HIGH ORDER EDGE FINITE ELEMENTS

A Thesis

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The Finite Element Method (FEM) is an extremely powerful and versatile computational technique. The standard (nodal) FEM, however, cannot be used universally in computational electromagnetics. In particular, the method fails to solve for the full vector Maxwell equations. The introduction of the vector FEM overcame most of the problems associated with nodal FEM. The present thesis extends some of the work done by Z. Ren and N. Ida on tetrahedral finite elements to hexahedral elements. The goal of the present thesis is twofold. First, to provide detailed mathematical and theoretical background, required to better understand the construction of the p-th order 1-form basis functions from a geometrical point of view. Second, to construct higher order basis functions for hexahedral elements, conforming in the space $H(\text{curl}, \Omega)$. To take full advantage of the insights the vector elements provide, one has to master the theory of differential forms, which is much more general tool than that of the vector analysis. The basis functions that are constructed in the present work are designed to model correctly the range space and null space of the exterior differential operator. They are hierarchical, so they can be used in p-refinement algorithms. In addition, by virtue of their construction the basis functions match the continuity conditions of the physical quantity modeled.
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1.1 Introduction

Very often in solving electromagnetic problems, the solution cannot be obtained analytically. In such cases a numerical solution of the problem is needed. In general the numerical methods that are most commonly used, can be classified according to their type of numerical formulation. For example, methods based on PDE formulations are the finite difference time domain (FDTD) as well as the finite element method (FEM). On the other hand methods such as the method of moments (MoM) and boundary element method (BEM), are based on the integral formulations. Another approach to classify the numerical techniques is based on the type of the computational domain: frequency or time domain. We should point out that the choice of one method or another is entirely based on the problem to be solved.

One of the earliest and simplest numerical techniques in CEM, is the FDTD method. It was originally proposed by Yee in 1966 [1]. FDTD has many advantages and this is the reason that the method is quite popular to this day. For example, the effects of numerical dispersion are kept under control, by requiring a sufficient number of grid points per wavelength. In addition the method can be easily used
in conjunction with specially designed models of boundary conditions (BC), such as absorbing boundary conditions (ABC) to take into account open regions in antenna design problems. The method, however, has certain limitations that do not allow the application of the method in some problems. For instance if the problem is electrically large, such as when many wavelengths span the computational domain, the method becomes extremely expensive to use, since the number of grid points that are required for the correct solution will be prohibitively large.

Approximately about the same time another method was developed, namely the MoM. Originally proposed for CEM in 1967 by Harrington [2]. The method is suitable for both differential and integral formulation of the problem, although it became popular by discretizing the electric field integral equation (EFIE). One of the advantages of the method is that it takes into account open region BC explicitly, by definition. The MoM, unlike PDE based methods, approaches the problem from different angle, by solving for the field sources. The major drawback of the method, however, is that knowledge for Green’s functions is required, which narrows the radius of applicability of the method in CEM to geometries that are not too complex.

The FEM is a powerful and quite general numerical technique for solving PDEs that overcomes most of the limitations of the methods above. The first that came up with the idea to break the solution region of a continuous problem into small regions and replacing the fields with their approximations within these small regions, was Courant [3]. It was, however, not before the late 50’s when the method was actually used in computations. The FEM was originally used in structural and aero-
nautical mechanics. Silvester [4] was the first to use the method in CEM. In the standard FEM procedure, the original boundary value problem is cast into a variational formulation in terms of an energy-related functional. The computational domain is then subdivided into discrete regions (elements) such as triangles and quadrilaterals for the 2D case, and tetrahedra and hexahedra in the 3D case. A trial solution is then defined over each region in terms of a basis function expansion. The resulting variational functional is minimized with respect to the unknown basis function coefficients through a process called the Galerkin method. Thus we obtain a set of algebraic equations which are then solved to obtain the basis function coefficients. We finally find a piecewise approximate solution of the original PDE.

The classic FEM uses nodal (scalar valued) basis functions. The method is very accurate in solving static EM problems, where the continuous electrostatic potential can be used. The method, however, fails in solving the full vector form of Maxwell’s equations for either the magnetic or electric fields. There is a class of problems in microwave engineering associated with the eigenvalues and eigenvectors that arise from calculation of resonant frequencies and modes, respectively. These kinds of problems occur in the study of waveguides and microwave resonant cavities and this is where the classical FEM shows its limitations. The problem with the nodal FEM is rooted in the fact that it enforces continuity of the field across the interface between two materials. A second problem associated with scalar valued FEM is that they exhibit so called spurious modes, which exist regardless of the number of elements we use to discretize the domain.
1.2 Overview

The first steps taken toward developing new types of elements were done by P.A Raviart and J.M. Thomas [5]. Their elements were conforming in the space $H(div, \Omega)$. Nedelec [6, 7], however, was the first to introduce a comprehensive mathematical framework for two new types of mixed finite elements build on 3D elements - tetrahedra and hexahedra - which are conforming in the spaces $H(curl, \Omega)$ and $H(div, \Omega)$. It should be noted that the notion of conformity is related to the property of these elements to match the corresponding field continuity conditions.

Nedelec’s paper was the cornerstone of the development of mixed finite elements. He, however did not report any basis functions. Due to this, many papers were published, proposing different expressions for basis functions on different topologies. Thus in [8] first order edge and facet basis functions were proposed, built on tetrahedra. First order edge basis on hexahedra were proposed in [9]. Further on, second order edge finite elements were presented in [10, 11] and 2.5 mixed finite elements in [12]. These works, however, present relatively low polynomial order finite elements. In order to increase the accuracy of the finite element solution, it has been realized that higher polynomial order should be investigated.

A common goal of all the work in the area of edge and facet finite elements is to reproduce correctly on discrete level the Helmholtz decomposition and the De Rham exact sequence. Helmholtz decomposition essentially represent the decomposition of an arbitrary vector field into curl-free and div-free fields. In order to model an
arbitrary vector field, one should be able to model the operators involved. Over the years, two major trends are observed in developing of high order edge and facet elements. In the majority of papers devoted to the developing of high order edge basis functions, the construction is done from a standard vector point of view. Some notable works are [13, 14, 15, 16, 17, 18, 19, 20]

Later on, we shall see that there are certain problems associated with the use of the standard vectors and vector calculus that need to be addressed [21]. In [8] the so called Whitney forms were introduced. They are based on Nedelec’s first order tetrahedral curl- and div- basis functions, which are discrete 1-forms and 2-forms, respectively. It becomes clear that differential forms are the natural choice for discretizing Maxwell’s equations. The discrete differential 1- and 2-forms are described by first order polynomials [8]. Hence, the second trend in constructing edge basis on various topologies, is the use of the far more general tool of differential forms and exterior algebra [22, 23, 24, 25, 26]. It should be mentioned, however, that the difference between using standard vectors or differential forms in the construction of high polynomial order elements is very often pedagogical. For instance, the De Rham diagram can be expressed in terms of vectors and vector operators or in terms of differential forms and exterior algebra, however, provide more geometrical insight and the coordinate independent operator - the exterior differential operator - unlike the vectors and vector calculus, which are defined in a particular coordinate system. More specifically, in [22, 23] a general and systematic description of high order elements, their
function spaces, and the assignment of the degrees of freedom on different subsimplices (edges, faces, volume of the reference element) was presented for the case of tetrahedral elements. A general description of high order finite elements, based on hexahedral and prismatic topologies is given in [27]. In [27] the dimensions of the functions spaces (as well as the dimension of the null and range spaces) are given. The exact distribution of the DoFs on the various subsimplexes is also provided. A unified framework was presented, based on the exterior algebra and differential forms, for constructing finite element spaces in [24].

There are a few aspects in the present construction of high order 1-form basis functions that we discuss here. The De Rham complex is becoming a standard tool in analyzing mixed finite element approximations. It relates the standard $H^1$ conforming, $H(\text{curl}, \Omega)$ conforming, and $H(\text{div}, \Omega)$ conforming elements. Here we try to put our construction in the context of some of the existing high order finite elements [28, 18, 29, 16, 15, 20, 30, 25]. While the analysis in [28, 20, 30] is provided in terms of standard vector calculus, a more geometrical approach to De Rham’s complex is provided in [22, 23]. A common goal of all high order constructions, as we mentioned earlier, is to provide a discrete model of the Helmholtz decomposition and De Rham’s exact sequence. In this context, the approach in [28, 18, 16, 20, 30, 25] is more or less similar. In the present construction we take the same approach, while we try to highlight the geometrical nature of the electromagnetic fields. In addition in [22, 23, 27] a detailed analysis was done on the discrete spaces $W^0_q, W^1_q, W^2_q$, providing the exact dimensions of the null space and range space of the exterior differential
operator. The space that concerns us for the construction of 1-form basis functions on a reference hexahedron, is $W^1_q$. This space was originally described in [6, 7] and later on studied in most works on high order edge finite elements. The space $W^1_q$, however, does not provide particular basis functions, rather it is seen as a template, and the actual construction of basis functions is done in conjunction with the various degrees of freedom. There are a few notable works which provide explicit expressions for the edge basis, built on a hexahedron [18, 15, 30, 25] and below we briefly compare the construction in the present thesis in the context of the existing ones. In a general construction of high order elements, there are few main points to be considered:

- The choice of the functional spaces
  
  - The spaces that are used in this thesis are Nedelec’s spaces [6, 7].
  
  - The spaces used by Graglia [15] are also identical with Nedelec’s spaces.
  
  - Demkowicz [17, 31, 18] uses spaces that are different from those of Nedelec in a sense that his spaces are more general. For uniform polynomial order $p$, his spaces are identical with Nedelec’s spaces. In other words, the spaces with variable polynomial order and element size, he has developed, are generalizations of Nedelec spaces.

- The choice of shape/basis functions
  
  - In the present thesis for the construction of the basis functions, standard Legendre and Lagrange polynomials are used.
– In the work of Graglia, Lagrange polynomials are used. He also uses the auxiliary interpolatory polynomials of Silvester on a canonical element to represent these polynomials.

– For the construction of the basis functions Demkowicz uses integrated Legendre polynomials (known also as Lobatto shape functions).

• Degrees of freedom - The number of degrees of freedom of different polynomial order elements in this thesis is identical with the number of DOFs in Graglia’s work. The difference is in the distribution of the DOFs on the various subsimplices. In the construction of Demkowicz the approach is much different due to the non uniform order of interpolation, so it is more general.

• The topology of the reference element - The use of a reference element in the
design of edge basis function is not required. There are a few works in which
the construction is done without the help of a reference element [29, 16, 15]. Its use, however, greatly simplifies the construction. In addition when we work
with a physical (unstructured) mesh, sometimes the elements are very complex
and the numerical integration could be problematic, while the integration over
a reference element is often a trivial task. In the present thesis, we use a master
element that spans the space $\hat{\Omega} = [-1,1]^3$ [30]. In comparison in [18, 25],
the reference element spans the space $\hat{\Omega} = [0,1]^3$. We use the former master
element, since the Legendre polynomial, used in the construction, are defined
over the interval $[-1,1]$ in 1D, so they are used over the reference element
without any modifications.

In the classical concept of finite elements, as we discussed earlier, the shape functions
result from the definition of degrees-of-freedom. In other words the logical flow is
that the function spaces of element shape functions in conjunction with DOFs gives
the shape functions. This is the approach that is taken in this thesis as well, which
is different from the approach taken in the work of Demkowicz. In his work, the
shape functions are defined first and the interpolation operators, are independent of
the choice of shape functions. Identification of DOF is a secondary issue and he
proposes to do so a-posteriori. In summary, we can say that Demkowicz’s work is
quite different and more general then the present construction in all aspects (spaces,
basis functions and assigning the DOFs). In addition to the mentioned fundamental
differences, there are some pedagogical differences between the present thesis and
the work of Demkowicz, namely the use of differential forms in the construction that provide a geometrical insight in the construction of high order elements. This is one of the main advantage in the present construction. As to the work of Graglia, it could be considered similar to the present construction since there are common points, in particular he uses Nedelec’s spaces, the DOFs are identical, but distributed in a different way (uniformly spaced).

1.3 Contribution

The construction in the present thesis of explicit expressions for p-th polynomial order 1-form basis functions intend to extend the work done in [22, 23], where the discrete function spaces, the assignment of the degrees of freedom on the various subsimplexes, and high order 1-form basis functions on a tetrahedral topology were presented. As a next step, in [27] the discrete function spaces and the assignment of the degrees of freedom on the various subsimplexes on a prism and a hexahedron was presented. In [27], however no basis functions were provided. In this context, the primary contribution of the present thesis is twofold:

- First, we construct higher order 1-form basis functions on hexahedral elements, conforming in the space \( H(\text{curl}, \Omega) \) of Nedelec type. In the notation of Hiptmair [24], 1-form finite element functions are discrete 1-form differential forms. In our notation we go further and define 1-form finite element functions as cochains defined on the various subsimplexes of a chain complex (discrete manifold). We
build the hierarchical 1-form basis functions by using the discrete version of
the space $H(\text{curl}, \Omega)$, which usually is denoted by $W^1_q$, in conjunction with the
degrees of freedom (DoF).

- In order to justify the construction of these high order 1-form basis functions,
the reader not familiar with the theory of differential forms, need to master
first the various mathematical notions from exterior algebra, exterior calculus,
differentiable topologies, and discrete geometry. The present thesis offers a
systematic detailed introduction, in order to better understand the construction
of 1-form basis functions from a geometrical point of view.

The main guideline is to relate the mathematical background to the actual construction
of explicit 1-form basis functions.

1.4 Outline

To achieve our goal in this thesis, we take the approach suggested by Faraday and
Maxwell’s wave theory. They view the electromagnetic energy as inseparably linked
to space, rather than being inserted into the space, which is the case with Newtonian
mechanics. This view allows us to take a geometrical approach to Maxwell’s equa-
tions and electromagnetic theory, such that we study static fields as 3-manifolds and
dynamic fields as 4-manifolds. In order to make full use of these insights, we replace
the 3D vector analysis by the far more general tool of differential forms. Thus the
equations of electromagnetics can be elegantly cast into the language of differential
forms. In this approach the electric scalar potential is a 0-form, the electric and magnetic field intensities are 1-forms, the electric and magnetic fluxes are 2-forms, and the scalar charge density is a 3-form.

In Chapter II of this thesis we start by studying the differential k-forms and the vector spaces where they are defined. In addition we introduce the basic operators of exterior (wedge) product, the exterior derivative operator, and the Hodge star operator. Finally in Chapter II, we show how these operators and differential forms are combined.

As we mentioned earlier in the study of electromagnetic fields and Maxwell’s equations, from a geometrical point of view, the notion of a manifold is used. Thus in order to use differential forms in the study of EM fields, we need to define them on manifolds. The problem is that differential forms exist in vector spaces and as we shall see, the manifolds are not ones. We address this issue in Chapter III. The solution of this problem is simply to assign a tangential vector space in a neighborhood of each point on the surface of the manifold. Then the differential forms are readily defined over these tangential vector spaces.

In Chapter IV we revisit Maxwell’s equations and translate them in the language of differential forms. We enlist the help of the so called Tonti diagrams to expose the mathematical structure of Maxwell’s equations.

In terms of FEM terminology, the physical mesh that discretizes the space, which is occupied by electromagnetic energy, geometrically is viewed as a discrete version of the original continuous manifold, studied in Chapter III. In order to dis-
cretize a continuous manifold, in Chapter V we introduce the notions of a simplex and a simplicial complex as well as a chain and a chain complex. Further on we introduce the notion of a cochain, which as we shall see is viewed as a discrete differential form. In our terminology a physical mesh should be considered as a chain complex, while the discrete differential k-forms (k-cochains) as a finite element basis function expansion, defined over the chain complex.

In Chapter VI we apply all the considerations above to construct q-th order edge basis functions based on discrete 1-forms over a reference hexahedron, which can be viewed as a chain complex of 1-chains.
CHAPTER II
DIFFERENTIAL FORMS

2.1 Vector spaces

A vector space $V$ defined over the set of real numbers $\mathbb{R}$, or the set of complex numbers $\mathbb{C}$, is a non-empty set of vectors such that: Let $a, b, 0 \in \mathbb{R}$ and $x, y, z, 0 \in V$:

1. $\forall x, y \in V, \exists z : z = x + y, z \in V$

2. $\forall x \in V$ and $a \in \mathbb{R}, \exists y : y = ax, y \in V$

3. $x + y = y + x$, commutative addition

4. $x + (y + z) = (x + y) + z$, associative addition

5. $a(bx) = ab(x)$, associative multiplication

6. $(a + b)x = ax + bx$, distributive scalar addition

7. $a(x + y) = ax + ay$, distributive vector addition

8. In addition a vector space over $\mathbb{R}$ must have the following properties:

   • $0 + x = x$, additive identity

   • $x + (-x) = 0$, additive inverse
- $1x = x$, multiplicative identity
- $-1x = -x$, multiplicative inverse identity
- $0x = \overline{0}$

From linear algebra we know that if an n-tuple $(\sigma_1, \ldots, \sigma_n)$, in an n-dimensional space $V$ for any positive integer $n$, spans the space and if linearly independent, it forms a basis of $V$. This means $\sum_{i=1}^{n} a^i \sigma_i = 0$ implies $a^i = 0$, where $\sigma_i \in V$ and $a^i, \in \mathbb{R}$.

2.2 Exterior product and k-vectors

In order to introduce the exterior product we shall start with something familiar. Let’s suppose we have two vectors $v$ and $u$ from $V(\mathbb{R}^3)$.

$$u = \sum_{i=1}^{3} u^i \sigma_i, \quad v = \sum_{i=1}^{3} v^i \sigma_i,$$

where $u, v, \sigma_i \in V$ and $u^i, v^i \in \mathbb{R}$. We shall evaluate the cross product of these two vectors. Before doing so, it should be pointed out that we are going to consider the general case when the basis vectors are arbitrarily oriented. The cross product of $u$ and $v$, in this case is [32]:

$$u \times v = (u^2 v^3 - u^3 v^2) \sigma_2 \times \sigma_3 + (u^1 v^3 - u^3 v^1) \sigma_1 \times \sigma_3 + (u^1 v^2 - u^2 v^1) \sigma_1 \times \sigma_2 \quad (2.2)$$

The elements of $V$, generated by $\sigma_i$, are called 1-vectors and they belong to the 1-vector space. Similarly, the elements generated by linear combinations of the basis vectors $\sigma_i \times \sigma_j$, are called 2-vectors and the space where they belong is called 2-vector
space, denoted by $V^2$ (to be more precise, the notation should be $V^2(\mathbb{R}^3)$, but most of the time we shall use the shorter notation above). The operation in Eq. 2.2 is called wedge product or exterior product and is denoted by '$\wedge$' [33, 34]. From now on we shall use this notation, instead of '$\times$' [32]. Here we should note that although we used the cross product to introduce the exterior product, they are quite different.

Below we explain the major differences:

- The cross product of two vectors is a vector from the same vector space, while the wedge product of two vectors is a vector that belongs to a different vector space. As we saw above the exterior product of two 1-vectors is a 2-vector.

- In the case of exterior product the vectors do not have to be orthonormal. In fact, they can be absolutely arbitrarily oriented.

Further, following the same idea of 1-vectors, 2-vectors, and so on, we can generalize to k-vectors from the k-vector space $V^k$. This space is defined in the same manner as the space $V$ from Section 2.1 so it must have the same properties and follow the same rules as the space $V$. The basis of this space is $\sigma_{i_1} \wedge \ldots \wedge \sigma_{i_k}$, where $1 \leq i_1 < \ldots < i_k \leq n$ [34]. The reasons for this particular ordering of the indices will be discussed a little bit later on in this chapter.

Let $a, b \in \mathbb{R}$, and $u, v_1, \ldots, v_k \in V^k, k \leq n$. Then the exterior product is defined by the following rules:

1. Multilinearity (k-linearity)

\[
(a u + b v_1) \wedge v_2 \wedge \ldots \wedge v_k = (a u \wedge v_2 \wedge \ldots \wedge v_k) + (b v_1 \wedge v_2 \wedge \ldots \wedge v_k) \quad (2.3)
\]
2. Skew symmetry (antisymmetry)

\[ \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \ldots \wedge \mathbf{v}_i \wedge \mathbf{v}_{i+1} \wedge \ldots \wedge \mathbf{v}_k = -\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \ldots \wedge \mathbf{v}_{i+1} \wedge \mathbf{v}_i \wedge \ldots \wedge \mathbf{v}_k \quad (2.4) \]

As a result of Eq. 2.4 for some pair of indices, \( i \neq j \) and \( \mathbf{v}_i = \mathbf{v}_j \), we have:

\[ \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \ldots \wedge \mathbf{v}_k = 0 \quad (2.5) \]

The dimension of \( \mathbf{V}^k \) over \( \mathbb{R}^n \) is:

\[ \text{dim} \mathbf{V}^k = \binom{n}{k} = \frac{n!}{k!(n-k)} \quad (2.6) \]

From Eq. 2.6 above, it becomes apparent that the dimensions of \( \mathbf{V}^k \) and \( \mathbf{V}^{n-k} \) are equal. The system of \( \mathbf{V}^i, i = 0, 1, \ldots, k \), together with the exterior product form the exterior algebra (also called Grassman algebra) of the vector space \( \mathbf{V} \).

2.3 Exterior product of multivectors

As we saw earlier, when we apply the exterior product on two 1-vectors \( \mathbf{v}_1 = \mathbf{\sigma}_1 \) and \( \mathbf{v}_2 = \mathbf{\sigma}_2 \), the result is a 2-vector \( \mathbf{\sigma}_1 \wedge \mathbf{\sigma}_2 \). Following the same idea, we try to extend the exterior product to multivectors. Let us take two vectors, say a k-vector and a p-vector. Applying the exterior product on these two will result in a (p+k)-vector. We write it as follows [34]:

\[ \mathbf{V}^k \wedge \mathbf{V}^p = \mathbf{V}^{p+k} \quad (2.7) \]

Now we can define the properties of the wedge product. Let us take three multivectors \( \mathbf{u} \in \mathbf{V}^p, \mathbf{v}, \mathbf{w} \in \mathbf{V}^k \), and the scalars \( a, b \in \mathbb{R} \)
1. Distributive law

\[ \mathbf{u} \wedge (a\mathbf{v} + b\mathbf{w}) = (a\mathbf{u} \wedge \mathbf{v}) + (b\mathbf{u} \wedge \mathbf{w}) \]  

(2.8)

2. Associative law

\[ \mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) = (\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w} \]  

(2.9)

3. Skew symmetry

\[ \mathbf{u} \wedge \mathbf{v} = (-1)^{pk} \mathbf{u} \wedge \mathbf{v}, \text{ if } p + k \leq n \]

\[ \mathbf{u} \wedge \mathbf{v} = 0, \text{ if } p + k > n \]

2.4 Covectors (forms)

Let us take a vector space \( \mathbf{V}^1 \) and \( \mathbf{u}, \mathbf{v} \in \mathbf{V}^1 \) and \( a \in \mathbb{R} \). The set of all mappings on \( \mathbf{V}^1 \phi : \mathbf{V}^1 \rightarrow \mathbb{R} \), form a vector space and are linear and antisymmmetric. We can define over \( \phi \) the operations of summation and multiplication by a scalar:

\[ \phi(\mathbf{u} + \mathbf{v}) + \psi(\mathbf{u} + \mathbf{v}) = (\phi(\mathbf{u}) + \phi(\mathbf{v})) + (\psi(\mathbf{u}) + \psi(\mathbf{v})) \]  

(2.10)

\[ \phi(a\mathbf{v}) + \psi(a\mathbf{v}) = a(\phi(\mathbf{v}) + \psi(\mathbf{v})) \]  

(2.11)

In addition to the operations defined above, a few more things should be noted:

1. If \( \phi \) and \( \psi \) are linear and antisymmetric on \( \mathbf{V}^1 \), then \( \phi + \psi \) is linear and antisymmetric on \( \mathbf{V}^1 \) as well.

2. If \( \phi \) is linear and antisymmetric on \( \mathbf{V}^1 \), then the value of \( a\phi(\mathbf{v}) \) is linear and antisymmetric on \( \mathbf{V}^1 \) as well.
3. Associative, distributive and commutative, properties are satisfied for covectors. These real valued mappings belong to a space that is called covector space and is dual to $V^1$. The covector space of $V^1$ is denoted by $V^1\ast$. In addition the spaces $V^1\ast$ and $V$ have equal dimensions. Furthermore, the linear antisymmetric mappings on $V$ form a basis of $V^1\ast$ (later in the section on differential forms we prove this statement and also call these forms elementary forms [33]).

