12] in 2004. They conducted an analytical study between different vibration-driven MEMS microgenerators design and functionality. Then their results were verified against full time-domain simulations. There were three different types of generators studied: velocity-damped resonant-generator (VDRG), Coulomb-damped resonant-generator (CDRG) and Coulomb-
force parametric-generator (CFPG). The CFPG was a nonlinear generator and did not operate in a resonant manner. It was found that this nonlinear generator was more applicable when the frequency was likely to vary and the amplitude of the vibration sources was small compared to the mass-to-frame displacement. For a machine which produced vibrations of 2nm at 2500Hz, the estimated power densities were near 800µW/cm³. The other two generators produced that same amount of power at resonant frequency, but produced more power when they were above and below resonant frequency.

Generating an ample amount of power from an energy harvester is a definite concern in energy harvesting, but implementing the power into everyday uses is an even larger concern. In 2004, [11] researched the different approaches to powering micro-sensor nodes. He discussed the advantages of exhaustible resources such as lithium batteries and fuel and the power scavenging sources such as solar cells, radio frequency, and thermal energy. A device that harvested power from vibrations or body motions was designed and it was concluded that with devices such as the one developed, the need for finite energy sources would decrease immensely. One year later, [1] researched different vibration excitation devices that had been developed. These devices included piezoelectric materials and electrostatic generators in which they were design to be used for low-power applications. Integrating the energy harvesters into real-life usages consisted of many applications such as inserting a piezoelectric material in to the sole of a shoe for energy storage, an ID encoder and a radio transmitter. This article provided insight on the many applications for
energy harvesting. One application that has obtained particular interest was the use of an energy harvester in a backpack. [16] developed a suspended-load backpack that converts mechanical energy from the vertical movement of carried loads to electricity. They discovered that a small amount of extra energy was required during the electricity generation, due to the change in gate or loading regime, which reduces the metabolic power needed for walking. This study led to an prototype that provided a electrical power output near 19.1W.

Predicting the available power output from a designed device has been a struggle throughout all of the energy harvesting research. In 2005, [8] wrote a thesis about modeling and the design of a piezoelectric energy harvester. He used linear constitutive relations for a piezoelectric material to predict the amount of voltage produced at sinusoidal and random vibrations. Then he compared his prediction to an actual model he designed subjected to the same vibrations. The compared results showed that the predicted voltage was always lower than the actual voltage produced when the frequency was near resonance. Then as the frequency moved away from resonance the correlation was excellent. It was also seen that at higher electrical loads the predicted voltage and power was again consistently lower than the actual amount of voltage and power produced by the model. This phenomenon was previously recorded by [6]. In the late 1980’s and early 1990’s, [6] studied a piezoceramic beam as different strains were applied to the beam. They found there were four ”nonidealities” when the linear relationship between the electric field and strain broke down: depoling of the ceramic, the dependence of the piezoelectric constant ($d_{3i}$) on strain, electric
field-strain hysteresis, and the frequency dependence and creep. Figure 2.2 displays this nonlinear relationship. Out of those four "nonidealities", the dependence of the piezoelectric constant on strain displayed the most nonlinear electric field and strain relationship.

In 2007, [9] showed that the single degree of freedom (SDOF) model used to obtain a general solution yielded inaccurate results. A correction factor was needed in the electro-mechanically coupled equations to improve the results. They studied a clamped-free beam with arbitrary translation and small rotation applied at the base of the beam. An Euler-Bernoulli beam model was used to obtain the general solution and then those solutions were compared to the SDOF model. Without the correction factor, the SDOF model could only be used when the beam had a tip mass with a sufficiently high tip mass-to-beam mass ratio. The ratio for this experiment was 1.33 and they found a correction factor of 1.0958 was needed to correctly predict the analytical results. It was also found that the correction factor depended on the mode of excitation, which added another difficulty in predicting the power output.

Since an accurate prediction of the power output is very difficult to obtain, improving the performance of the energy harvester has become of great interest. An alternative circuit to improve the performance of a vibration-based energy harvester was developed in 2007 by [15]. This was achieved by adding an inductor to the harvesting circuit and resulted in higher amounts of voltage being stored. The design was based off of [8], but with the new design the maximum power values were obtained at any frequency ratio. During the study though, singularities were found at low
damping ratios. Therefore, a bifurcation damping ratio was determined and the condition that the damping ratio had to be above the bifurcation damping ratio (for the singularities to be avoided) was established. The addition of the inductor changed the previously recorded results greatly and constant power was harvested at any frequency ratio. They also concluded that the value of the optimal power was independent of the coupling coefficient, frequency ratio, and optimal electric elements, which contradicted previous work. From their findings, when the coupling coefficient increased it did not necessarily mean the power also increased and optimal power was not achieved at all frequency ratios unless the damping ratio was higher than the bifurcation ratio. If this last condition was met it was shown that the addition of the inductor to the circuit greatly improved the power harvested from the previously used circuits.

Nonlinear energy harvesting has recently become of interest to researchers in the energy harvesting field. Whether this nonlinearity is included in the stiffness, damping, or even the constitutive equations, certain aspects of the power output are changed and in some cases the efficiency is improved. Recently, [19] studied a harmonically forced linear oscillator with an attached nonlinear energy sink experiencing an external forcing frequency near resonance. From previous research ([2]; [10]; [13]; [14]; [19]), it was known that essentially nonlinear local attachments could passively absorb energy from transiently loaded linear subsystems and act as a nonlinear energy sink (NES). When the system was near resonance, the response was quasiperiodic rather than a simple periodic response. Therefore, a model of a linear
oscillator and a strongly nonlinear attachment (pure cubic nonlinearity) was analyti-
cally investigated. Four types of bifurcation mechanisms were observed: Saddle-Node
bifurcation and generation of three periodic solutions, Hopf bifurcation of a single pe-
riodic solution, Hopf bifurcation of one of the periodic solution in the region of the
three periodic solutions and a Hopf bifurcation of two periodic solutions in the re-
region of three solutions. In the final analysis, the periodic and quasiperiodic regimes
that were analytically analyzed were verified numerically. At the ASME conference in
2007, [14] discussed the usage of strongly nonlinear vibrating attachments for harvest-
ing energy. Numerical simulations of linear and nonlinear SDOF energy harvesting
models were compared. The coupling between the mechanical and electrical compo-
nents of the model had a significant effect on the power output. It was found that
linear vibration-based energy harvesting devices did not perform well when ambient
vibrations of a lower frequency of the design frequency of the device were applied. As
for the nonlinear device, there was a broadband (multiple frequencies) passive energy
absorption due to the ability of the device to resonate at a wide range of frequen-
cies. This attribute then eliminated the need for tuning the device and the efficiency
of the nonlinear design was nearly two orders of magnitude greater than the linear
design. Since the use of nonlinearities significantly influences the efficiency of energy
harvesters, a more in-depth look at the nonlinearities in the piezoelectric effect and
in-particular the work conducted by [6] is discussed in the next section.
2.2 Nonlinearity In Piezoelectric Effect

One of the most important constants affecting the power output is the electro-mechanical or piezoelectric coupling coefficient \((d)\). This describes the efficiency in which the energy is converted by the material between the electrical and mechanical energy forms. [6] completed extensive research on the coupling coefficient. They found that there is a break down in the linear response of the piezoelectric material as a result of four “nonidealities”, but the field-strain relationship shows the most evidence of nonlinearity. The applied electrical field \((E)\) is related to the actuation strain \((\Lambda)\) by the piezoelectric coefficient \((d_{ij})\),

\[
\begin{bmatrix}
0 & 0 & d_{31} \\
0 & 0 & d_{31} \\
0 & 0 & d_{33} \\
0 & d_{13} & 0 \\
d_{13} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
E_3 \\
E_2 \\
E_3
\end{bmatrix}.
\]

If the strain is in the \(x_1-x_2\) plane, the relationship is then

\[
\Lambda_1 = d_{31}E_3.
\]

The subscripts on the piezoelectric coefficient describe how the atoms are poled and the direction the strain is applied. If the strain is in the 1-2 direction a \(d_{31}\) coefficient is used, while if the strain is in the 1-3 or 3-1 direction the \(d_{13}\) coefficient
Figure 2.1: Illustrated above is an example of $d_{33}$ and $d_{31}$, where the atoms are poled in the 3 direction and strained in the 3 and 1-2 direction.

is used. The $d_{33}$ coefficient is used when the strain is applied only in the 3-direction. Figure 2.1 illustrates these strains applied to a theoretical cantilever beam. [6] took this electrical field and strain relationship and compared it to experimentally tested PZT plates, see Fig. 2.2. Their results indicated that a secant piezoelectric coefficient, $d_{31}^*$, was needed to adequately relate the strain to electric field such that

$$d_{31}^* \equiv \frac{\Lambda_1}{E_3}. \quad (2.3)$$

The inclusion of $d_{31}^*$ still created a linear problem, therefore a nonlinear dependence was still needed. From previous studies, [7] found a nonlinear dependence on the level of induced strain as seen in Fig. 2.2. The induced strain is the strain that actually appears in the piezoelectric structure that is due to the piezoelectric actuation. With the inclusion of the secant piezoelectric coefficient, estimated from Fig. 2.2 at large strains, a more efficient description of the piezoelectric properties was obtained.
Figure 2.2: The above graph displays the $d_{31}^*$ dependence on induced strain. Note, the experimental fit is taken from [6].

