A THREE SECTOR, INTEGRATED APPROACH TO ECONOMIC GROWTH MODELING: PRODUCTION, HUMAN CAPITAL, & HEALTH EDUCATION

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Joseph James Tucker

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A THREE SECTOR, INTEGRATED APPROACH TO ECONOMIC GROWTH MODELING: PRODUCTION, HUMAN CAPITAL, & HEALTH EDUCATION

Joseph James Tucker

Thesis

Approved: 

Advisor
Dr. Gerald Young

Co-Advisor
Dr. Curtis Clemons

Co-Advisor
Dr. Katherine Sheppard

Department Chair
Dr. Joseph Wilder

Accepted: 

Dean of the College
Dr. Ronald F. Levant

Dean of the Graduate School
Dr. George R. Newkome

Date
ABSTRACT

The intent of this thesis is to model the interactions and interpret the ramifications of time invested in health education on the economy of a developing country. By constructing a three-sector economy, consisting of production, production education, and health education and by employing a Hamiltonian maximization technique, the aim of this thesis is to determine daily, non-leisure time allotments that encourage economic growth as well as delineate the time allotments that permit economic stagnation. This thesis presents an integrated approach to economic growth modeling by utilizing non-linear output modeling, incorporating logistic equation based technological, life-span, and productivity growth equations, and by combining inputs such as capital and life expectancy in order to yield one output. The interpretations that this model yields give insight pertaining to questions concerning foreign aid and international investment. The resulting conclusions indicate the validity of time spent in health education for increasing economic growth, given sufficient initial conditions, namely that productivity and human skill level are adequately high enough to maintain capital output despite less time being spent in production.
ACKNOWLEDGEMENTS

This being one of the more arduous undertakings I have ever endeavored upon, it is not without the help of a great many that I have succeeded. Thus it is with the utmost appreciation and sincerest of gratitude that I offer my acknowledgments and thanks.

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CHAPTER I

INTRODUCTION

1.1 Overview

In introductory economic courses one of the principle queries is, ”Why are the rich so rich and the poor so poor?” However it is not until higher academic levels that any reasonable speculation is given into the matter. In economic terms poor and rich are defined by the level of GDP per capita in a country, as a country grows that level increases over time. GDP stands for Gross Domestic Product and is essentially a measure of how much output a country creates and uses. But what causes this growth; what factors determine how fast or slow a country’s economy will grow? There is a wide spectrum of models and theories to answer this economic enigma.

This thesis examines the existing models of economic growth, evaluates their efficiencies and inadequacies, integrates a life expectancy term that more effectively models growth in accordance with economic theory into existing models, formulates and analyzes the governing equations of this new model. As will be shown in Section 1.2, the main elements of economic growth, human and physical capital and technology, have been modeled in a variety of ways. The motivation of this thesis and the basis
for adding life expectancy is derived from an investigation to validate whether international health aid and education are economically viable and moreover profitable endeavors for investors, non-profit organizations, government aid, private enterprises, and other groups that have been actively involved in spurring growth in developing countries.

There is a vast amount of literature dedicated to economic growth, both in mathematical and economic writing; thus a literature survey is conducted to identify existing models. These models will be evaluated, critiqued, and any changes that this thesis proposes will be expounded upon in Section 1.2. Likewise, existing models that implement life expectancy and health education will be presented in Section 1.3, as will the resulting differences between this modeling and theirs. In order to succinctly present the analysis this model yields, some assumptions must be made; the assumptions, their economic interpretations, and their consequential limitations will be denoted in Section 1.4. An exhaustive list of parameters is displayed in Section 1.5. The original model is presented in Chapter 2 followed by a transformed version of the model in Section 2.2. The governing equations for the original model are formed in Chapter 3.

In order to understand the real-world interpretations of the equations and to continually check that these interpretations are congruent with existing economic theory, we allot sections dedicated to the explanation of equations, solutions, and variables. The economic nuances of the models are explored in Section 2.3. The interpretations of
the governing equations will be presented in Section 3.2. Chapter 4 pertains to the solution procedure for the original governing equations and their respective steady state analysis; this chapter belabors the economic interpretations of the steady state analysis. Chapter 5 explicates the need for a transformed model, its governing equations, solution procedure, and steady state analysis. Chapter 6 denotes the linear stability analysis of the transformed model, as well as areas of further research. Lastly, Chapter 7 denotes the concluding statements of the model.

The full derivations from the modeling will be discussed in the Appendices. Appendix A will cover the logistic equation solution procedure used in the modeling sections of Chapter 2 as well as the limiting cases that will recover other existing models. Appendix B will denote the Hamiltonian formulations for obtaining the governing equations. Appendix C will show the derivations for the steady state solutions. Lastly, Appendix D will display the numerical code used to find an equilibrium points as well as eigenvalues and eigenvectors for those equilibrium points.

1.2 Literature Survey

An appropriate start for the literature survey is Charles Jones’ *Introduction to Economic Growth* which summarizes standardized new growth models [1]. Growth theory in economics is synonymous with the work of Nobel laureate Robert Solow. Jones articulates the many iterations of the Solow Model; showing the original model from Solow’s papers and the many changes involving investment, technology, productiv-
ity, and socio-political stability [2]. The Solow Model utilizes a Cobb-Douglas form production function and a capital accumulation equation, treats technology as an exogenous force that pushes the economy forward as technology increases, and treats socio-political stability as a coefficient that hinders or helps an economy’s growth. The Solow model has been the foundational economic growth model for many years. Jones focuses on the decision making process of where to invest. How does an economy decide to invest surplus time, money, and abilities? The choice of whether to spend now or invest in the next period is crucial to maximizing economic growth. Moreover, the choice of whether to invest in physical capital or human capital (where increased human capital increases productivity) affects the economic growth of a nation. Thus economic growth modeling evolves from the Solow model to AK models and new growth theories; one of the forefront economists of this genre is David Romer, of the University of California at Berkeley. Romer writes extensively on human capital. In fact the Romer model incorporates choosing between investments in human capital, research and development, or physical capital to maximize growth and utility [3]. More of Romer’s work is dedicated to validating his model and the augmented Solow model. In his *A Contribution to the Empirics of Economic Growth*, Romer and co-authors dictate the validity of their existing models through a substantial regression analysis [4]. The analysis confirms that the models do sufficiently capture the patterns of economic growth within developed countries. It is from articles such as this,
as well as Temple’s *The New Growth Evidence* [5] and Romer’s *The Origins of Endogenous Growth* [6], that the majority of the values of the parameters throughout this thesis are authenticated.

The Romer and Solow models succeed in accurately modeling the economic growth of developed countries. However, these models fail to accurately portray the economic stagnation of developing countries. Basic assumptions on which the Solow and Romer models are founded concerning developed countries are not necessarily true for developing countries. Consider health and life expectancy; it is straightforward to suppose that an individual would allot different amounts of surplus to different investments if his life expectancy were 38 years of age rather than 78. In fact an individual with such a low life expectancy may have no surplus with which to invest at all! Consequentially, it is not best to model a developed country the same as a developing country; some of the basic assumptions must be different. In previous research, the work has been criticized through regression analysis confirming that the Neo-classical and Solow models do not adequately model the growth of a developing country [7]. Thus more detailed models must be analyzed in order to understand (and ideally help induce growth in) developing countries.

Many economists have risen to the challenge of designing more accurate models for developing countries. Perhaps the best known was Robert Lucas, also a Nobel laureate, who modified the existing work of Uzawa [8] in his paper *On the Mechanics of*
Economic Development [9]. This paper studied economic growth by introducing three models with varying inputs of physical capital, technology, human capital through schooling, and human capital through learning-by-doing. Two works derived from Lucas’ original are the primary papers that this thesis utilizes: Fabrizio Zilibotti’s A Rostovian Model of Endogenous Growth and Underdevelopment Traps [10] and Michal Kejak’s Stages of Growth in Economic Development [11].

Fabrizio Zilibotti, of the London School of Economics, followed Lucas’ work with a paper that combined both Solow and Romer type growth to demonstrate differing states of an economy, the Solow model being used to demonstrate steady-state growth, with the Romer model’s portrayal endogenous growth. For endogenous growth, Zilibotti utilizes a technique to allow technology to grow through capital accumulation. Zilibotti’s logistic equation and solution are as follows:

\[
\frac{dA}{dt} = \phi \frac{a - A}{a} \frac{dK}{dt},
\]
\[
A = \frac{a}{1 + \left(\frac{a}{A_0} - 1\right)e^{-\phi Kt}}.
\]

Here \(A\) is technology, \(K\) is capital, \(t\) is time, and \(\phi\) is simply a positive, diffusion parameter. Note that we have changed Zilibotti’s notation for congruency of terms within this thesis. This is Zilibotti’s Equations (2) and (3). At \(t = 0\), \(A_t\) approaches \(A_0\); technology approaches its initial level. As time approaches \(\infty\), \(A_t\) approaches \(a\), the threshold of technology. However the truly interesting observation is that as capital increases, technology grows as well. This is the endogenous engine of growth.
that Zilibotti introduces into the growth model. In this thesis a similar, endogenous approach is used for life expectancy.

Another fascinating and pertinent technique that Zilibotti employs is his production function:

$$y = DA(K)k + Z[A(K)k]^\theta. \quad (1.3)$$

This is Zilibotti’s Equation (8). Here $D$ and $Z$ are parameters, $A(K)$ is Zilibotti’s (3) from above, $k$ is aggregate capital, and $\theta$ is a parameter for the non-linear term associated with production growth. This non-linear production function differs from past models through its embedded nature, since technology is a function of capital. Furthermore, its non-linearities avoid a zero-growth solution by giving the model a mechanism to escape underdevelopment traps and maintain growth. This Equation is used in Equation (2.11) to show the relationship between capital and production. If simplifications are made to this thesis’ model, one can recover the Zilibotti model; for a review of this procedure, please refer to Section A.2.

Likewise, Michal Kejak further enhanced Lucas’ work by incorporating human capital and productivity endogenously into the model. Kejak also utilizes a Hamiltonian
solution procedure. Presently it is critical to observe Kejak’s Equations (1-3):

$$\frac{dB}{dt} = \psi \frac{B_H - B}{B_H} \frac{dH}{dt},$$  \hspace{1cm} (1.4)  

$$B(H) = \frac{B_H}{1 + \left( \frac{B_H}{B_0} - 1 \right) e^{-\psi H}},$$  \hspace{1cm} (1.5)  

$$\frac{dh}{dt} = B(H)(1 - u)h.$$  \hspace{1cm} (1.6)  

Here $B$ is productivity at time $t$. Consequentially $B_0$ and $B_H$ are initial productivity and frontier productivity respectively. Kejak states that the parameter $\psi$ is a measure of the speed of diffusion and $H$ is the average level of human capital in the economy ([11] pg. 4). As $H$ increases, productivity approaches the frontier level of productivity. Likewise, as $H$ decreases, productivity decreases to the initial level of productivity. Equations 1.4 and 1.5 will be used directly in this thesis’ modeling. Both this model and Zilibotti’s model can be reverted to Kejak’s, as demonstrated in Section A.3.

Kejak’s Equation (1.6) is particularly interesting and relevant to this model because it introduces the concept of time allotment. Here $(1 - u)$ represents the amount of nonleisure time spent investing in human capital and therefore not spent in production. By optimizing time allotments Kejak’s model maximizes economic growth. In this paper’s model, $e_p$ is equivalent to Kejak’s $(1 - u)$ with the principle variance being that in addition to allotting time for investment in productivity, time allocated for investment in health (health education time) is incorporated into the model through $e_q$. Consequentially, this paper shares many similar derivations and equa-
tions with the Kejak model. Perhaps one of the most significant similarities with the Kejak model is the transformation of the model from a case-specific form to a more generalized model. As will be highlighted later, both Kejak and this model yield case-specific solutions that, although relevant, do not sufficiently interpret the spectrum of economic combinations and possibilities that a country could undertake. Thus, utilizing a change of variables and resolving the model, Kejak is able to analyze a greater domain of possible steady state points. Likewise, we employ Kejak’s change of variables to the same effect [11].

In a brief summation, we observe that Zilibotti employs a non-linear production function and an endogenous technological growth equation to existing growth models. Kejak proceeds by adding human skill level to Zilibotti’s modeling (although Kejak does disregard the non-linear production function). This thesis employs the endogenous technological equation, human skill level, and the non-linear production function as well as introducing an endogenous life expectancy variable through time spent in health education.

It is necessary to cite some of the other authors whose works contribute to this area of study. In *Capital Mobility and Underdevelopment Traps*, Charles Vellutini models the progress of two nations, one advanced and one developing; and how their interactions affect economic growth. Vellutini’s model demonstrates the convergence of growth rates as the advanced country ”helps” the developing country [12]. Likewise,
Daron Acemoglu of MIT has been a frequent author in this field, writing a wide range of articles pertaining to overall economic issues of developing countries from how productivity differences affect developing countries to the spread of disease and life expectancy [13, 14].

Thus after a brief review of the current work of economists in the field of development, this paper presupposes the speculation on how health education affects economic growth. It has been shown in Kejak’s work that investment in production education (human capital) is pivotal towards leaving economic stagnation and entering positive growth. In order to further Kejak’s work, this model incorporates health education and life expectancy to existing models to show the economic returns of a healthy society.

Before proceeding it is pertinent to review other articles that denote the present standard of health in developing countries and to understand the ramifications of that standard. In a review of three papers covering the effects of AIDS in South Africa [15], Mozambique [16], and Uganda [17], the results were conclusive: the AIDS epidemic has put an economic handicap on all three countries. The authors note the direct and indirect effects of AIDS, ranging from reducing time available to invest in human capital to reducing population size and thus labor population. Although these articles focus mainly on the affects of the AIDS epidemic, the authors all agree that AIDS is not the sole economic, nor health related hardship facing these countries.
Therefore the present query of the economic effects of health education takes further form to investigate the economic viability of international investment in health education. Is there incentive for foreign countries to invest in health education in developing countries? Is there a more effective way to deal with international aid and investment? Can health education spur a growth miracle that has been pointedly absent from Sub-Saharan Africa thus far? It is the intent of this thesis to begin to answer questions such as these.

1.3 Health Model Comparisons

We allot this portion of the Introduction to ascertaining the comparisons and contrasts between our model of economic growth incorporating health education and other existing models. R. Aisa’s and F. Pueyo’s *Endogenous Longevity, Health and Economic Growth: A Slow Growth for a Longer Life?* depicts the relationship between investment in health education (or in their terms longevity) and the rate of growth [18]. They principally observe that for less developed countries, both health education (thus longer living) and economic growth go hand in hand, They assert that the reason for this effect is that in developing countries health education exists to enable life. Whereas, in more developed countries, longevity increases and economic growth can assume rival positions. That is, there exists a tradeoff between living longer and growing faster as an economy.
Furthermore some of the primary differences between their paper and ours is their utilization of population, probability of dying, insurance policies, and describing health as a function of government spending. Some of the similarities occur within the structure of the functions, such as the production function and their governing equation for capital. But the aim of this thesis (to ascertain the effect of health education on economic growth and consumption maximization) and their aim (to assess the relationship between investments in longevity and growth rates) are strikingly different.

Thus we consider a paper with similar goals, Zon’s and Muysken’s *Health and Endogenous Growth* [19]. There are a number of similarities between our model and their model including their description of population, health level, human capital, and time allotment. However some of the principal differences are found in the assumptions of the paper. Their paper differentiates between ‘old’ and ‘young’ populations within the economy where the ‘young’ are both consumers and producers and the ‘old’ are solely consumers. Furthermore, they employ different age groups with different education levels, and define a utility function that incorporates longevity as an input towards utility. In our modeling, utility is solely based on consumption maximization.

Both papers do allot for some health sector that provides health services as an endogenous part of the model. Our paper assumes some ‘miraculous’ exogenous health education system (which is not an outrageous assumption for developing countries where groups often do come in to provide free health services). However, we ac-
knowledge the ingenuity behind an endogenous health care service whether that be by government expenditures or by instituting health care systems.

1.4 Assumptions

The model in this paper will seek to maximize the consumption and utility of the decision maker in a developing country by solving for the optimization of time allotments. So, the main problem of this paper is to determine how the individuals in a developing country should spend their time within a three-sector market while trying to maximize their lifetime consumption. The three sectors are assumed to be a production sector, a health-education sector, and a human capital education sector. The product of time spent in the production sector is output that is used to feed and fund the community, thus bringing utility. Time spent in the health education sector yields a higher life expectancy, and time spent in the human capital sector yields higher productivity in the production sector.

For simplicity, we assume that all members of this country are identical in their functional descriptions. That is to say that the modeling used for one member of the country is identical to the modeling used for the entire populous. This assumption detracts little from the authenticity of the model, since we can assume that our modeling is the average value, amount, rate, etc. for the modeled country. One glaring assumption is that this model neglects socio-political instability. Factors of corruption and instability could be factored into the model through varying coefficients, but
this model neglects that procedure in order to focus solely on the effects of health education in the economy. Although limiting the scope of the model, this assumption stands in order to keep the model simple and solvable.

Also, although the modeled country is presumed to be a developing country, this model may apply to different regions, sections, states, communities, or cities of a developed country given the following assumptions. It is assumed that there is some level of capital that is necessary to sustain healthy life, denote this level $K_n$. For any level of capital less than this amount it is assumed and modeled that life expectancy will drop. For life expectancy, it is assumed that there is a maximum average age, $Q$, that is attainable by an individual in a country; the actual value of $Q$ is assumed as the highest average life expectancy of a developed country. For this model, it is assumed that the economy in question has an initially low life expectancy; this is both a fair and accurate assumption given the life expectancies of developing countries from the CIA’s World Factbook [20].

1.5 Exhaustive Parameter List

Following is a comprehensive list of all the variables and parameters used in this thesis, their definition, and respective units of measure.
Table 1.1: Definition of Variables

<table>
<thead>
<tr>
<th>Variable</th>
<th>Definition</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>Time</td>
<td>Years</td>
</tr>
<tr>
<td>$A$</td>
<td>Technology as a function of time</td>
<td>unitless</td>
</tr>
<tr>
<td>$K$</td>
<td>Aggregate capital</td>
<td>Output</td>
</tr>
<tr>
<td>$k$</td>
<td>Capital</td>
<td>Output</td>
</tr>
<tr>
<td>$q$</td>
<td>Life expectancy</td>
<td>Years</td>
</tr>
<tr>
<td>$e_q$</td>
<td>Time spent in health education</td>
<td>unitless</td>
</tr>
<tr>
<td>$B$</td>
<td>Productivity level</td>
<td>1/Years</td>
</tr>
<tr>
<td>$H$</td>
<td>Aggregate skill level</td>
<td>unitless</td>
</tr>
<tr>
<td>$h$</td>
<td>Skill level</td>
<td>unitless</td>
</tr>
<tr>
<td>$e_p$</td>
<td>Time spent in production education</td>
<td>unitless</td>
</tr>
<tr>
<td>$u$</td>
<td>Time spent in production</td>
<td>unitless</td>
</tr>
<tr>
<td>$c$</td>
<td>Consumption</td>
<td>Output/Years</td>
</tr>
<tr>
<td>$w$</td>
<td>Wage rate</td>
<td>Output/Years</td>
</tr>
<tr>
<td>$F$</td>
<td>Cobb-Douglas Production function</td>
<td>Output/Years</td>
</tr>
<tr>
<td>$L$</td>
<td>Labor</td>
<td>unitless</td>
</tr>
<tr>
<td>$\Pi$</td>
<td>Profit function</td>
<td>Output</td>
</tr>
</tbody>
</table>
Table 1.1: Definition of Variables (continued)

<table>
<thead>
<tr>
<th>Variable</th>
<th>Definition</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>Shadow variable</td>
<td>unitless</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Costate variable</td>
<td>unitless</td>
</tr>
<tr>
<td>$x$</td>
<td>Capital to skill level ratio</td>
<td>Output</td>
</tr>
<tr>
<td>$v$</td>
<td>Consumption to capital ratio</td>
<td>1/Years</td>
</tr>
</tbody>
</table>