It is straightforward to extend the idea of covectors and covector space to the more general case of $k$-covectors and $k$-covector space. Let $\varphi : V \times V \times \ldots \times V \to \mathbb{R}$, satisfying the following rules:

- **Multilinearity** - $\varphi$ is a $k$-form and $v_i = au + bw$

  \[
  \varphi(v_1, \ldots, (au + bw), \ldots, v_k) = \varphi(v_1, \ldots, au, \ldots, v_k) + \varphi(v_1, \ldots, bw, \ldots, v_k)
  \]  

  (2.12)

- **Skew symmetry** - if any two arguments of $\varphi$ are interchanged, the sign of $\varphi$ alternates

  \[
  \varphi(v_1, \ldots, v_i \ldots v_j \ldots v_k) = -\varphi(v_1, \ldots, v_j \ldots v_i \ldots v_k)
  \]  

  (2.13)

From Eq. 2.13 above follows that: $\varphi(v_1, \ldots, v, v, \ldots, v_k) = 0$.

To summarize, the $k$-vectors and the $k$-covectors share the same properties, due to the duality of the spaces to which they belong. In particular, a $k$-covector space is a vector space by means of the definition from Section 2.1. In other words
Table 2.1: Summary of vectors and covectors and the spaces where they belong

<table>
<thead>
<tr>
<th>Space</th>
<th>Element</th>
<th>Notation of elements</th>
<th>Notation of spaces</th>
</tr>
</thead>
<tbody>
<tr>
<td>real space/0-vector space</td>
<td>scalar</td>
<td>( \mathbb{R}/V^0 )</td>
<td>c</td>
</tr>
<tr>
<td>1-vector space</td>
<td>1-vector</td>
<td>(</td>
<td>V^1)</td>
</tr>
<tr>
<td>2-vector space</td>
<td>2-vector</td>
<td>( V^2 )</td>
<td>( \sum_{i=1}^{n} x^i (\sigma_i \wedge \sigma_j) )</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>k-vector space</td>
<td>k-vector</td>
<td>( V^k )</td>
<td>( \sum_{i=1}^{n} x^i (\sigma_{i_1} \wedge \sigma_{i_2} \wedge \ldots \wedge \sigma_{i_k}) )</td>
</tr>
<tr>
<td>0-covector space</td>
<td>0-form</td>
<td>( V^0^* )</td>
<td>f(c)</td>
</tr>
<tr>
<td>1-covector space</td>
<td>1-form</td>
<td>( V^1^* )</td>
<td>( \sum_{i=1}^{n} \varphi(x^i \sigma_i) )</td>
</tr>
<tr>
<td>2-covector space</td>
<td>2-form</td>
<td>( V^2^* )</td>
<td>( \sum_{i=1}^{n} \varphi(x^i \sigma_i \wedge \sigma_j) )</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>k-covector space</td>
<td>k-form</td>
<td>( V^k^* )</td>
<td>( \sum_{i=1}^{n} \varphi(x^i (\sigma_{i_1} \wedge \ldots \wedge \sigma_{i_k})) )</td>
</tr>
</tbody>
</table>

The elements of the dual space (forms) must satisfy the same requirements, such as distributivity, associativity, commutativity, and have the same properties as their counterpart k-vectors. Finally, if the forms \( \varphi_1, \ldots, \varphi_k \) are the basis of \( V^1^* \), it can be shown that the basis of the k-covector space \( V^k^* \) is given by the linear combinations of the exterior product

\[
\varphi_{i_1} \wedge \ldots \wedge \varphi_{i_k}, \ 1 \leq i_1 < \ldots < i_k \leq n
\]  

(2.14)
2.5 Differential forms

Notation: In Section 2.4 we considered the covectors (exterior forms) generated by a linear combination of the basis given with Eq. 2.14. In this section we study a particular case of forms called differential forms. In other words, differential forms are exterior forms with special basis

\[ \varphi_{i_1} \wedge \ldots \wedge \varphi_{i_k} = dx_{i_1} \wedge \ldots \wedge dx_{i_k} \equiv dx_I, \quad (2.15) \]

where \( I \in E_p \) is the group of combinations of the indices:

\[ E = \{ I = (i_1, i_2, \ldots, i_k) \}, \quad 1 \leq i_1 < \ldots < i_k \leq n \quad (2.16) \]

Let us take a k-vector \( u \in V^k \). In terms of notations, very often we use \( \varphi \) for shorthand notation of \( dx_{i_1} \wedge \ldots \wedge dx_{i_k} \). In addition, since differential k-forms operate over k-vectors, the notation of \( \varphi(u) \) is used as equivalent to \( dx_{i_1} \wedge \ldots \wedge dx_{i_k}(u) \)

**Definition:** (Elementary differential k-forms on \( \mathbb{R}^n \)) An elementary k-form on \( \mathbb{R}^n \) is an expression of the form \( dx_{i_1} \wedge \ldots \wedge dx_{i_k} \), where \( 1 \leq i_1 < \ldots < i_k \leq n \) and \( 0 \leq k \leq n \). These forms evaluated on k-vectors \( v_1, \ldots, v_k \), give the determinant of a \( k \times k \) matrix, obtained by selecting \( i_1, i_2, \ldots, i_k \) rows of the matrix, whose columns are the vectors \( v_1, \ldots, v_k \).

Let us take a 3-vector in \( \mathbb{R}^4 \) (k=3, n=4) and apply the differential form \( \varphi = dx_1 \wedge dx_2 \wedge dx_4 \) on it. So we have:
Following the definition above, we select the first, second, and the forth rows to form a $3 \times 3$ matrix. The evaluation of the determinant of this matrix will give us a number. Now we recall the definition of a form (covector): multilinear, antisymmetric mapping of the form: $\varphi : V^k \to \mathbb{R}$. This is exactly what we did in our example: we took a 3-vector from $V^3$ on $\mathbb{R}^4$ and applied a 3-form on it to get a number that belongs to $\mathbb{R}$ ($\varphi : V^3(\mathbb{R}^4) \to \mathbb{R}$)

\[
\det = \begin{vmatrix}
1 & -1 & 1 \\
-2 & 1 & 1 \\
1 & 2 & 2
\end{vmatrix} = 2 + (-4) + (-1) - 1 - 2 - 4 = -10
\]

Next we explain the particular ordering of the indices $1 \leq i_1 < \ldots < i_k \leq n$ and why the requirement $k \leq n$ is necessary. As we saw above, the forms are closely related to evaluation of determinants. This is why their properties (at least partially) are inherited from the theory of determinants [34].

1. $1 \leq i_1 < \ldots < i_k \leq n$. There is a great deal of redundancy in the expression for elementary differential forms, above.

(a) If we take $\varphi = dx_1 \wedge dx_2 \wedge dx_1$, which is a differential 3-form and apply it to a 3-vector $\varphi(v_1, v_2, v_3)$, the result is 0. The resulting determinant will
always be 0 since two of its rows are equal. To express the same, but in more general terms, if any two indices of a k-form are equal, this means that \( dx_{i_1} \wedge \ldots \wedge dx_{i_k} = 0 \)

(b) As we mentioned earlier, due to the antisymmetric property of the covectors we have: \( dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k} = -dx_{i_2} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} \). More generally, if the indices \( i_1, \ldots, i_k \) and \( j_1, \ldots, j_k \) are the same indices, but taken in different order, so that \( j_1 = i_{\sigma(1)}, \ldots, j_k = i_{\sigma(k)} \), for some permutations \( \sigma \) of \( \{1, \ldots, k\} \). Then \( dx_{j_1} \wedge \ldots \wedge dx_{j_k} = sgn(\sigma)dx_{i_1} \wedge \ldots \wedge dx_{i_k} \), where \( sgn(\sigma) \) is the signature of the permutations: \( sgn(\sigma) = \det M_\sigma \), where \( M_\sigma \) is the permutation matrix, which is +1 for even number of permutations and -1 for odd number of permutations. To eliminate the redundancy we say that the elementary k-form is in the form

\[
dx_{i_1} \wedge \ldots \wedge dx_{i_k}, \text{ with } 1 \leq i_1 < \ldots < i_k \leq n
\]

Doing so, we choose a particular permutation, which is one of many possible.

2. There are no elementary differential k-forms on \( \mathbb{R}^n \) with \( k > n \): there are no maps \( \varphi \) that take k-vectors with \( k > n \) on \( \mathbb{R}^n \) and return a number, such that \( \varphi(v_1, \ldots, v_k) \) is multilinear and antisymmetric. If \( v_1, \ldots, v_k \) are vectors on \( \mathbb{R}^n (V \subseteq \mathbb{R}^n) \) and \( k > n \), then the vectors are not linearly independent. This means that at least one is a linear combination of the others. For example, let \( v_k = \sum_{i=1}^{k-1} a^i v_i \) So if \( \varphi \) is a k-form on \( \mathbb{R}^n \), the evaluation of
the k-vectors \( v_1, \ldots, v_k \), gives:
\[
\varphi(v_1, \ldots, v_k) = \sum_{i=1}^{k-1} a_i \varphi(v_1, v_i, v_1, \ldots, v_i) = \sum_{i=1}^{k-1} a_i \varphi(v_1, v_i)
\]
The determinant evaluated on these vectors will always be 0, since two columns always coincide.

Since it is difficult to imagine what the forms look like, we can consider some geometric interpretations. To do so, we can take another simple example. Let us apply a 2-form on a 2-vector from \( V^2 \) on \( \mathbb{R}^3 \).

\[
dx_1 \wedge dx_3 \left( \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \right) = \begin{vmatrix} v_1 & u_1 \\ v_3 & u_3 \end{vmatrix} = v_1 u_3 - v_3 u_1
\]

This can be understood geometrically as taking the vectors \( u \) and \( v \) and projecting them on the \((x_1, x_3)\) plane, so we get the vectors:

\[
\begin{bmatrix} v_1 \\ v_3 \end{bmatrix} \text{ and } \begin{bmatrix} u_1 \\ u_3 \end{bmatrix}.
\]

In addition, the determinant in this example gives the signed area of the parallelogram, which these vectors span. So, now it becomes clearer, why the result is always 0 when \( k > n \). The area of a parallelogram in \( \mathbb{R}^1 \) is 0 and so is a volume in \( \mathbb{R}^2 \).

In the previous section it was noted that k-forms form a vector space, and the elementary differential forms form a basis of this vector space. A space is a vector space if the operations addition and scalar multiplication are defined.
**Definition:** (Addition of differential forms) Let \( \varphi \) and \( \psi \) be two differential k-forms. Then \( \varphi(v_1, \ldots, v_k) + \psi(v_1, \ldots, v_k) = (\varphi + \psi)(v_1, \ldots, v_k) \)

**Definition:** (Multiplication of differential forms by a scalar) If \( \varphi \) is a differential k-form and \( a \) is a scalar, then: \( (a\varphi)(v_1, \ldots, v_k) = a(\varphi(v_1, \ldots, v_k)) \)

The above two definitions are sufficient to show that the space of differential k-forms is a vector space on \( \mathbb{R}^n \). Next we are going to show that the elementary differential forms form a basis of that vector space. Since the differential forms are particular cases of covectors, we will keep the same notations. Refer to Table 4.2 for more details.

**Definition:** The space of differential k-forms on \( \mathbb{R}^n \) is denoted by \( \mathbf{V}^k(\mathbb{R}^n) \)

**Theorem:** The elementary differential k-forms form a basis of \( \mathbf{V}^k(\mathbb{R}^n) \)

The above theorem means that every multilinear and antisymmetric map \( \varphi \) of k-vectors in \( \mathbf{V}^k(\mathbb{R}^n) \) at a point P, can be uniquely written as:

\[
\varphi = \sum_{1 \leq i_1 < \ldots < i_k \leq n} a^{i_1, i_2, \ldots, i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}, \text{ or }
\]

\[
\varphi = \sum_{1 \leq i_1 < \ldots < i_k \leq n} a^{i_1, i_2, \ldots, i_k} dx_{i_1},
\]

and the coefficients (constants), are given by:

\[
a^{i_1, i_2, \ldots, i_k} = \varphi(\sigma_{i_1}, \ldots, \sigma_{i_k})
\]

**Proof:** We assume that the determinant exists and is uniquely characterized by its properties of multilinearity, antisymmetry and normalization. Here we use a particular case that will convey the idea of the proof and avoid unnecessary complexity. For
example let us take a 2-form in $\mathbb{R}^3$ (k=2, n=3), which should be sufficient for the proof.

$$
\varphi \left( \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \right) = \varphi(v^1\sigma_1 + v^2\sigma_2 + v^3\sigma_3, u^1\sigma_1 + u^2\sigma_2 + u^3\sigma_3) = \\
= (v^1u^2 - v^2u^1)\varphi(\sigma_1, \sigma_2) + (v^1u^3 - v^3u^1)\varphi(\sigma_1, \sigma_3) + (v^2u^3 - v^3u^2)\varphi(\sigma_2, \sigma_3)
$$

Similar, but messier computation will show the same result for any k and n. Finally, the dimension of $V^k (\mathbb{R}^n)$ is equal to the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Now let $U$ denote an open domain in $\mathbb{R}^n$. A k-form on $U$ is obtained by choosing at each point P (with coordinates the n-tuples $(x^1, \ldots, x^n)$) of $U$ a k-form at that point, and doing that smoothly. Thus a k-form $\omega$ has the representation

$$\omega = \sum a_I dx_I,$$

where $a_I \equiv a_{i_1, \ldots, i_k}(x^1, x^2, \ldots, x^n)$

$a_I$ are smooth functions on $U$ and differentiable as often as we want. The exterior algebra applies to each point of $U$ and so may be applied on differential forms on $U$ itself. Thus if $\omega$ is a p-form and $\eta$ is a k-form on $U$, then $\omega \wedge \eta$ is a (k+p)-form on $U$, only if $(k + p) \leq n$. It is easy to check this. Let

$$\omega = \sum a_I dx_I \text{ and } \eta = \sum b_H dx_H,$$

then

$$\omega \wedge \eta = \sum a_I b_H dx_I dx_H,$$

so that the coefficients of $\omega \wedge \eta$ are again smooth functions.
2.6 Exterior derivative of differential forms

*Note:* Sometimes the ‘∧’ is omitted in the expressions for differential forms. So when we write $dx_1dx_2$, it should be understood $dx_1 \wedge dx_2$.

Let us denote $D^k(U)$ the set of all k-forms on $U$. We shall now define an operator $d$, which takes a k-form $\omega$ to a $(k+1)$-form $d\omega$. For example in vector calculus the gradient operator takes a real function $f(c)$, where $c \in \mathbb{R}$, and returns a vector. In the language of differential forms we replace the gradient with the exterior differential operator $d$, which takes a 0-form and returns a 1-form.

$$ df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz, \text{ where } f \in D^0(U) \tag{2.17} $$

Next, let us take a 1-form on $U$:

$$ \omega = Pdx + Qdy + Rdz $$

and find its exterior derivative:

$$ d\omega = d(P)dx + d(Q)dy + d(R)dz = $$

$$ \left( \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz \right)dx + \left( \frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy + \frac{\partial Q}{\partial z}dz \right)dy + \left( \frac{\partial R}{\partial x}dx + \frac{\partial R}{\partial y}dy + \frac{\partial R}{\partial z}dz \right)dz $$

We simplify the expression above using the properties $dx dx = 0$, $dy dy = 0$, $dz dz = 0$, and $dx_1 dx_2 = -dx_2 dx_1$. Finally we obtain:

$$ d\omega = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)dydz + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)dzdx + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)dx dy \tag{2.18} $$

We recognize the Eq. 2.18 as evaluation of a *curl* on a 1-vector. This, translated in the language of differential forms means that we apply the exterior derivative operator...
d on 1-form to get a 2-form. Finally, let us consider a 2-form and see what the result would be, after applying the exterior derivative operator. Let

$$\alpha = Adydz + Bdzdx + Cdxdy,$$

then

$$d\alpha = \left( \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} \right) dxdydz$$  \hspace{1cm} (2.19)

As we can see from Eq. 2.19, the result is a 3-form. From the examples we considered we can conclude that in the calculus of differential forms, all the operators of vector calculus are replaced by a single operator - the exterior derivative operator $d$. The proof of the existence and uniqueness of the operator $d$ can be found in [33]. Later on we shall see that the $d$ operator is completely independent of the coordinate system, which is one of its most important properties. Below, we give the formal definition for the general set of k-forms:

$$d : D^p(U) \rightarrow D^{p+1}(U),$$ \hspace{1cm} (2.20)

such that

- $d(\omega + \eta) = d\omega + d\eta$
- $d(\lambda \wedge \mu) = d\lambda \wedge \mu + (-1)^{\deg \lambda} \lambda \wedge d\mu$
- for each $\omega$, $d(d\omega) = 0$
- for each function $f : df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$
2.7 Change of variables (pullback)

We start with two domains: $U$ is a domain in $\mathbb{R}^m$ and $W$ is a domain in $\mathbb{R}^n$. Let $\phi$ is a smooth map: $\phi: U \rightarrow W$. We denote by $x = x^1, x^2, \ldots, x^m$ the coordinate of $U$ in $\mathbb{R}^m$ and $y = y^1, y^2, \ldots, y^n$ the coordinates of $W$ in $\mathbb{R}^n$. Then we can write

$$y^i = y^i(x^1, x^2, \ldots, x^m),$$

which shows that a point with coordinates $x$ is transformed by $\phi$ to a point with coordinate $y$. In addition the functions $y^i(x)$ are smooth. Now let us take a function $g$, which could be any real-valued function on $W$:

$$g: W \rightarrow \mathbb{R}$$

We can combine the function $g$ with $\phi$ to obtain a function (real-valued), from $U$ to $\mathbb{R}$, which is:

$$\phi^* g = g \circ \phi$$

Thus

$$\phi^*: D^0(W) \rightarrow D^0(U)$$

This means that from the map $\phi$, we constructed a new map $\phi^*$ that takes a 0-form on $W$ to a 0-form on $U$. Our goal is to construct such a map $\phi^*$ that takes a k-form on $W$ to a k-form on $U$.

$$\phi^*: D^k(W) \rightarrow D^k(U)$$

The key is to construct $\phi^*$ for 1-forms first. Let us take a 1-form $\omega$ on $W$

$$\omega = \sum a_i(y)dy^i$$
and taking into account that

\[ y^i = y^i(x) \text{ and } dy^i = \frac{\partial y^i}{\partial x^j} dx^j \]

we write

\[ \phi' \omega = \sum a_i(y(x)) \frac{\partial y^i}{\partial x^j} dx^j \]

in other words

\[ \phi' : D^1(W) \to D^1(U) \]

Next we can summarize for the general case of k-forms

\[ \phi' : D^k(W) \to D^k(U) \]

Finally, we list the basic properties of \( \phi' \)

- \( \phi'(\omega + \eta) = \phi'\omega + \phi'\eta \)
- \( \phi'(\lambda \wedge \mu) = (\phi'\lambda) \wedge (\phi'\mu) \)
- If \( \omega \) is a k-form on \( W \), \( d(\phi' \omega) = \phi'(d\omega) \)

The case, however, that interests us is a special case of the above. Here \( U \) and \( W \) are both in \( \mathbb{R}^n \), and \( \phi \) is such that it is 1-to-1 mapping of \( U \) into \( W \) with both \( \phi \)
and \( \psi = \phi^{-1} \) smooth. For a k-form \( \omega \), we have:

\[
\psi' \omega = \sum_K \left( \sum_I a_I \det(J) \right) dy_J, \quad I, K \in E_p
\]

where \( J \) is the jacobian matrix:

\[
J = \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(y_1, y_2, \ldots, y_n)}
\]

Further on, since

\[
dx_i = \sum_{i=1}^{n} \frac{\partial x_i}{\partial y_i} dy_j
\]

we can write

\[
dx_I = \sum_K \det(J) dy_K, \quad I, K \in E_p,
\]

Finally, we state one of the most important properties of the exterior derivative, namely the independance of the operator of the coordinate system.

\[
d(\phi' \omega) = \phi'(d\omega)
\]  

(2.21)

2.8 Inner product spaces

So far we did not mention anything about metric in the spaces we studied. This is because we did not need to. If we go back to the vector spaces that we are most familiar with, we recall that the metric defines an inner product for vectors. This also extends to forms. Given a metric, one can define the product of two k-forms in \( V^k_* \), which will measure in a way the projection of one on the other. Once we have defined the inner product on the space of interest, we can define an operator, denoted
by *, called Hodge (star) operator. It maps a k-form to a complementary (dual) (n-k)-form. We start with introducing an inner product space (note that the space \( \mathbb{R}^n \) endowed with inner product is the Euclidean space \( E^n \)). \( V = E^n \) has a metric if a real valued bilinear map \( g(u, v) \) is defined for every pair of vectors \((u, v)\) in \( V \). This map is called an inner product. The inner product has the following properties:

- \((u, v) = (au, v) = (u, av)\) - bilinear
- \((u, v) = (v, u)\) - symmetry
- if for fixed \( u \), \((u, v) = 0\), for all \( v \), then \( u = 0 \) - nondegenerate
- \((u, v + w) = (u, v) + (u, w)\) - distributive

It is a basic fact that every inner product space \( V \), has an orthonormal basis. The proof can be found in [33] The orthonormal basis of \( V \) consist of a basis \( e_1 \ldots, e_n \), such that:

\[
(e_i, e_j) = \pm \delta^{ij}
\]

There is another basic property that we consider below.

**Definition:** Let \( f \) be a linear function on \( V \), \( f : V \to \mathbb{R} \). Then there is a unique vector \( \lambda \) in \( V \) such that

\[
f(\mu) = (\mu, \lambda)
\]

for all \( \mu \) in \( V \)
The idea of inner product of two vectors \( \mathbf{u}, \mathbf{v} \in \mathbf{V} \) extends naturally to inner product of \( k \)-vectors \( \lambda, \mu \in \mathbf{V}^k \). Let \( \lambda = \lambda_1 \wedge \ldots \wedge \lambda_k \) and \( \mu = \mu_1 \wedge \ldots \wedge \mu_k \):

\[
(\lambda, \mu) = |(\lambda_i, \mu_j)|
\]

This definition works, because the determinant on the right side is an alternating multilinear function of \( \mu \)'s and \( \lambda \)'s. It should be noted also that \( (\lambda, \mu) = (\mu, \lambda) \), because interchanging the rows and columns of a matrix, does not change its determinant.

2.9 Hodge (star) operator

Now after we defined the inner product in our space \( \mathbf{V} \), we can go on and define the Hodge operator. Let the space \( \mathbf{V} \) have an inner product. We shall take a definite orientation of the space \( \mathbf{V} \), which remains fixed. The Hodge operator is a linear transformation on \( \mathbf{V}^k \) onto \( \mathbf{V}^{n-k} \) and it depends on the orientation of the space and the inner product. Let \( \lambda \in \mathbf{V}^k \) and \( \mu \in \mathbf{V}^{n-k} \). The mapping \( \mu \to \lambda \wedge \mu \) is a linear transformation of \( \mathbf{V}^{n-k} \) into the one dimensional space \( \mathbf{V}^n \). We may write also \( \lambda \wedge \mu = f_\lambda(\mu)e \), where \( f_\lambda(\mu) \) is a linear function on \( \mathbf{V}^{n-k} \). It could be shown that there is a unique \((n-k)\)-vector, which we denote \( \ast \lambda \) to indicate its dependance on \( \lambda \), such that

\[
\lambda \wedge \mu = (\ast \lambda, \mu)e
\]
Here \( e \equiv e_1, \ldots, e_n \) is the orthonormal basis of \( V \). This expression defines the \( * \) operator for \( V \). In order to compute \( *\lambda \) for generators of \( V^k \), let us take a \( \lambda = e_1 \wedge \ldots \wedge e_k \), then

\[
\lambda \wedge e_K = (\ast \lambda, e_K)e,
\]

(2.22)

where \( e \) is an orthonormal basis and \( K \) runs over \( n-k \) indices. The left-hand side above vanishes, unless \( K = \{k + 1, \ldots, n\} \), hence

\[
\ast \lambda = ce_{k+1} \wedge \ldots \wedge e_n, \text{ then}
\]

\[
e = \lambda \wedge e_K = c(e_K, e_K)e
\]

\[
c = (e_K, e_K) = \pm 1
\]

we finally obtain

\[
\ast \lambda = (e_K, e_K)e_K
\]

(2.23)

An example of using the Hodge operator is given in Appendix A.0.1
CHAPTER III

SMOOTH MANIFOLDS AND TANGENTIAL VECTOR SPACES

3.1 What is a manifold?

A manifold, informally speaking, is a topological space $M$, which looks locally like the Euclidean space, but may not be true for the whole. A simple example is a spherical surface $S^2$ in $\mathbb{R}^3$. It is a two dimensional space but not in $\mathbb{R}^2$. $S^2$ is not a vector space, since it cannot be drawn on one map, but it can be covered by just a few maps, and hence the surface of the Earth, for example, is a (two-dimensional) manifold. For every given point $P$ on $S^2$ there is a neighborhood of $P$ that can be seen locally as $\mathbb{R}^2$, on which the properties of a vector space hold.