The research conducted by [6] gave great insight on why there were inconsistencies in predicting the power output of designed energy harvesters. Unfortunately, the influence of the nonlinearities in the components on the systems overall performance was not addressed. Therefore, the following chapters will concentrate on the effects of nonlinear piezoelectric coupling models on the power generation of an electro-mechanical energy harvester.
CHAPTER III

ELECTRO-MECHANICAL MODEL

3.1 One-Dimensional Model

The weakly nonlinear energy harvesting system in this thesis was based off a spring-mass damper system with a base excitation, Fig. 1.1. Along with the spring and damper there was an electrical component from the piezoelectric material, Fig. 3.1. This resulted in a mechanical response of

\[ m(\ddot{u} + \ddot{z}) + g(z, \dot{z}, Q) = 0, \text{ with} \]

\[ g(z, \dot{z}, Q) = k z(1 + a z^2) + b \dot{z} - \frac{d(z)}{C} Q. \]

The electro-mechanical constitutive relation for the piezoelectric material was

\[ V = -\frac{d(z)}{C}z + \frac{Q}{C}. \]

In Eqns. 3.1 to 3.3, \( m \) is the mass, \( Q \) is the charge, \( V \) is the voltage, \( d(z) \) is the piezoelectric coupling coefficient, \( C \) is the piezoelectric capacitance, \( k \) is the stiffness of the spring, \( b \) is the total damping of the system and \( a \) is a unitless coefficient. Then
assuming a resistive electrical load, \( V = -RQ \), the coupled mechanical equations were derived to be

\[
\begin{align*}
    m\ddot{z} + k z(1 + a z^2) + b\dot{z} - \frac{d(z)}{C} Q &= -m\ddot{u}, \\
    R\dot{Q} - \frac{d(z)}{C} z + \frac{Q}{C} &= 0.
\end{align*}
\]

(3.4) \hspace{1cm} (3.5)

3.2 Nonlinear Energy Harvesting System

The coupled mechanical equations were first non-dimensionalized with a goal of being able to independently vary the nonlinearity, damping, coupling and the excitation of Eqns. 3.4 and 3.5. Therefore, scaling the coordinates and time as \( z = Lx, \ u = Lv \) and \( Q = n q \) and \( t = \Omega \tau \) in which

\[
\begin{align*}
    \frac{d}{dt} &= \Omega \frac{d}{d\tau} \quad \text{and} \\
    \frac{d^2}{dt^2} &= \Omega^2 \frac{d^2}{d\tau^2},
\end{align*}
\]

(3.6)
the nondimensional equation of motion reduce to Eqns. 3.7 and 3.8.

\[
m\Omega^2 L\ddot{x} + b\Omega L\dot{x} + k L x(1 + \alpha L^2 x^2) - \frac{1}{C} d(Lx)q n = -m\Omega^2 L\ddot{v}
\] (3.7)

\[
R\Omega n\dot{q} - \frac{1}{C} d(Lx) Lx + \frac{n}{C} q = 0.
\] (3.8)

A more familiar form of the coupled equations of motion were achieved by dividing through the mechanical equation by \(m\Omega^2 L\) and the electrical equation by \(n/C\), which resulted in

\[
\ddot{x} + \frac{b}{m\Omega} \dot{x} + x(1 + \alpha L^2 x^2) - d(Lx) \frac{n}{C m\Omega^2 L} q = -\ddot{v} \quad \text{and}
\] (3.9)

\[
RC\Omega \dot{q} + d(Lx) \frac{L}{n} x + q = 0.
\] (3.10)

The non-dimensional piezoelectric coefficient was \(\hat{d}(x) = (L/n)d(Lx)\) and the small parameter \(\epsilon\) was defined as \(\epsilon = n^2/(CkL^2)\). Then by letting \(\Omega^2 = k/m\), \(\epsilon\alpha \equiv aL^2\), \(2\epsilon\varsigma \equiv b/(m\Omega)\), and \(\rho \equiv RC\Omega\), equations 3.11 and 3.12 were obtained.

\[
\ddot{x} + 2\epsilon\varsigma \dot{x} + x(1 + \epsilon\alpha x^2) - \epsilon\beta(1 + \delta|x|)q = \epsilon\gamma \sin (\omega\tau - \phi)
\] (3.11)

\[
\rho \dot{q} - \beta(1 + \delta|x|) x + q = 0
\] (3.12)

In equations 3.11 and 3.12, it was assumed \(u(t) = U_0 \sin (\Omega t)\) so that the nondimensional excitation \(-\ddot{v}\) could be written as \(\ddot{v} = -\epsilon\gamma \sin (\omega\tau - \phi)\) where \(\omega \equiv \Omega/\sqrt{k/m}\).

The scaling assumptions described above were base on a piezoelectric energy harvester developed by [8]. Specific properties of that particular harvester were used to calculate and verify the appropriate scalings used to non-dimensionalize Eqns. 3.4 and 3.5. Table 3.1 lists these properties and the corresponding scaling assumptions calculated for the one dimensional model discussed in this paper.
Table 3.1: This table lists the energy harvester’s properties from [8] and the corresponding scaling for this paper.

<table>
<thead>
<tr>
<th>Energy Harvester Properties</th>
<th>du Toit ([8])</th>
<th>Corresponding Scaling</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_n$</td>
<td>671</td>
<td>$\omega_n^2$ 450 x 10^3</td>
</tr>
<tr>
<td>beam width</td>
<td>31.8 mm</td>
<td>$2\epsilon\zeta$ 4.70 x 10^{-4}</td>
</tr>
<tr>
<td>beam thickness</td>
<td>0.680 mm</td>
<td>$\epsilon\alpha$ 4.90 x 10^{-9}</td>
</tr>
<tr>
<td>beam length</td>
<td>55.0 mm</td>
<td>$\epsilon$ 1.32 x 10^{-2}</td>
</tr>
<tr>
<td>Stiffness ($k$)</td>
<td>4.76 MN/m</td>
<td>$\rho$ 0.317</td>
</tr>
<tr>
<td>beam displacement ($x$)</td>
<td>0.70 µm</td>
<td>$d(x)$ -269.5</td>
</tr>
<tr>
<td>Resistance ($R$)</td>
<td>11.0 kΩ</td>
<td></td>
</tr>
<tr>
<td>Capacitance ($C$)</td>
<td>43.0 nF</td>
<td></td>
</tr>
<tr>
<td>Mass ($m$)</td>
<td>10 g</td>
<td></td>
</tr>
<tr>
<td>Current</td>
<td>0.182 µA</td>
<td>$n$ 3.64 x 10^{-6}</td>
</tr>
<tr>
<td>Voltage</td>
<td>0.2 V</td>
<td></td>
</tr>
<tr>
<td>Time</td>
<td>0.317 s</td>
<td></td>
</tr>
</tbody>
</table>

To compensate for the nonlinearity in the piezoelectric coupling coefficient, the non-dimensionalized piezoelectric coefficient, $\hat{d}(x)$, was approximated to equal $\beta(1 + \delta|x|)$, where $\beta$ is the linear coupling coefficient and $\delta$ is the nonlinear coupling coefficient. Then the method of averaging and the Poincaré-Lindstedt method, in which $\tau^* = \omega\tau$, $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + ...$, $q = q_0 + \epsilon q_1 + \epsilon^2 q_2 + ...$ were utilized to obtain analytical approximations. Note, for near-resonant excitation $\omega = 1 + \epsilon\sigma$, so that

$$\frac{d}{d\tau} = (1 + \epsilon\sigma) \frac{d}{d\tau^*},$$

$$\frac{d^2}{d\tau^2} = (1 + \epsilon\sigma)^2 \frac{d^2}{d\tau^{*2}}.$$  \hspace{1cm} (3.13)

and for simplicity reasons in the later equations the * was dropped from $\tau^*$. The
order one equations were derived to be

\[ \ddot{x}_0 + x_0 = 0 \quad \text{and} \quad (3.15) \]

\[ \rho \dot{q}_0 - \beta (1 + \delta |x_0|) x_0 + q_0 = 0. \quad (3.16) \]

In the mechanical equation, the solution for \( x_0 \) (Eqn. 3.15) was calculated by assuming \( x_0 = \exp (m\tau) \). Then, this assumption was substituted back into Eqn. 3.15 to solve for \( m \). It was found that \( m = \pm i \). Therefore, \( x_0 = A \sin (\tau) + B \cos (\tau) \) and by using the initial condition that \( \ddot{x}(0) = 0 \), the final solution was \( x_0 = A \sin (\tau) \) with the amplitude \( A \) still unknown. This value for \( x_0 \) was then able to substituted into the order one electrical equation (Eqn. 3.16) to give

\[ \rho \dot{q}_0 + q_0 = \beta [1 + \delta |A \sin (\tau)|] A \sin (\tau) \equiv F(\tau). \quad (3.17) \]

The initial charge of Eqn. 3.17 was derived by solving the homogeneous equation,

\[ \dot{q}_0 + \frac{1}{\rho} q_0 = 0. \quad (3.18) \]

Equation 3.18 was solved using the same methods as the solution for \( x_0 \), which gave \( m = -1/\rho \). Therefore, the final solution for the initial charge was

\[ q_0(\tau) = q_{01}(\tau) \exp \left( \frac{-\tau}{\rho} \right), \quad (3.19) \]

with \( q_{01} \) still unknown. Solving for the particular solution to Eqn. 3.17 was a little more difficult.

