LOW VELOCITY IMPACT ANALYSIS OF COMPOSITE LAMINATED PLATES

A Dissertation

Presented to

The Graduate Faculty of The University of Akron

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

Daihua Zheng

December, 2007
LOW VELOCITY IMPACT ANALYSIS OF COMPOSITE LAMINATED PLATES

Daihua Zheng

Dissertation

Approved:

Advisor
Dr. Wieslaw K. Binienda

Committee Member
Dr. Atef F. Saleeb

Committee Member
Dr. Ernian Pan

Committee Member
Dr. Craig C. Menzemer

Committee Member
Dr. Xiaosheng Gao

Committee Member
Dr. Jerry Young

Accepted:

Department Chair
Dr. Wieslaw K. Binienda

Dean of the College
Dr. George K. Haritos

Dean of the Graduate School
Dr. George R. Newkome

Date

ii
ABSTRACT

In the past few decades polymer composites have been utilized more in structures where high strength and light weight are major concerns, e.g., aircraft, high-speed boats and sports supplies. It is well known that they are susceptible to damage resulting from lateral impact by foreign objects, such as dropped tools, hail and debris thrown up from the runway. The impact response of the structures depends not only on the material properties but also on the dynamic behavior of the impacted structure. Although commercial software is capable of analyzing such impact processes, it often requires extensive expertise and rigorous training for design and analysis. Analytical models are useful as they allow parametric studies and provide a foundation for validating the numerical results from large-scale commercial software. Therefore, it is necessary to develop analytical or semi-analytical models to better understand the behaviors of composite structures under impact and their associated failure process.

In this study, several analytical models are proposed in order to analyze the impact response of composite laminated plates. Based on Meyer’s Power Law, a semi-analytical model is obtained for small mass impact response of infinite composite laminates by the method of asymptotic expansion. The original nonlinear second-order ordinary differential equation is transformed into two linear ordinary differential equations. This is achieved by neglecting high-order terms in the asymptotic expansion. As a result, the semi-analytical solution of the overall impact response can be applied to contact laws
with varying coefficients. Then an analytical model accounting for permanent
deformation based on an elasto-plastic contact law is proposed to obtain the closed-form
solutions of the wave-controlled impact responses of composite laminates. The analytical
model is also used to predict the threshold velocity for delamination onset by combining
with an existing quasi-static delamination criterion. The predictions are compared with
experimental data and explicit finite element LS-DYNA simulation. The comparisons
show reasonable agreement.

Furthermore, an analytical model is developed to evaluate the combined effects of
prestresses and permanent deformation based on the linearized elasto-plastic contact law
and the Laplace Transform technique. It is demonstrated that prestresses do not have
noticeable effects on the time history of contact force and strains, but they have
significant consequences on the plate central displacement. For a impacted composite
laminate with the presence of prestresses, the contact force increases with the increasing
of the mass of impactor, thickness and interlaminar shear strength of the laminate. The
combined analytical and numerical investigations provide validated models for elastic
and elasto-plastic impact analysis of composite structures and shed light on the design of
impact-resistant composite systems.
ACKNOWLEDGEMENTS

First and foremost, I would like to express my sincere appreciation to my advisor, Professor Wieslaw K. Binienda, for his valuable guidance, continuous support and encouragement throughout this study.

The helpful advices and discussions from Dr. Robert K. Goldberg from NASA Glenn Research Center are gratefully acknowledged. The useful discussions and guidance regarding the method of asymptotic expansion from Dr. Jerry Young are really appreciated. Acknowledgements are also extended to my committee members, Dr. Atef F. Saleeb, Dr. Ernian Pan, Dr. Craig C. Menzemer, Dr. Xiaosheng Gao, for reviewing my work and helpful recommendations.

Special thanks are given to my fellow graduate students, Dr. Jingyun Cheng and Dr. Mijia Yang, for their useful discussions related to the topic of impact on composite structures. A sincere thank you extends to Dareen Moore for checking the manuscript and giving various useful comments. The sincere friendship and support from my friends in Akron, Yan Zhang, Xuetao Li, Mingkun Sun and Kedar Pathak, to name a few, always give me energy and impetus to finish this dissertation.

My deepest gratitude goes to my family, especially my wife Ziwei Yu who provides more love and support than I could ever expect.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>LIST OF TABLES</td>
<td>ix</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>ix</td>
</tr>
<tr>
<td>CHAPTER</td>
<td></td>
</tr>
<tr>
<td>I. BACKGROUND</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Research Motivation</td>
<td>2</td>
</tr>
<tr>
<td>1.3 Objectives</td>
<td>3</td>
</tr>
<tr>
<td>1.4 Literature Review</td>
<td>4</td>
</tr>
<tr>
<td>1.4.1 Dynamic Response of Composite Structures under Impact</td>
<td>5</td>
</tr>
<tr>
<td>1.4.2 Contact Laws for Composite Structures under Impact</td>
<td>7</td>
</tr>
<tr>
<td>1.4.3 Damage and Failure Modes of Composite Structures under Impact</td>
<td>9</td>
</tr>
<tr>
<td>1.4.4 Impact Responses of Laminates under Prestress</td>
<td>10</td>
</tr>
<tr>
<td>1.4.5 Composite Material Model in LS-DYNA</td>
<td>12</td>
</tr>
<tr>
<td>1.4.6 Cohesive Element Modeling of Delamination in ABAQUS</td>
<td>15</td>
</tr>
<tr>
<td>1.5 Thesis Outline</td>
<td>18</td>
</tr>
<tr>
<td>II. ELASTIC IMPACT RESPONSE OF INFINITE LAMINATES</td>
<td>20</td>
</tr>
</tbody>
</table>

vi
2.1 Classification of Impacts......................................................................................... 21

2.1 Derivation of the Governing Equations................................................................. 22

2.2 Hertzian Contact Law ............................................................................................ 26

2.3 Generalized Contact Law...................................................................................... 32

2.4 Conclusions............................................................................................................. 40

III. ELASTO-PLASTIC IMPACT RESPONSE OF INFINITE LAMINATES .............. 41

3.1 Nonlinear Elasto-plastic Contact Law ................................................................. 42

3.2 Linearized Elasto-plastic Contact Law ................................................................. 46

3.3 Validation of the Elasto-plastic Contact Law ....................................................... 49

3.5 Parametric Study of Closed-form Solution......................................................... 56

3.5.1 Effect of Mass of the Impactor ......................................................................... 57

3.5.2 Effect of Velocity of the Impactor .................................................................... 57

3.5.3 Effect of Plate Thickness .................................................................................. 57

3.5.4 Effect of Interlaminar Shear Strength.............................................................. 58

3.5.5 Effect of the Size of Impactor .......................................................................... 58

3.6 Conclusions............................................................................................................. 58

IV. PREDICTION OF THRESHOLD VELOCITY FOR
DELAMINATION ONSET .................................................................................. 66

4.1 Delaminations due to Impact .............................................................................. 66

4.2 Prediction of Threshold Velocity for Delamination Onset.................................... 68
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1 Composite Material Models in LS-DYNA</td>
<td>13</td>
</tr>
<tr>
<td>3.1 Geometrical and Material Properties of the HTA/6376 Laminate</td>
<td>51</td>
</tr>
<tr>
<td>4.1 Material Properties for Experimental Validation</td>
<td>72</td>
</tr>
<tr>
<td>4.2 Shear Strength of the Laminate ($S_s$)</td>
<td>72</td>
</tr>
<tr>
<td>4.3 Measured and Predicted Delamination Threshold Velocities</td>
<td>73</td>
</tr>
<tr>
<td>4.4 Details of the Models in LS-DYNA</td>
<td>75</td>
</tr>
<tr>
<td>4.5 Material Properties for Finite Element Validation</td>
<td>83</td>
</tr>
<tr>
<td>4.6 Strength Data for T300/5208</td>
<td>84</td>
</tr>
<tr>
<td>5.1 Properties of Composite Plate and Impact Parameters</td>
<td>97</td>
</tr>
<tr>
<td>Figure</td>
<td>Page</td>
</tr>
<tr>
<td>----------------------------------------------------------------------</td>
<td>--------</td>
</tr>
<tr>
<td>1.1 Delaminations Caused by Impact (Davies and Olsson, 2004)</td>
<td>2</td>
</tr>
<tr>
<td>1.2 Spring-mass Models (Abrate, 2001)</td>
<td>6</td>
</tr>
<tr>
<td>1.3 Damage of Prestressed Laminate under Impact (Robb et al., 1995)</td>
<td>11</td>
</tr>
<tr>
<td>1.4 Schematic Representation of Cohesive Failure Models</td>
<td>15</td>
</tr>
<tr>
<td>1.5 Damage Evolution Based on Effective Displacement (ABAQUS, 2000)</td>
<td>18</td>
</tr>
<tr>
<td>2.1 Classification of Different Impacts (Olsson, 2000)</td>
<td>22</td>
</tr>
<tr>
<td>2.2 Comparisons between Different Impact Responses (Olsson, 2003)</td>
<td>22</td>
</tr>
<tr>
<td>2.3 Structural Model for Small Mass Impact (Olsson, 2003)</td>
<td>24</td>
</tr>
<tr>
<td>2.4 Approximated Shape of Impact Affected Area</td>
<td>25</td>
</tr>
<tr>
<td>2.5 Indentation of Composite Laminate (Choi and Lim, 2004)</td>
<td>26</td>
</tr>
<tr>
<td>2.6 Nondimensional Indentation Histories as a Function of $\lambda_c$</td>
<td>30</td>
</tr>
<tr>
<td>2.7 Nondimensional Contact Force Histories as a Function of $\lambda_c$</td>
<td>30</td>
</tr>
<tr>
<td>2.8 Nondimensional Velocity Histories as a Function of $\lambda_c$</td>
<td>31</td>
</tr>
<tr>
<td>2.9 Static Indentation Tests for Carbon/epoxy Laminates (Liou, 1997)</td>
<td>33</td>
</tr>
<tr>
<td>2.10 $\alpha_0$ Histories as a Function of $\eta$ and $\lambda_s$</td>
<td>37</td>
</tr>
<tr>
<td>2.11 $\alpha_1$ Histories as a Function of $\eta$ and $\lambda_s$</td>
<td>37</td>
</tr>
<tr>
<td>2.12 Comparison of Numerical and Asymptotic Results (Over-damping)</td>
<td>38</td>
</tr>
</tbody>
</table>
2.13 Comparison of Numerical and Asymptotic Results.................................................. 38
(Under-damping & Critical-damping) .............................................................................. 38

2.14 Effect of Contact Law Exponent on Indentation History ( $\lambda_x = 4.0$ ) ............. 39

2.15 Effect of Contact Law Exponent on Indentation History ( $\lambda_y = 0.0$ ) .............. 39

3.1 Force-Indentation Response in Contact Loading of Laminate .............................. 43

3.2 Reduction of Laminate Tensile Strength after Contact Loading ......................... 43

3.4 Contact Force History for Impacted Plate with Damage ..................................... 51

3.5 Comparison of Contact Force History ( $\lambda_x=2.07$ ) ........................................... 53

3.6 Comparison of Contact Law ( $\lambda_x=2.07$ ) ........................................................ 54

3.7 Comparison of Center Deflection of Plate ( $\lambda_x=2.07$ ) .................................... 54

3.8 Comparison of Contact Force History ( $\lambda_x=3.30$ ) ........................................... 55

3.9 Comparison of Contact Law ( $\lambda_x=3.3$ ) .......................................................... 55

3.10 Comparison of Center Deflection ( $\lambda_x=3.3$ ) .................................................. 56

3.11 Comparison of Contact Force from Different Contact Laws ............................... 56

3.12 Effect of Impactor Mass on Contact Force ........................................................... 59

3.13 Effect of Impactor Mass on Plate Central Displacement ..................................... 60

3.14 Effect of Impactor Mass on Velocity Time History .............................................. 60

3.15 Effect of Impactor Velocity on Contact Force .................................................... 61

3.16 Effect of Impactor Velocity on Plate Central Displacement ............................... 61

3.17 Effect of Impactor Velocity on Velocity Time History ......................................... 62

3.18 Effect of Plate Thickness on Contact Force ....................................................... 62

3.19 Effect of Plate Thickness on Plate Central Displacement .................................... 63
3.20 Effect of Plate Thickness on Velocity Time History .................................................. 63
3.21 Effect of Interlaminar Shear Strength on Contact Force ........................................ 64
3.22 Effect of Interlaminar Shear Strength on Plate Central Displacement .................. 64
3.23 Effect of Impactor Radius on Contact Force ............................................................ 65
3.24 Effect of Impactor Radius on Plate Central Displacement .................................... 65
4.1 Representation of Impact-Induced Damage (Philippe et. al, 1998) ....................... 67
4.2 Shapes of Impact-induced Delaminations (Abrate, 1998) .................................... 68
4.3 Damage Size as a Function of Impact Force for Plates ............................................ 69
4.5 Different Mesh Sizes of the Plate ............................................................................. 76
4.6 Convergence of Contact Force with Different Mesh Sizes .................................... 76
4.7 Failure Surface of MAT 162 Material Model (Xiao et al., 2007) .......................... 81
4.8 One Element under Simple Tension ......................................................................... 84
4.9 Stress vs. Strain for X-tension and X-compression ................................................ 84
4.10 Stress vs. Strain for Z-tension ................................................................................ 85
4.11 Stress vs. Strain for Shear-Compression Interaction ............................................. 85
4.12 Mesh of One Layer of the Composite Laminate ..................................................... 86
4.13 Configuration of the Composite Laminate ............................................................. 87
4.14 Delaminations in Composite Laminate ................................................................... 87
5.1 Convergence Study for Contact Force ................................................................. 98
5.2 Convergence Study for Displacement ................................................................. 98
5.3 Convergence Study for Strain $\varepsilon_x$ ................................................................. 99
5.4 Comparison of Contact Force Time History ....................................................... 100
5.5 Comparison of Plate Central Displacement Time History .................................. 100
5.6 Contact Force Histories with Prestresses (Chiu et al., 1997)................................. 102
5.7 Effect of Prestress on Contact Force........................................................................ 103
5.8 Effect of Prestress on Central Displacement ......................................................... 104
5.9 Effect of Prestress on Strain $\varepsilon_x$ ...................................................................... 104
5.10 Effect of Prestress on the Strain $\varepsilon_y$ ................................................................. 105
5.11 Effect of Impact Velocity on Contact Force......................................................... 106
5.12 Effect of Impact Velocity on Central Displacement.............................................. 106
5.12 Effect of Plate Thickness on Contact Force......................................................... 107
5.13 Effect of Plate Thickness on Central Displacement ............................................ 108
5.14 Effect of Impactor Mass on Contact Force............................................................ 108
5.15 Effect of Impactor Mass on Central Displacement............................................... 109
5.17 Effect of Fiber Shear Strength on Contact Force.................................................. 110
5.18 Effect of Fiber Shear Strength on Central Displacement........................................ 110
CHAPTER I

BACKGROUND

1.1 Introduction

In the past few years, laminated composite materials have been used extensively due to their high-strength and light-weight features. However, their behavior under foreign object impact is of great concern since the impact-induced damage can significantly reduce the strength of the structure. A preliminary step in understanding the effect of impact is to develop models for predicting the overall responses of the structure. Since the impact response depends not only on the material properties but also on the dynamic behavior of the impacted structure, it is very important to have a basic understanding of the structural response under impact as well as how it is affected by different parameters.

Usually the impact-induced damage is caused by the interaction of local indentation and global deformation of the structure. Impact damage in composite laminates includes a couple of common damage characteristics, such as matrix cracks, delaminations and fiber ruptures. Delaminations are particularly serious since they can occur at relatively low loads and have a main influence on the flexural stiffness degradation and buckling failure of composite laminated plates. Delaminations in impacted laminates are primarily driven by interlaminar shear stresses. Delaminations are also a main energy absorption mechanism of polymer composite materials, which consists of the creation of
fracture areas at the weaker interfaces between the individual composite layers. A typical distribution of delaminations in composite laminates is shown in Figure 1.1. It indicates that the major axes of delaminations are usually oriented in the direction of fibers at the interface of plies.

![Diagram of delaminations](image)

(1) Compressive failure  
(2) Tensile failure  
(3) Shear-driven delaminations

Figure 1.1 Delaminations Caused by Impact (Davies and Olsson, 2004)

1.2 Research Motivation

An efficient approach to study the effect of impact on composite structures is to separately analyze two aspects of impact, impact damage resistance and impact damage tolerance. Impact damage resistance addresses the response and damage of the structures caused by impact. On the other hand, impact damage tolerance deals with the effect of existing impact damage on the strength and stability of the structures. The impact damage resistance of composite laminates is investigated throughout this study.

Significant research efforts on impact damage resistance of composite structures have been conducted in the last few decades. The state of arts of this subject has been
thoroughly reviewed by Abrate (1991; 1994; 1998). Most of the studies are based on the Hertzian contact law, which was originally developed for static loading on an isotropic linear elastic half-space. Permanent deformation associated with damage was not accounted for in the Hertzian contact law. It was shown that, even at low contact loads, the permanent deformation causing damages around the contact zone is present (Swanson and Rezaee, 1990). Therefore, a contact law incorporating effects of damage is needed for accurately modeling of contact force-deformation during impact.

Even though some commercial software, such as LS-DYNA (2006) and ABAQUS Explicit (2000), is capable of analyzing such impact processes, it often requires expertise and rigorous training in using such software packages for design and analysis. In this regard, analytical models are useful as they allow parametric studies and provide a foundation for validating the numerical results from large-scale commercial software. Thus, it is necessary to develop analytical or semi-analytical approaches to better understand the behaviors of composite structures under impact and associated failure processes.

1.3 Objectives

Motivated by above statement, there are two main goals of this study: (1) to investigate the dynamic response of composite laminates under low and medium velocity impact; (2) to predict the initiation of damages of composite structures and validate the results through existing experimental data and numerical simulations. The detailed objectives within the two goals are:
(1) Develop a semi-analytical solution for small mass impact response of laminated plates using generalized elastic contact law;

(2) Develop closed-form solutions for small mass impact responses of laminated plates using elasto-plastic contact law;

(3) Develop analytical predictions of delamination onset for small mass impact of laminated composite plates;

(4) Develop closed-form solutions for studying the combined effects of shear deformation, permanent indentation and prestress in impact loading of laminates.

Corresponding to the above objectives, four chapters (II to V) are presented in this study. The contributions of this study and recommendations for further research are summarized in Chapter VI.

1.4 Literature Review

Significant research efforts have been conducted in the area of impact damage resistance of composite laminates. The research on impact damage resistance of composites can be classified into the following three categories: (1) Dynamic response of composite laminates; (2) Contact mechanics of composite structures under impact loading; (3) Damage and failure modes of composite structures under impact loading. In the following sections, the research outcomes and progress corresponding to the above three categories of impact of composite structures are reviewed.
1.4.1 Dynamic Response of Composite Structures under Impact

A detailed review of the impact mechanics and dynamics of composite structures has been made by Abrate (1998; 2001). Depending upon how the structure was modeled, dynamic impact responses of composite structures can be classified as: spring-mass models, energy balance models, complete models, and model for impact on an infinite plate. Spring-mass models are simple and can provide accurate analysis for some types of impacts encountered during tests on small size specimens. There are two-degree-of-freedom (TDOF) and single-degree-of-freedom (SDOF) models for impacts on composite plates and beams, as shown in Figure 1.2(a) and Figure 1.2(b). For the TDOF model, it consists of one spring representing the linear stiffness of the structure (\( K_{nx} \)), another spring for the nonlinear membrane stiffness (\( K_m \)), nonlinear contact stiffness (\( k \)), effective mass of the structure (\( M_2 \)) as well as the mass of the projectile (\( M_1 \)). For the SDOF model, the global deformation of the structures is neglected. The local deformation is taken into account by using the nonlinear contact stiffness (\( k \)).