The aim of this chapter is to construct the exterior algebra and the differential forms, considered in Chapter II, on manifolds. This last statement, however, is not quite precise. In Chapter II we constructed the exterior algebra and the differential forms on vector spaces (spaces of covectors), which are subspaces, in general, of $\mathbb{R}^n$. Now, since the manifolds are topological spaces, which are not vector spaces, the previously discussed exterior algebra cannot be constructed directly on them. What we are going to do, is to define a tangent vector space at each point on the manifold and then construct the differential forms over them.
3.2 Topological manifold

A topological space $M$ is a topological manifold of dimension $n$ (n-manifold) if it has the following properties [35]:

1. $M$ is a Hausdorff space: for every pair of points $P, Q \in M$, there are disjoint open subsets $U, V \subset M$, such that $P \in U$ and $Q \in V$ (given any two distinct points, there exist neighborhoods of these points that do not intersect)

2. $M$ satisfies the second countable axiom: there exist a countable basis for the topology of $M$

3. $M$ is locally Euclidean of dimension $n$: every point $P$ on $M$ has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$. The locally Euclidean property means that for each $P \in M$, we can find the following:

   - an open set $U \subset M$ containing $P$
   - an open set $\tilde{U} \subset \mathbb{R}^n$
   - a homeomorphism $\varphi : U \rightarrow \tilde{U}$, which is a continuous and bijective map

(see Fig. 3.1)

The requirements above help to ensure that the manifolds behave in a way we expect from our experience with Euclidean spaces. Particularly, Hausedorff axiom is essential to prove limits of sequences converge to at most one point. Additionally, the second countable axiom is essential to prove the existence of a partition of unity, which is indispensable for the study of the topology of manifolds. Partitions of unity are useful
Figure 3.1: A coordinate chart

because they often allow one to extend local constructions to the whole space. We will come back to the concept of the partition of unity at the end of this chapter when we study the integration of forms over a manifold $M$. One of the most important consequences of the second countable, is that every open cover in $M$ has a finite subcover, which means that the topological spaces we study are compact.

Let $M$ be a topological n-manifold. A coordinate chart on $M$ is a pair $(U, \varphi)$, where $U$ is an open subset of $M$ and $\varphi : U \to \tilde{U}$ is a homeomorphism (topological isomorphism) from $U$ to an open subset $\tilde{U} = \varphi(U) \subset \mathbb{R}^n$ (Fig. 3.1). The definition of a topological manifold implies that each point $P \in M$ is contained in the domain of some chart $(U, \varphi)$. 

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Given a chart \((U, \varphi)\), we call the set \(U\) a coordinate neighborhood of each of its points. The map \(\varphi\) is called local coordinate map, and the component functions of \(\varphi\) are called local coordinates on \(U\).

3.3 Smooth manifold

The definition of manifold that we gave above is sufficient for studying the topological properties of a manifold. If we want to find derivatives of functions or curves, the definition above, however, is not sufficient [35, 36]. So to make sense of derivatives of functions, curves and maps we should add an extra structure to the topological manifold, defined above, by introducing a new kind of manifold called smooth manifold. The study of smooth manifolds is based on the calculus of maps between Euclidean spaces. Let \(U\) and \(V\) be open sets of Euclidean spaces \(\mathbb{R}^n\) and \(\mathbb{R}^m\) respectively. Then a map \(g : U \rightarrow V\) is said to be smooth if each component functions of \(g\) has continuous partial derivatives of all orders. In addition, if \(g\) is a bijective and has a smooth inverse map, it is called a diffeomorphism.

Let \((U, \varphi)\) and \((V, \psi)\) be two charts on a \(n\)-topological manifold \(M\), such that \(U \cap V \neq \emptyset\). Then the composite map \(\psi \circ \varphi^{-1} : \varphi(U \cap V) \rightarrow \psi(U \cap V)\) is a composition of homeomorphisms and therefore it is a homeomorphism itself. This composition is called transition map (see Fig. 3.2). Two charts \((U, \varphi)\) and \((V, \psi)\) are said to be smoothly compatible if either \(U \cap V = \emptyset\) or the transition map \(\psi \circ \varphi^{-1}\) is a diffeomorphism. Since \(\varphi(U \cap V)\) and \(\psi(U \cap V)\) are open subsets on \(\mathbb{R}^n\), smoothness of this map is to be interpreted in the ordinary sense of having continuous partial
derivatives of all orders. Let us take a point $P \in U \cap V$ and suppose that $U$ has coordinate system $x = x_1, \ldots, x_n$ and $V$ has coordinate system $y = y_1, \ldots, y_n$. We express the $V$ coordinates $y$ of the point $P$ in terms of the $U$ coordinates $x$ of the same point, using the transition map [33]:

$$y_i = y_i(x_1, \ldots, x_n), \ (i = 1, \ldots, n).$$

We note, however, that we can express the $U$ coordinates $x$ of the point $P$ in terms of the $V$ coordinates $y$ as well:

$$x_j = x_j(y_1, \ldots, y_n), \ (j = 1, \ldots, n).$$

Substituting yields:

$$y_i = y_i(x_1(y), \ldots, x_n(y)), \ (i = 1, \ldots, n).$$
Next we differentiate using the chain rule:

\[
\delta_k^i = \sum \frac{\partial y_i}{\partial x_j} \frac{\partial x_j}{\partial y_k},
\]

which could be written in matrix form:

\[
\begin{vmatrix}
\frac{\partial y_i}{\partial x_j}
\end{vmatrix} \cdot \begin{vmatrix}
\frac{\partial x_j}{\partial y_k}
\end{vmatrix} = I
\]

We take the determinants by the product rule:

\[
\frac{\partial y_1, \ldots, y_n}{\partial x_1, \ldots, x_n} \cdot \frac{\partial x_1, \ldots, x_n}{\partial y_1, \ldots, y_n} = 1.
\]

It follows that the Jacobian

\[
\frac{\partial y_1, \ldots, y_n}{\partial x_1, \ldots, x_n} \neq 0
\]

is different from 0 at each point.

We now define an atlas for \( M \) to be a collection of charts whose domains cover \( M \). An atlas \( \mathcal{A} \) is called a smooth atlas if any two charts in \( \mathcal{A} \) are smoothly compatible with each other. In order to define a smooth structure on \( M \), we need a smooth atlas, and to define a function \( f : M \to \mathbb{R} \) to be smooth iff \( f \circ \varphi^{-1} \) is smooth (in the ordinary sense of functions defined on open subsets of \( \mathbb{R}^n \)) for each coordinate chart \( (U, \varphi) \) in the atlas. The problem here is that in general there will be more than one possible choice of atlas that give the same smooth structure on \( M \). This is why we define a maximal atlas. A smooth atlas \( \mathcal{A} \) on \( M \) is maximal if it is not contained in any other strictly larger smooth atlas. In other words, every chart that is smoothly compatible with every chart in \( \mathcal{A} \) is already in \( \mathcal{A} \).
Now we can define the main concept of this section. A smooth structure on a topological n-manifold $M$ is a maximal atlas [35]. A smooth manifold is a pair $(M, A)$, where $M$ is a topological manifold and $A$ is a smooth structure on $M$. From now on we will simply say manifold and that should imply a smooth structure on a topological manifold. In addition it is worth mentioning that a smooth manifold is very often denoted by $C^\infty$ and a k-differentiable manifold is denoted by $C^k$.

3.4 Orientable manifold

A manifold is called orientable or two sided if $U \cap V \neq \emptyset$ and it is possible to choose the local coordinates in $U \cap V$ such that each Jacobian is positive. To demonstrate this we are going to take an example [33]. Let us consider a 2-sphere $S^2$ with six coordinate neighborhoods. We set

$$S^2 = \{(x, y, z), \text{ where } x^2 + y^2 + z^2 = 1\} \hspace{1cm} (3.2)$$

The neighborhoods are

$$U^+_1 = \{x > 0\}, \text{ coordinate system } y, z$$

$$U^-_1 = \{x < 0\}, \text{ coordinate system } z, y$$

$$U^+_2 = \{y > 0\}, \text{ coordinate system } z, x$$

$$U^-_2 = \{y < 0\}, \text{ coordinate system } x, z$$

$$U^+_3 = \{z > 0\}, \text{ coordinate system } x, y$$

$$U^-_3 = \{z < 0\}, \text{ coordinate system } y, x$$
On the intersection of $U_1^+ \cap U_2^+$ we have the coordinate transformation

\[
\begin{align*}
\begin{cases}
  z = z \\
  y = \sqrt{1 - x^2 - z^2}
\end{cases}
\end{align*}
\right., x > 0, y > 0
\tag{3.3}
\]

Let us denote $\sqrt{1 - x^2 - z^2} = A$, so we find

\[
\frac{\partial(y, z)}{\partial(z, x)} = \begin{vmatrix}
  -x & -x \\
  \sqrt{A} & \sqrt{A} \\
  1 & 0
\end{vmatrix} = \frac{x}{\sqrt{A}} > 0
\]

On the intersection of $U_1^+ \cap U_3^-$ we have the coordinate transformation

\[
\begin{align*}
\begin{cases}
  z = -\sqrt{1 - x^2 - y^2} \\
  y = y
\end{cases}
\end{align*}
\right., x > 0, z < 0
\tag{3.4}
\]

Let us denote $-\sqrt{1 - x^2 - y^2} = B$, so we find

\[
\frac{\partial(y, z)}{\partial(y, x)} = \begin{vmatrix}
  1 & 0 \\
  \frac{y}{\sqrt{B}} & \frac{x}{\sqrt{B}}
\end{vmatrix} = \frac{x}{\sqrt{B}} > 0
\]

On the intersection of $U_2^- \cap U_3^-$ we have the coordinate transformation

\[
\begin{align*}
\begin{cases}
  z = -\sqrt{1 - x^2 - y^2} \\
  x = x
\end{cases}
\end{align*}
\right., y < 0, z < 0
\tag{3.5}
\]

Let us denote $-\sqrt{1 - x^2 - y^2} = C$, so we find

\[
\frac{\partial(x, z)}{\partial(y, x)} = \begin{vmatrix}
  0 & 1 \\
  \frac{y}{\sqrt{C}} & \frac{x}{\sqrt{C}}
\end{vmatrix} = -\frac{y}{\sqrt{C}} > 0
\]

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And so on. Thus the two sphere is a two manifold, and our choice of local coordinates proves it to be orientable. The sphere $S^2$ has two opposite orientations and similarly an orientable n-manifold has two opposite orientations. A definite one of these is determined by the order in which local coordinates $x_1, \ldots, x_n$ are given, up to even permutation of this order. Odd permutation of local coordinates gives the opposite orientation.

3.5 Tangent and Cotangent vector spaces of a smooth manifold.

In the previous sections, the exterior algebra as well as the differential forms were introduced in the space $\mathbb{R}^n$. They, however, cannot be directly applied to a smooth manifold, because a manifold is not a vector space. To eliminate this obstacle we have somehow to assign a tangent vector space to each point on $M$. Now the theory previously described can be applied on these tangent spaces.

3.5.1 Tangent vectors and tangent vector spaces

We first introduce the notion of tangent vectors and the tangent spaces of a manifold $M$, by finding the tangent vectors of a set of curves that pass through a point $P$ in $M$. So the tangent plane is determined by the family of tangent lines to curves lying on the surface of the manifold $M$, passing through a point $P \in M$. The notion of a tangent vector to a smooth curve in $\mathbb{R}^n$ is familiar from elementary calculus. It is simply the vector whose components are the derivatives of the component functions of the curve. Here we extend this notion to curves in manifolds.
Let $M$ be a smooth manifold. We define a curve in $M$ to be a continuous map $\gamma : J \to M$, where $J \subset \mathbb{R}$, is an interval. For our purposes, the term 'curve' will always refer to a map from an interval into $M$ (which usually is called parametrization). Next, let $\gamma$ be a smooth curve in $M$. We define a tangent vector to $\gamma$ at $t_0 \in J$ to be the vector

$$\gamma'(t) = \left. \frac{d\gamma(t)}{dt} \right|_{t=t_0} \in T_{\gamma(t_0)}M,$$

where

$$\left. \frac{d}{dt} \right|_{t=t_0}$$

is the standard coordinate basis for $T_{t_0}\mathbb{R}$. Below we list some other common notations for the tangent vector $\gamma'$

$$\gamma(t_0), \quad \frac{d\gamma}{dt}(t_0), \quad \gamma_*(\left. \frac{d}{dt} \right|_{t_0}),$$

where the last notation is called pushforward. The tangent vector acts on functions by:

$$\gamma'(t_0)f = \left( \frac{d\gamma(t)}{dt} \right|_{t=t_0} f = \left. \frac{d}{dt} \right|_{t=t_0} (f \circ \gamma(t)) = \left. \frac{d(f \circ \gamma)}{dt} \right|_{t_0} (t_0)$$

In other words, $\gamma'(t_0)$ is the derivative of $\gamma(t_0)$, obtained by taking the derivative of a function $f$, along $\gamma$

Next we want to write a coordinate representation of the tangent vectors at a point $P$ on $M$. Let $\gamma(t = 0) = P \in M$. Let us also take a coordinate chart $(U, \varphi)$, where $U$ is a neighborhood around $P$. A tangent vector at $P$ is the tangent vector of the curve $\gamma(t)$ at $t = 0$, given by the expression above. A differentiable function $f$
under the coordinate \( \varphi \) is given by

\[
f \circ \varphi = f(x_1, x_2, \ldots, x_n)
\]  (3.7)

and the curve \( \gamma(t) \) is written as

\[
\varphi^{-1} \circ \gamma(t) = (x_1(t), x_2(t), \ldots, x_n(t))
\]  (3.8)

Now we are in a position to express the tangent vector at P in terms of local coordinates:

\[
\gamma'(0) f = \left. \frac{d(f \circ \gamma(t))}{dt} \right|_{t=0} = \left. \frac{d(f(x_1(t), \ldots, x_n(t)))}{dt} \right|_{t=0} = \sum_{i=1}^{n} \left. \frac{\partial f}{\partial x_i} \frac{dx_i}{dt} \right|_{t=0} = \left( \sum_{i=1}^{n} x'_i(0) \frac{\partial}{\partial x_i} \right) f
\]  (3.9)

Hence the tangent vector of the curve \( \gamma(t) \) at P in coordinate representation is:

\[
\gamma'(0) = \sum_{i=1}^{n} x'_i(0) \frac{\partial}{\partial x_i}
\]  (3.10)

The set of tangent vectors of all curves that pass through P, makes up an n-dimensional vector space called the tangent space of the manifold \( M \) at point P, denoted \( T_P M \), and the \( \frac{\partial}{\partial x_i} \bigg|_P \), \( i = 1, 2, \ldots, n \), form a coordinate basis of \( T_P M \) The tangent vector at P can be written then as a linear combination of the coordinate bases:

\[
X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i},
\]  (3.11)

where the i-th component of the tangent vector \( X_i \) is:

\[
X_i = \frac{dx_i}{dt}
\]  (3.12)

The tangent space \( T_P M \) is a linear space. If we denote by \( \mathcal{F}^p \) the space of all real valued functions at a point P on \( M \), then a tangent vector X at P is a mapping:

\[
X : \mathcal{F}^p \to \mathbb{R}
\]
Satisfying the following linear properties:
\[ \forall f, g \in F^p, \alpha, \beta \in \mathbb{R} \]

1. \[ X(\alpha f + \beta g) = \alpha X(f) + \beta X(g) \]

2. \[ X(f.g) = g(P).X(f) + f(P).X(g) \]

Thus the tangent vector \( X \) assigns to each smooth function \( f \) on \( M \) a real number \( X(f) \), which is the directional derivative of the function \( f \) along the tangent vector \( X \).

### 3.5.2 Cotangent vectors and cotangent vector spaces

Let \( T_PM \) be the tangent space of \( M \) at \( P \). The dual space of \( T_PM \), denoted by \( T_{P*}M \) is called the cotangent space of \( M \) at \( P \). It is the set of differentiable functions that map the tangent vectors to scalars. The elements of \( T_{P*}M \) are called tangent covectors.

In the previous section we showed that a smooth chart \( (U, \phi) \) at \( P \) determines coordinates \( (x_i) \) in \( U \), and induces the coordinate basis \( (\partial/\partial x_i)|_P \) for \( T_PM \). In this section we show that \( dx_i \) form a basis for \( T_{P*}M \), which is the dual basis of \( (\partial/\partial x_i)|_P \) for \( T_PM \). Let \( f \) be a smooth real-valued function on a smooth manifold \( M \). A tangent covector in \( T_{P*}M \) is a linear functional on \( T_PM \) associated with \( f \). We define a covector field \( df_P : T_PM \rightarrow \mathbb{R} \), called the differential of \( f \) at \( P \), by

\[ df_P(X) = df_P(\gamma'(0)) = (f \circ \gamma)'(0) = X(f), \quad (3.13) \]

where \( X \) is

\[ X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}, \quad X \in T_PM, \text{ and } dx_i(X) = X(x_i) \quad (3.14) \]
Now taking into account Eq. 3.13 and Eq. 3.14 we can write:

\[ df_p(X) = X(f) = \sum_{i=1}^{n} X(x_i) \frac{\partial f}{\partial x_i} \bigg|_P = \sum_{i=1}^{n} \left. \frac{\partial f}{\partial x_i} \right|_P dx_i(X) \quad (3.15) \]

Hence

\[ df = \frac{\partial f}{\partial x_i}(P)dx_i \quad (3.16) \]

Thus in the coordinate chart \((U, \varphi)\), the \(i\)th dual basis vector is the differential \(dx_j\) of the \(j\)th component of \(\varphi\). So the basis for \(T_P \ast M\) is given by \(dx_i, i = 1, \ldots, n\), which is dual to the basis \(\left. \frac{\partial}{\partial x_i} \right|_P\) for \(T_PM\) in a given coordinate chart. They satisfy the relation:

\[ dx_i \left( \left. \frac{\partial}{\partial x_j} \right|_P \right) = \delta^i_j \quad (3.17) \]

A tangent co-vector, is expressed by:

\[ \xi = \sum_{i=1}^{n} \xi_i dx_i \quad (3.18) \]

and its \(i\)th component is:

\[ \xi_i = \frac{\partial f}{\partial x_i} \quad (3.19) \]

The result of applying a tangent covector \(\xi = \sum_{i=1}^{n} \xi_i dx_i\) to a tangent vector 
\(X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}\) is given by \(\xi(X) = \xi_i X_i = \left. \frac{\partial f}{\partial x_i} \right|_P dx_i\).

In summary the cotangent space \(T_P \ast M\) at a point \(P\) is given by all linear maps from the tangent vector space \(T_PM\) to the space of real numbers \(\mathbb{R}\). Therefore, at \(P\), the cotangent space consists of 1-forms, that linearly map vectors from the tangent space to the space of the real numbers.
3.6 Change of coordinates

From the previous section, under the local coordinate chart \((U_i, \varphi_i)\) of a manifold \(M\), the tangent and cotangent vectors are given by Eq. 3.11 and Eq. 3.18, respectively, and the \(i\)th components of the tangent and cotangent vectors are given by Eq. 3.12 and Eq. 3.19. If \((u_j, \varphi_j)\) is another coordinate chart \((u_i \cap u_j \neq \emptyset)\), the components of \(X\) and \(\xi\) are respectively \(X'_i\) and \(\xi'_i\). They are related to the coordinate chart \((u_i, \varphi_i)\) by:

\[
X'_j = \sum_{i=1}^{n} X_i \frac{\partial y_j}{\partial x_i} \quad (3.20)
\]

and

\[
\xi_i = \sum_{i=1}^{n} \xi'_j \frac{\partial y_j}{\partial x_i} \quad (3.21)
\]

where

\[
\frac{\partial y_j}{\partial x_i} = \frac{\partial (\varphi_j \circ \varphi_i^{-1})}{\partial x_i} \quad (3.22)
\]

is the Jacobian of this transformation.

3.7 Tangent and cotangent bundles

In Sections 3.5 we have introduced the concepts of tangent vectors and tangent covectors at a point \(P\). The process of defining tangent vectors and covectors could be viewed as assigning a linear property to a point on a smooth manifold \(M\), which in general, is not a linear space (it cannot be described by a unique coordinate system). Repeating the process to all points on \(M\) will assign to each point on \(M\) a linear
vector tangent space. Gluing them all together will form a tangent bundle $TM$ or cotangent bundle $T^*M$ on $M$, respectively. We write as follows:

$$TM = \bigsqcup_{P \in M} T_P M$$  \hspace{1cm} (3.23)$$

and

$$T^*M = \bigsqcup_{P \in M} T^*_P M,$$  \hspace{1cm} (3.24)$$

where $\bigsqcup$ is the disjoint union. Eq. 3.23 says that the tangent bundle $TM$ is the union of the tangent spaces over all points in $M$ and its elements are 1-vectors over the manifold $M$. Similarly, Eq. 3.24 means that the cotangent bundle $T^*M$ is the union of the cotangent spaces over all points in $M$ and its elements are 1-forms over the manifold $M$. After having assigned the 1-vector/1-covector space to the manifold, we can generalize the above definitions to a p-vector space and a p-covector space on a manifold. Namely, a tangent p-vector space is obtained by tensor products of $T_P M$, p times:

$$\bigwedge^p(T_P M) = T_P M \times \cdots \times T_P M$$

and a tangent p-covector space is obtained by skew-symmetric tensor products of $T^*_P M$, p times

$$\bigwedge^p(T^*_P M) = T^*_P M \otimes \cdots \otimes T^*_P M$$

Finally, the exterior p-tangent and p-cotangent bundles are

$$\bigwedge^p(TM) = \bigsqcup_{P \in M} \bigwedge^p(T_P M)$$

$$\bigwedge^p(T^*M) = \bigsqcup_{P \in M} \bigwedge^p(T^*_P M)$$
3.8 Differential forms on smooth manifolds

Having assigned a linear property to a smooth manifold, we finally can define the
differential forms on \( M \) and this will allow us to apply the exterior algebra and
exterior calculus, already introduced in Chapter II. The smooth real valued functions
on \( M \) are also called 0-forms and their space is denoted by \( F^0(M) \). A 1-form at
a point \( P \) is defined as \( \sum a_i dx^i \), where \( a_i \) is a constant, for each local coordinate
system \( (x^i) \) valid in a neighborhood \( U \) on \( M \), such that \( P \in U \). Let us take another
neighborhood \( V \) with coordinate system \( y^j \). Let also \( U \cap V \neq \emptyset \) and \( P \in U \cap V \).
Then a 1-form at \( P \) in \( V \) is \( \sum b_i dy^i \). The transformation from coordinate system of
\( V \) to the one of \( U \) is given by \( \sum a_j dx^j \), where \( a_j = \sum b_i \partial y^i / \partial x^j \).

Having this we can use sums of exterior products of 1-forms, in general, to
construct a p-form at \( P \). Finally, we can define a p-form on \( M \), which is the totality
of the p-forms, assigned smoothly at each point \( P \) on \( M \). If \( U \) is given in local
coordinates \( (x^i) \), then a differential k-form is written as follows:

\[
\omega = \sum a_I(x) dx^I, \quad (3.25)
\]

where \( a_I(x) \) are a smooth functions on \( U \) and \( I = \{i_1, \ldots, i_k\} \). Let \( \omega = \sum b_K(y) dy^K \),
is a differential k-form with respect to a second coordinate system, which overlaps
with the first one. The coordinate change from \( (y^i) \) system to \( (x^i) \), can be performed
in the following way: We substitute \( y^i \) with \( y^i = y^i(x) \) and \( dy^i \) with \( \sum \partial y^i / \partial x^j dx^j \) Before
proceeding to the next section, where we integrate these newly defined differential
forms on $\mathbf{M}$, it is worth mentioning that all the rules that we considered in Chapter II are readily verified here, as well.