Equation 3.17 can be considered a first order ordinary differential equation (ODE) with an excitation, \( F(\tau) \), that consists of an impulse magnitude of \( F(t) \Delta t \).
and therefore
\[ \Delta F(\tau, t) = F(t) \Delta t \delta(\tau - t). \quad (3.20) \]
Equation 3.20 assumes small time increments, \( \Delta t \), beginning at \( \tau = t \). It is known that a superposition of impulses gives an approximate solution to the ordinary differential equation, but an exact solution is obtained when \( \Delta t \to 0 \) and thus
\[ F(\tau) \cong F(t) \Delta t \delta(\tau - t). \quad (3.21) \]
The system’s response to the impulse is then
\[ x(\tau) = \int_{0}^{\tau} f(t)g(\tau - t)dt \quad (3.22) \]
and the excitation response is
\[ x(\tau) = \int_{0}^{\tau} F(\tau - t)g(t)dt. \quad (3.23) \]
Using the method described above the response of the system (particular solution) in this study was
\[ q(\tau) = \int_{0}^{\tau} \frac{F(t)}{\rho} \exp\left(\frac{-(\tau - t)}{\rho}\right)dt = \int_{0}^{\tau} \frac{F(t)}{\rho} \exp\left(\frac{(t - \tau)}{\rho}\right)dt, \quad (3.24) \]
but \( q_{01} \) from Eqn. 3.19 was still unknown. Therefore knowing \( q(\tau) = q(\tau + T) \), a solution was derived by the following steps.
\[
q_{01} \exp\left(\frac{-\tau}{\rho}\right) + \int_{0}^{\tau} \frac{F(t)}{\rho} \exp\left(\frac{(t - \tau)}{\rho}\right)dt = q_{01} \exp\left(\frac{-(\tau + T)}{\rho}\right) \\
+ \int_{0}^{\tau+T} \frac{F(t)}{\rho} \exp\left(\frac{(t - \tau - T)}{\rho}\right)dt \\
q_{01} \exp\left(\frac{-\tau}{\rho}\right) - q_{01} \exp\left(\frac{-\tau}{\rho}\right) \exp\left(\frac{-T}{\rho}\right) = \int_{0}^{\tau+T} \frac{F(t)}{\rho} \exp\left(\frac{t}{\rho}\right) \exp\left(\frac{-\tau}{\rho}\right) \exp\left(\frac{-T}{\rho}\right)dt \\
- \int_{0}^{\tau} \frac{F(t)}{\rho} \exp\left(\frac{t}{\rho}\right) \exp\left(\frac{-\tau}{\rho}\right)dt \\
- \int_{0}^{\tau} \frac{F(t)}{\rho} \exp\left(\frac{t}{\rho}\right) \exp\left(\frac{-\tau}{\rho}\right)dt \\
= q_{01} \exp\left(\frac{-\tau}{\rho}\right) - q_{01} \exp\left(\frac{-\tau}{\rho}\right) \exp\left(\frac{-T}{\rho}\right) \\
+ \int_{0}^{\tau+T} \frac{F(t)}{\rho} \exp\left(\frac{t}{\rho}\right) \exp\left(\frac{-\tau}{\rho}\right) \exp\left(\frac{-T}{\rho}\right)dt \\
- \int_{0}^{\tau} \frac{F(t)}{\rho} \exp\left(\frac{t}{\rho}\right) \exp\left(\frac{-\tau}{\rho}\right)dt \\
= 0.
\]
\[ q_{01} \exp \left( -\frac{\tau}{\rho} \right) \left[ 1 - \exp \left( -\frac{T}{\rho} \right) \right] = \]
\[ \frac{1}{\rho} \exp \left( -\frac{\tau}{\rho} \right) \left[ \exp \left( -\frac{T}{\rho} \right) \int_0^{\tau+T} F(t) \exp \left( \frac{t}{\rho} \right) dt - \int_0^{T} F(t) \exp \left( \frac{t}{\rho} \right) dt \right] \]
\[ \rho q_{01} \left[ 1 - \exp \left( -\frac{T}{\rho} \right) \right] = \exp \left( -\frac{T}{\rho} \right) \int_0^{\tau+T} F(t) \exp \left( \frac{t}{\rho} \right) dt - \int_0^{T} F(t) \exp \left( \frac{t}{\rho} \right) dt \]

(3.25)

Then assuming \( \tau = 0 \),
\[ q_{01} = \frac{1}{\rho(\exp \left( \frac{T}{\rho} \right) - 1) \int_0^{T} F(t) \exp \left( \frac{t}{\rho} \right) dt. \] (3.26)

Since \( q_{01} \) was known, Eqn. 3.26 was substituted back into Eqn. 3.19 and a solution for \( q_0(\tau) \) was derived, (Eqn. 3.28).
\[ q_0(\tau) = \frac{\exp \left( -\frac{\tau}{\rho} \right)}{\rho} \left( \frac{1}{\exp \left( \frac{T}{\rho} \right) - 1} \int_0^{T} F(t) \exp \left( \frac{t}{\rho} \right) dt + \int_0^{\tau} F(t) \exp \left( \frac{t}{\rho} \right) dt \right) \]

(3.27)

Note, Eqn. 3.27 is only valid assuming \( F(\tau) = \beta \hat{d}(x_0(\tau)) x_0(\tau) = F(\tau + T) \) where \( T \) is the known period of \( 2\pi \). Evaluating the integrals in Eqn. 3.27 gave a final solution for \( q_0 \) to be
\[ q_0(\tau) = \beta A \left[ \frac{-1}{(1 + \rho^2)} \left( \rho \cos(\tau) - \sin(\tau) \right) \right] + \beta \delta A^2 \left[ \frac{1}{2(1 + 4\rho^2)} \left( 1 + 4\rho^2 - \cos(2\tau) - 2\rho \sin(2\tau) \right) \right] - \frac{\exp \left( -\frac{\tau}{\rho} \right)}{(1 + \frac{1}{(2\rho)^2})(1 + \exp \left( -\frac{\tau}{\rho} \right))}. \] (3.28)

The order \( \epsilon \) equation was then used to find the unknown amplitude \( A \), but only the mechanical equation (Eqn. 3.29) was needed.
\[ \ddot{x}_1 + x_1 = -2\zeta \dot{x}_0 - \alpha x_0^3 - 2\sigma \ddot{x}_0 + \beta (1 + \delta |x_0|) q_0 + \gamma \sin(\tau - \phi) \] (3.29)
A Fourier series expansion was then applied to the coupling, \( \hat{d}(x_0(\tau))q_0 \), to give a better approximation of the amplitude, \( A \). The expansion was based off of \( f(t) = f(t + T) \) so that

$$ f(t + T) = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k \cos \left( \frac{2\pi T}{T} k t \right) + b_k \sin \left( \frac{2\pi T}{T} k t \right), \quad \text{where} \quad (3.30) $$

$$ a_i = \frac{2}{T} \int_0^T f(t) \cos \left( \frac{2\pi T}{T} k t \right) dt \quad \text{and} \quad b_i = \frac{2}{T} \int_0^T f(t) \sin \left( \frac{2\pi T}{T} k t \right) dt. $$

Using Eqn. 3.30 resulted in the following expansion in which \( f(t) = \hat{d}(x_0(\tau))q_0 \).

$$ f(t + T) = \frac{1}{\pi} \int_0^T f(t) dt + \sum_{k=1}^{\infty} \frac{2}{T} \cos \left( \frac{2\pi T}{T} k t \right) \int_0^T f(t) \cos \left( \frac{2\pi T}{T} k t \right) dt + \sum_{k=1}^{\infty} \frac{2}{T} \sin \left( \frac{2\pi T}{T} k t \right) \int_0^T f(t) \sin \left( \frac{2\pi T}{T} k t \right) dt \quad (3.31) $$

The integrals in Eqn. 3.31 were evaluated with assistance from MatLab. Appendix C gives the detailed steps taken to obtain

$$ f(t + T) = a_0 + \frac{A \beta^2}{\pi} \left( \cos \left( \frac{t}{\rho} \right) a_1 + \sin \left( \frac{t}{\rho} \right) b_1 \right), \quad \text{where} \quad (3.32) $$

$$ a_0 = \delta^2 A^2 \left[ \frac{2 \left( \frac{4}{1 + \frac{1}{\rho^2}} \right)}{3 \rho^2 \left( 1 + \frac{1}{(2\rho)^2} \right) \left( 1 + \frac{1}{\rho^2} \right)} \right] + \delta A \left[ \frac{\pi \rho^2}{(1 + \rho^2)^2} - \frac{2\rho(1 - \exp(-\frac{\pi}{\rho}))}{\left( 1 + \frac{1}{(2\rho)^2} \right) \left( 1 + \exp(-\frac{\pi}{\rho}) \right)} \right] + \left[ \frac{4}{(1 + \rho^2)} \right], \quad (3.33) $$
$$a_1 = \delta^2 A^2 \left[ \frac{1}{2} \left( -\rho \pi \right) \frac{\left( \exp \left( -\frac{\pi}{\rho} \right) - 1 \right)}{\left( 1 + \frac{1}{(2\rho)^2} \right)^2 \left( 1 + \exp \left( -\frac{\pi}{\rho} \right) \right)} \right]$$

$$+ \delta A \left[ \frac{-4\rho}{(1 + \rho^2)} \right] + \left[ \frac{-\pi \rho}{(1 + \rho^2)} \right]$$

and

$$b_1 = \delta^2 A^2 \left[ \frac{\pi(3 + 8\rho^2)}{4(1 + 4\rho^2)} - \frac{\rho(1 - \exp \left( -\frac{\pi}{\rho} \right))}{\left( 1 + \frac{1}{(2\rho)^2} \right)^2 \left( 1 + \exp \left( -\frac{\pi}{\rho} \right) \right)} \right]$$

$$+ \delta A \left[ \frac{4 \left( 4 + \frac{1}{\rho^2} \right)}{3\rho^2 \left( 1 + \frac{1}{(2\rho)^2} \right) \left( 1 + \frac{1}{\rho^2} \right)} \right] + \left[ \frac{\pi}{(1 + \rho^2)} \right].$$

Equation 3.32 was then substituted into Eqn. 3.29 to obtain

$$\ddot{x}_1 + x_1 = -2\zeta A \cos(\tau) - \alpha \frac{A^3}{4} \left[ 3 \sin(\tau) - \sin(3\tau) \right]$$

$$+ 2\sigma A \sin(\tau) + \gamma \sin(\tau - \phi) + a_0 + \frac{A\beta^2 a_1}{\pi} \cos(\tau) + \frac{A\beta^2 b_1}{\pi} \sin(\tau).$$

Knowing the trig identity \(\sin(\tau - \phi) = \sin(\tau) \cos(\phi) - \cos(\tau) \sin(\phi)\), the secular terms (time dependent terms) from Eqn. 3.36 were collected. This resulted in the following equations

$$\gamma \sin(\phi) = \frac{\beta^2 a_1}{\pi} A - 2\zeta A$$

and

$$\gamma \cos(\phi) = \frac{3}{4} \alpha A^3 - 2\sigma A - \frac{\beta^2 b_1}{\pi} A.$$ (3.37)

(3.38)