Table 1.2: Definition of Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_0$</td>
<td>Initial life expectancy</td>
<td>Years</td>
</tr>
<tr>
<td>$Q_0$</td>
<td>Lowest life expectancy</td>
<td>Years</td>
</tr>
<tr>
<td>$Q$</td>
<td>Maximum life expectancy</td>
<td>Years</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Health education diffusion</td>
<td>unitless</td>
</tr>
<tr>
<td>$\epsilon_{q0}$</td>
<td>Initial health education allotment</td>
<td>unitless</td>
</tr>
<tr>
<td>$B_F$</td>
<td>Frontier productivity</td>
<td>1/Years</td>
</tr>
<tr>
<td>$B_O$</td>
<td>Lowest productivity</td>
<td>1/Years</td>
</tr>
<tr>
<td>$B_L$</td>
<td>Low productivity</td>
<td>1/Years</td>
</tr>
<tr>
<td>$B_H$</td>
<td>High productivity</td>
<td>1/Years</td>
</tr>
</tbody>
</table>
Table 1.2: Definition of Parameters (continued)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi$</td>
<td>Productivity diffusion</td>
<td>unitless</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Capital diffusion</td>
<td>1/Output</td>
</tr>
<tr>
<td>$K_n$</td>
<td>Capital necessary for healthy living</td>
<td>Output</td>
</tr>
<tr>
<td>$k_0$</td>
<td>Initial capital</td>
<td>Output</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Depreciation of capital</td>
<td>1/Years</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Elasticity measure</td>
<td>unitless</td>
</tr>
<tr>
<td>$a$</td>
<td>Technology threshold level</td>
<td>unitless</td>
</tr>
<tr>
<td>$A_0$</td>
<td>Initial technology level</td>
<td>unitless</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Technology diffusion</td>
<td>1/(Years*Output)</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Time discount factor</td>
<td>1/Years</td>
</tr>
<tr>
<td>$r$</td>
<td>Interest rate</td>
<td>1/Years</td>
</tr>
<tr>
<td>$Z$</td>
<td>Non-linear output coefficient</td>
<td>Output(^{1-\theta})/Years</td>
</tr>
<tr>
<td>$D$</td>
<td>Linear output coefficient</td>
<td>1/Years</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Non-linear Output coefficient</td>
<td>unitless</td>
</tr>
<tr>
<td>$G$</td>
<td>Cobb-Douglas multi-factor productivity constant</td>
<td>Output(^{\alpha})/Years</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Capital share of production</td>
<td>unitless</td>
</tr>
<tr>
<td>$x_0$</td>
<td>Initial ratio of capital to skill level</td>
<td>Output</td>
</tr>
<tr>
<td>$x_n$</td>
<td>Necessary ratio of capital to skill for healthy life</td>
<td>Output</td>
</tr>
</tbody>
</table>
2.1 The Initial Modeling

This model’s equations governing technological growth and life expectancy will be introduced, then equations that are common to both Kejak, Zilibotti, and standard growth model formulations. As delineated by Solow, technological innovation is the driving force of economic growth. This model considers progressive, technological growth, not revolutionary growth. That is to say, this model assumes a premier level of technology exists somewhere and that the modeled economy approaches the technological threshold; this model neglects such events as Industrial Revolutions. Technology is incorporated into the model as follows:

\[
\frac{dA}{dt} = \left( \frac{a - A}{a} \right) A\phi \left[ q \frac{dK}{dt} + dq \frac{dt}{K} \right],
\]

where \( A \) is level of technology at time \( t \), and \( a \) is the technology threshold of the world (that is the most state-of-the-art production technology). In this model, \( K \) stands for aggregate capital, and \( q \) represents life expectancy, with their respective dot functions being the time derivatives. The parameter, \( \phi > 0 \), models the diffusion of a change in the technological level into the economic system. Solving this differential equation
(Refer to A.1) gives:

\[
A(t) = \frac{a}{1 + \left(\frac{a}{A_0} - 1\right) e^{-\phi[q(t)K(t)-q(0)K(0)]}}. \tag{2.2}
\]

Observe that \( A_0 \) is the initial level of technology and that \( q(0) \) and \( K(0) \) are the respective initial values of life expectancy and capital at time \( t = 0 \); consequently \( A(0) = A_0 \). As average life expectancy increases (assuming a constant capital function), the level of technology in the economy approaches the technology threshold. The explanation for this modeling is as follows: the longer a person lives, the more time he/she has to create and apply innovative technology. Thus as the average life expectancy of a country increases, the populace will begin to invest more time in applying technology on the whole. Likewise, as capital increases (assuming a constant life expectancy function) technology approaches the state-of-the-art level. This reasoning follows from Zilibotti’s work on aggregate capital and technology [10].

It is important to ascertain the nature of the life expectancy function since changes in the value of \( q \) affect \( A \), the state of technology. The differential equation for life expectancy is as follows:

\[
\frac{dq}{dt} = \frac{Q - q}{Q} \frac{d}{dt} \left[ \gamma e_q + \eta(K - K_n) \right]. \tag{2.3}
\]

Here, \( Q \) is the maximum life expectancy; consider this to be the highest, average life expectancy in a developed country. Non-leisure, non-production time spent investing in health education is represented as \( e_q \); the positive parameter \( \gamma \) represents the diffusion effect of that time. As assumed, \( K_n \) is the amount of aggregate capital
necessary to support life. Thus \( \eta > 0 \) is again a parameter measuring the effect of sufficient and insufficient amounts of capital in the economy. Solving this equation one finds,

\[
q(t) = \frac{Q}{1 + \left( \frac{Q}{Q_0} - 1 \right) e^{-[\gamma(e_q-e_q(0))+\eta(K(t)-K(0))]}},
\]

(2.4)

Here \( Q_0 \) is the initial life expectancy of the country. Observe that for \( e_q = e_q(0) \) and for \( K(t) = K(0) \) the life expectancy of the economy is equal to \( Q_0 \), the initial life expectancy of the country. Here \( K(0) \) is the initial aggregate capital value of the country and \( e_q(0) \) is the initial amount of time allotted to health education. Note that as \( K \) and \( e_q \) increase, \( q \) approaches \( Q \), the threshold life expectancy, and in contrast as \( K \) and \( e_q \) decrease, life expectancy falls.

Continuing with Kejak’s model of human capital and productivity [11] and in congruence with Lucas’s assumption that time invested in human capital is a driving force of technical progress [9], the following equation is posed:

\[
\frac{dB}{dt} = \left( \frac{\psi}{B_F} \right) \frac{B_F - B}{B} \frac{dH}{dt},
\]

(2.5)

Here, as with the Kejak model, \( B \) is a measure of productivity at a given time \( t \). \( B_F \) is the frontier productivity, the highest productivity available throughout the world market. It is important to note that \( B_F \geq B \geq B_0 \) for all time, \( t \). Here, \( H_t \) is the average level of human capital; thus \( \frac{dH}{dt} \) is the change in human capital over time. \( \psi \) is a positive parameter for the speed of diffusion; this parameter will come to the
forefront when faced with the question of where and how to give international aid.

Equation (2.5) can be solved (Refer to A.1) yielding the logistic equation:

\[
B(H) = \frac{B_F}{1 + \left(\frac{B_F}{B_0} - 1\right) e^{-\psi H}}.
\]  

(2.6)

Here, \(B_0\) is the initial productivity of the country given its initial level of human capital. Before moving on it is important to note some of the qualities of Equations (2.5) and (2.6). Note that as human capital increases (\(H \to \infty\)), \(B\) approaches \(B_F\). Also, note that for low levels of human capital (\(H \to 0\)), \(B\) approaches \(B_0\). Furthermore, observe that for low values of productivity, \((B_t << B_F)\) that the growth rate for productivity \(\frac{\dot{B}_t}{B_t}\) is larger; and in contrast for high values of productivity the growth rate of productivity is smaller. This decreasing marginal effect on productivity growth is expected and congruent with economic theory.

Again following the Kejak form, "generalize the linear Uzawa-Rosen formulation of the production function for human capital assuming that the level of productivity in the education sector depends on the developmental level of a society expressed by the average level of human capital ([11], pg.4)." That is to say we express human skill level in terms of productivity and time spent in production education.

\[
\frac{dh}{dt} = B(H) e_p h. \tag{2.7}
\]

Here \(h\) is the skill level of the population, assuming that the entire population shares the same level, and \(B(H)\) is the previously presented productivity measure. The variable \(e_p\) represents the proportion of non-leisure time spent in production education.
We now introduce a formal presentation of our time allotment scheme; assign \( u \) to be the proportion of time spent in production and \( e_q \) to correspond to the amount of non-leisure, non-production time spent in health production. We sum these daily proportions of time allotments to be the equivalent of one work-day.

\[
1 = u + e_p + e_q. \tag{2.8}
\]

Again for clarity, it is assumed that all the members of this economy have skill level \( h \), that they spend a percentage, \( u \) amount of their work-day in the production sector of the economy. For the remainder of the work-day \( e_q \) is the percentage amount of time spent in the health education sector and \( e_p \) is the percentage amount of time spent in the human capital sector of the economy. Clearly it follows that \( 0 \leq u, e_q, e_p \leq 1 \).

(For an exhaustive list of the many parameters and variables refer to Section 1.5).

The above equations guide the workings of the economy; they will be used to determine how to allot non-leisure time and provide a framework from which to answer genuine economic questions. Before addressing the query of how best to alleviate dire economic need, the assumptions and corresponding time allotments of the economic individual, who in this model is the assumed mold for all individuals, must be analyzed. It is assumed that the individual in this model seeks to maximize their consumption over a period of time, their lifespan. Thus two more equations are presented, the maximization problem and the differential equation for capital.
\[ \max \int_0^Q \frac{c^{1-\sigma}}{1-\sigma} e^{-\delta t} dt \quad (2.9) \]

\[ \{c, k, h, u, e_q, e_p\} \lim Q \to \infty. \]

Here it is important to reiterate the above stated goal to maximize consumption over lifespan by choosing the optimal time allotments \(u, e_q, e_p\) as well as variables \(c, k,\) and \(h\). Here \(c\) is consumption at time \(t\), and \((1 - \sigma)\) is a measure of elasticity. The \(e^{-\delta t}\) is a time discounting factor for the model. This time discount factor, \(\delta\), models a consumers propensity to invest in the next period or not. A larger \(\delta\) implies that an individual would rather spend the entirety of their wage during the period in which they receive their income. In contrast, a smaller \(\delta\) implies that an individual is willing to invest some of their wage into some skill-set or capital that will benefit him or her in the next period. \(\delta\) proves to be an important parameter and will be further elucidated upon throughout the interpretation stages.

Note that the end-goal of this economy is not to maximize consumption and lifespan or to maximize education and production. This modeling postulates that the sole purpose of the economy is to maximize consumption over lifespan. That is to say that in order to maintain a straightforward utility function we assume that the citizens within the economy do not seek to maximize health, life expectancy, or education. These factors may be increased or changed in order to maximize consumption, but only as a means to an end. Moreover, this maximization is subject to two constrain-
ing functions, capital and human skill level. The constraint on human skill level is Equation (2.7). Whereas, the equation for capital is as follows:

\[ \frac{dk}{dt} = kr + wuh - c. \]  

(2.10)

In contrast to Equation (2.1), \( k \) is capital (not aggregate capital). Also, \( c, u, \) and \( h \) are respectively consumption, time in production, and skill level from their above equations. The new variables \( r \) and \( w \) are the investment return rate of capital and the wage for workers. It is imperative to note that

\[ kr - k\rho \approx y(t) = DAk + Z(Ak)\theta, \]  

(2.11)

that is capital adjusted by its investment rate minus the depreciation of capital, \( k\rho \), is approximately equal to the Gross Domestic Product (GDP) of the economy, \( y(t) \); where \( D \) and \( Z \) are parameters for that individual country and \( 0 \leq \theta \leq 1 \) is a parameter associated with the non-linear factor of output. The distinction between \( k \) and \( K \) will be delineated later as it pertains to the way derivatives are found within the Hamiltonian solution procedure.

This modeling formation of maximizing consumption over some life expectancy with constraints from capital and human skill level will yield a system of 5 equations with 5 unknowns \( c, k, h, e_p \) and \( e_q \), through a Hamiltonian maximization procedure, as will be shown in the proceeding chapter. Again observe that the maximization problem is constrained by human skill level (the ability to work efficiently) and capital (the
tools that are available for the economy). The remainder of this thesis is dedicated to obtaining and interpreting that system.

2.2 The Transformed Model

A steady state study will be shown in Chapter 4. At an equilibrium point, skill level, for example, does not change. Hence a transformation to the original model is induced so that the fixed points reflect dynamic paths. Throughout the rest of this thesis, we will present the original model, its interpretations, and limitations, followed by a similar presentation for the transformed model on which $h, k$, and $c$ change. The following is a presentation of the transformations made to the original model.

It has been established that this model uses the same transformation of variables that Kejak used in his modeling [11]. Thus following Kejak, we create variables $x$ and $v$, such that

$$x = \frac{k}{h},$$

(2.12)

$$v = \frac{c}{k}.$$  

(2.13)

Therefore $x$ is capital per human skill level and $v$ can be expressed as consumption per unit of capital. We will then seek a system of 4 equations, with the 4 unknowns being $x, v, e_q$ and $e_p$. Again the motivation for this change is congruent with Kejak’s formulation and reasoning: transforming the variables into ratios gives us an oppor-
tunity for dynamic values of $k$, $h$, and $c$. However, the models for $A$ and $q$ must also be adapted in order to accommodate this transformation.

The following changes will create a more general form of our existing model and set-up the new 4-variable system. Consider our original technology equation, Equation (2.1):

$$\frac{dA}{dt} = \left( \frac{a-A}{a} \right) A \phi \left[ q \frac{dk}{dt} + \frac{dq}{dt} k \right]. \quad (2.14)$$

We impose a direct change of $k$ to $x$, obtaining:

$$\frac{dA}{dt} = \left( \frac{a-A}{a} \right) A \phi \left[ q \frac{dx}{dt} + \frac{dq}{dt} x \right]. \quad (2.15)$$

Note that the magnitude of $\phi$ in Equation (2.14) is different from the magnitude of $\phi$ in Equation (2.15) due to the introduction of $h$. This change gives us a new technology logistic equation:

$$A(t) = \frac{a}{1 + \left( \frac{a}{A_0} - 1 \right) e^{-\phi[q(t)x(t)−q(0)x(0)]}}. \quad (2.16)$$

Our economic interpretation for $A$ is as follows. Observe that as $h$ increases the level of technology actually decreases (assuming that $k$ is constant). We suggest that as human skill level increases the need for newer technology decreases because the populous becomes more effective with the present technology. Although the level of technology may decrease, the application of lesser technologies is sufficient due to the
enhanced skill level. It is not guaranteed that $k$ will stay constant. Referring back to Equation (2.10), $k$ increases as $h$ increases, $\frac{dk}{dt} = kr + wuh - c$. A positive change in $h$ may not result in a decrease in the technology level; it may even result in an increase!

The model for life expectancy, Equation (2.3), is also modified. We seek to directly transform $k$ into $x$, as follows:

$$\frac{dq}{dt} = \frac{Q - q}{Q} \frac{d}{dt} \left[ \gamma e_q + \eta (K - K_n) \right],$$  \hspace{1cm} (2.17)

$$\frac{dq}{dt} = \frac{Q - q}{Q} \frac{d}{dt} \left[ \gamma e_q + \eta (x - x_n) \right],$$  \hspace{1cm} (2.18)

where $x_n$, a necessary capital to human skill level ratio, is similar to $K_n$. Also note that as with $\phi$, the value of $\eta$ changes from Equation (2.17) to Equation (2.18). This new equation can be solved to yield:

$$q(t) = \frac{Q}{1 + \left( \frac{Q}{Q_0} - 1 \right) e^{-[\gamma (e_q - e_q(0)) + \eta (x(t) - x(0))]}},$$  \hspace{1cm} (2.19)

where $x(0)$ is some initial capital to human skill level ratio. Again, we seek theoretical, economic affirmation for this change. We assume that time spent in health education does not affect productivity and in contrast that time spent in production education does not affect life expectancy. Therefore, it is sufficient to state that as $h$ increases life expectancy might decrease due to less time being spent in health education and more time being spent in production education. Furthermore, as with our rationale for the technology equation, as $h$ increases, $k$ may very well increase or stay the same.
These small changes to our original modeling are sufficient to allow the analysis to explore a wider set of economic possibilities. Now new fixed points allow us to more fully explore different time allotments throughout the day, differing choices of capital and consumption expenditures, and changes in human capital and skill level. And it is by the above derivations that we have obtained these dynamic solutions.

2.3 Initial Implications of the Model

Before analyzing the model it is worthwhile to again stress the functionality and aims of the modeling in a less-analytical format. We created a model that allows technology, the key element necessary for economic growth, to grow as a function of life expectancy and capital. We created a function for life expectancy that was dependent on time spent in health education and capital as well. We then constructed a model of productivity and skill level pertaining to the production sector that was dependent upon time spent in production education.

All these equations depend upon where time is allotted. Equation (2.8) models a full day; wherein a person within this economy chooses to spend their time, either in production, production education, or health education. It is the aim of this thesis to determine if time spent in health education is viable in an country where the population seeks to maximize utility by maximizing consumption over their lifespan. Is it time well spent?
To address this question, we employed a utility function that spans over a decision-maker’s lifetime and created a model for the growth of capital that is dependent upon time spent in production. However, a greater insight to the model is observed by looking at ratios rather than individual variables (as shown in Chapter 4). This allows us the freedom to augment the variables while keeping the ratios constant, and ultimately allows us to answer our original query of the viability of health education.
CHAPTER III

THE OPTIMIZATION PROBLEM - HAMILTONIAN APPROACH

3.1 Constructing the Hamiltonian

In order to solve our original model, we seek five governing differential equations; one for \( c, k, e_p, e_q, \) and \( h \). We utilize a Hamiltonian technique to reconstruct the restrained optimization problem into a system of differential equations. The Hamiltonian is constructed by taking the utility equation and adding to it the two constraining equations multiplied by some shadow variable, \( \lambda \) or \( \mu \). In order to maximize the utility subject to the restraints, we define the Hamiltonian operator as follows:

\[
HAM(c, e_q, e_p, k, h; \lambda, \mu) = \left[ \frac{c^{1-\sigma}}{1-\sigma} e^{-\delta t} \right] + \lambda \left[ kr + wuh - c \right] + \mu \left[ B(H) e_p h \right]. \tag{3.1}
\]

This optimization technique as shown in The Structure of Economics: A Mathematical Analysis ([21] pgs. 617-622) can be rewritten by solving Equation (2.8) for \( u \) and by adding zero in the form of \((+k\rho - k\rho)\) to yield:

\[
HAM = \left[ \frac{c^{1-\sigma}}{1-\sigma} e^{-\delta t} \right] + \lambda \left[ (kr + k\rho) - k\rho + w(1 - e_p - e_q) h - c \right] + \mu \left[ B(H) e_p h \right]. \tag{3.2}
\]

In order to maximize the utility we take partial derivatives of the Hamiltonian as follows:

\[
HAM_c = 0 = c^{-\sigma} e^{-\delta t} - \lambda, \tag{3.3}
\]
\( HAM_\lambda = \frac{dk}{dt} = (kr + k\rho) - k\rho + w(1 - e_p - e_q)h - c, \quad (3.4) \)

\( HAM_\mu = \frac{dh}{dt} = B(H)e_p h, \quad (3.5) \)

\( HAM_k = -\frac{d\lambda}{dt} = \lambda[DA + ZA^\theta k^{\theta-1}\theta - \rho], \quad (3.6) \)

\( HAM_h = -\frac{d\mu}{dt} = \lambda w(1 - e_p - e_q) + \mu B(H)e_p, \quad (3.7) \)

\( HAM_{e_p} = 0 = \mu B(H)h - \lambda wh, \quad (3.8) \)

\( HAM_{e_q} = 0 = \mu B(H)h - \lambda wh, \quad (3.9) \)

where \( \frac{dA}{dq} \) and \( \frac{dq}{de_q} \) are:

\[
\frac{dA}{dq} = \frac{a\phi K(t)(\frac{a}{A_0} - 1)e^{-\phi[qK-q(0)K(0)]}}{[1 + (\frac{a}{A_0} - 1)e^{-\phi[qK-q(0)K(0)]}]^2}, \quad (3.10)
\]

\[
\frac{dq}{de_q} = \frac{Q\gamma(\frac{Q}{Q_0} - 1)e^{-[\gamma e_q - e_q(0) - \gamma(K(t)-K(0))]}\frac{1}{[1 + (\frac{Q}{Q_0} - 1)e^{-[\gamma e_q - e_q(0)]}]^2}}. \quad (3.11)
\]

Before solving this system of equations there are two important ideas to note: the production and profit equations, and the concept of aggregate capital and skill level.

Following with Kejak’s solution method, we utilize a Cobb-Douglas production function \( F(k, L) = Gk^\alpha L^{1-\alpha} \) that models the production center. \( k \) is capital, \( L \) is labor, and \( G \) is a multi-productivity constant parameter. Note that labor, \( L \), can be defined as \( L = uh; \) that is labor is equivalent to time spent in production multiplied by the skill level of the population. Continuing, \( \alpha \) is a parameter between zero and one that denotes the capital share of production, with \( 1 - \alpha \) corresponding to the labor share of production. This production equation is then utilized to construct a profit equation

\[ \Pi = F(k, L) - wL - [DAK + Z(AK)^\theta], \] where \( w \) is the working wage rate.
The system is undetermined without an additional constraint, thus if we add the constraint of profit maximization, we can analyze the system of equations. By finding the partial derivatives of the profit equation with respect to labor and capital and setting them equal to zero (thus maximizing profit) we define $w$, the wage rate, and further constrain the system of equations thus permitting a solution to be found. Refer to Appendix (B.2) for the full derivations. One noteworthy insight from these derivations is the relationship between the parameters $D, Z$ and $G$. These parameters are not independent of each other and the maximization of profit yields their codependent relationship.