3.9 Integration of differential forms on smooth manifolds

The most intuitive and informal definition of differential forms is: “Differential forms are the things that appear under the integral sign” [33]. More precisely, they are multidimensional integrands on oriented domains. Thus, it looks natural as a next step to look at performing the integration of differential forms. This will bring us a step closer to the applications of the differential forms in Electromagnetic Theory and the Finite Element Method (FEM), since in the formulation of the problem we often integrate field quantities over manifolds. The former are defined as differential forms of various degrees. We start by evaluating the integral of differential forms on an oriented k-dimensional manifold $\mathbf{M}$, which is covered by a chart $(U, \varphi)$ with positive orientation. The mapping $\varphi(U) = (x_1, \ldots, x_k)$ defines a coordinate system on $\mathbf{M}$. Thus, as we saw above, a differential k-form on a smooth manifold $\mathbf{M}$, can be written as $\omega = f(\mathbf{x})dx_1dx_2\ldots dx_p$, where $f(\mathbf{x})$ is a smooth function and we assume that the integral of $f(\mathbf{x})$ on $\varphi(U)$ exists. The integral of a differential form $\omega$ over $\mathbf{M}$, is given by:

$$\int_{\mathbf{M}} \omega = \int_{\varphi(U)} f(\mathbf{x})dx_1dx_2\ldots dx_p \quad (3.26)$$

The right hand side is the usual multiple integral in $\mathbb{R}^n$. A short reminder here that $dx_1dx_2\ldots dx_p$ actually is $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_p$. From the above definition,
it becomes more obvious that the differential form could be seen as integrands of multiple integrals. For example, 1-forms are integrands for line integrals, 2-forms are integrands for surface integrals, and so on. It is very important to show that the definition of integral of differential forms on a manifold \( M \), is independent of the choice of coordinate system. Let us take a second chart \((V, \psi)\) on \( M \) with coordinates \( \psi = (y_1, \ldots, y_p) \). A differential form under this coordinate system can be written as \( \omega = g(y)dy_1 \ldots dy_p \). Now following the definition from Eq. 3.26 we write:

\[
\int_M \omega = \int_{\psi(V)} g(y)dy_1dy_2\ldots dy_p
\] (3.27)

From the previous section, we recall that the expressions of a differential form under two coordinate systems are related by the Jacobian, namely

\[
g(y) = f(x^i(y))det(J),
\]

where the Jacobian is given by

\[
J = \frac{\partial(x^1, \ldots, x^p)}{\partial(y^1, \ldots, y^p)}.
\]

Hence, we have:

\[
\int_M \omega = \int_{\phi(U)} \omega = \int_{\phi(U)} f(x)dx_1dx_2\ldots dx_p = \\
= \int_{\psi(V)} f(x^i(y))det(J)dy_1dy_2\ldots dy_p = \\
= \int_{\psi(V)} g(y)dy_1dy_2\ldots dy_p = \int_{\psi(V)} \omega = \int_M \omega
\]

Here we should note that the sign of the integral of \( \omega \) on two different coordinate charts does not change under the change of variables, if the orientations are the same on
\( \psi(V) \) and \( \varphi(U) \) (positive determinant of the Jacobian). Oterwise it takes the oposite sign (negative determinant of J). Now we are going to consider the general case of an n-dimensional oriented manifold \( M \). Let \( (U_i, \varphi_i), i=1, \ldots, m \), be the coordinate charts on \( M \). In order to define the integral of differential forms on \( M \), we introduce a series of differentiable functions \( \lambda_1, \ldots, \lambda_m \) such that \( \sum_{i=1}^{m} \lambda_i = 1, \ 0 \leq \lambda_i \leq 1 \).

In addition the support of \( \lambda_i \) is contained in the covering of \( U_i \), \( \text{supp} \lambda_i \in U_i \). \( \lambda_i \) is called the partition of unity (more information in the context of the Finite Element Method can be found in [37]) that is subordinate to the covering \( U_i \). The integral of the differential form \( \omega \) on \( M \) now is defined by:

\[
\int_M \omega = \sum_{i=1}^{m} \int_{U_i} (\lambda_i \omega)
\]

(3.28)

It is easy to show that this definition makes sense because

\[
\int_M \omega = \int_M (\sum_{i=1}^{m} \lambda_i) \omega = \sum_{i=1}^{m} \int_M (\lambda_i \omega) = \sum_{i=1}^{m} \int_{U_i} (\lambda_i \omega)
\]

Next, we have to show that the above definition is independent of the partition. To do so we consider another coordinate chart \( (V_j, \psi_j), j=1, \ldots, m \) of \( M \). Let us suppose that \( V_j \) and \( U_i \) have the same orientation. Let also \( \gamma_1, \ldots, \gamma_m \) be another series of smooth functions, such that \( \sum_{j=1}^{m} \gamma_j = 1 \) and \( 0 \leq \gamma_j \leq 1 \), then \( \text{supp} \gamma_j \in V_j \) is another partition of unit. We write

\[
\int_M \omega = \sum_{j=1}^{m} \int_{V_j} (\gamma_j \omega)
\]

(3.29)
Since the support of $\lambda_i$ is in $U_i$ and the support of $\gamma_j$ is in $V_j$, then the support of $\lambda_i\gamma_j$ is in $U_i \cap V_j$. Thus

$$\int_M \omega = \sum_{i=1}^m \int_{U_i} (\lambda_i \omega) = \sum_{i=1}^m \int_{U_i} (\lambda_i (\sum_{j=1}^m \gamma_j) \omega) = \sum_{i=1}^m \sum_{j=1}^m \int_{U_i \cap V_j} (\lambda_i \gamma_j \omega)$$

similarly,

$$\int_M \omega = \sum_{j=1}^m \int_{V_j} (\gamma_j \omega) = \sum_{j=1}^m \int_{V_j} (\gamma_j (\sum_{i=1}^m \lambda_i) \omega) = \sum_{i=1}^m \sum_{j=1}^m \int_{U_i \cap V_j} (\lambda_i \gamma_j \omega)$$

This clearly shows that different partitions, give the same result. Finally, we want to mention the linear property of the integration of differential forms. For any differential form $\omega_1, \omega_2$ on $M$ and $a_1, a_2 \in \mathbb{R}$, we write the linear property as follows:

$$\int (a_1 \omega_1 + a_2 \omega_2) = a_1 \int \omega_1 + a_2 \int \omega_2 \quad (3.30)$$

3.9.1 The generalized Stokes’ theorem

Let $M$ be an oriented manifold with boundary of dimension $p$, and $\omega$ be a differential $(p-1)$-form on $M$. The generalized Stokes’ theorem states:

$$\int_M d\omega = \int_{\partial M} \omega \quad (3.31)$$

It follows immediately that if $M$ has no boundary ($\partial M = 0$), the right hand side in this case is zero. So the Stokes’ theorem gives a more visual idea about the duality between the differential operator $d$ and the boundary operator $\partial$. Now we go back to the well known vector calculus, in the three dimensional space. We recall some of the most fundamental theorems of the vector calculus, which will help us to realize that they are nothing else but particular cases of the generalized Stokes’ theorem.

$$\int_L \text{grad}(\mathbf{x}) dL = \int_{\partial L} \mathbf{x} \quad (3.32)$$
\[
\int_S \text{curl}(\mathbf{x})dS = \int_{\partial S} (\mathbf{x})dL \\
\int_V \text{div}(\mathbf{x})dV = \int_{\partial V} (\mathbf{x})dS
\] (3.33)

For example let us take Eq. 3.32 and relate it to a familiar relation from Electromagnetic Theory, namely:

\[
\int_L e = \int_L dv = \int_{\partial L} v,
\]

where \(e\) is the 1-form electric field intensity and \(v\) is the 0-form electric scalar potential. This could be visualized as in Fig. 3.3a (the planes represent equipotential surfaces and the potential changes its value as we move along \(l\)).
CHAPTER IV

MATHEMATICAL STRUCTURE OF MAXWELL’S EQUATIONS

4.1 Introduction

So far, in Chapter II and Chapter III, we studied the differential forms over n-dimensional vector spaces \( V^n \), and over n-dimensional manifolds \( M \) respectively. We looked at them from more general mathematical points of view, as a p-form in n-space. In this chapter we consider one of their particular applications, namely studying the Maxwell’s equations and EM theory. Since in electromagnetics we have only scalar functions, circulating fields and flux fields, we limit ourselves to differential forms up to order 3 (\( 0 \leq p \leq 3 \)). In most textbooks on the subject of Electromagnetic Theory, Maxwell’s equation are written in terms of vectors and vector fields. Vector calculus and Maxwell’s equations in vector form have served engineers for many years now and they probably will continue to be an important part in the analysis of problems related to the EM phenomena. Why do we study differential forms then? The answer is partially contained into a couple of problems that are related to the nature of vectors and the way they are defined [38, 36]. In that context let us first consider the two magnetic vectors \( \mathbf{H} \) and \( \mathbf{B} \). The magnetic field intensity \( \mathbf{H} \) is associated with integration along a line, while \( \mathbf{B} \) is associated with integration over a surface. In
addition the use of these vectors requires a particular orientation of the coordinate
system. To avoid ambiguity usually we pick right handed coordinate system. The
problem is if we decide arbitrarily to change the coordinate system from, say, right
handed to a left handed one [21]. Under such coordinate transformation the vectors
\( \mathbf{H} \) and \( \mathbf{B} \) behave in different ways. Schematicly this is shown on Fig. 4.1. Such
difference in the behaviour of \( \mathbf{H} \) and \( \mathbf{B} \) would not matter if they were independent.
They, however, as we well know, are related through the constitutive relation:

\[
\mathbf{B} = \mu \mathbf{H}
\]

So \( \mathbf{H} \) and \( \mathbf{B} \) cannot both be unique physical quatities, independent of the choice

![Figure 4.1: Change in the behaviour of H and B under coordinate transformation](image)

of the coordinate system. The reasonable explanation is that although regarded as
vectors they both have different physical meaning. This is why some authors try
to distinguish between them by calling \( \mathbf{H} \) a polar vector and \( \mathbf{B} \) an axial vector. The
second problem is that the vectors of both kinds are defined as entities at a point,
which involves a limiting process. For instance, the vector $\mathbf{H}$ [A/m] is defined as a limit when the length shrinks to a point. An axial vector like $\mathbf{B}$ is defined as a limit when the area shrinks to a point. The problem here is that the limits are not directly observable, nor computable. Moreover, there is no such thing as a vector at a point. However, quanteties like $\mathbf{H} \cdot dl$ and $\mathbf{B} \cdot ds$ are observable and are given by a number. These considerations lead us to a very important conclusion; electromagnetic phenomena are not independent of space. Physics and geometry are not independent [21]. Maxwell’s fundamental discovery was, as we have seen in every textbook on EM theory that electric and magnetic energy are distributed in space and are dependent on time. In their text, Baldomir and Hammond [21], explain that Faraday and Maxwell make no sharp distinction between energy in free space and energy in space occupied by matter. Furthermore, both Maxwell and Faraday, view energy as linked to space, rather being “inserted” into the space. The former was the case with the old theory, where the attention was on the action between material bodies contained in the space. Since Maxwell’s equations are mathematical description of a physical phenomenon - Electromagnetism - in this chapter we have two main objectives. First to reveal the mathematical structure of Maxwell’s equations, in an attempt to better understand the physical phenomena behind them. Second, to show the relationship between electromagnetism and the geometry of the space and time. Hence we will need a mathematical tool that will allow us to relate energy with space and time. This implies that a geometrical approach to Maxwell’s equations should be taken. The differential forms that we studied in Chapter II and Chapter III, seem to be the
mathematical apparatus required to describe the electromagnetic fields. Moreover it
greatly simplifies the calculations and exposes certain symmetries in the equations
that are not immediately obvious when using vectors. Finally, the use of differential
forms avoids the ambiguities and contradictions that may arise in vector calculus and
vector algebra.

4.2 Maxwell’s equations in vector form

The classical vector representation of Maxwell’s equations is as follows [39]:

Faraday’s system:
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \] (4.1)
\[ \nabla \cdot \mathbf{B} = 0 \] (4.2)

Ampere’s system
\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \] (4.3)
\[ \nabla \cdot \mathbf{D} = \rho \] (4.4)

The reason for this separation in Faraday’s and Ampere’s systems will become ap-
parent later in this chapter. The two systems are connected by constitutive relations,
which will be discussed in the next sections.

Now we define the vector and scalar potentials, which are based on the irrota-
tional and solenoidal properties of the fields. If we take a solenoidal field we can define
a vector potential, while if the field is irrotational we can define a scalar potential.
Scalar and vector potentials can be used even if the fields are neither irrotational,
nor solenoidal. To do so we use the Helmholtz theorem to decompose the field into a sum of irrotational term and solenoidal term.

\[ \mathbf{G} = -\nabla U - \nabla \times \mathbf{C} \]  

(4.5)

Starting with Eq. 4.2 \( \nabla \cdot \mathbf{B} = 0 \), we can write \( \nabla \times \mathbf{A} = \mathbf{B} \), where \( \mathbf{A} \) is the magnetic vector potential. We can apply this to Eq. 4.1

\[ \nabla \times \mathbf{E} = -\frac{\partial (\nabla \times \mathbf{A})}{\partial t} = -\nabla \times \frac{\partial \mathbf{A}}{\partial t} \]

Then

\[ -\nabla \times \mathbf{E} - \nabla \times \frac{\partial \mathbf{A}}{\partial t} = 0 \]

Rearranging the above expression gives

\[ \nabla \times (\mathbf{E} - \frac{\partial \mathbf{A}}{\partial t}) = 0 \Rightarrow \mathbf{E} - \frac{\partial \mathbf{A}}{\partial t} = \nabla U \]

So, we finally get

\[ \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \]  

(4.6)

which satisfies the Helmholtz theorem.

Next we define the scalar potentials. Under the condition that the fields are irrotational, we can define electric and magnetic scalar potentials

if \( \nabla \times \mathbf{E} = 0 \) \( \Rightarrow \) \( \mathbf{E} = -\nabla V \)

if \( \nabla \times \mathbf{H} = 0 \) \( \Rightarrow \) \( \mathbf{H} = -\nabla \phi \)

Finally we just mention that under different conditions other potentials could be derived, such as the electric vector potential if \( \nabla \cdot \mathbf{D} = 0 \).
4.3 Interface conditions

The Maxwell equations from the previous section can be rewritten in integral form as follows [39, 38]:

\[ \oint_l E \cdot dl = \frac{\partial}{\partial t} \int_s B \cdot ds \]  
(4.7)

\[ \oint_B B \cdot ds = 0 \]  
(4.8)

\[ \oint_H H \cdot dl = \int_s J \cdot ds + \frac{\partial}{\partial t} \int_s D \cdot ds \]  
(4.9)

\[ \oint_D D \cdot ds = \int_v \rho \cdot dv \]  
(4.10)

In fact the differential form of Maxwell’s equations from the previous section is derived from the above integral equations, using Stokes’ theorem and the divergence theorem. Although Maxwell’s differential equations are used in computation of electromagnetic fields, the above integral equations give a more visual idea, of what the fields look like and they carry somewhat more physical meaning. To derive the interface conditions we use the above integral equations and to avoid repetitions we denote the fluxes \( B \) and \( D \) by \( F \), and the circulations \( H \) and \( E \) by \( C \). It is well known that the behaviour of the vector fields at the interface between two materials is important. We are interested in the exact behavior of the vector fields across the interface. Let us start with an arbitrary flux vector \( F \), (see Fig. 4.3,a) that passes though the cylindrical region. We write:
\[
\int_v (\nabla \cdot \mathbf{F}) dv = \oint_s \mathbf{F} \cdot d\mathbf{s} = (F_2 - F_1) \Delta S + S
\]

where \(S\) is the surface integral over the cylindrical wall. If we let the height of the cylinder \(\Delta h\) to go to zero, the surface integral \(S\) will vanish and we get:

\[
\lim_{\Delta h \to 0} (\nabla \cdot \mathbf{F}) \Delta S \Delta h = \hat{n} \cdot (F_2 - F_1) \Delta S
\]

where \(\hat{n}\) is the normal unit vector with positive direction from region 1 to region 2. Thus \(\Delta S\) cancels on both sides:

\[
\lim_{\Delta h \to 0} (\nabla \cdot \mathbf{F}) \Delta h = \hat{n} \cdot (F_2 - F_1)
\]

(4.11)

The left hand side goes to zero, provided that the divergence \(\nabla \cdot \mathbf{F}\) is well defined and consequently we will have normal continuity, allowing the tangential component...

Figure 4.2: The interface between two materials
to be discontinuous. We follow the same process for the surface integral:

\[ \int_s (\nabla \times \mathbf{C}) \cdot d\mathbf{s} = \oint_l \mathbf{C} \cdot d\mathbf{l} = (C_2 - C_1) \Delta l \]

We let again \( \Delta h \to 0 \) and we obtain:

\[ \lim_{\Delta h \to 0} (\nabla \times \mathbf{C}) \Delta h = \hat{n} \times (C_2 - C_1) \quad (4.12) \]

The left hand side of the equation above will vanish, provided that the curl of the field \( \mathbf{C} \) is well defined. If this is the case, then we will have a tangential continuity, allowing discontinuous normal components.

4.4 Maxwell’s equations in the language of differential forms

In Chapter II and Chapter III of this work we considered the differential forms in a more general way, namely the differential forms were p-dimensional in n-dimensional vector space or n-manifold. For applications in electromagnetics, however, we do not need such generality. This is why we will focus our attention on differential forms of order up to three, in \( \mathbb{R}^3 \) (\( 0 \leq p \leq 3 \)). As we mentioned in Section 4.1, the vectors \( \mathbf{E} \) and \( \mathbf{H} \), and \( \mathbf{D} \) and \( \mathbf{B} \) are not the same. In the language of differential forms we distinguish between them explicitly as follows: the scalar functions are 0-forms, the \( \mathbf{E} \) and \( \mathbf{H} \) are 1-forms, the \( \mathbf{D} \) and \( \mathbf{B} \) are 2-forms and finally the charge distribution \( \rho \) is a 3-form. In addition the various potential functions will be rewritten by differential forms as well. In what follows we introduce all the physical and mathematical quantities involved, by differential forms, and work out in details the process of deriving them.
In addition give their vector form counterparts for the sake of completeness with the risk of some repetitions.

We start with the static electric field intensity $E$, which can be derived by using the electric scalar potential $V$. Its vector form is:

$$E = -\nabla V = -\hat{x} \frac{\partial V}{\partial x} - \hat{y} \frac{\partial V}{\partial y} - \hat{z} \frac{\partial V}{\partial z} = \hat{x}E_x + \hat{y}E_y + \hat{z}E_z \quad (4.13)$$

The electric field intensity in terms of differential forms is derived with the help of the exterior differential operator, applied to the 0-form electric scalar potential:

$$e = dv = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz = E_x dx + E_y dy + E_z dz \quad (4.14)$$

There are a few points that need to be discussed here.

- First, $V$ and $v$ are essentially the same physical quantities but mathematically expressed in a different way. To distinguish between them, we use lower case letters to denote differential forms.

- We recall that the exterior differential operator $d$ replaces all the vector operators grad, curl and div. Essentially it increases the order of the forms. In the case above it replaces the gradient by taking a 0-form and returning a 1-form

  $$d : v \rightarrow e$$

- Finally, we notice that the vector basis set $(\hat{x}, \hat{y}, \hat{z})$ in $\mathbb{R}^3$ is replaced by a new basis set $(dx, dy, dz)$, which is the basis of the space $V^1 \ast (\mathbb{R}^3)$ or simply $V^1 \ast$
Further on, to better understand the meaning of Eq. 4.14, we take the integrand of the integral equation for the electric field intensity in $\mathbb{R}^3$:

$$\oint_l \mathbf{E} \cdot d\mathbf{l} = (\hat{x}E_x + \hat{y}E_y + \hat{z}E_z) \cdot (\hat{x}dx + (\hat{y}dy + (\hat{z}dz) = E_xdx + E_ydy + E_zdz = e$$

It becomes apparent that the quantity $e$ is exactly what appears under the integral sign. This means that a 1-form is a 1-dimensional integrand over a line. Identical steps are taken to get the differential form expression for the 1-form magnetic vector potential, magnetic field intensity, and the electric vector potential, so we just list them below:

$$a = A_xdx + A_ydy + A_zdz$$

(4.15)

$$h = H_xdx + H_ydy + H_zdz$$

(4.16)

$$t = T_xdx + T_ydy + T_zdz$$

(4.17)

Next we define the two forms $b$, $d$ and $j$, starting with the vector equation $\mathbf{B} = \nabla \times \mathbf{A}$ we write the vector form of $\mathbf{B}$:

$$\mathbf{B} = \nabla \times \mathbf{A} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

So

$$\mathbf{B} = \nabla \times \mathbf{A} = \hat{x}B_x + \hat{y}B_y + \hat{z}B_z$$

(4.18)

The 2-differential form is derived by taking the exterior derivative of the 1-form $a$:

$$da = d(A_xdx + A_ydy + A_zdz)$$
\[ da = \left( \frac{\partial A_x}{\partial x} dx + \frac{\partial A_x}{\partial y} dy + \frac{\partial A_x}{\partial z} dz \right) \wedge dx + \left( \frac{\partial A_y}{\partial x} dx + \frac{\partial A_y}{\partial y} dy + \frac{\partial A_y}{\partial z} dz \right) \wedge dy + \left( \frac{\partial A_z}{\partial x} dx + \frac{\partial A_z}{\partial y} dy + \frac{\partial A_z}{\partial z} dz \right) \wedge dz \] (4.19)

We can use the properties of the exterior differential operator to simplify the above equation, namely, \( dx_i \wedge dx_i = 0 \) and \( dx_i \wedge dx_j = -dx_j \wedge dx_i \) (\( i \neq j \)) We finally get:

\[ da = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) dy \wedge dz + \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) dz \wedge dx + \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dx \wedge dy \] (4.20)

\[ b = da = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy \] (4.21)

Exactly the same process establishes the 2-forms \( d \) and \( j \) and we list the final result below.

\[ j = J_x dx \wedge dz + J_y dz \wedge dx + J_z dx \wedge dy \] (4.22)

\[ d = D_x dy \wedge dz + D_y dz \wedge dx + D_z dx \wedge dy \] (4.23)

At last we establish the 3-form obtained by applying the exterior derivative on a 2-form. First in vector form the charge density \( \rho \) is:

\[ \nabla \cdot D = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho \] (4.24)

To derive the 3-form, we write:

\[ d(d) = d(D_x dy \wedge dz + D_y dz \wedge dx + D_z dx \wedge dy) \Rightarrow \]

\[ \Rightarrow d(d) = d(D_x dy \wedge dz) + d(D_y dz \wedge dx) + d(D_z dx \wedge dy) \Rightarrow \]
We finally get:

\[ d(d) = \left( \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) dx \wedge dy \wedge dz = \rho dx \wedge dy \wedge dz \]  

(4.25)

The time dependent Maxwell’s equations are shown below, to give an idea what they look like in the language of differential forms.

Faraday’s system

\[ db = 0 \ [\text{Wb}] \]

\[ de = -\partial_t b \ [\text{V}] \]

And

Ampere’s system

\[ d(d) = \rho \ [\text{C}] \]

\[ dh = j - \partial_t d \ [\text{A}] \]

From the equations above, and in particular their units, it becomes clear that Maxwell’s equations in differential forms have pure physical meaning. The units are summarized in the following two tables:

Finally we write the differential form expressions for the potential functions:

\[ da = b \]  

(4.26)

\[ e = -dv - \partial_t a \]  

(4.27)
Table 4.1: Summary of the units for the Faraday’s system

<table>
<thead>
<tr>
<th>Diff. Form</th>
<th>Faraday’s system</th>
<th>Faraday’s system</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-forms</td>
<td>v = V</td>
<td>—</td>
</tr>
<tr>
<td>1-forms</td>
<td>e = E \cdot dl</td>
<td>a = A \cdot dl</td>
</tr>
<tr>
<td>2-forms</td>
<td>—</td>
<td>b = B \cdot ds</td>
</tr>
<tr>
<td>3-forms</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>units</td>
<td>V</td>
<td>Wb</td>
</tr>
</tbody>
</table>

\[ dt = d \] \quad (4.28)

\[ h = -d\phi + \partial_t dt \] \quad (4.29)

In the next few sections we study in details these equations using diagrams called, Tonti’s diagrams. The value of these diagrams is in that a very simple structure is revealed that can be used for practical purposes. Special attention is paid to the constitutive equations since, as we shall see, they carry simultaneously geometrical and physical information of the system. In addition they serve as a bridge between the Faraday’s system and Ampere’s system, which will be clearly seen in the Tonti diagrams.
4.4.1 Electrostatics

We start the discussion of Maxwell’s equation with the electrostatic case. The Tonti diagram for this case is shown on Fig. 4.3. On the diagram it is easy to see the symmetrical structure. We distinguish the Faraday’s system part (left side), Ampere’s system part (right side) and the constitutive relation that connects them. It should be noted that Tonti’s diagrams are very easy to read. For now we need to know that the vertical arrows show the transition to higher order forms as a result of the application of the exterior differential operator. In addition the constitutive relation, represented by the inclined arrow, is a little different from what we are familiar with $D = \epsilon E$. It was mentioned earlier that the vectors $D$ and $E$ are not the same. They represent different physical quantities, although we call them both vectors. By using differential forms, this problem is naturally avoided with the help of the Hodge...
Figure 4.3: Tonti’s Electrostatic diagram
operator. We recall from Section 2.9 that if \( \eta \) is a \( p \)-form and \( \xi \) is an \( (n-p) \)-form we can write \((\eta \wedge \xi) = (\ast \eta, \xi)dx^n\), where \( n \) runs from 1 to \( N \) and \( \ast \eta \) is an \( (n-p) \)-form. For applications in electromagnetic theory we have \( 0 \leq p \leq 3 \), \( N=3 \), and \( n \) runs from 1 to 3. It becomes clear that the spaces \( V^0 \ast \) and \( V^3 \ast \), as well as \( V^1 \ast \) and \( V^2 \ast \) have the same dimensions and are dual to each other, so do their elements - the corresponding forms.

