ADAPTIVE SLIDING MODE CONTROL WITH APPLICATION TO A MEMS VIBRATORY GYROSCOPE

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ABSTRACT

Gyroscopes are commonly used sensors for measuring angular velocity in many areas of applications such as navigation, homing, and control stabilization. Vibratory gyroscopes are the devices that transfer energy from one axis to the other through Coriolis forces. Fabrication imperfections result in some cross stiffness and cross damping effects that may hinder the measurement of angular velocity of MEMS gyroscope. Other noise sources such as thermal, mechanical noise also affect the performance. The angular velocity measurement and minimization of the cross coupling between two axes are challenging problems in the control of vibrating gyroscopes.

This dissertation develops adaptive sliding mode control strategies for a MEMS z-axis gyroscope. The proposed adaptive sliding mode controllers for MEMS z-axis gyroscope make real-time estimates of the angular velocity as well as all unknown gyroscope parameters including coupling stiffness and damping parameters. Therefore, fabrication imperfection and time varying noise and disturbance can be compensated for. These estimates are updated using the tracking error between the reference model trajectory and mass’ real trajectory. The reference model trajectory is designed to satisfy the persistence of excitation condition to enable parameter estimates to converge to their true values. The indirect adaptive sliding mode controller and direct adaptive sliding mode controller with proportional and integral sliding surface are proposed for MEMS
gyroscope. In the presence of unmeasured velocity states, an adaptive sliding mode controller with a sliding mode observer that can reconstruct the unmeasured states is developed to estimate the angular velocity and the linear damping and stiffness coefficients of the gyroscope in real time despite parameter variations and external disturbance. Moreover, the adaptive sliding mode control for two axes angular sensor is extended to triaxial angular sensor and a novel concept for an adaptively controlled triaxial angular velocity sensor device that is able to detect rotation in the three orthogonal axes, using a single vibrating mass is proposed. The numerical simulations of MEMS gyroscope show the effectiveness of all the proposed adaptive sliding mode control schemes. It is shown that the proposed adaptive sliding mode control schemes offer several advantages such as consistent estimates of gyroscope parameters including angular velocity and large robustness to parameter variations and disturbance.
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CHAPTER I
INTRODUCTION

1.1 Introduction of MEMS Gyroscope

Gyroscopes are commonly used sensors for measuring angular velocity in many areas of applications such as stability and navigation control in spacecraft and airplane, rollover detection for automotive, consumer electronics and robotics, etc. Gyroscopes are the devices that transfer energy from one axis to other axis through Coriolis acceleration. The operation principle of the vast majority of all existing micromachined vibratory gyroscopes relies on the generation of a sinusoidal Coriolis force due to the combination of vibration of a proof-mass and an orthogonal angular-rate input. The proof mass is generally suspended above the substrate by a suspension system consisting of flexible beams. The overall dynamical system is a typical two degrees-of freedom (2-DOF) mass-spring-damper system, where the rotation-induced Coriolis force causes the energy transfer to the sense-mode proportional to the angular velocity input.

Micromachined gyroscopes can be a potential alternative to expensive and bulky conventional inertial sensors. High-performance gyroscopic sensors including precision fiber-optic gyroscopes, ring laser gyroscopes, and conventional rotating wheel gyroscopes are too expensive and too large for use in most emerging applications. With micromachining process, micro-sized gyroscope will provide high accuracy rotation
measurements. Thus, miniaturization of vibratory gyroscopes with innovative microfabrication processes and gyroscope designs is expected to become an attractive solution to current inertial sensing market needs, as well as open new market opportunities.

Recent advances in micro-machining technology have made the design and fabrication of MEMS gyroscopes possible. These devices are several orders of magnitude smaller than conventional mechanical gyroscopes, and can be fabricated in large quantities by batch processes. With their dramatically reduced cost, size, and weight, MEMS gyroscopes potentially have a wide application spectrum in the aerospace industry, military, automotive industry and consumer electronics market. The automotive industry applications are diverse, including high performance navigation and guidance systems, ride stabilization, advanced automotive safety systems like yaw and tilt control, roll-over detection and prevention, GPS augmentation such as MEMS inertial navigation sensor embedded GPS and next generation airbag and anti-lock brake systems. The military applications are micro airplanes and satellite controls etc. A wide range of consumer electronics applications include image stabilization in video cameras, video games, virtual reality products, inertial pointing devices, 3D mouse and camcorder image stabilization and computer gaming industry.

1.2 Original Contribution of the Dissertation

The design and fabrication of MEMS gyroscopes have been the subject of extensive research over the past few years. Moreover, advances in the fabrication techniques allow the detection and control electronics to be integrated on the same silicon chip together with the mechanical sensor elements. The cost of MEMS gyroscopes is decreasing while
their accuracy is continuously being improved. There are certain factors which may affect the performance of gyroscopes; these are known as the coupling parameters. Coupling parameters are the terms which exist in the drive and sense motion equations in the form of stiffness and damping terms multiplied by position and velocity states, thereby relating the drive and sense motion equations. Coupled damping parameters are usually small and are not so important compared to the coupled stiffness terms.

Most angular velocity sensors employ symmetric structures to reduce any asymmetries in the device dynamics. However, due to the limitations of fabrication, it is unlikely that the principle stiffness axes or damping axes will be perfectly aligned with the geometric axes of device. The asymmetric stiffness terms may arise from such things that as the centre of mass may not coincide perfectly with the geometric centre or the supporting springs may have unequal stiffness. The quadrature error forces are proportional to the mass displacement and are therefore 90 degree out of phase (quadrature phase) with the Coriolis forces which are proportional to the mass velocity. The quadrature phase makes asymmetric stiffness forces distinguishable from Coriolis forces. Like asymmetric stiffness, asymmetric (or cross) damping arises from fabrication imperfections. It is the misalignment of the principle damping axes from the geometric axes of the device. It also causes erroneous forces to act on the mass, however they are in phase with the Coriolis force since they are both proportional to the mass's velocity. By demodulating with respect to both a sine and a cosine signal the quadrature error can be distinguished from the combined Coriolis and asymmetric damping signals.
In most of the reported micromachined vibratory rate gyroscopes, the proof mass is driven into resonance in the drive direction by an external sinusoidal electrostatic or electromagnetic force. When the gyroscope is subjected to an angular rotation, a sinusoidal Coriolis force is induced in the direction orthogonal to the drive-mode oscillation at the driving frequency. Ideally, it is desired to utilize resonance in both the drive and the sense modes in order to attain the maximum possible response gain, and hence sensitivity. In the conventional micromachined rate gyroscopes, the mode-matching requirement renders the system response very sensitive to variations in system parameters due to fabrication imperfections and fluctuations in operating conditions. Inevitable fabrication imperfections affect both the geometry and the material properties of MEMS devices, and shift the drive and sense-mode resonant frequencies. The dynamical system characteristics are observed to deviate drastically from the designed values and also from device to device, due to slight variations in photolithography steps, etching processes, deposition conditions or residual stresses.

Conventional gyroscopes based on exact or close matching of the drive and sense modes are extremely sensitive to variations in oscillatory system parameters that shift the natural frequencies and introduce quadrature errors, and require compensation by advanced control architectures. The motion of a conventional mode-matched z-axis gyroscope does not satisfy the condition of persistent excitation and, all major fabrication imperfections cannot be identified and compensated for in real time.

Fabrication imperfections always result in some cross stiffness and cross damping effects. The angular velocity measurement and minimization of the cross coupling between two axes are challenging problems in the control of MEMS gyroscopes.
Since the asymmetric stiffness and damping can be time varying, an improvement in performance can be achieved if it is estimated and compensated for using real-time control strategies. This dissertation proposes an adaptive controller that allows equal movement in the drive and sense axes and the controller driving both axes. A series of adaptive sliding mode controllers to estimate the angular velocity and all unknown gyroscope parameters are developed. The angular velocity and all unknown gyroscope parameters including coupling damping and stiffness parameter are consistently estimated. The indirect adaptive sliding mode controller and direct adaptive sliding mode controller with proportional and integral sliding mode surface are developed. The proposed direct adaptive sliding mode controller with PI surface differs from the previous sliding mode techniques in the sense that the sliding surface is based on the proportional and integral sliding mode control. In the presence of the unmeasured states, this dissertation proposes an adaptive sliding mode controller with PI sliding surface in conjunction with a sliding mode observer which reconstructs estimates of the internal unmeasured states. The closed-loop stability of the adaptive sliding mode controller with a sliding mode observer is established. The proposed adaptive sliding mode controller can estimate in real-time the angular velocity and all the gyroscope parameters in the presence of unmeasured states. The contribution of this work comes as the development and design of a robust sliding mode observer which is integrated into an adaptive variable structure controller for a MEMS gyroscope. Moreover, adaptive sliding mode control for two axes angular sensor is extended to triaxial angular sensor and a novel concept for an adaptively controlled triaxial angular velocity sensor device that is able to detect rotation in three orthogonal axes, using a single vibrating mass is proposed.
1.3 Outline of the Dissertation

The outline of dissertation is organized as follows:

Chapter 1 is an introduction to MEMS gyroscopes and briefly points out the major contribution of this dissertation.

Chapter 2 describes the fundamentals and dynamics of MEMS gyroscopes. The non-dimensional version of the dynamics of MEMS gyroscope is derived.

Chapter 3 gives the literature review of the control of MEMS gyroscope such as open-loop and closed-loop modes, as well as control techniques to enhance the performance of these modes of operation.

Chapter 4 proposes indirect adaptive sliding mode control of a MEMS gyroscope.

Chapter 5 develops a general direct adaptive sliding mode control with proportional and integral sliding surface for a linear time in-varying system. Both single input and output inputs cases are discussed.

Chapter 6 derives direct adaptive sliding mode control with proportional and integral sliding surface of a MEMS vibratory gyroscope.

Chapter 7 presents a sliding mode observer-based adaptive sliding mode control algorithm which avoids the direct measurement of the velocity of the proof mass.

Chapter 8 extends adaptive sliding mode control from two axes angular velocity sensor to triaxial angular velocity sensor and proposes a novel concept for an adaptively controlled triaxial angular velocity sensor device.

Chapter 9 summarizes the major achievements of this dissertation and suggests future research directions.
2.1 MEMS Vibratory Angular velocity Sensors

Prior to the emergence of the MEMS angular velocity sensors, the conventional devices for measuring angular velocity are the spinning mass gyroscope, the ring laser gyroscope and the fibre optic gyroscope. Recently much research effort has gone into MEMS angular velocity sensing and the performance of MEMS angular velocity sensors has improved by an order of magnitude every two years.

While there are many different structural configurations that have been used for MEMS angular velocity sensing including vibratory masses, rings, stars, tuning forks, posts, beams and butterflies, they all operate using the same basic principle. The structure is driven into oscillation in a primary mode of vibration and when the device is subjected to an angular velocity, energy from the primary mode is transferred to a secondary mode causing it to oscillate. This transfer of energy to the secondary mode is due to the Coriolis effects and is indicative of the angular velocity input.

The MEMS vibratory gyroscope is shown in Figure 2.1 and Figure 2.2 including drive axis displacement, the angular velocity input and the resulting sensing axis motion. The physical structure consists of a two degree of freedom (DOF) spring mass damper system. The proof mass is driven into oscillation in one axis and the mass' displacement
is sensed in a perpendicular axis. This perpendicular vibration is caused by a transfer of energy from the primary to the secondary vibration axis, through rotation induced Coriolis force acting on the mass. Typically for MEMS devices the motion in the sense axis is an order of magnitude smaller than that of the drive axis. Next we will discuss the physical structure and mechanical and electrical design of MEMS sensing and actuation.

Figure 2.1: Schematic diagram of angular velocity sensor

Figure 2.2: Vibratory MEMS gyroscope
2.2. Mechanical Design.

All existing micromachined vibratory gyroscopes operate on the principle of detecting the sinusoidal Coriolis force induced on a vibrating proof-mass in the presence of an angular-velocity input. Since the induced Coriolis force is orthogonal to the drive-mode vibration, the proof-mass is required to be free to oscillate in two orthogonal directions, and is desired to be constrained in other vibrational modes. The proof-mass is generally suspended above the substrate by a suspension system consisting of thin flexible beams, usually formed in the same structural layer as the proof-mass. The guided-end cantilever beam is shown in Fig. 2.3 [83].

![Guided-end cantilever beam](image)

The translational stiffness in the orthogonal direction to the beam axis is given as

\[ k = \frac{Etw^3}{L^3} \]  

where \( E \) is the Young’s Modulus, \( t \) the beam length, thickness, and width are \( L \), \( t \), and \( w \), respectively. The spring constants are determined with the assumption that the axial strains in the other beams are negligible.
The major damping mechanism in the gyroscope structure is the viscous effects of the air surrounding the vibratory structure which are confined between the proof mass surfaces and the stationary surfaces. The damping of the structural material is usually orders of magnitude lower than the viscous damping, and is generally neglected. The resulting damping in the gyroscope is dominated by the internal friction of the air between the proof-mass and the substrate, and between the comb-drive and sense capacitor fingers. These viscous damping effects can be captured by using two general damping models: Couette flow damping and squeeze film damping shown in Figs 2.4-2.5 [83].

Couette flow damping occurs when two plates of an area $A$, separated by a distance $y_0$, slide parallel to each other. The Couette flow damping coefficient can be approximated as

$$C_{\text{Couette}} = \mu_p p \frac{A}{y_0}$$  \hspace{1cm} (2.2)

where $\mu_p$ is the viscosity constant for air, $p$ is the air pressure, $A$ is the overlap area of the parallel plate and $d$ is the plate distance.

Squeeze film damping occurs when two parallel plates approach each other and squeeze the fluid film inside. Squeeze film damping effects are more complicated, and can exhibit both damping and stiffness effects depending on the compressibility of the fluid. Using the Hagen-Poiseuille law, squeeze film damping can be modeled as

$$C_{\text{Squeeze}} = \mu_p p \frac{7A z_0^2}{y_0^3}$$  \hspace{1cm} (2.3)

where $z_0$ is the width of the plate.
2.3. Electrical Design

In the micro-domain, capacitive sensing and actuation offer several benefits when compared to other sensing and actuation means (piezoresistive, piezoelectric optical, magnetic, etc.) with their ease of fabrication and integration, good DC response and noise performance, high sensitivity, low drift, and low temperature sensitivity.

The electrostatic actuation and sensing components of micromachined devices can be modeled as a combination of moving parallel-plate capacitors shown in Fig. 2.6 [83]. In the most general case, the capacitance between two parallel plates is expressed as

\[
C = \varepsilon_0 \varepsilon \frac{A}{y_0} = \varepsilon_0 \varepsilon \frac{x_0 z_0}{y_0}
\]  

(2.4)
where $\varepsilon_0 = 8.854 \times 10^{12} \text{ F/m}$, $\varepsilon$ is the dielectric constant, $A = x_0 z_0$ is the total overlap area and $y_0$ is the electrode gap.

Figure 2.6: The capacitance between two plates

In parallel-plate electrodes, the electrostatic force is generated due to the electrostatic conservative force field between the plates. Thus, the force can be expressed as the gradient of the potential energy stored on the capacitor. Considering a voltage $V$ applied, the electrostatic force generated by the capacitor is given by

$$F = \frac{1}{2} \frac{\partial C}{\partial y_0} V^2 = -\frac{1}{2} \frac{\varepsilon_0 \varepsilon_0 z_0}{y_0^2} V^2. \quad (2.5)$$

The bulk-micromachining implementation of parallel-plate actuator is shown in Fig. 2.7 [83].

Fig. 2.7: Bulk-micromachining implementation of parallel-plate actuator
2.4. Dynamics of the MEMS Gyroscope:

This section discusses the dynamics of the MEMS gyroscope. A $z$-axis MEMS gyroscope is depicted in Fig. 2.8. The typical MEMS vibratory gyroscope includes a proof mass suspended by a spring, an electrostatic actuation and sensing mechanism for forcing an oscillatory motion and sensing the position of the proof mass. The proof mass is suspended with the help of four springs that are fixed to the gyroscope table. As such the proof mass is free to move in $x$, $y$ directions that are parallel and perpendicular to the gyroscope table respectively. In this design the external force is the electrostatic force. Coriolis force acting in the $y$ direction is generated whenever the proof mass undergoes rotation about the $z$ axis.

The conventional mode of operation reduces to driving one of the modes of the gyroscope into a known oscillatory motion and then detecting the Coriolis acceleration coupling along the sense axis, which is orthogonal to the driven mode. The response of the sense mode of vibration provides information about the applied angular velocity. More specifically, the proof mass is driven into a constant amplitude oscillatory motion along the $x$-axis (drive axis) by the $x$-axis control $u_x$. When the gyroscope is subjected to an angular rotation, a Coriolis force is generated along the $y$-axis (sense axis), whose magnitude is proportional to the oscillation velocity of the drive axis and the magnitude of $z$-axis component of angular velocity $\Omega_z$. This force excites the proof mass into an oscillatory motion along the $y$-axis.

The inertial frame and rotating frame is shown in Fig. 2.9. In the Fig. 2.9, $\vec{r}_A$ is position vector of inertial frame $A$, $\vec{r}_{AB}$ is relative position vector of inertial frame $A$
with respect to rotating frame \( B \), \( \vec{r}_b \) is the position vector of rotating frame \( B \). Their relation can be expressed as
\[
\vec{r}_A = \vec{r}_B + \vec{r}_{/B}
\]
\[
= \vec{r}_B + x i + y j + z k.
\] (2.6)

We assume that:

(i). The table where the proof mass is mounted is moving with a constant velocity;

(ii). The gyroscope is rotating at a constant angular velocity \( \Omega_z \) over a sufficiently long time interval;

(iii). The centrifugal forces \( m\Omega_z^2 x, m\Omega_z^2 y \) are assumed to be negligible;

(iv). Gyroscope undergoes rotation about the \( z \) axis only, and therefore the Coriolis forces are generated in a direction perpendicular to the drive and rotation axes.