It is imperative to denote here the difference between aggregate quantities and normal quantities. In our initial formulations we express capital, $k$ in two forms: $k$ and $K$. (Similarly, this argument follows for human skill level $h$ and $H$.) The fundamental difference between these two variables is $K$ represents aggregate capital, which is the total capital available throughout the whole of the economy, while $k$ represents solely the capital used within the economy. The $K$ quantity is independent of $k$ itself; thus when the Hamiltonian takes the derivative with respect to $k$, terms such as $A(K)$ are not affected since we are taking the derivative with respect to capital, not aggregate capital. However, when solving the system of equations at equilibrium the aggregate values equal the normal values; so at equilibrium $K = k$. (Likewise at equilibrium, human skill level is equal to aggregate skill level, $h = H$.) It is necessary to make this distinction because it is not until equilibrium that these variables are equivalent.
This being noted, we can proceed with the solution procedure for solving the system of equations. This system can be reduced by eliminating $\lambda$ and $\mu$ (Refer to Appendix B) to obtain the following governing equations for $c, h, k, e_p$, and $e_q$.

\[
\frac{dc}{dt} = \frac{DA + ZA^\theta k^{\theta-1} - \rho - \delta}{c}, \tag{3.12}
\]

\[
\frac{dh}{dt} = B(H)e_p, \tag{3.13}
\]

\[
\frac{dk}{dt} = DA + ZA^\theta k^{\theta-1} + Gk^{\alpha-1}L^{1-\alpha}(1-\alpha) - \rho - \frac{c}{k}, \tag{3.14}
\]

\[
\frac{d}{dt}(DAk + Z(Ak)^\theta) = wh, \tag{3.15}
\]

\[
\frac{de_p + de_q}{1 - e_p - e_q} = \frac{dh}{h} - \frac{dk}{k} + \frac{1}{\alpha} [DA + ZA^\theta k^{\theta-1} - \rho] - \frac{1}{\alpha}(1 - e_q)B(H) + \frac{1}{\alpha}B'(H)e_p h. \tag{3.16}
\]

Thus we have established a system of five governing equations (four differential and one algebraic) with five unknowns.

3.2 Economic Interpretations

Equation (3.12) states that the average rate of change of consumption is equivalent to the marginal effect of capital on gross output minus the depreciation rate of capital minus a time discounting factor, all scaled by a relative risk aversion coefficient, $\sigma$. Thus as the time discount factor increases (as people become more inclined to invest in savings, capital, or other things beside consumption), the average rate of consumption decreases. If people within a country decide to invest their money, rather than
spend it, consumption decreases. Furthermore, as the rate of depreciation of capital increases (as capital goods lose their value/effectiveness), the rate of consumption decreases. This also logically holds indicating that as a capital good starts to depreciate, an individual is inclined to spend less on consumption and more on capital in order to maintain his/her present level of capital. Lastly, as elasticity increases (as a person’s propensity to consume changes to an adverseness towards consumption) the average rate of consumption falls.

Equation (3.13) states that the average rate of change of human skill level is directly proportional to the productivity scaled by production education. This equation, mirroring Kejak’s Equation (3), continues to reinforce the thought that time spent in production education yields higher skill levels and thus higher outputs. Indeed Kejak reaches this same conclusion in his work, stating that after a productivity slowdown due to investing time in production education, there is a period of productivity, production and output growth ([11] pg. 24).

In the same fashion, Equation (3.15) can be interpreted as the marginal effect of health education on production is equivalent to one work hour. That is to say that the value of one work hour, wage times skill level, is equal in value to the time spent in health education. From this result alone, the groundwork for questioning the viability of health education is laid. It has been established that time spent in health education is equivalent in value to time spent working at a certain skill level. Al-
though more work must be done to fully reveal the ramifications of health education, it is worthwhile to acknowledge this initial discovery.

An overview of the capital governing equation, Equation (3.14), reveals that capital is equivalent to output plus a scaled production minus depreciation of capital minus consumption. Therefore the average rate of change of capital is defined by the above equivalence divided by capital itself. As technology increases, average rate of change of capital increases. As capital or labor increase, average rate of change of capital increases. The average rate of capital decreases as depreciation increases and likewise decreases as consumption increases.

The left hand side of Equation (3.16) can be understood as the negative, average rate of change of time spent in production. To obtain this definition simply differentiate Equation (2.8) with respect to time and switch signs throughout the equation. Thus the average rate of change of time spent in production is equivalent to the average rate of change of capital minus the average rate of change of skill level minus the scaled marginal effect of capital on output plus depreciation of capital plus the scaled productivity minus the rate of change of productivity multiplied by time spent in production. This explanation is similar to the interpretation of Kejak’s Equation (20) and primarily implies that time spent in education (either production or health education) has effects that filter throughout the economic system [11]. Determining the nature of those effects is the aim of this thesis.
4.1 Analyzing the Original Model

In the following sections we will determine the steady state solutions to Equations (3.12) through (3.16), estimate numerically the effects of the parameters on the steady state solutions, interpret the steady state of this model, and denote its limitations. Many of the parameters that have been casually introduced will now be thoroughly investigated as to their magnitude, sign, range, and definition.

4.2 Steady State Solutions

Using Equations (3.12) through (3.16), a steady state solution can be obtained. This is done by setting all time derivatives equivalent to zero. Hence,

\begin{align*}
0 &= \frac{DA + ZA^\theta k^{\theta-1} \theta - \rho - \delta}{\sigma}, \quad (4.1) \\
0 &= B(H)e_p h, \quad (4.2) \\
0 &= DA + ZA^\theta k^{\theta-1} + Gk^{\alpha - 1}L^{1-\alpha}(1 - \alpha) - \rho - \frac{c}{k}, \quad (4.3) \\
\frac{d(DAk + Z(Ak)^\theta)}{de_q} &= wh \quad (4.4) \\
0 &= \frac{0}{h} - \frac{0}{k} - \frac{1}{\alpha} \left[ DA + ZA^\theta k^{\theta-1} \theta - \rho \right] - \frac{1}{\alpha} (1 - e_q)B(H) + \frac{1}{\alpha}B'(H)e_p h. \quad (4.5)
\end{align*}
Starting with Equation (4.2), rewrite the equation as follows utilizing the definition of productivity from Equation (2.6):

\[
\frac{dh}{dt} = B(H)e_p h = \frac{B_F}{1 + \left(\frac{B_F}{B_0} - 1\right)e^{\psi H}} e_p h
\]  

(4.6)

In order for \( \frac{dh}{dt} \) to be equal to zero, \( h = 0, e_p = 0, \) or \( B_0 = 0 \) must be true. Immediately \( h = 0 \) is eliminated as an option since \( h \) is defined as skill level, thus \( h > 0 \) for mathematical clarity. The case \( B_0 = 0 \) is also eliminated as an option after a brief set of derivations in Appendix C.1 reveals that this initial condition does not support the model (It implies that productivity never grows, so no one is ever productive; a trivial case). This leaves the only valid option to be \( e_p = 0 \). Thus the people in this steady state spend no time in production education. Hence human skill level, \( h \), does not change. Now, consider Equation (4.5) under steady state conditions. With all time derivatives equal to zero as well as setting \( e_p = 0 \), this equation becomes,

\[
0 = \frac{1}{\alpha} \left[ DA + ZA^\theta k^{\theta-1} - \rho \right] - \frac{1}{\alpha} (1 - e_q) B(H).
\]  

(4.7)

However observe that from the steady state Equation (4.1) that

\[
0 = \frac{DA + ZA^\theta k^{\theta-1} - \rho - \delta}{\sigma} \Rightarrow DA + ZA^\theta k^{\theta-1} - \rho - \delta = 0 \Rightarrow \delta = DA + ZA^\theta k^{\theta-1} - \rho.
\]  

(4.8)

Thus the steady state Equation (4.5) can be simplified to

\[
0 = \frac{1}{\alpha} \delta - \frac{1}{\alpha} (1 - e_q) B(H).
\]  

(4.9)

We will follow the work of Kejak in assuming \( B(H) \) as a constant. \( B(H) \) as a measure of productivity is dependent upon \( h \), a variable we just determined to be
constant and fixed; also note that $e_p = 0$ implies that no time is being spent in production education. Thus since $B(H)$ is a logistic function, bounded by $B_F$ from above and $B_0$ from below, we consider a case of high productivity and a case of low productivity. For $B(H)$ small, $(B(H) = B_L)$, we assume that this is before any interaction and investment in production education and is close in value to $B_0$. For $B(H)$ large, we assume some growth miracle has occurred and productivity is now higher, $(B(H) = B_H = B_F)$. We do denote a third case, but it is more for mathematical precision than an actual case itself. $B(H) = B(H^*)$, that is the productivity is equal to some ideal value that causes the instant transformation from $B_L$ to $B_H$.

Considering these three cases, we will analyze the steady state solutions.

However, before we can analyze any solutions, we must continue working with the original system, now holding $B(H)$ to be a constant. We solve for $e_q$ utilizing Equation (4.9) as follows:

$$0 = \frac{1}{\alpha} \left( \delta - (1 - e_q)B(H) \right),$$

$$0 = \delta - (1 - e_q)B(H),$$

$$(1 - e_q) = \frac{\delta}{B(H)},$$

$$e_q = 1 - \frac{\delta}{B(H)}.$$

We want to ensure that $e_q$ is between zero and one (The fraction of time spent in health education during a day). Thus we must ensure that $0 \leq (1 - \frac{\delta}{B(H)}) \leq 1$. This implies that $\frac{\delta}{B(H)} > 0$ and that $\delta \leq B(H)$. This assumption is reasonable recalling
that $\delta$ is a time discount factor, thus it is reasonably small and positive. Moreover $B(H)$ is a positive value that is larger than any reasonably set discount factor.

Now that $e_p = 0$ and $e_q = 1 - \frac{\delta}{B(H)}$, we solve for the steady state solutions for $k, c,$ and $h$. In order to obtain $k$ and $c$, we utilize the Cobb-Douglas production function, the profit equation, and the definition of labor ($L = uh$). The full derivations are shown in Appendix C.1. By manipulating these equations and simplifying we can solve for $k$ and $c$ in terms of $h$:

$$k = \frac{\delta}{B(H)} h \left( \frac{G\alpha}{\rho + \delta} \right)^{\frac{1}{1-\alpha}},$$  \hspace{1cm} (4.14)

$$c = \frac{\delta}{B(H)} h \left[ \frac{G\alpha}{\rho + \delta} (DA - \rho) + G \frac{G\alpha}{\rho + \delta} \frac{1}{\alpha} (1 - \alpha) \right] + Z(\theta).$$  \hspace{1cm} (4.15)

By computing the derivatives of the profit function, Equations (B.7) and (B.8), and through some algebraic reformatting to Equation (3.9), we obtain an algebraic equation that is solved numerically for $h$,

$$\frac{dA}{dq} \frac{dq}{de_q} = A \frac{1 - \alpha}{\alpha} \frac{B(H)}{\delta}.$$  \hspace{1cm} (4.16)

4.3 Numerical Solutions: Parameter Settings

We break the numerical solutions portion into two distinct sections, the first delineating the parameter settings and scale, the second explaining the solution procedure and the economic significance of the solutions. The code itself is presented in Appendix D.1.
Here it is opportune to dictate the scale and value of the parameters used in this numerical solution. We begin by expounding on the technological level and output parameters, continue by investigating the age and life expectancy values, then consider their respective diffusion parameters. We then investigate the discount factors as well as the high and low productivity constants. Finally we investigate the Cobb-Douglas associated parameters as well as some other miscellaneous factors.

Technology level is an abstract idea. We have established an initial level of technology, \( A_0 \), as well as a frontier level of technology, \( a \), which serve primarily as numerical markers for the level of technological aptitude, availability, and application. Recall that Zilibotti’s main engine of growth is the endogenous growth of technology, therefore in congruence with Zilibotti’s work we set \( A_0 = 0.015 \) and \( a = 0.075 \), equivalent to Zilibotti’s [10]. Here we have set lower and upper bounds on technology, and we permit for countries with varying levels of technology (such as developing countries with lower technological levels, thus \( A \) close to \( A_0 \)). Note that the threshold technology level is five times the initial technology level, thus relatively their is a significant amount of room for technological growth.

The output function, Equation (2.11), also was one of the more pertinent equations of Zilibotti’s work. We now examine the parameters associated with that function, \( D, Z \) and \( \theta \). Here \( D \) and \( Z \) are simply scaling parameters of the output function. Set \( D = 1 \) and \( Z = 10 \), to keep the consumption steady state solution within a reasonable
scale, and for numerical accuracy. The values of 10 and 1 are somewhat arbitrary and can be fitted to model varying levels and measures of output; Since output is not clearly defined, $D$ and $Z$ can change to fit the desired measure of output. We set $\theta = .72$, based off numerical trial and error.

Now consider the values for life expectancy. Perhaps, the most straightforward values, we set $Q = 85$ years, $q_0 = 40$ years, and $Q_0 = 35$ years. Recall that $Q_0$ is the lowest possible life expectancy whereas $q_0$ is the initial life expectancy of the country. These values are consistent with the present values throughout the world. For example present life expectancy in Mozambique is indeed 40.9 years, while the life expectancy for the United States is 78 years [20].

Next we discuss the diffusion parameters. For the technology equation, Equation (2.2), $\phi$ governs the diffusion effects for changes in life expectancy and capital. Again, we keep our parameter value consistent with Zilibotti’s setting $\phi = (0.01/5.5)$ [10]. It is noteworthy to comment on the presentation of $\phi$ here, mainly the division by 5.5; Zilibotti’s corresponding parameter is set to 0.01, however we scale ours by 5.5 in order to more easily facilitate the eventual transformed models parameters. The same holds for $\eta$.

The value 5.5 is no coincidence. When the model is transformed (Refer to Section 5.1) the $k$ component of the technology equation and life expectancy equation is modified
to a $k/h$ form. Thus we scale the present diffusion variables by 5.5, a low value for $h$ within the original modeling. This allows for the results from the transformed model to mirror the results from the original model. Note that $\gamma$ is not scaled because it corresponds to $e_q$ not $k$.

We found no basis from which to set the $\gamma$ and $\eta$ diffusion parameters in the existing literature. Thus after much trial and error reasonable scales for these values were found. We found that $\gamma$ yielded realistic results within our model when ranging from $1.2 \leq \gamma \leq 6.5$. Likewise we found that $\eta$ yielded the most realistic results when ranging from $0.075 \leq \eta \leq 0.175$. The diffusion effect for changes in life expectancy, $q$, based on time spent in health education, $e_q$, is set to $\gamma = 2$; while the diffusion effect for a change in life expectancy based off a change in capital, $k$, is set to $\eta = (0.1/5.5)$.

The above parameters play a significant role in determining the effect of health education on economic growth, however they are not the only parameters to consider, recall $\rho$ and $\delta$. In Chapter 2.1, $\delta$ is explained as a time discount factor. For large values of $\delta$, the economy is prone to spend in the present time period; whereas for smaller values of $\delta$ the economy is open to investing in future time periods. We construct the bounds for this time discount factor, $0 < \delta < 1$, and set $\delta = 0.08$. This is to suggest that our country is opposed to investment into future time periods, thus we have a reasonably high time discount factor. We set the rate of depreciation of capital, $\rho = 0.05$; a commonly set value for the depreciation rate ([4], pg. 413).
In the steady state derivations, we obtained a solution for $e_q$ analytically, that was dependent on productivity. Also, we established $B(H)$ as a constant and created both low and high values for productivity. Departing slightly from Kejak’s parameter values, we set $B_L = 0.0876$ and set $B_H = 0.095$. Although these parameters are seemingly close, the relative shift in magnitude is comparable to Kejak’s shift in productivity. With these set parameters, we can evaluate the high-low steady state values of $e_q$, using Equation (4.13). Thus for productivity low, time spent in health education $e_q = 0.0867$ and for productivity high, time spent in health education $e_q = 0.1578$. Consider that the construction of the model is centered on utility maximization, thus for low values of $B$, the skill is not present to work efficiently, thus people have to work longer to see gains in the economy and consequentially there is less time available for health education. However for higher values of productivity, they are able to spend less time working (since the time spent in production is that much more productive) and more time is permitted for health education.

Our model yields that for an economy with a low productivity level, they would spend approximately 8 – 9% of their time learning about health. Whereas for an economy with higher productivity, they would spend 15 – 16% of their time in health education. As an initial inspection these seem like innane values, however with a deeper understanding of the ideas and aspects of what actually incorporates health education, the above settings begin to make sense. Health education is a fully integrated
part of life within most developed countries, while on the contrary most developing
countries do not fully incorporate health education into other forms of education.
Basic lessons such as bacteria transferal, boiling water, the concept of germs are not
normally taught nor understood. Thus using the above percentages, we see that 15%
of the workday, could be appointed to time spent boiling water, learning about sex-
ually transmitted diseases, learning and pursuing vaccinations, and the like. Thus it
is no longer fully irrational to spend as much as 16% of the workday learning how to
be healthy.

Lastly we investigate the terms of the Cobb-Douglas production function. The multi-
factor productivity constant $G$ is set to $G = 5$. Again, this parameter is simply a
measure of the synergy of both labor and capital producing together and is somewhat
arbitrary. Another prominent parameter, $\alpha$ is set to $\alpha = .25$. Recall that $\alpha$ represents
the production share of capital within an economy, thus with $\alpha = .25$ and $(1-\alpha) = .75$
it implies that our economy is labor intensive. This value, slightly lower than Romer’s
approximation of $\alpha = .33$ in *A Contribution to the Empirics of Economic Growth*, is
chosen to better model a developing country ([4], pg.417). With the parameters set
and explained, we can move on to the numerical simulation.

4.4 Numerical Solutions and Interpretations

The results from the numerical solution of Equations (4.13), (4.14), (4.15), and (4.16)
are presented in Table 4.1. The pertinent thing to take note of from Table 4.1 is the
Table 4.1: Equilibrium Values for the Original Model

<table>
<thead>
<tr>
<th>$B(H)$</th>
<th>$e_p$</th>
<th>$e_q$</th>
<th>$k$</th>
<th>$h$</th>
<th>$q$</th>
<th>$A$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_L = .0876$</td>
<td>0</td>
<td>0.0867</td>
<td>113.923</td>
<td>6.101</td>
<td>43.948</td>
<td>0.046979</td>
<td>123.023</td>
</tr>
<tr>
<td>$B_H = .095$</td>
<td>0</td>
<td>0.15789</td>
<td>108.715</td>
<td>6.314</td>
<td>44.957</td>
<td>0.043084</td>
<td>111.946</td>
</tr>
</tbody>
</table>

change in value of the parameters from $B_L$ to $B_H$. As productivity changes from low to high, how does this change affect the results? We now examine each variable independently, commenting on its range and change with productivity in order to extract the relevant results from this analysis.

Consider time spent in health education, $e_q$. As productivity changes, $e_q$ nearly doubles from about 9% to roughly 16% of the day. We have already noted that it is reasonable for a developing country to invest such a large portion of their day in health education. We surmise that as productivity increases, the effectiveness of time spent in the work-place also increases, thus more can be accomplished in less time. Thus there is ample time in the day to invest more in health education and prolonging longevity.

We believe that it is this decreased time spent in the work-place that leads to a decrease in capital. Although the decrease in capital (as productivity increases) is not as relatively significant as the change in time spent in health education, it is still
noteworthy to assume that a decreased time spent in the work-place creates a slightly
diminished need for capital. Also, since less time is spent in production, less output
is produced, thus there is slightly less consumption as well. Likewise with technology,
which is a function of both capital and life expectancy, as capital decreases, technol-
ogy decreases as well. Note that life expectancy has increased, which should lead to a
positive growth in technology. However, we deduce that the change in life expectancy
is not as significant as the change in capital for this particular set of parameters, thus
technology decreases.

But life expectancy does indeed increase! So as expected, for more time spent in
health education, the life expectancy of a country increases. Even though the in-
crease is only a year as productivity increases from $B_L$ to $B_H$, it is worthwhile to
note that life expectancy has increased by $4 - 5$ years from our initial 40 year life
expectancy.

It is important to consider the continuity of our steady state results as a function of
the parameters $\gamma, \eta, \phi, \delta, \rho, B(H)$, and $G$. Do small changes in the parameters cause
unprecedented changes to the steady state or is there some continuous curve that our
steady state solutions remain upon? Table 4.2 yields the altered steady state with
a 1% increase to all of the base parameters. As expected we observe that there is a
high degree of proximity between the values shown in Table 4.2 and Table 4.1. This
proximity suggests that the steady state solutions are continuous functions of the
parameters and permits us to consider varying changes along the continuous solution path.