Before continuing, we rewrite the well known cross and dot product in the language of differential forms. Let \( \alpha \) and \( \beta \) be 1-forms and \( \ast \beta \) be a 2-form, so that:

\[
\alpha = a_1 dx + a_2 dy + a_3 dz
\]

\[
\beta = b_1 dx + b_2 dy + b_3 dz
\]

\[
\ast \beta = b_1 dy \wedge dz + b_2 dz \wedge dx + b_3 dx \wedge dy
\]

We start with taking the exterior product of \( \alpha \) and \( \beta \) which should be a 2-form:

\[
\alpha \wedge \beta = (a_2 b_3 - a_3 b_2)dy \wedge dz + (a_3 b_1 - a_1 b_3)dz \wedge dx + (a_1 b_2 - a_2 b_1)dx \wedge dy
\]

Then the dual \( \ast (\alpha \wedge \beta) \) of \( (\alpha \wedge \beta) \) is a \( (3-2) \)-form and defines the cross product.

\[
\ast (\alpha \wedge \beta) = (a_2 b_3 - a_3 b_2)dx + (a_3 b_1 - a_1 b_3)dy + (a_1 b_2 - a_2 b_1)dz
\]

(4.30)

In a similar manner we find that \( \alpha \wedge \ast \beta \) is a 3-form

\[
\alpha \wedge \ast \beta = (a_1 b_1 + a_2 b_2 + a_3 b_3)dx \wedge dy \wedge dz
\]

Then the dual \( \ast (\alpha \wedge \ast \beta) \) of \( (\alpha \wedge \ast \beta) \) is a \( (3-3) \)-form and defines the inner product.

\[
(\alpha, \beta) = \ast (\alpha \wedge \ast \beta) = (a_1 b_1 + a_2 b_2 + a_3 b_3)1,
\]

(4.31)
where we use the map $\ast : 1 \leftrightarrow dx \wedge dy \wedge dz$ Now we go back to the constitutive relation from Tonti’s electrostatic diagram $d = \epsilon \ast e$. The aim is to find $\ast e$, which should be a 2-form. We use Eq. 4.14 and follow the example from Appendix A.0.1 to establish $\ast e$:

$$\ast e = E_x dy \wedge dz + E_y dz \wedge dx + E_z dx \wedge dy \quad (4.32)$$

And the expression for the constitutive relation takes the form:

$$d = \epsilon \ast e = (\epsilon E_x)dy \wedge dz + (\epsilon E_y)dz \wedge dx + (\epsilon E_z)dx \wedge dy \quad (4.33)$$

It is worth also mentioning that the Hodge operator works both ways, so that we can write the constitutive relation in the following form:

$$e = \frac{1}{\epsilon}(\ast d) = \left(\frac{1}{\epsilon} D_x\right)dx + \left(\frac{1}{\epsilon} D_y\right)dy + \left(\frac{1}{\epsilon} D_z\right)dz \quad (4.34)$$

Now we can use the above considerations and calculate the electrostatic energy, which is an exterior product of $e$ and $d$, which happens to be a 3-form:

$$\frac{1}{2} e \wedge d = \frac{1}{2} e \wedge \epsilon(\ast e) = \epsilon(E_x^2 + E_y^2 + E_z^2)dx \wedge dy \wedge dz = \frac{1}{2} \epsilon(e, e) \ast 1,$$

where

- $(e, e)$ is the inner product which, when evaluated gives a number.

- $\ast 1 = dx \wedge dy \wedge dz$ is a 3-form in $\mathbb{R}^3$, which is the volume element in $\mathbb{R}^3$

Thus

$$\frac{1}{2} e \wedge d = \frac{1}{2} \epsilon(e, e) \ast 1, \quad (4.35)$$
Eq. 4.35 associates the electrostatic energy with the volume element. In other words, through the constitutive relation, the geometry of the space is directly associated with the energy distribution. In a similar manner we can calculate the energy in terms of the 2-form $\mathbf{d}$:

$$\frac{1}{2} \mathbf{e} \wedge \mathbf{d} = \frac{1}{2} \epsilon (\mathbf{d}, \mathbf{d}) \ast 1,$$  \hfill (4.36)

The Eq. 4.35 and Eq. 4.36, however, represent the local energy distribution. To obtain the global energy of the system we need to integrate over the entire manifold, including its boundaries. This is written as:

$$\frac{1}{2} \epsilon (\mathbf{e}, \mathbf{e}) \ast 1 = \frac{1}{2} \epsilon \int_M \mathbf{e} \wedge \ast \mathbf{e}$$  \hfill (4.37)

Eq. 4.37 is called the Hilbert inner product. Next we list the corresponding Maxwell’s equations:

**Faraday’s system**

$$\mathbf{e} = -d\mathbf{v}$$  \hfill (4.38)

$$d\mathbf{e} = 0$$  \hfill (4.39)

**Ampere’s system**

$$\mathbf{d} = dt$$  \hfill (4.40)

$$d\mathbf{d} = \rho$$  \hfill (4.41)
Now let us suppose we have a Hilbert inner product given by $\langle \alpha, d\beta \rangle$, where $\alpha$ and $d\beta$ are $p$-forms and $\beta$ is a $(p-1)$-form. In the context of electrostatics we can take $\alpha = e$ and $\beta = v$, so we obtain $\langle e, -dv \rangle$. In order to obtain an integral in terms of $v$, we need to integrate by parts, which requires an adjoint operator that we denote by $\delta$. This operator is defined by:

$$\langle \alpha, d\beta \rangle = \langle \delta \alpha, \beta \rangle$$ (4.42)

or in the case of electrostatics:

$$\langle e, -dv \rangle = \langle \delta e, -v \rangle$$ (4.43)

Since $e$ and $-dv$ are of the same order, as the definition of inner product requires, the order of $\delta e$ and $-v$ must be the same as well. So we observe that $\delta e$ is a 0-form and the $\delta$ operator reduces the order of the differential forms. This is called a co-differential operator and is dual to the differential operator $d$. We follow the following simple steps to derive an expression for $\delta$ [36]:

- Let $\omega$ be a $p$-form, then $\ast \omega$ is a $(n-p)$-form
- The exterior derivative of $\ast \omega$, is $d(\ast \omega)$ which is a $(n-p+1)$-form
- The dual of, $d(\ast \omega)$ is $\ast [d(\ast \omega)]$ and is $(n-(n-p+1))=p-1$ form. The expression we are looking for is $\delta = \ast d\ast$. In addition the sign in front of the operator may change.

Finally, we define $\delta$ as: $\delta = (-1)^{p(n-p+1)+s} \ast d\ast$, where $p$ is the degree of the form to which we apply the operator, $n$ is the dimension of the space and $s$ is the number of
negative signs in the "metric" of the space. To get an idea, what is the exact meaning of this, let us take the ordinary Euclidean space, where a distance is given by:

\[ dS^2 = dx^2 + dy^2 + dz^2 \]

or

\[
\begin{pmatrix}
  dx & dy & dz
\end{pmatrix}
\begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  dx \\
  dy \\
  dz
\end{pmatrix},
\]

where

\[
 g_{ij} = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{pmatrix}
\]
defines the metric of the standard Euclidean space, which does not have any negative signs and consequently \( s=0 \). We can now use the co-differential operator and write a single expression that can replace the Tonti diagram above. We start with:

\[
\rho = d(d) = d(\epsilon \ast e) = d(\epsilon \ast (-dv)) = -\epsilon (d \ast dv)
\]

Let us divide both sides by \(-\epsilon\) and then apply the Hodge operator on both sides. We get:

\[
-\frac{1}{\epsilon} \rho = d \ast dv
\]

\[
-\frac{1}{\epsilon} \ast \rho = \ast d \ast dv
\]

Using the definition of the co-differential operator, we calculate for \( p=1, n=3 \) and \( s=0 \):

\[
\delta = (-1)^{1(3-1+1)+0} \ast d \ast = (-1) \ast d \ast
\]

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Finally we obtain:

\[ \delta dv = \frac{1}{\epsilon} * \rho \]  

(4.44)

Next let us obtain the gradient, curl, and divergence in the language of differential forms. We start with calculating the exterior derivative of the magnetic vector potential \( \mathbf{a} = A_x dx + A_y dy + A_z dz \). We start with Eq. 4.20. Its dual is:

\[
*da = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) dx + \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) dy + \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dz 
\]  

(4.45)

Eq. 4.45 is equivalent to \( \nabla \times \mathbf{A} \) (curl of \( \mathbf{A} \)) and \( *d \equiv \nabla \times \). To define the divergence we start with the dual of the magnetic vector potential:

\[
*\mathbf{a} = A_x dy \wedge dz + A_y dz \wedge dx + A_z dx \wedge dy
\]

Then we calculate the exterior derivative of \( *\mathbf{a} \):

\[
d * \mathbf{a} = \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) dx \wedge dy \wedge dz
\]

Applying the Hodge operator on both sides finally gives the expression for the divergence of \( \mathbf{A} \) (\( \nabla \cdot \mathbf{A} \)):

\[
*d * \mathbf{a} = + \delta \mathbf{a} = \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right),
\]

(4.46)

where \( \delta \equiv *d * \equiv \nabla \cdot \). Finally we establish the gradient of a 0-form:

\[
dv = \frac{\partial V_x}{\partial x} dx + \frac{\partial V_y}{\partial y} dy + \frac{\partial V_z}{\partial z} dz
\]  

(4.47)

To explore the value of the expressions above we take one of the second order vector identities:

\[
\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})
\]  

(4.48)
Now we use Eqs. 4.45, 4.46, 4.47 to rewrite Eq. 4.48 by differential forms.

\[ \nabla^2 a = d(*d*a) - *d(*da) = d(-\delta a) - (+\delta da) = (-d\delta - \delta d)a \]

\[ \nabla^2 \equiv \triangle \equiv -d\delta - \delta d \quad (4.49) \]

Eq. 4.49 can be considered as a generalized Laplacian and applied to a 0-form \( v \) we obtain:

\[ \triangle v = -d\delta v - \delta dv = -\delta dv \]

taking into account that \( \delta = (-1) * d* \), since the co-differential operator is applied to a 1-form \( -dv \), we get:

\[ \triangle v = -(-1) * d * dv = *d*(-e) = (-1) * d\frac{\epsilon}{\epsilon} = (-1) * \rho \frac{\epsilon}{\epsilon} \]

Finally we write:

\[ \triangle v = (-1) * \frac{\rho}{\epsilon}, \quad (4.50) \]

which we recognize as Poisson’s equation for the electric scalar potential.

4.4.2 Magnetostatics

Let us start this section by rewriting the magnetostatic Maxwell’s equations using differential forms. The Magnetostatic Tonti diagram is given below.

We recall that the magnetic 1-form \( h \) is given by Eq. 4.16 and the magnetic 2-form \( b \) is given by Eq. 4.21. From the Tonti diagram, we observe once again that the constitutive relation connects the Ampere’s (left) and Faraday’s (right) systems. It also provides geometrical information through the Hodge operator as well as physical
information for the system through \( \mu \). As in the previous section we calculate the local magnetostatic energy by means of the wedge product of the 2-form \( b \) and the 1-form \( h \):

\[
\frac{1}{2} h \wedge b = \frac{1}{2} \mu h \wedge *h = \frac{1}{2} \mu (H_x^2 + H_y^2 + H_z^2) dx \wedge dy \wedge dz = \frac{1}{2} \mu (h, h) * 1,
\]

where \(*1\) is the volume element such that \(* : dx \wedge dy \wedge dz \leftrightarrow 1\). The total magnetic energy of the system is obtained by integration of the local energy over the entire manifold, including its boundary:

\[
\frac{1}{2} \mu (h, h) * 1 = \frac{1}{2} \mu \int_M h \wedge *h,
\]

which is the Hilbert inner product. Below we list Maxwell’s equations for magnetostatic fields.

Faraday’s system
\[ b = da \quad (4.52) \]
\[ db = 0 \quad (4.53) \]

Ampere’s system

\[ h = -d\phi \quad (4.54) \]
\[ dh = -j \quad (4.55) \]

It should be noted from Tonti’s diagram, the absence of 0-forms in magnetostatics. Under certain conditions, however, we can introduce a magnetic scalar potential \( \phi \). Since we need \( dh = 0 \) to hold, in order to define \( \phi \), this should be considered as a particular case. This is shown in Eq. 4.54. Another approach, which is more general, is to introduce an arbitrary 0-form \( \chi \) as \( a' = a + d\chi \). This can be done because it does not change anything in the equations:

\[ da' = da + d(d\chi) = da = b \]

Let us take a look at the equation \( dh = j \). We note that due to the Poincaré lemma we have \( d(dh) = d(j) = 0 \), which is the continuity equation for the current. Further on we can write:

\[ j = dh = d\frac{1}{\mu} \ast b \Rightarrow d(\ast b) = \mu j \Rightarrow \ast d(\ast b) = \mu \ast j = \delta b, \]

but \( b = da \), so we finally write:

\[ \delta da = \mu \ast j \quad (4.56) \]

Now we take the vector Laplacian \( \triangle a \):

\[ \triangle a = -d\delta a - \delta da \Rightarrow \delta da = -\triangle a - d\delta a = \mu \ast j \]
Let us compare this equation with the similar one for electrostatics. Here we have the extra term $d\delta a$. According to the Helmholtz theorem, on the other hand both the curl and the divergence of a vector should be specified. We know that the curl of $a$ is $da = b$. The divergence, however, which is $\delta a$, is not specified, and we are free to choose, as long as we do not violate any of the related equations. There are probably many different choices, but for static fields, probably the best is to pick $\delta a = 0$. The final expression obtained is:

$$\triangle a = -\mu * j$$

Eq. 4.57 relates both sides of Tonti’s diagram for magnetostatics in one equation. The choice of $\delta a = 0$ is called Coulomb gauge and is the best, since it simplifies the equations. The need for choosing a gauge in magnetostatics draws our attention to a critical difference between electricity and magnetism. This difference is even more important, as we shall see in electrodynamics. Finally, we need to keep in mind that Tonti’s diagrams, we studied by now are local representation of the fields and do not necessarily apply to the topology of the whole domain or manifold.

4.4.3 Electrodynamics

Let us start this section directly with the electridynamic equations, since we already have all the quantities needed. Maxwell’s equations and Tonti’s diagram are given below:

$$dd = \rho$$

$$db = 0$$
Figure 4.5: Tonti’s diagram for electrodynamics
\[ de = -\partial_t b \quad (4.60) \]
\[ dh = j + \partial_t d \quad (4.61) \]

Now we take Eq. 4.61:
\[ dh = j + \partial_t d = j + \epsilon \partial_t \ast e \quad (\ast) \left( \frac{1}{\mu} \right) \quad (4.62) \]

We use the constitutive relation \( b = \mu \ast h \) and apply the Hodge operator on both sides as well as multiply both sides by \( \frac{1}{\mu} \):
\[ \ast b \frac{1}{\mu} = h \quad (4.63) \]

Using Eq. 4.63 we can substitute in Eq. 4.62:
\[ d(\ast b) \frac{1}{\mu} = j + \epsilon \partial_t \ast e \quad (\ast) \left( \frac{1}{\mu} \right) \Rightarrow \]
\[ \Rightarrow \ast d(\ast b) = \mu \ast j + \epsilon \mu \partial_t e \Rightarrow \]
\[ \Rightarrow \ast d(\ast d a) = \mu \ast j + \epsilon \mu \partial_t e, \]

Finally we get:
\[ \delta d a = \mu \ast j + \epsilon \mu \partial_t e = -\Delta a - d\delta a \quad (4.64) \]

or
\[ \mu \ast j + \epsilon \mu \partial_t (-d v - \partial_t a) = -\Delta a - d\delta a = \]
\[ \mu \ast j - d(\epsilon \mu \partial_t v) - \epsilon \mu \partial_t^2 a = -\Delta a - d\delta a \quad (4.65) \]

We will come back to the above equation a bit later on. Next we take the equation \( dd = \rho \):
\[ dd = \rho \Rightarrow d\epsilon(\ast e) = \rho \quad . \ast \Rightarrow *d \ast e = \frac{1}{\epsilon} \ast \rho \Rightarrow \delta e = \frac{1}{\epsilon} \ast \rho \]

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Now we substitute for $e$ from Eq. 4.27:

$$
\delta(-\partial_t a - dv) = \frac{1}{\epsilon} \star \rho \Rightarrow -\partial_t \delta a - \delta dv = \frac{1}{\epsilon} \star \rho \quad (4.66)
$$

Now we use Eq. 4.49 on the electric scalar potential and write from Eq. 4.66:

$$
\delta dv = -\triangle \phi = \frac{1}{\epsilon} \star \rho + \partial_t \delta a \quad (4.67)
$$

Let us take Eq. 4.65 and Eq. 4.67 and put them together.

$$
\begin{align*}
\triangle a - \epsilon \mu \partial_t^2 a + d \delta a - d(\epsilon \mu \partial_t v) &= -\mu \star j \\
\triangle \phi + \partial_t \delta a &= -\frac{1}{\epsilon} \star \rho
\end{align*}
\quad (4.68)
$$

It has to be mentioned that the two equations in Eq. 4.68 are coupled. They are linked by the equation for the local conservation of charge:

$$
d \mathbf{j} + \partial_t \rho = 0 \quad (4.69)
$$

In other words these two equations have to be studied together. In order to solve Eq. 4.68 we need to choose $\delta a$. As we saw earlier there are two common choices. If we pick the Coulomb gauge ($\delta a = 0$) we obtain:

$$
\begin{align*}
\triangle a - \epsilon \mu \partial_t^2 a - d(\epsilon \mu \partial_t v) &= -\mu \star j \\
\triangle \phi &= -\frac{1}{\epsilon} \star \rho
\end{align*}
\quad (4.70)
$$

The Coulomb gauge simplifies the equation for the electric vector potential, but complicates the equation for the magnetic vector potential. This is why a more common choice is the Lorenz gauge:

$$
\delta a - \mu \epsilon \partial_t v = 0, \quad (4.71)
$$
which produces the symmetrical equations:

\[
\begin{align*}
\triangle a - \epsilon \mu \partial_t^2 a &= -\mu * j \\
\triangle \phi + \epsilon \mu \partial_t^2 v &= -\frac{1}{\epsilon} * \rho
\end{align*}
\]  

(4.72)

The two equations above are the familiar nonhomogeneous wave equations written in terms of the magnetic vector and electric scalar potentials. The homogeneous, source-free equations are obtained by simply setting \( j = 0 \) and \( \rho = 0 \). Identical steps are taken to derive the wave equations in terms of electric and magnetic field intensities

\[
\begin{align*}
\triangle a - \epsilon \mu \partial_t^2 a &= 0 \\
\triangle \phi + \epsilon \mu \partial_t^2 v &= 0
\end{align*}
\]  

(4.73)

4.4.4 Magnetodynamics

Magnetodynamic fields are also called quasi-static fields, or eddy current fields, indicating that fields are dominated by induced currents and the absence of dielectric effects. In these settings, we assume that electric charge densities do not exist and the electric effects are negligible, compared with the magnetic effects. In terms of physical quantities this means that \( \epsilon = 0 \) and consequently \( d = 0 \) as well. With these considerations taken into account, we build Tonti’s diagram shown below.

We also list the relevant Maxwell equations:

\[
d e = -\partial_t b
\]  

(4.74)

\[
d b = 0
\]  

(4.75)

\[
d h = j
\]  

(4.76)

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Figure 4.6: Tonti’s diagram for magnetodynamics
The constitutive relation here are slightly different:

\[ \mathbf{b} = \mu \ast \mathbf{h} \]  
\[ \mathbf{j} = \sigma \ast \mathbf{e} \]

(4.77)  
(4.78)

We take, as in the previous section, the equation for \( d\mathbf{h} \), which in this case is Eq. 4.76 and using Eq. 4.63, we can write:

\[ d \ast \mathbf{b} = \mu \mathbf{j} \]

(4.79)

Applying the Hodge operator on both sides gives:

\[ *d * \mathbf{b} = \mu * \mathbf{j} \Rightarrow \delta \mathbf{b} = \mu * \mathbf{j} \Rightarrow \delta \mathbf{a} = \mu * \mathbf{j} = \]

(4.80)

Taking into account Eq. 4.78 and that \( *.*=1 \) in \( \mathbb{R}^3 \), we write:

\[ \delta \mathbf{a} = \mu * \mathbf{j} = \mu * \sigma * \mathbf{e} = \mu \mathbf{e} = \mu (-\sigma dv - \sigma \partial_t \mathbf{a}) \]

(4.81)

And finally, substituting in Eq. 4.49, we obtain:

\[ -\triangle \mathbf{a} - d \delta \mathbf{a} = \mu (-\sigma dv - \sigma \partial_t \mathbf{a}) \]

(4.82)

Since \( \delta \mathbf{a} \) is the divergence, we are free to choose and it seems that the Coulomb gauge would be the best choice. Then \( \delta \mathbf{a} = 0 \). Also, we note that the term \( \sigma dv \) represents current and it does not depend on time. In other words it would exist even if the field was static. The other term in the brackets is the eddy current density. So we may assume that \( \sigma dv = \mathbf{j}_s \) is the source current, which for many applications is zero.

\[ \triangle \mathbf{a} - \mu \sigma \partial_t \mathbf{a} = \mathbf{j}_s \]

(4.83)
5.1 Introduction

In Chapter 3 we explained that the manifolds and in particular smooth manifolds are not vector spaces and as a consequence we could not construct the exterior algebra and exterior calculus on them directly. To overcome this obstacle, however, we introduced the tangential k-vectors in the tangent space at a point P on the manifold $M$. Similarly, we introduced the tangential k-covectors (k-forms) in the cotangent space at a point P on the manifold $M$. Our main objective here is to juxtapose this chapter to the chapters previously considered, by means of creating discrete versions of all objects and operations, studied by now. Doing so we will try to keep clear separation of the topologic (metric-independent) and geometric (metric dependent) components of the quantities involved. More precisely, here we present a proper discretization of the differential forms on simplicial complexes. We will show, in addition, that this geometric structure is designed to preserve all the fundamental differential properties.
5.2 Notion of Euclidean simplex

At the end of the Chapter 3 we stated the Stokes’ theorem in the language of differential forms as follows [21]:

\[ \int_M d\omega = \int_{\partial M} \omega \]  \hspace{1cm} (5.1)

where \( \partial M \) is the boundary of the manifold \( M \). Eq. 5.1 gives us a hint for the existing relation between the exterior differential operator and the boundary operator. It is easier to see this relation if we rewrite this equation in terms of inner product between the differential form \( \omega \) and the manifold \( M \):

\[ (d\omega, M) = (\omega, \partial M) = \text{number} \]  \hspace{1cm} (5.2)

This reveals a very important property of the differential forms, namely that the exterior differential operator \( d \) can be replaced by the boundary operator \( \partial \). In other words, Eq. 5.2 gives the relationship between the local geometry and the global structure of the system, described by the manifold. This relationship has great practical usefulness in the finite element method of field calculation.

A smooth manifold can be discretized into geometrical idealized objects called simplexes. Informally speaking a k-simplex describes the simplest mesh element, hence the name. We will start first by defining the Euclidean simplices and later on in one of the next sections, we will define the simplices on a manifold as a triplet \( \sigma_k = (S_k, U, \phi) \) [33], where \( S_k \) is a Euclidean k-simplex, \( U \) is a neighborhood of \( S_k \), and \( \phi \) is a map from \( U \) to \( M \). The Euclidean simplexes are defined as follows:

- 0-simplex = \( S_0 = (P_0) \) - point
Figure 5.1: Simplexes. a) 0-simplex; b) 1-simplex; c) 2-simplex; d) 3-simplex

- 1-simplex = \( S_1 = (P_0, P_1) \) - line - directed closed segment of straight line that is completely determined by its ordered pair of vertices \((P_0, P_1)\)

- 2-simplex = \( S_2 = (P_0, P_1, P_2) \) - triangle - a closed triangle with vertices taken in definite order. It is completely determined by its ordered triple of vertices \((P_0, P_1, P_2)\)

- 3-simplex = \( S_3 = (P_0, P_1, P_2, P_3) \) - tetrahedron - It is completely determined by its ordered quadruple \((P_0, P_1, P_2, P_3)\), and so on.