Both sides of Eqns. 3.37 and 3.38 were squared and then the two equations were added together to obtain a sixth order polynomial for the amplitude (\(A\)),

$$\gamma^2 = A^2 \left( \frac{\beta^2 a_1^2}{\pi^2} - 4\zeta \frac{\beta^2 a_1}{\pi} + 4\zeta^2 \right)$$

$$+ A^2 \left( \frac{9}{16} \alpha^2 A^4 - 3\alpha \sigma A^2 - \frac{3\alpha \beta^2 b_1}{2\pi} A^2 + 4\sigma^2 + \frac{4\sigma \beta^2 b_1}{\pi} + \frac{3\beta^2 b_1^2}{\pi^2} \right).$$ (3.39)
and a tangent equation for the phase shift, $\phi$,
\[ \phi = \tan^{-1}\left(\frac{\beta^2 a_1 - 2\zeta}{\frac{3}{2} \alpha A^2 - 2\sigma - \frac{\beta^2}{\pi} b_1}\right). \] (3.40)

If the nonlinearity on the piezoelectric coupling coefficient was ignored ($\delta = 0$), Eqn. 3.39 would reduce to
\[ \gamma^2 = A^6 \left[ \frac{9}{16} \alpha^2 \right] + A^4 \left[ -3\alpha \sigma - \frac{3\alpha \beta^2}{2(1 + \rho^2)} \right] + A^2 \left[ \frac{\beta^4(1 + \rho^2)}{(1 + \rho^2)^2} - \frac{(4\zeta \rho + 4\sigma)\beta^2}{(1 + \rho^2)} + 4\zeta^2 + 4\sigma^2 \right]. \] (3.41)

Once the amplitude, $A$, was known, the electrical equation (Eqn. 3.5) was solved for $\dot{Q}$ to obtain the following equation
\[ \dot{Q} = n \dot{q} \]
\[ = \frac{n}{\rho} \left\{ \beta [1 + \delta] A \sin (\tau)] A \sin (\tau) - q_0(\tau) \right\}. \] (3.42)

Equation 3.42 was then substituted into the previously defined voltage ($V = -R\dot{Q}$) equation. This substitution and defining power as $P = V^2/R$ gave the electrical generation of the system to be
\[ P = R\dot{Q}^2 \]
\[ = Rn^2 \dot{q}^2 \]
\[ = R(L^2 kC \dot{q}^2) \]
\[ = R \left( L^2 kC \left[ \frac{V\omega^2}{\gamma} \right] \right) \dot{q}^2 \]
\[ = \frac{\rho}{C\omega_n} \left( L^2 kC \left[ \frac{V\omega^2}{\gamma} \right] \right) \dot{q}^2 \]
\[ P = L^2 \sqrt{k m} \left[ \frac{V\omega^2}{\gamma} \right] \dot{q}^2. \] (3.43)
From Eqn. 3.43, the non-dimensionalized power was derived as

$$\hat{P} = \rho \frac{V \omega^2}{\gamma} q^2. \quad (3.44)$$

The dependance of the non-dimensionalized power upon the coupling coefficient and the nonlinearity in the piezoelectric material was then analytically and numerically investigated using MatLab simulations and the results are presented in the next chapter.
CHAPTER IV
RESULTS AND ANALYSIS

In order to obtain an understanding of how the power depended on the ambient vibration characteristics, the coupling coefficient and the nonlinearities in the system, programs were created to analytically and numerically simulate an energy harvesting system. Three specific systems were investigated: a linear system, a nonlinear stiffness system and a nonlinear piezoelectric coefficient system. Along with the three types of systems, the electro-mechanical coupling coefficient’s, $\beta$, influence on the power output was also investigated. In this chapter, the results from the analytical and numerical simulations are presented and discussed.

4.1 NLpiezoP.m Program

The solutions obtained from the method of averaging procedure were implemented into a MatLab program named NLpiezoP.m (see Appendix D for complete program). Important variables such as $\epsilon$, $\rho$, $\zeta$, $\beta$, $\delta$, and $\alpha$ were passed into the program as parameters, to adjust from a linear system to a nonlinear system easier. These parameters along with the tuning frequency ($\sigma$) were inserted into Eqn. 3.39 and the positive real root for the amplitude ($A$) was calculated. The amplitude was then used to solve for $q_0$ by Eqn. 3.28. This solution was inserted into Eqn. 3.44 to obtain the
non-dimensional power. Depending upon which variable was of interest, plots were created of the average non-dimensional power and maximum non-dimensional power to see the variable’s influence. The maximum non-dimensional power was plotted using a program named Pmax.m, which was also used to calculate the numerical solutions for the maximum power.

4.2 Numerical Simulation Programs

In order to validate the analytical solutions, an exact solution to the differential equations was calculated using MatLab. This was achieved by converting Eqns. 3.11 and 3.12 into three first order differential equations and implementing these equations into a program named deriv.m (see Appendix E.1). Then another program, poweraverage.m (see Appendix E.2), called deriv.m and used an initial value solver that was an explicit one-step Runge Kutta medium-order (4th-to-5th order) solver to obtain values for the derivatives. In this process, the solver excepted and produced a time $t$ and solution $x$. These values of $t$ and $x$ were exported into another program named Averagepower.m (see Appendix E.3). In this program, the difference between the work that occurred one whole period apart divided by the whole period was averaged. Then the exported averaged power was placed in an array with the corresponding forcing frequency in poweraverage.m. Once the total array was obtained, it was imported into the program Pmax.m where the maximum power and the corresponding forcing frequency were calculated. The combination of these four programs made it possible to vary the forcing frequency ($\omega$) while still varying the
coupling ($\beta$) and the coupling nonlinearity ($\delta$). Plots were then used to display the important findings in the study. The following sections discuss the results obtained from the numerical and analytical simulations for the different systems studied.

4.3 Linear Energy Harvesting System

The first system that was studied was a linear system. This system set $\alpha$ and $\delta$ equal to zero, while $\beta = 1$, and was used as a control to see how changing the stiffness ($\alpha$), the coupling coefficient ($\beta$) and the nonlinearity in the piezoelectric coupling ($\delta$) effected the power. For the analytical results, Fig. 4.1 displays how the average non-dimensional power peaks near $\sigma = -0.1$. Then as the tuning frequency, $\sigma$, strays

![Average Power as $\sigma$ Diffs](image)

Figure 4.1: A plot of a linear energy harvesting system’s power is displayed.
Figure 4.2: The above graph shows a linear system’s power output for the numerical simulation.

away from this value the average non-dimensional power decreases. The numerical simulation results, Fig. 4.2, correspond very closely to the analytical results. Notice, the highest power achieved is near 0.85 which is approximately 1.5 less than the analytical results. The resonant frequency is approximately the same also, since $\omega = 1 + \epsilon \sigma$ and $\epsilon = 0.1$.

4.4 Weakly Nonlinear Stiffness Energy Harvesting System

The second system that was analyzed included a weakly nonlinear spring ($\alpha \neq 0$). Figure 4.3 shows as $\alpha$ increases, the maximum average power does not change. Notice
when $\alpha = 0.2$ the parabola no longer is symmetric about $-0.1$, but has begun to tilt towards the right. Then as $\alpha$ increases the tilt angle increases. From these patterns, it was concluded that as the nonlinearity in the stiffness increases the resonant frequency shifts to the right. This conclusion was validated by the numerical results seen in Fig. 4.4. The same trend of the peak leaning to the right was seen, but as the nonlinearity in the stiffness increased it seemed the maximum average power began to decrease slightly. In the analytical results the maximum average power does not change, but this could be due to the number of points plotted from the simulation.

![Average Power as $\alpha$ Diffs](image-url)

Figure 4.3: The above plot displays the non-dimensional power dependence on the linearity of the stiffness of the system.
Figure 4.4: The nonlinear stiffness ($\alpha$) influence on the average non-dimensional power output is displayed above.

4.5 Effects of Electro-mechanical Coupling Coefficient

In order to understand how the power output depended upon the electro-mechanical coupling and the nonlinearity in the coupling, plots were created for different coupling coefficients ($\beta$) and nonlinear coupling coefficients ($\delta$) as the frequency ($\omega$) or tuning frequency ($\sigma$) varied. Figure 4.5 displays the analytical results of the influence of the coupling coefficient value on the average non-dimensionalized power as the tuning frequency varied. Notice when $\beta = 0.75$ the average power is about 0.93 and when the coupling is increased by just 0.25 the average power increased to 1.0. Another
Figure 4.5: The above graph displays the average non-dimensional power as the frequency is varied for different coupling coefficient values.

increase in the coupling of 0.25 gave a coupling coefficient of 1.25 and the average power decreased to approximately 0.95. Then as the coupling becomes even larger the average power continues to reduce. These results were validated by the numerical results seen in Fig. 4.6. When $\beta$ was small (0.75) the maximum power was less than 0.7. Then as the coupling increased to 1.0 the maximum average power also increase just as it did in the analytical results. When the coupling was increased to 1.25 the maximum average power increase, unlike in the analytical results where it has decreased with $\beta = 1.25$. As the coupling increased more though, the trend of the maximum average power decreasing occurred just as it did in the analytical
Figure 4.6: The above graph shows the numerical average non-dimensional power output as the coupling coefficient, $\beta$, and forcing frequency are varied.

results. Physically, this trend makes sense. The coupling is pulling off energy from the mechanical system and theoretically the more energy that can be pulled off the more power that can be produce. This does not consider the effect on the mechanical systems amplitude, which depicts the amount of energy available. As more and more energy is pulled from the mechanical system, the amplitude of the oscillations are decreased or dampened and thus the amount of energy available is also decreased.

Therefore, there must be an optimal coupling coefficient value that maximizes the output power without dampening out the mechanical oscillations. To calculate this particular coupling coefficient value, Fig. 4.7 was created that plots the maximum
power as the coupling coefficient, $\beta$, varied. From Fig. 4.7, it is seen that the optimal coupling coefficient value is around 1.0. The numerical results for the optimal coupling coefficient are also plotted in Fig. 4.7. These results slightly differ from the analytical results. The optimal coupling coefficient is near 1.25 and the maximum power is always slightly lower than the analytical power (0.84). There are some similarities though, such as the general slope of the line before and after the optimal coupling value. As the coupling increases from zero the slope is very sharp, but after the optimal coupling value the slope seems to flatten out as $\beta$ becomes larger and larger. Thus, the maximum power is never completely dampened out to zero.