Next, we observe the changes in the steady state for $B(H) = B_L$ as one parameter is fluctuated leaving the rest of the parameters constant. The aim of this exercise is to more fully understand the responses of the economy to a change in the parameters. The full results are expressed in Table 4.3 with the interpretations following. An entry of (+) represents that as the parameter changed the variable increased from the original steady state for $B_L$; while an entry of (-) signifies a decrease from the original steady state. Furthermore an entry of 'nc' signifies that no significant change was observed from the original steady state as the parameter changed.

Consider the variable $\alpha$, the fractional allocation of capital within an economy. Thus it is expected that as $\alpha$ increases capital should increase as well; and since an increase in capital causes an increase in technology level, life expectancy, and consumption it is also straightforward that these variables should increase with $\alpha$. Lastly, human

<table>
<thead>
<tr>
<th>$B(H)$</th>
<th>$c_p$</th>
<th>$c_q$</th>
<th>$k$</th>
<th>$h$</th>
<th>$q$</th>
<th>$A$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_L = .0885$</td>
<td>0</td>
<td>0.0868</td>
<td>114.025</td>
<td>5.965</td>
<td>44.074</td>
<td>0.047913</td>
<td>124.749</td>
</tr>
<tr>
<td>$B_H = .0960$</td>
<td>0</td>
<td>0.15833</td>
<td>108.857</td>
<td>6.179</td>
<td>45.127</td>
<td>0.042248</td>
<td>113.922</td>
</tr>
</tbody>
</table>
Table 4.3: Parameter Fluctuations and Steady State Response: Case $B = B_L$

<table>
<thead>
<tr>
<th>Parameter Change</th>
<th>$e_p$</th>
<th>$e_q$</th>
<th>$k$</th>
<th>$h$</th>
<th>$q$</th>
<th>$A$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Increase $\alpha$</td>
<td>nc</td>
<td>nc</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Increase $G$</td>
<td>nc</td>
<td>nc</td>
<td>nc</td>
<td>-</td>
<td>nc</td>
<td>nc</td>
<td>nc</td>
</tr>
<tr>
<td>Decrease $\rho$</td>
<td>nc</td>
<td>nc</td>
<td>nc</td>
<td>-</td>
<td>nc</td>
<td>nc</td>
<td>-</td>
</tr>
<tr>
<td>Decrease $\delta$</td>
<td>nc</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Increase $\eta$</td>
<td>nc</td>
<td>nc</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Increase $\gamma$</td>
<td>nc</td>
<td>nc</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Increase $\phi$</td>
<td>nc</td>
<td>nc</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Increase $q_0$</td>
<td>nc</td>
<td>nc</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Increase $A_0$</td>
<td>nc</td>
<td>nc</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
skill level, $h$, decreases as $\alpha$ increases. Consider the following economic justification, as the market share of capital increases, labor must decrease, likewise skilled labor will also decrease; thus a decrease in $h$.

A change in the multi-factor productivity constant $G$ causes little change throughout the system, only minimally decreasing $h$. However decreasing $\rho$ and $\delta$ cause a myriad of changes throughout the model. A decrease in $\rho$, the depreciation rate of capital, decreases both $h$ and $c$, human skill level and consumption. However, the truly noteworthy variable here is $\delta$, the time discount factor. As $\delta$ decreases, as people are more inclined to invest in the future, we observe that $e_q$ rises! We explain this by the following. First it is clear from Equation (4.13) that a decrease in $\delta$ causes an increase in the steady state value of $e_q$. Second we assume that if adequate health training is provided that addresses the immediate concern of the people then they will choose to invest in this option. To elaborate, consider the difference between enabling life (health education for a developing country) or prolonging death (health education and care for a developed country, where life is presently enabled with ease). It is this difference that causes a rise in $e_q$. We assume the health education has an immediate effect. In turn $q$ does increase, but the remaining variables (save of course $e_p$) decrease due to the prolonged time now spent in health education.

The ramifications of a change in the diffusion parameters are fairly straightforward. As one would expect, an increase in $\eta$ the diffusion parameter for a change in $e_q$ causes
life expectancy, $q$, to rise. However in turn, all other parameters (except $e_q$ and $e_p$) decrease. Recall that this steady state is solely one point that the economy could converge to or diverge from and that the strict restrictions of the steady state (all time derivatives equal to zero) are more than likely the cause of the decrease in parameters. When $\gamma$ increases, all the parameters (again save $e_q$ and $e_p$) increase, as is expected for a positive change in diffusion. Similar to $\eta$, an increase in $\phi$ serves to increase technology as well as consumption, while decreasing $q$, $h$, and $k$. For higher values of $\phi$ we observe that skill level decreases since technology serves as substitute for skill.

One of the more interesting elements of this parameter experiment to analyze is the initial condition parameters. What if the country we considered did not have an initially low life expectancy? What if the initial level of technology was not substantially low? How do these changes affect the results of the model? For an increase in the initial life expectancy, $q_0$, we note that all variables increase (once again save $e_q$ and $e_p$), however what is not shown here is that this increase is relatively small. Given that the initial level of life expectancy is now higher, the increases of $c$, $k$, $h$, and $A$ are essentially negligible. This holds with economic theory that for an initially low level of life expectancy health education will cause significant growth. However, for a higher level of longevity health education will only cause minor growth to life expectancy, due to the logistic nature of the function and the diminishing returns of health education. In contrast, with a higher initial level of technology, $A_0$, all the variables that change, decrease. However we note that although all parameters
decrease in value from the original steady state, since we have changed the initial conditions by augmenting $A_0$, some of the parameters do show an increase from the augmented initial condition. It is again, due to a relatively small amount of growth within this system that has caused these values to remain smaller than the original steady state values. Therefore, as with a change in initial life expectancy, the growth in this case is relatively small. Thus our original assumption holds. This modeling best serves developing countries and fails to adequately model the ramifications of health education on a developed country.

Recall that for the steady state analysis we determined $e_p = 0$; no time spent in production education. We assume that there is some productivity jump, but there is no engine for that jump within the model. Economists call such behavior an economic growth miracle. We find this solution to be dissatisfactory. Since $e_p = 0$ indicates no endogenous change in skill level, we seek a steady state solution that permits time spent in production education, $e_p$ to be non-zero. Thus we have constructed the transformed model. To further support the transformed model, we again cite two relevant articles, Health and Endogenous Growth by A. Zon and J. Muysken [19] and Endogenous Longevity, Health and Economic Growth: A Slow Growth for a Longer Life? by R. Aisa and F. Pueyo [18]. Both these papers articulate the give-and-take relationship of health education and production education; sometimes these two elements are complementary, sometimes they are rival elements. Regardless, both papers pronounce the need for both educational elements within a growing economy.
CHAPTER V
STEADY STATE SOLUTIONS FOR THE TRANSFORMED MODEL

5.1 Constructing the Transformed Model

Now, we consider the transformations discussed in Section 2.2. Our aim is to reduce the system of five variables into a system of four variables by utilizing ratios as variables, $x = \frac{k}{h}$ and $v = \frac{c}{k}$. Note that although we will solve for $x$ and $v$, we will also need to backsolve for $h$, $c$, and $k$. We do this by solving for $x$ and $v$, then finding $h$ from Equation (3.13), which is described independently.

Steady state solutions in the transformed model correspond to dynamic paths for $k$ and $c$ due to the time dependent nature of $h$. For the majority of the solution procedure Equations (3.3)-(3.8) remain unchanged. Even though technology now has a human skill level component, we assume that, as with the derivative of capital, that $h$ is counted as aggregate human skill level for that derivative and thus is not considered for a part of the derivations. However the derivatives of Equation (3.9), Equation (3.10), and Equation (3.11), are fundamentally changed since they directly involve the new technology and life expectancy equations, Equations (2.16) and (2.19). Since $A$ is dependent upon $q$, and $q$ is dependent upon $e_q$, the $\frac{dA}{dq}$ and $\frac{dq}{de_q}$ components of
Equation (3.9) become

\[ HAM_{eq} = 0 = \lambda \left[ Dk \frac{dA}{dq} \frac{dq}{de_q} + Zk^\theta \theta A^{\theta-1} \frac{dA}{dq} \frac{dq}{de_q} \right] - \lambda w h, \]  

(5.1)

\[ \frac{dA}{dq} = \frac{a \phi x(t)(\frac{Q}{Q_0} - 1)e^{-\phi[qz(q)-q(0)x(0)]}}{[1 + (\frac{Q}{Q_0} - 1)e^{-\phi[qz(q)-q(0)x(0)]}]^2}, \]  

(5.2)

\[ \frac{dq}{de_q} = \frac{Q \gamma (Q_0 - 1)e^{-\gamma[e_q - e_q(0)] + \eta(x(t) - x(0))}}{[1 + (\frac{Q}{Q_0} - 1)e^{-\gamma[e_q - e_q(0)]}]^2}. \]  

(5.3)

One might note that \( k \) remains in the above Equation (5.1), however since the steady state solution procedure remains the same despite our change of variables, capital will eventually be replaced using the constraint given from the profit maximization equation \( \frac{d\Pi}{dk} \) shown in Appendix B.5. There, Equation (C.20) is the reduced form of Equation (3.9), as shown in Equation (5.4), only now the derivative of technology with respect to life expectancy and the derivative of life expectancy with respect to time spent in health education are based off the transformed Equations (2.16) and (2.19):

\[ \frac{dA}{dq} \frac{dq}{de_q} = A \frac{1 - \alpha}{\alpha} \frac{1}{1 - e_p - e_q}. \]  

(5.4)

Thus Equation (5.4) becomes one of the four equations we need in order to solve our system of four unknowns \((x, v, e_q, e_p)\). To find the other three governing equations, we transform the governing Equations (3.12)-(3.16). The transformation process, which is fully derived in Appendix B.5, begins by finding the derivatives and growth rates of \( v \) and \( x \), which gives

\[ \frac{dx}{dt} - \frac{dk}{dt} \frac{k}{h^2} \frac{h}{k} = \frac{dk}{dt} \frac{k}{h} - \frac{dh}{dt} \frac{k}{h}, \]  

(5.5)

\[ \frac{dv}{dt} - \frac{dk}{dt} \frac{k}{c^2} \frac{c}{k} = \frac{dk}{dt} \frac{k}{c} - \frac{dc}{dt} \frac{k}{c}. \]  

(5.6)
By substituting in the original governing equations, Equations (3.12), (3.13), and (3.14), and by transforming the variables where applicable we obtain the remaining three governing equations:

\[
\frac{\dot{x}}{x} = DA \left(1 + \frac{1}{\theta}\right) + G \left(1 - \frac{e_p - e_q}{x}\right)^{1-\alpha} \left(1 - \alpha + \frac{\alpha}{\theta}\right) - \rho - v - B(H)e_p, \quad (5.7)
\]

\[
\frac{\dot{v}}{v} = \left(\frac{\alpha}{\sigma} - 1 + \alpha - \frac{\alpha}{\theta}\right) G \left(1 - \frac{e_p - e_q}{x}\right)^{1-\alpha} + v + \left(1 - \frac{1}{\sigma}\right) \rho - \frac{\delta}{\sigma} - DA \left(1 + \frac{1}{\theta}\right),
\]

\[
\frac{d e_p}{d t} + \frac{d e_q}{d t} = -\frac{d x}{x} + \frac{1}{\alpha} \left[G\alpha\left(1 - \frac{e_p - e_q}{x}\right)^{1-\alpha} - \rho\right] - \frac{1}{\alpha}(1 - e_q)B(H). \quad (5.9)
\]

Thus our four unknowns \(x, v, e_p, e_q\) can be solved using the four Equations (5.4), (5.7), (5.8), and (5.9).

### 5.2 Numerical Solutions of the Transformed Model

The remaining tasks are as follows: find the steady state solutions of the transformed model, linearize the model about the steady state solutions in order to determine the stability of these critical points, and apply an extensive interpretation to the given results. The solution procedure for finding the steady state values for the transformed four variable \((x, v, e_q, e_p)\), problem using Equations (5.4), (5.7), (5.8), and (5.9) is as follows. Set all the time derivatives to zero. Use Equation (5.4) to solve for \(e_p\) explicitly in terms of \(e_q\) and \(x\). Plug this term into Equations (5.7), (5.8), and (5.9) in order to reduce the four variable system into a system of only three equations and three unknowns. However, the time derivative for \(e_p\) must be solved for using the Chain rule and the explicit formulation for \(e_p\). Furthermore, when numerically solving for
Table 5.1: Base Steady State of the Transformed Model

<table>
<thead>
<tr>
<th>$e_q$</th>
<th>$e_p$</th>
<th>$u$</th>
<th>$x$</th>
<th>$v$</th>
<th>$q$</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0149</td>
<td>0.1797</td>
<td>0.8054</td>
<td>15.4616</td>
<td>0.5130</td>
<td>40.3605</td>
<td>0.05002</td>
</tr>
</tbody>
</table>

the time derivative $e_q$, one must algebraically rearrange Equation (5.9). The system then can be solved numerically and undergo a linear stability analysis as well. The derivations, found in Appendix C.2, fully reveal the above solution procedure.

Thus using Maple to solve Equation (5.4) for $e_p$, then using the remaining Equations (5.7), (5.8), and (5.9) to find $v, e_q,$ and $x$ yields the results shown in Table 5.1. This base set of parameters differs slightly from the base set of parameters given in Section 4.3. This is primarily due to the fact that the $\eta$ and $\phi$ values are now augmented by $h$, thus no longer divided by 5.5. Also the overall value of $\phi$ is slightly increased in order to present a broader scope of findings as shown in Table 5.4. We used the following base set of parameters to obtain this solution: $\phi = 0.02, \gamma = 2.00, \eta = 0.1, \delta = .08, \rho = .05, \alpha = .25, \theta = .72, \sigma = .4$, and $B(H) = .0876$, with initial conditions $x_0 = 13$ and $e_{q0} = .01$. Again, refer to Appendix C.2 for the derivations and to Appendix D.2 for the Maple code. Note as well that $q$ and $A$ are displayed in the table; although not explicitly stated within the four equations, $q$ and $A$ are implicitly woven in to the system of equations and denoting their changes and values is worthwhile to observe for further interpretation.
Table 5.2: Effects of Parameter Variations on the Base State Values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>% Change</th>
<th>$e_q$</th>
<th>$e_p$</th>
<th>$u$</th>
<th>$x$</th>
<th>$v$</th>
<th>$q$</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.07</td>
<td>-12.5%</td>
<td>0.12273</td>
<td>0.19546</td>
<td>.68181</td>
<td>14.40426</td>
<td>0.47135</td>
<td>42.69588</td>
<td>0.046928</td>
</tr>
<tr>
<td></td>
<td></td>
<td>+725.340%</td>
<td>+8.770%</td>
<td>-15.348%</td>
<td>-6.840%</td>
<td>-8.118%</td>
<td>+5.786%</td>
<td>-6.184%</td>
</tr>
<tr>
<td>0.081</td>
<td>+1.25%</td>
<td>0.004348</td>
<td>0.17748</td>
<td>0.81817</td>
<td>15.56573</td>
<td>0.51715</td>
<td>40.13573</td>
<td>0.050252</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-70.760%</td>
<td>-1.235%</td>
<td>+1.582%</td>
<td>+0.674%</td>
<td>+0.809%</td>
<td>-0.559%</td>
<td>+0.457%</td>
</tr>
<tr>
<td>$B(H)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.95</td>
<td>+8.447%</td>
<td>0.10165</td>
<td>0.14061</td>
<td>.75774</td>
<td>14.68314</td>
<td>0.51140</td>
<td>42.39253</td>
<td>0.049492</td>
</tr>
<tr>
<td></td>
<td></td>
<td>+583.554%</td>
<td>-21.754%</td>
<td>-5.921%</td>
<td>-5.035%</td>
<td>-0.311%</td>
<td>+5.035%</td>
<td>-1.058%</td>
</tr>
<tr>
<td>$e_{q0}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>.005</td>
<td>-50%</td>
<td>0.014011</td>
<td>0.18187</td>
<td>.80412</td>
<td>15.42523</td>
<td>0.51313</td>
<td>40.458808</td>
<td>0.050038</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-5.778%</td>
<td>+1.207%</td>
<td>-0.163%</td>
<td>-0.235%</td>
<td>+0.027%</td>
<td>+0.244%</td>
<td>+0.033%</td>
</tr>
</tbody>
</table>

From Table 5.1, we will now analyze percent changes in the steady state caused by changes in the parameters. From the following analysis, we will determine the cases that best answer our original query of how health education affects the economy, then further analyze those cases in the concluding section.

Table 5.2 reveals a variety of insights into the nature of the model. Most notable, is the time discount factor, $\delta$. Observe that for a 12.5% decrease in the time discount factor (that is citizens within the economy begin to increase their value of future utility), $e_q$ shoots up by 725% and $e_p$ also shoots up by roughly 9%. In response to
less time working, $x$ and $v$ drop (the capital to skill level ratio and the consumption to capital ratio), while life expectancy increases. Thus as $\delta$ decreases, people within the economy become less likely to consume in the present and more likely to invest and consume in the future periods. Therefore, $\delta$ does indeed contribute to the effective value of education. For lower time discount rates, education becomes a more valuable commodity. The contrast can be shown for the 1.25% increase in $\delta$. Thus one primary element of a successful third-world education program would have to be ensuring that the people involved cared more and more about the future.

It is important to note the changes in $v$ as well. Recall that $v = c/k$, thus $v$ points us to increases and decreases in consumption within the model. Note that for lower $\delta$, we observe a decrease of roughly 7% in $v$. This can be attributed to a decrease in consumption, increase in capital, or both. However for this particular change, a lower time discount factor, we suggest a decrease in consumption. The argument being that the decision makers within this economy have apportioned more time for education (both health and production) and less time for present production, hoping for later payoffs of longevity and more effective production in the future.

The increase in productivity ($B(H)$) causes an expected decrease in time spent in production education, and consequentially allowing people to invest more in health education without a detrimental decrease in capital or consumption. Note that even though the decrease in $x$ is significant at 5%, this can be attributed to a large increase
in skill level, and a minor decrease in capital (recall that \( x = (k/h) \)). For a more in-depth look at how productivity affects economic growth, refer to Kejak [11].

Lastly, the decrease in the initial amount of time spent in health education, \( e_{q0} \), is simply a means to ensure that the parameters are set for a developing country. Note that there is actually very little change in the steady state solution, even for a large change in the amount of time spent in health education. The lack of significant change indicates that the modeled economy is indeed developing, since it is plausible that despite different initial conditions, that a poor country (or economy) will remain poor.

Table 5.3 displays the preference parameters; the Cobb-Douglas coefficient, \( \alpha \), a measure of elasticity, \( \sigma \), and the output function parameter, \( \theta \). The table implies that \( \theta \) has little effect on the majority of the variables, save \( v \), where an increase in \( \theta \) causes \( v \) to decrease (thus causing consumption to decrease; recall \( v = c/k \)). And in contrast a decrease in \( \theta \) causes consumption to increase. Considering this case trivial we move on to the two other significant parameters.