Another approach for analyzing the impact is to consider the balance of the energy in the system. According to this model, the initial kinetic energy of the projectile is completely used to deform the structure during impact. When the structure reaches its maximum deflection, the velocity of the projectile becomes zero. Both spring-mass model and energy balance model assumed that the structure behaves in a quasi-static manner.
In the complete models for impact responses of structures, the dynamic behavior of the entire structure is accounted for as opposed to the above two models. Sun and Chattapadhyay (1975) used the plate equations developed by Whitney and Pagano (1970) to study a simply-supported orthotropic plate subjected to central impact. Christoforou and Swanson (1991) obtained an analytical solution of shear deformable composite plate based on the double Fourier series expansions and Laplace transform techniques. Mittal and Khalili (1994) used Fourier transform and Hertz-Sveklo’s contact theory to study the transverse impact of orthotropic composite plates. An analytical model which includes the combined effects of shear deformation, rotary inertia and nonlinear Hertzian contact law was proposed by Pierson and Vaziri (1996) to study the impact response of composite laminated plates.

In most of the theoretical models for the impact analysis of composite laminates, the permanent deformations were typically not included due to the adoption of Hertzian contact law. The Hertzian contact law is adequate in low contact loads. At higher contact loads, matrix cracking and delamination can induce a decreasing indentation since the reduced stiffness will result in a redistribution of the contact pressure around the contact.
zone. During unloading, some of the impact-induced deformation can not be recovered and Hertzian contact law is not appropriate in those cases. Therefore, the inclusion of permanent indentation in the contact law is important in order to accurately model the impact behavior of composite laminates under high contact loads.

1.4.2 Contact Laws for Composite Structures under Impact

The static indentation of a spherical impactor on fiber-reinforced composite laminates has been shown to follow a Hertzian type contact law (Willis, 1966; Cairns and Lagace, 1987). During the initial stage of impact loading, Hertzian contact law is adequate. It has been shown experimentally that even at low contact loads, the permanent deformation causing damages around the contact zone is present (Swanson and Rezaee, 1990).

Damage effects were accounted for by using a modified unloading curve which was based on experimental data (Yang and Sun, 1982). Swanson and Rezaee (1990) investigated the effect of damage on lateral contact loads of a laminate. The laminate was supported by rigid flatten so that the contact loads were introduced without bending effects. It was shown that Hertzian contact law can adequately model the loading phase before reaching some large loads (36 KN). When the contact force reaches some level, the permanent indentation can not be recovered during unloading phase. The contact law will deviate significantly from Hertzian contact law.

Christoforou (1993) proposed an analytical solution for the contact between a rigid sphere and a thin composite laminate supported on a rigid substrate. Contact-induced damages were included in the contact force-deformation relation by using an elastic-perfectly plastic constitutive law and a maximum shear failure criterion. The derived
contact force-deformation including damage effects is in good agreement with the experimental results (Swanson and Rezaee, 1990). Cairns (1991) developed a simple elasto-plastic contact law for composites to account for the effect of permanent deformation. The material is assumed to be elastic until a critical indentation is reached. The contact area is divided into a plastic zone and an elastic zone as the loading increases. By assuming elastic-perfectly plastic response for the through-the-thickness constitutive behavior of the composite laminate, closed-form representation of the contact law was obtained for both elastic and plastic contact condition.

Recently, Yigit and Christoforou (1994) developed a more realistic elasto-plastic contact law including the permanent indentation effects. The contact law was derived by combining the classical Hertzian contact theory (Goldsmith, 1960) and elastic-plastic indentation theory for metallic bodies by Johnson (1985). In this contact law, the contact is assumed to be Hertzian in the first phase. In the second phase, the elastic-plastic behavior is assumed after the “yielding” point is exceeded. Due to the brittle failure behavior of fiber composites, “yielding” in the material indicates a combination of different damage modes such as matrix crack and fiber tensile failure or buckling. The third phase is assumed to be elastic Hertzian behavior again for unloading. This contact law has been successfully applied to the impact analysis of composite beams and plates (Christoforou and Yigit, 1998; Yigit and Christoforou, 1995) and validated against experimental data by Poe and Illg (1989). The elasto-plastic contact law proposed by Yigit and Christoforou (1994) was used throughout this study.
1.4.3 Damage and Failure Modes of Composite Structures under Impact

The impact damage and impact resistance of composite laminates have been extensively investigated. Impact damage is caused by the interaction of local indentation and global deflection and involves several common damage features such as matrix cracks, delaminations and fiber ruptures. Impact-induced delaminations have been found to reduce the residual compressive strength of composite structures significantly. Delaminations are mainly induced by interlaminar shear stress, which are enhanced by matrix cracks and ply stiffness mismatch (Davies et al., 1992). It was reported by experimental studies that delaminations occur only at interfaces between plies with different fiber orientations. Impact-induced delamination of composites has been performed by Chang and his colleagues (Choi et al., 1991a; Choi et al., 1991b; Wu and Chang, 1989; Choi and Chang, 1992) by detailed experimental and numerical 2D and 3D analyses of the damage process.

The delamination area is usually plotted against the initial kinetic energy of the impactor (Cantwell 1988a; Cantwell 1988b; Wu and Springer, 1988). After a threshold value is reached, the size of delaminations increase linearly with the kinetic energy of the impactor. Since the experimental results scatter from one specimen from another, it is difficult to determine experimentally the threshold value of the kinetic energy. It was indicated by several studies that the damage is initiated when the contact force reaches a critical value (Sjoblom, 1987; Lindsay and Wilkins, 1991). The critical contact force is also the same for static indentation tests and low-velocity impact tests. (Davies et. al, 1994) proposed an approach to deal with the prediction of the threshold value of the contact force that corresponds to damage initiation. Zhou (1995) showed that damage
initiation dominated by delamination can be predicted from a simplified theoretical model based on the impact response.

1.4.4 Impact Responses of Laminates under Prestress

Most of the structures will be under some level of prestress when impacted by foreign accidental objects. It has been shown by several studies that the application of tensile preload can greatly reduce the strength of composite laminates (Butcher, 1976a; Butcher 1976b; Sankar and Sun, 1985). In the experiments, uniaxial tension produced a larger damage area than when uniaxial compression or no preload was applied (Chiu et al., 1997). Under compressive preload the critical velocity, which causes catastrophic failure, decreases with increasing preload (Herszberg et al., 1997). Low-velocity impact tests on E-glass composite laminates were conducted with several levels of tensile and compressive prestresses as well as an unstressed state (Robb et al., 1995). The absorbed energy was discovered to be greatest when a compressive prestress was present and at a minimum in the tension/tension quadrant. Equi-biaxial tension/tension preload produced the greatest peak load while equi-biaxial tension/compression loading returning the minimum value. It was also shown that the type of prestress applied to the test specimen gives characteristic shapes to the damage areas, as shown in Figure 1.4.
Sun and Chattopadhyay (1975) studied the response of symmetric cross-ply laminated composite plates subjected to impact loading in the presence of biaxial pre-stresses. The resulting nonlinear integral equation was solved by a stepwise numerical integration scheme. Sun and Chen (1985) conducted a finite element study for composite plates modeled as homogeneous orthotropic layers. A nine-node isoparametric plate finite element program was developed by incorporating a plastic contact law. The plastic contact law takes into account the permanent indentation and differs during loading and unloading process.

Christoforou and Swanson (1991) obtained an analytical solution for shear deformable specially-orthotropic composite plates. Double Fourier series expansion and Laplace transform technique were used in their study. In order to obtain the analytical solution, a linearized contact stiffness needs to be used based on an estimated value of the peak
contact force. An iteration of solution is required to obtain an estimate of the peak contact force. Permanent indentation or damage effects were not considered in their study. The simultaneous effects of preloads and permanent indentation on the impact response of laminated composite plates have not been reported in the literature.

1.4.5 Composite Material Model in LS-DYNA

Numerical finite element analyses were utilized to validate the proposed analytical and semi-analytical models. The commercial explicit nonlinear finite element software LS-DYNA was used throughout this study. The available composite material models in LS-DYNA (2007) are summarized as shown in Table 1.1.
Table 1.1 Composite Material Models in LS-DYNA

<table>
<thead>
<tr>
<th>Model Description</th>
<th>Damage</th>
<th>Shell Element</th>
<th>Solid Element</th>
<th>Shear</th>
<th>Strain Rate</th>
<th>Failure Criteria</th>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAT_2 (orthotropic elastic)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td>*MAT_ADD_EROSION</td>
</tr>
<tr>
<td>MAT_21 (thermal orthotropic)</td>
<td></td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MAT_23 (thermal orthotropic)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MAT_22 (composite damage)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>Chang-Chang</td>
<td>Laminate</td>
</tr>
<tr>
<td>MAT_40 (nonlinear orthotropic)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td>Failure Strain</td>
<td>Unstable</td>
</tr>
<tr>
<td>MAT_54 (enhanced composite damage)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td>Chang-Chang</td>
<td>Laminate</td>
</tr>
<tr>
<td>MAT_55 (enhanced composite damage)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td>Tsai-Wu</td>
<td>Laminate</td>
</tr>
<tr>
<td>MAT_58 (laminated composite fabric)</td>
<td>✓</td>
<td></td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
<td>Hashin</td>
</tr>
<tr>
<td>MAT_59 (composite failure)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MAT_81, MAT_82 (mat_plasticity with damage)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MAT_114 (layered linear plasticity)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MAT_116 (composite layup)</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Elastic</td>
</tr>
<tr>
<td>MAT_117 (composite matrix)</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Elastic</td>
</tr>
<tr>
<td>MAT_118 (composite direct)</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Elastic</td>
</tr>
<tr>
<td>MAT_161, MAT_162 (composite_(DMG)_msc)</td>
<td>✓</td>
<td></td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>Hashin</td>
<td>Delamination</td>
</tr>
</tbody>
</table>
In summary, most composites material models were implemented for shell elements, while only MAT_81, MAT_82, MAT_114 and MAT_161/162 considered the strain rate effects. The favorite failure criteria in those material models are Chang-Chang, Tsai-Wu and Hashin criteria. MAT_22 and MAT_54 provided Chang-Chang fiber and matrix tensile and compressive failure modes due to in-plane stresses in unidirectional lamina. MAT_55 considered Tsai-Wu failure criteria. MAT_58 is based on the Hashin criteria and continuum damage mechanics. In those 2D failure models, the failure modes due to out-of-plane shear and normal stresses are neglected. MAT_59 can be used for both shell and solid elements. This material model is close to MAT_54, but it does not need maximum strains. MAT_116, MAT_117 and MAT_118 are used to model the elastic response of composites where a pre-integration is used to compute the extensional, bending and coupling stiffness coefficients for use with the Belschko-Tsay resultant shell formulation. MAT_116 was based on standard composite lay-up theory while MAT_117 and MAT_118 were based on laminated shell theory. The stiffness coefficients are in material and element coordinate system for MAT_117 and MAT_118, respectively.

It can be seen from Table 1.1 that only MAT_161/162 can simulate delamination in composites. MAT_161 is a composite lamina model based on 3D stresses field. The failure model can be used to effectively simulate fiber failure, matrix damage and delamination behavior under all loading conditions – opening, closure and sliding of failure surfaces. The progressive failure model adopted the failure criteria developed by Hashin (1980) and included the effect of highly constrained pressure on composite failure. The model with DMG option (MAT_162) generalized from MAT_161 by adopting the damage mechanics approach. MAT_162 can better characterize the softening behavior of
the material after damage initiation. This material model can be used to simulate the progressive failure of unidirectional composite laminates as well as woven fabric composite structures.

1.4.6 Cohesive Element Modeling of Delamination in ABAQUS

ABAQUS has the capability to model delamination in composites due to impact using cohesive elements. The two most significant cohesive modeling approaches available in the literature are the intrinsic exponential potential-based law (Xu and Needleman, 1995) and the extrinsic linear law (Camacho and Ortiz, 1996). In the intrinsic method the failure criterion is included in the constitutive model of the cohesive elements. The cohesive tractions increase from zero to a maximum value (failure point) and then decrease gradually back to zero values. In the extrinsic method the failure criterion is external to cohesive elements. Once the cohesive tractions reach a critical value, the interface between two volumetric elements is allowed to open according to a prescribed traction-separation law. Additional nodes coupled by a cohesive law will be introduced along the failed interface. The schematic representation of intrinsic and extrinsic approach is shown in Figure 1.4 (Geubelle and Baylor, 1998).

Figure 1.4 Schematic Representation of Cohesive Failure Models
In ABAQUS, the intrinsic and extrinsic approaches have been implemented as traction-separation and continuum responses. The traction-separation response modeling in ABAQUS is suitable for delamination simulation and assumes a linear elastic behavior followed by damage initiation and evolution. The elastic behavior is expressed in terms of an elastic constitutive matrix relating the nominal stresses to the nominal strains across the interface. Once a damage initiation criterion is met, material damage can occur according to a user-defined damage evolution law.

Several damage initiation criteria are available in ABAQUS: maximum nominal stress criterion, maximum nominal strain criterion, quadratic nominal stress criterion and quadratic nominal strain criterion. The damage evolution law represents the rate at which the material stiffness is degraded once the corresponding initiation criterion is reached. A scalar damage variable \( D \) is used to represent the overall damage in the material and captures the combined effects of all the active mechanisms. It initially has a value of 0 and monotonically evolves from 0 to 1 upon further loading after the initiation of damage. Damage evolution is based on fracture energy or effective displacement of the cohesive elements.

For damage evolution based on effective displacement, linear or exponential softening after damage initiation can be used, as shown in Figure 1.5. To describe the evolution of damage under a combination of normal and shear deformation across the interface, an effective displacement was introduced and defined as (Camanho and Davila, 2002):

\[
\delta_e = \sqrt{<\delta_i>^2 + \delta_{II}^2 + \delta_{III}^2}
\] (1.1)
where $\delta_i$ and $\delta_{II}$ are the tangential normal and shear relative displacement of the element, respectively, $\delta_{III}$ is the out-of-plane relative displacement of the element. \{ \} in Eq. (1.1) is Macaulay bracket, which means that only positive tangential normal relative displacement will be considered.

For damage evolution based on fracture energy, there are two analytical functions considering mode mix: power law failure criterion and Benzegagh-Kenane (BK) form (Benzegagh and Kenane, 1996). The BK fracture criterion is particularly useful when the critical fracture energies along the first and the second shear directions are the same. The power law failure criterion and BK form analytical functions are shown in Eq. (1.2) and Eq. (1.3), respectively.

\[
\left( \frac{G_i}{G_{IC}} \right)^\alpha + \left( \frac{G_{II}}{G_{IC}} \right)^\alpha + \left( \frac{G_{III}}{G_{III}} \right)^\alpha = 1 \tag{1.2}
\]

\[
G_i^c + (G_{II} - G_{IC}) \left( \frac{G_{II} + G_{III}}{G_T} \right)^\eta = G_c \tag{1.3a}
\]

\[
G_T = G_i + G_{II} + G_{III} \tag{1.3b}
\]

\[
G_c = G_{IC} + G_{II} + G_{III} \tag{1.3c}
\]

where $\alpha$ and $\eta$ are power law exponents which are different for different materials. $G_{IC}$, $G_{II}$ and $G_{III}$ are the fracture toughness for mode I, mode II and mode III fracture, respectively.
1.5 Thesis Outline

The thesis is organized as follows: Chapter II proposes a semi-analytical model to analyze small mass impact response of an infinite composite plate using the method of asymptotic expansion. Chapter III describes the closed-form solutions of the small mass impact responses of infinite composite plates based on an elasto-plastic contact law. Permanent indentation associated with damage effects was accounted for in the contact law. Chapter IV presents a closed-form prediction of delamination onset, which is based on quasi-static delamination threshold load criterion as well as the peak contact force.
prediction from the model in Chapter III. Chapter V presented the effect of shear deformation and prestress that is neglected in previous models. Finally, the conclusions and contributions of this research as well as future research opportunities are discussed in Chapter VI.
CHAPTER II

ELASTIC IMPACT RESPONSE OF INFINITE LAMINATES

Impacts are often classified as low-velocity and high-velocity impacts. However, a more relevant classification is proposed as boundary-controlled and wave-controlled impacts. Generally, boundary-controlled and wave-controlled impacts are typically associated with large-mass and small-mass impact responses. A mass-ratio based criteria governing boundary-controlled and wave-controlled impact response has been derived in detail by Olsson (2000). The derivation shows that wave-controlled impacts occur when the mass of the impactor is less than one fourth of the mass of the largest possible area, which waves do not interfere with the boundaries. Large-mass impacts have been investigated thoroughly and can be modeled using the familiar spring-mass system. Due to the more localized deformation, small-mass impacts can cause higher impact forces and earlier damage initiation.

In the original study for small-mass impact done by Olsson (1992), the Hertzian contact law was used. In the Hertzian contact law, the power law coefficient in the force-indentation relation is taken as 3/2. However, in practical engineering problems, the contact law coefficient can be varied from 1 to 3/2. In this study, a generalized contact law with power law coefficients fitting from experiment was applied to the governing equation for small mass impact. The original nonlinear equation was decoupled into two linear equations based on the method of asymptotic expansion. Closed-form solutions can
be derived for the first equation, which is linear and homogeneous. The second linear equation is non-homogeneous and was solved numerically. The overall impact responses can be obtained semi-analytically based on asymptotic expansion and this has been found to agree well with the numerical solutions of nonlinear governing equations. The proposed methodology is useful for providing guidance to numerical simulation of small-mass impact on composite structures.

2.1 Classification of Impacts

It is well-known that an impact initiates elastic waves propagating from the point of impact. Material damping and energy dissipation related to wave propagation will result in a decaying response. Therefore, the duration of impact plays a key rule in determining the type of impact responses. If the impact duration is in the order of the transition time for dilatational waves, the response will be dominated by through-the-thickness waves, as shown in Figure 2.1(a). For longer impact duration, the response will be governed by flexure and shear waves, as shown in Figure 2.1(b). If impact duration is much longer than the time for the waves reaching the structure boundaries, the resulting response will be quasi-static. This happens because the deflection and load would have similar relation as in a static loading, as shown in Figure 2.1(c).

Typically, the response in Figure 2.1(a) is related to ballistic impact. The responses in Figure 2.1(b) and Figure 2.1(c) are usually associated with impact by runway debris and impact from drop weights, respectively. In most cases, the response in Figure 2.1(a) will cause easily detectable impact damage. The responses in Figure 2.1(b) and Figure 2.1(c) can cause non-visible impact damage, i.e. barely visible impact damage (BVID), which is
the focus of this study. The responses in Figure 2.1(b) and Figure 2.1(c) are also designated as wave-controlled and boundary-controlled impacts, respectively.

For boundary-controlled impact, the entire structure is deformed during the impact with the contact force and deformation in phase, as shown in Figure 2.2(a). However, for wave-controlled impact, the deformation is localized to the region around the impact point with the contact force and deformation out-of-phase, as shown in Figure 2.2(b). Wave-controlled impact will be the focus of Chapter III and Chapter IV.