The velocity vector of the point vector with respect to the inertial frame can be derived as
\[
\vec{v}_A = \vec{v}_B + \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k + x \frac{di}{dt} + y \frac{dj}{dt} + z \frac{dk}{dt}.
\] (2.7)

Since \( \frac{di}{dt} = \Omega_z \times i, \frac{dj}{dt} = \Omega_z \times j, \frac{dk}{dt} = \Omega_z \times k \), substituting them into (2.7) yields
\[
\vec{v}_A = \vec{v}_B + \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k + x(\Omega_z \times i) + y(\Omega_z \times j) + z(\Omega_z \times k)
\]
\[
= \vec{v}_B + \vec{v}_{/A} + \Omega_z \times \vec{r}_{/B}
\] (2.8)

where: \( \vec{v}_{/A} \) and \( \vec{v}_B \) are the velocities with respect to the inertial frame,
\[
\vec{v}_{/B} = \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k
\] is the relative velocity vector of inertial frame \( A \) with respect to rotating frame \( B \).
The property \( \frac{d(\Omega \times \vec{r}_{\gamma/\beta})}{dt} = \frac{d\Omega}{dt} \times \vec{r}_{\gamma/\beta} + \Omega \times \vec{v}_{\gamma/\beta} + \vec{\Omega} \times \Omega \times \vec{r}_{\gamma/\beta} \) can be proved as

\[
\frac{d(\Omega \times \vec{r}_{\gamma/\beta})}{dt} = \frac{d\Omega}{dt} \times (xi + yj + zk) + \Omega \times \left( \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k + x \frac{di}{dt} + y \frac{dj}{dt} + z \frac{dk}{dt} \right)
\]

\[
= \frac{d\Omega}{dt} \times \vec{r}_{\gamma/\beta} + \Omega \times \vec{v}_{\gamma/\beta} + \Omega \times \vec{\Omega} \times \vec{r}_{\gamma/\beta}
\]

(2.9)

The property \( \frac{d\vec{v}_{\gamma/\beta}}{dt} = \vec{a}_{\gamma/\beta} + \Omega \times \vec{v}_{\gamma/\beta} \) can be proved as

\[
\frac{d\vec{v}_{\gamma/\beta}}{dt} = \frac{d}{dt} \left( \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k \right)
\]

\[
= \frac{d^2x}{dt^2}i + \frac{d^2y}{dt^2}j + \frac{d^2z}{dt^2}k + \frac{dx}{dt} \frac{di}{dt} + \frac{dy}{dt} \frac{dj}{dt} + \frac{dz}{dt} \frac{dk}{dt}
\]

\[
= \frac{d^2x}{dt^2}i + \frac{d^2y}{dt^2}j + \frac{d^2z}{dt^2}k + \Omega \times \left( \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k \right)
\]

(2.10)

\[
= \frac{d^2x}{dt^2}i + \frac{d^2y}{dt^2}j + \frac{d^2z}{dt^2}k + \Omega \times \vec{v}_{\gamma/\beta}
\]

Differentiating (2.8) and using (2.9) and (2.10) yields

\[
\vec{a}_A = \vec{a}_B + \vec{a}_{\gamma/\beta} + 2\Omega \times \vec{v}_{\gamma/\beta} + \frac{d\Omega}{dt} \times \vec{r}_{\gamma/\beta} + \Omega \times (\vec{\Omega} \times \vec{r}_{\gamma/\beta})
\]

(2.11)

where: \( \vec{a}_A \), \( \vec{a}_B \), are the accelerations with respect to the inertial frame,

\[
\vec{a}_{\gamma/\beta} = \frac{d^2x}{dt^2}i + \frac{d^2y}{dt^2}j + \frac{d^2z}{dt^2}k
\]

is the relative acceleration vector of inertial frame \( A \) with respect to rotating frame \( B \).

Multiplying (2.11) by mass \( m \) gives
\[ m\ddot{\mathbf{A}} = m\ddot{\mathbf{B}} + m\ddot{\mathbf{A}}_{/B} + 2m\Omega_z \times \mathbf{v}_{/B} + m \left( \frac{d\Omega}{dt} \times \mathbf{r}_{/B} \right) + m\Omega_z \times (\Omega_z \times \mathbf{r}_{/B}) \]  

(2.12)

where \( 2m\Omega_z \times \mathbf{v}_{/B} \) is the Coriolis force and \( m\Omega_z \times (\Omega_z \times \mathbf{r}_{/B}) \) is the Centrifugal force.

The Coriolis force acting on the proof mass along \( x \) direction is derived as

\[ F_{\text{coriolis}_{-x}} = 2m\Omega_z k \times \dot{y} = 2m\Omega_z \dot{y}i. \]  

(2.13)

The Coriolis force acting on the proof mass along \( y \) direction is derived as

\[ F_{\text{coriolis}_{-y}} = 2m\Omega_z k \times \dot{x} = 2m\Omega_z \dot{x}j . \]  

(2.14)

By using the property of \( k \times (k \times i) = -i \), the Centrifugal force acting on the proof mass along \( x \) direction can be derived as

\[ F_{\text{centrifugal}_{-x}} = m\Omega_z k \times (\Omega_z k \times xi) = -m\Omega_z^2 xi. \]  

(2.15)

By using the property of \( k \times (k \times j) = -j \), the Centrifugal forces acting on the proof mass along \( y \) direction can be derived as

\[ F_{\text{centrifugal}_{-y}} = m\Omega_z k \times (\Omega_z k \times yj) = -m\Omega_z^2 yj. \]  

(2.16)

Therefore from the above analysis and assumptions, the dynamics of gyroscope become

\[ m\ddot{x} + d_{xx}\dot{x} + d_{xy}\dot{y} + k_{xx}x + k_{xy}y = u_x + 2m\Omega_z \dot{y} \]  

(2.17)

\[ m\ddot{y} + d_{xy}\dot{x} + d_{yy}\dot{y} + k_{xy}x + k_{yy}y = u_y - 2m\Omega_z \dot{x} \]  

(2.18)

Fabrication imperfections contribute mainly to the asymmetric spring and damping terms, \( k_{xy} \) and \( d_{xy} \). The \( x \) and \( y \) axes spring and damping terms \( k_{xx}, k_{yy}, d_{xx}, \) and \( d_{yy} \) are mostly known, but have small unknown variations from their nominal values. The mass of proof mass can be determined very accurately, \( u_x \) and \( u_y \) are the control forces in the \( x \)
and y direction.

The technique of nondimensionalizing follows that of [22]. Dividing (2.17) and (2.18) by the reference mass and rewriting the gyroscope dynamics in vector forms result in

\[ \ddot{q} + \frac{D}{m} \dot{q} + \frac{K_a}{m} q = \frac{u}{m} - 2\Omega \dot{q} \]  

(2.19)

where \( q = \begin{bmatrix} x \\ y \end{bmatrix} \), \( u = \begin{bmatrix} u_x \\ u_y \end{bmatrix} \), \( \Omega = \begin{bmatrix} 0 & -\Omega_z \\ \Omega_z & 0 \end{bmatrix} \), \( D = \begin{bmatrix} d_{xx} & d_{xy} \\ d_{xy} & d_{yy} \end{bmatrix} \), \( K_a = \begin{bmatrix} k_{xx} & k_{xy} \\ k_{xy} & k_{yy} \end{bmatrix} \).

Using non-dimensional time \( t^* = w_0 t \), and dividing both sides of (2.19) by reference frequency \( w_0^2 \) and the reference length \( q_0 \) give the final form of the non-dimensional equation of motion as

\[ \frac{\ddot{q}}{q_0} + \frac{D}{m w_0^2 q_0} \frac{\dot{q}}{q_0} + \frac{K_a}{m w_0^2 q_0} q = \frac{u}{m w_0^2 q_0^2} - \frac{2\Omega}{w_0} \frac{\dot{q}}{q_0} \]  

(2.20)

Defining a set of new parameters as follows:

\[ q^* = \frac{q}{q_0}, d_{xy}^* = \frac{d_{xy}}{m w_0^2}, \Omega^* = \frac{\Omega}{w_0}, \]  

(2.21)

\[ u_x^* = \frac{u_x}{m w_0^2 q_0}, u_y^* = \frac{u_y}{m w_0^2 q_0}, \]  

and

\[ w_x = \sqrt{\frac{k_{xx}}{m w_0^2}}, w_y = \sqrt{\frac{k_{yy}}{m w_0^2}}, w_{xy} = \frac{k_{xy}}{m w_0^2}. \]  

(2.22)

Ignoring the superscript (*) for notational clarity, the nondimensional representation of (2.17) and (2.18) is

\[ \ddot{q} + D \dot{q} + K_a q = u - 2\Omega \dot{q} \]  

(2.24)
where $K_b = \begin{bmatrix} w_x^2 & w_{xy} \\ w_{xy} & w_y^2 \end{bmatrix}$.

![Figure 2.8: A simplified model of MEMS z-axis gyroscope](image)

Figure 2.8: A simplified model of MEMS z-axis gyroscope

![Figure 2.9: Inertial frame and rotating frame](image)

Figure 2.9: Inertial frame and rotating frame
CHAPTER III
CONTROL OF A VIBRATORY MEMS GYROSCOPE

The emergence of MEMS technology is opening up new market opportunities and applications in a wide variety of areas. One of the most important emerging fields in MEMS design is the control-methodology-based MEMS. This approach utilizes feedback control to compensate fabrication/design flaws and thus ensure the system robust performance. In this approach, not only a control algorithm is integrated into a system, but also the MEMS structure design is altered accordingly.

Control refers to forcing a system to behave in a desired manner. The integration of control systems and MEMS is becoming more common since VLSI logic circuits can be fabricated on the same substrate as the mechanical elements. The roles that control plays in MEMS is varying, some of these include:

- Providing precise control over MEMS actuation elements as electrostatic plates.
- Improving response time and device input range and accuracy through force-feedback schemes.
- Providing real-time estimates of the unknown parameters of a MEMS system.
- Compensating for the effects of fabrication imperfections.
- Compensating for time varying effects such as the effect of temperature.
This chapter gives an overview of the main approaches and examples of how the control methods have been used for MEMS gyroscope applications.

3.1 Noise and Control

There are some noises such as sensing noise, noise existing in the various angular velocity sensors. Noise can enter at various stages of the position and/or velocity sensing process, from the sensing element itself, any associated sensing circuits and wiring. The most common noise source is parasitic capacitances that form between capacitive sensing elements. Significant improvements have been found by using an insulating substrate such as glass. Phase differential sensing schemes have been shown to be robust against variations in sensing element scale factor [50], [57] and [72]. An alternative tunneling-based sensing element only for angular velocity sensors operating in the force-balancing mode has been employed to improve mass displacement sensitivity [61].

Variations in the temperature of the structure also perturb the dynamical system parameters due to the temperature dependence of Young’s Modulus and thermally induced localized stresses. Thermal mechanical noise appears as a white noise force acting on the vibrating proof mass wherever there is damping. Thermal mechanical noise limits the ultimate achievable resolution of all vibratory angular velocity sensors. A practical correlation filter can greatly reduce thermal noise in gyroscope output signals compared to using a conventional low pass filter [67]. Another approach to de-noising gyroscope signals by Wang and Huang uses wavelet packet analysis [70].
3.2 Open-Loop Operation

An angular velocity sensor that operates in open-loop mode employs no feedback control in either the drive or sense axes. The sensitivity is increased if the drive and sense axes have matched resonant frequencies due to mechanical amplification in the Coriolis axis. Any mismatch in the resonant frequencies between the drive and sense axes can be tuned by applying a force offset to one of the axes. This has the effect of altering its stiffness. Alper and Akin [53] presented an open loop angular velocity sensor with a symmetric structure that provided matched and decoupled resonant modes.

Most angular velocity sensors by Mochida et al [64] and Alper and Akin [53] employ symmetric structures to reduce any asymmetries in the device dynamics. Asymmetric stiffness causes unwanted forces that act on the mass, causing the proof mass to trace a circular path instead of a linear one. This result in a zero rate output (ZRO) called quadrature error [73]. ZRO is an erroneous output signal from the device when it is subject to zero angular velocity input. One way to reduce the ZRO is to employ mismatched modes to uncouple the two axes and better isolate the driving motion from the piezoresistive sensing elements. The other solution to increase bandwidth is to add a negative feedback loop to null the motion of the mass in the sense axis. Acar et. al. [51] use two coupled masses each with two degrees of freedom and separate resonant peaks and operating the device in the flatter and wider operating region between the two peaks for more robust performance. This sort of approach was further developed by Acar and Shkel [52] in which eight interconnected vibrating masses in a ring arrangement are used thereby resulting in a very wide bandwidth that is robust against temperature fluctuations and has small ZRO due to the decoupling nature of the multiple mismatched modes.
Differential phase demodulation has been shown to be insensitive to variations in drive amplitude [50], [57] and [72], however its use is limited to open-loop operation. Oboe et. al [76] describes the design of the control loops in a z-axis, MEMS vibrational gyroscope operating in a vacuum enclosure.

3.3 Force-balancing Control

Past MEMS gyroscope research focused on the development of the micro-electromechanical fabrication processes, sensor modeling, and the active control of the sensor dynamics for model identification and angular velocity sensing (14], [15], and [16]). Some control algorithms have been proposed to control the MEMS gyroscope. In terms of automatic control, force-balancing feedback control scheme using sigma-delta modulation [36] has been proposed for conventional mode of operation.

In force-balancing mode [66], instead of using the sense axis motion resulting from the Coriolis force as output, it is used as an input to a negative feedback loop. This feedback loop acts to null the motion in the sense axis using the control signal. The demodulated feedback force then becomes the output of the device, since it is proportional to the angular velocity input in an ideal system. More complex control strategies are required to identify and compensate for the errors and the angular velocity separately. Force-balancing mode is an integral part of the tunneling based angular velocity sensor proposed by Kenuba et al [61] which uses force-balancing mode to maintain the tunneling gap.
Leland et al. [19] proposed one adaptive controller which estimates both the cross stiffness and input angular velocity and compensates for their effects on the sense axis in a force balancing mode. Dong et al. [84] proposed a novel active disturbance rejection control by an extended state observer to estimate the rotation rate of MEMS gyroscope. Zheng et al. [85] introduced a novel oscillation controller for the drive axis of a MEMS gyroscope that is a traditional PD controller plus a linear extended state observer. Since the controller design does not require exact information of system parameters, it is very robust against structural uncertainties of the drive axis.

Sungsu Park and Horowitz [14] presented adaptive add-on control algorithms for the conventional mode of operation of MEMS z-axis gyroscopes. This scheme is realized by adding an outer loop to a conventional force-balancing scheme that includes a parameter estimation algorithm. The parameter adaptation algorithm estimates the angular velocity, identifies and compensates the quadrature error.

3.4 Adaptive Control

Adaptively controlled systems can be characterized as those that can adapt to changes in the system dynamics over time. The adaptive mode of operation is better suited for medium-cost gyroscopes that are used in high-performance applications.

Leland [16] proposed an adaptive controller for an open loop device that tunes the frequency of vibration of the drive axis to match that of a fixed reference drive signal. This offers advantages over using a Phase locked loop (PLL), which tunes the driving signal frequency to match the resonant frequency because the dynamics of the system are
dictated by the time invariant reference model instead of the natural resonant frequency of the device, which may be time varying. Leland [18] added adaptive control to regulate the drive oscillation amplitude and compensate for quadrature error due to asymmetric stiffness and null the sense axis vibration in force-balancing mode. Leland et. al [17] present an adaptive controller for the drive axis of a vibrational gyroscope. A piezoelectrically driven gyroscope is operated as an oscillator circuit, by using a destabilizing positive feedback with automatic gain control. Recently Leland et. al [19] incorporated time varying input rate in a Lyapunov based adaptive controller which estimates and compensates for both input angular velocity and cross stiffness terms by using a polynomial approximation techniques.

Asymmetric stiffness and damping arise from fabrication imperfections. It is not easy to separate the Coriolis signal from the asymmetric damping signal. Park [22] has shown to be able to distinguish from one another. He used a model reference adaptive control approach. Closkey et. al [63] used off-line lattice-filter based algorithms to estimate high-order linear MIMO models for their cloverleaf vibrating post angular velocity sensor dynamics. All the stiffness terms were successfully identified, however it was impossible to identify the cross damping terms, as they are indistinguishable from the angular velocity terms. Later work on the same cloverleaf design proposed evolutionary optimization computation to tune the mismatch in resonant modes in an off-line fashion [59].

Salah et. al [74] proposes a new adaptive controller to control both axes of a z-axis MEMS gyroscope and to facilitate time varying angular velocity sensing. An off-line adaptive least square estimation strategy is first developed that accurately estimates
the unknown model parameters. An online active controller/observer is then developed for time-varying angular velocity sensing.

Feng et al. [86] present an adaptive estimator-based technique to estimate the angular motion by providing the Coriolis force as the input to the adaptive estimator and to improve the bandwidth of microgyrosopes. Jagannathan et al. [21] presents an adaptive force-balancing control (AFBC) scheme with actuator limits for a MEMS Z-axis gyroscope. The proposed AFBC scheme controls the vibratory modes of the proof mass while ensuring that the control input satisfies the magnitude constraints and the performance of the gyroscope is enhanced in the presence of fabrication uncertainties. Park and Horowitz [24] presented a discrete time version of the observer-based adaptive control system for micro-electro-mechanical systems gyroscopes, which can be implemented using digital processors.

3.5 Adaptive Sliding Mode Control.

Sliding mode control is a robust control technique which has many attractive features such as robustness to parameter variations and insensitivity to disturbances. The sliding mode controller is composed of an equivalent control part that describes the behavior of the system when the trajectories stay on the sliding manifold and a variable structure control part that enforces the trajectories to reach the sliding manifold and prevent them leaving the sliding manifold. It has some limitations such as chattering or high frequency oscillation in practical applications. Adaptive control is an effective approach to handle parameter variations. Adaptive methods are used to automatically
adjust the response of the controller to compensate for changes in the response of the plant. Therefore adaptive sliding mode control has the advantages of combining the robustness of variable structure methods with the tracking capability of adaptive control. The block diagram of an adaptive sliding mode control system is shown in Fig. 3.1. In the figure, sliding mode control is combined with linear controller which is adjusted by adaptive law. The tracking error between plant state and reference model is fed back to the sliding mode controller and adaptive controller.

Utkin [1] [2] introduced the variable structure system and showed that variable structure control is insensitive to parameters perturbations and external disturbances. Narendra et. al [3], Astrom [4], Ioannou and Sun [5], Tao [6] described the model reference adaptive control. In the last few years, many applications have been developed using sliding mode control and adaptive control. The adaptive control law, merging parameter identification and sliding mode was proposed and analytically studied by Fradkov and Andrievsky [79-82]. Lee et. al [7] developed a variable structure augmented adaptive controller for a gyro platform. Wang et. al [8] presented an adaptive sliding mode controller for a microgravity isolation system. Song et. al [9] proposed a smooth robust compensator. Sam et. al [10] presented a class of proportional and integral sliding mode control with application to active suspension system. Chou et. al [11] and Lin et. al [12] proposed an integral sliding surface and derived an adaptive law to estimate the upper bound of uncertainties. Hsu et. al [13] developed a design of input/output based variable structure adaptive control. Some adaptive laws to estimate the upper bound of uncertainties and disturbance are discussed in [37-49].
Batur et al [20] developed a sliding mode control for a MEMS gyroscope system. A model reference adaptive feedback controller and a sliding mode controller have been considered to guarantee the stability of the MEMS device. Park [22] proposed an adaptive controller for a two axes driven MEMS gyroscope which drive both axes of vibration and controls the entire operation of the gyroscope. Park and Horowitz [23] proposed an adaptive control scheme which estimates the component of the angular velocity vector and compensates for friction forces, and fabrication imperfections. John et al [58] extends the Park’s method [22] to triaxial angular sensors and presents a novel concept for an adaptively controlled triaxial angular sensor.

![Block diagram of adaptive sliding mode control system](image-url)

Figure 3.1: Block diagram of adaptive sliding mode control system
A surface-micromachined dual axis gyroscope based on rotational resonance of a 2 \( \mu \)m thick polysilicon rotor disk has been reported in [22]. Since the disk is symmetric in two orthogonal axes, the sensor can sense rotation equally about two axes. This is suitable for an adaptive mode of operation, which allows equal movements in the drive and sense axes and drives both axes.