Recall that \( \alpha \) is the Cobb-Douglas coefficient. Our setting of \( \alpha = 0.25 \) implies that 25% of our production comes from capital, while the other 75% comes from skilled labor. This is consistent for a developing economy where capital is generally in short supply and human laborers (both skilled and unskilled) are readily available. Thus our percent changes in \( \alpha \) convey an interesting message. As \( \alpha \) increases (as the pro-
Table 5.3: Effect of Elasticity, Utility, and Cobb-Douglas Parameter Variations on the Base State

<table>
<thead>
<tr>
<th>Parameter</th>
<th>% Change</th>
<th>$e_q$</th>
<th>$e_p$</th>
<th>$u$</th>
<th>$x$</th>
<th>$v$</th>
<th>$q$</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.255</td>
<td>+2.00%</td>
<td>0.00076146</td>
<td>0.214993</td>
<td>0.784251</td>
<td>15.58034</td>
<td>0.50462</td>
<td>40.01408</td>
<td>0.049820</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-94.916%</td>
<td>+19.640%</td>
<td>-2.627%</td>
<td>+0.768%</td>
<td>-1.632%</td>
<td>-0.858%</td>
<td>-0.403%</td>
</tr>
<tr>
<td>$\alpha$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.23</td>
<td>-8.00%</td>
<td>0.073190</td>
<td>0.0033921</td>
<td>0.89289</td>
<td>14.96979</td>
<td>0.54877</td>
<td>41.79210</td>
<td>0.050546</td>
</tr>
<tr>
<td></td>
<td></td>
<td>+392.161%</td>
<td>-81.124%</td>
<td>+10.859%</td>
<td>-3.181%</td>
<td>+6.974%</td>
<td>+3.547%</td>
<td>+1.048%</td>
</tr>
<tr>
<td>$\theta$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.85</td>
<td>+18.056%</td>
<td>0.014875</td>
<td>0.17971</td>
<td>0.80543</td>
<td>15.46157</td>
<td>0.49467</td>
<td>40.36050</td>
<td>0.050021</td>
</tr>
<tr>
<td></td>
<td>≤</td>
<td>.005%</td>
<td>≤</td>
<td>.005%</td>
<td>≤</td>
<td>.005%</td>
<td>≤</td>
<td>.005%</td>
</tr>
<tr>
<td>$\theta$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.65</td>
<td>-9.722%</td>
<td>0.014875</td>
<td>0.17971</td>
<td>0.80543</td>
<td>15.46157</td>
<td>0.52590</td>
<td>40.36050</td>
<td>0.050021</td>
</tr>
<tr>
<td></td>
<td>≤</td>
<td>.005%</td>
<td>≤</td>
<td>.005%</td>
<td>≤</td>
<td>.005%</td>
<td>≤</td>
<td>.005%</td>
</tr>
<tr>
<td>$\sigma$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.45</td>
<td>+12.5%</td>
<td>0.0066654</td>
<td>0.17798</td>
<td>0.81535</td>
<td>15.54280</td>
<td>0.51623</td>
<td>40.18467</td>
<td>0.050203</td>
</tr>
<tr>
<td></td>
<td>-55.176%</td>
<td>-0.955%</td>
<td>+1.232%</td>
<td>+0.525%</td>
<td>+0.631%</td>
<td>-0.436%</td>
<td>+0.363%</td>
<td></td>
</tr>
<tr>
<td>$\sigma$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.35</td>
<td>-12.5%</td>
<td>0.023272</td>
<td>0.18139</td>
<td>0.79534</td>
<td>15.37862</td>
<td>0.50970</td>
<td>40.54066</td>
<td>0.049829</td>
</tr>
<tr>
<td></td>
<td>+56.500%</td>
<td>+0.940%</td>
<td>-1.253%</td>
<td>-0.536%</td>
<td>-0.643%</td>
<td>+0.446%</td>
<td>-0.385%</td>
<td></td>
</tr>
</tbody>
</table>
duction share of capital grows) the need for health education drops and the need for production education rises. For a decrease in $\alpha$ (as human labor’s production share grows) the need for health education increases and the need for production education decreases. As is congruent with the work of economist E. F. Schumacher [22], the need for intermediate tools is presented to help ensure health within a developing country. Intermediate tools being some form of capital that increases productivity, but allows for a large amount of human work. The measure of elasticity $\sigma$ behaves as expected. For an increase in $\sigma$, the amount of time spent in both health education and production education decreases, due to the increased value of time spent working. In contrast, for a decrease in elasticity, health and production education increase.

Lastly, Table 5.4 shows the effects of the diffusion parameters. These parameters measure the rate at which a change in, technology, life expectancy, or capital permeates throughout the economy. Consider $\eta$, the change in capital (or in this case capital to skill level ratio) parameter; An increase in $\eta$, which makes changes in capital more influential towards life expectancy, causes an increase in time spent in health education, a decrease in time spent in production education, and an increase in time spent in production. Most notable however is the increase in $x$, which is presumably an increase in capital.

Recall that $\phi$ is the diffusion parameter for technology as life expectancy and the capital to skill level ratio change. We investigate solely a decrease in $\phi$. As $\phi$ decreases,
Table 5.4: Effects of Diffusion Parameter Variations on the Base State

<table>
<thead>
<tr>
<th>Parameter</th>
<th>% Change</th>
<th>$e_q$</th>
<th>$e_p$</th>
<th>$u$</th>
<th>$x$</th>
<th>$v$</th>
<th>$q$</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta$</td>
<td>-25.00%</td>
<td>0.021826</td>
<td>0.16233</td>
<td>0.81584</td>
<td>15.75549</td>
<td>0.51142</td>
<td>39.81839</td>
<td>0.051116</td>
</tr>
<tr>
<td>0.075</td>
<td></td>
<td>+46.743%</td>
<td>-9.666%</td>
<td>+1.293%</td>
<td>+1.901%</td>
<td>-0.307%</td>
<td>-1.343%</td>
<td>+2.188%</td>
</tr>
<tr>
<td>$\eta$</td>
<td>+25.00%</td>
<td>0.0093482</td>
<td>0.19352</td>
<td>0.79713</td>
<td>15.23022</td>
<td>0.51422</td>
<td>40.81787</td>
<td>0.049225</td>
</tr>
<tr>
<td>0.125</td>
<td></td>
<td>-37.170%</td>
<td>+7.693%</td>
<td>-1.031%</td>
<td>-1.496%</td>
<td>+2.384%</td>
<td>+1.133%</td>
<td>-1.592%</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>-25.00%</td>
<td>0.0083055</td>
<td>0.19613</td>
<td>0.79556</td>
<td>15.18682</td>
<td>0.51738</td>
<td>39.51853</td>
<td>0.041551</td>
</tr>
<tr>
<td>1.50</td>
<td></td>
<td>-44.183%</td>
<td>+9.144%</td>
<td>-1.225%</td>
<td>-1.777%</td>
<td>+0.854%</td>
<td>-2.086%</td>
<td>-16.933%</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>+25.00%</td>
<td>0.018770</td>
<td>0.16997</td>
<td>0.81126</td>
<td>15.62584</td>
<td>0.51039</td>
<td>40.96714</td>
<td>0.055074</td>
</tr>
<tr>
<td>2.50</td>
<td></td>
<td>+26.191%</td>
<td>-5.415%</td>
<td>+7.238%</td>
<td>+1.062%</td>
<td>-0.509%</td>
<td>+1.503%</td>
<td>+10.101%</td>
</tr>
<tr>
<td>$\phi$</td>
<td>-50.00%</td>
<td>0.0048443</td>
<td>0.20478</td>
<td>0.79037</td>
<td>15.04326</td>
<td>0.52452</td>
<td>39.051061</td>
<td>0.024691</td>
</tr>
<tr>
<td>0.01</td>
<td></td>
<td>-67.422%</td>
<td>+13.959%</td>
<td>-1.870%</td>
<td>-2.705%</td>
<td>+2.242%</td>
<td>-3.244%</td>
<td>-50.639%</td>
</tr>
</tbody>
</table>
time spent in health education drops, while time spent in production education increases, and actual time spent in production decreases. The capital-skill level ratio decreases (presumably from an increase in skill level) and life expectancy drops. Essentially the interpretation for this change of steady-state is as follows: for lower diffusion values, $\phi$, investing in health education is not worthwhile. The laborers of a country would rather invest time in production education and work effectively than live longer and work longer. However as $\phi$ increases (as we return to our base state), the incentive to invest in health education is reintroduced. Now time spent in health education increases life expectancy, which more effectively increases technology, which is key (according to Solow) towards economic growth. Thus we view $\phi$ as an incentive towards health education and a vital part of inducing continued economic growth. This is consistent with the findings of Zilibotti [10].

The diffusion parameter $\gamma$ is a gauge of the effectiveness of health education. So for lower values of $\gamma$, the opportunity costs of time spent in health education are not worth the small increase of life expectancy. However, for higher values of $\gamma$, the opportunity costs of health education (i.e. not spending time in production or production education) are superceded by the value of increased life expectancy and all the advantages that come with longevity, which for our model includes increased technology.

This description is congruent with the steady-state comparisons shown in Table 5.4. One can note that for increases in $\gamma$, time spent in health education does indeed
increase, as well as technology and life expectancy. Observe that time spent in production education decreases, but that overall time spent in production itself increases. We seek to further analyze the conditions surrounding this final steady state solution and will do so in the proceeding section by means of a linear stability analysis.
CHAPTER VI
LINEAR STABILITY ANALYSIS OF THE EQUILIBRIUM STATE

6.1 An Underdeveloped Economy

A linear stability analysis of the system of equations for varying values of $\gamma$, thus differing levels of effectiveness of health education, yields the results shown in Tables 6.1 through 6.5. Only the eigenvectors corresponding to the positive eigenvalue are shown. The other two eigenvectors have negative real parts. We consider only eigenvectors with positive real parts because that eigenvector will determine the overall direction of the steady state as it travels along an unsteady path.

In this case the positive eigenvalue is associated with an eigenvector that points primarily in the $x$ direction in an $x,v,e_q$ system. As is apparent from the tables, as $\gamma$ increases (as time spent in health education becomes more and more worthwhile) the norm or magnitude of the eigenvector begins to decrease, while the real part of the eigenvalue begins to increase. We examine varying $\gamma$ values under two scenarios, first $B_L$ then $B_H$, to demonstrate the differences between increases in effective health education and increases in productivity.
Table 6.1: Positive Real Part of the Eigenvalue and Corresponding Eigenvector in the \((x, v, e_q)\) Coordinate Frame for \(\gamma = 1.2\)

<table>
<thead>
<tr>
<th>(B_L = 0.0876)</th>
<th>(B_H = .095)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steady State</td>
<td>Steady State</td>
</tr>
<tr>
<td>(x = 14.89879)</td>
<td>(x = 14.37227)</td>
</tr>
<tr>
<td>(v = 0.52186)</td>
<td>(v = 0.51949)</td>
</tr>
<tr>
<td>(e_q = 0.00134127)</td>
<td>(e_q = 0.09434)</td>
</tr>
<tr>
<td>(e_p = 0.213542)</td>
<td>(e_p = 0.158882)</td>
</tr>
<tr>
<td>(u = 0.78512)</td>
<td>(u = 0.74678)</td>
</tr>
<tr>
<td>(q = 38.74462)</td>
<td>(q = 39.99037)</td>
</tr>
<tr>
<td>(A = 0.0329968)</td>
<td>(A = 0.0320774)</td>
</tr>
</tbody>
</table>
Table 6.2: Positive Real Part of the Eigenvalue and Corresponding Eigenvector in the \((x, v, e_q)\) Coordinate Frame for \(\gamma = 1.5\).

<table>
<thead>
<tr>
<th>(B_L = 0.0876)</th>
<th>(B_H = .095)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steady State</td>
<td>Steady State</td>
</tr>
<tr>
<td>(x = 15.18681)</td>
<td>(x = 14.564139)</td>
</tr>
<tr>
<td>(v = 0.517378)</td>
<td>(v = 0.515264)</td>
</tr>
<tr>
<td>(e_q = 0.0083055)</td>
<td>(e_q = 0.0988647)</td>
</tr>
<tr>
<td>(e_p = 0.19613)</td>
<td>(e_p = 0.147575)</td>
</tr>
<tr>
<td>(u = 0.79556)</td>
<td>(u = 0.753560)</td>
</tr>
<tr>
<td>(q = 39.51853)</td>
<td>(q = 41.077545)</td>
</tr>
<tr>
<td>(A = 0.041551)</td>
<td>(A = 0.0408452)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(B_L = 0.0876)</th>
<th>(B_H = .095)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steady State</td>
<td>Steady State</td>
</tr>
<tr>
<td>(x = 15.18681)</td>
<td>(x = 14.564139)</td>
</tr>
<tr>
<td>(v = 0.517378)</td>
<td>(v = 0.515264)</td>
</tr>
<tr>
<td>(e_q = 0.0083055)</td>
<td>(e_q = 0.0988647)</td>
</tr>
<tr>
<td>(e_p = 0.19613)</td>
<td>(e_p = 0.147575)</td>
</tr>
<tr>
<td>(u = 0.79556)</td>
<td>(u = 0.753560)</td>
</tr>
<tr>
<td>(q = 39.51853)</td>
<td>(q = 41.077545)</td>
</tr>
<tr>
<td>(A = 0.041551)</td>
<td>(A = 0.0408452)</td>
</tr>
</tbody>
</table>
Table 6.3: Positive Real Part of the Eigenvalue and Corresponding Eigenvector in the \((x, v, e_q)\) Coordinate Frame for \(\gamma = 2.0\)

<table>
<thead>
<tr>
<th>(B_L = 0.0876)</th>
<th>(B_H = .095)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steady State</td>
<td>Steady State</td>
</tr>
<tr>
<td>Eigenvector:</td>
<td>Eigenvector:</td>
</tr>
<tr>
<td>(x = 15.46157)</td>
<td>(x = 14.68314)</td>
</tr>
<tr>
<td>(v = 0.51300)</td>
<td>(v = 0.51140)</td>
</tr>
<tr>
<td>(e_q = 0.014875)</td>
<td>(e_q = 0.10165)</td>
</tr>
<tr>
<td>(e_p = 0.17971)</td>
<td>(e_p = 0.14061)</td>
</tr>
<tr>
<td>(u = 0.80543)</td>
<td>(u = 0.75774)</td>
</tr>
<tr>
<td>(q = 40.36050)</td>
<td>(q = 42.39253)</td>
</tr>
<tr>
<td>(A = 0.050021)</td>
<td>(A = 0.049492)</td>
</tr>
</tbody>
</table>
Table 6.4: Positive Real Part of the Eigenvalue and Corresponding Eigenvector in the \((x, v, e_q)\) Coordinate Frame for \(\gamma = 2.5\)

<table>
<thead>
<tr>
<th></th>
<th>(B_L = 0.0876)</th>
<th>(B_H = .095)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steady State</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x)</td>
<td>15.62584</td>
<td>14.703245</td>
</tr>
<tr>
<td>(v)</td>
<td>0.51039</td>
<td>0.509321</td>
</tr>
<tr>
<td>(e_q)</td>
<td>0.018770</td>
<td>0.102121</td>
</tr>
<tr>
<td>(e_p)</td>
<td>0.16997</td>
<td>0.139434</td>
</tr>
<tr>
<td>(u)</td>
<td>0.81126</td>
<td>0.758445</td>
</tr>
<tr>
<td>(q)</td>
<td>40.96714</td>
<td>43.433827</td>
</tr>
<tr>
<td>(A)</td>
<td>0.055074</td>
<td>0.0546233</td>
</tr>
</tbody>
</table>
Table 6.5: Positive Real Part of the Eigenvalue and Corresponding Eigenvector in the \((x, v, e_q)\) Coordinate Frame for \(\gamma = 3.0\)

| \(B_L = 0.0876\) | \(B_H = .095\) |
|------------------|--|------------------|--|
| **Steady State** | **Eigenvector:** | **Steady State** | **Eigenvector:** |
| \(x = 15.735206\) | 5.09999965 | \(x = 14.680051\) | 10.1273087 |
| \(v = 0.508655\) | 1.00834182 | \(v = 0.508010\) | 1.16281837 |
| \(e_q = 0.0213490\) | -.23880473 | \(e_q = 0.101579\) | -.60551257 |
| \(e_p = 0.163522\) | **Eigenvalue:** | \(e_p = 0.140788\) | **Eigenvalue:** |
| \(u = 0.815129\) | 0.30627878 | \(u = 0.757633\) | 0.29157952 |
| \(q = 41.456680\) | \(q = 44.327821\) | \(Q = 0.0584307\) | \(A = 0.0580154\) |
What this implies is a push and pull between time spent in health education and time spent in production (or as the tables demonstrate, investment in capital). For lower values of $\gamma$, when time spent in health education doesn’t yield any impressive results on life expectancy, the maximization of utility pushes the citizen of this economy to work more and spend less time in health education. However as $\gamma$ increases (imagine a new teacher comes into the country or area and is able to more effectively teach good health principles), the push towards capital and time spent in production decreases, and people are encouraged to spend more time in health education. Note however that the eigenvalues do increase as $\gamma$ increases; this is due to the positive consequence of time spent in health education. Life expectancy is now longer, technology increases, and overall people live longer and are able to work for more of their lives. Thus there still is a push towards time spent in production, just a relatively smaller push.

One can observe the push towards investing in the future (again a key element according to Solow to induce economic growth), the increase of technology (another staple element for growth), and a continued, albeit lesser, push towards producing in the present, pending the time spent in health education is effective. We now proceed to further elucidate these simulations, to see under what conditions they change and if they fail to function for varying parameters. For example, if productivity is higher, is there still a need for effective health education? If the initial parameters of the country are different (perhaps an already developed country) will the need for effec-
tive health education be as apparent? The remainder of this section is dedicated to questions such as these.

Consider the latter columns of the tables corresponding to $B_H = 0.095$. Here we observe the outcome of an economy with higher productivity, with varying levels of health education effectiveness. For lower values of $\gamma$ we observe imaginary numbers in the eigenvalues and eigenvectors. Here we observe an interesting phenomenon, for $\gamma$ low enough, despite higher productivity, the solution path points in the direction of decreasing $x$. Thus from the first row, we can surmise that according to our model large $\gamma$ is a necessary component in order to incur growth in an initially poor and unhealthy country. Larger $\gamma$ can be interpreted as a shock from $Q_0$ to $Q$.

Again it is notable to observe the changes made to $v$. As $\gamma$ increases in magnitude, $v$ decreases. Although the scale of the changes is small, there is still a notable decrease. We ascribe this decrease to the choice of spending more time in education and less in production. There are also noteworthy revelations to be found in the $v$ component of the eigenvector. In Table 6.3 we observe a negative eigenvector in the $v$ direction, however for values of $\gamma > 2.00$ we observe positive eigenvector components in the $v$ direction. This implies that for $\gamma$ sufficiently high consumption will begin to increase, even though there is an initial drop due to choosing to invest less time in production and more into education.
Furthermore in congruence with Kejak’s work on productivity, we detect a shift in sign from the leading component of the eigenvector when $\gamma$ is less than or equal to 2.00. We observe that for lower levels of effectiveness of health education, the tendency of the economy is to push inwards towards a lower $x$ value. This is partly due to the increased $h$ for $B_H$, but can also be ascribed as a push towards health education, despite its low returns. The ideal situation develops with higher values of $\gamma$ and for a higher productivity setting, the economy pushes towards growth in $x$. The setting here being, time spent in health education is worthwhile, as is time spent in production; thus less time can be spent in health education with the same or greater result as for lower values of $\gamma$. In order to obtain this scenario, one could imagine both effective health teachers and on-the-job trainers introduced into the economy in order to jump the values of $\gamma$ and $B_H$.

For $B(H) = B_L$ and $\gamma = 1.2$, the imaginary eigenvalues and eigenvectors are again displayed showing that the economy is potentially in a zero-growth trap. This validates health education as a means of stimulus for economic growth, since the negative ramifications for low values of time spent in education are present and given the right conditions perilous.

In summary, the linear stability analysis has demonstrated the effectiveness and necessity of time spent in health education, which increases life expectancy, which increases technology and time available to work. We consider the positive consequences
of placing effective educators or educational systems into a developing country on that countries economic growth.

6.2 Semi-Developed Country

While the model was designed to describe a developing country, we now determine if the model is valid for a semi-developed country or developed country. That is, if the base parameters are drastically changed to more adequately model a developed country, what changes would be observable within the steady states and eigenvalues/vectors? In order to reasonably speculate on this question, one must first denote the variables that we intend to change in order to move our economy from developing to developed. We select the following variables to change in order to test the robustness of the model, \( q_0, \phi, \gamma, \delta, \alpha, \eta \) and \( B(H) \). For significant changes ranging from a 25% decrease in \( \delta \), to a 75% increase in initial life expectancy, the numerical simulation fails to model the country effectively. Life expectancy and technology assume unreasonable values. Thus the model does not adequately represent fully developed countries like the US or Japan, where initial life expectancies are much higher and the time discount factors are much lower. That is to say that developed countries are generally healthier and intrinsically encourage its citizens to invest towards the future. However it is possible to consider the model using different, moderately higher base parameters.
Table 6.6 denotes the change in parameters made to signify the difference between a developing country and a semi-developed country. Observe first the changes in the parameters. The diffusion variables, $\phi$, $\gamma$, and $\eta$ all increase, implying that a semi-developed country has more efficient means of education and diffusion. The initial life expectancy is higher by ten years; we switch from low productivity to high productivity; and lastly we lower the depreciation rate, inferring that a semi-developed country values the future more. The results are as expected, we observe a large increase in time spent in education and life expectancy, decreases in capital and increases in human skill level. Moreover, note the relative changes in life expectancies. The developing country with an initial life expectancy of 40 years increases only to 40.361 years (approximately a 1% increase), whereas the semi-developed country with an initial life expectancy of 50 years increases to 52.467 years (a 5% increase).