![Figure 2.1 Classification of Different Impacts (Olsson, 2000)](image)

(a) Boundary-controlled                                     (b) Wave-controlled

![Figure 2.2 Comparisons between Different Impact Responses (Olsson, 2003)](image)

(a) Boundary-controlled                                     (b) Wave-controlled

2.1 Derivation of the Governing Equations

The formulation of the governing equations is based on the original work done by Olsson (1992). A basic assumption is that the time involved is short that the bending
waves do not reflect back from the boundaries of the plate, i.e. in the “early state” of the impact phenomenon. It has been analytically shown, as may be intuitively expected, that the boundary conditions have no influence on the dynamic response of the structure during the “early state” (Mittal, 1987). In particular, if the impact lasts for a period much less than the time taken for the return of the fastest flexure waves to the point of impact, then the structure can be assumed infinite. When shear deformation is neglected, the response under small mass impact can be obtained from the structural model as shown in Figure 2.3.

The analysis of the structural model is based on Kirchhoff’s plate theory (classical plate theory) for specially orthotropic composite plates. For a plate with zero damping, the response in a point \((x, y)\) to a point load in \((x_0, y_0)\) is represented as

\[
W_p(x, y, t) = \int_0^t F(\tau) \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{W_{mn}(x, y)W_{mn}(x_0, y_0)}{m_p \omega_{mn}} \sin[\omega_{mn}(t - \tau)]d\tau \tag{2.1}
\]

where \(W_{mn}\) is the normalized eigenfunctions of the plate, \(m_p\) is the mass of the plate per unit weight and \(\omega_{mn}\) is the natural frequencies. For a specially orthotropic composite plate, \(\omega_{mn}\) is denoted as

\[
\omega_{mn} = \frac{\pi^2}{\sqrt{m_p}} \sqrt{D_{11}\left(\frac{m}{a}\right)^4 + 2(D_{12} + 2D_{66})\left(\frac{m}{a}\right)^2\left(\frac{n}{b}\right)^2 + D_{22}\left(\frac{n}{b}\right)^4} \tag{2.2}
\]

where \(D_y\) are called the bending stiffnesses which can be calculated from laminate theory. \(a\) and \(b\) are the effective side length of plate affected by the flexure wave, as shown in the following sections.

In an orthotropic composite plate, the flexure wave propagates elliptically and the center of the wave is located in the impact point, as shown in Figure 2.4(a). Therefore,
the area affected by impact can be approximated with a simply supported rectangular plate with side lengths $a$ and $b$, as shown in Figure 2.4(b). The velocity of plane flexure wave in the direction $\theta$ within the thin plates is given by Skudrzuk (1968)

$$C_F(\theta) = \left( \frac{D(\theta)}{m} \right)^{\frac{1}{4}} \sqrt{\omega}$$  \hspace{1cm} (2.3)

where $m$ and $\omega$ are the effective mass and fundamental frequency of the plate. The ratio $a/b$ is equal to the ratio of wavelengths which are the inverse of wave numbers. For an orthotropic composite laminated plate, the ratio $a/b$ can be obtained from Eq. (2.3) as

$$a/b = C_{Fx}/C_{Fy} = \left( \frac{D_{11}}{D_{22}} \right)^{\frac{1}{4}}$$  \hspace{1cm} (2.4)

The normalized eigenfunctions of the plate are

$$W_{mn} = \frac{2}{ab} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$  \hspace{1cm} (2.5)

Figure 2.3 Structural Model for Small Mass Impact (Olsson, 2003)
After some algebraic manipulations, the center deflection of the plate at the impact point \( (0, 0) \) can be expressed as

\[
w_p(0, 0, t) = \frac{1}{8 \sqrt{m_p D^*}} \int_0^t F(\tau) d\tau
\]  

(2.6)

where \( m_p \) and \( D^* \) is the effective mass and stiffness of the plate. A sufficient approximation of \( D^* \) is defined as (Olsson, 2003):

\[
D^* = \sqrt{D_{11} D_{22} (A+1)/2} \\
A = (D_{12} + 2D_{66})/\sqrt{D_{11} D_{22}}
\]  

(2.7)

By neglecting the vibrations of the impactor, the displacement of the impactor and the initial conditions can be expressed as

\[
w_i(t) = V_0 t - \frac{1}{m_i} \int_0^t \int_0^\tau F(\xi) d\xi d\tau \\
w_i(0) = 0, \dot{w}_i(0) = V_0
\]  

(2.8)

where \( m_i \) and \( V_0 \) are mass and the initial velocity of the impactor, respectively.

As shown in Figure 2.5, indentation is defined as the difference between the displacement of the impactor and the plate and is given by
\[ \alpha = w_i - w_p \] \hfill (2.9)

Differentiating the above equation twice with respect to time, the indentation is governed by the differential equation

\[ \frac{d^2 \alpha}{dt^2} + \frac{1}{8 \sqrt{m_p D^6}} \frac{dF}{dt} + \frac{F}{m_i} = 0 \] \hfill (2.10)

\[ \alpha(0) = 0, \dot{\alpha}(0) = V_0 \]

Figure 2.5 Indentation of Composite Laminate (Choi and Lim, 2004)

2.2 Hertzian Contact Law

The static indentation for the contact of a spherical impactor and fiber-reinforced composites has been shown to follow a Hertzian-type contact law (Willis, 1966; Cairns and Lagace, 1987). In the original study done by Olsson (1992), the Hertzian contact law was utilized to describe the occurring contact phenomenon. According to the Hertzian theory developed for static loading on an isotropic linear elastic half-space, the relationship between the contact force and indentation can be described as

\[ F(\alpha) = k_h \alpha^{3/2} \] \hfill (2.11)
where $k_h$ is the contact stiffness and can be expressed as

$$k_h = \frac{4}{3} Q_\alpha \sqrt{R}$$

(2.12)

with

$$1/Q_\alpha = 1/Q_{zi} + 1/Q_{zp}$$

(2.13a)

$$Q_{zk} = E_{zk}/(1-\nu_{rzk}\nu_{zrk}), k = i, p$$

(2.13b)

In Eq. (2.12) and Eq. (2.13), $R$ is the impactor nose radius, and $E_{zk}$ is the modulus in the thickness direction of body $k$, and $\nu_{rzk}$ and $\nu_{zrk}$ are the different Poisson’s ratios for body $i$ (impactor) and $p$ (plate). When the impactor is much stiffer than the plate, Eq. (2.13a) simplifies to

$$1/Q_\alpha = 1/Q_{zp}$$

(2.14)

As shown recently by Swanson (2005), Hertzian contact law for impact on isotropic half-space can be extended to impact on transversely isotropic plate with finite thickness. The extension can be obtained if the isotropic modulus is replaced by a combination of transversely isotropic properties. Thus, for normal contact on transversely isotropic plates with finite thickness, the relationship between contact load and the indentation can be described as

$$F(\alpha) = \beta k_h \alpha^{3/2}$$

(2.15)

where $k_h$ is the contact stiffness defined in Eq. (2.11), except that the effective modulus $Q_{zp}$ is replaced for transversely isotropic materials.

In Eq. (2.15), $\beta$ is an empirical constant that accounts for contact force reduction in a plate with finite thickness, relative to half-space contact loading of orthotropic materials.
The effect of finite thickness of the plate on the indentation stiffness was studied numerically by Suemasu (1994). It was shown that the contact stiffness is smaller in finite thickness plates than the stiffness predicted from contact loading on a half-space. It was shown by Turner (1979) that the effective modulus for transversely isotropic normal contact can be expressed as

$$Q_{zp} = \frac{2}{\alpha_3}$$  \hspace{1cm} (2.16)

with

$$\alpha_1 = \sqrt{\frac{E_s / E_z - \nu_{xz}^2}{1 - \nu_{xy}^2}}$$  \hspace{1cm} (2.17a)

$$\alpha_2 = \frac{1 + \left(\frac{E_s}{2G_{xz}} - 1\right) - \nu_{xz}(1 + \nu_{xy})}{1 - \nu_{xy}^2}$$  \hspace{1cm} (2.17b)

$$\alpha_3 = \sqrt{\frac{\alpha_1 + \alpha_2}{2}} \left(\frac{1 - \nu_{xy}}{G_{xy}}\right)$$  \hspace{1cm} (2.17c)

In Eqs. 2.17, the equivalent orthotropic properties $E_s$, $E_z$, $G_{xy}$, $G_{xz}$, $\nu_{xy}$ and $\nu_{xz}$ can be determined from the stiffness or compliance matrix of a laminate (Tsai and Hahn, 1980).

Inserting Eq. (2.15) into Eq. (2.10) gives

$$\frac{d^2 \alpha}{dt^2} + \frac{1}{8m_p D^2} \cdot \frac{3}{2} \beta k_h \cdot \alpha^{1/2} \cdot \frac{d\alpha}{dt} + \frac{k_h}{m_i} \cdot \alpha^{3/2} = 0$$  \hspace{1cm} (2.18)

$$\alpha(0) = 0, \dot{\alpha}(0) = V_0$$

Introducing the non-dimensional variables

$$\overline{\alpha} = \frac{\alpha}{V_0 T_e}, \overline{T} = \frac{t}{T_e}$$  \hspace{1cm} (2.19)
Eq. (2.18) becomes
\[
\frac{d^2 \bar{\alpha}}{dt^2} + \lambda_e \frac{3}{2} \frac{d\bar{\alpha}}{dt} + \bar{\alpha}^{3/2} = 0
\]  
(2.20)

where the coefficients
\[
\lambda_e = \left( \frac{\beta k_h}{8 \sqrt{m_p}} \right)^{2/5} \frac{V_0^{1/5} \cdot m_i^{3/5}}{\beta V_0} \quad \text{and} \quad T_e = \left( \frac{m_i}{\beta k_h \sqrt{V_0}} \right)^{2/5}
\]
(2.21)

Inserting Eqs. (2.19) and (2.21) into Eq. (2.15), the physical contact force can be obtained as
\[
F = \beta k_i^{2/5} \left( MV_0^2 \right)^{3/5} \cdot \bar{\delta}^{3/2}
\]
(2.22)

The momentum of the impact is
\[
I = \int_0^\tau F(\tau) d\tau = MV_0 \int_0^\tau \bar{\delta}^{3/2} d\tau
\]
(2.23)

Inserting Eq. (2.23) into Eq. (2.6), the physical central displacement of the plate can be obtained as
\[
w_i(0,0,t) = \frac{MV_0^{3/2}}{8 \sqrt{m_p D^*}} \int_0^\tau \bar{\delta}^{3/2} d\tau
\]
(2.24)

Eq. (2.20) is a second-order nonlinear ordinary differential equation. The \( \bar{\alpha}(\bar{\tau}) \) is solved numerically for different values of \( \lambda_e \). The dimensionless indentation, contact force histories, as well as velocity, have been plotted for different values of \( \lambda_e \), as shown in Figure 2.6, Figure 2.7 and Figure 2.8.
Figure 2.6 Nondimensional Indentation Histories as a Function of $\lambda_e$

Figure 2.7 Nondimensional Contact Force Histories as a Function of $\lambda_e$
It is observed that for an infinite plate with the Hertzian contact law, the impact is governed by a single parameter $\lambda_e$. The $\lambda_e$ combines the effect of the contact stiffness, impact velocity, the mechanical properties and geometrical properties of both the plate and the projectile. In Figure 2.6, the highest contact force is obtained for $\lambda_e = 0$, where the plate is very rigid and the problem of the impact can be considered as impact on a half-space. As $\lambda_e$ increases, the contact force history becomes more asymmetrical. Also the contact duration increases as the deformation of the plate becomes more significant. As shown in Figure 2.8, the rebound velocity of the impactor is equal to the initial velocity for $\lambda_e = 0$. This means that the plate is rigid and does not absorb any energy during the impact process. However, when $\lambda_e > 0$, the deformation of the plate absorbs more energy and the behavior of the impactor is similar to that after an inelastic impact.

Figure 2.8 Nondimensional Velocity Histories as a Function of $\lambda_e$
From Figure 2.6 and Figure 2.7, the non-dimensional impact time is always $\pi$ for the case $\lambda_c = 0$ and can be explained by the following: while $\lambda_c = 0$, the non-dimensional governing equation Eq. (2.20) can be written as

$$\frac{d^2 \alpha}{dt^2} + \alpha^{3/2} = 0$$

$$\alpha(0) = 0, \dot{\alpha}(0) = 1$$

(2.25)

where Eq. (2.25) represents a non-damping dynamic system with non-dimensional frequency $\bar{\omega} = 1$. Therefore, the contact duration for the case $\lambda_c = 0$ is half of the period of the dynamic system and equal to $\pi$. When $\lambda_c \neq 0$, Eq. (2.20) can be seen as a dynamic system with viscous damping and the frequency will be reduced as

$$\bar{\omega}_d = \bar{\omega}\sqrt{1 - \xi^2}$$

(2.26)

where $\xi$ is the damping ratio. With the above equations, the contact duration increases with the increase of the single parameter $\lambda_c$.

2.3 Generalized Contact Law

Considering the restrictions of Eq. (2.11), efforts have been made to replace the Hertzian contact law by a relation more applicable for contact laws fitting from experimental data. Figure 2.9 shows the results from some static indentation tests on Carbon/epoxy laminates. The least-square approach gives the best curve fitting with contact law exponents that are different from 3/2 in Hertzian contact law. It can also be seen that the contact law’s stiffness and exponent change with the thickness of the laminate. A generalized contact law is needed for wide range applications of the original small-mass impact model.
It is shown that the load-indentation relation between a solid and an elastic half-space can be expressed by the Meyer power law (Hill et al., 1989)

\[ F = k_h \alpha^q \]  

(2.27)

where \( k_h \) is the contact stiffness, \( \alpha \) is the indentation and the exponent \( q \) is bound by \( 1 < q < \frac{3}{2} \). The applicability of Eq. (2.27) was investigated by Anderson and Nilsson (1995). The variables \( k_h \) and \( q \) are obtained by fitting experimental data from static indentation. Taking into account the effect of finite thickness of the plate, Eq. (2.27) becomes

\[ F = \beta k_h \alpha^q \]  

(2.28)

Substituting Eq. (2.28) into Eq. (2.10), using the same nondimensionalization procedure, the nondimensional indentation is governed by the nonlinear differential equation as
\[
\frac{d^2 \tilde{\alpha}}{dt^2} + \lambda_g \cdot q \cdot \tilde{\alpha}^{q-1} \cdot \frac{d\tilde{\alpha}}{dt} + \tilde{\alpha}^q = 0 \tag{2.29}
\]

where

\[
\tilde{\alpha}(0) = 0, \tilde{\alpha}(0) = 1
\]

and

\[
\lambda_g = (\beta k_i V_0^{q-1} m_i^q)^{\frac{1}{q+1}} \left(8 \sqrt{m_i D^*} \right) \quad \text{and} \quad T_g = \left(\frac{m_i}{\beta k_i V_0^{q-1}} \right)^{\frac{1}{q+1}} \tag{2.30}
\]

For Hertzian contact, the results obtained by Olsson (1992) can be recovered from Eq. (2.29). An approximated solution to Eq. (2.29) using the method of asymptotic expansion is proposed here to account for contact laws with different exponents \(q\). Because of the exponent’s range we assume

\[
q = 1 + \varepsilon \tag{2.32}
\]

where \(\varepsilon << 1\). Substituting Eq. (2.32) into Eq. (2.29), it becomes

\[
\frac{d^2 \tilde{\alpha}}{dt^2} + \lambda_g \cdot (1 + \varepsilon) \cdot \tilde{\alpha}^{\varepsilon} \cdot \frac{d\tilde{\alpha}}{dt} + \tilde{\alpha}^{1+\varepsilon} = 0 \tag{2.33}
\]

\[
\tilde{\alpha}(0) = 0, \tilde{\alpha}(0) = 1
\]

Performing power-series expansion for all dependent variables and a Taylor expansion of the \(\tilde{\alpha}^{\varepsilon}\) and \(\tilde{\alpha}^{1+\varepsilon}\) components in Eq. (2.33), reorganizing the terms, it can be shown that

\[
\tilde{\alpha} = \alpha_0 + \varepsilon \alpha_1 + \varepsilon^2 \alpha_2 + \cdots \tag{2.34}
\]

\[
\tilde{\alpha}^{\varepsilon} = 1 + \varepsilon \ln(\alpha_0) + \varepsilon^2 \left(\frac{\alpha_1}{\alpha_0} + \frac{(\ln \alpha_0)^2}{2!}\right) + \cdots \tag{2.35}
\]
\[ \tilde{a}^{+e} = \alpha_0 + \epsilon (\alpha_1 + \alpha_0 \ln \alpha_0) + \epsilon^2 \left( \alpha_2 + \alpha_1 \ln \alpha_0 + \frac{\alpha_0 \ln^2 \alpha_0}{2!} \right) + \cdots \] (2.36)

Substituting Eq. (2.35) and Eq. (2.36) into Eq. (2.33) and examining the governing equation by the asymptotic sequence, a uniformly valid asymptotic expansion can be obtained

\[
O(1): \begin{cases} 
\frac{d^2 \alpha_0}{dt^2} + \lambda_g \frac{d\alpha_0}{dt} + \alpha_0 = 0 \\
\alpha_0(0) = 0, \dot{\alpha}_0(0) = 1
\end{cases} \tag{2.37}
\]

\[
O(\epsilon): \begin{cases} 
\frac{d^2 \alpha_1}{dt^2} + \lambda_g \frac{d\alpha_1}{dt} + \alpha_1 = \frac{d^2 \alpha_0}{dt^2} \ln \alpha_0 - \lambda_g \frac{d\alpha_0}{dt} \\
\alpha_0(0) = 0, \dot{\alpha}_0(0) = 0
\end{cases} \tag{2.38}
\]

In Eq. (2.37) and Eq. (2.38), the second order approximation from the expansion is neglected. Both Eq. (2.37) and Eq. (2.38) are linear second-order ordinary differential equations. Eq. (2.37) is the equation of motion for single degree of freedom system with viscous damping. By introducing variable \( \eta = \frac{\lambda_g}{2} \) and distinguishing different range of values of \( \eta \), the closed-form solution of homogeneous Eq. (2.37) can be obtained (Abrate, 1998)

Under-damping: \( \eta < 1 \) \[ \alpha_0 = e^{-\eta T} \sin(\sqrt{1-\eta^2} \cdot T) / \sqrt{1-\eta^2} \] (2.39a)

Critical-damping: \( \eta = 1 \) \[ \alpha_0 = T e^{-T} \] (2.39b)

Over-damping: \( \eta > 1 \) \[ \alpha_0 = e^{-\eta T} \sinh(\sqrt{\eta^2-1} \cdot T) / \sqrt{\eta^2-1} \] (2.39c)

Eq. (2.38) is a non-homogeneous, second-order, linear ordinary differential equation. Due to the \( \ln \alpha_0 \) term on the right-hand side of equation a closed-form solution cannot be obtained. The solutions of \( \alpha_0 \) and \( \alpha_1 \) for different values of \( \eta \) and \( \lambda_g \) are shown in
Figure 2.10 and Figure 2.11, respectively. The leading order term $\alpha_0$ has the characteristics of solution and $\alpha_1$ is the first-order correction. As $\eta$ and $\lambda$ increase, both $\alpha_0$ and $\alpha_1$ become more asymmetrical with respect to non-dimensional impact time.