In general, a k-simplex is the closed convex hull of \((k+1)\) independent distinct points \((P_0, \ldots, P_k)\):

\[
S_k = \sum_{i=0}^{k} t_i P_i, \tag{5.3}
\]

where \(t_i \geq 0\), \(\sum t_i = 1\) and the masses \(t_i\) are located at the points \(P_i\)
5.3 Orientation of a simplex

The orientation of an Euclidean $k$-simplex is determined by the specific ordering of the $(k+1)$ vertices. In the definition of 1,2,3-simplexes in Subsection 5.2, we used ordered pair, triple, quadruple respectively. Choosing such particular ordering of the vertices, simply means that we fix some default orientation of the simplex. Since there are two possible orientations, denoted by +1 (the ‘+’ sign is usually omitted) and -1. Here we assume that the default orientation is the positive one. We note also that if two orderings differ by an even permutation of the $(k+1)$ vertices, the result is a positive orientation as well. On the other hand if two orderings differ by an odd permutation the result is a negative orientation. For example, the orientation associated with the simplexes in Fig. 5.1 are:

- Oriented line segment: $(P_0, P_1) = -(P_1, P_0) = S_1$

- Oriented triangle

  $$(P_0, P_1, P_2) = -(P_0, P_2, P_1) =$$

  $$(P_2, P_0, P_1) = -(P_2, P_1, P_0) =$$

  $$(P_1, P_2, P_0) = -(P_1, P_0, P_2) = S_2$$

- In general, the orientation is given by:

  $$(P_{i_0}, \ldots, P_{i_k}) = sgn(\sigma)(P_{j_0}, \ldots, P_{j_k}),$$

The idea here was to describe discrete geometrical objects that later could be defined on a smooth manifold, in an attempt to correctly discretize the given smooth manifold.
Here, we briefly compare the notion of orientation of a simplex and that of a manifold. The differential definition of orientation in Chapter 3, uses the notion of atlases, determined by the sign of the Jacobian. On the other hand, the orientation of an edge (1-simplex) could be one of two directions; the triangle (2-simplex) could be rotated clockwise or counterclockwise; a volume (3-simplex) could be right-handed helix or left-handed one. As we can see, we do not need the notion of atlases to define the orientation of simplices, which significantly simplifies the computations, as we shall see later on in this chapter.

5.4 Boundary of a simplex

Any (k-1)-simplex, spanned by a strict subset of \((P_1, \ldots, P_k)\) is called a \((k - 1)\)-face of \(S_k\). That is a \((k - 1)\)-face is simply a \((k - 1)\)-simplex, whose k-vertices are all from the \((k+1)\) vertices of the k-simplex \(S_k\). The union of all \((k - 1)\)-faces is called the boundary of the k-simplex. The boundary \(\partial S_k\) of a k-simplex \(S_k\), in practice, is given by the formal sum of simplices of one dimension lower with integer coefficients [33, 21]:

\[
\partial S_k = \partial(P_0, P_1, \ldots, P_k) = \sum_{i=0}^{n} (-1)^i(P_0, P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_k) \quad (5.4)
\]

Now, let us refer again to Fig. 5.1 and using Eq. 5.4 we calculate:

\[
\partial S_1 = \partial(P_0, P_1) = (P_1) - (P_0)
\]

\[
\partial S_2 = \partial(P_0, P_1, P_2) = (P_1, P_2) - (P_0, P_2) + (P_0, P_1)
\]

\[
\partial S_3 = \partial(P_0, P_1, P_2, P_3) = (P_1, P_2, P_3) - (P_0, P_2, P_3) + (P_0, P_1, P_3) - (P_0, P_1, P_2) \quad (see \, fig.5.2)
\]
5.5 Simplices on a smooth manifold

Now we consider a manifold $M$ and we will define a $k$-simplex in $M$. As we mentioned earlier, a preliminary $k$-simplex in $M$ (we will call it ‘preliminary’ for now) consists of three components: an Euclidean $k$-simplex $S_k$, a $k$-dimensional neighborhood $U$ of $S_k$ in Euclidean space, and a smooth mapping $\phi$,

$$\phi : U \rightarrow M \quad (5.5)$$

We denote this preliminary $k$-simplex by

$$(S_k, U, \phi)$$
A second simplex 

$$(T_k, V, \psi)$$

will be considered equal to the first one if their images are equal:

$$\phi\left(\sum_{i=0}^{k} t_i P_i\right) = \psi\left(\sum_{i=0}^{k} t_i Q_i\right) \quad (t_i \geq 0, \sum t_i = 1),$$

where

$$S_k = (P_0, \ldots, P_k) \quad T_k = (Q_0, \ldots, Q_k)$$

In other words we set up a linear equivalence between $S_k$ and $t_k$ that preserves the ordering of the vertices

$$S_k \leftrightarrow T_k$$

then $\phi(P) = \psi(Q)$, where P and Q are corresponding points. This could also be expressed by the following commutative diagram:

![Figure 5.3: Commutative diagram, expressing the map of $t_k$ and $S_k$ from $U$, $V$, respectively, to $M$](image)

The totality of these preliminary simplices, $(S_k, U, \phi)$ which in this way are identified as a single one, make up an object which we call a k-simplex in $M$, denoted by $\sigma_k$. 

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Finally, we mention that the boundary of such a k-simplex, as well as the orientation are determined in the same way as in the case of an Euclidean simplex.

5.6 Simplicial complex

A simplicial complex is a collection of simplices which satisfies both: every face of each simplex of $\mathcal{K}$ is in $\mathcal{K}$, and the intersection of any two simplices in $\mathcal{K}$ is either empty, or a single common face. Examples of simplicial complexes are shown on Fig. 5.4.

![Figure 5.4: Examples of simplicial complexes](image)

Computer graphics and the mesh generation algorithms, for example, make heavy use of simplicial complexes. Triangular meshes in 2D and tetrahedra in 3D are examples of such simplicial complexes. In the finite element method the generation of such meshes is integral part of the preprocessing of the domain of interest (discretizing the domain).
5.7 Discrete manifolds

An $n$-dimensional discrete manifold $\mathcal{M}$ is an $n$-dimensional simplicial complex that satisfies the following condition: for each simplex, the union of all the adjacent simplices form an $n$-dimensional ball, or half a ball if the simplex is on the boundary. As a result, each $(n-1)$-simplex has exactly two adjacent $n$-simplices, or only one, if it is on the boundary [40]. In practice a discrete manifold is simply a triangulation of the smooth manifold. For example in 2D, a discrete manifold cannot have isolated edges or isolated vertices and each of their edges is adjacent to two faces, except the edges on the boundary, which are adjacent to only one face. The same can be written for the 3D case, following the simple condition above.

Figure 5.5: Simplicial complex consisting of all vertices and edges (left). This simplicial complex, however is not yet a discrete manifold. if we add the faces $f_0$ and $f_1$, then we form a discrete 2-manifold with one boundary (right).
5.8 Notion of chains and boundary operator

A k-chain is a linear combination of k-simplices. We start with chains, which are linear combinations of Euclidean simplices:

\[ c = \sum_{i=0}^{k} a_i S_i, \]  

(5.6)

where \( a_i \) are constants and the \( S_i \) are Euclidean simplices. Its boundary is defined by

\[ \partial c = \sum_{i=0}^{k} a_i (\partial S_i), \]  

(5.7)

It can be shown that the boundary of the boundary of a chain vanishes, namely \( \partial[\partial c] = 0 \) Next, we take a simplex \( \sigma_k \), represented by \( (S_k, U, \phi) \). Let us denote the faces of the Euclidean k-simplex \( S_k \) by \( q_0, q_1, \ldots, q_k \) or just \( q_i \). Each of these faces is a (k-1)-simplex. Since there are only two possible orientations, we can replace the coefficients \( a_i \), by +1 or -1:

\[ \partial S_k = \sum_{i=0}^{k} \pm q_i \]  

(5.8)

We go further on by restricting \( \phi \) to act only on the faces \( q_i \) ((k-1)-simplices). In addition we define a neighborhood \( V_i \) of the \( q_i \)s. So now we have:

\[ \phi : V_i \to M, \text{ and} \]

\[ \tau = (q_i, V_i, \phi) \]  

(5.9)

Finally, we can write the boundary of a k-simplex embedded in a manifold \( M \), which is a (k-1)-simplex in \( M \) as well:

\[ \partial \sigma_k = \sum_{i=0}^{k} \pm \tau_i \]  

(5.10)
Eq. 5.10 represents a \((k-1)\)-chain in \(M\). In general a \(k\)-chain in \(M\) is a linear combination of \(k\)-simplexes in \(M\) as well. It should be noted that a simplex can occur in the chain more than once. A chain is a formal sum in the form:

\[
c = \sum_{i=0}^{k} a_i \sigma_i
\]  

(5.11)

with constant coefficients \(a_i\), which show how many times a simplex occurs in the chain.

Now let \(K\) be a simplicial complex in \(M\). A \(k\)-chain is a subset of \(k\)-simplices in \(K\). We can define addition and multiplication by a constant, in particular the sum of two \(k\)-chains \(a\) and \(b\) is the symmetric difference of the two sets:

\[
a + b = a \cup b - a \cap b
\]  

(5.12)

which is commutative. The set of all \(k\)-chains on \(M\) together with the addition and multiplication by a constant make up a space, denoted by \(C_k\).

We shall check Eq. 5.12. Let \(K\) be a 2-simplicial complex. First, some notations. The positive orientation is assumed to be clockwise. We introduce local numbering for each 2-simplex (see Table 5.1).

Next we recall the boundary of a 2-simplex is a 1-chain. The totality of all 1-chains form the 1-chain over the simplicial complex \(K\). We proceed by finding the boundary of each 2-simplex, using Eq. 5.4 and the indices are running over the local numbering of the nodes, specified in Table 5.1.
Figure 5.6: Applying the boundary operator on simplicial complex of 2-simplices, results in an 1-chain. Note that the triangle efc is not a 2-simplex, but a hole.

Table 5.1: Local numbering of each 2-simplex from Fig. 5.6, within the 2-simplicial complex.

<table>
<thead>
<tr>
<th>2-simplex</th>
<th>Node 1</th>
<th>Node 2</th>
<th>Node 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_2^1$</td>
<td>a</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>$S_2^2$</td>
<td>b</td>
<td>c</td>
<td>e</td>
</tr>
<tr>
<td>$S_2^3$</td>
<td>c</td>
<td>f</td>
<td>d</td>
</tr>
<tr>
<td>$S_2^4$</td>
<td>e</td>
<td>f</td>
<td>g</td>
</tr>
</tbody>
</table>
\[ \partial S_2^1 = -cd + ad - ac \]
\[ \partial S_2^2 = -ce + be - bc \]
\[ \partial S_2^3 = -fd + \text{cd} - cf \]
\[ \partial S_2^4 = -fg + eg - ef \]

Now we sum up all the terms from above (1-simplices) and we see that the edge cd appears twice with opposite orientation, hence it cancels and the result is consistent with Eq. 5.12. We finally get:

\[ C_1 = ad - ac - ce + be - bc - fd - cf - fg + eg - ef \]

which is, in other words, the union of all 1-chains, minus the intersection of two of them (in this case).

Now we set
\[ \partial c = \sum a_i \partial \sigma_i. \quad (5.13) \]

Thus
\[ \partial : \quad C_k(M) \to C_{k-1}(M) \quad (5.14) \]

The basic property of the boundary operator \( \partial \) follows readily from the corresponding Euclidean case: for a k-chain \( c \),

\[ \partial(\partial c) = 0 \]
5.9 Notion of cochains

A k-cochain $\omega$ is the dual of a k-chain. In other words, $\omega$ is a linear mapping that takes a k-chain to $\mathbb{R}$ [36],[40] namely:

$$\omega : C_k \to \mathbb{R}$$

(5.15)

$$c \to \omega(c) = \omega(\sum_{i=0}^{k} a_i \sigma_i) = \sum_{i=0}^{k} a_i \omega(\sigma_i)$$

(5.16)

Now it is clear that a cochain corresponds to one value per simplex. The difference, however, between k-chains and k-cochains is that the k-cochains are evaluated on each k-simplex. So a k-cochain can be thought of as a field that can be evaluated on each simplex of an oriented simplicial complex $\mathcal{K}$.

It is becoming apparent that the newly introduced k-cochains are nothing more, but discrete analog of differential forms that we introduced first on standard vector spaces in Chapter 2 and later in Chapter 3 we defined them on smooth manifolds. Thus these k-cochains deserve to be called discrete differential forms. Indeed, a continuous k-form was defined as a linear mapping from k-dimensional sets to $\mathbb{R}$. In the discrete settings that is a linear mapping from a chain to a number that we called a cochain. For example, a 0-form can be evaluated at each point of the mesh, a 1-form can be evaluated on each curve of the mesh and a 2-form can be evaluated on each surface, etc.

If we now restrict the integration to take place only on k-submanifold (k-chain), which is the sum of k-simplices in the triangulation, we get a k-cochain, thus we have natural discretization of k-forms. If we integrate the k-form on each
k-simplex, and assign the resulting number to that simplex, we obtain a k-cochain on the k-simplicial complex. Thus we end up with a scalar field, which is a discrete representation of the original continuous differential k-form.

5.10 Integration of a form on a chain

We can now naturally extend the notion of evaluation of a differential form $\omega$ on a chain, simply using the linearity:

$$\int_c \omega = \int_{\sum_i c_i \sigma_i} \omega = \sum_i c_i \int_{\sigma_i} \omega$$  \hspace{1cm} (5.17)

So, the integration of $\omega$ on each k-simplex $\sigma_k$, simply assigns a number on each simplex. The totality of all numbers represent the approximated differential form $\omega$:

$$\omega[i] = \int_{\sigma_k} \omega$$

In that context the Eq. 5.17 is a mapping from a k-form $\omega$ to a k-cochain. Before continuing we need to explain the notion of standard simplex. When it comes to defining integrals, it is often convenient to introduce standard models. In the case of simplices we define the standard k-simplex $\bar{S}_k$ in Euclidean space, and $\phi$ as a smooth mapping of the neighborhood $U$ of $\bar{S}_k$ into $M$:

$$\bar{S}_k = (R_0, \ldots, R_n)$$  \hspace{1cm} (5.18)

For example:

$$R_0 = 0$$

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\[ R_1 = (1, 0, \ldots, 0) \]

\[ R_2 = (0, 1, \ldots, 0), \text{ etc} \]

For instance Fig. 5.7 shows a standard Euclidean 3-simplex.

![Figure 5.7: Standard Euclidean 3-simplex $S_3$](image)

Let $\omega$ be a k-form defined on a domain $U$ of $E^k$, which includes $S_k$. Now we wish to define

\[ \int_{S_k} \omega \]  \hfill (5.19)

We do this by writing $\omega$ in a unique way as follows:

\[ \omega = A(x_1, \ldots, x_k)dx_1dx_2\ldots dx_k \]

Then we get an ordinary n-fold integration

\[ \int_{S_k} \omega = \int_{S_k} A(x)dx_1dx_2\ldots dx_k \]

Let us now go back to Eq. 5.17. We recall that a k-form $\omega$ can be integrated only over a k-manifold (or submanifold). Now in the discrete case we do not have a continuous
manifold but only a mesh (very often triangles in 2D) to work with. The mesh on the other hand is simply a chain, which is embedded in the manifold (the chain sometimes is called discrete submanifold). So we have a chain $c$ and k-form $\omega$, both in $M$. In Eq. 5.17 we represent the k-simplex $\sigma_k$ in the form: $(\bar{S}_k, U, \phi)$, where $\bar{S}_k$ is the standard k-simplex in $E^k$, and $\phi$ is a smooth mapping of the neighborhood $U$ of $\bar{S}_k$ into $M$.

We finally obtain the sought definition which is a form of pullback (see Fig. 5.8):

$$\int_{\sigma_k} \omega = \int_{\bar{S}_k} \phi^* \omega$$

(5.20)

![Diagram](image)

Figure 5.8: The figure shows the compound function $\phi^* \omega$ acting on elements from $U$ and sending them to $R$.

5.11 Discrete exterior derivative and the coboundary operator

By now we defined the discrete differential forms as cochains. It seems natural as a next step to define a discrete version of the exterior derivative operation. Stokes’ theorem in terms of differential forms, stated earlier in the beginning of this chapter,
can be used to define the exterior derivative in a discrete setting that we have here. We start with the Stokes’ theorem, written in terms of discrete k-forms and the formula becomes even simpler: it states that $d$ applied to an arbitrary discrete form $\omega$ is evaluated on arbitrary simplex $\sigma$ as follows:

$$\int_\sigma d\omega = \int_{\partial \sigma} \omega$$

(5.21)

Eq. 5.21 states that the integral over a k-dimensional set turns into a boundary integral over a (k-1)-dimensional set. So each time we encounter an exterior derivative of a form, say $\omega$, we replace any evaluation over a simplex $\sigma$, by direct evaluation of the form itself over the boundary of the simplex $\partial \sigma$, which is a chain of simplices of one dimension lower (k-1). In this way the exterior derivative is readily defined.

We now recall the connection between the exterior differential operator $d$ and the boundary operator $\partial$, given by Eq. 5.2, which gives a relation between the local and global geometry of the structure. Here we extend this relation to be applicable to our discrete setting. Let us denote $\int_\sigma \omega = (\omega, \sigma)$, $\int_\sigma d\omega = (d\omega, \sigma)$, and $\int_{\partial \sigma} \omega = (\omega, \partial \sigma)$. Since $\int_\sigma d\omega = \int_{\partial \sigma} \omega$, and we find the relation that we were looking for, namely, $(d\omega, \sigma) = (\omega, \partial \sigma)$. This means that applying the exterior differential on the left side of a discrete k-form is equivalent to applying the boundary operator on the right hand side of a k-simplex. Finally, by linearity, we can write a general expression of Stokes’ theorem, now extended to arbitrary chains:

$$\int_c d\omega = \int_{\sum_i c_i \sigma_i} d\omega = \int_{\sum_i c_i \sigma_i} \omega = \int_{\sum_i c_i \partial \sigma_i} \omega = \sum_i c_i \int_{\partial \sigma_i} \omega$$
6.1 Variational formulation

Variational calculus is a field of mathematics which deals with functions of functions, the so called functionals. The interest is in those functions that will make the functional assume a maximum or minimum value. The fundamental lemma of the variational calculus states that if:

\[ \int_{\alpha}^{\beta} M(x)h(x)dx = 0 \]  

(6.1)

for all \( h(x) \) with continuous partial derivatives, then

\[ M(x) = 0 \]  

(6.2)

on the interval \([\alpha, \beta]\)

Now we take an elliptic PDE in a general form, such as:

\[ \mathcal{L}u - f = 0 \]  

(6.3)

for some general operator \( \mathcal{L} \) and function \( u \). Next let us take a basis function expansion that approximates \( u \), in the form:

\[ u \approx u_h = \sum_{i=1}^{N} c_i \psi_i, \]  

(6.4)
where $\psi_i$ denote the basis functions and since they are finite number, substituting them into the PDE above will generate a residual error:

$$\mathcal{L}u_h - f = R \quad (6.5)$$

Our goal in the numerical scheme is to determine the coefficients $c_i$ such that they minimize the residual $R$ [41]. This can be done by requiring the integral of the residual $R$, over some domain $\Omega$ to evaluate to zero:

$$\int_{\Omega} R d\Omega = 0 \quad (6.6)$$

The problem here is that we have $N$ coefficients and only one equation. One solution to this problem is the so called method of weighted residuals. The essence of the method is to introduce $N$ weighting functions $w_i$ and to write an integral for each of them by weighing the residual. Setting up all integrals to zero, results in $N$ equation for the $N$ unknown expansion coefficients:

$$\int_{\Omega} R w_i d\Omega = 0, \ i = 1 \ldots N \quad (6.7)$$

The Galerkin method is a particular case of the method, considered above. Here we choose the weighting functions to be the basis functions $\psi_i$ we used to approximate $u$:

$$\int_{\Omega} R \psi_i d\Omega = 0, \ i = 1 \ldots N \quad (6.8)$$

The basis functions $\psi_i$ are required to be members of a complete set of functions. Since a complete set of functions can exactly represent any function, the basis expansion $u_h$ is capable of representing the exact solution, if the number of the terms is large enough.
6.2 Variational formulation in magnetostatics

In Section 4.4.2 for the case of magnetostatic fields we used the equation:

\[ dh = -j \]  \hspace{1cm} (6.9)

and using the properties of differential forms we derived:

\[ \nabla a = \delta da = -\mu * j \]  \hspace{1cm} (6.10)

Eq. 6.10 represent the weak formulation of the Ampere’s law, given by 6.9 (called also Ampere’s theorem by some authors). Let us take a specific example. Consider a bounded region \( \Omega \) and its boundary \( \partial \Omega \), which is devided into two disjoint regions \( \partial \Omega_b \) and \( \partial \Omega_h \), such that \( \partial \Omega = \partial \Omega_b \cup \partial \Omega_h \) [42, 43]. The boundary conditions on the boundary are as follows: \( n \cdot b = 0 \) on \( \Omega_b \) and \( n \times h = 0 \) on \( \Omega_h \). We should note here that Eq. 6.10 is a local representation. To find an equation that takes into account the entire domain we integrate as follows:

\[ \int_{\Omega} \delta da = -\mu \int_{\Omega_s} *j \]  \hspace{1cm} (6.11)

Where \( \Omega_s \) is the domain of the source. Further on, we can rewrite Eq. 6.11 as:

\[ \int_{\Omega} *d * da = -\mu \int_{\Omega_s} *j \]  \hspace{1cm} (6.12)

The term \( *d * da \) in vector calculus is equivalent to \( \nabla \times (\nabla \times A) \). Let us now take the wedge product of an arbitrary 1-form \( a' \), on both sides of the integral above

\[ \int_{\Omega} (\ast da') \wedge (\ast da) = -\mu \int_{\Omega_s} a' \wedge \ast j, \ for \ \forall a' \in \Omega_b \]  \hspace{1cm} (6.13)
The right hand side can be rewritten in terms of a vector potential $*j = *dt$. Then we integrate by parts and finally obtain:

$$\int_{\Omega} (*da') \wedge (*da) = -\mu \int_{\Omega_s} *da' \wedge t$$  \quad (6.14)$$

Eq. 6.14 can be used for the finite element solution. If we now assume:

$$a \approx a_h = \sum_{i=1}^{N} c_i \psi_i$$  \quad (6.15)$$

and according to Galerkin’s method we pick the weighting function $a'$ to be approximated by the same basis functions as $a$:

$$a' \approx \sum_{j=1}^{N} \psi_j$$  \quad (6.16)$$

We can write:

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \left( \int_{\Omega} (*d\psi_j) \wedge c_i (*d\psi_i) \right) = \sum_{i=1}^{N} \left( -\mu \int_{\Omega_s} *d\psi_j \wedge t \right).$$ \quad (6.17)$$

We end up with a $N \times N$ matrix equation, where the left hand side is the so called stiffness matrix and the right hand side is the mass matrix. In the next few sections, our goal will be to define proper function discrete spaces. The basis functions $\psi_i$ will be described by polynomials that will belong to polynomial spaces which must be in complete agreement with these discrete function spaces.

6.3 Function spaces

The solution of Maxwell’s equations belong to the spaces of square integrable scalar and vector fields $L^2(\Omega)$ and $L^2(\Omega)$. 108
\[ L^2(\Omega) = \{ u | \int_{\Omega} u^2(x) \, dx < \infty \} \]  
\[ L^2(\Omega) = \{ u | \int_{\Omega} \|u^2(x)\| \, dx < \infty \} \]  

The two definitions above say that a given function \( u(x) \) is square integrable if its \( L^2 \) norm is finite. These are Hilbert spaces, which usually describe physical fields of finite energy. On these spaces the vector calculus operators \( \text{grad}, \text{curl}, \text{div} \), define the following scalar and vector subspaces:

\[ H^1(\Omega) = \{ u \in L^2(\Omega) | \partial_x u, \partial_y u, \partial_z u \in L^2(\Omega) \} \]  
\[ H(\text{curl}, \Omega) = \{ u \in L^2(\Omega) | \text{curl}(u) \in L^2(\Omega) \} \]  
\[ H(\text{div}, \Omega) = \{ u \in L^2(\Omega) | \text{div}(u) \in L^2(\Omega) \} \]  

In the above formulas \( H^1(\Omega) \) is the Sobolev space for which all first order partial derivatives are also square integrable. The Hilbert spaces \( H(\text{curl}, \Omega) \) and \( H(\text{div}, \Omega) \) are spaces of vectors for which only some combinations of the spatial derivatives need to be square integrable. Namely, for \( H(\text{curl}, \Omega) \) (see Eq. 4.20), we need:

\[
\left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \in L^2 \\
\left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \in L^2 \\
\left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \in L^2 
\]

Note that \( \partial A_x/\partial x, \partial A_y/\partial y, \partial A_z/\partial z \) should not be square integrable. In the case of \( H(\text{div}, \Omega) \) (see Eq. 4.25) we need

\[
\left( \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \in L^2 
\]
Note that $\partial D_x/\partial y, \partial D_x/\partial z, \partial D_y/\partial x, \partial D_y/\partial z, \partial D_z/\partial x, \partial D_z/\partial y$ should not be square integrable. Therefore the spaces $H(curl, \Omega)$ and $H(div, \Omega)$ fall between the spaces $H^1$ and $L^2$. These spaces are connected in the 3D domain $\Omega$ by the following structure called De Rham’s complex [22]

$$H^1(\Omega) \xrightarrow{\text{grad}} H(curl, \Omega) \xrightarrow{\text{curl}} H(div, \Omega) \xrightarrow{\text{div}} L^2(\Omega)$$

In the De Rham’s complex above it is assumed that the domain $\Omega$ is not “too complex”. Under this assumption, the complex above is an exact sequence. In addition it is important to point out that De Rham’s complex is based on Poincaré’s lemma, namely $d(d\omega) = 0$. In terms of vector calculus these are the well known null identities $\text{curl} (\text{grad} (f)) = 0$ and $\text{div} (\text{curl} (u)) = 0$

In the study of differential forms, we define an exact form. That is, for each $k$-form $\omega$ there is a $(k-1)$-form $\alpha$, such that $\omega = d\alpha$. Further, a $k$-form is called closed, if $d\omega = 0$. Based on these definitions we conclude that every exact form is closed; however in the general case we cannot say that every closed form is exact. All these considerations are summarized on the so called De Rham’s diagram below.