3.6. Sliding Mode Observer

In practical MEMS gyroscopes the axial velocities may be unmeasurable. The use of velocity sensors may add to the cost, size and weight of the vibrating platform. It is necessary to design observer to estimate the unmeasured states. A high-gain observer was investigated by Esfandiari and Khalil [25] for the design of output feedback controller due to its ability to robustly estimate the unmeasured states while asymptotically attenuating disturbances. Krstic et al. [26] proposed an adaptive backstepping nonlinear observer for a nonlinear system. The basic sliding mode observer structure consists of switching terms added to a conventional Luenberger observer. The sliding mode design method enhances robustness over a range of system uncertainties and disturbances. Slotine et al.[27] and Walcott and Zak [28] designed sliding mode observers with additional switching function to address plant uncertainties using the Lyapunov stability theorem. Sliding mode controller and observer for microgravity isolation system was investigated in [29]. Batur et al.[30] proposed a sliding mode observer and controller design for a hydraulic motion control system. Edwards and Spurgeon [31-33] described the sliding mode observer theory and application examples. A class of nonlinear extended state observers was proposed by J. Han [34] as a unique observer design. It is rather
independent of the mathematical model of the plant, thus achieving inherent robustness. A number of robust state observers have the form of standard observer plus a strengthening function to reflect the unknown perturbation such as model uncertainty and external disturbance. Moura et al. [35] suggested a function to estimate plant uncertainties and added it to the state estimator.

3.7 Proposed Adaptive Sliding Mode Controller for MEMS Gyroscope

The motivation of this dissertation is to propose a novel sliding mode adaptive controller to estimate the angular velocity and all gyroscope parameters. Our controller is different from the ones proposed in [22-23] in that a sliding mode control algorithm is incorporated into the adaptive control system and the feasibility of an adaptive sliding mode control with a sliding mode observer in the presence of the model uncertainty and external disturbance is investigated. The proposed adaptive sliding mode controllers in this dissertation are as follows:

1. Indirect adaptive sliding mode controller for a MEMS gyroscope.
2. Direct adaptive sliding mode control with PI sliding surface for general system.
3. Direct adaptive sliding mode control with PI sliding surface for a MEMS gyroscope.
4. Sliding mode observer based adaptive sliding mode control for a MEMS gyroscope.
5. Adaptive sliding mode control for a triaxial angular velocity sensor.
CHAPTER IV

INDIRECT ADAPTIVE SLIDING MODE CONTROL FOR A MEMS GYROSCOPE

This chapter presents a new sliding mode adaptive controller for MEMS z-axis gyroscope. The proposed adaptive sliding mode control algorithm can estimate the angular velocity and the linear damping and stiffness coefficients in real time. The stability of the closed-loop system can be established with the proposed control strategy. The numerical simulation for MEMS gyroscope is investigated to verify the effectiveness of the proposed adaptive sliding mode control scheme. It is shown that the proposed adaptive sliding mode control scheme offers several advantages such as real-time estimation of gyroscope parameters including angular velocity and large robustness to parameter variations and external disturbance.

4.1 Problem Statement

This section proposes an adaptive sliding mode control strategy for MEMS gyroscopes. The control target is to achieve real-time compensation for fabrication imperfections and closed-loop identification of the angular velocity. The block diagram of an indirect adaptive sliding mode control for a MEMS gyroscope is shown in Fig. 4.1, the tracking error between reference state and gyroscope state comes to the indirect
adaptive sliding mode controller. The adaptive sliding mode controlled is proposed to control the MEMS gyroscope. Angular velocity can be estimated by adaptive estimator.

Referring to (2.24), we consider the dynamics with parametric uncertainties and external disturbance as

\[ \ddot{q} + (D + 2\Omega + \Delta D)\dot{q} + (K_b + \Delta K_b)q = u + d \] (4.1)

where \( \Delta D \) is the unknown parameter uncertainties of the matrix \( D + 2\Omega \), \( \Delta K_b \) is the unknown parameter uncertainties of the matrix \( K_b \), \( d \) is an uncertain extraneous disturbance and/or unknown nonlinearity of the system.

Rewriting (4.1) as

\[ \ddot{q} + (D + 2\Omega)\dot{q} + K_b q = u + f \] (4.2)

where \( f \) represents the matched lumped uncertainty and disturbance which is given by

\[ f = d - \Delta D\dot{q} - \Delta K_b q . \] (4.3)

![Figure 4.1: Block diagram of an indirect adaptive sliding mode control for a MEMS gyroscope](image)
We make the following assumption:

The lumped uncertainty and disturbance $f$ is bounded such as $\|f\| \leq \alpha_1 \|q\| + \alpha_2 \|\dot{q}\| + \alpha_3$, where $\alpha_1$, $\alpha_2$ and $\alpha_3$ are known positive constants.

4.2. Adaptive Sliding Mode Controller Design

Suppose that a reference trajectory is generated by an ideal oscillator and the control objective is to make the trajectory of the gyroscopes to follow that of the reference model. The reference model is defined as

$$\ddot{q}_m + K_m q_m = 0$$

where $K_m = \text{diag}\{w_1^2, w_2^2\}$.

The tracking error is defined as

$$e = q - q_m.$$  \hfill (4.5)

The sliding surface is defined as

$$s(t) = \dot{e} + \lambda e$$

where $\lambda$ is a positive definite constant matrix to be selected, i.e. $\lambda = \text{diag}\{\lambda_1, \lambda_2\}$.

The derivative of the sliding surface is

$$\dot{s} = \ddot{e} + \lambda \dot{e}$$

$$= \ddot{q} - \ddot{q}_m + \lambda (\dot{q} - \dot{q}_m)$$

$$= u + f - (D + 2\Omega)\dot{q} - K_b q + \lambda (\dot{q} - \dot{q}_m) + K_m q_m.$$  \hfill (4.7)

Substituting $D = \begin{bmatrix} d_{xx} & d_{xy} \\ d_{yx} & d_{yy} \end{bmatrix}$, $\Omega = \begin{bmatrix} 0 & -\Omega_x \\ \Omega_x & 0 \end{bmatrix}$ and $K_b = \begin{bmatrix} w_x^2 & w_{xy} \\ w_{yx} & w_y^2 \end{bmatrix}$ into (4.7) yields
\[
\dot{s} = u + f - \left[ \begin{array}{ccc}
    d_{xx} & d_{xy} & -2\Omega_z \\
    d_{xy} + 2\Omega_z & d_{yy} & 0 \\
    w_x^2 & w_{xy} & 0 \\
\end{array} \right] \left[ \begin{array}{c}
    \dot{q}_1 \\
    \dot{q}_2 \\
    \dot{q}_1 \\
\end{array} \right] - \left[ \begin{array}{ccc}
    w_y^2 & w_{xy} & 0 \\
\end{array} \right] \left[ \begin{array}{c}
    q_1 \\
    q_2 \\
\end{array} \right] + \lambda(\dot{q} - \dot{q}_m) + K_m q_m. \tag{4.8}
\]

Rewriting (4.8) yields
\[
\dot{s} = u + f - \left[ \begin{array}{cccc}
    \dot{q}_1 & \dot{q}_2 & 0 & -2\dot{q}_2 \\
    0 & \dot{q}_1 & \dot{q}_2 & 2\dot{q}_1 \\
    0 & q_1 & q_2 & 0 \\
\end{array} \right] \left[ \begin{array}{c}
    d_{xx} \\
    d_{xy} \\
    d_{yy} \\
    \Omega_z \\
    w_x^2 \\
    w_{xy} \\
    w_y^2 \\
\end{array} \right] + \lambda(\dot{q} - \dot{q}_m) + K_m q_m. \tag{4.9}
\]

Defining
\[
Y = \left[ \begin{array}{cccc}
    \dot{q}_1 & \dot{q}_2 & 0 & -2\dot{q}_2 \\
    0 & \dot{q}_1 & \dot{q}_2 & 2\dot{q}_1 \\
    0 & q_1 & q_2 & 0 \\
\end{array} \right], \tag{4.10}
\]

\[
\theta^* = \left[ \begin{array}{cccc}
    d_{xx} & d_{xy} & d_{yy} & \Omega_z \\
    w_x^2 & w_{xy} & w_y^2 \\
\end{array} \right]^T, \text{ and } Q = \lambda(\dot{q} - \dot{q}_m) + K_m q_m \tag{4.11}
\]

Then, (4.9) becomes
\[
\dot{s} = u + f - Y\theta^* + Q \tag{4.12}
\]

where \(Y(q, \dot{q}, q_d, \dot{q}_d)\) is an \(2 \times 7\) matrix of known functions and \(\theta^*\) contains unknown system parameter. We assume both positions and velocities are measurable.

Setting \(\dot{s} = 0\) to solve equivalent control \(u_{eq}\) gives
\[
\begin{align*}
    u_{eq} &= Y\theta^* - Q - f. \tag{4.13}
\end{align*}
\]

The adaptive controller \(u\) is proposed as
\[
\begin{align*}
    u &= Y\theta - Q + u_s \\
    &= Y\theta - Q - \rho \text{sgn}(s) \tag{4.14}
\end{align*}
\]
where \( u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \), \( s = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \), \( \theta \) is the estimation of \( \theta^* \).

\[
\begin{aligned}
u_s &= \begin{bmatrix} u_{s1} \\ u_{s2} \end{bmatrix} = -\rho \text{sgn}(s) = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix} \begin{pmatrix} \text{sgn}(s_1) \\ \text{sgn}(s_2) \end{pmatrix}
\end{aligned}
\]

is the sliding mode signal.

Substituting (4.14) into (4.12) yields

\[
\dot{s} = Y\tilde{\theta} + f - \rho \text{sgn}(s).
\] (4.15)

where \( \tilde{\theta} = \theta - \theta^* \).

Define a Lyapunov function to analyze the stability of (4.15) as

\[
V = \frac{1}{2} s^T s + \frac{1}{2} \tilde{\theta}^T \tau^{-1} \tilde{\theta
\] (4.16)

where \( \tau = \tau^T \) are positive definite matrix.

Differentiating \( V \) with respect to time yields

\[
\dot{V} = s^T \dot{s} + \tilde{\theta}^T \tau^{-1} \tilde{\theta}
\]

\[
= s^T (Y\theta - Q - Y\theta^* + Q - \rho \text{sgn}(s) + f) + \tilde{\theta}^T \tau^{-1} \tilde{\theta}
\]

\[
= s^T Y\tilde{\theta} - \rho \|s_1\| + \|s_2\| + s^T f + \tilde{\theta}^T \tau^{-1} \tilde{\theta}
\]

\[
= -\rho \|s_1\| + \|s_2\| + s^T f + (s^T Y\tilde{\theta} + \tilde{\theta}^T \tau^{-1} \tilde{\theta}).
\] (4.17)

To make \( \dot{V} \leq 0 \), we choose an adaptive law

\[
\dot{\theta}(t) = \hat{\theta}(t) = -\alpha Y^T s(t)
\] (4.18)

with \( \theta(0) \) being arbitrary. This choice yields

\[
\dot{V} = -\rho \|s_1\| + \|s_2\| + s^T f \leq -\rho \|s\| + \|f\|
\]

\[
\leq -\|s\| (\rho - \alpha_1 \|q\| - \alpha_2 \|\hat{q}\| - \alpha_3) \leq 0.
\] (4.19)

With the choice of \( \rho \geq \alpha_1 \|q\| + \alpha_2 \|\hat{q}\| + \alpha_3 + \eta \), where \( \eta \) is a positive constant, \( \dot{V} \)
becomes negative semi-definite, i.e., $\dot{V} \leq -\eta \|s\|$. This implies that the trajectory reaches the sliding surface in finite time and remains on the sliding surface. $\dot{V}$ is negative definite implies that $s$ and $\tilde{K}$ converge to zero. $\dot{V}$ is negative semi-definite ensures that $V$, $s$ and $\tilde{\theta}$ are all bounded. It can be concluded from (4.15) that $\dot{s}$ is also bounded.

LaSalle’s invariant set theorem can be used to prove that $\lim_{t \to \infty} s(t) = 0$. $\dot{V} = 0$ implies that $s = 0$ and there is no other solution but $s = 0$. According to LaSalle’s invariant set theorem, defining $R = \{ s \in R^n \mid \dot{V}(s) = 0 \}$, then if $R$ contains no other trajectories other than $s = 0$, the origin 0 is asymptotically stable. Therefore the sliding surface $s = 0$ is an invariant set which implies that any trajectory starting from an initial condition within the set remains in the set all the time, that is $s(t)$ will asymptotically converge to zero, $\lim_{t \to \infty} s(t) = 0$.

Barbalat’s lemma can also be used to prove that $\lim_{t \to \infty} s(t) = 0$. The inequality $\dot{V} \leq -\eta \|s\|$ implies that $s$ is integrable as $\int_0^\infty \|s\| \; dt \leq \frac{1}{\eta} [V(0) - V(t)]$. Since $V(0)$ is bounded and $V(t)$ is nonincreasing and bounded, it can be concluded that $\lim_{t \to \infty} \int_0^t \|s\| \; dt$ is bounded. Since $\lim_{t \to \infty} \int_0^t \|s\| \; dt$ is bounded and $\dot{s}$ is also bounded, according to Barbalat’s lemma, $s(t)$ will asymptotically converge to zero, $\lim_{t \to \infty} s(t) = 0$.

Remark 1. Definition of Persistence of Excitation (PE): A vector $v \in R^q \geq 1$ is said to be persistence of excitation if there exist positive constants $\alpha$ and $T$ such that for all $t > 0$, $\int_0^{t+T} v(\tau)v^T(\tau) \; d\tau \geq \alpha d$. 

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PE is a notion of a time signal that contains sufficient richness so that the \( v(\tau)v^T(\tau) \) matrix is nonsingular. It requires that \( v(\tau) \) varies in such a way with time that the integral of the matrix is positive definite over any time interval \([t, t+T]\).

After we have proved that \( s(t) \) and \( e(t) \) all asymptotically converge to zero. To make conclusions about \( \tilde{\theta} = 0 \), other than the fact that they are bounded, we need to make the persistence of excitation argument. From the adaptive law \( \hat{\theta}(t) = \hat{\theta}(t) = -\tau Y^T s(t) \), according to the persistence excitation theory [5], if \( Y \) is persistently exciting signal, then \( \hat{\theta}(t) = -\tau Y^T s(t) \) guarantees that \( \tilde{\theta} \to 0 \), \( \theta \) will converges to its true values. Because \( s \to 0 \) implies \( e \to 0 \), tracking error \( e = q - q_m \) determines that

\[
q_1 = A_1 \sin(w_1t), \quad \dot{q}_1 = w_1 A_1 \cos(w_1t), \quad q_2 = A_2 \sin(w_2t), \quad \dot{q}_2 = w_2 A_2 \cos(w_2t).
\]

It can be shown that there exist some positive scalar constants \( \alpha \) and \( T \) such that for all \( t > 0 \),

\[
\int_{t}^{t+T} Y^T Y d\tau \geq \alpha d.
\]

where

\[
Y^T = \begin{bmatrix}
\dot{q}_1^2 & \dot{q}_1 \dot{q}_2 & 0 & -2\dot{q}_1 \dot{q}_2 & \dot{q}_1 \dot{q}_1 & \dot{q}_1 \dot{q}_2 & 0 \\
\dot{q}_1 \dot{q}_2 & \dot{q}_1^2 + \dot{q}_2^2 & \dot{q}_1 \dot{q}_2 & -2\dot{q}_2^2 + 2\dot{q}_1^2 & \dot{q}_1 \dot{q}_2 & \dot{q}_2 \dot{q}_1 + \dot{q}_1 \dot{q}_1 & \dot{q}_2 \dot{q}_2 \\
0 & \dot{q}_2 \dot{q}_1 & \dot{q}_2^2 & 2\dot{q}_1 \dot{q}_2 & 0 & \dot{q}_2 \dot{q}_1 + \dot{q}_1 \dot{q}_1 & \dot{q}_2 \dot{q}_2 \\
-2\dot{q}_1 \dot{q}_2 & -2\dot{q}_2^2 + 2\dot{q}_1^2 & 2\dot{q}_1 \dot{q}_2 & 4\dot{q}_2^2 + 4\dot{q}_1^2 & -2\dot{q}_1 \dot{q}_2 & -2\dot{q}_2 \dot{q}_2 + 2\dot{q}_1 \dot{q}_2 & 2\dot{q}_1 \dot{q}_2 \\
q_1 \dot{q}_1 & q_1 \dot{q}_2 & 0 & -2q_1 \dot{q}_2 & q_1^2 & q_1 q_2 & 0 \\
q_2 \dot{q}_1 & q_2 \dot{q}_2 & 0 & -2q_2 \dot{q}_2 & q_2^2 & q_2 q_2 & 0 \\
0 & \dot{q}_1 \dot{q}_2 & q_2 \dot{q}_2 & -2q_2 \dot{q}_2 & 0 & q_1 q_2 & q_2^2
\end{bmatrix}
\]

From (4.5) and (4.6) it can be shown that \( Y^T Y \) has full rank if \( w_1 \neq w_2 \), i.e. the excitation frequencies on \( x \) and \( y \) axes should be different. In other words, excitation of proof mass
should be persistently exciting [22]. Since $\bar{\theta} \to 0$, then the unknown angular velocity as well as all other unknown parameters can be consistently estimated. It has been shown that angular velocity $\Omega_z$ and all gyroscope parameter such as $d_{xx}$, $d_{xy}$, $w_x^2$, $w_y$ and $w_y^2$ converge to their true values.

In summary, if persistently exciting drive signals, $x_m = A_t \sin(w_1 t)$ and $y_m = A_2 \sin(w_2 t)$ are used, then $\bar{\theta}(t)$, $s(t)$ and $e(t)$ all converge to zero asymptotically. Consequently the unknown angular velocity can be determined as $\lim_{t \to \infty} \hat{\Omega}_z \approx \Omega_z$.

However it is difficult to establish the convergence rate.

Remark: In the adaptive control system design, the persistent excitation condition is an important factor to estimate the angular velocity $\Omega_z$ correctly. The reference trajectory that the gyroscope must follow is generated such that the resonance frequency of the $x$-axis is different from that of the $y$-axis which satisfies the persistent excitation condition.

4.3. Comparison with Standard Adaptive Controller.

A standard adaptive controller which has an addition term $K_f s$ is proposed as

$$u = Y\dot{\theta} - Q - K_f s$$

(4.20)

where $\dot{\theta}$ is the estimated parameter of $\dot{\theta}$, $K_f$ is positive definite matrix.