The approximate solution of Eq. (2.30) using the defined asymptotic expansion is written as

$$\bar{\alpha} = \alpha_0 + \varepsilon \alpha_1$$

(2.40)

Eq. (2.40) is the closed-form solution of the homogeneous equation along with the first correction term. The generality of Eq. (2.40) is that, for any given nondimensional impact parameter, the solution of nondimensional indentation and contact force history can be obtained for any contact law exponent $g$ with one pair of solution for $\alpha_0$ and $\alpha_1$. When $\varepsilon = 0.5$, the solution for purely elastic contact law in Olsson’s paper can be recovered. With a purely elastic contact law ($\varepsilon = 0.5$), the comparison between the results from numerical solution and asymptotic expansion for different values of $\lambda_0$ is shown in Figure 2.12 and Figure 2.13. The proposed method of asymptotic expansion can capture the characteristics and values of the contact force reasonably well.
Figure 2.10 $\alpha_0$ Histories as a Function of $\eta$ and $\lambda_g$

Figure 2.11 $\alpha_1$ Histories as a Function of $\eta$ and $\lambda_g$
Figure 2.12 Comparison of Numerical and Asymptotic Results (Over-damping)

Figure 2.13 Comparison of Numerical and Asymptotic Results (Under-damping & Critical-damping)
Figure 2.14 Effect of Contact Law Exponent on Indentation History ($\lambda_g = 4.0$)

Figure 2.15 Effect of Contact Law Exponent on Indentation History ($\lambda_g = 0.0$)
Figure 2.14 and Figure 2.15 show the effect of the contact law exponent $q$ on the time history of local indentation for the case of $\lambda_g = 4.0$ (over-damping) and $\lambda_g = 0.0$ (critical-damping), respectively. By increasing of the contact law exponent, the indentation increases and thus the contact force time history. For the over-damping case, the time to reach the maximum indentation or contact force increases as the contact law exponent increases. However, for critical-damping case, the time to reach the maximum indentation or contact force almost keeps the same as the contact law exponent increases.

2.4 Conclusions

In this chapter, a semi-analytical solution was proposed to investigate the impact response of infinite plate under generalized contact law. The solution is based on asymptotic expansion of the non-dimensional indentation in the derived nonlinear ordinary differential equation. The nonlinear ordinary differential equation was transformed into two linear ordinary differential equations by neglecting the high-order terms. The $O(1)$ ordinary differential equation can be solved in closed-form. Due to the $\ln \alpha$ term, the $O(\varepsilon)$ ordinary differential equation cannot be solved in closed-form and is solved numerically. The overall solution of the problem is obtained approximately based on the principle of superposition. The proposed semi-analytical solution is applied to the generalized contact law with a changing contact law exponent, and is found by fitting experimental data from static indentation test.
CHAPTER III

ELASTO-PLASTIC IMPACT RESPONSE OF INFINITE LAMINATES

In Chapter II the solution of small mass impact on an infinite plate was derived semi-analytically based on elastic Hertzian contact law, and a generalized contact law. The Hertzian contact law was obtained from the elasto-static analysis of contact between a spherical impactor and an elastic half-space, where permanent deformations due to damage were not accounted for. During the initial stages of impact loading, Hertzian-type contact laws are adequate for most cases. However, it has been shown that even at low contact loads, the permanent deformations that cause damages around the contact zone are present. In this Chapter, an elasto-plastic contact law considering permanent deformation was substituted into the governing equations derived for small-mass impact on an infinite plate. The effect of finite thickness of the plate as well as the transversely isotropic material properties of the composite laminates is also included in the formulation.

Closed-form solutions of the impact responses during small mass impact were obtained using a linearized version of the elasto-plastic contact law. It is indicated that the impact responses are governed by a single parameter which combines the effect of contact
stiffness, impact velocity, the mechanical and geometrical properties of the laminate and projectile as well as the permanent indentation.

3.1 Nonlinear Elasto-plastic Contact Law

Swanson and Rezaee (1990) investigated the effect of damage on lateral contact loads of a laminate that was supported by a rigid flatten so that the contact loads were introduced without bending effects. A 50.8mm×250mm×2.4mm [0/±68/0]_s Im7/55A Carbon/Epoxy flat laminate was statically indented by a steel spherical indentor with diameter of 15.88mm. The typical force-deflection response from the experiment is shown in Figure 3.1. A new feature in this data is the pronounced softening at very high loads and contact deflection. Presumably this softening is associated with significant damage of the laminate, as seen from the depth of penetration and size of the residual ‘crater’ after unloading. The results of the laminate strength under subsequent in-plane tensile loading are shown in Figure 3.2. It can be seen that a significant of loss of strength is related to high contact loadings. In this Figure, the “open hole strength” is from an experiment performed in same-size laminate with a hole of 15.88mm in diameter. The specimen with the hole corresponds to a lower limit of the strength retention. It was shown that Hertzian contact law can adequately model the loading phase till some large loads (36 KN). After this, the contact laws deviate significantly from the Hertzian contact law.

Cairns (1991) developed a simple elasto-plastic contact law for composites to account for the effect of permanent deformation. The material is assumed to behave elastically until a critical indentation is reached. As the loading increase, the contact area is divided
into a plastic zone and an elastic zone and a new contact law is obtained. The permanent indentation as a result of contact loading was compared with measured permanent indentation with good results. However, unloading was not considered in the simple elasto-plastic contact law.

![Figure 3.1 Force-Indentation Response in Contact Loading of Laminate](image1)

**Figure 3.1** Force-Indentation Response in Contact Loading of Laminate

![Figure 3.2 Reduction of Laminate Tensile Strength after Contact Loading](image2)

**Figure 3.2** Reduction of Laminate Tensile Strength after Contact Loading

In this study, the contact law accounting for permanent deformation and unloading proposed by Yigit and Christoforou (1994) is used to analyze the small mass impact response of an infinite laminate. The contact law assumes the contact consists of three phases. The contact is assumed to be Hertzian for the first phase. In the second phase, the
elastic-plastic behavior is assumed where the “yielding” point is exceeded. In case of fiber composites, “yielding” means a combination of different damage modes such as matrix crack and fiber tensile failure or buckling. The third phase is assumed to be elastic Hertzian behavior again. The quasi-static contact law for the three phases is given as following:

Phase I: Elastic Loading

\[ F(\alpha) = K_h \alpha^{3/2} \quad 0 \leq \alpha \leq \alpha_{cr} \]  

(3.1)

Phase II: Elasto-Plastic Loading

\[ F(\alpha) = K_y (\alpha - \alpha_{cr}) + K_h \alpha_{cr}^{3/2} \quad \alpha_{cr} \leq \alpha \leq \alpha_m \]  

(3.2)

Phase III: Elastic Unloading and Reloading

\[ F(\alpha) = K_h (\alpha_{cr}^{3/2} - \alpha_m^{3/2} + \alpha_{cr}^{3/2}) + K_y (\alpha_m - \alpha_{cr}) \]  

(3.3)

where \( \alpha_{cr} \) is the critical indentation when material yields and \( \alpha_m \) is the maximum indentation when the unloading or restitution begins. \( K_h \) is the contact stiffness accounting for the finite thickness of the plate as well as the transversely isotropic properties of the laminate, as shown in Eqs. (2.15) to (2.17).

The critical indentation \( \alpha_{cr} \) can be obtained from the contact stress distribution based on the maximum shear failure criterion. According to Hertzian contact theory, the contact stress distribution within contact radius can be represented as

\[ p(r) = p_0 \sqrt{1 - \frac{r^2}{c^2}} \]  

(3.4)

where \( p_0 \) is the maximum normal stress in the contact zone and given by
\[ p_0 = \frac{3F}{2\pi c^2} \]  

(3.5)

In Eq. (3.5), \( c \) is the contact radius. For a spherical indenter

\[ c = \sqrt{Ra} \]  

(3.6)

From plate theory, the maximum shear stress is given by Olsson (2006)

\[ \tau_{\text{max}} = \frac{3F}{4\pi hr_i} \]  

(3.7a)

\[ r_i = (3/4)^{1/4} c \]  

(3.7b)

The critical indentation can be obtained by combining Eqs. (3.4) to (3.7) and is given as

\[ \alpha_{cr} = 0.68 \cdot (2S_u)^2 \pi^2 R \]  

\[ Q_\alpha^2 \]  

(3.8)

where \( S_u \) is the interlaminar shear strength of laminated composites, \( R \) is the radius of the impactor and \( Q_\alpha \) is the effective modulus of the system.

The slope of the elasto-plastic indentation curve at \( \alpha = \alpha_{cr} \) is give as

\[ K_y = \frac{3}{2} K_b \sqrt{\alpha_{cr}} \]  

(3.9)

As shown in Chapter II, the governing equation for small mass impact on an infinite plate can be written as

\[ \frac{d^2 \alpha}{dt^2} + \frac{1}{8\sqrt{m_{ip}D^*}} \frac{dF}{dt} + \frac{F}{m_i} = 0 \]  

(3.10)

\( \alpha(0) = 0, \dot{\alpha}(0) = V_0 \)

Substituting Eqs. (3.1)-(3.3) into Eq. (3.10), the governing equation of the indentation considering permanent indentation and damage effects can be obtained. For the elastic loading phase the governing equation is
\[
\frac{d^2\alpha}{dt^2} + \frac{1}{8\sqrt{m_P D^*}} \cdot \frac{3}{2} k_{h} \cdot \alpha^{1/2} \cdot \frac{d\alpha}{dt} + \frac{k_{h}}{m_{i}} \cdot \alpha^{3/2} = 0
\]

\[\alpha(0) = 0, \dot{\alpha}(0) = V_{0}, 0 \leq \alpha \leq \alpha_{cr}\]  (3.11)

For the elasto-plastic loading phase the governing equation is

\[
\frac{d^2\alpha}{dt^2} + \frac{1}{8\sqrt{m_P D^*}} \cdot K_{y} \cdot \frac{d\alpha}{dt} + \frac{K_{cr}}{m_{i}} \cdot (\alpha - \alpha_{cr}) + \frac{k_{h}}{m_{i}} \cdot \alpha^{3/2} = 0
\]

\[\alpha(t_{cr}) = \alpha_{cr}, \dot{\alpha}(t_{cr}) = V_{cr}, \dot{\alpha}(t_{m}) = 0, \alpha_{cr} \leq \alpha \leq \alpha_{m}\]  (3.12)

For the elastic unloading phase the governing equation is

\[
\frac{d^2\alpha}{dt^2} + \frac{1}{8\sqrt{m_P D^*}} \cdot K_{h} \cdot \alpha^{1/2} \cdot \frac{d\alpha}{dt} + \frac{K_{h}}{m_{i}} \cdot (\alpha^{3/2} - \alpha_{m}^{3/2} + \alpha_{cr}^{3/2}) + \frac{K_{y}}{m_{i}} \cdot (\alpha - \alpha_{cr}) = 0
\]

\[\alpha(t_{m}) = \alpha_{m}, \dot{\alpha}(t_{m}) = 0, \alpha_{f} \leq \alpha \leq \alpha_{m}\]  (3.13)

In Eqs. (3.11) and (3.12), \(t_{cr}\) and \(t_{m}\) are the time to reach critical indentation and maximum indentation, respectively. \(\alpha_{cr}\), \(\alpha_{m}\) and \(\alpha_{f}\) are the critical indentation, maximum indentation and permanent indentation, respectively. The permanent indentation \(\alpha_{f}\) is represented as

\[\alpha_{f} = \alpha_{m} - \alpha_{cr}\]  (3.14)

For this study, a FORTRAN program “PlateImpact_Plastic” has been written. The boundary-value problem for the three phases is solved using IMSL library which is based on Runge-Kutta-Verner fifth-order and sixth-order method. The detailed code is shown in the Appendix.

3.2 Linearized Elasto-plastic Contact Law

As shown in Eqs. (3.11) to (3.13), due to the nonlinearity of the elastic and elasto-plastic contact law, an analytical solution is typically not possible. One method of contact
stiffness linearization is to obtain an equivalent linear stiffness for a single degree-of-freedom lumped model that will result in the same maximum contact force (Bucinell and Nuismer et al., 1991). Since the linearized contact stiffness only depends on the maximum contact force, this method may not be adequate for situations where the details of the contact law and permanent deformations are important.

Another method for the situation where permanent deformation is present is to linearize each phase of the contact law separately with very good results (Yigit and Christoforou, 1995). However, for most composite materials, damage occurs early in the loading phase and most of the local response is dominated by the second phase of the contact law (Yigit and Christoforou, 1994). Therefore, it is worthwhile to consider only the linearization of the elasto-plastic phase that is already linear. The linearized contact law can be written as

\[ F(\alpha) = K_\alpha \alpha \]  
(3.15)

Substituting the linearized contact law into the governing equation for indentation Eq. (3.10), the governing equation becomes

\[ \frac{d^2 \alpha}{dt^2} + \frac{1}{8 \sqrt{m_p D'}} \cdot K_y \cdot \frac{d \alpha}{dt} + \frac{K_y}{m_i} \cdot \alpha = 0 \]
(3.16)

\[ \alpha(0) = 0, \dot{\alpha}(0) = V_0 \]

Introducing the same nondimensional variables

\[ \bar{\alpha} = \frac{\alpha}{V_0 T_p}, \bar{T} = \frac{t}{T_p} \]
(3.17)

Eq. (3.16) can be written as
\[
\frac{d^2 \tilde{\alpha}}{dt^2} + \lambda_p \cdot \frac{d\tilde{\alpha}}{dt} + \tilde{\alpha} = 0
\]  \hspace{1cm} (3.18)

\[
\tilde{\alpha}(0) = 0, \tilde{\alpha}(0) = 1
\]

with the coefficients

\[
\lambda_p = \frac{1}{8} \sqrt{\frac{K_m}{m_i y}} \text{ and } T_p = \sqrt{\frac{m_i}{K_y}}
\]  \hspace{1cm} (3.19)

Eq. (3.18) is the equation of motion for single degree of freedom system with viscous damping (Abrate, 1998). By introducing variable \( \eta = \lambda_p / 2 \) and distinguishing different range of values of \( \eta \), the closed-form solution of Eq. (3.16) can be obtained

Under-damping: \( \eta < 1 \)

\[
\tilde{\alpha} = e^{-\eta \tau} \sin(\sqrt{1-\eta^2} \cdot \tau) / \sqrt{1-\eta^2}
\]  \hspace{1cm} (3.20a)

Critical-damping: \( \eta = 1 \)

\[
\tilde{\alpha} = \eta e^{-\eta \tau}
\]  \hspace{1cm} (3.20b)

Over-damping: \( \eta > 1 \)

\[
\tilde{\alpha} = e^{-\eta \tau} \sinh(\sqrt{\eta^2-1} \cdot \tau) / \sqrt{\eta^2-1}
\]  \hspace{1cm} (3.20c)

The physical indentation time history can be obtained by multiplying dimensionless quantity with the physical constants in Eq. (3.17)

\[
\alpha(t) = V_0 \cdot T_p \cdot \tilde{\alpha}(t / T_p)
\]  \hspace{1cm} (3.21)

The physical contact force time history can be obtained as

\[
\eta < 1 \quad F(t) = K_y V_0 T_p \cdot e^{-\eta \tau / T_p} \cdot \sin(\sqrt{1-\eta^2} \cdot t / T_p) / \sqrt{1-\eta^2}
\]  \hspace{1cm} (3.22a)

\[
\eta = 1: \quad F(t) = K_y V_0 t \cdot e^{-t / T_p}
\]  \hspace{1cm} (3.22b)

\[
\eta > 1: \quad F(t) = K_y V_0 T_p \cdot e^{-\eta \tau / T_p} \sinh(\sqrt{\eta^2-1} \cdot t / T_p) / \sqrt{\eta^2-1}
\]  \hspace{1cm} (3.22c)

It has been shown in Chapter II that the plate center deflection at impact point \((0, 0)\) for small mass impact on an infinite plate can be obtained as
From Eq. (3.23), the physical central displacement of the plate can be written as

$$w_p(0,0,t) = \frac{1}{8\sqrt{m_p D^*}} \int_0^t F(\tau) d\tau$$  \hspace{1cm} (3.23)

From Eqs. (3.20a) to (3.20c), the central displacement of the plate for under-damping case is

$$w_p(0,0,t) = \frac{mV_0}{8\sqrt{m_p D^*}} \left[ 1 - e^{-\eta t/T_p} \cdot \left( \cos(\sqrt{1-\eta^2} \cdot t/T_p) + \frac{\eta \sin(\sqrt{1-\eta^2} \cdot t/T_p)}{\sqrt{1-\eta^2}} \right) \right]$$  \hspace{1cm} (3.25a)

The center displacement of the plate for critical-damping case is

$$w_p(0,0,t) = \frac{mV_0}{8\sqrt{m_p D^*}} \left[ 1 - e^{-i t/T_p} \cdot (1 + t/T_p) \right]$$  \hspace{1cm} (3.25b)

The center displacement of the plate for over-damping case is

$$w_p(0,0,t) = \frac{mV_0}{8\sqrt{m_p D^*}} \left[ 1 - e^{-\eta t/T_p} \cdot \left( \cosh(\sqrt{\eta^2 - 1} \cdot t/T_p) + \frac{\eta \sinh(\sqrt{\eta^2 - 1} \cdot t/T_p)}{\sqrt{\eta^2 - 1}} \right) \right]$$  \hspace{1cm} (3.25c)

The residual velocity of the impactor can be obtained by following

$$\dot{w}_i = \dot{\alpha} + \dot{w}_p$$  \hspace{1cm} (3.26)

3.3 Validation of the Elasto-plastic Contact Law

The predictions of the contact force history based on elastic Hertzian and elasto-plastic contact law are compared with experimental data when the material has occurred significant damage. In the experiment, a [45/-45/90/0]_4s quasi-isotropic laminate is impacted with a drop test rig with different energies (Davies et. al, 1994). Figure 3.3
shows the ultrasonic C-scan of the increasing delamination patterns for the impacted laminate.

The details of the impact problem are shown in Table 3.1. Because the mass of the impactor is larger than 1/4 of the effective mass of the plate, this is a boundary-controlled impact and the infinite plate theory derived above can not be applied. Instead, the impact responses are obtained based on first-order shear deformation theory considering the shear deformation (Zheng and Binienda, 2007). The comparison of the predictions of the contact force based on elastic and elasto-plastic contact law and corresponding experiment is shown in Figure 3.4. It can be seen that the use of elasto-plastic contact law can capture the impact response more accurately when the laminate experiences significant damage.