Very often the kernel of the $d$ operator is denoted by $Z^i(\Omega)$ (the set of all closed forms), where $0 \leq i \leq 2$, and the image of $d$ is denoted by $B^i(\Omega)$, where $0 \leq i \leq 2$. From Fig. 6.1 we can conclude that in a general $k$-manifold, the kernel of the $k$-th differential operator $d$ is a direct sum of the image of the $(k-1)$-th operator $B^{k-1}(\Omega)$ and the subspace $H_k$ Eq. 6.23.

$$Z^k(\Omega) = B^{k-1}(\Omega) \oplus H_k \quad (6.23)$$
Every such set of closed forms $Z^k(\Omega)$ is the k-th cohomology group [22]. The subspaces $H_1$ and $H_2$, in Fig. 6.1, need now to be explained. The space $H_1$ is the set of all curl-free fields, which are not gradients and the space $H_2$ is the set of all div-free fields, which are not curls. Topologically the dimension of $H_1$ gives the number of loops in $\Omega$ and the dimension $H_2$ gives the number of cavities in $\Omega$. As a particular case, when $H_1 = 0$, the manifold $\Omega$ is called simply connected and $\Omega$ is contractible if $H_2 = 0$. In the case when both conditions hold, simply connected and contractible, the manifold is called trivial. We are most interested in studying trivial domains, since the reference elements on which we build basis functions, are trivial manifolds. Based on Eq. 6.23 and the discussion above we conclude that the image of the (k-1)-
th operator $B^{k-1}$ is the kernel of the k-th operator. Due to this fact we will denote $B^{k-1}$ for the case of trivial domains as $Y^{k-1}$. This is just a change in notation to distinguish between the image of an operator in trivial and non trivial manifolds. This is shown in Eq. 6.24 and also is depicted in De Rham’s diagram in Fig. 6.2a.

$$Z^k = B^{k-1} \equiv Y^{k-1}$$

(6.24)

We observe that DeRham’s structure takes quite symmetrical form for trivial domains. In addition, from Fig. 6.2a we note that each continuous space $D^k$ can be decomposed into a disjoint union of the kernel of the k-th operator, which is equal to

Figure 6.2: De Rham’s diagrams for a trivial 3D manifold

(a) Cohomology complex, continuous spaces on a trivial manifold

(b) Cohomology complex, discrete spaces on a trivial manifold

mains.
the image of the (k-1)-th operator, and the image of the k-th operator:

\[ D^k(\Omega) = Z^k(\Omega) \oplus Y^k(\Omega) \]  \hspace{1cm} (6.25)

Another way to look at DeRham’s diagram is shown on Fig 6.3. It shows how the spaces are connected by the exterior derivative operator (the top sequence and in reverse order, the bottom sequence), as well as the duality of the different spaces (using the Hodge operator). This structure in particular reveals even more. If we take a look in the first loop we find a familiar operator, namely grad-div, commonly seen in the scalar wave equation. The second loop is the curl-curl operator, commonly found in the vector wave equation. All these considerations may be used as a roadmap to construct discrete spaces that will mimic the continuous ones.

![DeRham's diagram](image)

Figure 6.3: DeRham’s diagram

6.4 Discrete spaces

The challenge is to construct discrete spaces that approximate correctly the continuous spaces described above. This is done by discretizing the domain using suitable
discrete elements (finite elements). These finite elements should model the range space (image of the differential operator) and the null space (the kernel of the differential operator). The choice of the type of finite element depends on the field to be approximated. For example the electric filed intensity, being a 1-form (circulation) must be modeled with tangential (also called edge elements).

In the context of the previous section, in the case of discrete differential forms, the local function spaces are denoted by $W_q^k$. These spaces are built on a trivial manifold, which from a practical point of view is simply a reference element with regular geometry, denoted by $\hat{\Omega}$. The space $W_q^k$ can be decomposed in a similar way to its continuous counterpart $D_k$, in a kernel and image of the k-th operator:

$$W_q^k(\hat{\Omega}) = Z^k(\hat{\Omega}) \oplus Y^k(\hat{\Omega})$$  \hspace{1cm} (6.26)

Eq. 6.26 is discrete representation of the Helmholtz decomposition of a vector field. In other words, the discrete space $W_q^1$, has to be constructed by curl-free polynomials (the subspace $Z^1$, which contains all the gradients) and div-free polynomials (the space $Y^1$, that contains all the curls). For example in the case $k=2$, the dimensions of these spaces are given below. More details on the dimensions of the discrete spaces can be found in [6, 7, 22, 23, 27] The major difference in Eq. 6.26, in comparison with Eq. 6.25 is that we operate over discrete domain that has to model correctly the continuous one. As we can see in Fig. 6.2b the structure of DeRham’s diagram for discrete spaces is identical, which means that the differential operator in descrete form behaves in the same way as the continuous one.
Table 6.1: Dimensions of incomplete spaces for hexahedral elements of order k= 2.

<table>
<thead>
<tr>
<th></th>
<th>$Z^1(\hat{\Omega})$</th>
<th>$Y^1(\hat{\Omega})$</th>
<th>$W_2^1(\hat{\Omega})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>dimension</td>
<td>26</td>
<td>28</td>
<td>54</td>
</tr>
</tbody>
</table>

Table 6.2: DoF of incomplete 1-form hexahedral elements of order k= 2.

<table>
<thead>
<tr>
<th>DoF</th>
<th>$Z^1(\hat{\Omega})$</th>
<th>$Y^1(\hat{\Omega})$</th>
<th>$W_2^1(\hat{\Omega})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>On Edges</td>
<td>19</td>
<td>5</td>
<td>24</td>
</tr>
<tr>
<td>On Faces</td>
<td>6</td>
<td>18</td>
<td>24</td>
</tr>
<tr>
<td>In Volume</td>
<td>1</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>Total</td>
<td>26</td>
<td>28</td>
<td>54</td>
</tr>
</tbody>
</table>

6.5 Hexahedral reference element

Definition:(Finite Element) [44, 30]: Finite Element is the triplet $C = (\hat{\Omega}, W_q^k, \Sigma)$, where:

- $\hat{\Omega}$ is the domain in $\mathbb{R}^3$ (in the case of a 3D element).
- $W_q^k$ is the discrete space that models its continuous counterpart $D^k$ by polynomials of degree at most $q$. 

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• $\Sigma = A_1, A_2, \ldots, A_N$, is a set of linear functionals (forms) called degrees of freedom, commonly abbreviated by DoF, such that:

$$A_i = W^k_q \rightarrow \mathbb{R}, \ i = 1, 2, \ldots, N \quad (6.27)$$

The degrees of freedom should also satisfy the following properties:

− Unisolvence - $A_i$ is dual to the finite element space $W^1_q$; that is, there exists a set $l_j \subset W^1_q$ such that $A_i(l_j) = \delta_{ij}$ [19, 20, 25]

− Invariance - degrees of freedom remain unisolvent under a change of the coordinate system; this implies they are not affected by the pullback operation.

− Locality - the trace of a basis function on a subsimplex is determined by degrees of freedom associated only with that subsimplex.

Having established the properties of differential forms as well as the properties of their discrete version, we now revisit some of the notions in the context of the edge finite elements (we deal with 1-chains/cochains and 1-vectors/forms). Let us recall that we can think of 1-chain (which is a discrete version of a 1-vector) as a boundary of a 2-simplex. Thus, 1-chain is a simplicial complex of 1-simplices. Note in the expression below, that the 1-simplices play the role of a discrete basis. That is, 1-chain is a linear combination of 1-simplices:

$$c = \partial \sigma^2 = \sum a_i \partial \sigma^2_i = \sum a_i \sigma^1_i = \sigma^1, \quad (6.28)$$

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where \( c \) is a 1-chain, \( \sigma^2 \) and \( \sigma^1 \) are 2- and 1-simplicial complexes and \( a_i \) are constant coefficients. We also recall:

\[
\int_c \omega = \int_{\sum_i a_i \sigma_i} \omega = \sum_i a_i \int_{\sigma_i} \omega
\]  

which evaluates the form \( \omega \) on each subsimplex \( \sigma_i \) and assigns a number to it (in the case of edge elements this simply means we assign a value to each edge of the element):

\[
e \approx e_h = \int_c e = \int_{\sum_i a_i \sigma_i} e = \sum_{j=1}^{\text{dim}(W^1_q)} \sum_{i=1}^{\text{dim}(W^1_q)} \int_{\sigma_i} A_i(l_j) w_i = \sum_{j=1}^{\text{dim}(W^1_q)} \sum_{i=1}^{\text{dim}(W^1_q)} \int_{\sigma_i} \psi_{ij},
\]

The basis function expansion that we are looking for has the form:

\[
\psi_{ij} = \sum_{j=1}^{\text{dim}(W^1_q)} \sum_{i=1}^{\text{dim}(W^1_q)} A_i(l_j) w_i
\]

Eq. 6.31 above contains the constants \( A_i(l_j) \) which serve as weights to the vector basis \( w_i \). In addition we should note that the hierarchal vector basis functions \( w_i \) could be used directly to solve electromagnetic problems [16]. However, as the polynomial order increases, the issue of matrix ill-conditioning becomes important [45, 46]. The basis functions become increasingly similar to one another, and consequently the condition number of the stiffness matrix built from them deteriorates, affecting accuracy. This problem does not seem to appear in interpolatory bases, where a high degree of linear independence seems to follow from the interpolation property. However, with hierarchal bases it is an issue that must be addressed. We can use the interpolatory functions to enforce the linear independence of the vector basis functions.
Now, we outline the procedure to construct the new basis function expansion $\psi$ that should be a combination of non-uniform interpolatory shape functions and hierarchical vector basis. The DoFs applied on normalized Lagrange polynomials $(A_i(l_j))$ form a unit matrix since $A_i(l_j) \neq 0$ only when $i = j$.

$$V_{ij} = A_i(l_j); \quad l_j \in W_q^1,$$

(6.32)

where $V_{ij}$ conveys the same idea as Vandermonde matrix [47]. To demonstrate how the Vandermonde matrix is used to build a new basis as a linear combination of some primitive basis we shall take an example of a square in 2D with four DoF.

In Fig. 6.4 the coordinates are as follows: $(x_3, y_3) = (1, 1), (x_4, y_4) = (-1, 1), (x_1, y_1) = (-1, -1), (x_2, y_2) = (1, -1)$. Next let us take $[X] = [1 \ x \ y \ xy]$ to be the primitive basis. Then we build the matrix equation:

$$[B][V] = [I],$$

(6.33)

where $[B]$ is a vector containing the unknown coefficients and $[V]$ is the Vandermonde matrix. We find:

$$[B] = [V]^{-1}$$

(6.34)

and the new basis is:

$$[L] = [X][V]^{-1}$$

(6.35)

This is a linear mapping which expresses the new basis in terms of some primitive basis and will have a rank equal to the dimension of the primitive basis. We have
to find the inverse of the Vandermonde matrix in order to enforce the unisolvency property on the new basis. Let us take a look in what the Vandermonde matrix look like. The determinant of the Vandermonde matrix has to be non-singular, otherwise it is impossible to enforce unisolvency to the new basis $\psi$:

$$V_{ij} = A_i(l_j) = \begin{vmatrix} 1 & x_1 & y_1 & x_1y_1 \\ 1 & x_2 & y_2 & x_2y_2 \\ 1 & x_3 & y_3 & x_3y_3 \\ 1 & x_4 & y_4 & x_4y_4 \end{vmatrix},$$

where $l$ is a polynomial in the form $l_j = a_1 + a_2x_j + a_3y_j + a_4x_jy_j$ and $A_i$ is linear form (DoF).

Let us now go back to the higher order edge basis functions. The primitive basis could be generated by many ways, but probably the easiest one is the tensor
product of 1D Lagrange polynomials. The Vandermonde matrix will have a form similar to the example above. The inverse of $V$ gives us the unisolvent property.

$$[L] = [X][V]^{-1}, \quad (6.36)$$

where $[X]$ is a vector containing the primitive basis used to construct the new interpolation basis at the interpolating points. Therefore, the new basis will be a linear combination of the primitive basis that spans the space $W^1_q$. Exactly the same procedure is followed to construct the vector basis $W$ at the interpolation points, so that at each interpolation point we have the numerical value given by the interpolatory basis and direction given by the hierarchical vector basis:

$$[\Psi] = [L]^T[W] \quad (6.37)$$

We have to keep in mind that all basis functions will be constructed on a master element in local coordinates, which implies that for actual problem these functions need to be mapped to the physical mesh by appropriate local to global transformation.

6.5.1 Geometry of the reference hexahedron

The first step in hierarchical higher order finite element methods (FEM) is to design a suitable reference (master) element. The reference hexahedral element, used to build the edge basis functions, is shown in Fig. 6.5. In order to make the integration over the element as simple as possible, we adopt a standard Cartesian coordinate system with origin at $(0,0,0)$.

$$\hat{\Omega} = \left\{ x, y, z \in \mathbb{R}^3; \ -1 \leq x, y, z \leq 1 \right\} \quad (6.38)$$
Throughout the remaining sections this is going to be the coordinate system that we shall use. The topology of the reference element is determined by 8 vertices, 12 edges, 6 faces and the volume. The totality of these “nodes” form a discrete manifold. The connectivity that determines a positive orientation of edges and faces is given in Table 6.3 and Table 6.4.

![Figure 6.5: The reference hexahedron.](image)

**6.5.2 Polynomial space $W_q^1$**

In this section we define the polynomial space $W_q^1$ that will be used to construct the 1-form basis functions. We refer to DeRham’s complex when needed, since the DeRham complex serves as a road map in the construction. In order for a polynomial space to be well defined, it has to model properly the behaviour of the exterior differential differential
Table 6.3: Local connectivity of the edges with positive orientation.

<table>
<thead>
<tr>
<th>Edge ID</th>
<th>Vertices</th>
<th>Edge ID</th>
<th>Vertices</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$P_1P_3$</td>
<td>7</td>
<td>$P_2P_3$</td>
</tr>
<tr>
<td>2</td>
<td>$P_4P_5$</td>
<td>8</td>
<td>$P_6P_7$</td>
</tr>
<tr>
<td>3</td>
<td>$P_2P_6$</td>
<td>9</td>
<td>$P_1P_2$</td>
</tr>
<tr>
<td>4</td>
<td>$P_3P_7$</td>
<td>10</td>
<td>$P_4P_3$</td>
</tr>
<tr>
<td>5</td>
<td>$P_1P_4$</td>
<td>11</td>
<td>$P_5P_6$</td>
</tr>
<tr>
<td>6</td>
<td>$P_5P_8$</td>
<td>12</td>
<td>$P_8P_7$</td>
</tr>
</tbody>
</table>

operator. In other words, the discrete space should model correctly the image and the kernel of the operator. We should note here that a well defined polynomial space by itself is not sufficient to construct basis functions. To build concrete basis functions, the polynomial space is used in conjunction with the degrees of freedom. For example this space among some others were originally proposed by Nedelec [6] but no basis function were reported in his work. The polynomial space for 1-forms on a hexahedral element, described in [6] and its dimension is:

$$W_q^1(\hat{\Omega}) = Z_p^1 \oplus Y_p^1 = Q_{q-1,q,q} \times Q_{q,q-1,q} \times Q_{q,q,q-1},$$

(6.39)

$$\dim(W_q^1(\hat{\Omega})) = 3q(q + 1)(q + 1)$$

(6.40)

In order to ensure the proper conformity across element to element interfaces, it is crucial that the basis functions (and consequently the degrees of freedom) be as-
Table 6.4: Local connectivity of the faces.

<table>
<thead>
<tr>
<th>Face ID</th>
<th>Vertices</th>
<th>Local Orientation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$P_1P_2P_3P_4$</td>
<td>$+z$</td>
</tr>
<tr>
<td>1</td>
<td>$P_5P_6P_7P_8$</td>
<td>$+z$</td>
</tr>
<tr>
<td>1</td>
<td>$P_1P_2P_6P_5$</td>
<td>$-y$</td>
</tr>
<tr>
<td>1</td>
<td>$P_4P_3P_7P_8$</td>
<td>$-y$</td>
</tr>
<tr>
<td>1</td>
<td>$P_1P_4P_8P_5$</td>
<td>$+x$</td>
</tr>
<tr>
<td>1</td>
<td>$P_2P_3P_7P_6$</td>
<td>$+x$</td>
</tr>
</tbody>
</table>

associated with the various sub-simplices of the element (nodes, edges, faces). This property is referred to as locality [24]. For 1-forms, locality implies that the edge basis functions should have non-vanishing tangential components only along one edge. The face basis functions will have non-vanishing tangential components along only one face with no tangential components along any of the edges. Finally, the volume basis functions will have no tangential components along either of the edges or faces.

The total number of DoF is equal to the dimension of the space. The DoF, based on the locality property, are distributed on the various subsimplices as follows: $3x4q$ DoF on the edges; $6x2q(q-1)$ DoF on the faces; and $3q(q-1)(q-1)$ in the volume of the element. Some tabulated degrees of freedom are presented in the Table 6.5. More details can be found in [27].

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Table 6.5: 1-form DoFs associated with the various simplices of the reference hexahedron for polynomial orders $q=1\ldots5$.

<table>
<thead>
<tr>
<th>Degrees of Freedom</th>
<th>for $q=1$</th>
<th>for $q=2$</th>
<th>for $q=3$</th>
<th>for $q=4$</th>
<th>for $q=5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>On the edges</td>
<td>12</td>
<td>24</td>
<td>36</td>
<td>48</td>
<td>60</td>
</tr>
<tr>
<td>On the faces</td>
<td>0</td>
<td>24</td>
<td>72</td>
<td>144</td>
<td>240</td>
</tr>
<tr>
<td>In the volume</td>
<td>0</td>
<td>6</td>
<td>36</td>
<td>108</td>
<td>240</td>
</tr>
<tr>
<td>total</td>
<td>12</td>
<td>54</td>
<td>144</td>
<td>300</td>
<td>540</td>
</tr>
</tbody>
</table>

Finally, it has been pointed out in [48] that $dW_1^1 \in W_q^2$ and $dW_0^0 \in W_q^1$. Indeed

$$dW_1^1(\hat{\Omega}) = Q_{q,q-1,q-1} \times Q_{q-1,q-1,q} \times Q_{q-1,q-1,q} \in dW_q^2(\hat{\Omega})$$ (6.41)

$$dW_0^0(\hat{\Omega}) = Q_{q,q,q} = Q_{q-1,q,q} \times Q_{q,q-1,q} \times Q_{q,q,q-1} \in dW_q^1(\hat{\Omega})$$ (6.42)

This is required to satisfy the DeRham’s exact sequence property, discussed earlier.

6.5.3 Interpolatory 1-form basis functions $L$

1. Lagrange interpolation polynomials. The element interpolation functions can be conveniently identified by the Lagrange $q$-th order interpolating polynomials.

We first describe them on a 1D domain $-1 \leq x \leq 1$. The general form of $q$-th order Lagrange interpolating functions in 1D is:

$$l_i(x) = \frac{(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_{q+1})}{(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_{q+1})}$$ (6.43)
For \( i = 1, \ldots, q + 1 \), the \( l_i \) polynomials satisfy the cardinal interpolation property:

\[
l_i(x_j) = A_i(x_j) = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j
\end{cases}
\]  

(6.44)

where \( A_i \) are linear forms described by Eq. 6.27. A function of interest \( \psi(x) \) defined over the 1D domain can be approximated in terms of specified nodal values at the interpolatory DoF and Lagrange polynomials by:

\[
\psi(x) = \sum_{i=1}^{q+1} l_i(x)w(x_i) = \sum_{i=1}^{q+1} \delta_{ij}w(x_i) = w(x_j),
\]

(6.45)

where \( x_j \) is the \( j \)-th DoF on the reference domain.

2. Non-uniform distribution of the interpolation points. It is well known that the choice of the interpolation points in Lagrange polynomials is very important. Evenly spaced points are notoriously bad from both conditioning and interpolation point of view. This fact is closely related to an object from the interpolation theory, called Lebesgue constant defined by:

\[
\bigwedge(X) = \max_{x \in [-1,1]} \sum_{i=1}^{q+1} |L_i^q(x, X)|, 
\]

(6.46)

where \( X = \{x_1, \ldots, x_{q+1}\} \) is a set of interpolation points. As one can see, the Lebesgue constant depends only on the choice of interpolation points \( X \) and it grows with the increase of the order \( q \). The proper choice of points determines exactly how fast it will grow. It turns out that evenly spaced points is the worst case scenario with exponential growth of the condition number of the resulting stiffness matrices, as a function of \( q \) [46]. The best case would be achieved with
specially generated sets of points, non-uniformly distributed over the reference domain. Using Chebishev points is one way to work around this problem. They are generated by:

\[ x_i = \cos \left( \frac{\pi(i - 1)}{q} \right), \quad i = 1, 2, \ldots, q + 1 \]  

(6.47)

Another approach is to use Gauss-Lobatto points, which are the roots of:

\[ (1 - x^2)\mathcal{L}_q'(x), \]  

(6.48)

where \( \mathcal{L}_q(x) \) is the q-th Legendre polynomial. For our purposes we are going to use the Chebishev points. To give a visual idea for the origin of the problem we show in Fig. 6.6 the Lagrange polynomials for q=4 with evenly spaced interpolation points.

Figure 6.6: Lagrange interpolation using low order polynomials (q=4) with evenly spaced interpolation points.
the polynomial order, say $q=10$ (see Fig. 6.7) with evenly spaced points, some oscillations occur near the end of the interval. This is known as the Runge effect. One can see from the plots that unless we use low order polynomials for the interpolation, the evenly spaced point may cause problems with accuracy. In Fig. 6.8 in comparison we show the plot of Lagrange polynomials of the same order as in Fig. 6.7 but using Chebishev points.

![Figure 6.7](image)

Figure 6.7: Lagrange interpolation using higher order polynomials $q=10$ with evenly spaced interpolation points.

3. Explicit expressions for the interpolation basis. In this section we define the explicit formulae for the interpolatory basis functions. We shall use tensor products of 1D Lagrange polynomials which we studied above. Since our reference domain is the reference hexahedron in $\mathbb{R}^3$, we first need to generate the interpolation points using the Chebishev formula (Eq. 6.47). The aim is to
Figure 6.8: Lagrange interpolation using higher order polynomials $q=10$ with non-uniform Chebishev interpolation points.

keep the Lebesgue constant, as a function of $q$, small.