This continues to cement the concept of the importance of education towards economic growth. For the semi-developed country, over half of the day is spent in education. Now although this number might be inflated from a realistic standpoint, it continues to serve as a flag signaling the importance of investment in the future both in health education prolonging life expectancy and production education increasing human skill level in the workforce. However as stated above, the model does cease to function for fully developed countries. Thus we suggest the following adaptations, improvements, and areas of further research in order to construct a fuller, more functional model.
Table 6.6: Changes in Base Parameters and Variables for a Semi-developed Country and Developing Country

<table>
<thead>
<tr>
<th></th>
<th>Developing Country</th>
<th>Semi-Developed Country</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi = 0.2$</td>
<td>$q_0 = 40$</td>
<td>$\phi = 0.3$</td>
</tr>
<tr>
<td>$\gamma = 2$</td>
<td>$\delta = 0.8$</td>
<td>$\gamma = 3$</td>
</tr>
<tr>
<td>$\alpha = 0.25$</td>
<td>$\eta = 0.1$</td>
<td>$\alpha = 0.3$</td>
</tr>
<tr>
<td>$B(H) = 0.0876$</td>
<td></td>
<td>$B(H) = 0.095$</td>
</tr>
<tr>
<td>$e_p = 0.180$</td>
<td>$e_q = 0.015$</td>
<td>$e_p = 0.384$</td>
</tr>
<tr>
<td>$x = 15.462$</td>
<td>$v = 0.513$</td>
<td>$x = 14.033$</td>
</tr>
<tr>
<td>$q = 40.361$</td>
<td>$A = 0.0500$</td>
<td>$q = 52.467$</td>
</tr>
</tbody>
</table>

6.3 Further Research

One important aspect of utility that has been purposefully excluded from this modeling is time spent in leisure activities. The overall assumption here is that most developing countries do not have a large amount of time allotted towards leisure. Furthermore, our model solely considers the allotment of time throughout the work day, and in doing so allows for leisure after the work day. However, it is not counted in the utility function. Thus to further improve the model, one could incorporate time spent in leisure as a part of the utility function, as well as a variable that permeates throughout the whole of the model.

Another fault of the utility function is that it does not fully display the negative attributes of not choosing health education. Within the model there is no engine or
function that decreases utility or is detrimental to the economy if the citizens of the modeled country were to choose against spending time in health education. Again, in order to improve the accuracy of the model, the utility function could be adapted in order to show increased utility for higher longevity and a decreased utility for a lower lifespan.

Lastly, there are many parameters that remain unexplored throughout the modeling. Investigating how shocks to the parameters change the system could unveil more information concerning how to spur on economic growth. We investigated solely the topic of health education and life expectancy to answer our original questions, however one could choose to investigate changes in the time discount factor, $\delta$, the Cobb-Douglas parameter, $\alpha$, or the diffusion rates $\phi$ and $\eta$.

In our linear stability analysis, we assume that a shock to $\gamma$ is derived from a change in the education system, educator, or style of teaching health education in a way that increases or decreases its effectiveness significantly. However, it is harder to imagine what a shock to the time-discount factor may look like. Furthermore it is harder to interpret and implement a shock to the time-discount factor. Thus in order to further the research in these areas one would have to investigate not only the stability and growth results from numerical changes in the model, but also the interpretation and implementation of such changes.
CHAPTER VII

CONCLUSIONS

The original question proposed by this modeling was to determine the effect and effectiveness of an introduction of health education into existing growth models. Much of the prior research delineated the usefulness of production education and human capital; there is also a wide-spectrum of work denoting the health need in many developing countries. Our model continues that work by including life expectancy, \( q \), into the logistic equation for technology, \( A \); implying that as life expectancy increases the level of technology should increase. Furthermore, we allow \( q \) to vary dynamically as a function of capital and health education, making life expectancy an endogenous part of the model. We also combine the existing growth models of Kejak and Zilibotti to include production education and Equation (2.11) for GDP. Thus the question arose of how to invest (how to change \( \gamma, \phi \) or \( \eta \)) effectively in a developing country. What areas yield positive economic changes if implemented properly? Moreover, is health education a viable engine of economic growth?

Regarding the latter question on health education our resulting answer is a Yes with reservations. As shown from the steady state analysis as a function of \( \gamma \), increases in the effectiveness of health education (increases in \( \gamma \)) do indeed cause economic
growth, where growth is defined by increasing $x$. Recall that an increase in $x$ could be caused by an increase in $k$, capital, a decrease in $h$, human skill level, or an increase in both $k$ and $h$. Moreover, recall that from Equation (2.10) as $k$ increases, $h$ may increase as well. Thus we consider growth in this case to be increases in capital. However there are certain conditions that need to be met in order to truly spur the country towards economic growth. These conditions include, but are not limited to a low enough discount rate, sufficient time spent in production and production education, and an effective means of distribution of health education (the parameter $\gamma$ must be sufficiently high). However, there is a clear pattern of growth, as defined by increases in $x$, resulting with increases in the effectiveness of health education.

In an applicable sense, this is consistent with what is presently true about developing countries. In practice, it is worthwhile to invest in health education. Companies that create factories and plants overseas do well when they encourage healthy practices from their workers [14]; health education has been at the forefront of relief efforts, church work, and international aid [16]. Our model predicts results consistent with these observations.

Recall from the interpretations in Section 3.2 that the value of one work hour was equivalent to the value of time spent in health education. Therefore $e_q$ and consequently $q$ are important variables to consider when modeling economic growth for developing countries. The variable $e_q$, time spent in health education, has a broad
interpretation. This could range from time spent improving the irrigation of a local village and educating the villagers about the importance of clean water to classes that discuss the spread of STD’s, including AIDS. Thus for an exogenous entity to invest in a developing country via health education, one could send over a teacher, curriculum, facility, or some form of health capital (i.e. toothbrushes, glasses, etc.) that effectively augments the $\gamma$ variable so that time spent in health education is time well spent towards increasing life expectancy and maximizing utility. An important question to answer in future research is to find which of these exogenous methods are most effective in increasing $\gamma$.

Life expectancy, $q$, does not have a broad interpretation; it is simply how long on average individuals live within a certain country. However its unique modeling in Equation (2.4) and introduction into Equation (2.2), the logistic equation for technology, are valuable additions to the economic community and growth modeling. Recall that the optimization problem within this thesis was to maximize utility (consumption) over one’s lifetime. Given this utility function and maintaining certain parameter conditions, health education proves to be an important variable.

We must note that although steady-state consumption initially drops in the modeling, the utility maximization was undertaken through lifetime consumption. Therefore life expectancy has ramifications on consumption, both current and future! Another important question is to identify the global maximum of the utility function and the
parametric changes in $\gamma, \phi$, and $\eta$ to drive your steady state toward the global maximum given your initial conditions.

In conclusion, we state that given favorable parametric conditions and an effective means of distribution, health education is a sound and needed means of initializing economic growth within many developing countries. This is not to say that it is the sole instigator of growth, rather that it is a facet of economic growth that must be considered in many developing countries today.

There are other ways to invest in order to increase economic growth within a developing country. The diffusion affect that changes in capital have on life expectancy, $\eta$, was shown to be an ineffective means of increasing $x$. From Table 5.4, we observe that an increase in $\eta$ actually causes a decrease in the steady-state value of $x$. This implies that increases in the diffusion of the capital stock do not necessarily increase $x$, nor cause growth. A more effective method, as shown by Zilibotti as well, is to increase the effectiveness of increases in the technological level, $A$. Thus, to shock or change the parameter, $\phi$ [10]. Referring to Table 5.4, our results, consistent with Zilibotti’s, show that for an decreased $\phi$, a decrease in $x$. Likewise for an increase in $\phi$, $x$ increases. Technology has proved itself an established step for economic growth. A shock to the parameter $\phi$ could be the introduction of cell phone towers, computers, and computational education, or an adapted form of a new agricultural tool.
Summarily, we have created a model that incorporates life expectancy into existing models using health education, integrates production education models, and predicts results consistent with current international observations. We believe that health education and life expectancy are worthwhile fields for present economic and empirical research, especially in Sub-Saharan African countries where life expectancy is relatively low.
BIBLIOGRAPHY


APPENDICES
A.1 Solving the Logistic Equation

In Section 2.1 the differential equations for productivity (2.5), technology (2.1), and life expectancy (2.3) are all very similar and can all be solved using a similar technique that yields their respective logistic equations. The explicit derivations for the technology differential equation will be shown here, and any significant differences from the other equations will be noted.

Recall the technology differential equation:

$$\frac{dA}{dt} = \left(\frac{a - A}{a}\right) A \phi \left[q \frac{dk}{dt} + dq \frac{dk}{dt}\right]. \quad (A.1)$$

Now observe that this can be rewritten as:

$$\frac{dA}{dt} = \left(\frac{a - A}{a}\right) A \phi \frac{dk}{dt}(qk). \quad (A.2)$$

Using algebra to group like terms in order to integrate we find:

$$\frac{a}{A(a - A)}dA = \phi \frac{dk}{dt}(qk)dt. \quad (A.3)$$

Integrating the right side is straightforward from the Fundamental Theorem of Calculus and given the initial condition that at time $t = 0, A = A_0$. However the left
side requires integration by partial fractions which results in the following:

\[
\int \left[ \frac{1}{A} + \frac{1}{a-A} \right] dA = \phi(qk - q(0)k(0)). \tag{A.4}
\]

Solving the left side gives:

\[\ln \left( \frac{A}{a-A} \right) + C = \phi(qk - q(0)k(0)). \tag{A.5}\]

Taking the exponential of both sides and shifting C to the right side we obtain:

\[\frac{A}{a-A} = e^{\phi(qk-q(0)k(0))} e^{-C}. \tag{A.6}\]

Rewriting \(e^{-C}\) as simply \(C\) since it is a constant and using algebra to solve for \(A\) we find:

\[A = a \frac{a}{1 + \frac{1}{C} e^{-\phi(qk-q(0)k(0))}}. \tag{A.7}\]

Solving for \(C\) using \(A = A_0\) when \(t = 0\) we determine:

\[C = \frac{A_0}{a-A_0} \Rightarrow \frac{1}{C} = \left( \frac{a}{A_0} - 1 \right). \tag{A.8}\]

Finally we have the logistic equation:

\[A(t) = \frac{a}{1 + \left( \frac{a}{A_0} - 1 \right) e^{-\phi[q(t)k(t)-q(0)k(0)]}}. \tag{A.9}\]

The productivity derivations are similar to these given that \(B(0) = B_0\). The life expectancy derivations also follow a similar solution procedure but have different initial conditions. For life expectancy \(e_q(0)\) and \(K(0)\) enter the final equation.
A.2 Recovering Zilibotti

This model utilized Fabrizio Zilibotti’s framework to establish a new model that incorporated health education and life expectancy into the existing model [10]. Zilibotti’s model does not allow for time allotments $e_p, e_q$ and $u$; thus to recover Zilibotti’s work we remove all variances of time allotments, either by setting the allotted time variable, $(e_p, e_q, u)$, equivalent to 1, 0, or some constant value, depending on how it relates to the model.

Since Zilibotti does not consider life expectancy, we will take $q$ to be a constant. Thus $\dot{q}$ is equivalent to zero. Furthermore, this model’s Equation (2.1) becomes:

$$\frac{dA}{dt} = \left( \frac{a - A}{a} \right) A\phi \frac{d[q]}{dt} .$$

(A.10)

Simplifying further, life expectancy can be absorbed by $\phi$ in order to obtain Zilibotti’s Equation (2). Note that there may be a difference in notation, but that the fundamental structure of the equation is the same. Moreover, this implies a recovery of Zilibotti’s Equation (3) as well,

$$\frac{dA}{dt} = \left( \frac{a - A}{a} \right) A\phi \frac{dk}{dt} .$$

(A.11)

This model’s utility Equation (2.9), and output Equation (2.11), are identical to Zilibotti’s Equations (8) and (9). Thus when the Hamiltonian is considered, this
model’s Hamiltonian, Equation (3.2), is simplified as follows:

\[ HAM = \left[ \frac{c_t^{1-\sigma}}{1-\sigma} e^{-\delta t} \right] + \lambda \left[ (kr + k\rho) - k\rho + w(1 - e_p - e_q)h - c \right] + \mu [B(H)e_p h]. \]  

(A.12)

Zilibotti ignores depreciation of capital, thus \( k\rho \) is reduced to zero. We maintain the equivalence of output and \( kr \) as noted in Equation (2.11), and ignore skill level. Thus \( h = 0 \). Furthermore we set \( e_q = e_p = 0 \), neglecting time spent in education. This will yield:

\[ HAM = \left[ \frac{c_t^{1-\sigma}}{1-\sigma} e^{-\delta t} \right] + \lambda \left[ DAk + Z(Ak)^{\theta} - c \right], \]  

(A.13)

which is similar in structure to Zilibotti’s Equation (12) given changes in notation. To obtain Zilibotti’s governing equations, one would solve the Hamiltonian to yield

\[ \frac{dc}{dt} = DA + ZA^\theta k^{\theta - 1} - \frac{c}{\sigma}, \]  

(A.14)

\[ \frac{dk}{dt} = DA + ZA^\theta k^{\theta - 1} - \frac{c}{k}, \]  

(A.15)

which recovers Zilibotti’s governing Equations (13) and (14). Thus through basic simplifications, this model can be reduced to recover the Zilibotti model.

A.3 Recovering Kejak

In structure the model of this paper is most similar to Michal Kejak’s model in Stages of Growth in Economic Development [11]. The main difference is that this thesis incorporates time allotted towards health education and treats life expectancy as a
variable. Consequentially, in order to recover the Kejak model, one lets \( e_q = 0 \), treats life expectancy, \( q \), as a constant, and sets \( \theta = 1 \).

Kejak treats technology as an exogenous factor, hence \( A \) also becomes a constant. In contrast, Equations (2.5)-(2.7) match the structure of Kejak’s Equations (1)-(3) with changes in notation. In fact, there is a commonality between most of Kejak’s equations and the equations of this model, the utility function, the output equation, and even the Hamiltonian all relate closely to Kejak’s. The Hamiltonian, Equation (3.2), once reverted back to Kejak-form is as follows:

\[
HAM = \left[ \frac{c^1 - \sigma}{1 - \sigma} e^{-\delta t} \right] + \lambda [(k r + w(1 - e_p) h - c] + \mu [B(H)e_p h].
\]  

(A.16)

Recall that \( (1 - e_p) = u \) yields the Kejak Hamiltonian. Thus the governing equations for skill level, \( h \), and consumption, \( c \) match Kejak’s governing Equations (18) and (19) given respective changes in notation. However the governing equations for capital and production differ due to the presence of \( e_q \) in this model’s equations.

Thus in order to recover the Kejak model from this model’s Equation (3.14) one must simplify as follows:

\[
\frac{dk}{dt} = DA + ZA^\theta k^{\theta - 1} + Gk^{\alpha - 1}L^{1 - \alpha}(1 - \alpha) - \rho - \frac{c}{k}.
\]  

(A.17)

Recall that Kejak treats technology as an exogenous source. Thus for \( \theta = 1 \), the above equation can be simplified to:

\[
\frac{dk}{dt} = DA + ZA + Gk^{\alpha - 1}L^{1 - \alpha}(1 - \alpha) - \rho - \frac{c}{k}.
\]  

(A.18)
However, note that $DA + ZA$ is simply the derivative of capital which when maximized by the profit function is equal to $F_k(\alpha)$ and that $Gk^{\alpha-1}L^{1-\alpha}(1-\alpha)$ can be reduced to $F_k(1-\alpha)$. Therefore the above is reduced again to obtain:

$$\frac{dk}{dt} = F_k(\alpha) + \frac{F_k(1-\alpha) - \rho - c}{k},$$  \hspace{1cm} (A.19)

$$\frac{dk}{dt} = F_k - \rho - \frac{c}{k},$$  \hspace{1cm} (A.20)

which is Kejak’s governing equation for capital.

Lastly, in order to revert this model’s Equation (3.16) back to the original Kejak framework, the following derivations must be undertaken:

$$\frac{de_q + de_p}{1 - e_p - e_q} = \frac{dh}{h} - \frac{dk}{k} - \frac{1}{\alpha} [DA + ZA^\theta K^{\alpha-1}\theta - \rho] - \frac{1}{\alpha}(1 - e_q)B(H) + \frac{1}{\alpha}B'(H)e_ph.$$  \hspace{1cm} (A.21)

Set $e_q = 0, \theta = 1$. The above equation becomes:

$$\frac{de_p}{1 - e_p} = \frac{dh}{h} - \frac{dk}{k} - \frac{1}{\alpha} [DA + ZA - \rho] - \frac{1}{\alpha}B(H) + \frac{1}{\alpha}B'(H)e_ph.$$  \hspace{1cm} (A.22)

Furthermore substituting in the Kejak $h$ and $k$ equations we obtain:

$$\frac{de_p}{1 - e_p} = B(H)e_p - \frac{F}{k} - \rho - \frac{c}{k} - \frac{1}{\alpha} \left[ F_k(\alpha) - \rho \right] - \frac{1}{\alpha}B(H) + \frac{1}{\alpha}B'(H)e_ph,$$  \hspace{1cm} (A.23)

which can be simplified to

$$\frac{de_p}{1 - e_p} = B(H)e_p - \rho - \frac{c}{k} + \frac{1}{\alpha}\rho \frac{1}{\alpha}B(H) + \frac{1}{\alpha}B'(H)e_ph,$$  \hspace{1cm} (A.24)

which is similar in structure to Kejak’s Equation (20) given changes in notation and the fact that Kejak assumes $B(H)$ to be a constant, thus $B'(H) = \frac{dB}{dH} = 0$.  

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APPENDIX B

HAMILTONIAN DERIVATIONS

In Section 3.1, the Hamiltonian is presented and then solved. The deeper workings of solving this tedious problem will now be delineated.

B.1 Consumption Governing Equation

Rewrite Equation (3.3) to find

\[ c^{-\sigma} e^{-\delta t} = \lambda. \]  

(B.1)

Substitute this result into Equation (3.6) to give

\[-[\sigma c^{-\sigma} e^{-\delta t} \frac{dc}{dt} + c^{-\sigma} (-\delta) e^{-\delta t}] = c^{-\sigma} e^{-\delta t} [DA + ZA^\theta k^{\theta-1} \theta - \rho],\]  

(B.2)

which simplifies to

\[ \sigma c^{-1} \frac{dc}{dt} + \delta = [DA + ZA^\theta k^{\theta-1} \theta - \rho], \]  

(B.3)

which when rearranged yields Equation (3.12):

\[ \frac{dc}{dt} c = \frac{DA + ZA^\theta k^{\theta-1} \theta - \rho - \delta}{\sigma}. \]  

(B.4)
B.2 Capital Governing Equation

From Equation (3.5), the governing equation, Eqn. (3.13), is found through a straightforward division. However in order to solve for the capital governing equation, some further assumptions and substitutions must be made. Assume a one sector production firm in the economy with a Cobb-Douglas production function with factors of capital, $k$, labor, $L$, and a multi-productivity parameter $G$:

$$F(k, L) = G K^\alpha L^{1-\alpha}, \quad (B.5)$$

with profits,

$$\Pi = F(k, L) - wL - [DAk + Z(Ak)^\theta]. \quad (B.6)$$

Thus in order to maximize profits the following conditions must hold

$$\frac{d\Pi}{dL} = 0 = F_L - w, \quad (B.7)$$

$$\frac{d\Pi}{dk} = 0 = F_K - DA - ZA^\theta k^{\theta-1} \theta. \quad (B.8)$$

This maximization adds another constraining relationship on $e_p, e_q, h, A$ and $k$. Divide Equation (3.4) through by $k$ (recall that $k = K$) to find

$$\frac{d}{dt} \frac{k}{k} = DA + ZA^\theta k^{\theta-1} - \rho + \frac{w(1 - e_q - e_p)h}{k} - \frac{c}{k}. \quad (B.9)$$

It is helpful to reintroduce the labor equality

$$L = uh = (1 - e_q - e_p)h. \quad (B.10)$$

Thus Equation (B.9) becomes

$$\frac{d}{dt} \frac{k}{k} = DA + ZA^\theta k^{\theta-1} - \rho + \frac{wL}{k} - \frac{c}{k}. \quad (B.11)$$
However $w$ can be determined by rearranging Equation (B.7) as follows

$$w = F_L = GK^\alpha L^{-\alpha}(1 - \alpha).$$  \hfill (B.12)

Again, the Equation (B.11) becomes

$$\frac{dk}{dt} = DA + ZA^\theta k^{\theta - 1} - \rho + \frac{GK^\alpha L^{-\alpha}(1 - \alpha)L}{k} - \frac{c}{k}. \hfill (B.13)$$

At equilibrium $k = K$, thus our final governing equation for $k$ becomes

$$\frac{dk}{dt} = DA + ZA^\theta k^{\theta - 1} + Gk^{\alpha - 1}L^{1 - \alpha}(1 - \alpha) - \rho - \frac{c}{k}. \hfill (B.14)$$

B.3 Health Education Governing Equation

By adding $\lambda wh$ to both sides of Equation 3.9, then by dividing through by $\lambda$ we can write

$$wh = \left[ DK \frac{dA}{dq} \frac{dq}{de} + Zk^\theta A^{\theta - 1} \frac{dA}{dq} \frac{dq}{de} \right]. \hfill (B.15)$$

We can further simplify this equation by factoring out the derivative of technology with respect to life expectancy as well as the derivative of life expectancy with respect to health education, yielding

$$wh = \frac{dA}{dq} \frac{dq}{de} \left[ DK + Zk^\theta A^{\theta - 1} \right]. \hfill (B.16)$$

We observe that the derivatives will simplify to the derivative of technology with respect to health education and that the bracketed quantity is equivalent to the
derivative of production with respect to technology.

\[ wh = \frac{dA}{de_q} \frac{d(DAK + Z(AK)^\theta)}{dA}, \]  

which again can be reduced to

\[ wh = \frac{d(DAK + Z(AK)^\theta)}{de_q}. \]  

Thus the marginal effect of health education on production is equivalent to the worth of one skilled work hour at equilibrium.