Figure 3.3 Increasing Delamination Patterns with Force from C-Scan
Table 3.1 Geometrical and Material Properties of the HTA/6376 Laminate

Plate: \([45/-45/90/0]_4\) \text{HTA/6376 carbon/epoxy composite plate, clamped edges}
Plate Size: 125 mm \(\times\) 75 mm \(\times\) 4 mm
\(E_{11} = 156\text{ GPa}, E_{22} = 9.09\text{ GPa}, G_{12} = 6.96\text{ GPa}, G_{23} = 3.24\text{ GPa}, \nu_{12} = 0.228\)
\(D_{11} = 290.006\text{ Nm}, D_{12} = 114.374\text{ Nm}, m_p = 6.48\text{ kg/m}^2, D_{22} = 318.628\text{ Nm}\)
\(D_{66} = 126.004\text{ Nm}, \alpha = 0.007\)
Impactor: \(R = 6.35\text{ mm}, V_0 = 2.36\text{ m/s}, m_i = 2.25\text{ kg}, \text{Incident Energy} = 6.24\text{ J}\)

![Elastic vs Elasto-plastic force history](image)

Figure 3.4 Contact Force History for Impacted Plate with Damage

3.4 Application Examples: Effect of Different Contact Laws

A symmetric cross-ply \([0/90/0/90/0]_s\) \text{T300/934 graphite/epoxy composite plate} is analyzed to illustrate the effect of different contact laws on the impact responses. The material properties, geometry, and impact conditions are shown in Table 3.2 (Cairns and Lagace, 1989). This is a wave-controlled impact problem therefore the theory derived in Section 3.1 for small mass impact can be applied.
For the first case, the initial velocity of the projectile is 3.0 m/s. The calculated impact parameter for elastic and elasto-plastic contact law is $\lambda_e = 2.07$ and $\lambda_p = 1.71$, respectively. The comparison of the impact force history, contact law and plate deformation for impact with different contact laws is shown in Figure 3.5, Figure 3.6 and Figure 3.7, respectively. The use of elasto-plastic contact law reduces the contact force and increases the contact duration. There is significant amount of permanent deformation during elasto-plastic impact.

In the second case, the initial velocity of the projectile is increased to 30 m/s by keeping all other parameters the same as before. The calculated impact parameter for elastic and elasto-plastic contact law is $\lambda_e = 3.30$ and $\lambda_p = 1.71$, respectively. The comparison of impact force history, contact law, and plate deflection from elastic and elasto-plastic impact is shown in Figure 3.8, Figure 3.9 and Figure 3.10, respectively. It can be seen from Figure 3.8 to Figure 3.10 that the use of the elasto-plastic contact law significantly changes the contact force history and plate center deflection when the velocity of projectile reaches medium velocity. The plate central displacements associated with permanent indentations are significantly less compared to the elastic contact solutions because of the smaller contact force resulting at the contact region.

Figure 3.11 shows the comparison of contact force from elastic Hertzian contact law, elasto-plastic contact law, as well as linearized elasto-plastic contact law. It can be seen that the closed-form solution of the peak impact force is pretty close to the numerical solution from elasto-plastic contact law. The impact duration from closed-form solution also agrees well with the numerical solutions.
Table 3.2 Geometrical and Material Properties of the Composite Plate

| Plate: \([0/90/0/90/0]_s\) T300/934 graphite/epoxy composite plate, simple supported |
|---|---|
| Plate Size: 200 mm × 200 mm |
| \(E_{11} = 137\ \text{GPa},\ E_{22} = 10.6\ \text{GPa},\ G_{12} = 5.4\ \text{GPa},\ G_{23} = 3.5\ \text{GPa},\ \nu_{12} = 0.4\) |
| \(D_{11} = 154.9\ \text{Nm},\ D_{12} = 4.760\ \text{Nm},\ m_p = 4.132\ \text{kg/m}^2,\ D_{22} = 91.4\ \text{Nm}\) |
| \(D_{66} = 8.970\ \text{Nm}\) |
| Projectile 1: \(R = 6.35\ \text{mm},\ V_0 = 3.0\ \text{m/s},\ m_i = 8.30\ \text{g}\) |
| Projectile 2: \(R = 6.35\ \text{mm},\ V_0 = 30\ \text{m/s},\ m_i = 8.30\ \text{g}\) |
| Impact Parameter: \(K_h = 1.033E9\ \text{N/m}^{3/2},\ \lambda_{el} = 2.07,\ \lambda_{p2} = 3.30,\ \lambda_p = 1.71\) |

Figure 3.5 Comparison of Contact Force History (\(\lambda_{el} = 2.07\))
Figure 3.6 Comparison of Contact Law ($\lambda_e = 2.07$)

Figure 3.7 Comparison of Center Deflection of Plate ($\lambda_e = 2.07$)
Figure 3.8 Comparison of Contact Force History ($\lambda_{h}=3.30$)

Figure 3.9 Comparison of Contact Law ($\lambda_{h}=3.3$)
3.5 Parametric Study of Closed-form Solution

Parametric studies on the closed-form solution are carried out for the composite laminate impact problem in Table 3.2. A [0/90/0/90/0]s T300/934 graphite/epoxy simple-
supported composite plate was impacted by a spherical steel ball of 12.7 mm diameter with an impact velocity of 3.0 m/s. Unless otherwise mentioned, the above impact condition is used in the following sections.

3.5.1 Effect of Mass of the Impactor

With constant energy, the effect of the mass of the impactor on contact force, plate central displacement, and the residual velocity of the impactor are shown in Figure 3.12, Figure 3.13 and Figure 3.14, respectively. Smaller mass impactor will induce larger impact force and shorter impact duration. The central displacement of the plate increases with the increase of the impactor mass. From the velocity time history of the impactor, it can be seen that the plate can absorb the kinetic energy of the impactor at a higher speed when impacted by smaller mass.

3.5.2 Effect of Velocity of the Impactor

The effect of velocity of the impactor on contact force, plate central displacement and the velocity time history of the impactor are shown in Figure 3.15, Figure 3.16 and Figure 3.17, respectively. Under constant energy, when the impactor velocity increases, the contact force will increase and the central displacement will decrease. For second and third case, the impactor was bounced off from the plate after first impact. The duration for first contact also decreases with the increase of the impactor velocity.

3.5.3 Effect of Plate Thickness

The thickness of the laminate also plays an important role in the impact responses. The effect of plate thickness on contact force, the plate central displacement and the velocity
time history of the impactor are shown in Figure 3.18, Figure 3.19 and Figure 3.20, respectively. With the same amount of impact energy, the thicker plate has larger contact force and smaller central displacement produced under small mass impact. Similarly to section 3.5.2, the impactor was rebounded from the plate after first impact for the second and third case.

3.5.4 Effect of Interlaminar Shear Strength

Considering the critical indentation and the slope of elasto-plastic contact law are related to the interlaminar shear strength, the effect of interlaminar shear strength on the contact force and plate central displacement are studied. They are shown in Figure 3.21 and Figure 3.22. Accordingly an increase of interlaminar shear strength will introduce larger contact force, while the plate central deflection remains the same.

3.5.5 Effect of the Size of Impactor

The effect of the size of the spherical impactor on the contact force and plate central displacement was also investigated, as shown in Figure 3.23 and Figure 3.24, respectively. The increase of the radius of the impactor will produce larger contact force. The plate central displacement will remain the same. The trend is similar to the effect of interlaminar shear strength. This can be explained by Eq. (2.12), Eq. (3.8) and Eq. (3.9).

3.6 Conclusions

In this Chapter, a linearized elasto-plastic contact law including permanent indentation and damage effects is used to derive the closed-form approximation of the small mass wave-controlled impact response of composite laminate. It was shown that consideration
of permanent indentation within impact analysis can significantly change the impact responses when the velocity of the impactor reaches medium velocity. Based on the closed-form solution of the impact responses, the parametric studies were conducted in order to investigate effect of the impactor characteristics, the laminate thickness and interlaminar shear strength on the impact responses of a composite laminate. The closed-form solutions of the impact responses are helpful in providing guidance to numerical simulation of impact on composite structures.

Figure 3.12 Effect of Impactor Mass on Contact Force
Figure 3.13 Effect of Impactor Mass on Plate Central Displacement

Figure 3.14 Effect of Impactor Mass on Velocity Time History
Figure 3.15 Effect of Impactor Velocity on Contact Force

Figure 3.16 Effect of Impactor Velocity on Plate Central Displacement
Figure 3.17 Effect of Impactor Velocity on Velocity Time History

Figure 3.18 Effect of Plate Thickness on Contact Force
Figure 3.19 Effect of Plate Thickness on Plate Central Displacement

Figure 3.20 Effect of Plate Thickness on Velocity Time History
Figure 3.21 Effect of Interlaminar Shear Strength on Contact Force

Figure 3.22 Effect of Interlaminar Shear Strength on Plate Central Displacement
Figure 3.23 Effect of Impactor Radius on Contact Force

Figure 3.24 Effect of Impactor Radius on Plate Central Displacement
CHAPTER IV

PREDICTION OF THRESHOLD VELOCITY FOR DELAMINATION ONSET

Experimental and numerical results indicate that threshold forces can be used as a qualitative indicator of delamination onset under small mass impacts (Beks, 1996). In Chapter III, the closed-form approximations of wave-controlled impact responses are derived based on a linearized elasto-plastic contact law. Based on the characteristics of the contact force time history, the peak contact force during impact is obtained from the closed-form solutions of the impact responses. In this Chapter, the closed-form prediction of the peak contact force is combined with an existing quasi-static delamination threshold load criterion in order to analytically predict the threshold velocity for delamination onset. The dynamic inertia effects during impact are also included in the modified quasi-static load criterion. The predictions are validated against published experimental results and numerical simulations with good agreement.

4.1 Delaminations due to Impact

Impact damage is caused by the interaction of local indentation and global deflection. When the effect of global deformation is negligible or when the back surface of the impacted panel is supported, pure indentation damage may occur. Hull and Shi (1993) provided an extensive review of impact damage observations. The sequence of impact-induced damage in laminated composites usually involves initial matrix cracks oriented towards the impact point, followed by delaminations and eventually fiber fracture or additional transverse matrix cracks. The basic mechanisms for this phenomenon are
depicted schematically in Figure 4.1. Matrix cracks caused by tensile, compressive and shear stresses are usually distributed within the entire damage region. Fiber compressive and shear failure are observed locally in the contact area, while local fiber tensile failure typically occurs on the opposite surface, and in the area of large matrix cracks.

Delaminations are particularly serious because they are formed at relatively low contact loads and play an important role on the flexural stiffness and buckling failure of composite structures. Different orientations of the lamina within a laminate can promote delamination of two adjacent plies due to the stiffness mismatch at their interface. The delamination areas are influenced directly by the changes in the impact energy. A typical distribution of delaminations due to impact is shown in Figure 4.2. At the interfaces between plies with different fiber orientation, the delaminations are usually ‘peanut shaped’ with their major axis oriented in the direction of the fibers in the low lamina at the interface. It is important to recognize that the size of delaminations increases as the ply misalignment angle increases (Liu, 1998).

![Figure 4.1 Representation of Impact-Induced Damage (Philippe et. al, 1998)](image)
4.2 Prediction of Threshold Velocity for Delamination Onset

Typically, the initial matrix cracking is due to high contact stresses that are generated at relatively low contact loads. The onset of delamination initiates at higher impact loads. Therefore, the peak contact force becomes a vital impact parameter and determines the criticality of small mass impacts (Christopherson et al., 2005).

Davies and Robinson (1992) proposed an approach to deal with the prediction of the threshold contact force that corresponds to damage initiation. It was shown that when the damage area is plotted versus the maximum impact force, there is a clear sudden increase in damage size once the load reaches a critical value. This method has been used to successfully predict the onset of delamination damage for several quasi-isotropic graphite-epoxy laminates. The simple model for estimating this critical load is shown in Figure 4.3. At the threshold value of impact force there is an unstable crack propagation leading to a large size of delamination. This will cause the impact force to drop suddenly
in the response representing the loss of transverse stiffness. After this, the delamination size increases linearly with the force and it indicates stable growth of delamination. For quasi-static axisymmetric delamination growth in a centrally loaded clamped plate, the threshold load for delamination can be derived from this model as (Davies et. al, 1994)

\[ F_{\text{del}}^{\text{stat}} = \pi \sqrt{\frac{32 D^* G_{\text{IIc}}}{3}} \]  

(4.1)

where \( G_{\text{IIc}} \) is the critical energy release rate for mode II fracture, \( D^* \) is the effective stiffness of the laminate as defined in Eq. (2.7), and \( F_{\text{del}}^{\text{stat}} \) is the quasi-static delamination threshold load. The independence of delamination size and boundary conditions implies that the delamination threshold load is applicable to impact situations where initial effects may be neglected (Olsson, 2003).

By considering the kinetic energy of a circular area with \( n \) delaminations, the inertia effects can be included in the delamination threshold load (Olsson et al., 2006)

\[ F_{\text{del}}^{\text{dyn}} = F_{\text{del}}^{\text{stat}} \sqrt{1 - \frac{7 \pi^2}{216}} \approx 1.213 F_{\text{del}}^{\text{stat}} \]  

(4.2)

where \( F_{\text{del}}^{\text{dyn}} \) is the delamination threshold load considering inertia effects.

![Figure 4.3 Damage Size as a Function of Impact Force for Plates](image-url)
Based on the closed-form solution of the contact force in Chapter III, the peak contact force can be obtained from the characteristics of the contact force time history curve. Figure 4.4 shows the non-dimensional indentation as a function of non-dimensional time for over-damping, critical-damping and under-damping, respectively. It can be seen that the peak value of contact force can be obtained by

\[ F_{\text{peak}} = K_y \alpha_{\text{max}} = K_y \alpha(t) \bigg|_{\frac{d\alpha}{dt}=0} \]  

(4.3)

After algebraic manipulation the peak contact force can be obtained for three different cases

\[ \eta < 1: F_{\text{peak}} = K_y \cdot V_0 \cdot T_p \cdot e^{\frac{\eta \cdot \text{Arcsin}(\eta)}{\eta^2 - 1}} \]  

(4.4a)

\[ \eta = 1: F_{\text{peak}} = K_y \cdot V_0 \cdot T_p \cdot \frac{1}{e} \]  

(4.4b)

\[ \eta > 1: F_{\text{peak}} = K_y \cdot V_0 \cdot T_p \cdot \frac{\left(-1 + \eta \cdot \bar{\xi}\right) \cdot \left(-1 + 2\eta \cdot \bar{\psi}\right)}{\sqrt{\eta^2 - 1}} \]  

(4.4c)

where \( \bar{\xi} = \eta + \sqrt{\eta^2 - 1} \) and \( \bar{\psi} = \eta - \sqrt{\eta^2 - 1} \). \( \eta \) is half of \( \lambda_p \) defined in Eq. (3.19).

Figure 4.4 Non-dimensional Indentation Time History for Three Cases
Combining Eq. (4.2) and Eq. (4.4), the threshold velocity for a given plate and projectile is predicted with closed-form solutions:

\[
\eta < 1: V_{th} = \frac{F_{del}^{\text{dyn}}}{\left( K_y \cdot T_p \cdot e \right)^{\frac{\eta \cdot \text{arc \cos} (\eta)}{\sqrt{1-\eta^2}}}} \tag{4.5a}
\]

\[
\eta = 1: V_{th} = \frac{F_{del}^{\text{dyn}}}{\left( K_y \cdot T_p \cdot \frac{1}{e} \right)} \tag{4.5b}
\]

\[
\eta > 1: V_{th} = \frac{F_{del}^{\text{dyn}}}{\left( K_y \cdot T_p \cdot \frac{(-1 + \eta \xi) \cdot (-1 + 2\eta \xi)^{\frac{1}{2+2\eta \psi}}}{\sqrt{\eta^2 - 1}} \right)} \tag{4.5c}
\]

4.3 Application Examples: Delamination Onset Prediction

The closed-form predictions of the delamination threshold velocity for small mass impacts are validated against several experimental studies on symmetric composite laminates. The laminates have been impacted by small mass steel or aluminum impactors. Table 4.1 shows the material properties for the laminates and impactor from literature (Olsson, 2003). The interlaminar shear strength values of the laminates are obtained from manufacture’s data and available experimental data (Choi and Chang, 1992; Christoforou and Yigit, 1998). Interlaminar shear strength is usually defined as the shear stress at rupture, where the plane of fracture is located along the lamina interfaces of the composite laminate. Harding and Li (1992) employed double-lap shear specimens to measure the interlaminar shear strength of glass/epoxy and carbon/epoxy laminates at different loading rates. (Pahr et al., 2002) gave a review on a number of experimental approaches for the measurement of interlaminar shear strength. It was shown that the interlaminar shear strength values measured from experiment represent a lower bound of the true interlaminar shear strength of the laminate.
Table 4.3 illustrates the comparison between predicted and measured delamination threshold velocities in various experimental studies. The effects of plate finite thickness and effective modulus of transversely isotropic material properties were included in all of the calculations (Zheng and Binienda, 2007). Since the ratio of plate thickness to contact radius is significantly larger than 2.0 for all of the composite laminates, the empirical constant $\beta$ was chosen to be 1.0 for all the cases (Swanson, 2005). Good agreement with experimental observations is obtained. The predictions from current theory compares well with the predictions from asymptotic solutions by Olsson (2006). The current approach for prediction of delamination onset is based on elasto-plastic contact law which includes the effect of permanent indentation due to damage.

Table 4.1 Material Properties for Experimental Validation

<table>
<thead>
<tr>
<th>Material</th>
<th>$E_{11}$ (GPa)</th>
<th>$E_{22}$ (GPa)</th>
<th>$G_{12}$ (GPa)</th>
<th>$G_{23}$ (GPa)</th>
<th>$\nu_{12}$</th>
<th>$\nu_{23}$</th>
<th>$\rho$ (kg/m$^3$)</th>
<th>$t_{ply}$ (mm)</th>
<th>$G_{IIc}$ (J/m$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum</td>
<td>71</td>
<td>71</td>
<td>27</td>
<td>27</td>
<td>0.30</td>
<td>0.30</td>
<td>2790</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Steel</td>
<td>206</td>
<td>206</td>
<td>79</td>
<td>79</td>
<td>0.30</td>
<td>0.30</td>
<td>7850</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>HTA/6376C</td>
<td>137</td>
<td>10.4</td>
<td>5.2</td>
<td>3.5</td>
<td>0.30</td>
<td>0.51</td>
<td>1620</td>
<td>0.130</td>
<td>500</td>
</tr>
<tr>
<td>T300/5208</td>
<td>132</td>
<td>10.8</td>
<td>5.6</td>
<td>4.4</td>
<td>0.24</td>
<td>0.50</td>
<td>1600</td>
<td>0.127</td>
<td>300</td>
</tr>
<tr>
<td>AS4/2220-3</td>
<td>123</td>
<td>11.1</td>
<td>6.3</td>
<td>3.7</td>
<td>0.29</td>
<td>0.50</td>
<td>1600</td>
<td>0.129</td>
<td>510</td>
</tr>
<tr>
<td>XAS/914C</td>
<td>145</td>
<td>9.5</td>
<td>5.6</td>
<td>3.6</td>
<td>0.31</td>
<td>0.50</td>
<td>1600</td>
<td>0.125</td>
<td>416</td>
</tr>
<tr>
<td>AS4/PEEK</td>
<td>137</td>
<td>10.6</td>
<td>5.4</td>
<td>3.5</td>
<td>0.40</td>
<td>0.50</td>
<td>1600</td>
<td>0.135</td>
<td>1959</td>
</tr>
</tbody>
</table>

Table 4.2 Shear Strength of the Laminate ($S_u$)

<table>
<thead>
<tr>
<th>Fiber Volume Ratio</th>
<th>AS4/PEEK</th>
<th>AS4/2220-3</th>
<th>T300/5208</th>
<th>HTA/6376C</th>
<th>XAS/914C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_u$ (MPa)</td>
<td>157</td>
<td>110</td>
<td>101</td>
<td>93</td>
<td>95</td>
</tr>
</tbody>
</table>

72
4.4 Finite Element Validation

(Minh et al., 2005) gave an excellent review of available explicit finite element software for impact analysis of composite structures. Different modeling strategies for impact damage in composite materials in software MSC.Dytran, Pam-Shock and LS-DYNA were compared. The treatment for nonlinear material property due to post-failure degradation in MSC.Dytran is divided into three parts: initiation of failure, selection of elastic properties for degradation as well as the degradation of selected elastic properties at a defined strain rate or strain. However, Pam-Shock uses a more complicated bi-phase model that integrates the above three parts. In this study, LS-DYNA was used for the simulation of impact-induced delamination in composite laminates.