![Figure 4.7](image.png)

**Figure 4.7:** Plotted above are the numerical and analytical results for the maximum non-dimensional power output as the coupling coefficient varies.
The nonlinearity in the coupling coefficient was investigated using the same approach used for $\beta$. Figure 4.8 displays the analytical results for the average non-dimensional power dependence upon the nonlinear coefficient, $\delta$. When the coupling is linear, $\delta = 0$, the maximum average power is 1.0. Then as the nonlinear coefficient ($\delta$) increases to 0.25 the average power increases to about 1.1. As the nonlinearity increases even more, the maximum average power begins to decrease. Notice when the nonlinear coefficient was 2.5 the maximum average power was about 0.85 which is less than the linear system’s results.

![Figure 4.8](image)

Figure 4.8: The influence of a nonlinear coupling coefficient on the average non-dimensional power output as the frequency varies is displayed above.
Figure 4.9: The numerical results of the nonlinear coupling coefficient’s, $\delta$, influence on the average power are displayed above.

The same results were seen for the numerical simulation. In Fig. 4.9 when $\delta = 0.0$, the maximum average power was near 0.8. Then as the nonlinearity became larger, the maximum average power began to increase. Just as before when the coupling ($\beta$) was varied, the numerical results give an increase in the maximum average power when $\delta = 0.5$, while the analytical results have the power decreasing. The overall trend of the maximum average power decreasing after a certain value of nonlinearity is consistent though in both the analytical and numerical simulations.

The nonlinear coupling results offer a possible explanation for the variability in the power output prediction noted by previous researchers. [8] noted that the
power was under-predicted by the identified linear model while [17] found the opposite results, that the linear model over-predicted the power output. In both conclusions, the nonlinearity in the harvesting system was put forth as a possible source for the discrepancy. If the nonlinearity in the coupling was small then the linear model would under predict the power output and vise versa for larger nonlinearity in the coupling.

Another graph was created to find the largest allowable amount of nonlinearity in the coupling to achieve the maximum power output. Figure 4.10 displays these results in which the coupling, $\beta$, was set equal to 1 and the nonlinear coefficient, $\delta$ varied. From this graph, it is seen that the maximum power output for the analytical result is produced when the nonlinearity coefficient is near 0.35, which produces a power output of about 1.1. Then as $\delta$ becomes larger than 1.0 the maximum power output becomes less than the linear coupling power output. The numerical result gives a maximum power of near 0.98 when the nonlinear coefficient is about 0.5. Then when $\delta$ becomes larger than approximately 1.8, the maximum power is lower than the linear coupling power output. Both the analytical and numerical result’s trend are quite similar. The numerical solution’s slope after the maximum power is slightly smaller and always has about 0.18 less maximum power than the analytical results. This again confirms the previous trends in predicting the power output of the energy harvesting system. Thus, by taking into consideration the nonlinearity in the piezoelectric constitutive equations a more precise prediction of the power output can be obtained.
4.6 A More In-depth Study on Maximum Power Dependence

In Figs. 4.6 and 4.9 it was seen that the maximum average power does not occur at the same forcing frequency, $\omega$. Therefore, 3-dimensional plots were created obtain a better idea of the coupling coefficient, coupling nonlinearity and the forcing frequency influence the maximum power. In Fig. 4.11, the numerical solutions reveal that as $\beta$ increases the frequency at which the maximum power is generated decreases. At maximum power, approximately $0.83$, $\omega \approx 0.93$ and $\beta \approx 1.25$. Similar results are found when the nonlinearity in the coupling was plotted against the frequency and maximum power, while $\beta = 1.0$. Figure 4.12 displays how the rate of the resonant fre-
frequency shift seems increase after the largest maximum power occurs for the numerical results. This shift is still to a lower frequency, as it was before for the coupling.

Figure 4.11: The above graph shows the relationship between the maximum power output, coupling coefficient and forcing frequency.

Figure 4.12: The relationship between the nonlinearity in the coupling, the maximum power and the forcing frequency is displayed.
The dependence of the maximum power on the interaction between the electro-mechanical coupling strength and its nonlinearity is still not clear. Therefore, Fig. 4.13 and Fig. 4.14 display two dimensional plots of the maximum power as the coupling varies for different values of nonlinearity. As seen in Fig. 4.13, as the nonlinear piezoelectric coefficient increases the value of $\beta$ at which the maximum power is generated decreases. Also, there are strange trends that appear in the numerical solutions (Fig. 4.14). The numerical results had a tendency to become unstable when taking all values of coupling ($1 \geq \beta \leq 3$) into consideration and increasing the nonlinearity. Notice, in Fig. 4.14 as the nonlinearity increases the span of evaluated coupling values decrease. This is due to the system becoming unstable. By looking at

Figure 4.13: Displayed above is the effect of nonlinearity in the coupling and the coupling on maximum power output.
the stability of singular points as the coupling is varied, a very good approximation were the system became unstable is achieved.

A program was created to plot the singular points as $\beta$ was varied from 0 to 5 and when any point crossed the positive real axis (system becomes unstable) the value of the coupling was displayed. Figure 4.15 displays the singular points as the coupling varied. The stable system is plotted in red, while the unstable system is blue. When the system became unstable the value of $\beta$ was 3.18. Therefore, this explains why the span of the evaluated coupling had to decrease as the nonlinearity increase (since the nonlinearity acts as damping). This instability will be investigated more in future work. For now, a 3-dimensional plot is created for the analytical result of the

Figure 4.14: The influence of nonlinearity in the coupling while the actual coupling varies to produce the maximum power output is shown above.
Figure 4.15: Singular points as coupling is varied are plotted above. The red points represent a stable system, while the blue points represent an unstable system.

maximum power while simultaneously varying the coupling, $\beta$, and the nonlinearity, $\delta$. Figure 4.16 displays this analytical result which again shows this interesting trend, as seen in Figs. 4.13 and 4.14, when the nonlinearity in the coupling and the coupling are both varied.
Figure 4.16: The relationship between the nonlinearity in the coupling, the actual coupling and the maximum power is displayed.
CHAPTER V

CONCLUSION

The goal of this study was to understand the effects of nonlinear piezoelectric coupling on the power output in a vibration based energy harvester while considering the effects of nonlinear stiffness as well. This goal was achieved by studying a one-dimensional spring-mass damper system using perturbation analysis in which the Poincaré-Lindstedt approach and the method of averaging were utilized. The results obtained from this analysis indicated that the inclusion of nonlinear stiffness in the system did not alter the maximum power output. They did indicate a relationship between the resonant frequency and the degree of nonlinearity in the stiffness, such as an increase in the nonlinearity also increases the resonant frequency in order to obtain optimal power production.

The coupling results were much more complicated. They indicated an optimal amount of coupling was needed to obtain the greatest amount of power production. If the coupling was too small the power production was an inadequate amount and if it was too large the power again decreased below a usable amount. As nonlinearity was introduced, it was seen that the power production was increased, but as the
nonlinearity increased, the power generated was reduced below a linear system’s production. This gave reasoning for the problems acquired while predicting the power output of the designed energy harvesters in earlier research.

Overall, the results from this study suggest that the effects of nonlinear piezoelectric coupling influence the system’s performance significantly. When the nonlinearity is small the system’s performance is actually improved, but as the nonlinearity increases the performance worsens. This influence of nonlinearity in the electro-mechanical coupling coincides with the inconsistencies in predicting the amount of power generated from the harvesting systems found in previous research. Therefore, by including the role of nonlinearities in the electro-mechanical coupling during the design process, the power output prediction can be improved and ultimately the system’s performance can be maximized.
BIBLIOGRAPHY


APPENDIX A

DERIVATION OF STEADY-STATE AND TRANSIENT RESPONSE

It is assumed in that the vibration applied to the system is a harmonic excitation. The response to sine harmonic or cosine harmonic excitation that is applied at \( t = 0 \) is composed of a transient portion and a steady-state portion. In Eqn. 1.3 it was assumed the known forcing function \( u(t) \) was \( U \sin(\omega_b t) \). Knowing the initial conditions are zero \( (z(0) = 0 \text{ and } \dot{z}(0) = 0) \), the generalized solution for the response is developed.

A.1 Particular Solution (Steady State Solution)

The particular solution to Eqn. 1.3 is

\[
z(t) = Z_0 \sin(\omega_b t - \phi),
\]

but \( Z_0 \) and \( \phi \) are unknown. By substituting Eqn. A.1 into Eqn. 1.3, a solvable equation for \( Z_0 \) and \( \phi \) are achieved. The following equations give the steps taken to obtain the solutions for \( Z_0 \) and \( \phi \).

To find \( Z_0 \):

\[
\left[ \omega_n^2 - \omega_b^2 \right] Z_0 \sin(\omega_b t - \phi) + 2\zeta\omega_n\omega_b Z_0 \cos(\omega_b t - \phi) = \omega_b^2 U \sin(\omega_b t)
\]

\[
Z_0 \left[ (\omega_n^2 - \omega_b^2) \sin(\omega_b t - \phi) + 2\zeta\omega_n\omega_b \cos(\omega_b t - \phi) \right] = \omega_b^2 U \sin(\omega_b t)
\]
Knowing \( a \sin(\omega t - \phi) + b \cos(\omega t - \phi) = \sqrt{a^2 + b^2} \sin(\omega t) \)

\[
Z_0 \left[ \sqrt{\left(\omega_n^2 - \omega_b^2\right)^2 + (2\zeta\omega_n\omega_b)^2} \right] = \omega_b^2 U \sin(\omega_b t)
\]

\[
Z_0 = \frac{\omega_b^2 U}{\sqrt{\left(\omega_n^2 - \omega_b^2\right)^2 + (2\zeta\omega_n\omega_b)^2}} \tag{A.2}
\]

To find \( \phi \):

Knowing: \( \sin(\omega_b t) = \sin(\omega_b t - \phi) \cos(\phi) + \cos(\omega_b t - \phi) \sin(\phi) \)

\[
Z_0 \left[ (\omega_n^2 - \omega_b^2) \sin(\omega_b t - \phi) + 2\zeta\omega_n\omega_b \cos(\omega_b t - \phi) \right]
\]

\[
= \omega_b^2 U \left[ \sin(\omega_b t - \phi) \cos(\phi) + \cos(\omega_b t - \phi) \sin(\phi) \right]
\]