Then (4.12) becomes

$$\dot{s} = u + f - Y\dot{\theta}^* + Q$$

$$= Y\tilde{\theta} - K_f s + f$$

(4.21)
Define a Lyapunov function

\[ V = \frac{1}{2} s^T s + \frac{1}{2} \tilde{\theta}^T \tau^{-1} \tilde{\theta} \]  

(4.22)

where \( \tau = \tau^T > 0, \) \( \tilde{\theta} = \theta - \theta^* . \)

Differentiating \( V \) with respect to time yields

\[ \dot{V} = s^T \dot{s} + \tilde{\theta}^T \tau^{-1} \tilde{\theta} \]

\[ = s^T (Y \tilde{\theta} - K_f s + f ) + \tilde{\theta}^T \tau^{-1} \tilde{\theta} \]  

(4.23)

\[ = -s^T K_f s + s^T f + (s^T Y \tilde{\theta} + \tilde{\theta}^T \tau^{-1} \tilde{\theta}). \]

To make \( \dot{V} \leq 0 \), we choose an adaptive law as

\[ \dot{\tilde{\theta}}(t) = \dot{\theta}(t) = -\tau Y^T s(t). \]  

(4.24)

If \( f = 0 \), therefore \( \dot{V} \) becomes

\[ \dot{V} = -s^T K_f s \leq 0 \]  

(4.25)

which implies that the stability of closed-loop system can be guaranteed. If \( f \neq 0 \), the stability of closed-loop system cannot be guaranteed.

Remake 1. Such an adaptive controller would be inadequate to address the control system where there exist appreciable non-parametric uncertainties which include unmodelled dynamics, external disturbance and other imperfections in the estimates of gyroscopes parameters. Therefore, the proposed adaptive sliding mode controller incorporates the capability to maintain stable performance in the presence of model uncertainties and external disturbance.
Remark 2. In order to eliminate the control discontinuities, a smooth sliding mode control \( \tanh(as) \) that can reduce chattering problem is introduced. The parameter \( a \) determines the slope of \( \tanh(as) \) function at \( s = 0 \).

Therefore the smooth sliding mode controller is proposed as

\[
u = Y\theta - Q - \rho \tanh(as)
\] (4.26)

Remark 4:
If gyroscope system does not have same coupling damping and spring constant, that is the gyroscope system dynamics can be written as

\[
\ddot{q} + D\dot{q} + K_q q = u - 2\Omega\dot{q}
\] (4.27)

where \( q = \begin{bmatrix} x \\ y \end{bmatrix} \), \( u = \begin{bmatrix} x \\ y \end{bmatrix} \), \( \Omega = \begin{bmatrix} 0 & -\Omega_z \\ \Omega_z & 0 \end{bmatrix} \), \( D = \begin{bmatrix} d_{xx} & d_{xy} \\ d_{yx} & d_{yy} \end{bmatrix} \), \( K_b = \begin{bmatrix} w_x^2 & w_{xy} \\ w_{yx} & w_y^2 \end{bmatrix} \).

The dynamics of sliding surface can be derived as

\[
\dot{s} = u + f = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -2q_2 & q_1 & q_2 \\ 0 & 0 & 0 & \Omega_z & 0 & 0 \end{bmatrix} + \lambda(\dot{q} - \dot{q}_m) + K_m q_m
\] (4.28)

Define

\[
Y = \begin{bmatrix} \dot{q}_1 & \dot{q}_2 & 0 & 0 & -2q_2 & q_1 & q_2 \\ 0 & 0 & \dot{q}_1 & \dot{q}_2 & 2q_1 & 0 & 0 \end{bmatrix}
\] (4.29)
$\theta^* = \begin{bmatrix} d_{xx} & d_{xy} & d_{yx} & d_{yy} & \Omega_z & w^2_x & w_{xy} & w^2_{yx} \end{bmatrix}^T$ (4.30)

$Q = \lambda(\dot{q} - \dot{q}_m) + K_m q_m,$ (4.31)

Similarly as Lyapunov analysis before, the adaptive law is derived as

$\hat{\theta}(t) = \hat{\theta}(t) = -\tau Y^T s(t)$ (4.47)

Therefore, all the system parameter including unsymmetrical coupling damping and spring parameters such as $d_{xy}, d_{yx}, w_{xy}$ and $w_{yx}$ can be consistently estimated.

Remark 3. The motion of a mode-unmatched gyroscope, in which the resonance frequency of the $x$-axis is different from that of the $y$-axis, has sufficient persistence of excitation to permit the identification of all major fabrication imperfections as well as angular velocity. A MEMS gyroscope, suitable for the adaptive mode of operation requires equal movements in the $x$ and $y$ axes. Thus, there is no specific drive and sense axis in the sense of conventional MEMS gyroscopes. It should be noted that a conventional gyroscope structure is normally designed based on the assumption that the movement of the proof mass in the drive axis ($x$-axis) is relatively large, but the movement in the sense axis ($y$-axis) is very small. The proposed gyroscope design consists of a proof mass, four hairpin type spring suspensions and several pairs of parallel electrodes for actuation and sensing located at both $x$ and $y$ axes.

Conclusion: With the control law (4.15) and the parameter adaptation law (4.18), if the gyroscope is controlled to follow the mode-unmatched reference model, the persistent excitation condition is satisfied, i.e. $w_1 \neq w_2$, and all unknown gyroscope parameters, including the angular velocity, are estimated correctly.
4.4 Simulation Studies

We evaluated the proposed adaptive sliding mode control on a lumped MEMS gyroscope model (2.18-2.19) using MATLAB/SIMULINK. The control objective is to design an adaptive state tracking sliding mode controller so that a consistent estimate of $\Omega_z$ can be obtained. We allowed $\pm 2\%$ parameter variations for the spring and damping coefficients with respect to their nominal values. We further assumed $\pm 1\%$ magnitude changes in the coupling terms $d_{xy}$ and $\omega_{xy}$. The disturbance is assumed to be random variable with zero mean and unity variance. The MEMS gyroscope parameters are shown in Table 4.1. The Matlab/Simulink block of adaptive sliding mode control of MEMS gyroscope and Matlab/Simulink sub-block of sliding surface and tracking error, system dynamics of MEMS gyroscope, adaptive estimator, sliding mode controller are depicted in Figs. 4.2-4.6 respectively.

The initial condition is $\theta(0) = 0.95\theta^*$, the angular velocity is $\Omega = 5.0$ rad/s. The sliding gain of (4.15) is $\rho = \text{diag}[200 \quad 200]$, the adaptive gain of (4.18) is chosen as $\tau = \text{diag}[200 \quad 200 \quad 200 \quad 200 \quad 200 \quad 200]$. The sliding mode parameter of (4.16) is $\lambda = \text{diag}[4 \quad 4]$, in the smooth sliding mode controller $\tanh(as)$, $a = 5$.

The tracking error and sliding surface are shown in Fig.4.7 and Fig. 4.8. Fig.4.9 and Fig.4.10 show that the estimation parameters converge to their true values with persistently exciting sinusoidal reference signals. Fig. 4.11 and Fig. 4.12 compare the angular velocity estimation between smooth sliding mode controller and bang-bang type sliding mode controller $\text{sgn}(s)$ under the sinusoidal reference signal. Both figures show that estimation of angular velocity converges to its true value. It is observed that the
estimated angular velocity with smooth sliding mode controller has better convergence performance. Fig. 4.13 and Fig. 4.14 compare the sliding mode control force between smooth sliding mode controller (4.26) and bang-bang type sliding mode controller (4.14). It is shown that adaptive sliding mode system with smooth sliding mode controller can reduce chattering significantly
Table 4.1: MEMS Gyroscope Parameters

<table>
<thead>
<tr>
<th>SYMBOL</th>
<th>VALUE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td>$0.57e-8$ Kg</td>
</tr>
<tr>
<td>$d_{xx}$</td>
<td>$0.429e-6$ N s/m</td>
</tr>
<tr>
<td>$d_{xy}$</td>
<td>$0.0429e-6$ N s/m</td>
</tr>
<tr>
<td>$d_{yy}$</td>
<td>$0.687e-3$ N s/m</td>
</tr>
<tr>
<td>$k_{xx}$</td>
<td>80.98 N/m</td>
</tr>
<tr>
<td>$k_{xy}$</td>
<td>5 N/m</td>
</tr>
<tr>
<td>$k_{yy}$</td>
<td>71.62 N/m</td>
</tr>
<tr>
<td>$w_1$</td>
<td>4.11 kHz</td>
</tr>
<tr>
<td>$w_2$</td>
<td>5.11 kHz</td>
</tr>
<tr>
<td>$w_0$</td>
<td>3 kHz</td>
</tr>
<tr>
<td>$q_0$</td>
<td>1 µm</td>
</tr>
</tbody>
</table>
Figure 4.2: Matlab/Simulink block of adaptive sliding mode control for a MEMS gyroscope

Figure 4.3: Matlab/Simulink sub-block of sliding surface and tracking error
Figure 4.4: Matlab/Simulink sub-block of system dynamics

Figure 4.5: Matlab/Simulink sub-block of adaptive estimator
Figure 4.6: Matlab/Simulink sub-block of sliding mode controller

Figure 4.7: Convergence of the tracking error
Figure 4.8: Convergence of the sliding surface

Figure 4.9: Adaptation of damping coefficients of gyroscope
Figure 4.10: Adaptation of spring constants of gyroscope

Figure 4.11: Convergence of the estimated angular velocity with smooth sliding mode controller
Figure 4.12: Convergence of the estimated angular velocity with bang-bang type sliding mode controller

Figure 4.13: Smooth sliding mode control force of the adaptive sliding mode controller
4.5 Concluding Remarks

This chapter investigates the design of adaptive control with sliding mode controller for the MEMS gyroscope. New adaptive sliding mode controller was formulated for MEMS gyroscopes with two unmatched oscillatory modes which have persistence of excitation to permit the identification of all gyroscope parameters including the linear damping and stiffness coefficients and angular velocity. The proposed control structure can establish the stability of closed-loop system. Numerical simulations show that the proposed adaptive sliding mode control has satisfactory performance and robustness in the presence of input disturbances and model variations.
CHAPTER V
DIRECT ADAPTIVE SLIDING MODE CONTROL WITH PROPORTIONAL AND INTEGRAL SLIDING SURFACE FOR GENERAL SYSTEM

This chapter presents an adaptive tracking controller with a proportional and integral (PI) sliding surface. A new adaptive sliding mode controller based on model reference adaptive control is proposed to deal with the state tracking problem for a class of linear dynamic systems. First, a proportional and integral sliding surface instead of a conventional sliding surface is chosen. Then an adaptive sliding mode controller is derived. It is shown that the stability of the closed-loop system can be guaranteed with the proposed adaptive sliding mode control strategy. The adaptive design is extended to the multiple inputs system. The numerical simulations are performed to show the effectiveness of the proposed adaptive sliding mode control scheme with proportional and integral sliding mode action.
5.1 Problem Statement

Consider the system with parametric uncertainties

\[ \dot{x}(t) = (A + \Delta A)x(t) + Bu + f(t) \]  

(5.1)

where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R} \) and \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n} \) are unknown constant parameter matrices and the state vector \( x(t) \) is available for measurement, \( \Delta A \) is the unknown parameter uncertainties of the matrix \( A \), \( f(t) \) is an uncertain extraneous disturbance.

The reference model is given by

\[ \dot{x}_m(t) = A_m x_m(t) + B_m r(t) \]  

(5.2)

where \( x_m(t) \in \mathbb{R}^n \), \( r(t) \in \mathbb{R} \), and \( B_m \in \mathbb{R}^n \) are known constant matrices.

We make the following assumptions:

1. All eigenvalues of \( A_m \) are in the open left-half complex plane, and \( r(t) \) is bounded and piecewise continuous;

2. There exists a constant vector \( k_1^* \in \mathbb{R}^n \) and a non-zero constant scalar \( k_2^* \in \mathbb{R} \) such that the following equations are satisfied \( A + B k_1^* T = A_m, \ B k_2^* = B_m \). These conditions are called the matching conditions;

3. \( \Delta A \) and \( f(t) \) have matched and unmatched terms. There exists unknown matrices of appropriate dimension \( D, G \) such that \( \Delta A(t) = BD(t) + \Delta \tilde{A}(t) \) and \( f(t) = BG(t) + \tilde{f}(t) \), and, where \( BD(t) \) is matched uncertainty and \( \Delta \tilde{A}(t) \) is unmatched uncertainty,
$BG(t)$ is matched disturbance and $\bar{f}(t)$ is unmatched disturbance.

From this assumption, (5.1) can be rewritten as

$$\dot{x}(t) = Ax(t) + Bu(t) + \Delta Ax(t) + f(t)$$

$$= Ax(t) + Bu(t) + BDx(t) + \Delta \tilde{A}x(t) + BG + \bar{f}(t)$$

$$= Ax(t) + Bu(t) + Bf_m + f_u$$

(5.3)

where $Bf_m(t, x, u)$ represents the lumped matched uncertainty and disturbance which is given by

$$f_m(t, x, u) = Dx(t) + G.$$  

(5.4)

The term $f_u(t, X)$ represents the system lumped unmatched uncertainty and disturbance which is given by

$$f_u(t, x) = \Delta \tilde{A}x(t) + \bar{f}(t).$$  

(5.5)

4. The matched and unmatched lumped uncertainty and disturbance $f_m$ and $f_u$ are bounded by known positive parameters $\alpha_{m1}$, $\alpha_{m2}$ and $\alpha_{u1}$, $\alpha_{u2}$ such as

$$f_m(t, x, u) \leq \alpha_{m1} \|x\| + \alpha_{m2}$$

and

$$\|f_u(t, x)\| \leq \alpha_{u1} \|x\| + \alpha_{u2}.$$  

(5.6)

5. The sign of the parameter $k^*_{2}$ is known and the sign of the $\lambda B$ is known, where $\lambda \in R^n$; $|\lambda B| > \beta$, where $\beta$ is known positive parameter.

The control objective is to determine the plant input $u(t)$ so that all signals are bounded and the plant state $x(t)$ tracks the reference model state $x_m(t)$ as close as possible for any given reference input $r(t)$. 

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In this section the tracking error and sliding surface will be defined and adaptive laws for the sliding model control system will be derived. The block diagram of a direct adaptive sliding mode control for a general system is shown in Fig. 5.1.

The tracking error and its derivative are

\[ e(t) = x(t) - x_m(t) \]  

\[ \dot{e} = Ax + Bu + Bf_m + f_u - (A_m x_m + B_mr) \]

\[ = A_m e + (A - A_m) x + Bu + Bf_m + f_u - B_m r \]  

The proportional and integral sliding surface \( s(t) = 0 \) is defined as

\[ s(t) = \lambda e - \int_0^t \lambda A_m e d\tau . \]  

where \( \lambda \) is a constant matrix.

The derivative of the sliding surface is
\[ \dot{s} = \lambda \dot{e} - \lambda A_m e \]
\[ = \lambda A_m e + \lambda (A - A_m) x + \lambda Bu + \lambda B f_m + \lambda f_u - \lambda B_m r - \lambda A_m e \]
\[ = \lambda (A - A_m) x + \lambda Bu + \lambda B f_m + \lambda f_u - \lambda B_m r. \quad (5.10) \]

Setting \( \dot{s} = 0 \) to solve equivalent control \( u_{eq} \) gives

\[ u_{eq} = - (\lambda B)^{-1} \lambda [(A - A_m) x - B_m r] - f_m - (\lambda B)^{-1} \lambda f_u \]
\[ = - (\lambda B)^{-1} \lambda (A - A_m) x + (\lambda B)^{-1} \lambda B_m r - f_m - (\lambda B)^{-1} \lambda f_u \]
\[ = k_1^* x(t) + k_2^* r(t) - f_m - (\lambda B)^{-1} \lambda f_u. \quad (5.11) \]

The adaptive sliding mode state tracking controller structure is proposed as

\[ u(t) = k_1^T (t) x(t) + k_2 (t) r(t) - \rho \text{sgn}(s) \quad (5.12) \]

where \( k_1(t) \) and \( k_2(t) \) are the estimates of parameter \( k_1^* \) and \( k_2^* \) respectively.

The design task is to choose adaptive laws to update controller parameters so that the model reference control objective is still achievable. The main functions of adaptive update laws for \( k_1(t) \) and \( k_2(t) \) are to ensure the stability of the closed-loop system.

Define the parameter errors as

\[ \tilde{k}_1(t) = k_1(t) - k_1^*, \quad (5.13) \]
\[ \tilde{k}_2(t) = k_2(t) - k_2^*. \quad (5.14) \]

Substituting (5.13) and (5.14) into (5.3), we get

\[ \dot{x}(t) = Ax(t) + B[k_1^T (t) x(t) + k_2 (t) r(t)] + Bf_m + f_u - B \rho \text{sgn}(s) \]
\[ = A_m x(t) + B_m r(t) + B_m \left[ \frac{1}{k_2^*} \tilde{k}_1^T(t) x(t) + \frac{1}{k_2^*} \tilde{k}_2(t) r(t) \right] + Bf_m + f_u - B \rho \text{sgn}(s). \quad (5.15) \]

Substituting (5.15) into (5.7), we have the tracking error equation

\[ \dot{e}(t) = A_m e(t) + B_m \left[ \frac{1}{k_2^*} \tilde{k}_1^T(t) x(t) + \frac{1}{k_2^*} \tilde{k}_2(t) r(t) \right] + Bf_m + f_u - B \rho \text{sgn}(s). \quad (5.16) \]
The sliding surface dynamics is

\[
\dot{s}(t) = \lambda \dot{e}(t) - \lambda A_m e(t) = \lambda A_m e(t) + \lambda B_m \left[ \frac{1}{k_2^*} \tilde{k}_1^T(t)x(t) + \frac{1}{k_2^*} \tilde{k}_2^T(t)r(t) \right] + \lambda B_f m + \lambda f_u - \lambda B \rho \text{sgn}(s) - \lambda A_m e(t) \tag{5.17}
\]

\[
= B_m \left[ \frac{1}{k_2^*} \tilde{k}_1^T(t)x(t) + \frac{1}{k_2^*} \tilde{k}_2^T(t)r(t) \right] + \lambda B_f m + \lambda f_u - \lambda B \rho \text{sgn}(s).
\]

The state vector of the closed-loop system is defined as

\[
e_c(t) = \left( s(t), \tilde{k}_1^T(t), \tilde{k}_2^T(t) \right)^T \in \mathbb{R}^{2n+1}. \tag{5.18}
\]

Define a Lyapunov function

\[
V(e_c) = \frac{1}{2} s^2 + \frac{1}{2k_2^*} \tilde{k}_1^T \tau^{-1} \tilde{k}_1 + \frac{1}{2k_2^*} \tilde{k}_2^T \gamma^{-1} \tilde{k}_2.
\tag{5.19}
\]