### B.4 Production Education Governing Equation

Consider Equation (3.8), which can be rewritten and simplified to yield

\[ \mu B(H) = \lambda w. \]  

If this equality is inserted into Equation (3.7) we obtain

\[ \frac{-d\mu}{dt} = \mu B(H)(1 - e_p - e_q) + \mu B(H)e_p, \]  

which simplifies to

\[ \frac{-d\mu}{dt} = \mu B(H)(1 - e_q). \]  

This equation will be used in the following derivations. Recall Equation (3.8). Divide through by \( h \) and then take a time derivative to obtain

\[ \frac{d\lambda}{dt}w + \frac{dw}{dt}\lambda - \frac{-d\mu}{dt}B(H) - B'(H)\frac{dh}{dt}\mu = 0. \]
Applying Equations (B.21) and (3.5) and solving Equation (B.19) for $\mu$ we find

$$
\frac{d\lambda}{dt} w + \frac{dw}{dt} \lambda - \mu B(H) (1 - e_q) B(H) - B'(H) B(H) e_p h \frac{\lambda w}{B(H)} = 0. \quad (B.23)
$$

Simplifying then dividing through by $\lambda w$ we obtain

$$
\frac{d\lambda}{dt} + \frac{dw}{dt} (1 - e_q) B(H) - B'(H) e_p h = 0. \quad (B.24)
$$

Use Equation (3.4) to simplify this equation further:

$$
\frac{dw}{dt} \left[ DA + Z A^\theta K^{\theta-1} \theta - \rho \right] - (1 - e_q) B(H) - B'(H) e_p h = 0. \quad (B.25)
$$

The main task remaining is to solve for $\frac{dw}{dt}$. This is done by applying a time derivative to Equation (B.12), as follows:

$$
\frac{dw}{dt} = G (1 - \alpha) (\alpha K^{\alpha-1} \frac{dK}{dt} L^{-\alpha} - \alpha L^{-\alpha-1} \frac{dL}{dt} K^{\alpha}). \quad (B.26)
$$

Dividing through by $w$,

$$
\frac{dw}{dt} \frac{1}{w} = G (1 - \alpha) K^{\alpha} L^{-\alpha} \frac{dK}{dt} L^{-\alpha} - \alpha L^{-\alpha-1} \frac{dL}{dt} K^{\alpha} \frac{1}{G (1 - \alpha) K^{\alpha} L^{-\alpha}}. \quad (B.27)
$$

Simplifying,

$$
\frac{dw}{dt} \frac{1}{w} = \alpha \frac{dK}{dt} K - \alpha \frac{dL}{dt} L. \quad (B.28)
$$

At equilibrium $K = k$, thus $\frac{dK}{K}$ is equivalent to Equation (3.16). However $\frac{dL}{L}$ must be solved for using Equation (B.10),

$$
\frac{dt}{L} = \frac{\frac{dt}{L} \left( 1 - e_p - e_q \right) h}{(1 - e_p - e_q) h} = \frac{dt}{(1 - e_p - e_q) h} \left( 1 - e_p - e_q \right) - \left( \frac{de_p}{dt} + \frac{de_q}{dt} \right) h \frac{1}{(1 - e_p - e_q) h}. \quad (B.29)
$$

Simplifying this expression, substituting it into the above equations, and solving for the time derivatives of health education and production education we obtain Equation
\( \frac{de_p}{dt} + \frac{de_q}{dt} = \frac{dh}{h} - \frac{dk}{k} - \frac{1}{\alpha} [DA + ZA^\theta K^\theta - \theta - \rho] \) 
\( = -\frac{1}{\alpha} (1 - e_q) B(H) + \frac{1}{\alpha} B'(H) e_p h. \) 

(B.30)

B.5 Transforming the Governing Equations

In this section, the process of transforming the case-specific, original model into the general model is continued and the derivations for the system of four governing equations used for the steady state analysis are fully expressed. Recall Section 5.1. We now continue the derivations started there by finding the time derivatives of \( x \) and \( v \), solve for their respective growth equations, substitute in the original governing equations, and then simplify.

Therefore, it is straightforward to apply the change of variables to the existing steady state solutions. We seek to find four equations for the four unknowns, \( x, v, e_p \) and \( e_q \). We start by finding the derivatives of \( x \) and \( v \).

\[ \frac{dx}{dt} = \frac{\frac{dk}{dt} h - \frac{dh}{dt} k}{h^2}, \]  
\[ \frac{dv}{dt} = \frac{\frac{dc}{dt} k - \frac{dk}{dt} c}{k^2}. \]  

(B.31)  
(B.32)
Thus the derivative rates of change become

\[
\frac{dx}{dt} = \frac{d}{dt}h - \frac{d}{dt}k \frac{h}{k} = \frac{dk}{h} - \frac{dh}{k}, \tag{B.33}
\]

\[
\frac{dv}{dt} = \frac{dc}{dt}k - \frac{dk}{c} = \frac{dc}{c} - \frac{dk}{k}. \tag{B.34}
\]

We can now substitute our existing \( \frac{dh}{dt} \), \( \frac{dk}{dt} \) and \( \frac{dc}{dt} \) expressions into these equations to obtain differential equations for \( x \) and \( v \).

For \( x \) we find

\[
\frac{dx}{dt} = DA + ZA^\theta k^{\theta - 1} + Gk^{\alpha - 1}L^{1 - \alpha}(1 - \alpha) - \rho - \frac{c}{k} - B(H)e_p. \tag{B.35}
\]

Immediately, we can transform the \( \frac{c}{k} \) term into a \( v \). However the rest of the transformation will require some algebraic manipulations. Observe the \( Gk^{\alpha - 1}L^{1 - \alpha}(1 - \alpha) \) term. Recall our \( L = uh = (1 - e_p - e_q)h \) equivalency. Substituting this term into the expression yields: \( Gk^{\alpha - 1}((1 - e_p - e_q)h)^{1 - \alpha}(1 - \alpha) \). We now obtain an inverted \( x \) raised to the \( 1 - \alpha \). Reformatting the entire expression now yields:

\[
\frac{dx}{dt} = DA + ZA^\theta k^{\theta - 1} + G\left(\frac{1 - e_p - e_q}{x}\right)^{1 - \alpha}(1 - \alpha) - \rho - v - B(H)e_p. \tag{B.36}
\]

However there is still a \( k \) term in this equation. In order to transform this \( k \), recall our profit maximization with respect to capital Equation (B.8), apply our definition of \( L \) to the equation, substitute in \( x \), and solve for an equivalency for our \( k \) term,
$ZA^\theta k^{\theta-1}$. As follows:

\begin{align}
G\alpha k^{-1} - DA - ZA^\theta k^{\theta-1} \theta &= 0, \quad (B.37) \\
G\alpha k^{-1} L^{1-\alpha} &= DA + ZA^\theta k^{\theta-1} \theta, \quad (B.38) \\
G\alpha k^{-1} L^{1-\alpha} &= DA + ZA^\theta k^{\theta-1} \theta, \quad (B.39) \\
G\alpha((1 - e_p - e_q) x)^{1-\alpha} &= DA + ZA^\theta k^{\theta-1} \theta, \quad (B.40) \\
G\alpha \left( \frac{(1 - e_p - e_q)}{x} \right)^{1-\alpha} &= DA + ZA^\theta k^{\theta-1} \theta, \quad (B.41) \\
\frac{1}{\theta} \left( G\alpha \left( \frac{(1 - e_p - e_q)}{x} \right)^{1-\alpha} - DA \right) &= ZA^\theta k^{\theta-1}. \quad (B.42)
\end{align}

Now, input this term into our existing equations and simplify in order to obtain a differential equation for $x$, Equation (5.7):

\begin{align}
\frac{dx}{dt} x &= DA + \frac{1}{\theta} \left( G\alpha \left( \frac{(1 - e_p - e_q)}{x} \right)^{1-\alpha} - DA \right) \quad (B.44) \\
&+ G \left( \frac{1 - e_p - e_q}{x} \right)^{1-\alpha} (1 - \alpha - \rho - v - B(H)e_p) \\
\frac{dx}{dt} x &= DA \left( 1 + \frac{1}{\theta} \right) + G \left( \frac{1 - e_p - e_q}{x} \right)^{1-\alpha} \left( 1 - \alpha + \frac{\alpha}{\theta} \right) - \rho - v - B(H)e_p. \quad (B.45)
\end{align}

With the above simplifications it is now straightforward to transform the $v$ equation.

For simplicity and to avoid redundancy, we avoid repeating the above equivalencies and yield only the simplified form (the derivations follow the above simplifications).
Thus we obtain Equation (5.8):

\[
\frac{dv}{dt} = \frac{DA + ZA^\theta k^{\theta-1}\theta - \rho - \delta}{\sigma} \tag{B.46}
\]

\[
-DA - ZA^\theta k^{\theta-1} - GK^{\alpha-1}L^{1-\alpha}(1 - \alpha) + \rho + \frac{c}{k},
\]

\[
\frac{dv}{dt} = \left(\frac{\alpha}{\sigma} - 1 + \alpha - \frac{\alpha}{\theta}\right)G \left(\frac{1 - e_p - e_q}{x}\right)^{1-\alpha}
\]

\[
+ v + \left(1 - \frac{1}{\sigma}\right)\rho - \frac{\delta}{\sigma} - DA \left(1 + \frac{1}{\theta}\right). \tag{B.47}
\]

Lastly, Equation (3.16) must be transformed.

\[
\frac{de_p}{dt} + \frac{de_q}{dt} = \frac{dh}{dt} + \frac{1}{\alpha} [DA + ZA^\theta K^{\theta-1}\theta - \rho] \tag{B.48}
\]

\[
-\frac{1}{\alpha}(1 - e_q)B(H) + \frac{1}{\alpha}B'(H)e_p h.
\]

This transformation is also straightforward, utilizing Equations (B.40) and (5.5), and recalling that \(B(H)\) is being set as a constant, implying that \(B'(H) = 0\).

\[
\frac{de_q}{dt} + \frac{de_p}{dt} = -\frac{dx}{dt} + \frac{1}{\alpha} \left[G\alpha(\frac{1 - e_p - e_q}{x})^{1-\alpha} - \rho\right] - \frac{1}{\alpha}(1 - e_q)B(H). \tag{B.49}
\]

Therefore, we obtain Equation (5.9).
APPENDIX C

STEADY STATE DERIVATIONS

C.1 Steady State Derivations for the Original Model

Recall from Section 4.2 that in the steady state analysis of the original model $e_p = 0$ and $e_q = 1 - \frac{\delta}{B(H)}$. The remaining work of this section requires solving for $k$ and $c$ in terms of $h$, and lastly simplifying the algebraic solution to terms of $h$, in order to solve for $h$ numerically. We begin by solving for $k$.

Recall the Cobb-Douglas production function, $F = Gk^\alpha L^{1-\alpha}$ and the profit function, $\Pi = F - wL - DAk - Z(Ak)^\theta$. Using these equations we obtain the following derivations:

$$\frac{\partial \Pi}{\partial k} = 0 = \frac{\partial F}{\partial k} - DA - ZA^\theta k^{\theta-1} \theta, \quad (C.1)$$

$$0 = G\alpha k^{\alpha-1} L^{1-\alpha} - DA - ZA^\theta k^{\theta-1} \theta, \quad (C.2)$$

$$G\alpha k^{\alpha-1} L^{1-\alpha} = DA + ZA^\theta k^{\theta-1} \theta. \quad (C.3)$$

Substituting $L = uh = (1 - e_p - e_q)h$ and $DA + ZA^\theta k^{\theta-1} \theta = \rho + \delta$ we obtain the
following, which we can then use to solve for $k$ in terms of $h$,

$$G{\alpha}{k^{\alpha-1}}((1-e_p-e_q)h)^{1-\alpha} = \rho + \delta,$$  \hspace{1cm} (C.4)

$$k^{\alpha-1} = ((1-e_p-e_q)h)^{1-\alpha}\frac{\rho+\delta}{G{\alpha}},$$  \hspace{1cm} (C.5)

$$k = ((1-e_p-e_q)h)(\frac{\rho+\delta}{G{\alpha}})^{\frac{1}{\alpha-1}},$$  \hspace{1cm} (C.6)

$$k = \frac{\delta}{B(H)}h(\frac{G{\alpha}}{\rho+\delta})^{\frac{1}{\alpha-1}},$$  \hspace{1cm} (C.7)

where we set $e_p$ and $e_q$ to be their steady state solutions. We can now substitute $k$ into Equation (3.14) to solve for $c$. Again we will utilize the definition of $L$ as well as $k$.

$$\frac{dk}{dt} = DA + ZA^{\theta}k^{\theta-1} + Gk^{\alpha-1}L^{1-\alpha}(1-\alpha) - \rho - \frac{c}{k},$$  \hspace{1cm} (C.8)

$$0 = DAk + Z(Ak)^{\theta} + Gk^{\alpha}L^{1-\alpha}(1-\alpha) - k\rho - c,$$  \hspace{1cm} (C.9)

$$c = DAk + Z(Ak)^{\theta} + Gk^{\alpha}L^{1-\alpha}(1-\alpha) - k\rho.$$  \hspace{1cm} (C.10)

Substituting in $k$ and simplifying, we obtain:

$$c = \frac{\delta}{B(H)}h\left[\frac{G{\alpha}}{\rho+\delta}(DA - \rho) + G(\frac{G{\alpha}}{\rho+\delta})^{\frac{1}{\alpha-\theta}}(1-\alpha)\right] + Z(AK)^{\theta}.$$  \hspace{1cm} (C.11)

Now that we have solved for $e_p$ and $e_q$ directly and for $k$ and $c$ in terms of $h$, we have only $h$ to solve for. Recall again from the partial derivative of the profit function that

$$w = GK^{\alpha}L^{-\alpha}(1-\alpha) = GK^{\alpha}(h(1-e_p-e_q)^{-\alpha}(1-\alpha)).$$

Furthermore we can express our algebraic governing equation, Equation (3.15), as the following:

$$\frac{d(DAK + Z(AK)^{\theta})}{de_q} = wh,$$  \hspace{1cm} (C.12)

$$DK\frac{dA}{dq}\frac{dq}{de_q} + Z(K)^{\theta}\theta A^{\theta-1}\frac{dA}{dq}\frac{dq}{de_q} = wh,$$  \hspace{1cm} (C.13)

$$\frac{dA}{dq}\frac{dq}{de_q}\left[DK + ZK^{\theta}A^{\theta-1}\theta\right] = GK^{\alpha}(h(1-e_p-e_q)^{-\alpha}(1-\alpha)h).$$  \hspace{1cm} (C.14)
Recall again from the profit function that $G\alpha K^{\alpha -1}L^{1-\alpha} = DA + ZA^\theta K^{\theta -1}\theta$. With some rearranging this becomes:

$$G\alpha K^{\alpha -1}L^{1-\alpha} - \frac{A}{K} [DK + ZA^{\theta -1}K^{\theta}] = 0,$$  \hspace{1cm} (C.15)

$$DK + ZA^{\theta -1}K^{\theta} = \frac{K}{A} G\alpha K^{\alpha -1}L^{1-\alpha}.$$  \hspace{1cm} (C.16)

Plugging this into our revised algebraic governing equation and simplifying we obtain:

$$\frac{dA}{dq} dq \frac{d}{dq} dq \frac{d}{dq} dq \left[ \frac{K}{A} G\alpha K^{\alpha -1}L^{1-\alpha} \right] = G\alpha K^{\alpha} (h(1 - e_p - e_q))^{-\alpha} (1 - \alpha)h,$$  \hspace{1cm} (C.17)

$$\frac{dA}{dq} dq \frac{d}{dq} dq \frac{d}{dq} dq \frac{d}{dq} dq \left[ \frac{K}{A} G\alpha K^{\alpha -1}(h(1 - e_p - e_q))^{1-\alpha} \right] = G\alpha K^{\alpha} (h(1 - e_p - e_q))^{-\alpha} (1 - \alpha)h,$$  \hspace{1cm} (C.18)

$$\frac{dA}{dq} dq \frac{d}{dq} dq \frac{d}{dq} dq \frac{d}{dq} dq = A \frac{1 - \alpha}{\alpha} \frac{1}{1 - e_p - e_q},$$  \hspace{1cm} (C.19)

$$\frac{dA}{dq} dq \frac{d}{dq} dq \frac{d}{dq} dq \frac{d}{dq} dq = A \frac{1 - \alpha B(H)}{\alpha \delta},$$  \hspace{1cm} (C.20)

which must be solved numerically for $h$.

Case: Initial Productivity Equals Zero

In order to consider the $B_0 = 0$ case it is necessary to evaluate the effect of this equivalency on $B(H)$ and $B'(H)$. Refer to Equation (2.6):

$$B(H) = \frac{B_F}{1 + \left( \frac{B_F}{B_0} - 1 \right) e^{-\psi H}} = B_F \left[ 1 + \left( \frac{B_F}{B_0} - 1 \right) e^{-\psi H} \right]^{-1}.$$  \hspace{1cm} (C.21)

Thus, taking the derivative with respect to $H$, we find $B'(H)$ to be

$$B'(H) = B_F \left[ 1 + \left( \frac{B_F}{B_0} - 1 \right) e^{-\psi H} \right]^{-2} \left( \frac{B_F}{B_0} - 1 \right) \psi e^{-\psi H}.$$  \hspace{1cm} (C.22)
However, taking the limit as $B_0$ approaches zero reveals the following,

$$\lim_{B_0 \to 0} B'(H) = \lim_{B_0 \to 0} \frac{B_F \left( \frac{B_F}{B_0} - 1 \right) \psi e^{-\psi H}}{1 + \left( \frac{B_F}{B_0} - 1 \right) e^{-\psi H}}^2 = 0. \quad (C.23)$$

Thus for $B_0 = 0$, then $B(H) = B'(H)) = 0$ which is both a trivial solution for this problem and an economic impossibility. Therefore this case is ignored.

C.2 Steady State Derivations for the Transformed Model

The solution procedure for solving the transformed four variable/equation steady state problem is as follows. Set all the time derivatives equivalent to zero. Use Equation (5.4) to solve for $e_p$ explicitly in terms of $e_q$ and $x$. Plug this term into Equations (5.7), (5.8), and (5.9) in order to reduce the four variable system into a system of only three equations and three unknowns. However, the time derivative for $e_p$ must be solved for using the Chain rule and the explicit formulation for $e_p$. Furthermore, when numerically solving for the time derivative $e_q$, one must algebraically rearrange Equation (5.9). The system then can be solved numerically and undergo a linear stability analysis as well.

As stated, we algebraically rearrange Equation (5.4) to obtain:

$$\frac{dA}{dq} \frac{dq}{de_q} = A \frac{1 - \alpha}{\alpha} \frac{1}{1 - e_p - e_q}, \quad (C.24)$$

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\[
\phi x \left( \frac{a}{A_0} - 1 \right) e^{-\phi[\psi x - q(0)x(0)]} \frac{Q\gamma \left( \frac{Q}{Q_0} - 1 \right) e^{-[\gamma e_q - e_q(0)] + \eta(x(t) - x(0))}}{[1 + \left( \frac{a}{A_0} - 1 \right) e^{-\phi[\psi x - q(0)x(0)]}] \left[ 1 + \left( \frac{Q}{Q_0} - 1 \right) e^{-[\gamma e_q - e_q(0)]} \right]^2} = \frac{1 - \alpha}{\alpha} \frac{1}{1 - e_p - e_q},
\]

\[
e_p = 1 - e_q \quad (C.26)
\]

Now that we have obtained an explicit solution for \(e_p\), we continue to solve for the time derivative of \(e_p\) by parts. Note that we leave the substitution of \(e_p\) into the three remaining differential equations to the Maple algorithm. (To read the algorithm, see Appendix D.2). Observe that since \(e_p\) is dependent upon \(x\) and \(e_q\) that we must take the time derivatives of \(e_p\) with respect to both of those variables, as follows:

\[
\frac{d e_p}{d t} = \frac{d[e_p]}{d e_q} \dot{e}_q + \frac{d[e_p]}{d x} \dot{x}. \quad (C.27)
\]

Again, we proceed to only write the main derivations leaving the substitutions for the computer. Thus the \(\frac{d e_p}{d t}\) and \(\frac{d e_p}{d x}\) components are known from our earlier derivations.