Split up into two equations

1. \[
Z_0 \left[ (\omega_n^2 - \omega_b^2) \sin(\omega_b t - \phi) \right] = \omega_b^2 U \cos(\phi) \sin(\omega_b t - \phi)
\]

\[
Z_0 = \frac{\omega_b U}{(\omega_n^2 - \omega_b^2)} \cos(\phi)
\]

2. \[
Z_0 \left[ 2\zeta\omega_n\omega_b \cos(\omega_b t - \phi) \right] = \omega_b^2 U \sin(\phi) \cos(\omega_b t - \phi)
\]

\[
Z_0 = \frac{\omega_b^2}{2\zeta\omega_n\omega_b} \sin(\phi)
\]

Substitute in for \( Z_0 \)

\[
\frac{\omega_b^2 U}{(\omega_n^2 - \omega_b^2)} \cos(\phi) = \frac{\omega_b^2 U}{2\zeta\omega_n\omega_b} \sin(\phi)
\]

\[
\frac{2\zeta\omega_n\omega_b}{\omega_n^2 - \omega_b^2} = \tan(\phi) \tag{A.3}
\]

Dividing the top and bottom by \( \omega_n^2 \) gives:

\[
\tan(\phi) = \frac{2\zeta r}{1 - r^2}
\]

60
Therefore,

\[ z(t) = \frac{r^2 U}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \sin (\omega_b t - \phi) \]  

(A.4)

A.2 Damped Homogeneous Solution (Transient Solution)

Beginning with a homogenous equation of motion,

\[ \ddot{z} + 2\zeta \omega_n \dot{z} + \omega_n^2 = 0, \]  

(A.5)

and assuming \( x = \exp (mt) \), the resulting equation was solvable for \( m \).

\[ m^2 \exp (mt) + 2\zeta \omega_n m \exp (mt) + \omega_n^2 \exp (mt) = 0 \]

\[ m^2 + 2\zeta \omega_n m + \omega_n^2 = 0 \]

Using the quadratic formula:

\[ m = \frac{-2\zeta \omega_n \pm \sqrt{(2\zeta \omega_n)^2 - 4\omega_n^2}}{2} \]

\[ = \frac{-2\zeta \omega_n \pm \sqrt{4\omega_n^2(\zeta^2 - 1)}}{2} \]

\[ = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1}. \]

The resulting equation for \( z(t) \) is

\[ z(t) = \exp \left( [-\zeta \omega_n \pm \omega_d]t \right) \]  

(A.6)

\[ = \exp (-\zeta \omega_n t) \left[ A \sin (\omega_d t) + B \cos (\omega_d t) \right] , \]
where \( \omega_d = \omega_n \sqrt{\zeta^2 - 1} \) is defined as the damped natural frequency. The boundary conditions of \( z(0) = z_0 \) and \( \dot{z}(0) = v_0 \) give

\[
\begin{align*}
  z(0) &= z_0 = B \\
  \dot{z}(0) &= v_0 = -\zeta \omega_n [B] + [A \omega_d] \\
  \Rightarrow A &= \frac{v_0 + \zeta \omega_n z_0}{\omega_d}
\end{align*}
\]

Knowing \( a \sin (\omega t) + b \cos (\omega t) = \sqrt{a^2 + b^2} \sin (\omega t + \theta) \), the homogeneous solutions is

\[
  z(t) = A_0 \exp (-\zeta \omega_n t) \sin (\omega_d t + \theta), \quad (A.7)
\]

where

\[
  A_0 = \sqrt{z_0^2 + \left( \frac{v_0 + \zeta \omega_n z_0}{\omega_d} \right)^2}. \quad (A.8)
\]

The value of \( \theta \) is found by setting \( t = 0 \) in Eqn. A.7 to obtain

\[
  z(t) = A_0 \sin (\theta) \\
  \Rightarrow A_0 = \frac{z(t)}{\sin (\theta)}.
\]

Then defining \( \dot{z}(t) \) as

\[
  \dot{z}(t) = -\zeta \omega_n A_0 \exp (-\zeta \omega_n t) \sin (\omega_d t + \theta) + \omega_d A \exp (-\zeta \omega_n t) \cos (\omega_d t + \theta) \quad \text{and} \quad (A.9)
\]

solving for \( \dot{z}(t) \) when \( t = 0 \) gives

\[
  \dot{z}(0) = -\zeta \omega_n \frac{z_0}{\sin (\theta)} \sin (\theta) + \omega_d \frac{z_0}{\sin (\theta)} \cos (\theta) \\
  = -\zeta \omega_n z_0 + \omega_d z_0 \frac{1}{\tan (\theta)}. \quad (A.10)
\]
Therefore,

$$\theta = \tan^{-1}\left(\frac{z_0\omega_d}{v_0 + \zeta \omega_n z_0}\right)$$  \hspace{1cm} (A.11)

The total solution is then

$$z(t) = A_0 \exp(-\zeta \omega t) \sin(\omega_d t + \theta) + Z_0 \sin(\omega_b t - \phi).$$  \hspace{1cm} (A.12)

It is often common to ignore the transient response when the system is largely damped ($\zeta > 1$) and depending upon the application of the system it is sometimes ignored as well.
APPENDIX B

FrequencyRatio.m PROGRAM

%comparison of how nondimensional power is related to the damping ratio and
% r = \frac{w}{\omega_n}
clf
r = 1;
% e = 0.3;
% for r = 0:0.001:2.5;
for e = 0:0.01:1;
Pavg = e.*r.^2./((1-r.^2).^2+4*e.^2.*r.^2);
Amp = r.^2/((1-r.^2).^2+4*e.^2.*r.^2);
w=1;
wn=1;
m=0.1;
k=0.1;
Cm=0.1;
F=1;
Ce = 10e-4:0.01:10;
DimPavg = (Ce.*w^2/2).*F^2./((k-w^2*m)^2+((Cm+Ce).*w).^2);
figure(1)
subplot(1,2,1),loglog(Ce,DimPavg,'-b')
xlabel('Electrical Damping (c_e)')
ylabel('Average Power')
title('Power Dependance on Electrical Damping')
subplot(1,2,2),plot(Ce,DimPavg,'-b')
xlabel('Electrical Damping (c_e)')
ylabel('Average Power')
title('Power Dependance on Electrical Damping')

64
% B = [e Pavg];
% save Amplitude1.dat B -ascii -append
end
% A = load('Amplitude1.dat');
% % B = load('Amplitude2.dat');
% % C = load('Amplitude3.dat');
% % D = load('Amplitude4.dat');
% figure(2)
% % plot(A(:,1),A(:,2),'b-',B(:,1),B(:,2),'r-',C(:,1),C(:,2),'k-',D(:,1),D(:,
% % 2),'g-')
% plot(A(:,1),A(:,2),'b-')
% xlabel('Damping Ratio ($\zeta$)')
% ylabel('Non-dimensional Average Power')
% title('Average Power as Damping Ratio Differs')
% box on
APPENDIX C

DETAILED STEPS TAKEN TO OBTAIN $F(t + T)$

In the process of evaluating $q_0$, it was found there were basically two integrals that were used throughout the analysis. The first type was labeled $I(t)$ in which the integration consisted of $\sin(t)$ multiplied by $\exp(t/\rho)$ and the second type was labeled $J(t)$ in which the integration consisted of $|\sin(t)|$ multiplied by $\exp(t/\rho)$. Notice that before the integral could be evaluated the sign of the $\sin(t)$ term had to be taken into consideration. For example, in $J(t)$ integral of $|\sin(\tau)|\exp(-\tau/\rho)$ could not be evaluated simply from 0 to $2\pi$. Instead, either the integral had to be split up into $\int_0^\pi - \int_{2\pi}^\pi$ or use the property that $\int_0^{2\pi} |\sin(\tau)| = 2 \int_0^\pi \sin(t)$ to account for the absolute value of the sine term. For simplicity the $2 \int_0^\pi \sin(t)$ method was used throughout the whole derivation of $F(\tau + T)$. When these integrals were evaluated, Eqns. C.1 and C.2 were obtained. Note, $t$ was a dummy variable.

$$I(t) = \int \sin(t) \exp\left(\frac{t}{\rho}\right) dt$$

$$= \frac{-\rho}{(1 + \rho^2)} \exp\left(\frac{t}{\rho}\right) (\rho \cos(t) - \sin(t)) \quad \text{(C.1)}$$

$$J(t) = \int \sin^2(t) \exp\left(\frac{t}{\rho}\right) dt$$

$$= \frac{-\rho}{2(1 + 4\rho^2)} \exp\left(\frac{t}{\rho}\right) \left(1 + 4\rho^2 - \rho \cos(2t) - 2\rho \sin(2t)\right) \quad \text{(C.2)}$$
Using this knowledge, \( q_0 \) (Eqn. 3.27) could be split up into two integral equations, \( Q_{01} \) and \( Q_{02} \).