Differentiating \(V\) with respect to time yields

\[
\dot{V} = s \dot{s} + \frac{1}{k_2^*} \tilde{k}_1^T \tau^{-1} \dot{\tilde{k}}_1 + \frac{1}{k_2^*} \tilde{k}_2^T \gamma^{-1} \dot{\tilde{k}}_2
\]

\[
= s \lambda B_m \left[ \frac{1}{k_2^*} \tilde{k}_1^T(t)x(t) + \frac{1}{k_2^*} \tilde{k}_2^T(t)r(t) \right] - \lambda B \rho \text{sgn}(s)s + \lambda B_f m s + \lambda f_u s
\]

\[
+ \frac{1}{k_2^*} \tilde{k}_1^T \tau^{-1} \dot{\tilde{k}}_1 + \frac{1}{k_2^*} \tilde{k}_2^T \gamma^{-1} \dot{\tilde{k}}_2
\tag{5.20}
\]

\[
= -\lambda B \rho |s| + \lambda B f m s + \lambda f_u s + \left[ s \lambda B_m \frac{1}{k_2^*} \tilde{k}_1^T(t)x(t) + \frac{1}{k_2^*} \tilde{k}_1^T \tau^{-1} \dot{\tilde{k}}_1 \right]
\]

\[
+ \left[ s \lambda B_m \frac{1}{k_2^*} \tilde{k}_2^T(t)r(t) + \frac{1}{k_2^*} \tilde{k}_2^T \gamma^{-1} \dot{\tilde{k}}_2 \right].
\]

To make \(\dot{V} \leq 0\), we choose the adaptive laws as

\[
\dot{\tilde{k}}_1(t) = \tilde{k}_1(t) = -\text{sgn}(k_2^*) \tau(t)s(t) \lambda B_m \tag{5.21}
\]

\[
\dot{\tilde{k}}_2(t) = \tilde{k}_2(t) = -\text{sgn}(k_2^*) \rho(t)s(t) \lambda B_m \tag{5.22}
\]
with \( \tau = \tau^T > 0, \gamma > 0, k_1(0) \) and \( k_2(0) \) being arbitrary. With this adaptive law choice,

\[
\rho > \alpha_{a_1} \|x\| + \alpha_{a_2} + \frac{\|x\|}{\beta} \left( \alpha_{a_1} \|x\| + \alpha_{a_1} \right) \text{ and } \lambda B \text{ positive, then}
\]

\[
\dot{V} = -\lambda B \rho \|s\| + \lambda B f_m s + \lambda f_s s \\
\leq -\lambda B \rho \|s\| + \lambda B |f_m| \|s\| + \|\lambda f_s\| \|s\| \\
= -\lambda B \|s\| \left( \rho - |f_m| - \frac{\|\lambda f_s\|}{\lambda B} \right) \\
< -\lambda B \left[ \rho - \alpha_{a_1} \|x\| - \alpha_{a_2} - \frac{\|\lambda f_s\|}{\lambda B} \left( \alpha_{a_1} \|x\| + \alpha_{a_1} \right) \right] < 0. \tag{5.23}
\]

With \( \rho \geq \alpha_{a_1} \|x\| - \alpha_{a_2} - \frac{\|\lambda f_s\|}{\lambda B} \left( \alpha_{a_1} \|x\| + \alpha_{a_1} \right) + \eta \), where \( \eta \) is a positive constant, \( \dot{V} \) becomes negative semi-definite, i.e., \( \dot{V} \leq -\theta \|s\| \). This implies that the trajectory reaches the sliding surface in finite time and remains on the sliding surface. \( \dot{V} \) is negative definite implies that \( s \) and \( \tilde{k}_1 \) and \( \tilde{k}_2 \) converge to zero. \( \dot{V} \) is negative semi-definite ensures that \( V, s \) and \( \tilde{k}_1 \) and \( \tilde{k}_2 \) are all bounded. It can be concluded from (5.17) that \( \dot{s} \) is also bounded.

LaSalle’s invariant set theorem can be used to prove that \( \lim_{t \to \infty} s(t) = 0 \). Barbalat’s lemma can also be used to prove that \( \lim_{t \to \infty} s(t) = 0 \). The inequality \( \dot{V} \leq -\theta \|s\| \) implies that \( s \)

is integrable as \( \int_0^\infty \|s\| \, dt \leq \frac{1}{\eta} \left[ V(0) - V(t) \right] \). Since \( V(0) \) is bounded and \( V(t) \) is nonincreasing and bounded, it can be concluded that \( \lim_{t \to \infty} \int_0^t \|s\| \, dt \) is bounded. Since \( \lim_{t \to \infty} \int_0^t \|s\| \, dt \) is bounded and \( \dot{s} \) is also bounded, according to Barbalat lemma, \( s(t) \) will asymptotically converge to zero, \( \lim_{t \to \infty} s(t) = 0 \).
To make conclusions about the parameter errors $\tilde{k}_1 = 0$ and $\tilde{k}_2 = 0$ other than the fact that they are bounded, we need to make the persistence of excitation argument. From

$$\dot{\hat{k}}_1(t) = \hat{k}_1(t) = -\text{sgn}(\hat{k}_2^*) \xi(t) s(t) \hat{\lambda}_B m,$$

and

$$\dot{\hat{k}}_2(t) = \hat{k}_2(t) = -\text{sgn}(\hat{k}_2^*) \eta(t) s(t) \hat{\lambda}_B m,$$

according to the persistence excitation theory [5], if $x$ and $r$ are persistently exciting signals, then it can be guaranteed that $\tilde{k}_1 \to 0$ and $\tilde{k}_2 \to 0$. It can be shown that the controller parameters converge to their true values if persistence of excitation condition is satisfied.

Conclusion: The adaptive controller (5.12) with the adaptive laws (5.21-5.22), applied to the system (5.3), guarantee that all closed-loop signals are bounded and the sliding mode control system will asymptotically converge to the sliding surface $s(t) = 0$. The adaptive designs can be extended to the multiple inputs system.

Remark 3. Convergence rate analysis

The averaging technique is a useful tool to evaluate the local stability and the convergence rate of the adapted parameters.

Let $G_u$ and $G_v$ be stable transfer functions and let $u$ and $v$ denote the steady-state responses to the input $r = r_0 \sin wt$.

The average of the product $uv$ is given by [3]

$$\text{avg}(uv) = \frac{r_0^2}{2} \text{Re}(G_u(iw)G_v(-iw)).$$

From (5.2), the Laplace transform of $x_m(t)$ is derived as

$$x_m = (SI - A_m)^{-1} B_m r$$

where capital S is used here to indicate the Laplace transform therefore it avoids
confusion with the sliding surface expression. For the situation where \( s \to 0 \) i.e. the system is on sliding surface, assuming \( f_u = 0 \) and \( f_m = 0 \), from (5.16), the Laplace transform of tracking error \( e(t) \) is derived as

\[
e = (SI - A_m - B\tilde{k}_1)^{-1} B[k_1 (SI - A_m)^{-1} B_m + \tilde{k}_2] r.
\]  

(5.26)

Assuming \( k_2^* \) is positive and Rewriting \( \hat{k}_1(t) \) and \( \hat{k}_2(t) \) from (5.21) and (5.22) yield

\[
\dot{\hat{k}}_1(t) = \dot{k}_1(t) = -\pi(t)s\lambda B_m = -\tau(e + x_m)s\lambda B_m
\] 

\[
= -\tau(e + x_m)(\lambda e - \int_0^\lambda \lambda A_m e d\tau)\lambda B_m
\]  

(5.27)

\[
\dot{\hat{k}}_2(t) = \dot{k}_2(t) = -\gamma(t)s\lambda B_m
\] 

\[
= -\gamma(t)(\lambda e - \int_0^\lambda \lambda A_m e d\tau)\lambda B_m.
\]  

(5.28)

Substituting (5.25) and (5.26) into (5.27) and (5.28), and according to (5.24), the average equations can be derived as

\[
\frac{d\overline{k}_2}{dt} = -\frac{\gamma_0^2}{2}\lambda B_m \text{Re}\left\{\lambda (SI - A_m - B\overline{k}_1)^{-1} \cdot B[k_1 (SI - A_m)^{-1} B_m + \overline{k}_2] - \lambda A_m \frac{1}{S} (SI - A_m - B\overline{k}_1)^{-1} \right\}_{S=iw}
\] 

(5.29)

\[
\frac{d\overline{k}_1}{dt} = -\frac{\pi_0^2}{2}\lambda B_m \text{Re}\left\{((SI - A_m - B\overline{k}_1)^{-1} \cdot B[k_1 (SI - A_m)^{-1} B_m + \overline{k}_2] + (SI - A_m)^{-1} B_m)\right\}_{S=iw}
\] 

\[
\text{Re}\left\{\lambda (SI - A_m - B\overline{k}_1)^{-1} \cdot B[k_1 (SI - A_m)^{-1} B_m + \overline{k}_2] + (SI - A_m)^{-1} B_m\right\}_{S=iw} - \frac{\pi_0^2}{2}\lambda B_m \lambda A_m \text{Re}\left\{((SI - A_m - B\overline{k}_1)^{-1} \cdot B[k_1 (SI - A_m)^{-1} B_m + \overline{k}_2] + (SI - A_m)^{-1} B_m)\right\}_{S=iw}
\] 

\[
- \frac{1}{S} (SI - A_m - B\overline{k}_1)^{-1} \cdot B[k_1 (SI - A_m)^{-1} B_m + \overline{k}_2] \right\}_{S=iw}.
\]  

(5.30)
The stability of the equilibrium of the averaged equations can be investigated from the linearized averaged equations. The parameters $\gamma$, $\tau$ and $B$ determine the convergence rate and the parameters $w$ and $A_m$ influence the behavior of the convergence rate.

Remark 4. This remark shows the proof why the controller structure is proposed as

$$u(t) = k_1^T(t)x(t) + k_2(t)r(t) - \rho \text{sgn}(s)$$

when proportional and integral sliding surface is defined as $s(t) = \lambda e - \int_0^t \lambda A_m ed \tau$.

If the following controller structure is proposed,

$$u(t) = k_1^T(t)x(t) + k_2(t)r(t) + k_3^T(t)e(t) - \rho \text{sgn}(s) \tag{5.31}$$

where $k_1(t)$ and $k_2(t)$ are the estimate parameter of $k_1^*$ and $k_2^*$ respectively. The design task now is to choose adaptive laws to update these parameters so that the model reference control objective is still achievable. One of the functions of adaptive update laws for $k_1(t)$ and $k_2(t)$ is to ensure the stability of the closed loop system. Define the parameter errors as

$$\ddot{k}_1(t) = k_1(t) - k_1^* \quad \ddot{k}_2(t) = k_2(t) - k_2^* \quad \ddot{k}_3(t) = k_3(t) - k_3^* \tag{5.32}$$

Substituting $u(t)$ into (5.3), we get

$$\dot{x}(t) = Ax(t) + B[k_1^T(t)x(t) + k_2(t)r(t) + k_3^T(t)e(t)] + Bd - B\rho \text{sgn}(s)$$

$$= A_m x(t) + B_m r(t) + B_m \left[ \frac{1}{k_2^*} \ddot{k}_1(t)x(t) + \frac{1}{k_2^*} \ddot{k}_2(t)r(t) \right] + Bd - B\rho \text{sgn}(s) + Bk_3^T(t)e(t). \tag{5.33}$$

Substituting into (5.7), we have the tracking error equation

$$\dot{e}(t) = (A_m + Bk_3^T(t))e(t) + B_m \left[ \frac{1}{k_2^*} \ddot{k}_1(t)x(t) + \frac{1}{k_2^*} \ddot{k}_2(t)r(t) \right] + Bd - B\rho \text{sgn}(s). \tag{5.34}$$
We discuss the sliding surface in two cases.

Case 1. integral sliding surface.

The integral sliding surface is defined as \( s(t) = \lambda e - \int_0^t \lambda A_m e d\tau \), and its derivative is

\[
\dot{s}(t) = \lambda \dot{e}(t) - \lambda A_m e(t)
\]

\[
= \lambda B k_3^e e + \lambda B m \left[ \frac{1}{k_2^*} \tilde{k}_1^T(t)x(t) + \frac{1}{k_2^*} \tilde{k}_2(t)r(t) \right] + \lambda B d - \rho \text{sgn}(s).
\]

(5.35)

Define a Lyapunov like function candidate

\[
V = \frac{1}{2} s^2 + \frac{1}{2k_2^*} \tilde{k}_1^T \tau^{-1} \tilde{k}_1 + \frac{1}{2k_2^*} \tilde{k}_2^2 \gamma^{-1} + \tilde{k}_3^T p^{-1} \tilde{k}_3.
\]

(5.36)

Differentiating \( V \) with respect to time yields

\[
\dot{V} = s \dot{s} + \frac{1}{k_2^*} \tilde{k}_1^T \tau^{-1} \tilde{k}_1 + \frac{1}{k_2^*} \tilde{k}_2^2 \gamma^{-1} + \tilde{k}_3^T p^{-1} \tilde{k}_3
\]

\[
= -\lambda B p |s| + \lambda B d s + \left[ s \lambda B m \left[ \frac{1}{k_2^*} \tilde{k}_1^T(t)x(t) + \frac{1}{k_2^*} \tilde{k}_2^T \tau^{-1} \tilde{k}_1 \right] + \left[ \lambda B k_3 e + \tilde{k}_3^T \gamma^{-1} \tilde{k}_3 \right] \right.
\]

\[
+ \left[ s \lambda B m \frac{1}{k_2^*} \tilde{k}_2(t)r(t) + \frac{1}{k_2^*} \tilde{k}_2^2 \gamma^{-1} \tilde{k}_2 \right] + \left[ \lambda B k_3 e + \tilde{k}_3^T \gamma^{-1} \tilde{k}_3 \right].
\]

(5.37)

To make \( \dot{V} \leq 0 \), we can choose the adaptive laws as

\[
\dot{\tilde{k}}_1(t) = \tilde{k}_1(t) = -\text{sgn}(k_2^*) \tau(t) s \lambda B_m
\]

(5.38)

\[
\dot{\tilde{k}}_2(t) = \tilde{k}_2(t) = -\text{sgn}(k_2^*) \gamma(t) s \lambda B_m
\]

(5.39)

Since \( \lambda B k_3 e + \tilde{k}_3^T \tau^{-1} \tilde{k}_3 = \lambda B k_3^e + \lambda B k_3^* + \tilde{k}_3^T \tau^{-1} \tilde{k}_3 \), it can be seen that the adaptive law regarding \( \tilde{k}_3(t) \) can not be generated, therefore the stability of closed-loop system can not be easily guaranteed. It should be noted that not being able to find a Lyapunov function does not mean that the system will be unstable.
Case 2. sliding surface without integral term.

If sliding surface without integral term is \( s(t) = \lambda e \)

\[
\dot{s}(t) = \lambda \dot{e}(t) = \lambda A_m e(t) + \lambda B_m \left[ \frac{1}{k_2} \tilde{k}_1^T (t)x(t) + \frac{1}{k_2} \tilde{k}_2 (t)r(t) \right] + \lambda Bd - \lambda B \rho \text{sgn}(s). \quad (5.40)
\]

Define a Lyapunov function candidate same as in case 1

\[
\dot{V} = s \dot{s} + \frac{1}{|k_2|} \tilde{k}_1^T \tau^{-1} \tilde{k}_1 + \frac{1}{|k_2|} \tilde{k}_2^T \gamma^{-1} \tilde{k}_2

= s \lambda A_m e(t) + s \lambda B_m \left[ \frac{1}{k_2} \tilde{k}_1^T (t)x(t) + \frac{1}{k_2} \tilde{k}_2 (t)r(t) \right] - s \lambda B \rho \text{sgn}(s) + \lambda Bd

+ \frac{1}{|k_2|} \tilde{k}_1^T \tau^{-1} \tilde{k}_1 + \frac{1}{|k_2|} \tilde{k}_2^T \gamma^{-1} \tilde{k}_2

= s \lambda B_m \frac{1}{k_2} \tilde{k}_1^T (t)x(t) + \frac{1}{|k_2|} \tilde{k}_1^T \tau^{-1} \tilde{k}_1

+ s \lambda B_m \frac{1}{k_2} \tilde{k}_2 (t)r(t) + \frac{1}{|k_2|} \tilde{k}_2^T \gamma^{-1} \tilde{k}_2

+ \lambda e(t) \lambda A_m e(t) - \lambda B \rho |s| + \lambda Bd.
\]

To make \( \dot{V} \leq 0 \), we choose the adaptive laws same as in (5.38) and (5.39), thus

\[
\dot{V} = -\lambda B \rho |s| + \lambda Bd + \lambda e \lambda A_m e

\leq -\lambda B \rho |s| + \lambda B |d||s| + \lambda e \lambda A_m e

\leq -\lambda B |s| (\rho - d) + \lambda e \lambda A_m e < \lambda e \lambda A_m e.
\]

Here we assumed as before that \( \lambda B > 0 \), whether the term \( \lambda e \lambda A_m e \) negative or positive can not be determined, therefore \( \dot{V} < 0 \) can not be established, thus the stability of closed-loop system can not be guaranteed. In conclusion the controller as in (5.40) can not guarantee the stability of closed-loop system without an integral term. Obviously this conclusion only refers to the choice of Lyapunov function as in (5.41). It is conceivable that other choices of Lyapunov function can guarantee stability for the sliding surface without an integral term.
5.3 Adaptive Sliding Mode Controller with Multiple Inputs.

In this section, we will address the adaptive sliding mode control problem for system with multiple inputs. Consider the system with multiple inputs with parametric uncertainties

\[
\dot{x}(t) = (A + \Delta A)x(t) + Bu(t) + f(t)
\]  

(5.43)

where \(x(t) \in \mathbb{R}^n\), \(u(t) \in \mathbb{R}^m\) and \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{m \times n}\) are unknown constant parameter matrices, \(\Delta A\) is unknown parameter uncertainties of the matrices \(A\), \(f(t)\) is an uncertain extraneous disturbance and nonlinearity of the system.

The reference model is given by

\[
\dot{x}_m(t) = A_m x_m(t) + B_m r(t)
\]  

(5.44)

where \(x_m(t) \in \mathbb{R}^n\), \(r(t) \in \mathbb{R}^m\), \(A_m \in \mathbb{R}^{n \times n}, B_m \in \mathbb{R}^{m \times n}\) are known constant matrices.

Similarly as section 5.1, we make the following assumptions

1. All eigenvalues of \(A_m\) are in the open left-half complex plane, and \(r(t)\) is bounded and piecewise continuous;

2. There exists a constant matrix \(K_1^* \in \mathbb{R}^{n \times m}\) and a non-zero constant matrix \(K_2^* \in \mathbb{R}^{m \times m}\) such that the following equations are satisfied \(A + BK_1^* = A_m, BK_2^* = B_m\), which requires that \(B^TB\) is nonsingular.

3. The sign of the \(\lambda B\) is known, where \(\lambda \in \mathbb{R}^{m \times m}; \|\lambda B\| < \mu\), where \(\mu\) is known positive parameter, \(\gamma_{\min}(\lambda B) > \delta\), where \(\gamma_{\min}(\lambda B)\) is the eigenvalue of matrix \(\lambda B\) with minimum real part.

4. There is a known matrix \(Q \in \mathbb{R}^{m \times m}\) such that \(K_2^*Q\) is symmetric and positive definite.
\[ M = K_2^*Q = (K_2^*Q)^T = Q^T K_2^* > 0. \] (5.45)

5. \( \Delta A \) and \( f(t) \) have matched and unmatched terms. There exists unknown matrices of appropriate dimension \( D, \ G \) such that \( \Delta A(t) = BD(t) + \Delta \tilde{A}(t) \) and \( f(t) = BG(t) + \tilde{f}(t) \), where \( BD(t) \) is matched uncertainty and \( \Delta \tilde{A}(t) \) is unmatched uncertainty, \( BG(t) \) is matched disturbance and \( \tilde{f}(t) \) is unmatched disturbance.