Some example sets of points are:

$$[1 - 1] q = 1$$

$$[1 0 - 1] q = 2$$

$$[1 0.5 0.5 - 1] q = 3$$

$$[1 0.7 0 - 0.7 - 1] q = 4$$

Next using the Chebishev points for the corresponding order we build the 1D Lagrange polynomials. Let us take for $q=1$

$$l_0(x_0) = \frac{x + 1}{2} \quad (6.49)$$

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\[ l_1^1(x_1) = \frac{1 - x}{2} \quad (6.50) \]

and for \( q = 2 \)

\[ l_0^2(x_0) = \frac{x(x + 1)}{2} \quad (6.51) \]

\[ l_1^2(x_1) = (1 - x)(x + 1) \quad (6.52) \]

\[ l_2^2(x_2) = \frac{x(x - 1)}{2} \quad (6.53) \]

In terms of notation, we use \( l_2^2 \) to denote the Lagrange second order polynomial that interpolates the third point from the interpolation points (2+1 points). Usually the interpolation functions are of reduced gradient kind, which are a preferable choice for electromagnetic problems and especially when we deal with waveguiding structures and 3D resonant cavities. The use of complete polynomial functions may lead to spurious modes (non-physical solutions) and incorrect results [49]. The nature the interpolation functions provides good linear independence in the space (hexahedron). This implies that the interpolatory basis functions are members of the full polynomial space \( W^1_q \) and therefore are of maximum polynomial degree. The interpolatory functions may have the following general form: On the edges

\[
L^x_e = \sum_{i=0}^{q-1} l_0^q(y)l_0^q(z)l_i^{q-1}(x)\hat{x} + \sum_{i=0}^{q-1} l_0^q(y)l_0^q(z)l_i^{q-1}(x)\hat{x} + \sum_{i=0}^{q-1} l_0^q(y)l_0^q(z)l_i^{q-1}(x)\hat{x} + \sum_{i=0}^{q-1} l_0^q(y)l_0^q(z)l_i^{q-1}(x)\hat{x} \quad (6.54)
\]

\[
L^y_e = \sum_{i=0}^{q-1} l_0^q(x)l_0^q(z)l_i^{q-1}(y)\hat{y} + \sum_{i=0}^{q-1} l_0^q(x)l_0^q(z)l_i^{q-1}(y)\hat{y} + \sum_{i=0}^{q-1} l_0^q(x)l_0^q(z)l_i^{q-1}(y)\hat{y} + \sum_{i=0}^{q-1} l_0^q(x)l_0^q(z)l_i^{q-1}(y)\hat{y}
\]
There are four edges in each direction and \( q \) (from 0 to \( q-1 \)) basis functions for each edge (The total number of DoFs associated with edges is determined as follows: 3 spatial directions times four edges per direction times \( q \) basis functions per edge equals 3x4xq). On the faces

\[
L^x_i = \sum_{j=1}^{q-1} \sum_{i=0}^{q-1} l^q_0(x) l^q_0(y) l^q_t(z) l^q_t^{-1}(x) \hat{x} + \sum_{j=1}^{q-1} \sum_{i=0}^{q-1} l^q_0(x) l^q_0(y) l^q_t(z) l^q_t^{-1}(x) \hat{x} +
\]

\[
L^y_j = \sum_{j=1}^{q-1} \sum_{i=0}^{q-1} l^q_0(x) l^q_0(y) l^q_t(z) l^q_t^{-1}(y) \hat{y} + \sum_{j=1}^{q-1} \sum_{i=0}^{q-1} l^q_0(x) l^q_0(y) l^q_t(z) l^q_t^{-1}(y) \hat{y} +
\]

\[
L^z_j = \sum_{j=1}^{q-1} \sum_{i=0}^{q-1} l^q_0(x) l^q_0(y) l^q_t(z) l^q_t^{-1}(z) \hat{z} + \sum_{j=1}^{q-1} \sum_{i=0}^{q-1} l^q_0(x) l^q_0(y) l^q_t(z) l^q_t^{-1}(z) \hat{z} +
\]

\[
L_f = L^x_i \cup L^y_j \cup L^z_j
\]
There are two parallel faces normal to each spatial direction and $2q(q-1)$ basis functions on each face, resulting in a total of $2 \times 3 \times 2q(q-1)$ DoFs associated with the faces of the element. Finally, in the volume of the master hexahedron

$$L_v = \sum_{i=1}^{q-1} \sum_{j=1}^{q-1} \sum_{k=0}^{q-1} l_i^q(z) l_j^q(y) l_k^{q-1}(x) \hat{x} + \sum_{i=1}^{q-1} \sum_{j=1}^{q-1} \sum_{k=0}^{q-1} l_i^q(x) l_j^q(z) l_k^{q-1}(y) \hat{y} + \sum_{i=1}^{q-1} \sum_{j=1}^{q-1} \sum_{k=0}^{q-1} l_i^q(x) l_j^q(y) l_k^{q-1}(z) \hat{z}$$

(6.62)

There are $q(q-1)(q-1)$ basis function for each spatial direction (the total number of shape functions associated with DoFs in the volume of the element can be calculated by $3 \times q(q-1)(q-1)$)

$$L = L_e \cup L_f \cup L_v$$

(6.63)

### 6.5.4 Hierarchical 1-form vector basis functions $W$

To build the hierarchical 1-form vector basis functions we use the Legendre polynomials which are defined in 1D in the reference domain [-1,1]. Plot of the first few orders are shown in Fig. 6.9.

$$W^x_e = \sum_{i=0}^{q-1} l_0^1(y) l_0^1(z) \mathcal{L}_i(x) \hat{x} + \sum_{i=0}^{q-1} l_0^1(y) l_1^1(z) \mathcal{L}_i(x) \hat{x} + \sum_{i=0}^{q-1} l_1^1(y) l_0^1(z) \mathcal{L}_i(x) \hat{x} + \sum_{i=0}^{q-1} l_1^1(y) l_1^1(z) \mathcal{L}_i(x) \hat{x}$$

(6.64)

$$W^y_e = \sum_{i=0}^{q-1} l_0^1(x) l_0^1(z) \mathcal{L}_i(y) \hat{y} + \sum_{i=0}^{q-1} l_0^1(x) l_1^1(z) \mathcal{L}_i(y) \hat{y} + \sum_{i=0}^{q-1} l_1^1(x) l_0^1(z) \mathcal{L}_i(y) \hat{y} + \sum_{i=0}^{q-1} l_1^1(x) l_1^1(z) \mathcal{L}_i(y) \hat{y}$$

(6.65)
Figure 6.9: Legendre polynomials of order $q=0 \ldots 4$.

\[
W^z_e = \sum_{i=0}^{q-1} l_0^i(x) l_0^i(y) L_i(z) \hat{z} + \sum_{i=0}^{q-1} l_0^i(x) l_1^i(y) L_i(z) \hat{z} + \\
\sum_{i=0}^{q-1} l_1^i(x) l_0^i(y) L_i(z) \hat{z} + \sum_{i=0}^{q-1} l_1^i(x) l_1^i(y) L_i(z) \hat{z} \quad (6.66)
\]

\[
W_e = W^x_e \cup W^y_e \cup W^z_e \quad (6.67)
\]

The subset $W_e$ spans the subspace $\{Q_{q-1,1,1} \times Q_{1,q-1,1} \times Q_{1,1,q-1}\}$. On the faces

\[
W^x_f = \sum_{i=0}^{q-1} \sum_{j=0}^{q-2} l_0^i(y) L_j(z) l_1^j(x) \hat{x} + \sum_{i=0}^{q-1} \sum_{j=0}^{q-2} l_1^i(y) L_j(z) l_1^j(x) \hat{x} + \\
\sum_{i=0}^{q-1} \sum_{j=0}^{q-2} l_0^i(z) L_j(y) l_1^j(x) \hat{x} + \sum_{i=0}^{q-1} \sum_{j=0}^{q-2} l_1^i(z) L_j(y) l_1^j(x) \hat{x} \quad (6.68)
\]

\[
W^y_f = \sum_{i=0}^{q-1} \sum_{j=0}^{q-2} l_0^i(x) L_j(z) l_1^j(y) \hat{y} + \sum_{i=0}^{q-1} \sum_{j=0}^{q-2} l_1^i(x) L_j(z) l_1^j(y) \hat{y} + \\
\sum_{i=0}^{q-1} \sum_{j=0}^{q-2} l_0^i(x) L_j(y) l_1^j(z) \hat{y} + \sum_{i=0}^{q-1} \sum_{j=0}^{q-2} l_1^i(x) L_j(y) l_1^j(z) \hat{y} + \\
\sum_{i=0}^{q-1} \sum_{j=0}^{q-2} l_0^i(y) L_j(x) l_1^j(z) \hat{y} + \sum_{i=0}^{q-1} \sum_{j=0}^{q-2} l_1^i(y) L_j(x) l_1^j(z) \hat{y} + \\
\sum_{i=0}^{q-1} \sum_{j=0}^{q-2} l_0^i(y) L_j(x) l_1^j(y) \hat{y} + \sum_{i=0}^{q-1} \sum_{j=0}^{q-2} l_1^i(y) L_j(x) l_1^j(y) \hat{y} +
\]

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\[ \sum_{i=0}^{q-1} \sum_{j=0}^{q-2} l_i^0(z) \mathcal{L}_j(z) l_j^2(x) \mathcal{L}_i(y) \hat{y} + \sum_{i=0}^{q-1} \sum_{j=0}^{q-1} l_i^1(z) \mathcal{L}_j(z) l_j^2(x) \mathcal{L}_i(y) \hat{y} \] (6.69)

\[ W_f^i = \sum_{i=0}^{q-1} \sum_{j=0}^{q-2} l_i^0(y) \mathcal{L}_j(x) l_j^2(y) \mathcal{L}_i(z) \hat{z} + \sum_{i=0}^{q-1} \sum_{j=0}^{q-1} l_i^1(y) \mathcal{L}_j(z) l_j^2(y) \mathcal{L}_i(z) \hat{z} \] (6.70)

\[ W_f = W_f^x \cup W_f^y \cup W_f^z \] (6.71)

and finally in the volume of the master hexahedron

\[ W_v = \sum_{i=0}^{q-2} \sum_{j=0}^{q-2} \sum_{k=0}^{q-1} \mathcal{L}_i(y) l_j^2(y) \mathcal{L}_j(z) l_k^2(z) \mathcal{L}_k(x) \hat{x} + \sum_{i=0}^{q-2} \sum_{j=0}^{q-2} \sum_{k=0}^{q-1} \mathcal{L}_i(x) l_j^2(x) \mathcal{L}_j(z) l_k^2(z) \mathcal{L}_k(y) \hat{y} + \sum_{i=0}^{q-2} \sum_{j=0}^{q-2} \sum_{k=0}^{q-1} \mathcal{L}_i(x) l_j^2(x) \mathcal{L}_j(y) l_k^2(y) \mathcal{L}_k(z) \hat{z} \] (6.72)

\[ W = W_e \cup W_f \cup W_v \] (6.73)

And finally the basis functions \( \Psi = \sum_{i=1}^{N} L_i W_i \)

6.6 Conclusion and future work

In the present thesis arbitrary high order hierarchical 1-form basis functions were constructed on a reference hexahedron, using the theory of differential forms, unlike the constructions in [28, 16, 15, 20], where the construction is done in terms of vectors. Since the mathematical background required includes notions from various branches of mathematics, such as exterior algebra, exterior calculus, differentiable
topologies, and discrete geometry, before the actual construction we provide complete mathematical introduction. By design, the 1-form basis functions fit in the discrete polynomial space $W_q^1$. This space contains all polynomials of order $q$ that are supposed to model the continuous Hilbert space $H(curl, \Omega)$, as defined in [27]. It is important to mention again that the Hilbert space $H(curl, \Omega)$, requires only some combinations of first partial derivatives not to be equal to zero, while the rest first partial derivatives are allowed to be equal to zero. In comparison, the Sobolev space $H^1(\Omega)$, which is modeled by $W_q^0$ in scalar valued FEM, requires all first order partial derivatives to differ from zero. Further on, the 1-form basis functions in this thesis were designed to model the null space and the range space of the exterior differential operator $d$ and the mean to calculate the dimensions of these subspaces as a function of $q$, is provided in [27]. In addition due to their nature, the 1-form basis functions that we constructed, match the continuity conditions of the electric and magnetic field intensities on the interface between two materials, allowing the normal component of the fields to be discontinuous.

It is important, however, to point out here that Demkowicz in his work has developed spaces that generalize the Nedelec’s discrete spaces $W_q^0$, $W_q^1$, and $W_q^2$. In the spaces proposed by Nedelec all the basis functions are of uniform polynomial order, while the interpolation functions in Demkowicz’s work are of variable order. So the Nedelec spaces are a particular case of the spaces described by Demkowich. Another important point to mention again here is the pedagogical difference in the construction of high order finite elements - the use of differential forms. This provides
many advantages over the standard vector based formulation. The use of exterior calculus is quite useful in developing of high order elements and future research in the context of the work of Demkowicz for hp-elements might be an interesting direction to explore.

There are many directions that can be taken from here for future work. Since the construction of the 1-form basis functions is on a reference hexahedron in local coordinates, as a natural next step would be a construction of maps from a local to a global coordinate system. In other words, we need to map the reference hexahedron to a physical mesh. We already showed that the exterior differential operator is independent of the coordinate system and such map should be done by using the Jacobian of the transformation.

Another aspect of future work should be the actual implementation of the basis functions in Matlab or C++ for instance. Further on, additional study should be done of the condition number of the resulting mass and stiffness matrices. This is important, since in high order p-elements the Runge effect is becoming an issue. Investigation with different sets of non uniformly distributed points should be done. Also, it would be interesting to compare the interpolation presented in [15] and the interpolation using various non-uniformly distributed points. Additional study is required for establishing the rate of convergence and error estimation of the presented 1-form basis functions.
BIBLIOGRAPHY


APPENDIX

A.0.1 Hodge operator

*Example:* Let us find \(*e_1\) in \(V^2(\mathbb{R}^3)\), where \(e_1\) is a 1-vector in \(V^1(\mathbb{R}^3)\). This must be a 2-vector and hence must have the form

\[ *e_1 = \lambda_1(e_1 \wedge e_2) + \lambda_2(e_1 \wedge e_3) + \lambda_3(e_2 \wedge e_3) \]

We have to find \(\lambda_1, \lambda_2, \text{ and } \lambda_3\). We calculate \((*e_1, e_1 \wedge e_2) = \lambda_1\).

Next we write: \(\lambda_1(e_1 \wedge e_2 \wedge e_3) = (*e_1, e_1 \wedge e_2)e_1 \wedge e_2 \wedge e_3 = e_1 \wedge (e_1 \wedge e_2) = 0\)

\(\lambda_1 = 0\)

Similarly

\(\lambda_2(e_1 \wedge e_2 \wedge e_3) = (*e_1, e_1 \wedge e_3)e_1 \wedge e_2 \wedge e_3 = e_1 \wedge (e_1 \wedge e_3) = 0\)

\(\lambda_2 = 0\)

And

\(\lambda_3(e_1 \wedge e_2 \wedge e_3) = (*e_1, e_2 \wedge e_3)e_1 \wedge e_2 \wedge e_3 = e_1 \wedge (e_2 \wedge e_3) = 0\)

\(\lambda_3 = 1\)

Finally, we obtain

\[ *e_1 = e_2 \wedge e_3 \]
A.0.2 Second order vector basis functions

All interpolatory basis functions in the case $q=2$ are listed below.

\[
\begin{align*}
\mathbf{L}_1 &= -\frac{(x-1)(y-1)yz}{8} \\
\mathbf{L}_2 &= \frac{(x+1)(y-1)yz}{8} \\
\mathbf{L}_3 &= \frac{(x+1)(y-1)yz(z+1)}{8} \\
\mathbf{L}_4 &= -\frac{(x-1)(y-1)y(z+1)}{8} \\
\mathbf{L}_5 &= -\frac{(x-1)(y+1)(z-1)z}{8} \\
\mathbf{L}_6 &= \frac{(x+1)(y+1)(z-1)z}{8} \\
\mathbf{L}_7 &= \frac{(x+1)(y+1)yz(z+1)}{8} \\
\mathbf{L}_8 &= -\frac{(x-1)(y+1)y(z+1)}{8} \\
\mathbf{L}_9 &= \frac{(x-1)(y-1)(y+1)(z-1)z}{4} \\
\mathbf{L}_{10} &= \frac{(x-1)(y-1)y(z-1)(z+1)}{4} \\
\mathbf{L}_{11} &= \frac{(x-1)(y-1)y(z+1)(z+1)}{4} \\
\mathbf{L}_{12} &= \frac{(x-1)(y+1)(y-1)(z-1)(z+1)}{4} \\
\mathbf{L}_{13} &= -\frac{(x+1)(y-1)(y+1)(z-1)z}{4} \\
\mathbf{L}_{14} &= -\frac{(x+1)(y+1)(z-1)(z+1)}{4} \\
\mathbf{L}_{15} &= -\frac{(x+1)(y-1)(y+1)yz(z+1)}{4} \\
\mathbf{L}_{16} &= -\frac{(x+1)(y-1)y(z-1)(z+1)}{4} \\
\mathbf{L}_{17} &= \frac{(x-1)(y-1)(y+1)(z-1)(z+1)}{2} \\
\mathbf{L}_{18} &= \frac{(x+1)(y-1)(y+1)(z-1)(z+1)}{2} \\
\mathbf{L}_{19} &= -\frac{(x-1)x(y-1)(z-1)z}{8} \\
\mathbf{L}_{20} &= -\frac{(x-1)x(y-1)z(z+1)}{8} \\
\mathbf{L}_{21} &= -\frac{(x-1)x(y+1)z(z+1)}{8} \\
\mathbf{L}_{22} &= \frac{(x+1)x(y+1)(z-1)z}{8} \\
\mathbf{L}_{23} &= -\frac{x(x+1)(y-1)(z-1)z}{8} \\
\mathbf{L}_{24} &= \frac{x(x+1)(y+1)(z-1)z}{8} \\
\mathbf{L}_{25} &= \frac{x(x+1)(y+1)z(z+1)}{8} \\
\mathbf{L}_{26} &= -\frac{x(x+1)(y+1)z(z+1)}{8} \\
\mathbf{L}_{27} &= \frac{(x-1)(x+1)(y-1)(z-1)z}{4} \\
\mathbf{L}_{28} &= \frac{(x-1)x(y-1)(z-1)(z+1)}{4} \\
\mathbf{L}_{29} &= \frac{(x-1)(x+1)(y-1)z(z+1)}{4} \\
\mathbf{L}_{30} &= \frac{x(x+1)(y-1)(z-1)(z+1)}{4} \\
\mathbf{L}_{31} &= -\frac{(x-1)(x+1)(y+1)(z-1)z}{4} \\
\mathbf{L}_{32} &= -\frac{(x-1)x(y+1)(z-1)(z+1)}{4} \\
\mathbf{L}_{33} &= -\frac{x(x+1)(y+1)(z-1)(z+1)}{4} \\
\mathbf{L}_{34} &= -\frac{x(x+1)(y+1)z(z+1)}{4} \\
\mathbf{L}_{35} &= -\frac{(x-1)(x+1)(y-1)(z-1)(z+1)}{2} \\
\mathbf{L}_{36} &= \frac{(x-1)(x+1)(y+1)(z-1)(z+1)}{2}
\end{align*}
\]
\[
\begin{align*}
L_{37} &= -\frac{(x-1)xy(y-1)(z-1)}{8} \\
L_{38} &= \frac{(x-1)xy(y+1)(z+1)}{8} \\
L_{39} &= \frac{(x-1)xy(y+1)(z+1)}{8} \\
L_{40} &= -\frac{(x-1)xy(y+1)(z-1)}{8} \\
L_{41} &= -\frac{x(x+1)(y-1)y(z-1)}{8} \\
L_{42} &= -\frac{x(x+1)xy(y+1)(z-1)}{8} \\
L_{43} &= \frac{x(x+1)xy(y+1)(z+1)}{8} \\
L_{44} &= \frac{x(x+1)(y-1)y(z+1)}{8} \\
L_{45} &= \frac{(x-1)xy(y-1)(y+1)(z-1)}{4} \\
L_{46} &= \frac{(x-1)(x+1)xy(y+1)(z-1)}{4} \\
L_{47} &= \frac{x(x+1)(y-1)(y+1)(z-1)}{4} \\
L_{48} &= \frac{(x-1)(x+1)(y-1)y(z-1)}{4} \\
L_{49} &= -\frac{(x-1)xy(y-1)(y+1)(z+1)}{4} \\
L_{50} &= -\frac{(x-1)(x+1)xy(y+1)(z+1)}{4} \\
L_{51} &= -\frac{x(x+1)(y-1)(y+1)(z+1)}{4} \\
L_{52} &= -\frac{(x-1)(x+1)(y-1)y(z+1)}{4} \\
L_{53} &= -\frac{(x-1)(x+1)(y-1)(y+1)(z-1)}{2} \\
L_{54} &= \frac{(x-1)(x+1)(y-1)(y+1)(z+1)}{2}
\end{align*}
\]
All the hierarchical vector basis functions for \( q=2 \) are listed below

\[
\begin{align*}
W_1 &= \frac{y+1}{2} \frac{z+1}{2} \hat{l} \\
W_2 &= \frac{y+1}{2} \frac{z+1}{2} x \\
W_3 &= \frac{y+1}{2} \frac{1-z}{2} \hat{l} \\
W_4 &= \frac{y+1}{2} \frac{1-z}{2} x \\
W_5 &= \frac{1-y}{2} \frac{z+1}{2} \hat{l} \\
W_6 &= \frac{1-y}{2} \frac{z+1}{2} x \\
W_7 &= \frac{1-y}{2} \frac{1-z}{2} \hat{l} \\
W_8 &= \frac{1-y}{2} \frac{1-z}{2} x \\
W_{19} &= \frac{x+1}{2} \frac{z+1}{2} \hat{l} \\
W_{20} &= \frac{x+1}{2} \frac{z+1}{2} y \\
W_{21} &= \frac{x+1}{2} \frac{1-z}{2} \hat{l} \\
W_{22} &= \frac{x+1}{2} \frac{1-z}{2} y \\
W_{23} &= \frac{1-x}{2} \frac{z+1}{2} \hat{l} \\
W_{24} &= \frac{1-x}{2} \frac{z+1}{2} y \\
W_{25} &= \frac{1-x}{2} \frac{1-z}{2} \hat{l} \\
W_{26} &= \frac{1-x}{2} \frac{1-z}{2} y \\
W_{27} &= (z+1)(1-z) \frac{1}{2} \frac{1}{2} \hat{l} \\
W_{28} &= (z+1)(1-z) \frac{1}{2} \frac{1}{2} y \\
W_{29} &= (z+1)(1-z) \frac{1}{2} \frac{1}{2} \hat{l} \\
W_{30} &= (z+1)(1-z) \frac{1}{2} \frac{1}{2} y \\
W_{31} &= (x+1)(1-x) \frac{1}{2} \frac{1}{2} \hat{l} \\
W_{32} &= (x+1)(1-x) \frac{1}{2} \frac{1}{2} y \\
W_{33} &= (x+1)(1-x) \frac{1}{2} \frac{1}{2} \hat{l} \\
W_{34} &= (x+1)(1-x) \frac{1}{2} \frac{1}{2} y \\
W_{35} &= 1(x+1)(1-x) \frac{1}{2} \frac{1}{2} \hat{l} \\
W_{36} &= 1(x+1)(1-x) \frac{1}{2} \frac{1}{2} y \\
\end{align*}
\]
\[
\begin{pmatrix}
W_{37} = \frac{x+1}{2} \frac{y+1}{2} \frac{1}{2} \\
W_{38} = \frac{x+1}{2} \frac{y+1}{2} z \\
W_{39} = \frac{x+1}{2} \frac{1-y}{2} 1 \\
W_{40} = \frac{x+1}{2} \frac{1-y}{2} z \\
W_{41} = \frac{1-x}{2} \frac{y+1}{2} 1 \\
W_{42} = \frac{1-x}{2} \frac{y+1}{2} z \\
W_{43} = \frac{1-x}{2} \frac{1-y}{2} 1 \\
W_{44} = \frac{1-x}{2} \frac{1-y}{2} z \\
W_{45} = (y+1)(1-y)1 \frac{x+1}{2} 1 \\
W_{46} = (y+1)(1-y)1 \frac{x+1}{2} z \\
W_{47} = (y+1)(1-y)1 \frac{1-x}{2} 1 \\
W_{48} = (y+1)(1-y)1 \frac{1-x}{2} z \\
W_{49} = (x+1)(1-x)1 \frac{y+1}{2} 1 \\
W_{50} = (x+1)(1-x)1 \frac{y+1}{2} z \\
W_{51} = (x+1)(1-x)1 \frac{1-y}{2} 1 \\
W_{52} = (x+1)(1-x)1 \frac{1-y}{2} z \\
W_{53} = 1(y+1)(1-y)1(x+1)(1-x)1 \\
W_{54} = 1(y+1)(1-y)1(x+1)(1-x)z \\
\end{pmatrix}
\]
Figure A.1: Example of low order interpolatory basis functions associated with a DoFs on an edge of the hexahedron for $q=2$ polynomial order

Figure A.2: Example of low order interpolatory basis functions associated with a DoFs on a face of the hexahedron for $q=2$ polynomial order
Figure A.3: Example of low order interpolatory basis functions associated with a DoFs in the volume of the hexahedron for q=2 polynomial order

Figure A.4: Example of low order hierarchical vector basis functions associated with a DoFs in the volume of the hexahedron for q=2
Figure A.5: Example of low order hierarchical vector basis functions associated with a DoFs on an edge of the hexahedron for $q=1$

Figure A.6: Example of low order hierarchical vector basis functions associated with a DoFs on an edge of the hexahedron for $q=2$
Figure A.7: Example of low order hierarchical vector basis functions associated with a DoFs on a face of the hexahedron for q=1

Figure A.8: Example of low order hierarchical vector basis functions associated with a DoFs on a face of the hexahedron for q=2