\[
Q_{01}(t) = \frac{\exp(-\frac{t}{\rho})}{\rho} \left[ \int_0^t \sin(t) \exp\left(\frac{t}{\rho}\right) dt - \frac{1}{(\exp\left(\frac{\pi}{\rho}\right) + 1)} \int_0^{\pi} \sin(t) \exp\left(\frac{t}{\rho}\right) dt \right]
\]

\[
= \frac{\exp(-\frac{t}{\rho})}{\rho} \left[ (I(t) - I(0)) - \frac{(I(\pi) - I(0))}{(\exp(\frac{\pi}{\rho}) + 1)} \right]
\]

\[
= \frac{\exp(-\frac{t}{\rho})}{\rho} \left[ -\frac{\rho}{1 + \rho^2} \exp\left(\frac{t}{\rho}\right) (\rho \cos(t) - \sin(t)) \right]
\]

\[
= \frac{\exp(-\frac{t}{\rho})}{\rho} I(t) \tag{C.3}
\]

\[
Q_{02}(t) = \frac{\exp(-\frac{t}{\rho})}{\rho} \left[ \int_0^t \sin^2(t) \exp\left(\frac{t}{\rho}\right) dt - \frac{1}{(\exp\left(\frac{\pi}{\rho}\right) + 1)} \int_0^{\pi} \sin^2(t) \exp\left(\frac{t}{\rho}\right) dt \right]
\]

\[
= \frac{\exp(-\frac{t}{\rho})}{\rho} \left[ (J(t) - J(0)) - \frac{(J(\pi) - J(0))}{(\exp(\frac{\pi}{\rho}) + 1)} \right]
\]

\[
= \frac{\exp(-\frac{t}{\rho})}{\rho} \left[ \frac{\rho \exp\left(\frac{t}{\rho}\right)}{2(1 + 4\rho^2)} (1 + 4\rho^2 - \cos(2t) - 2\rho \sin(2t)) - \frac{4\rho^3 \exp\left(\frac{\pi}{\rho}\right)}{(\exp(\frac{\pi}{\rho}) + 1)} \right]
\]

\[
= \frac{\exp(-\frac{t}{\rho})}{\rho} \left[ J(t) - \frac{2J(0)}{1 + \exp\left(\frac{\pi}{\rho}\right)} \right] \tag{C.4}
\]

Then a condensed equation for \( q_0 \) was formed in Eqn. C.5, which is identical to Eqn. 3.28.

\[
q_0(t) = \beta A Q_{01}(t) + \beta \delta A^2 Q_{02}(t) \tag{C.5}
\]

Equation C.5 was then substituted into Eqn. 3.31 with \( k = 1 \) and the resulting equation was

\[
f(t + T) = \int_0^T \beta d(A \sin(\tau)) q_0(\tau) dt
\]

\[
+ \frac{2}{T} \cos\left(\frac{2\pi}{T} t\right) \int_0^T \beta d(A \sin(\tau)) q_0(\tau) \cos\left(\frac{2\pi}{T} t\right) dt
\]

\[
+ \frac{2}{T} \sin\left(\frac{2\pi}{T} t\right) \int_0^T \beta d(A \sin(\tau)) q_0(\tau) \sin\left(\frac{2\pi}{T} t\right) dt. \tag{C.6}
\]
Each integral in Eqn. C.6 was given a specific label and evaluated separately for

\( T = 2\pi \). The first integral was labeled \( a_0 \) and in expanded form the result was

\[
a_0 = 2 \int_0^\pi Q_{01}(t) dt + 2\delta A \int_0^\pi (Q_{02}(t) + \sin(t)Q_{01}(t)) dt + 2\delta^2 A^2 \int_0^\pi \sin(t)Q_{02}(t) dt
\]

\[
= 2 \int_0^\pi \frac{-1}{(1 + \rho^2)} (\rho \cos(t) - \sin(t))
\]

\[
+ 2\delta A \int_0^\pi \left( \frac{1}{2(1 + 4\rho^2)} (1 + 4\rho^2 - \cos(2t) - 2\rho \sin(2t)) - \frac{\exp\left(\frac{-t}{\rho}\right)}{(1 + \frac{1}{(2\rho)^2})(1 + \exp\left(\frac{-\pi}{\rho}\right))} \right) dt
\]

\[
+ \sin(t) \frac{-1}{(1 + \rho^2)} (\rho \cos(t) - \sin(t)) dt
\]

\[
+ 2\delta^2 A^2 \int_0^\pi \sin(t) \left( \frac{1}{2(1 + 4\rho^2)} (1 + 4\rho^2 - \cos(2t) - 2\rho \sin(2t)) - \frac{\exp\left(\frac{-t}{\rho}\right)}{(1 + \frac{1}{(2\rho)^2})(1 + \exp\left(\frac{-\pi}{\rho}\right))} \right) dt.
\]

(C.7)

The evaluation of Eqn. C.7 resulted in

\[
a_0 = \frac{\beta}{2\pi} \left[ \frac{4}{(1 + \rho^2)} + \delta A \left( \frac{\pi \rho^2}{(1 + \rho^2)} - \frac{2\rho(1 - \exp\left(\frac{-\pi}{\rho}\right))}{(1 + \frac{1}{(2\rho)^2})(1 + \exp\left(\frac{-\pi}{\rho}\right))} \right) + \delta^2 A^2 \left( \frac{2}{3\rho^2 \left( \frac{1}{(2\rho)^2} \right)} \right) \right].
\]

(C.8)
The second integral from Eqn. C.6 was labeled $a_1$ and in expanded form the integral appeared as

$$
a_1 = 2 \int_0^\pi Q_{01}(t) \cos(t) dt + 2\delta A \int_0^\pi (Q_{02}(t) + \sin(t)Q_{01}(t)) \cos(t) dt$$

$$+ 2\delta^2 A^2 \int_0^\pi \sin(t)Q_{02}(t) \cos(t) dt$$

$$= 2 \int_0^\pi \left( -\frac{1}{(1+\rho^2)} \left( \rho \cos(t) - \sin(t) \right) \right) \cos(t) dt$$

$$+ 2\delta A \int_0^\pi \left( \frac{1}{2(1+4\rho^2)} \left( 1 + 4\rho^2 - \cos(2t) - 2\rho \sin(2t) \right) \right. \frac{\exp(-\frac{t}{\rho})}{1 + \frac{1}{(2\rho)^2}} \left(1 + \exp\left(-\frac{\pi}{\rho}\right) \right)$$

$$+ \sin(t) \left( -\frac{1}{(1+\rho^2)} \left( \rho \cos(t) - \sin(t) \right) \right) \cos(t) dt$$

$$+ 2\delta^2 A^2 \int_0^\pi \sin(t) \left( \frac{1}{2(1+4\rho^2)} \left( 1 + 4\rho^2 - \cos(2t) - 2\rho \sin(2t) \right) \right. \frac{\exp(-\frac{t}{\rho})}{1 + \frac{1}{(2\rho)^2}} \left(1 + \exp\left(-\frac{\pi}{\rho}\right) \right)$$

$$\cos(t) dt. \quad (C.9)$$
Then after evaluating the integrals using MatLab Eqn. C.10 was obtained, which was simplified to achieve a final solution for \( a_1 \) (Eqn. C.11).

\[
a_1 = -\frac{2}{(1 + \rho^2)} \left[ \frac{\rho \pi}{2} \right] \\
+ \delta A \left[ \frac{1}{(1 + 4\rho^2)} \left( \frac{-8\rho}{3} \right) \right] \\
- \frac{2}{(1 + \frac{1}{(2\rho)^2})} \left( 1 + \exp \left( \frac{-\pi \rho}{\rho} \right) \right) \left( \frac{\rho}{(1 + \rho^2)} \left( 1 + \exp \left( \frac{-\pi \rho}{\rho} \right) \right) \right) \\
+ \frac{-2}{(1 + \rho^2)} \left( \frac{2\rho}{3} \right) \\
+ \delta^2 A^2 \left[ \frac{1}{(1 + 4\rho^2)} \left( \frac{-\rho \pi}{2} \right) \right] \\
- \frac{2}{(1 + \frac{1}{(2\rho)^2})} \left( 1 + \exp \left( \frac{-\pi \rho}{\rho} \right) \right) \left( \frac{\rho^2}{(1 + 4\rho^2)} \left( 1 - \exp \left( \frac{-\pi \rho}{\rho} \right) \right) \right) \tag{C.10}
\]

\[
a_1 = \delta^2 A^2 \left[ \frac{1}{2} \left( \frac{-\rho \pi}{(1 + 4\rho^2)} + \frac{(\exp \left( \frac{-\pi \rho}{\rho} \right) - 1)}{(1 + \frac{1}{(2\rho)^2})^2 \left( 1 + \exp \left( \frac{-\pi \rho}{\rho} \right) \right)} \right) \right] \\
+ \delta A \left[ \frac{-4\rho}{(1 + \rho^2)} \right] + \left[ \frac{-\pi \rho}{(1 + \rho^2)} \right] \tag{C.11}
\]
The last integral in Eqn. C.6 was labeled $b_1$ and in its expanded form the integral was

$$b_1 = 2 \int_0^\pi Q_{01}(t) \sin(t) \, dt + 2 \delta A \int_0^\pi (Q_{02}(t) + \sin(t)Q_{01}(t)) \sin(t) \, dt$$

$$+ 2 \delta^2 A^2 \int_0^\pi \sin(t)Q_{02}(t) \sin(t) \, dt$$

$$= 2 \int_0^\pi \left( \frac{-1}{(1 + \rho^2)} (\rho \cos(t) - \sin(t)) \right) \sin(t) \, dt$$

$$+ 2 \delta A \int_0^\pi \left( \frac{1}{2(1 + 4\rho^2)} (1 + 4\rho^2 - \cos(2t) - 2\rho \sin(2t)) \right) \sin(t) \, dt$$

$$- \frac{\exp \left( \frac{-t}{\rho} \right)}{\left(1 + \frac{1}{(2\rho)^2}\right)} \left(1 + \exp \left( \frac{-\pi}{\rho} \right) \right)$$

$$+ 2 \delta^2 A^2 \int_0^\pi \sin(t) \left( \frac{1}{2(1 + 4\rho^2)} (1 + 4\rho^2 - \cos(2t) - 2\rho \sin(2t)) \right) \sin(t) \, dt$$

$$- \frac{\exp \left( \frac{-t}{\rho} \right)}{\left(1 + \frac{1}{(2\rho)^2}\right)} \left(1 + \exp \left( \frac{-\pi}{\rho} \right) \right) \sin(t) \, dt. \tag{C.12}$$

Equation C.12 was evaluated with the aid of MatLab to obtain

$$b_1 = \frac{-2}{(1 + \rho^2)} \left[ \frac{-\pi}{2} \right] + \delta A \left[ \frac{1}{(1 + 4\rho^2)} \left( \frac{8}{3} + 8\rho^2 \right) \right]$$

$$- \frac{2}{\left(1 + \frac{1}{(2\rho)^2}\right)} \left(1 + \exp \left( \frac{-\pi}{\rho} \right) \right) \left( \frac{\rho^2}{(1 + \rho^2)} (1 + \exp \left( \frac{-\pi}{\rho} \right) \right)$$

$$+ \frac{2}{\left(1 + \frac{1}{(2\rho)^2}\right)} \left( \frac{2\rho^2}{4 + 2\rho^2} \pi \right)$$

$$- \frac{2}{\left(1 + \frac{1}{(2\rho)^2}\right)} \left( \frac{2\rho^3}{4 + 2\rho^2} \left(1 - \exp \left( \frac{-\pi}{\rho} \right) \right) \right) \]. \tag{C.13}$$