From this assumption, (5.43) can be rewritten as

\[
\dot{x}(t) = Ax(t) + Bu(t) + \Delta Ax(t) + f(t) \\
= Ax(t) + Bu(t) + BDx(t) + \Delta \tilde{A}x(t) + BG + \tilde{f}(t) \\
= Ax(t) + Bu(t) + Bf_m + f_u
\] (5.46)

where \( Bf_m(t,x,u) \) represents the system matched lumped uncertainty and disturbance and lies in the range space of \( B \), is given by

\[
f_m(t,x,u) = Dx(t) + G. \] (5.47)

The term \( f_u(t,X) \) represents the system lumped unmatched uncertainty and disturbance, is given by

\[
f_u(t,x) = \Delta \tilde{A}x(t) + \tilde{f}(t). \] (5.48)

4. The matched and unmatched lumped uncertainty and disturbance \( f_m \) and \( f_u \) are bounded by known positive parameters \( \alpha_{m1}, \alpha_{m2} \) and \( \alpha_{u1}, \alpha_{u2} \) such as

\[
\|f_m(t,x,u)\| \leq \alpha_{m1}\|x\| + \alpha_{m2} \text{ and } \|f_u(t,x)\| \leq \alpha_{u1}\|x\| + \alpha_{u2}.
\]

The tracking error is also defined as \( e(t) = x(t) - x_m(t) \) and its derivative is

\[
\dot{e} = Ax + Bu + Bf_m + f_u - (A_m x_m + B_m r) \\
= A_m e + (A_A - A_m)x + Bu + Bf_m + f_u - B_m r. \] (5.49)
The integral sliding surface is defined as

$$ s(t) = \lambda e - \int_0^t \lambda A_m e d\tau $$

(5.50)

where $\lambda \in \mathbb{R}^{m \times n}$ is a constant matrix and is chosen so that $\lambda B$ is nonsingular.

The derivative of the sliding surface is

$$ \dot{s} = \lambda \dot{e} - \lambda A_m e $$

$$ = \lambda A_m e + \lambda (A - A_m)x + \lambda Bu + \lambda Bf_m + \lambda f - \lambda B_m r - \lambda A_m e $$

$$ = \lambda (A - A_m)x + \lambda Bu + \lambda Bf_m + \lambda f - \lambda B_m r. $$

(5.51)

Setting $\dot{s} = 0$ to solve equivalent control $u_{eq}$ gives

$$ u_{eq} = -(\lambda B)^{-1} \lambda [(A - A_m)x - B_m r] - f_m - (\lambda B)^{-1} \lambda f $$

$$ = -(\lambda B)^{-1} \lambda (A - A_m)x + (\lambda B)^{-1} \lambda B_m r - f_m - (\lambda B)^{-1} \lambda f $$

$$ = K_1^T x(t) + K_2^* r(t) - f_m - (\lambda B)^{-1} \lambda f. $$

(5.52)

The adaptive controller $u(t)$ is proposed as

$$ u(t) = K_1^T(t)x(t) + K_2(t) r(t) - \rho \frac{s}{\|s\|} $$

(5.53)

where $K_1(t)$ and $K_2(t)$ are the estimates of $K_1^*$ and $K_2^*$ respectively, $\rho$ is constant, $\|\|$ is the Euclidean norm, $\rho \frac{s}{\|s\|}$ is the unit sliding mode signal.

Define the parameter errors as

$$ \tilde{K}_1(t) = K_1(t) - K_1^* $$

(5.54)

$$ \tilde{K}_2(t) = K_2(t) - K_2^* $$

(5.55)

Substituting (5.54), (5.55) into (5.46), we get
\[
\dot{x}(t) = Ax(t) + B[K_1^T(t)x(t) + K_2(t)r(t)] + Bf_m + f_u - B\rho \frac{s}{\|s\|}
\]

\[
= A_m x(t) + B_m r(t) + B_m \left[K_2^{-1}\tilde{K}_1^T(t)x(t) + K_2^{-1}\tilde{K}_2(t)r(t)\right] + Bf_m + f_u - B\rho \frac{s}{\|s\|}.
\] (5.56)

Substituting (5.54), (5.55) into (5.49), yields tracking error equation

\[
\dot{e}(t) = A_m e(t) + B_m \left[K_2^{-1}\tilde{K}_1^T(t)x(t) + K_2^{-1}\tilde{K}_2(t)r(t)\right] + Bf_m + f_u - B\rho \frac{s}{\|s\|}.
\] (5.57)

The dynamics of sliding surface is

\[
\dot{s}(t) = \lambda \dot{e}(t) - \lambda A_m e(t)
\]

\[
= \lambda A_m e(t) + \lambda B_m \left[K_2^{-1}\tilde{K}_1^T(t)x(t) + K_2^{-1}\tilde{K}_2(t)r(t)\right] + \lambda Bf_m + \lambda f_u - \lambda B\rho \frac{s}{\|s\|} - \lambda A_m e(t)
\] (5.58)

\[
= \lambda B_m \left[K_2^{-1}\tilde{K}_1^T(t)x(t) + K_2^{-1}\tilde{K}_2(t)r(t)\right] + \lambda Bf_m + \lambda f_u - \lambda B\rho \frac{s}{\|s\|}.
\]

The state vector of the closed-loop system is defined as:

\[
e_c(t) = \left(s^T(t), \tilde{K}_1^T(t), \tilde{K}_2(t)\right)^T \in R^{(m+1)n+m^2}.
\] (5.59)

Define a Lyapunov function

\[
V(e_c) = \frac{1}{2} s^T s + \frac{1}{2} tr[\tilde{K}_1^T M^{-1} \tilde{K}_1] + \frac{1}{2} tr[\tilde{K}_2^T M^{-1} \tilde{K}_2]
\] (5.60)

where \( M = M^T > 0 \) satisfies the assumption, \( tr[M] \) denoting the trace of \( M \).

Differentiating \( V \) with respect to time yields
\[ \dot{V} = s^T \dot{s} + tr[\tilde{K}_1 M^{-1} \dot{\tilde{K}}_1 ] + tr[\tilde{K}_2 M^{-1} \dot{\tilde{K}}_2 ] \\
= s^T \lambda B_m [\tilde{K}_1^{-1} \dot{\tilde{K}}_1 (t) x(t)] + s^T \lambda \dot{\tilde{K}}_2 (t) r(t) \\
- s^T \lambda B \rho \frac{s}{\|s\|} + s^T \lambda B f_m + s^T \lambda f_u + tr[\tilde{K}_1 M^{-1} \dot{\tilde{K}}_1 ] + tr[\tilde{K}_2 M^{-1} \dot{\tilde{K}}_2 ] \\
= -s^T \lambda B \rho \frac{s}{\|s\|} + s^T \lambda B f_m + s^T \lambda f_u + [tr[\tilde{K}_1 M^{-1} Q^T B_m \lambda^T s^T x^T ] + tr[\tilde{K}_1 M^{-1} \dot{\tilde{K}}_1 ] ] \\
+ [tr[\tilde{K}_2 M^{-1} Q^T B_m \lambda^T s^T r^T ] + tr[\tilde{K}_2 M^{-1} \dot{\tilde{K}}_2 ] ] \\
(5.61) \]

where we use the definition \( M = K_1^T Q = M^T > 0 \) and the properties

\[ tr[M_1 M_2 ] = tr[M_2 M_1 ], tr[M_3 ] = tr[M_3^T ] \] for any matrices \( M_1, M_2 \) and \( M_3 \).

To make \( \dot{V} \leq 0 \), we choose the adaptive laws as

\[ \dot{\tilde{K}}_1 (t) = \dot{\tilde{K}}_1 (t) = -Q^T B_m^T \lambda^T s^T x^T \]

\[ \dot{\tilde{K}}_2 (t) = \dot{\tilde{K}}_2 (t) = -Q^T B_m^T \lambda^T s^T r^T \]

with \( Q \) satisfying assumption and \( K_1 (0) \) and \( K_2 (0) \) being arbitrary. With this adaptive law choice, \( \gamma_{\min} (\lambda B) > \delta \) and \( \rho > \frac{\mu (\alpha_{m1} \| x \| \alpha_{m2}) + \| \lambda \| (\alpha_{u1} \| x \| + \alpha_{u1})}{\delta} \), we have

\[ \dot{V} = -s^T \lambda B \rho \frac{s}{\|s\|} + s^T \lambda B f_m + s^T \lambda f_u \\
\leq -\rho \gamma_{\min} (\lambda B) \| s \| \| s \| \| \lambda \| \| f_u \| + \| s \| \| \lambda \| \| f_u \| \leq -\rho \gamma_{\min} (\lambda B) \| s \| + \mu \| s \| \| f_u \| + \| s \| \| \lambda \| \| f_u \| (5.64) \]

\[ < -\gamma_{\min} (\lambda B) \| s \| \left[ \rho - \frac{\mu (\alpha_{m1} \| x \| + \alpha_{m2}) + \| \lambda \| (\alpha_{u1} \| x \| + \alpha_{u1})}{\delta} \right] < 0. \]

With \( \rho \geq \frac{\mu (\alpha_{m1} \| x \| + \alpha_{m2}) + \| \lambda \| (\alpha_{u1} \| x \| + \alpha_{u1})}{\delta} + \eta \), where \( \eta \) is a positive constant, \( \dot{V} \) becomes negative semi-definite, i.e., \( \dot{V} \leq -\eta \| s \| \). This implies that the trajectory reaches the sliding surface in finite time and remains on the sliding surface. \( \dot{V} \) is negative definite.
implies that \( s \) and \( \tilde{K}_1 \) and \( \tilde{K}_2 \) all converge to zero. \( \dot{V} \) is negative semi-definite ensures that \( V, s \) and \( \tilde{K}_1 \) and \( \tilde{K}_2 \) are all bounded. It can be concluded from (5.58) that \( \dot{s} \) is also bounded.

LaSalle’s invariant set theorem can be used to prove that \( \lim_{t \to \infty} s(t) = 0 \). Barbalat’s lemma can also be used to prove that \( \lim_{t \to \infty} s(t) = 0 \). The inequality \( \dot{V} \leq -\frac{1}{\eta} s \| s \| \) implies that \( s \) is integrable as \( \int_0^\infty \| s \| \, dt \leq \frac{1}{\eta} [V(0) - V(t)] \). Since \( V(0) \) is bounded and \( V(t) \) is nonincreasing and bounded, it can be concluded that \( \lim_{t \to \infty} \int_0^t \| s \| \, dt \) is bounded. Since \( \lim_{t \to \infty} \int_0^t \| s \| \, dt \) is bounded and \( \dot{s} \) is also bounded, according to Barbalat’s lemma, \( s(t) \) will asymptotically converge to zero, \( \lim_{t \to \infty} s(t) = 0 \).

After we have proved that \( s \) converges to zero, \( \lim_{t \to \infty} s(t) = 0 \), that is \( s(t) \) and \( e(t) \) all converge to zero asymptotically. To make conclusions about the parameter errors \( \tilde{K}_1 = 0 \) and \( \tilde{K}_2 = 0 \) other than the fact that they are bounded, we need to make the persistence of excitation argument. From the adaptive laws \( \dot{\tilde{K}}_1(t) = \dot{\tilde{K}}_2^T(t) = -Q^T B_m A^T s^T x^T \), \( \dot{\tilde{K}}_2(t) = \dot{\tilde{K}}_1(t) = -Q^T B_m A^T s^T r^T \), according to the persistence excitation theory [5], if \( x \) and \( r \) are persistent excitation signals, then it can be guaranteed that \( \tilde{K}_1 \to 0 \) and \( \tilde{K}_2 \to 0 \). It can be shown that the controller parameters converge to their true values if persistence of excitation condition is satisfied.
5.4. Simulation Studies

In this section, a numerical simulation for the adaptive control with integral sliding mode control will be presented. The adaptive sliding mode control scheme is applied to the controlled plant with single input. The control objective is to design a controller so that the trajectory of \( x(t) \) can track the state of reference model \( x_m(t) \) using the proposed adaptive sliding mode control.

The unknown plant is described by

\[
x = \frac{1}{s^2 + s - 2} \quad u
\]

The reference model is given by

\[
x_m = \frac{1}{s^2 + 1.4s + 1} \quad r
\]

Simulation parameters are: \( \rho = 5, \ a = 5, \ \lambda = [4 \ 1], \ \tau = \gamma = 1 \) and the initial conditions are: \( x(0) = 0.02 \) and \( x_m(0) = 0 \). \( \Delta A = 0 \), \( f(t) \) is a random signal with zero mean and unity variance.

Fig. 5.2 and Fig. 5.3 compare the tracking error, controller parameter estimation and control input between the robust control using \( \text{sgn}(s) \) and smooth sliding mode controller \( (\text{tanh}(as)) \) under the sinusoidal reference signal \( r(t) = 10 \sin t \). It is shown that adaptive sliding mode system with smooth sliding mode controller can reduce chattering significantly. However the control parameters \( k_1 \) and \( k_2 \) do not converge to their true values since the reference signal is not persistently exciting for identification of three parameters. Fig. 5.4 shows the control results with persistent sinusoidal reference signal \( r(t) = 10 \sin t + 13 \sin 3.3t \). It is observed that the controller parameters converge to their true values.
Fig. 5.5 and Fig. 5.6 compare the tracking error and the controller parameter estimation between strong smooth sliding mode controller with $\rho = 50$ and a weak smooth sliding mode controller with $\rho = 0.001$ under the sinusoidal reference signal $r(t) = 10 \sin t$ and a random disturbance $f$ in (5.1) with zero mean and unity variance. It is observed that the adaptive sliding mode control system with strong smooth sliding mode control has better disturbance rejection performance.
Figure 5.3: Smooth adaptive sliding mode controller performance under a sinusoidal reference \( r(t) = 10 \sin t \)

Figure 5.4. Smooth adaptive sliding mode controller performance under a persistent sinusoidal reference signal \( r(t) = 10 \sin t + 13 \sin 3.3t \).
Figure. 5.5. Adaptive sliding mode controller performance with strong smooth sliding controller under $r(t) = 10 \sin t$ and random disturbance.

Figure. 5.6. Adaptive sliding mode controller performance with weak smooth sliding mode controller under $r(t) = 10 \sin t$ and random disturbance.
5.5 Concluding Remarks

This chapter investigates the design of a model reference adaptive state feedback control with sliding mode property. A novel sliding mode controller with a proportional and integral sliding surface is proposed and an adaptive sliding mode controller is derived and its stability is proved. The proposed adaptive sliding mode control structure with integral sliding action can establish the stability of closed-loop system. The adaptive designs can be extended to the multiple input systems. Numerical simulations show that the proposed adaptive sliding mode control has satisfactory performance and robustness in the presence of input disturbances.
CHAPTER VI
DIRECT ADAPTIVE SLIDING MODE CONTROL WITH PI SLIDING SURFACE
FOR A MEMS GYROSCOPE

This chapter presents a new adaptive sliding mode controller for MEMS gyroscope; an adaptive tracking controller with a proportional and integral sliding surface is proposed. The adaptive sliding mode control algorithm can estimate the angular velocity vector and the linear damping and stiffness coefficients in real time. A proportional and integral sliding surface, instead of a conventional sliding surface is adopted. An adaptive sliding mode controller that incorporates both matched and unmatched uncertainties and disturbances is derived and the stability of the closed-loop system is established. The numerical simulation is presented to verify the effectiveness of the proposed control scheme. It is shown that the proposed adaptive sliding mode control scheme offers several advantages such as consistent estimation of gyroscope parameters including angular velocity and large robustness to parameter variations and external disturbances.
6.1 Problem Statement

This section proposes a new adaptive sliding mode control strategy for MEMS gyroscopes using a proportional and integral sliding surface, which will be referred to as the direct adaptive sliding mode control shown in Fig. 6.1, the proportional-integral sliding mode control algorithm is presented and solved in the presence of matched and mismatched parameter uncertainties and external disturbance in the gyroscope model.

![Diagram of a MEMS gyroscope control system](image)

Figure 6.1: Block diagram of a direct adaptive sliding mode control for a MEMS gyroscope

The control target for MEMS gyroscope is to maintain the proof mass to oscillate in the x and y direction at given frequency and amplitude as $x_m = A_1 \sin(w_1 t)$, $y_m = A_2 \sin(w_2 t)$. Using (2.24) and rewriting the gyroscope model in state space equation:

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\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-w_x^2 & -d_{xx} & -w_{xy} & -(d_{xy} - 2\Omega_z) \\
0 & 0 & 0 & 1 \\
-w_{xy} & -(d_{xy} + 2\Omega_z) & -w_y^2 & -d_{yy}
\end{bmatrix}
\]

\[
X + \begin{bmatrix}
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
u_x \\
u_y
\end{bmatrix}
\]

which is \( \dot{X} = AX + Bu \)

where \( X = \begin{bmatrix} x \ x \ y \ y \end{bmatrix}^T = [q_1 \ q_2 \ q_3 \ q_4]^T \).

The reference model is defined as

\[
\dot{q}_m + K_m q_m = 0
\]

where \( K_m = \text{diag}\{w_1^2, w_2^2\} \).

Rewriting the reference model in state space equation:

\[
\dot{X}_m = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-w_1^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -w_2^2 & 0
\end{bmatrix}
\begin{bmatrix}
x_m \\
x_m \\
y_m \\
y_m
\end{bmatrix}
\]

which is \( \dot{X}_m = A_m X_m \), where \( X_m = \begin{bmatrix} x_m \ x_m \ y_m \ y_m \end{bmatrix}^T = [q_{1m} \ q_{1m} \ q_{2m} \ q_{2m}]^T \).

Consider the system with multiple inputs with parametric uncertainties

\[
\dot{X}(t) = (A + \Delta A)X(t) + Bu + f(t)
\]

where \( x(t) \in \mathbb{R}^n, \ u(t) \in \mathbb{R}^m \) and \( A \in \mathbb{R}^{n \times n} \) is unknown constant parameter matrix, \( \Delta A \) is the unknown parameter uncertainties of the matrix \( A \), \( f(t) \) is an uncertain extraneous disturbance or unknown nonlinearity of the system .

We make the following assumptions.

1. \( \Delta A \) and \( f(t) \) have matched and unmatched terms. There exists unknown matrices of appropriate dimension \( D, G \) such that \( \Delta A(t) = BD(t) + \Delta \tilde{A}(t) \) and \( f(t) = BG(t) + \tilde{f}(t) \),
where $BD(t)$ is matched uncertainty and $\Delta A(t)$ is unmatched uncertainty, $BG(t)$ is matched disturbance and $\tilde{f}(t)$ is unmatched disturbance.