Table 4.3 Measured and Predicted Delamination Threshold Velocities

<table>
<thead>
<tr>
<th>Plate</th>
<th>Layup</th>
<th>$h$ (mm)</th>
<th>$G_{fc}$ (J/m²)</th>
<th>$m_p$ (Kg/m²)</th>
<th>$m_i$ (g)</th>
<th>$V_{exp}$ (m/s)</th>
<th>$V^a_{pred}$ (m/s)</th>
<th>$V^b_{pred}$ (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T300/5208</td>
<td>$(0/45/90)_v$</td>
<td>6.2</td>
<td>300</td>
<td>9.75</td>
<td>3.0</td>
<td>38</td>
<td>[Williams, 1984]</td>
<td>38</td>
</tr>
<tr>
<td>AS4/2220-3</td>
<td>$(0/45/90)_v$</td>
<td>6.2</td>
<td>510</td>
<td>9.91</td>
<td>3.0</td>
<td>55</td>
<td>[Williams, 1984]</td>
<td>46</td>
</tr>
<tr>
<td>XAS/914C</td>
<td>$[0_/±45]_2s$</td>
<td>2.0</td>
<td>416</td>
<td>3.2</td>
<td>0.9</td>
<td>30</td>
<td>[Cantwell, 1989]</td>
<td>34</td>
</tr>
<tr>
<td>XAS/914C</td>
<td>$[0_/90]_s$</td>
<td>0.5</td>
<td>416</td>
<td>0.8</td>
<td>0.9</td>
<td>33</td>
<td>[Cantwell, 1988]</td>
<td>32</td>
</tr>
</tbody>
</table>

$V^a_{pred}$ : prediction from reference (Olsson, 2006);
$V^b_{pred}$ : prediction from present study.
LS-DYNA has a variety of element types, which include four-node tetrahedron and eight-node solid elements, beam elements, three-node and four-node shell elements, eight-node solid shell elements, truss elements, membrane elements, discrete elements and rigid bodies. A variety of element formulations are also available for each element type. The most advantageous capability of LS-DYNA over other explicit finite element codes is its included contact algorithm. Several types of contact interfaces can be defined in LS-DYNA including surface to surface, nodes to surface, nodes tied to surface, and surface tied to surface contacts etc.

4.4.1 Verification of LS-DYNA

To verify the accuracy and convergence of LS-DYNA, a steel plate impacted by a rigid ball was studied and compared with the literature result (Karas, 1939). The steel plate with length 200 mm, width 200 mm and thickness 8 mm was clamped along four edges. The mass, velocity and radius of the steel ball impactor are 0.0329 kg, 1 m/s and 10 mm, respectively. Three different mesh sizes using Belyschko-Tsay shell elements were used to conduct the convergence test. Table 4.4 shows the details of three FEM models with different mesh size. The Young’s modulus, Poisson’s ratio and yield stress of the steel are 200 GPa, 0.3 and 325 MPa, respectively. LS-DYNA can provide the displacements and stress distributions for both impactor and plate. The contact force can be obtained based on Hertzian type contact law, as shown in Eq. (2.9) and Eq. (2.11). Different mesh sizes of the plate are shown in Figure 4.5. Figure 4.6 shows the results of the contact force from three different models and the comparison with the analytical solution. It can be
seen the numerical results obtained by LS-DYNA can converge to analytical solution upon the refinement of the mesh and properly chosen time step, as shown in section 4.4.2.

Table 4.4 Details of the Models in LS-DYNA

<table>
<thead>
<tr>
<th></th>
<th>Steel Ball</th>
<th>Plate</th>
<th>Mesh Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mesh 1</td>
<td>*Mat_Rigid</td>
<td>*MAT_PIECEWISE_LINEAR_PLASTICITY</td>
<td>20 x 20</td>
</tr>
<tr>
<td>Mesh 2</td>
<td>*Mat_Rigid</td>
<td>*MAT_PIECEWISE_LINEAR_PLASTICITY</td>
<td>40 x 40</td>
</tr>
<tr>
<td>Mesh 3</td>
<td>*Mat_Rigid</td>
<td>*MAT_PIECEWISE_LINEAR_PLASTICITY</td>
<td>80 x 80</td>
</tr>
</tbody>
</table>
Figure 4.5 Different Mesh Sizes of the Plate

Figure 4.6 Convergence of Contact Force with Different Mesh Sizes
4.4.2 Time Step Control

The time step in an explicit analysis is defined as the minimum stable time step in any deformable element in the overall mesh. The choice of the time step is crucial in a dynamic analysis since large time steps can result in unstable simulations, while a small time step can make the computation inefficient. LS-DYNA automatically calculates the largest time step which can be used during the simulation without triggering any numerical instability. Numerical instability will occur when the period of any mode of deformation is less than $\pi$ times the time step.

LS-DYNA will check all of the elements while determining the required time step in an analysis. The time step can be estimated roughly based on the following

$$\Delta t = 0.9 \frac{l_e}{c} \quad (4.6)$$

where $l_e$ is the character length of the smallest element and $c$ is the wave speed in the material. When calculating $l_e$, shell thickness and beam cross section dimensions are ignored. Rigid body elements are also not included. The speed of wave propagation in 3D continuum is given by

$$c = \sqrt{\frac{E(1-\nu)}{(1+\nu)(1-2\nu)\rho}} \quad (4.7)$$

4.4.3 Failure Criteria of Material Model

A significant amount of work has been done on modeling the failure mechanism of composites subjected to impact loading. (Johnson et al., 2001) reported a computational model to predict composite damage using a Continuum Damage Mechanics (CDM)
approach and a stack of shell elements tied by contact interface conditions. Based on the methodology developed by Hashin (1980) and the CDM model for unidirectional composites developed by (Matzenmiller et al., 1995), Yen (2002) developed material model 161/162 in LS-DYNA that captures the progressive failure modes of composite laminates (both unidirectional and plain weave laminates). These progressive failure models can be used to effectively simulate fiber failure, matrix damage, and delamination behavior under all conditions (opening, closure and sliding of failure surfaces). Furthermore, this progressive modeling approach is advantageous as it can enable one to predict delamination when the locations of delamination can not be predetermined.

Delamination is modeled within elements adjacent to ply interfaces in a composite laminate. The load carrying capacity of the element in the direction normal to the crack plane is assumed to be elastic and dependent on the opening and closure of such a crack. For an opening delamination (positive normal strain), the normal stress is reduced to zero. For a closing delamination (negative normal strain), the normal stress is assumed to regain the initial elastic capacity, while the shear stresses are assumed to be limited by a constant slide shear value. Under the crack closure condition, this approach can effectively model the interfacial friction behavior generated by the normal stress. This model allows one to effectively approximate the dynamic delamination behavior without the use of the time-consuming contact surface elements.

Three failure criteria are used for fiber failures: tension/shear, compression and crush under pressure. They are chosen in terms of quadratic strain forms as follows.

(1) Tensile/shear fiber mode:
\[
\left( \frac{E_a \langle \varepsilon_a \rangle}{S_{AT}} \right)^2 + \left( \frac{G_{ab} \varepsilon_{ab}}{S_{FS}^2} \right)^2 + \left( \frac{G_{ca} \varepsilon_{ca}}{S_{FS}^2} \right)^2 - r_1^2 = 0
\] 

(4.8)

(2) Compression fiber mode:

\[
\left( \frac{E_a \langle \varepsilon_a \rangle}{S_{AC}} \right)^2 - r_2^2 = 0
\] 

(4.9)

(3) Fiber crush mode:

\[
\left( \frac{E_c \langle \varepsilon_c \rangle}{S_{FC}} \right)^2 - r_3^2 = 0
\] 

(4.10)

where \( \langle \cdot \rangle \) is Macaulay bracket, \( S_{AT} \) and \( S_{AC} \) are the tensile and compressive strengths of the lamina in the fiber direction, respectively. \( S_{FS} \) and \( S_{FC} \) are the lamina strength associated with the fiber shear and crush failure, respectively.

Matrix mode failures must occur without fiber failure, and hence they will be on planes parallel to fibers. Two matrix damage functions are chosen for the failure plane perpendicular and parallel to the lamina plane.

(1) Perpendicular matrix mode:

\[
\left( \frac{E_b \langle \varepsilon_b \rangle}{S_{BT}} \right)^2 + \left( \frac{G_{bc} \varepsilon_{bc}}{S_{BT}^0 + S_B^{R}} \right)^2 + \left( \frac{G_{bc} \varepsilon_{bc}}{S_{BT}^0 + S_B^{R}} \right)^2 - r_4^2 = 0
\] 

(4.11)

(2) Parallel matrix mode:

\[
S^2 \left( \left( \frac{E_c \langle \varepsilon_c \rangle}{S_{CT}} \right)^2 + \left( \frac{G_{bc} \varepsilon_{bc}}{S_{BC}^0 + S_R^{C}} \right)^2 + \left( \frac{G_{bc} \varepsilon_{bc}}{S_{BT}^0 + S_B^{R}} \right)^2 \right) - r_5^2 = 0
\] 

(4.12)

where \( S_{BT} \) and \( S_{CT} \) are the transverse tensile strengths of the lamina, and \( S_{BC}^0 \), \( S_{AB}^0 \) and \( S_{Ca}^0 \) are the shear strength values of the corresponding tensile modes (\( \varepsilon_b > 0 \) or \( \varepsilon_c > 0 \)).
The damage strengths are assumed to be dependent on the compressive normal strains based on the Mohr-Coulomb theory when the elements are under compressive transverse strain \( (\varepsilon_b < 0 \text{ or } \varepsilon_c < 0) \), i.e.,

\[
S_R^B = E_B \tan(\phi) \langle -\varepsilon_b \rangle
\]

\[
S_R^C = E_C \tan(\phi) \langle -\varepsilon_c \rangle
\]  

where \( \tan(\phi) \) is similar to coefficient of friction and therefore \( \phi \) is a material constant.

The constant \( S \) is introduced to provide better correlation of delamination area with experiments. It can be determined by fitting the analytical prediction to the test data for the delamination area. The damage thresholds \( r_i \) (\( i = 1 \text{ to } 5 \)) have the initial values equal to 1 before the damage initiated, and are updated according to the damage accumulation of associated damage modes.

A set of damage variables are introduced to relate the onset and growth of damage to stiffness losses in the material. The damage functions are converted from the above failure criteria for fiber and matrix failure modes by neglecting the effect of Possion’s ratio. The stiffness losses are expressed in terms of the associated damage parameters:

\[
E_i = (1 - \omega_i) E_{i0}
\]  

\[
G_i = (1 - \omega_i) G_{i0}
\]

where \( E_{i0} \) and \( G_{i0} \) are the initial elastic moduli and shear moduli, respectively. \( E_i \) and \( G_i \) are the reduced elastic moduli and shear moduli, respectively. \( \omega_i \) is the damage parameter and given by

\[
\omega_i = 1 - e^{\frac{t_i - t_i^0}{\tau_i}}
\]
where \( r_j \) are the damage thresholds computed from the associated damage functions for fiber and matrix failure as well as delamination, \( m_i \) are the material damage parameters. The damage functions take into account the overall nonlinear elastic response of a lamina including the hardening and softening behavior beyond ultimate strength.

The piecewise failure surface of the material model is shown in Figure 4.7. It clearly shows how the material model works under multi-axial loading and different loading conditions. The horizontal and vertical axis refers to the out-of-plane normal stress and shear stress, respectively. When shear stress and out-of-plane compressive stress increase, the failure surface is captured by the fiber shear strength and crush strength.

![Figure 4.7 Failure Surface of MAT 162 Material Model (Xiao et al., 2007)](image)

4.4.4 Element Erosion and History Variables

In LS-DYNA impact simulations, cracking induced by impact is modeled by removing the element from the mesh. This is also named as element erosion method. Failure occurs when an element is subjected to a certain critical stress, strain or a user defined damage
value. The element erosion is necessary in order to avoid the excess distortion of a failed element, which will induce increased solution time.

In material model MAT_161/162, the erosion criterion is formulated on three different ways: (1) if fiber tensile failure is predicted in an element and the axial tensile strain is greater than E_LIMIT; (2) if compressive relative volume ratio in a failed element is smaller than ECRSH; (3) if tensile relative volume ratio in a failed element is larger than EEXPN.

The damage history variables for the associated failure modes are saved for each element and time step. Information regarding the damage history variables for the associated failure modes can be plotted in LS-Prepost. The delamination mode is the 12th history variable. The parallel and perpendicular matrix failure mode are the 10th and 11th history variable, respectively.

4.4.5 One-Element Validation

To better understand the material model before doing the full impact analysis, simulations were carried out on a single 8-node solid element representing a unidirectional composite layer. T300/5208 (0/±45/90)_s carbon/epoxy composite laminate impacting by an aluminum impactor in Table 4.1 was chosen for the one-element validation and impact simulation. The material properties of the laminate and impactor are shown in Table 4.4. The strength data for T300/5208 carbon/epoxy laminate required in material model MAT_162 were taken from quasi-static tests and are shown in Table 4.5 (Williams, 1984). In Table 4.6, X_t, X_c, Y_t and Y_c are the longitudinal tensile, longitudinal compressive, transverse tensile and transverse compressive strength,
respectively; $S_{FC}$ and $S_{FS}$ are fiber crush strength and fiber mode shear strength, respectively; $Z_i$ and $S_i$ are out-of-plane tensile strength and interlaminar shear strength, respectively. Four different types of loading conditions are applied on the solid element, i.e., x-tension, x-compression, z-tension and interaction between shear and compression in yz-plane. Figure 4.8 shows the simple tension in fiber direction with constant loading rate. After exceeding the tensile strength, the load was held for a while to get rid of the inertial effects in quasi-static simulations. The simulation results for x-tension and x-compression, z-tension and shear-compression interaction are shown in Figure 4.9, Figure 4.10, Figure 4.11, respectively. The softening and progressive failure behavior after reaching the strength can be clearly seen from all the cases. It can be seen from Figure 4.11 that the shear stress decreases till certain level of compressive strain then it almost keep as a constant since the damage strength increases with the increasing compressive strain according to Mohr-Coulomb friction theory as seen in Eq. (4.13). When the compressive strain is big, the numerators of Eq. (4.11) and (4.12) dominate the failure compared to the denominators and the material will have complete degradation or failure.

Table 4.5 Material Properties for Finite Element Validation

<table>
<thead>
<tr>
<th>Material</th>
<th>$E_{11}$ (GPa)</th>
<th>$E_{22}$ (GPa)</th>
<th>$G_{12}$ (GPa)</th>
<th>$G_{23}$ (GPa)</th>
<th>$\nu_{12}$</th>
<th>$\nu_{23}$</th>
<th>$\rho$ (kg/m³)</th>
<th>$t_{ply}$ (mm)</th>
<th>$G_{IIC}$ (J/m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum</td>
<td>71</td>
<td>71</td>
<td>27</td>
<td>27</td>
<td>0.30</td>
<td>0.30</td>
<td>2790</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>T300/5208</td>
<td>132</td>
<td>10.8</td>
<td>5.6</td>
<td>4.4</td>
<td>0.24</td>
<td>0.50</td>
<td>1600</td>
<td>0.127</td>
<td>300</td>
</tr>
</tbody>
</table>
Table 4.6 Strength Data for T300/5208

<table>
<thead>
<tr>
<th>Material</th>
<th>$X_t$ (MPa)</th>
<th>$X_c$ (MPa)</th>
<th>$Y_t$ (MPa)</th>
<th>$Y_c$ (MPa)</th>
<th>$S_u$ (MPa)</th>
<th>$Z_t$ (MPa)</th>
<th>$S_{FC}$ (MPa)</th>
<th>$S_{FS}$ (MPa)</th>
<th>$S_{LB}$ (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T300/5208</td>
<td>1286</td>
<td>1346</td>
<td>53.4</td>
<td>300</td>
<td>84.5</td>
<td>129</td>
<td>1655</td>
<td>828</td>
<td>103</td>
</tr>
</tbody>
</table>

Figure 4.8 One Element under Simple Tension

Figure 4.9 Stress vs. Strain for X-tension and X-compression
4.4.6 Impact Simulation Results

T300/5208 (0/±45/90)$_{6s}$ carbon/epoxy composite laminate impacting by an aluminum impactor in Table 4.1 was also chosen for the impact simulation to predict numerically the threshold velocity for delamination. The size of the plate is 0.1m $\times$ 0.1m and a cylindrical impactor was used. Due to the symmetry, only one quarter of the plate was
modeled. Although it is a wave-controlled impact and the effect of boundary conditions is negligible, simply-supported boundary conditions were applied along the edges of the plate. Since it is a localized impact, the mesh of the plate under the impactor was much finer than other parts of the plate. The detailed mesh of the plate is shown in Figure 4.12. The 48 layers of the composite laminate were modeled explicitly using *PART_COMPOSITE keyword in LS_DYNA, as shown graphically in Figure 4.13.

From the simulations, the laminate starts to delaminate when the velocity of the impactor increases to 32 m/s. When the impactor velocity reaches 35 m/s, many delaminations occur through the thickness of the laminate, as shown in Figure 4.14. It can be safely concluded that the threshold velocity of delamination for this particular case is around 32 m/s. The predictions agree well with experimental data and closed-form solutions, which are 38 m/s and 57 m/s, respectively. The close-from prediction is higher than both experiment and numerical results. This is due to the peak contact force is smaller when including the effect of permanent indentation. Thus the threshold velocity for delamination onset is higher for a constant quasi-static delamination threshold load.

(a) Overall Mesh                                    (b) Enlarged Mesh around the Corner

Figure 4.12 Mesh of One Layer of the Composite Laminate
4.5 Conclusion

In this Chapter, the closed-form solutions of the impact responses are combined with an existing quasi-static delamination criterion to provide the analytical predictions of the threshold velocity for delamination. The inertia effects during impact situations are also taken into account in the analysis. It is demonstrated that the closed-form predictions are
in good agreement with experimental values for several test cases from existing literature. The predictions are also validated against finite element simulations using material model MAT_162 in LS-DYNA. The analytical delamination threshold velocity would be very useful for designers who prefer closed-form solutions for the criticality of small-mass high-velocity impacts on composite structures.
CHAPTER V

ELASTO-PLASTIC IMPACT RESPONSE OF LAMINATES UNDER PRESTRESS

Usually, in addition to impact loading, a composite structure experiences pre-stresses produced either by service loads or by manufacturing/assembly process. A common example of initial stress effect is that the structural components of an aircraft which may experience bird-strike, while they are highly stressed due to a variety of service loads. Another example of pre-stresses is the impact loading of jet engine fan blades due to blade-out event when they are subjected to centrifugal forces in service. Most of the available literature deals with impact on structures without any pre-stresses.