A simplified version of Eqn. C.13 is

$$b_1 = \delta^2 A^2 \left[ \frac{\pi(3 + 8\rho^2)}{4(1 + 4\rho^2)} - \frac{\rho(1 - \exp \left( \frac{-\pi}{\rho} \right))}{\left(1 + \frac{1}{(2\rho)^2}\right)^2} \right]$$

$$+ \delta A \left[ \frac{4}{3\rho^2 (1 + \frac{1}{(2\rho)^2})} \left(4 + \frac{1}{\rho^2} \right) \right] + \left[ \frac{\pi}{(1 + \rho^2)} \right]. \tag{C.14}$$
APPENDIX D

NLPiezoP.m PROGRAM

function [Pmax] = NLpiezoP(par)
% par = [0.25, 2.00, 0.50, 1.00, 1.00, 1.00];
zeta = par(1);
gam = par(2);
alp = par(3);
rho = par(4);
bet = par(5);
delt = par(6);
N = 2048;
t = 2*pi*[0:N]/N;
% close all
sigmin = -3.0;
sigmax = 3.0;
NN = 64;
i=1;
    for i=1:NN;
        % for delta = 0:0.05:3;
        % for beta = 0:0.05:3;
        sig(i) = sigmin + (sigmax-sigmin)*(i-1.0)/(NN-1);
        % sig(i) = 0.0;
        a12 = del^2*( (1/2)*((-rho*pi/(1+4*rho^2))+(exp(-pi/rho)-1)...
                        /((1+1/((2*rho^2)^2))^2*(1+exp(-pi/rho)))) );
        a11 = del*(-4*rho/(1+r^2));
        a10 = (-pi*rho/(1+r^2));
        b12 = del^2*( (pi*(3+8*r^2)/(4*(1+4*r^2)))... 
                    -(rho*(1-exp(-pi/rho)))/(1+1/((2*r^2)^2)) )^2 ...
\[
\begin{align*}
\text{b11} &= \delta \left( \frac{4(4+1/(\rho^2))^2}{(3\rho^2)^2(1+1/(2\rho^2))(1+1/(\rho^2)^2)} \right) \\
\text{b10} &= \left( \frac{\pi}{1+\rho^2} \right)
\end{align*}
\]

\[
\begin{align*}
E &= \left( \beta^4 \frac{1}{\pi^2} \right) (a_{12}^2 + b_{12}^2) + \left( \frac{9}{16} \right) \alpha^2 \\
&\quad - \left( 3\alpha \beta \right) \frac{\delta}{2} a_{12} + \left( 3\alpha \beta \right) \frac{\delta}{2} b_{12}; \\
H &= \left( \beta^4 \frac{1}{\pi^2} \right) (2a_{12}a_{11} + 2b_{12}b_{11}) - (3\alpha \beta \frac{1}{2\pi}) b_{11}; \\
I &= \left( \beta^4 \frac{1}{\pi^2} \right) (2a_{12}a_{10}a_{11} + 2b_{12}b_{10} + b_{11}^2) \\
&\quad - 4\zeta \left( \beta^2 \frac{1}{\pi} \right) a_{12} - 3\alpha \beta^2 \frac{1}{2\pi} b_{12}; \\
J &= \left( \beta^4 \frac{1}{\pi^2} \right) (2a_{11}a_{10} + 2b_{11}b_{10}) - 4\zeta \beta^2 \frac{1}{\pi} b_{11} \\
&\quad + 4\zeta \beta^2 \frac{1}{\pi} b_{12}; \\
W &= \left( \beta^4 \frac{1}{\pi^2} \right) (a_{10}^2 + b_{10}^2) - 4\zeta \beta^2 \frac{1}{\pi} b_{10} \\
&\quad + 4\zeta \frac{1}{\pi} b_{12}; \\
M &= 0; \\
Nz &= -\gamma^2; \\
equ &= [E \; H \; I \; J \; W \; M \; Nz]; \\
A &= \text{roots}(equ); \\
A &= A(\text{imag}(A)==0); \\
Ao &= A(\text{sign}(A)==1); \\
\text{plot}(\text{sig}(i), Ao, 'ok')
\end{align*}
\]

\[
\begin{align*}
\text{for } j &= 1:\text{length}(Ao); \\
\text{qo1} &= \left( -1/(1+\rho^2) \right) (\rho \cos(t) - \sin(t)); \\
\text{qo2} &= \left( (1/(2(1+4\rho^2))) \right) (1+4\rho^2 \cos(2t) - 2\rho \sin(2t)) \\
&\quad \exp(-t./\rho) \left((1+i/(2\rho^2)^2)*(1+\exp(-i*pi/\rho)) \right); \\
\text{qo} &= \beta \alpha \delta \beta \delta \text{qo1} + \beta \alpha \beta \delta \beta \delta \text{qo2}; \\
\text{qdot} &= \beta \alpha \delta \beta \delta \text{qo} + \beta \alpha \beta \ delta \delta \text{qo} + \beta \delta \beta \delta \text{qo} / \rho; \\
\text{Power} &= \rho \delta \delta \text{qdot} \left((1+i/(2\rho^2)^2)*(1+\exp(-i*pi/\rho)) \right); \\
\text{Pavg(i)} &= \text{trapz(t(1:N/2),Power(1:N/2))/(pi)}; \\
\text{figure}(2); \\
\text{plot}(t,\text{Power}, 'k'); \\
\text{s(length(Pavg))} &= \text{sig}(i);
\end{align*}
\]
Pavg(end+1) = 0;

end

end

Pmax = max(Pavg);

figure(4);
plot(t,qo,'-r')
B = [delta Pmax];
save MaxPBeta05.dat B -ascii -append
save MaxPDelta.dat B -ascii -append
box on
hold on

figure(1);
size(s);
size(Pavg);
plot(s,Pavg(1:length(s)),'-b');
xlabel(\sigma')
ylabel('Average Non-dimensional Power')
title('Average Power as \sigma Differs')

D=load('MaxPDelta.dat');
Z = D';
Z = D;
save P_sigma_d020.dat Z -ascii
pstricksdata(Z(:,1),Z(:,2),'PmaxBeta.dat');
figure(2)
plot(Z(:,1),Z(:,2),'-b>')
plot(A(:,1),A(:,2),'-bo', B(:,1),B(:,2),'-r>')
xlabel('Piezoelectric Coupling (\delta)')
ylabel('Maximum Non-dimensional Power')
title('Maximum Power as \delta Differs; \sigma = 0, \zeta = 0.25, \beta = 1.0, \alpha = 0.5')
E.1 deriv.m

%plotting the power vs beta and sigma with out using method of averageing

%Power = Voltage^2/R = RQdot^2

function [xdot] = deriv(t,x,par)
% par = [2.5, 2.00, 0.00, 1.00, 1.00, 0.0 0.1];
zeta = par(1);
gamma = par(2);
alpha = par(3);
rho = par(4);
beta = par(5);
delta = par(6);
esilon = par(7);
omega = par(8);

xdot = zeros(length(x),1);

\ddot{x} + 2esilon*esilon*dot{x} + x(1+esilon*alpha*x^2) - esilon*beta...\
\dot{x} + 2esilon*esilon*dot{x} + x(1+esilon*alpha*x^2) - esilon*beta...\
\dot{x} + 2esilon*esilon*dot{x} + x(1+esilon*alpha*x^2) - esilon*beta...
xdot(3)=qdot
xdot(4)=Wdot

x = x(1);

xdot(1) = x(2);

xdot(2) = esilon*gamma*sin(omega*t)^2 - esilon*gamma*sin(omega*t)^2 - esilon*gamma*sin(omega*t)^2

xdot(3) = (1/rho)*(-x(3) + beta*(1+(delta*abs(x(1))))*x(3));

xdot(4) = rho*xdot(3)^2;
function [omega,Pavg]=poweraverage(par)
zeta = par(1);
gamma = par(2);
alpha = par(3);
rho = par(4);
beta = par(5);
delta = par(6);
epsilon = par(7);
C=8;
N=42;
for i=1:N;
    omega(i) = 1.00+3*epsilon - 6*epsilon*(i-1)/(N-1);
    par(8) = omega(i);
    Tspan = (2*pi/par(8)).*[0:0.1:20];
    [t x] = ode45(@deriv,Tspan, [0 0 0 0 0 0],[],par);
    [AP]=Averagpower(t,x, par);
    Pavg(i) = AP;
% AvgP = AP;
% end
[i,omega(i), Pavg(i)];
% B = [omega AP];
% save NumPavg_omega_beta300.dat B -ascii -append
% % % plot(omega,AP,'om')
% % % hold on
end
% D = load('NumPavg_omega_beta300.dat');
% % plot(D(:,1),D(:,2), '-r')
figure(1);
% plot(omega,Pavg,'k');
plot3(x(:,1),x(:,2),x(:,4),'-r')
% title('Average Non-dimensional Power as Forcing Frequency (\omega) Varies')
% xlabel('Forcing Frequency ($\omega$)')
% ylabel('Average Non-dimensional Power')
% drawnow;

E.3 Averagpower.m

function [AP]=Averagpower(t,x, par)
zeta = par(1);
gamma = par(2);
alpha = par(3);
rho = par(4);
beta = par(5);
delta = par(6);
epsilon = par(7);
b=0;
a=find(t>b*(2*pi/par(8)));
C=find(t==(b*(2*pi/par(8)))+(2*pi/par(8)));
for i=a(1):a(end-C);
    power(i)=(x(i+C,4)-x(i,4))/(2*pi/par(8));
    Pavg = sum(power);
end
AP = Pavg/(length(a));