From this assumptions, $\dot{X}(t) = (A + \Delta A)X(t) + Bu + f(t)$ can be rewritten as

$$
\dot{X}(t) = AX(t) + Bu(t) + \Delta AX(t) + f(t)
$$

$$
= AX(t) + Bu(t) + BDX(t) + \Delta \tilde{A}X(t) + BG + \tilde{f}(t)
$$

$$
= AX(t) + Bu(t) + Bf_m + f_u
$$

(6.5)

where $Bf_m(t, X, u)$ represents the system matched lumped uncertainty and disturbance and lies in the range space of $B$ which is given by

$$
f_m(t, X) = DX(t) + G.
$$

(6.6)

The term $f_u(t, X)$ represents the system lumped unmatched uncertainty and disturbance which is given by

$$
\dot{f}_u(t, X) = \Delta \tilde{A}X(t) + \tilde{f}(t).
$$

(6.7)

2. The matched and unmatched lumped uncertainty and external disturbance $f_m$ and $f_u$ are bounded such as $\|f_m(t, X)\| \leq \alpha_{m1}\|X\| + \alpha_{m2}$ and $\|f_u(t, X)\| \leq \alpha_{u1}\|X\| + \alpha_{u2}$, where $\alpha_{m1}, \alpha_{m2}, \alpha_{u1}, \alpha_{u2}$ are known positive constants.

3. There exists a constant matrix $K^*$ such that the following matching condition

$$
A + BK^*T = A_m
$$

can always be satisfied.

Remark 1. From the equation $A + BK^*T = A_m$, we obtain

$$
w_x^2 = k_{11}^* + w_1^2, \quad d_{xx} = k_{21}^*, \quad w_y = k_{31}^* = k_{12}^*, \quad d_{yy} = k_{42}^*.
$$

From $d_{xy} + 2\Omega_z = k_{22}^*$ and $d_{xy} - 2\Omega_z = k_{41}^*$, we get
\[ d_{xy} = \frac{1}{2}(k_{22}^* + k_{41}^*) \text{ and } \Omega_z = \frac{1}{4}(k_{22}^* - k_{41}^*). \]

It can be seen that the unknown gyroscope parameters \( d_{xx}, d_{xy}, d_{yy}, w_x^2, w_{xy}, w_y^2 \) and \( \Omega_z \) can be determined by the controller parameters \( K^* \). After we obtain the \( K^* \), we can calculate the angular velocity and gyroscope parameters.

6.2 Adaptive Sliding Mode Control Design.

The tracking error and its derivative are

\[
e(t) = X(t) - X_m(t) \quad (6.8)
\]

\[
\dot{e} = AX + Bu + Bf_m + f_u - A_m X_m - A_m e = (A - A_m)X + Bu + Bf_m + f_u. \quad (6.9)
\]

As a preliminary, let us denote the largest eigenvalue of a matrix by \( \beta_{\text{max}}(.) \). For a \( n \times 1 \) vector \( x \), we shall use the Euclidean norm \( \|x\| = \sqrt{x^Tx} \), while the norm of a \( m \times n \) matrix \( A \) is the corresponding induced norm \( \|A\| = \sqrt{\beta_{\text{max}}(A^TA)} \).

The proportional-integral sliding surface \( s(t) = 0 \) is defined as

\[
s(t) = \lambda e - \int_0^t \lambda(K_{\varepsilon} + B\lambda)e d\tau \quad (6.10)
\]

where \( \lambda \) is a constant full rank matrix which satisfies the condition that the matrix \( \lambda B \) is nonsingular diagonal matrix. The constant \( K_{\varepsilon} \) satisfies the condition that \( (A_m + BK_{\varepsilon}) \) is Hurwitz.
The matrix \( \lambda B \) is nonsingular diagonal matrix, as

\[
\lambda B = \begin{bmatrix}
\lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} \\
\lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
\lambda_{31} & \lambda_{34} \\
\lambda_{22} & \lambda_{24}
\end{bmatrix}
\]  

(6.11)

with \( \lambda_{12}, \lambda_{24} \) positive, \( \lambda_{22} = \lambda_{14} = 0 \), \( \lambda B \) is a nonsingular diagonal matrix.

The derivative of the sliding surface is

\[
\dot{s} = \lambda \dot{e} - \lambda (A_m + B K_e) e
= \lambda A_m e + \lambda (A - A_m) X + \lambda B u + \lambda B f_m + \lambda f_u - \lambda (A_m + B K_e) e
= \lambda (A - A_m) X + \lambda B u + \lambda B f_m + \lambda f_u - \lambda B K_e e.
\]  

(6.12)

Setting \( \dot{s} = 0 \) to solve equivalent control \( u_{eq} \) gives

\[
u_{eq} = -(\lambda B)^{-1} \lambda (A - A_m) X + K_e e - f_m - (\lambda B)^{-1} \lambda f_u
= K^* X(t) + K_e e - f_m - (\lambda B)^{-1} \lambda f_u
\]  

(6.13)

Substituting \( u_{eq} \) into equation (6.9) yields

\[
\dot{e}(t) = (A_m + B K_e) e(t) + (I - B (\lambda B)^{-1} \lambda) f_u.
\]  

(6.14)

which is the closed-loop error dynamic equation after system enters the sliding surface.

The adaptive control signal \( u \) is proposed as

\[
u(t) = K^T(t) X(t) + K_e e(t) - \rho (\lambda B)^{-1} \frac{S}{\|S\|}
\]  

(6.15)

where \( K^T = \begin{bmatrix} k_{11} & k_{21} & k_{31} & k_{41} \\
k_{12} & k_{22} & k_{32} & k_{42} \end{bmatrix} \), \( K(t) \) is estimate of \( K^* \).

Define the estimation error as

\[
\tilde{K}(t) = K(t) - K^*.
\]  

(6.17)

Substituting (6.17) into (6.5) yields
\[
\dot{X}(t) = AX(t) + BK^T(t)X(t) + BK_e e + Bf_m + f_u - B\rho(\lambda B)^{-1} \frac{S}{\|s\|}
\]
\[
= (A_m - BK^T)X(t) + BK^T(t)X(t) + BK_e e + Bf_m + f_u - B\rho(\lambda B)^{-1} \frac{S}{\|s\|}
\]
\[
= A_m X(t) + B\bar{K}^T(t)X(t) + BK_e e + Bf_m + f_u - B\rho(\lambda B)^{-1} \frac{S}{\|s\|},
\]

(6.18)

Then, we have the tracking error equation
\[
\dot{e}(t) = A_m e + B\bar{K}^T(t)X(t) + BK_e e + Bf_m + f_u - B\rho(\lambda B)^{-1} \frac{S}{\|s\|}
\]
\[
= (A_m + BK_e)e + B\bar{K}^T(t)X(t) + Bf_m + f_u - B\rho(\lambda B)^{-1} \frac{S}{\|s\|}.
\]

(6.19)

The sliding surface dynamics is
\[
\dot{s}(t) = \dot{\lambda}(A_m + BK_e)e(t) + \lambda B\bar{K}^T(t)X(t) + \lambda Bf_m + \lambda f_u - \lambda B\rho(\lambda B)^{-1} \frac{S}{\|s\|} - \dot{\lambda}(A_m + BK_e)e(t)
\]
\[
= \lambda B\bar{K}^T(t)X(t) + \lambda Bf_m + \lambda f_u - \rho \frac{S}{\|s\|}.
\]

(6.20)

Define a Lyapunov function
\[
V = \frac{1}{2} s^T s + \frac{1}{2} tr[\bar{K}M^{-1}\bar{K}^T]
\]
\[
= \frac{1}{2} \left( m_i \begin{bmatrix} m_i \end{bmatrix} + 0 \begin{bmatrix} 0 \end{bmatrix} \right),
\]

(6.21)

where \( M = M^T > 0 \), \( M \) is positive definite matrix, i.e., \( M = \begin{bmatrix} m_i & 0 \\ 0 & m_2 \end{bmatrix} \), \( tr[M] \) denoting the trace of a square matrix \( M \).

Differentiating \( V \) with respect to time yields

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\[ \dot{V} = s^T \dot{s} + \text{tr} \left[ \hat{K} M^{-1} \dot{\hat{K}} \right] \]

\[ = s^T \left[ \lambda B \dot{K}^T (t) X (t) + \lambda B f_m + \lambda f_u - \rho \frac{S}{\|S\|} \right] + \text{tr} \left[ \hat{K} M^{-1} \dot{\hat{K}} \right] \]

\[ = -s^T \rho \frac{S}{\|S\|} + s^T \lambda B f_m + s^T \lambda f_u + \left( s^T \lambda B \dot{K}^T (t) X (t) + \text{tr} \left[ \hat{K} M^{-1} \dot{\hat{K}} \right] \right) \]  

(6.22)

In order to make \( \dot{V} \) negative definite, the updating law is chosen as

\[ \dot{K} (t) = \dot{K} (t) = -MB^T \lambda s(t) X^T (t) \]

\[ = (6.23) \]

\[ \dot{\hat{K}}(t) = \dot{K}(t) = -X(0)s^T \lambda BM \]

(6.24)

with \( K(0) \) being arbitrary. This adaptive law yields

\[ \dot{V} = -\rho \|S\| + s^T \lambda B f_m + s^T \lambda f_u \]

\[ \leq -\rho \|S\| + \|s\| \|\lambda B\| f_m + \|s\| \|\lambda\| f_u \]

\[ \leq -\rho \|S\| + \|s\| \|\lambda B\| (\alpha_{m1} \|X\| + \alpha_{m2}) + \|s\| \|\lambda\| (\alpha_{u1} \|X\| + \alpha_{u2}) \]

\[ = -\|S\| \rho + \|\lambda B\| (\alpha_{m1} \|X\| + \alpha_{m2}) + \|\lambda\| (\alpha_{u1} \|X\| + \alpha_{u2}) \leq 0. \]

(6.25)

With \( \rho \geq \|\lambda B\| (\alpha_{m1} \|X\| + \alpha_{m2}) + \|\lambda\| (\alpha_{u1} \|X\| + \alpha_{u2}) + \eta \), where \( \eta \) is a positive constant, \( \dot{V} \) becomes negative semi-definite, i.e., \( \dot{V} \leq -\eta \|S\| \). This implies that the trajectory reaches the sliding surface in finite time and remains on the sliding surface. \( \dot{V} \) is negative definite implies that \( s \) and \( \tilde{K} \) converge to zero. \( \dot{V} \) is negative semi-definite ensures that \( V, s \) and \( \tilde{K} \) are all bounded. It can be concluded from (6.20) that \( \dot{s} \) is also bounded.

LaSalle’s invariant set theorem can be used to prove that \( \lim_{t \to \infty} s(t) = 0 \). Barbalat’s lemma can also be used to prove that \( \lim_{t \to \infty} s(t) = 0 \). The inequality \( \dot{V} \leq -\eta \|S\| \) implies that \( s \) is integrable as

\[ \int_{0}^{\infty} \|S\| dt \leq \frac{1}{\eta} \left[ V(0) - V(t) \right] \].

Since \( V(0) \) is bounded and \( V(t) \) is
nonincreasing and bounded, it can be concluded that \( \lim_{t \to \infty} \int_0^t \|s\| dt \) is bounded. Since
\[
\lim_{t \to \infty} \int_0^t \|s\| dt \text{ is bounded and } \dot{s} \text{ is also bounded, according to Barbalat lemma, } s(t) \text{ will asymptotically converge to zero, } \lim_{t \to \infty} s(t) = 0, \text{ that is } s(t) \text{ and } e(t) \text{ all converge to zero asymptotically.}
\]

To make conclusions about the parameter errors \( \tilde{K} = 0 \), other than the fact that they are bounded, we need to make the persistence of excitation argument. From the adaptive laws \( \dot{\tilde{K}}^T (t) = \dot{K}^T (t) = -MB^T \lambda^T sX^T \), according to the persistence excitation theory [5], if \( X \) is persistent excitation signal, then \( \dot{\tilde{K}}^T (t) = -MB^T \lambda^T sX^T \) guarantees that \( \tilde{K} \to 0 \), \( K \) will converges to its true values.

In order to show persistence of excitation, we need to show that there exist some positive scalar constants \( \alpha \) and \( T \) such that for all \( t > 0 \)
\[
\int_t^{t+T} XX^T d\tau \geq \alpha d.
\]

For simplicity, assuming \( A_1 = A_2 = 1 \), then \( \int_t^{t+T} XX^T d\tau \) becomes
\[
\int_t^{t+T} \begin{bmatrix}
\sin^2(w_1 t) & w_1 \sin(w_1 t) \cos(w_1 t) & \sin(w_1 t) \sin(w_2 t) & w_2 \sin(w_1 t) \cos(w_2 t) \\
w_1 \sin(w_1 t) \cos(w_1 t) & w_1^2 \cos^2(w_1 t) & w_1 \cos(w_1 t) \sin(w_2 t) & w_1 w_2 \cos(w_1 t) \cos(w_2 t) \\
\sin(w_1 t) \sin(w_2 t) & w_1 \cos(w_1 t) \sin(w_2 t) & \sin^2(w_2 t) & w_2 \sin(w_1 t) \cos(w_2 t) \\
w_2 \sin(w_1 t) \cos(w_2 t) & w_1 w_2 \cos(w_1 t) \cos(w_2 t) & w_2 \sin(w_2 t) \cos(w_1 t) & w_2^2 \cos^2(w_2 t)
\end{bmatrix} d\tau.
\]

For convenience of calculation, assume \( w_2 = 2w_1 \), selecting \( T = \frac{\pi}{w_1} \), then
\[\int_{t}^{t+T} XX^T d\tau = \begin{bmatrix}
\pi & 0 & 0 & 0 \\
0 & \pi & 0 & 0 \\
0 & 0 & \pi & 0 \\
0 & 0 & 0 & \pi
\end{bmatrix} \geq \alpha \tau .\]

Hence, the persistence excitation condition is satisfied with \(T = \frac{\pi}{w_i}\) and \(\alpha = \frac{\pi}{w_i}\). It can be shown that there always exist some positive scalar constants \(\alpha\) and \(T\) such that for all \(t > 0\), \(\int_{t}^{t+T} XX^T d\tau \geq \alpha \tau\) if \(w_1 \neq w_2\). If \(w_1 = w_2\), \(XX^T\) is a singular matrix, therefore we cannot find positive scalar constants \(\alpha\) such that for all \(t > 0\), \(\int_{t}^{t+T} XX^T d\tau \geq \alpha \tau\).

Remark 2. The parameter convergence can also be explained from the sliding mode concept. If non-adaptive controller is used, then the sliding mode dynamics is
\[\dot{s}(t) = \lambda Bf_m + \bar{\lambda}f_u - \rho \frac{s}{\|s\|}\]
which implies that when system arrives at the idea sliding surface and remain in the sliding surface, \(s \to 0\) and \(\dot{s} \to 0\), the sliding mode controller \(\rho \frac{s}{\|s\|}\) can compensate for the matched and unmatched disturbance term \(\lambda Bf_m + \bar{\lambda}f_u\). In adaptive case, the sliding dynamics is
\[\dot{s}(t) = \lambda B \bar{\lambda}K^T (t)X(t) + \lambda Bf_m + \bar{\lambda}f_u - \rho \frac{s}{\|s\|}\]
which implies that parameter estimation error term \(\lambda B \bar{\lambda}K^T (t)X(t) = 0\). It can be concluded that the parameter error term will not take effect in the sliding mode dynamics. This is a typical characteristic in the sliding mode control. Since \(\lambda B\) is nonsingular, then the
desired solution $\tilde{K} = 0$ implies that that $XX^T$ has full rank.

where $XX^T = \begin{bmatrix}
\dot{x}_1 x_1^2 & \dot{x}_1 x_1 x_2 & x_1 x_2 & x_1^2 \\
\dot{x}_1 x_2 & x_2^2 & x_1 x_2 & x_1^2 \\
\dot{x}_2 x_1 & \dot{x}_2 x_1 & x_2^2 & x_1^2 \\
\dot{x}_2 x_2 & \dot{x}_2 x_2 & x_2^2 & x_1^2 \\
\end{bmatrix}$.

It can be shown that $XX^T$ has full rank if $w_1 \neq w_2$, i.e. the excitation frequencies on $x$ and $y$ axes should be different. In other words, excitation of proof mass should be persistently exciting [22]. Since $\tilde{K} \to 0$, then the unknown angular velocity as well as all other unknown parameters can be determined from $A + BK^T = A_m$.

In summary, if persistently exciting drive signals, $x_m = A_1 \sin(w_1 t)$ and $y_m = A_2 \sin(w_2 t)$ are used, then $\tilde{K}$, $s(t)$ and $e(t)$ all converge to zero asymptotically. Consequently the unknown angular velocity can be determined as $\lim_{t \to \infty} \dot{\Omega}_z (t) = \Omega_z$. However it is difficult to establish the convergence rate.

Remark 3. It is not easy to guarantee that the stated inequality condition on $\rho$ can always be satisfied for all times and all state. However if $\alpha_{m1} \ll \alpha_{m2}$ and $\alpha_{s1} \ll \alpha_{s2}$, then the condition becomes state independent and knowing $\alpha_{m2}$ and $\alpha_{s2}$ can guarantee inequality on $\rho$. It is also critical to have $\rho$ as small as possible in order to avoid saturation in actuators.

Remark 4. If $(A_m, B)$ is a controllable pair, the closed-loop matrix $A_m + BK_r$ can be assigned to have arbitrary set of eigenvalues to control the tracking error settling rates. Therefore $K_r$ is chosen such that $(A_m + BK_r)$ is Hurwitz.
Remark 5. In order to eliminate the chattering, the discontinuous control component in (6.15) similarly can be replaced by a smooth sliding mode component to yield

\[
u(t) = K^T(t)X(t) + K_e e - (\lambda B)^{-1} \rho \frac{s}{\|s\| + \epsilon}
\] (6.27)

where \(\epsilon > 0\) is small constant. This creates a small boundary layer about the switching surface in which the system trajectory will remain. Therefore, the chattering problem can be reduced significantly.

Remark 6. Adaptive Control For the Estimation of the Upper Bound of \(f_m\)

The information of the upper bound of the uncertainties and disturbance must be known in advance in the previous research. Now we remove this limitation and derive an adaptive control algorithm to estimate the upper bound of the uncertainties and disturbance. Assuming \(f_a = 0\), \(\|f_m\| \leq \alpha_1 + \alpha_2 \|X\|\), where \(\alpha_1\) and \(\alpha_2\) are unknown positive constants.

The control input is

\[
u(t) = K^T(t)X(t) + K_e e(t) - \rho \frac{B^T \tilde{\lambda} s}{\|B^T \tilde{\lambda} s\|}
\] (6.28)

where \(\rho = \alpha_1 + \alpha_2 \|X\|\).

we have the tracking error equation

\[
\dot{e}(t) = (A_m + BK_e)e + B\tilde{K}^T(t)X(t) + Bf_m - B\rho \frac{B^T \tilde{\lambda} s}{\|B^T \tilde{\lambda} s\|}
\] (6.29)

and the derivative of \(s(t)\) is
\[ \dot{s}(t) = \lambda B \tilde{K}^T (t) X(t) + \lambda B f_m - \rho \lambda B \frac{B^T \lambda^T s}{\|B^T \lambda^T s\|} \quad (6.30) \]

Define a Lyapunov function

\[ V = \frac{1}{2} s^T s + \frac{1}{2} \text{tr}[\tilde{K} M^{-1} \tilde{K}^T ] + \frac{1}{2} q_1 \tilde{\alpha}_1^2 + \frac{1}{2} q_2 \tilde{\alpha}_2^2 \quad (6.31) \]

where \( M = M^T > 0, \tilde{\alpha}_1 = \alpha_1 - \bar{\alpha}_1, \tilde{\alpha}_2 = \alpha_2 - \bar{\alpha}_2, M \) is positive definite matrix.