(Khalili et. al, 2007) investigated the impact response of a large uni-axially reinforced graphite/epoxy composite plate with tensile prestresses in its plane. Sveklo’s contact theory for orthotropic materials was used to describe the interaction between the plate and the impactor. Shear deformation was neglected in their analysis. The effect of shear deformation on the impact response of elastic plates was addressed by using Mindlin plate theory and it was shown that for high velocities such effects are important (Chattopadhyay, 1977). Combined effects of shear deformation and permanent indentation on the impact response of elastic plates were considered by Chattopadhyay and Saxena (1991). It was found that additional energy was dissipated in the permanent indentation process when compared to the elastic case.
The simultaneous effects of preloads and permanent indentation on the impact response of laminated composite plates have not been solved analytically earlier in the literature. In this Chapter, an analytical solution is derived for a laminated composite plate subjected to pre-stresses in the plane of plate and a transverse impact load produced by an impactor. Combined effects of shear deformation, permanent indentation and prestresses were considered by using the linearized elasto-plastic contact law as in Chapter III and Laplace transform technique. The effects of mass and velocity of the impactor, plate thickness and interlaminar shear strength on the dynamic responses of a plate under prestresses were also studied.

5.1 Dynamic Response of Laminated Plates under Prestresses

It is assumed that the plate is a symmetrically cross-play laminate. The initial stress resultants can be added into the plate equations developed by Whitney and Pagano (1970):

\[ D_{11} \psi'_{x''xx} + D_{66} \psi'_{x''yy} + (D_{12} + D_{66}) \psi'_{x'yxy} - \kappa A_{55} (\psi_x' + w_x) = I_2 \dot{\psi}_x \]

\[ (D_{12} + D_{66}) \psi'_{x'xy} + D_{22} \psi'_{y''xx} + D_{66} \psi'_{y'xy} - \kappa A_{44} (\psi_y' + w_y) = I_2 \dot{\psi}_y \]

\[ \kappa A_{55} \psi''_{x'x'} + \kappa A_{44} \psi''_{y'y'} + (\kappa A_{55} + N_x^0)w_{xx} + (\kappa A_{44} + N_y^0)w_{yy} + q = I_1 \ddot{w} \]

where \( w \) is the transverse displacement at the plate mid-plane, \( \psi_x' \) and \( \psi_y' \) are the shear rotations of the plane sections in the \( x \) and \( y \) direction and \( \kappa \) is the shear correction factor and is often taken to be \( \pi^2 / 12 \). In Eq. (5.1),

\[ (A_g, D_g) = \int_{-h/2}^{h/2} Q_g (1, z^2) dz \]
\[(I_1, I_2) = \int_{-h/2}^{h/2} \rho(1, z^2) dz \quad (5.3)\]

where \(Q_i\) are the reduced stiffness for plane stress, \(\rho\) is the mass density, and \(h\) is the thickness of the plate.

A simply-supported rectangular laminated plate was considered with dimensions \(a\) and \(b\) for which the boundary conditions are given by

\[w = \psi\_x = 0 \quad \text{at} \quad x = 0, a\]
\[w = \psi\_y = 0 \quad \text{at} \quad y = 0, b \quad (5.4)\]

The solution of above dynamic problem is based on the expansions of the load, displacement and rotations in double Fourier series (Yang et. al, 2005). The double Fourier series expansions for the shear rotations and transverse displacement of a simply-supported rectangular plate subjected to impact loading are given by

\[\psi\_x (x, y, t) = \sum_m \sum_n A\_mn (t) \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\]
\[\psi\_y (x, y, t) = \sum_m \sum_n B\_mn (t) \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}\]
\[w(x, y, t) = \sum_m \sum_n W\_mn (t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\]
\[q(x, y, t) = \sum_m \sum_n Q\_mn (t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (5.5)\]

where \(A\_mn (t)\), \(B\_mn (t)\), \(W\_mn (t)\) and \(Q\_mn (t)\) are the time dependent coefficients to be determined.
For a uniform distributed load applied over a rectangular area \( u \times v \) with center at point \((\xi; \eta)\), \( Q_{mn}(t) \) can be represented as (Dobyns and Porter, 1981)

\[
Q_{mn}(t) = T_{mn} F(t) \quad (5.6)
\]

where

\[
T_{mn} = \frac{16}{\pi^2 m n u v} \sin \frac{m \pi \xi}{a} \sin \frac{n \pi \eta}{b} \sin \frac{m \pi u}{2a} \sin \frac{n \pi v}{2b} \quad (5.7)
\]

The influence of rotary inertia terms are small and can be neglected (Mindlin, 1951).

Substitution of Eq. (5.5) into Eq. (5.1) results in an independent set of three equations

\[
\begin{bmatrix}
A_{mn}(t) \\
B_{mn}(t) \\
W_{mn}(t)
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
Q_{mn}(t) - \rho h \ddot{W}_{mn}(t)
\end{bmatrix} \quad (5.8)
\]

where the coefficients of symmetric matrix \([L_{ij}]\) are given by

\[
\begin{align*}
L_{11} &= D_{11} \left( \frac{m \pi}{a} \right)^2 + D_{66} \left( \frac{n \pi}{b} \right)^2 + \kappa A_{55} \\
L_{12} &= (D_{11} + D_{66}) \left( \frac{m \pi}{a} \right) \left( \frac{n \pi}{b} \right) \\
L_{13} &= \kappa A_{55} \left( \frac{m \pi}{a} \right) \\
L_{22} &= D_{66} \left( \frac{m \pi}{a} \right)^2 + D_{22} \left( \frac{n \pi}{b} \right)^2 + \kappa A_{44} \\
L_{23} &= \kappa A_{55} \left( \frac{n \pi}{b} \right) \\
L_{33} &= (\kappa A_{55} + N^0_x) \left( \frac{m \pi}{a} \right)^2 + (\kappa A_{55} + N^0_y) \left( \frac{n \pi}{b} \right)^2
\end{align*}
\quad (5.9)
\]

Following the same procedure as the work done by Birman and Bert (1987), Eq. (5.8) can be reduced to a single ordinary differential equation by transformations
\[ \dot{A}_{mn}(t) = K_A W_{mn}(t) \]
\[ \dot{B}_{mn}(t) = K_B W_{mn}(t) \]  \hspace{1cm} (5.10)

where \( K_A \) and \( K_B \) can be obtained using Cramer's Rule

\[ K_A = \frac{L_{21}L_{23} - L_{43}L_{22}}{L_{11}L_{22} - L_{12}^2} \]
\[ K_B = \frac{L_{12}L_{13} - L_{11}L_{23}}{L_{11}L_{22} - L_{12}^2} \]  \hspace{1cm} (5.11)

Substitution of Eq. (5.10) into Eq. (5.8) will reduce the system of equation into the following

\[ \ddot{W}_{mn}(t) + \omega_{mn}^2 W_{mn}(t) = \frac{T_{mn} F(t)}{\rho h} \]  \hspace{1cm} (5.12)

where \( \omega_{mn} \) are fundamental frequencies of the plate and

\[ \omega_{mn} = \sqrt{\frac{L_{13}K_A + L_{23}K_B + L_{33}}{\rho h}} \]  \hspace{1cm} (5.13)

The above derivation is based on the First-order Shear Deformable Theory (FSDT). Neglecting the shear deformation (Classical Plate Theory; CPT), the same system of equation can be obtained as Eq. (5.12) with

\[ \omega_{mn} = \sqrt{\frac{K_C + K_D}{\rho h}} \]  \hspace{1cm} (5.14a)

where

\[ K_C = D_{11} \left( \frac{m\pi}{a} \right)^4 + 2(D_{12} + D_{66}) \left( \frac{m\pi}{a} \right)^2 \left( \frac{n\pi}{b} \right)^2 + D_{22} \left( \frac{n\pi}{b} \right)^4 \]  \hspace{1cm} (5.14b)

\[ K_D = N_x \left( \frac{m\pi}{a} \right)^2 + N_y \left( \frac{n\pi}{b} \right)^2 \]  \hspace{1cm} (5.14c)
It is assumed that the plate has zero initial displacement and velocity during the impact and

\[ W_{mn}(0) = 0; \quad \dot{W}_{mn}(0) = 0 \]  

(5.15)

Combining Eq. (5.12) and Eq. (5.15), the solution of the ordinary differential equation is obtained using the convolution integral:

\[ W_{mn}(t) = \frac{T_{mn}}{\rho \omega_{mn}} \int_0^t F(\tau) \sin \omega_{mn}(t - \tau) d\tau \]  

(5.16)

The displacement of the plate at any point is given by

\[ w(x, y, t) = \sum_m \sum_m \frac{T_{mn}}{\rho \omega_{mn}} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \int_0^t F(\tau) \sin \omega_{mn}(t - \tau) d\tau \]  

(5.17)

### 5.2 Contact Law and Impact Dynamics

The linearized version of the elasto-plastic contact law accounting for permanent indentation as shown in Chapter III is used:

\[ F(\alpha) = K_y \alpha \]  

(5.18)

where \( K_y \) is the linearized elasto-plastic contact stiffness given as

\[ K_y = 1.5 K_h \sqrt{\alpha_y} \]  

(5.19)

where \( K_h \) is the contact stiffness accounting for finite thickness of the plate as well as the transversely isotropic properties of the laminate, as shown in Eqs. (2.15) to (2.17). \( \alpha_y \) is the critical indentation for the local yielding of the plate to occur and give as

\[ \alpha_y = \frac{2.72 \pi^2 R S_n^2}{Q_{\alpha}^2} \]  

(5.20)

where \( S_n \) is the interlaminar shear strength of the laminated composites.
In Eq. (5.18), the relative approach $\alpha$ can be defined as

$$\alpha = w_i - w(a/2, b/2, t)$$ (5.21)

where $w_i$ is the displacement of the impactor and $w(a/2, b/2, t)$ is the transverse displacement in the center point of the plate, which can be obtained from Eq. (5.17) as

$$w(a/2, b/2, t) = \sum_n \sum_m \frac{T_{nn}}{\rho \omega_{mn}} \sin \frac{n\pi}{2} \sin \frac{n\pi}{2} \int_0^t F(\tau) \sin \omega_{mn}(t-\tau) d\tau$$ (5.22)

The displacement of the impactor can be obtained from the equation of motion of the impactor:

$$\begin{cases} M_i \ddot{w}_i = -F(t) \\ w_i(0) = 0, \dot{w}_i(0) = V_0 \end{cases}$$ (5.23)

Thus the displacement of the impactor can be obtained as

$$w_i(t) = V_0 t - \frac{1}{M_i} \int_0^t F(\tau)(t-\tau) d\tau$$ (5.24)

If the Hertzian contact law is used, the impact dynamic equation for the system can be obtained by combining Eqs. (2.9), (5.22) and (5.24)

$$V_0 t - \frac{1}{M_i} \int_0^t F(\tau)(t-\tau) d\tau - \left( \frac{F(t)}{K_h} \right)^{2/3} = w(a/2, b/2, t)$$ (5.25)

Due to the nonlinear term in the left-hand side of Eq. (5.25), it is can only be solved numerically by Newmark integration method or stepwise numerical integration scheme (Hughes, 1987).

In order to get the analytical solution, the linearized elasto-plastic contact law can be used in the impact dynamic equation of the system by combining Eq. (5.18), Eq. (5.21) and Eq. (5.22)
\[ V_0 t - \frac{1}{M_i} \int_0^t F(\tau)(t - \tau) d\tau = \frac{F(t)}{K_y} = \sum_m \sum_n \frac{T_{mn}}{\rho \omega_{mn}} \sin \frac{m\pi}{2} \sin \frac{n\pi}{2} \int_0^t F(\tau) \sin \omega_{mn}(t - \tau) d\tau \quad (5.26) \]

Defining Laplace Transform as

\[ \mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt \quad (5.27) \]

Taking the Laplace Transform of Eq. (5.26), after some algebraic simplifications it yields:

\[ F(s) = \frac{M_i V_0}{1 + \frac{M_i}{K_y} s^2 + \sum_m \sum_n T_{mn} M_i \sin \frac{m\pi}{2} \sin \frac{n\pi}{2} s^2 + \omega_{mn}^2} \quad (5.28) \]

Using the inverse Laplace transform and following a similar procedure as (Christoforou and Swanson, 1991) yields:

\[ F(t) = \sum_j F_j \sin(\omega_j t) \quad (5.29) \]

where \( \omega_j \) are the roots of the polynomial expression of the denominator in Eq. (5.28) and

\[ F_j = \frac{M_i V_0}{\omega_j \left[ \frac{M_i}{K_y} + \sum_m \sum_n T_{mn} M_i \sin \frac{m\pi}{2} \sin \frac{n\pi}{2} \omega_{mn}^2 \right]} \quad (5.30) \]

Similarly, taking the Laplace Transform of Eq. (5.16)

\[ W_{mn}(s) = T_{mn} \sum_j \frac{F_j \omega_j}{(s^2 + \omega_{mn}^2)(s^2 + \omega_j^2)} \quad (5.31) \]

Using the inverse Laplace Transform, Eq. (5.31) becomes

\[ W_{mn}(t) = T_{mn} \sum_j \frac{F_j}{\omega_{mn}(\omega_j^2 - \omega_{mn}^2)} (\omega_j \sin \omega_{mn} t - \omega_{mn} \sin \omega_j t) \quad (5.32) \]

The strains can be obtained using the corresponding Fourier expansions of the shear rotations.
\[ \varepsilon_x(x, y, t) = z \frac{\partial \psi_x(x, y, t)}{\partial x} = -z \frac{\pi}{a} K_a \sum_m \sum_n mW_{mn}(t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \]  
(5.33a)

\[ \varepsilon_y(x, y, t) = z \frac{\partial \psi_y(x, y, t)}{\partial y} = -z \frac{\pi}{b} K_b \sum_m \sum_n nW_{mn}(t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \]  
(5.33b)

5.3 Analytical Results

For validation of the analytical solution, the impact problem studied by Carvalho and Guedes Soares (1995) are used. It consists of a simply supported composite plate centrally impacted by a steel spherical ball. The material properties, geometry of the plate and the impact conditions are given in Table 5.1. The convergence of the present analytical method for the above problem without prestresses is studied, as shown in Figure 5.1 to Figure 5.3. Due to the vanishing terms in the Fourier series for center impact, the number of terms refers to non-vanishing terms. Figure 5.1 and 5.2 show that convergent solutions are obtained with 25 terms for contact force and displacement. Figure 5.3 shows that the calculation for strain takes significantly more terms and convergent solution can be obtained with 49 terms. In the following computations, 49 terms were used for the calculation of contact force, displacement and strains.

| Table 5.1 Properties of Composite Plate and Impact Parameters

| Plate: [0/90/0/90/0]s T300/934 carbon-epoxy |
| Plate Size: square – 1.0 m x 1.0 m |
| Plate Thickness: 10.76 mm (1.076 mm/layer) |
| \( E_{11} = 120 \text{ GPa}; \ E_{22} = 7.9 \text{ GPa}; \ G_{12} = G_{23} = 5.5 \text{ GPa}; \) |
| \( \nu_{12} = \nu_{23} = 0.3; \rho = 1580 \text{ kg/m}^3 \) |
| Impactor: 50.0 mm diameter steel ball; m = 500 g; impact velocity \( V_0 = 3.0 \text{ m/s} \) |
| Contact Parameters: Shear Strength \( S_u = 220 \text{ MPa} \) |
| \( K_h = 1.665 \times 10^9 \text{ N/m}^{3/2}; \ K_y = 2.33 \times 10^7 \text{ N/m} \) |
Figure 5.1 Convergence Study for Contact Force

Figure 5.2 Convergence Study for Displacement
Figure 5.3 Convergence Study for Strain $\varepsilon_x$

Figure 5.4 shows the comparison of the contact force history from CPT with and without permanent indentation, FSDT with and without permanent indentation. It can be seen that the combined effects of shear deformation and permanent indentation change the contact force variation as well as decreasing the maximum value of the contact force.

Figure 5.5 shows plate central displacement time history for different cases. It can be seen that all the curves follow the similar pattern with displacement continuously increasing until it reaches a plateau, then they increases to a peak value when the waves reflects from the boundary of the plate. The peak values of the displacement related to permanent indentations are less than the Hertzian impacts. This is due to the smaller contact force resulting from the contact region.

Figure 5.4 and 5.5 indicate small contribution of shear deformation for contact force and central displacement time histories. The effect of shear deformation is not pronounced at moderate impact velocities. As reported by Chattopadhyay (1977), for higher impact velocities, significant differences can result due to shear deformation. It
can be also seen that the contact force and plate central displacement from the elasto-plastic contact law are less than the ones from Hertzian contact law. The same conclusion was also made in Chapter III.

Figure 5.4 Comparison of Contact Force Time History

Figure 5.5 Comparison of Plate Central Displacement Time History
In the analytical computation of the effect of prestresses, nondimensional quantities were employed. The use of dimensionless variables is useful for the identification of the relationship between different physical parameters or properties of the impact. The following nondimensional variables are defined:

\[ \omega = \sqrt{ \frac{K_y}{M_i} } \]  
\[ \tau = \omega t \]  
\[ F^* (\tau) = \frac{F(t)}{M_i V_0 \omega} \]  
\[ N_x^* = \frac{N_x^0 h}{M_i V_0 \omega} \]  
\[ N_y^* = \frac{N_y^0 h}{M_i V_0 \omega} \]  
\[ w^* (\tau) = \frac{w(t) \omega}{V_0} \]

where \( \omega \) is the linearized contact frequency considering permanent indentation.

The analytical solution considering the permanent indentation and shear deformation was then used to study the effects of bi-axial pre-loads on the impact response of the above plate problem for the cases \( N_x^* = N_y^* = +0.2, 0.0, \) and \(-0.1\) where “+” and “-” mean tension and compression, respectively. Figure 5.7 to Figure 5.10 show the effect of prestress on the contact force, plate central displacement, strains for different cases.

It is evident from Figure 5.7 that the prestress does not have a significant effect on the contact force. This is consistent with Whittingham et al. (2004) who concluded that the peak contact force is almost independent of the magnitude of the pre-loads in low-
velocity impacts. The total momentum transferred from the impactor to the plate is almost the same for all three cases. After first impact, the peak contact force for tensile prestress is larger than the case with compressive prestress and the case free from prestress. This is in agreement with the experimental results (Chiu et al., 1997). In this test, a quasi-isotropic lay-up \([(0/45/90/-45)_s]\), composite laminate was impacted under the condition of various prestresses. The contact forces time history after first impact for tensile prestress, compressive prestress and prestress-free are shown in Figure 5.6.

![Figure 5.6 Contact Force Histories with Prestresses (Chiu et al., 1997)](image)

However, it can be seen from Figure 5.7 that the central displacement changes significantly with the presence of prestresses. The softening effect of the compressive prestress and the stiffening effect of the tensile prestress are clearly exhibited. The tensile prestress can reduce the flexure deflection and therefore increase the flexure stiffness. Vice versa, compressive stress will lead to the decrease of flexure stiffness.
It can be noted from Figure 5.8 and Figure 5.9 that the dynamic bending strains are not affected by the presence of prestress at the beginning of the impact. Significantly high bending strains $\varepsilon_x$ are obtained at the beginning of the impact when the maximum contact force occurs. For the cases of both tensile prestress and prestress-free cases, the maximum bending strains $\varepsilon_y$ occurs when the plate experiences its maximum deflection. Due to the anisotropy of the composite laminate, the x direction sustains most of the contact force at the beginning of the contact. When the plate deflection increases and impactor rebounds off from the plate, the y direction starts to takes more control of the contact force.