Below, we solve for the \(\frac{d e_p}{d e_q}\) portion by using the quotient rule and the chain rule:

\[
\frac{d e_p}{d e_q} = -1 + \frac{\alpha - 1}{\alpha x \phi \left( \frac{a}{A_0} - 1 \right) Q\gamma \left( \frac{Q}{Q_0} - 1 \right)} \frac{T'B - B'T}{B^2}, \quad (C.28)
\]
where $T$ and $B$ are respectively the top and bottom of the quotient rule, as follows:

$$T = \left[ 1 + \left( \frac{a}{A_0} - 1 \right) e^{-\phi[qx-q(0)x(0)]} \right] \left[ 1 + \left( \frac{Q}{Q_0} - 1 \right) e^{-\gamma[e_q-e_q(0)]-\eta[x-x(0)]} \right]^2, \quad (C.29)$$

$$T' = \left( \frac{a}{A_0} - 1 \right) e^{-\phi[qx-q(0)x(0)]} \left( -\phi x \frac{dq}{de_q} \right) \left[ 1 + \left( \frac{Q}{Q_0} - 1 \right) e^{-\gamma[e_q-e_q(0)]} \right]^2 \quad (C.30)$$

$$-2 \left[ 1 + \left( \frac{Q}{Q_0} - 1 \right) e^{-\gamma[e_q-e_q(0)]} \right] \left( \frac{Q}{Q_0} - 1 \right) \star e^{-\gamma[e_q-e_q(0)]} \gamma \left[ 1 + \left( \frac{a}{A_0} - 1 \right) e^{-\phi[qx-q(0)x(0)]} \right],$$

$$B = e^{-\phi[qx-q(0)x(0)]} e^{-\gamma[e_q-e_q(0)]-\eta[x(t)-x(0)]}, \quad (C.31)$$

$$B' = -e^{-\phi[qx-q(0)x(0)]-\gamma[e_q-e_q(0)]-\eta[x(t)-x(0)]} \left( \phi x \frac{dq}{de_q} + \gamma \right), \quad (C.32)$$

where $\frac{dq}{de_q}$ is stated in Equation (5.3). Substituting in all the above equations into Equation (C.28) we obtain the initial half of the $\frac{de_p}{dx}$ equation. We now solve for the remaining half by finding $\frac{de_p}{dx}$.

$$\frac{de_p}{dx} = \frac{\alpha - 1}{\alpha} \phi \left( \frac{a}{A_0} - 1 \right) Q \gamma \left( \frac{Q}{Q_0} - 1 \right) \frac{T' B - B'T}{B^2}, \quad (C.33)$$

where again $T$ and $B$ are respective to the top and bottom of the quotient rule, but take the slightly different forms of:

$$T = \left[ 1 + \left( \frac{a}{A_0} - 1 \right) e^{-\phi[qx-q(0)x(0)]} \right] \left[ 1 + \left( \frac{Q}{Q_0} - 1 \right) e^{-\gamma[e_q-e_q(0)]-\eta[x-x(0)]} \right]^2, \quad (C.34)$$

$$T' = \left( \frac{a}{A_0} - 1 \right) e^{-\phi[qx-q(0)x(0)]} \left( -\phi \left( \frac{dq}{dx} + q \right) \right) \star \left[ 1 + \left( \frac{1 + \frac{Q}{Q_0}}{Q_0 - 1} \right) e^{-\gamma[e_q-e_q(0)]-\eta[x-x(0)]} \right]^2$$

$$-2 \left( \frac{Q}{Q_0} - 1 \right) e^{-\gamma[e_q-e_q(0)]-\eta[x-x(0)]} \eta \star \left[ 1 + \left( \frac{a}{A_0} - 1 \right) e^{-\phi[qx-q(0)x(0)]} \right] \left[ 1 + \left( \frac{Q}{Q_0} - 1 \right) e^{-\gamma[e_q-e_q(0)]-\eta[x-x(0)]} \right], \quad (C.35)$$

$$B = xe^{-\phi[qx-q(0)x(0)]} e^{-\gamma[e_q-e_q(0)]+\eta[x(t)-x(0)]}, \quad (C.36)$$

$$B' = e^{-\phi[qx-q(0)x(0)]-\gamma[e_q-e_q(0)]-\eta[x(t)-x(0)]} \left( 1 + x(-\eta - \phi \left( \frac{dq}{dx} + q \right) \right). \quad (C.37)$$
Again substituting these terms into Equation (C.33), will yield the full form of \( \frac{de_p}{dx} \).

Furthermore substituting the partial derivatives of \( e_p \) into Equation (C.27) will result in \( e_p \) in terms of \( x, v, e_q, \frac{dx}{dt}, \) and \( \frac{de_q}{dt} \). Thus lastly we must algebraically rearrange Equation (5.9) in order to solve the system numerically. Observe the original form of the equation:

\[
\frac{de_q}{dt} + \frac{de_p}{de_q} = \frac{dx}{x} + \frac{1}{\alpha} \left[ \frac{G e_q (1 - e_p - e_q)}{x} \right] (1 - \alpha) - \frac{1}{\alpha} (1 - e_q) B(H). \tag{C.38}
\]

By multiplying both sides by \((1 - e_p - e_q)\) and then subtracting the \( \frac{de_p}{dt} \) from both sides and substituting it with Equation (C.27), we find:

\[
\frac{de_q}{dt} = (1 - e_p - e_q) \left[ \frac{dx}{x} + G \left( \frac{1 - e_p - e_q}{x} \right) (1 - \alpha) - \frac{\rho}{\alpha} \frac{1}{\alpha} (1 - e_q) B(H) \right] \tag{C.39}
\]

\[
-\frac{d[e_p]}{de_q} \frac{de_q}{dt} - \frac{d[e_p]}{dx} \frac{dx}{dt}.
\]

Next we algebraically rearrange to find \( \frac{de_q}{dt} \).

\[
\frac{de_q}{dt} = (1 - e_p - e_q) \left[ \frac{dx}{x} + G \left( \frac{1 - e_p - e_q}{x} \right) (1 - \alpha) - \frac{\rho}{\alpha} \frac{1}{\alpha} (1 - e_q) B(H) \right] \tag{C.40}
\]

\[
-\frac{d[e_p]}{de_q} \frac{de_q}{dt} - \frac{d[e_p]}{dx} \frac{dx}{dt},
\]

\[
\frac{de_q}{dt} = \frac{(1 - e_p - e_q) \left[ \frac{dx}{x} + G \left( \frac{1 - e_p - e_q}{x} \right) (1 - \alpha) - \frac{\rho}{\alpha} \frac{1}{\alpha} (1 - e_q) B(H) \right]}{(1 + \frac{d[e_p]}{de_q})} \tag{C.41}
\]

And we have obtained an explicit equation for \( \frac{de_q}{dt} \). Now the system can be solved numerically through Maple as shown in Appendix D.2.
APPENDIX D
NUMERICAL CODING

D.1 Original Mode Code - Determining Steady State Values

The following are two Matlab codes used to approximate the steady state solutions of the original model. The first program approximates a value for $k$ that satisfies the steady state equations, while the second program takes the approximated capital and uses it to compute the remaining variables.

D.1.1 Program: ksolver

```matlab
fq=@ (Q,Q0,gamma,eta,K,k0,eq,eq0) Q/(1+(Q/Q0-1)*exp(-(gamma*(eq-eq0)+eta*(K-k0))));

%%%Initialization
%%%Initial technology and Technology threshold from Zil
lila=0.075;
A0=0.015;
D=10;

%%%Min age, initial age, max age
```
Q0=35;
q0=40;
Q=85;

%Initial capital
k0=99;

%Distribution parameters
phi=.01/5.5;
gamma=2.00;
eta=.1/5.5;

%depreciation and time discount
delta=.08;
rho=.05;

%Productivity and health education
Bstar=.0876;
%Bstar=.095;
eq= 1-delta/Bstar
eq0=.01;

%parameters
G=5;
alpha=.25;

%Right Hand Side
RHS1 = (1-alpha)/alpha;
RHS2 = Bstar/delta;
RHS3 = 1/(phi*(lila/A0-1));
RHS4 = 1/(Q*gamma*(Q/Q0-1));
RHS = RHS1*RHS2*RHS3*RHS4;

K = 100:.01:120;
q = zeros(1,length(K));
LHS = zeros(1,length(K));
Plot = zeros(1,length(K));

for i = 1:length(K)
    q(i) = feval(fq,Q,Q0,gamma,eta,K(i),k0,eq,eq0);
    LHS1 = exp(-phi*(q(i)*K(i)-q0*k0))/(1+(lila/A0-1)*exp(-phi*(q(i)*K(i)-q0*k0)));
    LHS2 = exp(-gamma*(eq-eq0)-eta*(K(i)-k0))/(1+(Q/Q0-1)*exp(-gamma*(eq-eq0)-eta*(K(i)-k0)))^2;
    LHS(i) = K(i)*LHS1*LHS2;
    Plot(i) = LHS(i) - RHS;
end

plot(K,Plot)
D.1.2 Program: stdystO

This Matlab program inputs the estimated $k$ value and a value of productivity and outputs the steady state values of the remaining variables $h, q, A,$ and $c$.

function [h,c]=stdystO(k,Bstar)

%Initialization

%Initial technology and Technology threshold from Zil
lila=0.075;
A0=0.015;
D=10;
Z=10;

%MIn age, initial age, max age
Q0=35;
q0=40;
Q=85;

%Initial capital
k0=99;

%Distribution parameters
phi=.01/5.5;
gamma=2.00;
et=1/5.5;

%depreciation and time discount
delta=.08;
 rho=.05;

%Productivity and health education

eq = 1-delta/Bstar;

eq0=.01;

%parameters

theta=.67;

G=5;

alpha=.25;

\[ h = \frac{k \times Bstar}{\delta} \times \frac{G \times \alpha}{\rho + \delta} \times \frac{1}{1 - \alpha} \]

\[ q = \frac{Q}{1 + (Q/Q0-1) \times \exp\left(-\frac{\gamma(eq-eq0)+\eta(k-k0))}{\phi(q-k-k0)}\right)} \]

\[ A = \frac{lila}{1 + (lila/A0-1) \times \exp\left(-\phi(q-k-k0)\right)} \]

\[ c = D \times A \times k + Z \times (A \times k)^\theta + G \times (1-\alpha) \times k^\alpha \times (h*(1-eq))^{1-\alpha} - \rho \times k \]

D.2 Transformed Model Code - Steady State & Stability Analysis

The following is a Maple code, (with many of the output values suppressed) used to test the linear stability of the Transformed model. The parameters were changed in order to generate the tables and percent changes.

restart:

Digits:=15:
a := 0.75e-1:#Technology Threshold

A0 := 0.15e-1:#Initial level Technology

Q0 := 35:#Lowest possible life expectancy

q0 := 40:#Initial Life Expectancy

Q := 85:#Highest possible life expectancy

#phi := 0.1e-1;#Distribution Parameter for technology, Zil’s parameter

phi := 0.2e-1:#Distribution Parameter for technology

gama:=2.0:#Distribution parameter for eq for changes in q

eta := .1:#Distribution parameter for k for changes in q

delta := 0.8e-1:#Time discount factor

rho := 0.5e-1:#Depreciation of Capital

G := 5:#Cobb-Douglas multi-factor productivity constant .3¡G¡5.7

alpha := .25:#Cobb Douglas exponent

Z := 10:#Production function parameters

DD := 1:

theta := .72:#Relative Risk Aversion

sigma := 0.4:#.66666;#Elasticity Measure

x0 := 13:#Initial value for x

eq0 := 0.1e-1:#initial level of health education

q:=Q/(1+(Q/Q0-1)*exp(-gama*(eq-eq0)-eta*(x-x0))):  
A:= a/(1+(a/A0-1)*exp(-phi*(q*x-q0*x0)) ):  
M := 1+(a/A0-1)*exp(-phi*(q*x-q0*x0)):
\[ N := 1 + (Q/Q_0 - 1) \exp(-\gamma (eq - eq_0) - \eta (x - x_0)) \]

\[ ep := 1 - eq - (1 - \alpha) M N^2 / (\alpha \phi x (a/A_0 - 1) \exp(-1 \phi (q * x - q_0 * x_0)) Q \gamma (Q/Q_0 - 1) \exp(-\gamma (eq - eq_0) - \eta (x - x_0))) \]

\[ G * ((1 - ep - eq) / x)^{1 - \alpha} - \rho / \alpha - 1 / \alpha B * (1 - eq) = 0; \]

\[ dqdx := Q * \eta (Q/Q_0 - 1) \exp(-\gamma (eq - eq_0) - \eta (x - x_0)) / N^2; \]

\[ dqdeq := Q * \gamma (Q/Q_0 - 1) \exp(-\gamma (eq - eq_0) - \eta (x - x_0)) / N^2; \]

\[ depdeq := -1 + (\alpha - 1) N * (-a/A_0 - 1) \exp(-\phi (q * x - q_0 * x_0)) \phi x \gamma (Q/Q_0 - 1) \exp(-\phi (q * x - q_0 * x_0) - \gamma (eq - eq_0) - \eta (x - x_0)) + (\phi x \gamma + \gamma) M N / (\alpha \phi x (a/A_0 - 1) Q \gamma (Q/Q_0 - 1) \exp(-\phi (q * x - q_0 * x_0) - \gamma (eq - eq_0) - \eta (x - x_0))) \]

\[ depdx := (\alpha - 1) N (x (a/A_0 - 1) \exp(-1 \phi (q * x - q_0 * x_0)) (-\phi) (dqdx x + q) N - 2 \eta x M (Q/Q_0 - 1) \exp(-\gamma (eq - eq_0) - \eta (x - x_0)) - M N (1 - x (\eta + \phi (dqdx x + q))) / (\alpha \phi (a/A_0 - 1) Q \gamma (Q/Q_0 - 1) \exp(-\phi (q * x - q_0 * x_0) - \gamma (eq - eq_0) - \eta (x - x_0))) \]

### Generating the Coefficients for the Linearized System of Equations

\[ RX := x * (DD * A * (1 - 1/\theta)) + G * ((1 - ep - eq) / x)^{1 - \alpha} * (1 - \alpha + \alpha / \theta) - \rho - v - B * ep; \]

\[ MM[1,1] := \text{diff}(RX, x); \]

\[ MM[1,2] := \text{diff}(RX, v); \]

\[ MM[1,3] := \text{diff}(RX, eq); \]

\[ RV := v * ((\alpha / \sigma - 1 + \alpha - \alpha / \theta) * G * ((1 - ep - eq) / x)^{1 - \alpha} + v + (1 - 1/\sigma) * \rho = \delta / \sigma - DD * A * (1 - 1/\theta)); \]
\[ R Eq := ((1 - ep - eq) \times (RX / x + G((1 - ep - eq) / x) \times (1 - alpha)) - rho/alpha - B * (1 - eq) / (alpha) - depdx \times RX) / (1 + depdeq); \]

\[ MM[3,1] := \text{diff}(REq, x); \]
\[ MM[3,2] := \text{diff}(REq, v); \]
\[ MM[3,3] := \text{diff}(REq, eq); \]

\# Bstar := 0.095; Bstar := 0.0876:

fpts := fsolve\left(G \times ((1 - ep - eq) / x) \times (1 - alpha) - rho/alpha - 1 / (alpha) \times Bstar \times (1 - eq) = 0, \text{subs}(B = Bstar, RX = 0), \text{subs}(B = Bstar, RV) = 0, x = 0..16, v = 0..1, eq = 0..1\right);

eq = 0.0148754639676155, x = 15.4615651931067, v = 0.512996922090144
evalf(eval(ep, fpts[1], fpts[2], fpts[3]));
0.179706317249914

# fsolve\left(subs(B = Bstar, RX = 0), subs(B = Bstar, RV) = 0 \right);

NRX := subs(B = Bstar, RX) = 0;

NRV := subs(B = Bstar, RV) = 0;

NREq := subs(B = Bstar, REq) = 0;

# RX = 0; \times [x, v, eq];

####### Enter Desired Fixed Points

####### Different Cases are Commented out

####### Generating Constant Matrix by Substituting Fixed Points
NN[1,1]:=evalf(subs(B=Bstar, fpts[1],fpts[2],fpts[3],MM[1,1])):
NN[1,2]:=evalf(subs(B=Bstar, fpts[1],fpts[2],fpts[3],MM[1,2])):
NN[1,3]:=evalf(subs(B=Bstar,fpts[1],fpts[2],fpts[3],MM[1,3])):
NN[2,1]:=evalf(subs(B=Bstar, fpts[1],fpts[2],fpts[3],MM[2,1])):
NN[2,2]:=evalf(subs(B=Bstar, fpts[1],fpts[2],fpts[3],MM[2,2])):
NN[2,3]:=evalf(subs(B=Bstar, fpts[1],fpts[2],fpts[3],MM[2,3])):
NN[3,1]:=evalf(evalf(subs(B=Bstar, fpts[1],fpts[2],fpts[3],MM[3,1]))):
NN[3,2]:=evalf(subs(B=Bstar, fpts[1],fpts[2],fpts[3],MM[3,2])):
NN[3,3]:=evalf(subs(B=Bstar, fpts[1],fpts[2],fpts[3],MM[3,3])):
with(linalg): with(LinearAlgebra):
R := Matrix( [[NN[1,1],NN[1,2],NN[1,3]], [NN[2,1],NN[2,2],NN[2,3]], [NN[3,1],NN[3,2],NN[3,3]]] ) :
evl:=eigenvals(R);

-2.41324739697898, -0.469010396640360, 0.284523336607193

evec:=eigenvectors(R):

######## Numerically Solving the Nonlinear 3x3 System
RRX :=subs(B=Bstar, x=x(t),v=v(t),eq=eq(t), x * (DD * A * (1 − 1/theta) + G * ((1 − ep − eq)/x)(1 − alpha) * (1 − alpha + alpha/theta) − rho − v − B * ep)):
RRV :=subs(B=Bstar, x=x(t),v=v(t),eq=eq(t),v * ((alpha/sigma − 1 + alpha − alpha/theta)*G* ((1−ep−eq)/x)(1−alpha) + v + (1−1/sigma)*rho−delta/sigma− DD * A * (1 − 1/theta) )):
RREq :=subs(B=Bstar, x=x(t),v=v(t),eq=eq(t),((1 − ep − eq) *(RX/x + G * (1−
\( ep - eq)/x(1 - alpha) - rho/alpha - B*(1 - eq)/alpha - depdx*RX)/(1 + depdeq) 

### preparing rhs of equations to test fixed points ###

\[
\begin{align*}
TRX & := \text{subs}(B=B_{\text{star}}, x \ast (DD \ast A \ast (1 - 1/\theta) + G \ast ((1 - ep - eq)/x)(1 - alpha) \ast \\
& (1 + alpha + alpha/\theta) - rho - v - B \ast ep)) \\
TRV & := \text{subs}(B=B_{\text{star}}, v \ast ((alpha/\sigma - 1 + alpha - alpha/\theta) \ast G \ast ((1 - ep - eq)/x)(1 - alpha) \ast \\
& v + (1 - 1/\sigma) \ast rho - delta/\sigma - DD \ast A \ast (1 - 1/\theta))) \\
TREq & := \text{subs}(B=B_{\text{star}}, ((1 - ep - eq) \ast (TRX/x + G \ast ((1 - ep - eq)/x)(1 - alpha) \ast \\
& rho/alpha - B \ast (1 - eq)/alpha) - depdx \ast TRX)/(1 + depdeq) \\
\end{align*}
\]

### Testing fixed points ###

\[
\begin{align*}
\text{fpts[1];fpts[2];fpts[3];eep:=evalf(eval(ep, fpts[1],fpts[2], fpts[3]));} \\
\text{eq = 0.0148754639676155} \\
\text{v = 0.512996922090144} \\
\text{x = 15.4615651931067} \\
\text{0.179706317249914} \\
\text{qq:=evalf(subs(fpts[1],fpts[2],fpts[3],Q/(1+(Q/Q0-1)*exp(-gama*(eq-eq0)-eta*(x-x0)))));} \\
\text{40.3605007776550} \\
\text{AA:= evalf(subs(fpts[1],fpts[2],fpts[3],a/(1+(a/A0-1)*exp(-phi*(q*x-q0*x0)))));} \\
\text{0.0500214743566077} \\
\text{evalf(subs(fpts[1],fpts[2],fpts[3],TRX));} \\
\text{evalf(subs(fpts[1],fpts[2],fpts[3],TRV));} \\
\text{evalf(subs(fpts[1],fpts[2],fpts[3],TREq));}
\end{align*}
\]