![Figure 5.7 Effect of Prestress on Contact Force](image)

Figure 5.7 Effect of Prestress on Contact Force
Figure 5.8 Effect of Prestress on Central Displacement

Figure 5.9 Effect of Prestress on Strain $\varepsilon_x$
Figure 5.10 Effect of Prestress on the Strain $\varepsilon_y$

5.3.1 Effect of Impact Velocity

The cross-plied laminated plate with initial stresses $N_x^* = N_y^* = +0.2$ impacted by the impactor with velocities of 1, 3 and 15 m/s was investigated first. The effects of velocity on the time history of contact force and plate central displacement are shown in Figure 5.11 and Figure 5.12, respectively. It can be seen that the contact forces and plate responses corresponding to different velocities are similar except the amplitude which is proportional to the initial impact velocity. This is also verified by Eqs. (5.30) and (5.32) where the contact force and plate displacement have a linear relationship with the impactor velocity.
Figure 5.11 Effect of Impact Velocity on Contact Force

Figure 5.12 Effect of Impact Velocity on Central Displacement

5.3.2 Effect of Plate Thickness

The effect of the thickness of the plate under initial stresses $N_x^* = N_y^* = +0.2$ impacted by the impactor with velocities of 3 m/s is illustrated in Figure 5.13 and 5.14, where the calculated contact force and displacement time history are presented. As would be
expected, increasing the thickness of the plate will increase the contact force and decrease the central displacement and vice versa.

Figure 5.12 Effect of Plate Thickness on Contact Force

5.3.3 Effect of Mass of Impactor

In this case, the cross-ply laminate under initial stresses $N_x^* = N_y^* = +0.2$ impacted by the impactor with velocities of 3 m/s and same radius but of different weights. Figure 5.15 and 5.16 show the effect of impactor mass on the time history of contact force and central displacement. It is found that a larger mass also leads more severe dynamic responses of the plate, higher induced contact force and longer contact time.
Figure 5.13 Effect of Plate Thickness on Central Displacement

Figure 5.14 Effect of Impactor Mass on Contact Force
5.3.4 Effect of Material Properties

The effect of fiber shear strength included in the elasto-plastic contact law on the contact force and dynamic response of the plate are shown in Figure 5.17 and 5.18. As can be seen for a given impact energy, the maximum contact force increases with increasing fiber shear strength whereas the contact time decreases. Due to the higher contact force, the central displacement increases with increasing fiber strength. After the first impact, the displacement will not follow this trend any more.
Figure 5.17 Effect of Fiber Shear Strength on Contact Force

Figure 5.18 Effect of Fiber Shear Strength on Central Displacement

5.4 Conclusion

In this Chapter, the dynamic responses of laminated composite plates under prestress and impact are studied analytically based on the linearized elasto-plastic contact law, which accounts for the effect of permanent indentation. The effects of prestresses,
velocity and mass of the impactor, plate thickness and interlaminar shear strength of the laminate on the contact force and plate central displacement are investigated. It is found that initial stresses do not have noticeable effect on the contact force history but they have significant effects on the plate central displacement. Compressive prestress tends to increase the displacement and vice versa. It is also found that the impact responses are proportional to the velocity of the impactor. The contact force increases with the increasing of the mass of impactor, thickness and interlaminar shear strength of the laminate. Larger deflections of the plate are also introduced by a heavier mass and thinner plate thickness. The analytical solution is very useful for parametric studies of different impact conditions and can provide some guidance for large scale numerical simulation of complex structures.
6.1 Concluding Remarks and Contributions

In this study, the elastic and elasto-plastic wave-controlled impact responses of composite laminated plates are investigated. The elasto-plastic impact responses of composite laminates under prestresses are also studied analytically. The main conclusions and contributions are summarized as following:

(1) Based on a structural model for wave-controlled impact, Hertzian contact law was extended to generalized contact laws fitting from experiment to investigate impact responses of composite laminates. The original nonlinear governing equation was transformed into two linear equations using asymptotic expansion. Closed-form solution can be derived for the first linear homogeneous equation, which is the equation of motion for single degree of freedom system with viscous damping. The second linear non-homogeneous equation was solved numerically. The overall impact responses for wave-controlled impacts can be obtained semi-analytically and agree well with the numerical solutions of nonlinear governing equations. The proposed methodology is useful for providing guidance to numerical simulation of impact on complex composite structures with contact laws fitting from experimental data.
(2) A linearized elasto-plastic contact law including permanent indentation and damage effects was used to derive the closed-form solutions of the small mass wave-controlled impact response of composite laminate. It was shown that consideration of permanent indentation in impact analysis can significantly change the impact responses when the velocity of the impactor reaches medium velocity. The closed-form solutions of the impact responses are useful in parametric studies, which can provide guidance to numerical simulation of impact on composite structures.

(3) The analytical threshold velocity for delamination due to impact can be achieved based on the closed-form solutions of the impact responses combining with quasi-static delamination criterion. The closed-form solutions of the threshold velocity for delamination agree well with a wide range of experimental data. The predictions are also comparable with the results from finite element simulations in LS-DYNA. The proposed methodology would be very useful for designers who prefer closed-form solutions for the criticality of the impacts on composite structures.

(4) The combined effects of permanent indentation, shear deformation and prestresses are also solved analytically using Laplace transformation technique. It was found that initial stresses do not have noticeable effect on the contact force history but they have significant effects on the plate central displacement.

In summary, several elastic and elasto-plastic models for composite laminates under wave-controlled impact and prestresses are developed in this study. The effects of permanent indentation are particularly taken into account in those impact models. The combined theoretical and numerical investigations can provide validated models for impact analysis of elastic and elasto-plastic composite laminates. The proposed
methodologies are useful for designing model tests and numerical simulation of complex composite structures.

6.2 Recommendations for Future Research

The specific topics recommended for the future research effort are summarized as follows:

(1) It has been shown that the effects of permanent indentation and damage are more prominent when the impactor reaches medium velocity (around 40–70 m/s). The derivations are all based on the parameters determined from static indentation tests. As shown before, the slope of the elasto-plastic contact law used in the impact models is related to the interlaminar shear strength of the composite laminate. Based on the measurement from single-lap shear specimens under different impact loading rate using Split Hokinson Bar Apparatus (SHPB), it was shown that the interlaminar shear strength is rate-dependent (Hallet et al., 1999). Therefore, strain rate dependence needs to be incorporated into the analytical impact models.

(2) Composite sandwich structures are becoming more and more popular in structural applications. (Malekzadeh et. al, 2007) proposed a dynamic model based on improved higher order sandwich plate theory (IHSAPT) and two models representing the contact behavior between the impactor and the panel. This model can be applied to both boundary-controlled and wave-controlled impact. Permanent indentation was not considered in their study. The analytical models accounting for permanent indentation in this study can be extended to the impact analysis of composite sandwich structures.
(3) As shown in the literature review, ABAQUS has great features to model delamination in composite laminates. The closed-form prediction of delamination threshold from the analytical models can also be validated against the numerical predictions from ABAQUS.

(4) More experimental data are needed to validate the closed-form predictions of impact responses of composite laminates. More efficient experimental methodology is also needed to determine the initiation of delamination in composite laminates due to impact.


APPENDIX

Program PlateImpact_Plastic

! Use fifth-order Runge_Kuta method to solve 2nd-order ODE in Olsson's Paper
! Considering the elasto-plastic impact law
! USE MSIMSLMD
Use NUMERICAL_LIBRARIES
Implicit None
EXTERNAL FCN1, FCN2, FCN3, FCNJ
INTEGER::MXPARM, N
INTEGER::MABSE, MBDF, MSOLVE
PARAMETER (MABSE=0, MBDF=2, MSOLVE=2)
PARAMETER (MXPARM=50, N=2)
Common/Data1/Lame1
Common/Data2/Beta1, Beta2, Beta3
Common/Data3/Theta1, Theta2, Theta3

! SPECIFICATIONS FOR LOCAL VARIABLES
INTEGER::IDO, ISTEP, NOUT, NIN, J
Double precision::PARAM(MXPARM), T1, TEND, TOL, Y1(N), Y1PRIME(N), Amatr(1,1)
Double Precision::T2, Y2(N), Y2PRIME(N), T3, Y3(N), Y3PRIME(N)
Double precision::Ec, Kh, m1, M2, D11, D22, D12, D66, R, V0
Double precision::A, DStar, C1, C2, dt, Time1, Force1, Def
Double precision::Pi, Sy, AlphaY, Ky, AlphaM, Temp
Double Precision::Lame1, Lame2, Beta1, Beta2, Beta3, Theta1, Theta2, Theta3
Double Precision::AlphaY1, Ttotal, Indent, Velo, TRef, Length, Force, Temp1,

! SPECIFICATIONS FOR SUBROUTINES
CALL UMACH (-2, NOUT)
Call UMACH (1, NIN)
Open (NIN, file = 'CaseI-Plastic.in', FORM='FORMATTED', ACCESS='SEQUENTIAL',
STATUS='NEW')
Open (NOUT, file='CaseI-Plastic.out', FORM='FORMATTED', ACCESS='SEQUENTIAL',
STATUS='OLD')
Read(NIN,*)D11, D22, D12, D66, R, Length
Read(NIN,*)V0, Sy, Ec, m1, M2, dt
Pi = 4.d0*ATAN(1.d0)
Kh = 4.d0*Ec*DSQRT(R)/3.d0
A = (D12+2.d0*D66)/DSQRT(D11*D22)
DStar = (A+1.d0)*DSQRT(D11*D22)/2.d0
C1 = (3.d0/2.d0)*Kh/(8.d0*DSQRT(m1*DStar))
C2 = Kh/M2
Lame1 = (Kh**(2.d0/5.d0))*(V0**(1.d0/5.d0))*(M2**(3.d0/5.d0))/
(8.d0*DSQRT(m1*DStar))
Ttotal = (M2/(Kh*DSQRT(V0)))**(2.d0/5.d0)
AlphaY = 0.68d0*Sy*Pi*Pi*R/(Ec**2)
Ky = 1.5d0*Kh*DSQRT(AlphaY)

126
AlphaY1 = AlphaY/(V0*Ttotal)
Write(*,*)'Alpha = ', AlphaY, AlphaY1
Write(*,*)'Kh & Ky = ', Kh, Ky
Pause
Pi = 4.d0*ATAN(1.d0)
! Calculate the time of reflection from boundaries
TRef = length/(2*DSQRT(Pi)*((D11/m1)**(1.d0/4.d0))*((2*(A+1))**(1.d0/8.d0)))
TRef = TRef**2
write(*,*)'Lame1=', Lame1
write(*,*)'Ttotal=', Ttotal
write(*,*)'TRef =', TRef
write(*,*)'DStar= ', DStar
pause
! Set initial conditions
T1 = 0.d0
Y1(1) = 0.d0
Y1(2) = 1.d0
TOL = 1E-4
! Select absolute error control
PARAM(4) = 20000
PARAM(10) = MABSE !Set absolute error control
PARAM(12) = MBDF !Set BDF method
PARAM(13) = MSOLVE !Select chord method and a divided difference Jacobian
! Print header
WRITE (NOUT,999)
IDO = 1
ISTEP = 0
Def = 0.d0
Write(*,*)'Enter Elastc Loading.............'
Pause
J = 0
Do While(Y1(1).ge.0.d0.AND.(Y1(1)-AlphaY1).lt.TOL)
TEND = ISTEP*dt
Call FCN1 (N, T1, Y1, Y1PRIME)
CALL DIVPRK (IDO, N, FCN1, T1, TEND, TOL, PARAM, Y1)
! CALL DIVPAG (IDO, N, FCN1, FCNJ, Amatr, T1, TEND, TOL, PARAM, Y1)
If(Y1(1).lt.0.d0) Then
Y1(1) = 0.d0
End if
Force = DSQRT(Y1(1)**3.d0)
Force1 = Force*((Kh)**(2.d0/5.d0))*((M2*V0*V0)**(3.d0/5.d0))
! Convert to microseconds
Time1 = Ttotal*T1*1.0E6
! Convert to millimeter
! Def = Def + M2*V0*Force*dt/(8.d0*DSQRT(m1*DStar))*1000.d0
Def = Def + Force1*dt*Ttotal/(8.d0*DSQRT(m1*DStar))*1000.d0
Indent = Ttotal*V0*Y1(1)*1000.d0
Velo = Y1(2)*V0
WRITE (NOUT,'(I6,7F12.5)') ISTEP,  Time1, Y1, Velo, Force1, Indent, Def
! Final call to release workspace
IF (ISTEP .EQ. 20000) IDO = 3
999 FORMAT (4X,'ISTEP', 5X, 'Time', 6X, 'Y1', 9X, 'Y2', 9X, 'Velocity', &
   5X, 'Force1', 6X, 'Indent', 5X, 'Deflection')
ISTEP = ISTEP + 1
J = J + 1
127
IF(ISTEP.eq.2E8) then
    Exit
End IF
END DO
Write(*,*)Y1(1), Y1(2)
Write(*,*)ISTEP=", ISTEP
Write(*,*)Elastic Loading Finished.......'
Write(*,*)Total Number of Time Steps: ', J
Pause
WRITE(*,.91)PARAM(35)
91 FORMAT(4X, 'Number of FCN calls with IVPRK =', F6.0)
IDO = 1
T2 = T1
J = 0
Temp = Y1(2)
Y2(1) = Y1(1)
Y2(2) = Y1(2)
Write(*,*)'Enter Elasto-Plastic Loading.............'
Write(NOUT,*)'Enter Elasto-Plastic Loading.............'
Pause
Beta1 = Ky*Ttotal/(8*DSQRT(m1*DStar))
Beta2 = Ky*(Ttotal**2/M2)
Beta3 = (Kh*(AlphaY**(3.d0/2.d0)) - Ky*AlphaY)*Ttotal/(M2*V0)
Write(*,*)'Beta=', Beta1, Beta2, Beta3
Pause
Do While(Temp.ge.0.d0)
    TEND = ISTEP*dt
    Call FCN2 (N, T2, Y2, Y2PRIME)
    CALL DIVPRK (IDO, N, FCN2, T2, TEND, TOL, PARAM, Y2)
    CALL DIVPAG (IDO, N, FCN2, FCNJ, Amatr, T2, TEND, TOL, PARAM, Y2)
    IF(Y2(1).lt.0.d0) Then
        Y2(1) = 0.d0
    End If
    Force1 = Ky*(Y2(1)*V0*Ttotal-AlphaY) + Kh*DSQRT(AlphaY**3)
    Time1 = Ttotal*T2*1.0E6
    Indent = Ttotal*V0*Y2(1)*1000.d0
    Velo = Y2(2)*V0
    COR = COR + Ttotal*dt*Force1
    Def = Def + Ttotal*dt*Force1/(8.d0*DSQRT(m1*DStar))*1000.d0
    WRITE (NOUT,'(I6,7F12.5)') ISTEP,  Time1, Y2, Velo, Force1, Indent, Def
    Final call to release workspace
    IF (ISTEP .EQ. 2E8) IDO = 3
    ISTEP = ISTEP + 1
    J = J + 1
    Temp = Y2(2)
End Do
Write(*,*)Y2(1), Y2(2)
Write(*,*)ISTEP=", ISTEP
Write(*,*)Elasto-plastic Loading Finished.......'
Write(*,*)Total Number of Time Steps: ', J
Pause
WRITE(*,191)PARAM(35)
191 FORMAT(4X, 'Number of FCN calls with IVPRK =', F6.0)
Write(*,*)COR till maximum indentation = ',COR
! IDO = 1
AlphaM = Y2(1)*(V0*Ttotal)
Write(*,*)'AlphaY & AlphaM =', AlphaY, AlphaM
Write(*,*)'T2 = ', T2
Pause
T3 = T2
J = 0
Y3(1) = Y2(1)
Y3(2) = Y2(2)
Write(*,*)'Enter Elastic Unloading........'
Write(NOUT,*)'Enter Elastic Unloading........'
Pause
Theta1 = Kh*DSQRT(V0)*(Ttotal**(3.d0/2.d0))/(8*DSQRT(m1*DStar))
Theta2 = Kh*DSQRT(V0)*(Ttotal**(5.d0/2.d0))/M2
Theta3 = (Kh*(AlphaY**(3.d0/2.d0)-AlphaM**(3.d0/2.d0)) & + Ky*(AlphaM-AlphaY))*Ttotal/(M2*V0)
Write(*,*)'Theta=', Theta1, Theta2, Theta3
Pause
Do While(Force1.gt.0.d0)
  TEND = ISTEP*dt
  Write(*,*)'Y3=', Y3(1), Y3(2)
  Pause
  Call FCN3 (N, T3, Y3, Y3PRIME)
  ! CALL DIVPRK (IDO, N, FCN3, T3, TEND, TOL, PARAM, Y3)
  CALL DIVPAG (IDO, N, FCN3, FCNJ, Amatr, T3, TEND, TOL, PARAM, Y3)
  IF(Y3(1).lt.0.d0) Then
    Y3(1) = 0.d0
  End If
  Force1 = Kh*(AlphaY**(3.d0/2.d0)-AlphaM**(3.d0/2.d0)) & + Kh*((Y3(1)*V0*Ttotal)**(3.d0/2.d0)) + Ky*(AlphaM-AlphaY)
  Time1 = Ttotal*T3*1.0E6
  Indent = Ttotal*V0*Y3(1)*1000.d0
  Velo = Y3(2)*V0
  Def = Def + Ttotal*dt*Force1/(8.d0*DSQRT(m1*DStar))*1000.d0
  Temp1 = Kh*(AlphaY**(3.d0/2.d0)-AlphaM**(3.d0/2.d0))
  Write(*,*)'Temp1= ', Temp1
  Pause
  WRITE (NOUT,'(I6,7F12.5)') ISTEP,  Time1, Y3, Velo, Force1, Indent, Def
End Do
Write(*,*)Y3(1), Y3(2)
Write(*,*)ISTEP=', ISTEP
Write(*,*)'Elastic unloading Finished.......'
Write(*,*)'Total Number of Time Steps: ', J
Pause
WRITE(*,291)PARAM(35)
291  FORMAT(4X, 'Number of FCN calls with IVPRK =', F6.0)
END Program PlateImpact_Plastic
SUBROUTINE FCN1(N, T1, Y1, Y1PRIME)
    INTEGER::N
    Double precision::T1, Y1(N), Y1PRIME(N)
    Double precision::Lame1
    Common/Data1/Lame1
    Y1PRIME(1) = Y1(2)
    Y1PRIME(2) = -3.d0*Lame1*DSQRT(Y1(1))*Y1(2)/2.d0 &
    - DSQRT(Y1(1)**3.d0)
    RETURN
END

SUBROUTINE FCN2(N, T2, Y2, Y2PRIME)
    INTEGER::N
    Double precision::T2, Y2(N), Y2PRIME(N)
    Double precision::Beta1, Beta2, Beta3
    Common/Data2/Beta1, Beta2, Beta3
    Y2PRIME(1) = Y2(2)
    Y2PRIME(2) = -Beta1*Y2(2) - Beta2*Y2(1) - Beta3
    RETURN
END

SUBROUTINE FCN3(N, T3, Y3, Y3PRIME)
    INTEGER::N3
    Double precision::T3, Y3(N), Y3PRIME(N)
    Common/Data3/Theta1, Theta2, Theta3
    Y3PRIME(1) = Y3(2)
    Y3PRIME(2) = -3.d0*Theta1*DSQRT(Y3(1))*Y3(2)/2.d0 &
    - DSQRT(Y3(1)**3.d0)*Theta2 - Theta3
    RETURN
END

SUBROUTINE FCNJ(N, T3, Y3, DYPDY)
    ! This routine is not called here
    INTEGER::N3
    Double precision::T3, Y3(N), DYPDY(N,*)
    RETURN
END