Differentiating \( V \) with respect to time yields

\[ \dot{V} = s^T \dot{s} + \text{tr}[\tilde{K} M^{-1} \dot{\tilde{K}}^T ] + q_1 \tilde{\alpha}_1 \dot{\tilde{\alpha}}_1 + q_2 \tilde{\alpha}_2 \dot{\tilde{\alpha}}_2 \]

\[ = s^T \left[ \lambda B \tilde{K}^T (t) X(t) + \lambda B f_m - \rho \lambda B \frac{B^T \lambda^T s}{\|B^T \lambda^T s\|} \right] + \text{tr}[\tilde{K} M^{-1} \dot{\tilde{K}}^T ] + q_1 \tilde{\alpha}_1 \dot{\tilde{\alpha}}_1 + q_2 \tilde{\alpha}_2 \dot{\tilde{\alpha}}_2 \quad (6.32) \]

In order to make \( \dot{V} \) negative definite, the updating law is chosen as

\[ \dot{\tilde{K}}(t) = \tilde{K}(t) = -X(t) s^T \lambda BM \quad (6.33) \]

with \( K(0) \) being arbitrary. This adaptive law yields

\[ \dot{V} = -\rho \|B^T \lambda^T s\| + s^T \lambda B f_m + q_1 \tilde{\alpha}_1 \dot{\tilde{\alpha}}_1 + q_2 \tilde{\alpha}_2 \dot{\tilde{\alpha}}_2 \]

\[ = -\|B^T \lambda^T s\| (\alpha_1 + \alpha_2 \|X\|) + s^T \lambda B f_m + q_1 (\alpha_1 - \bar{\alpha}_1) \dot{\tilde{\alpha}}_1 + q_2 (\alpha_2 - \bar{\alpha}_2) \dot{\tilde{\alpha}}_2 \]

\[ \leq -\|B^T \lambda^T s\| (\alpha_1 + \alpha_2 \|X\|) + \|B^T \lambda^T s\| f_m \| + q_1 (\alpha_1 - \bar{\alpha}_1) \dot{\tilde{\alpha}}_1 + q_2 (\alpha_2 - \bar{\alpha}_2) \dot{\tilde{\alpha}}_2 \quad (6.34) \]

\[ \leq -\|B^T \lambda^T s\| (\alpha_1 + \alpha_2 \|X\|) + \|B^T \lambda^T s\| (\alpha_1 + \alpha_2 \|X\|) + q_1 (\alpha_1 - \bar{\alpha}_1) \dot{\tilde{\alpha}}_1 + q_2 (\alpha_2 - \bar{\alpha}_2) \dot{\tilde{\alpha}}_2 \]

\[ = (\alpha_1 - \bar{\alpha}_1) \|B^T \lambda^T s\| - q_1 \dot{\tilde{\alpha}}_1 + (\alpha_2 - \bar{\alpha}_2) \|B^T \lambda^T s\| \|X\| - q_2 \dot{\tilde{\alpha}}_2 \]

we choose the adaptive laws as

\[ \dot{\tilde{\alpha}}_1 = \frac{1}{q_1} \|B^T \lambda^T s\| \quad (6.35) \]

\[ \dot{\tilde{\alpha}}_2 = \frac{1}{q_2} \|B^T \lambda^T s\| \|X\| \quad (6.36) \]
Therefore, \( \dot{V} \leq 0 \) implies that the closed-loop system is stable, \( \lim_{t \to \infty} s(t) = 0 \).

It should be noted that the convergence of the adaptive parameters of upper bound to real ones cannot be guaranteed, since the system trajectory is restricted to the switching surface after arriving at the sliding mode, the adaptive parameters converge to some values depending on the values of \( \bar{\alpha}_1, \bar{\alpha}_2, q_1 \) and \( q_2 \).

In some time \( s \) will not be equal to zero all the time, this will decrease tracking accuracy of control system. The adaptive gains will slowly increase boundlessly. The dead-zone techniques can be used to remove this implementation problem. To eliminate the problem of integral wind-up in the adaptation of the upper bound of the unknown disturbance, the adaptive laws are modified as

\[
\dot{\hat{\alpha}}_1 = \frac{1}{q_1} \left( -\varphi_1 \hat{\alpha}_1 + \|B^T \lambda^T s\| \right) \quad \text{(6.37)}
\]

\[
\dot{\hat{\alpha}}_2 = \frac{1}{q_2} \left( -\varphi_2 \hat{\alpha}_2 + \|B^T \lambda^T s\| \|x\| \right) \quad \text{(6.38)}
\]

6.3. Simulation Studies.

We evaluated the proposed adaptive sliding mode control on a lumped MEMS gyroscope model (2.18-2.19) using MATLAB/SIMULINK. The control objective is to design an adaptive state tracking sliding mode controller so that a consistent estimate of \( \Omega_z \) can be obtained. The Matlab/Simulink block of adaptive sliding mode control with PI sliding surface of MEMS gyroscope and Matlab/Simulink sub-block of adaptive estimator system dynamics of MEMS gyroscope, and sliding mode controller are depicted in Figs. 6.2-6.5 respectively.
We allowed ±2% parameter variations for the spring and damping coefficients with respect to their nominal values. We further assumed ±1% magnitude changes in the coupling terms \( d_{xy} \) and \( \omega_{xy} \). The disturbance is assumed to be a random variable with zero mean and unity variance. The controller parameter \( K_c \) of (6.13) is chosen as

\[
K_c = \begin{bmatrix}
-10000 & -10000 & 1000 & 20000 \\
-1000 & -1000 & -1000 & -1000
\end{bmatrix}
\]

to place the eigenvalues of matrix \( (A_m + BK_e) \) in \{-80304, -29581, -1, -0.5\}. The sliding gain \( \rho \) of (6.16) is \( \rho = \text{diag}\{10000, 10000\} \), the adaptive gain in (6.25) is \( M = \text{diag}\{20, 20\} \), the sliding mode parameter \( \lambda \) is

\[
\lambda = \begin{bmatrix}
0 & 10 & 0 & 0 \\
0 & 0 & 0 & 10
\end{bmatrix}.
\]

The tracking error and sliding surface under step tracking are shown in Figs.6.6-6.7. Fig. 6.9 shows that the controller parameters converge to their true values with persistent excitation. Fig.6.8 verifies the step tracking performance of the angular velocity estimation. Fig 6.10 depicts the sliding mode control input with smooth sliding mode controller under step tracking. It is shown that the adaptive sliding mode system with the smooth sliding mode controller can reduce chattering significantly. Fig. 6.11 verifies the step tracking performance of the angular velocity estimation under non-step tracking.
Adaptive Control with Proportional-Integral Sliding Mode Controller for MEMS Gyroscope

Figure 6.2: Matlab/Simulink block of adaptive sliding mode control with PI sliding surface for a MEMS gyroscope

Figure 6.3: Matlab/Simulink sub-block of adaptive estimator

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Figure 6.4: Matlab/Simulink sub-block of system dynamics

Figure 6.5: Matlab/Simulink sub-block of sliding mode controller
Figure 6.6: Plot of the tracking error under change in $\Omega_z$

Figure 6.7: Plot of the sliding surface under change in $\Omega_z$
Figure 6.8: Adaptation of angular velocity under change in $\Omega_z$
Figure 6.9: Adaptation of controller parameters under change in $\Omega_z$
Figure 6.10: Smooth sliding mode control force of the adaptive sliding mode controller

Figure 6.11: Adaptation of angular velocity under no change in $\Omega_z$
6.4 Concluding Remarks

This chapter investigates the design of a model reference adaptive sliding mode state feedback control for MEMS gyroscopes. It is assumed that all states of the system are known. In Chapter 7 we show how this restrictive assumption can be avoided by use of a sliding mode observer. The controller proposed here uses a novel sliding mode algorithm consisting a proportional and integral sliding surface. An adaptive sliding mode controller is derived to control the axes of the gyroscope and to estimate the unknown angular velocity. Furthermore a smooth version of the adaptive sliding mode controller is used to reduce the control chattering. The proposed adaptive sliding mode control structure with proportional and integral sliding action can handle with both matched and unmatched uncertainties and disturbance, provided that upper bounds for these matched and unmatched uncertainties are available. Simulation results demonstrate that the use of the proposed proportional-integral sliding mode adaptive control technique is effective in estimating the gyroscope parameters and angular velocity in the presence of matched and unmatched disturbances.
CHAPTER VIII
SLIDING MODE OBSERVER-BASED ADAPTIVE SLIDING MODE CONTROL
WITH PI SLIDING SURFACE

This chapter presents an adaptive sliding mode control with a sliding mode observer for a MEMS z-axis gyroscope. The proposed adaptive sliding mode controller with a sliding mode observer which reconstructs the unmeasured velocity states can real-time estimate the component of the angular velocity and the linear damping and stiffness coefficients of the gyroscope despite parameter variations and external disturbance in the presence of unmeasured velocity states. An adaptive sliding mode controller with a proportional and integral sliding surface is derived and the stability of the closed-loop system can be guaranteed with the proposed adaptive sliding mode controller with the sliding mode observer. The numerical simulation for the MEMS Gyroscope model is investigated to verify the effectiveness of the proposed control algorithm.

7.1 Problem Statement

In practical MEMS gyroscope not all the state variables are available therefore estimates of unavailable states need to be constructed for use in the control law. In order to extend the practical applicability of the proposed control scheme, we can utilize the sliding mode observer to estimate the unmeasured states. Since the sliding mode observer
has many advantages over Luenberger observers such as robustness to parameter uncertainty, measurement noise and external disturbances, a sliding observer will be incorporated to provide estimates of the internal unmeasured states. We combine the adaptive sliding mode controller with the observer to synthesize an adaptive sliding mode controller that does not require full state measurement in this section.

Rewriting the gyroscope model in state space form we get:

\[
\dot{X} = AX + Bu
\]  \hspace{1cm} (7.1)

where

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-w_x^2 & -d_{xx} & -w_{xy} & -d_{xy} - 2\Omega_z \\
0 & 0 & 0 & 1 \\
-w_{xy} & -(d_{xy} + 2\Omega_z) & -w_y^2 & -d_{yy}
\end{bmatrix}
\]  \hspace{1cm} (7.2)

\[
B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T, \quad u = \begin{bmatrix} u_x \\ u_y \end{bmatrix}, \quad X = \begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{bmatrix} \hspace{1cm} (7.3)
\]

Consider the system in (7.1) with parametric uncertainties and external disturbance

\[
\dot{X}(t) = (A + \Delta A)X(t) + Bu + f(t)
\]  \hspace{1cm} (7.4)

where \(\Delta A\) is the unknown parameter uncertainties of the matrix \(A\), \(f(t)\) is an uncertain disturbance or unknown nonlinearity of the system. We assume that not all states \(X=[x \ \dot{x} \ y \ \dot{y}]^T\) are measurable; only position signals \(x\) and \(y\) are measurable, therefore the system output is

\[
Y(t) = CX(t)
\]  \hspace{1cm} (7.5)

where \(C = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \).
The control target for MEMS gyroscope is to maintain the proof mass to oscillate in the $x$ and $y$ direction at given frequency $w_1$ & $w_2$ and amplitude $A_1$ & $A_2$ as $x_m = A_1 \sin(w_1 t)$, and $y_m = A_2 \sin(w_2 t)$. Suppose that the control objective is to make the trajectory of the gyroscope to follow the reference model as

$$\dot{q}_m + K_m q_m = 0 \quad \text{(7.6)}$$

where $K_m = \text{diag}\{w_1^2, w_2^2\}$. Rewriting the reference model in the state space equation:

$$\dot{X}_m = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -w_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -w_2^2 & 0 \end{bmatrix} X_m \equiv A_m X_m \quad \text{(7.7)}$$

where $A_m$ is a known constant matrix.

We make the following assumptions:

1. $\Delta A$ and $f(t)$ have matched and unmatched components. There exist unknown matrices of appropriate dimension $D, G$ such that (i) $\Delta A(t) = BD(t) + \tilde{\Delta} A(t)$, where $BD(t)$ is the matched uncertainty and $\tilde{\Delta} A(t)$ is the unmatched uncertainty; (ii) $f(t) = BG(t) + \tilde{f}(t)$, where $BG(t)$ is the matched disturbance and $\tilde{f}(t)$ is the unmatched disturbance.

Therefore, (7.4) can be rewritten as

$$\dot{X}(t) = AX(t) + Bu(t) + \Delta AX(t) + f(t)$$

$$= AX(t) + Bu(t) + BDX(t) + \Delta A X(t) + BG + \tilde{f}(t)$$

$$= AX(t) + Bu(t) + Bf_m + f_u \quad \text{(7.8)}$$

where $Bf_m(t, X)$ represents the lumped matched uncertainty and disturbance which is given by
The term $f_u(t, X)$ represents the lumped unmatched uncertainty and disturbance which is given by

$$f_u(t, X) = \Delta \tilde{A}X(t) + \tilde{f}(t). \quad (7.10)$$

2. The lumped matched uncertainty and external disturbance $f_m$ is bounded such as

$$\|f_m(t, X)\| \leq \alpha_1 \|X\| + \alpha_2$$

where $\alpha_1, \alpha_2$ are known positive constants satisfying $\alpha_1 \|X\| << \alpha_2$. The lumped unmatched uncertainty and external disturbance $f_u = 0$.

3. There exists a constant matrix $K^*$ such that the following matching condition $A + BK^{*T} = A_m$ can always be satisfied.

7.2 Sliding Mode Controller and Sliding Mode Observer Design.

The tracking error is defined as

$$e(t) = X(t) - X_m(t) \quad (7.11)$$

and the derivative of tracking error is

$$\dot{e} = A_m e + (A - A_m) X + Bu + Bf_m. \quad (7.12)$$

The proportional-integral sliding surface is defined as

$$s(t) = \lambda e - \int_0^t \lambda (A_m + BK_\tau) e d\tau \quad (7.13)$$
where $\lambda$ is a constant matrix as $\lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24} \end{bmatrix}$ which satisfies the condition that $\lambda B$ is nonsingular, $s$ is a column vector as $s = [s_1, s_2]^T$, $K_\varepsilon$ is a constant matrix which satisfies the condition that $(A_m + BK_\varepsilon)$ is Hurwitz.

The derivative of the sliding surface is

$$\dot{s} = \lambda(A - A_m)X + \lambda Bu + \lambda Bf_m - \lambda BK_\varepsilon e.$$  

(7.14)

Setting $\dot{s} = 0$ to solve equivalent control $u_{eq}$ gives

$$u_{eq} = - (\lambda B)^{-1} \lambda(A - A_m)X + K_\varepsilon e - f_m$$

$$= K^* X(t) + K_\varepsilon e - f_m.$$  

(7.15)

Substituting $u_{eq}$ into equation (7.12) yields

$$\dot{e}(t) = (A_m + BK_\varepsilon)e(t).$$  

(7.16)

which is the closed-loop error dynamic equation after system enters the sliding surface.

Therefore, from (7.15), the control signal $u$ can be generated as

$$u = K^* X(t) + K_\varepsilon e - \rho_1 (\lambda B)^{-1} \frac{s}{\|s\|}$$  

(7.17)

where $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, the matrix $K^*$ is defined by $A + BK = A_m$ where $K^T = \begin{bmatrix} k_{11}^* & k_{21}^* & k_{31}^* & k_{41}^* \\ k_{12}^* & k_{22}^* & k_{32}^* & k_{42}^* \end{bmatrix}$, therefore $w_{xy} = k_{12}^* = k_{31}^*$.

The sliding mode component $u_s = \begin{bmatrix} u_{s1} \\ u_{s2} \end{bmatrix} = - \rho_1 (\lambda B)^{-1} \frac{s}{\|s\|}$, where $\rho_1$ is constant.
If the velocity states are unmeasurable, a sliding mode observer is designed that can provide estimates of unmeasured velocity states based on measured position states. Let \( \hat{x} \) be the estimate of \( x \) obtained from the observer. The observation error can be defined as

\[
\tilde{X} = \hat{X} - X = \begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
\tilde{x}_3 \\
\tilde{x}_4
\end{bmatrix} = \begin{bmatrix}
\hat{x}_1 - x_1 \\
\hat{x}_2 - x_2 \\
\hat{x}_3 - x_3 \\
\hat{x}_4 - x_4
\end{bmatrix}.
\]  

(7.18)

Consider the sliding mode observer structure given by

\[
\dot{\hat{X}}(t) = A\hat{X}(t) + Bu - L(C\hat{X} - Y) + Bv_0 \\
= A\hat{X}(t) + Bu - LC\tilde{X} + Bv_0.
\]  

(7.19)

The discontinuous vector is given by

\[
v_0 = \begin{cases} 
-\rho_2 \frac{FC\tilde{X}}{\|FC\tilde{X}\|} & \text{if } F\,C\tilde{X} \neq 0 \\
0 & \text{otherwise}
\end{cases}
\]  

(7.20)

where \( \rho_2 \) is a scalar.

If the system is observable, the observer gain matrix \( L \) can be chosen so that the closed-loop matrix \( (A - LC) \) is stable and has a Lyapunov matrix \( P = P^T > 0 \) satisfying

(i) \((A - LC)^TP + P(A - LC) = -Q\)  

(7.21)

for some positive definite design matrix \( Q = Q^T > 0 \) and,

(ii) the structural constraint \( FC = B^TP \), for some non-singular matrix \( F \).

The observation error dynamics is

\[
\dot{\tilde{X}} = (A - LC)\tilde{X} + Bv_0 - Bf_m.
\]  

(7.22)
For the gyroscope dynamics (7.1) and with the choice of $L = [l_1 \ l_2 \ l_3 \ l_4]^T$ and $F = [0.707 \ 0.707]^T$, the sliding mode observer can be constructed as

$$
\dot{\hat{x}}_1 = \hat{x}_2 - l_1 \hat{x}_1 \\
\dot{\hat{x}}_2 = -w_x^2 \hat{x}_1 - d_x \hat{x}_2 - w_{xy} \hat{x}_3 - (d_y - 2\Omega_c) \hat{x}_4 - l_2 \hat{x}_1 - \rho_2 \text{sgn}(\hat{x}_1 + \hat{x}_3) + u_x \\
\dot{\hat{x}}_3 = \hat{x}_4 - l_3 \hat{x}_1 \\
\dot{\hat{x}}_4 = -w_{xy} \hat{x}_1 - (d_y + 2\Omega_c) \hat{x}_2 - w_x^2 \hat{x}_3 - d_y \hat{x}_4 - l_4 \hat{x}_1 - \rho_2 \text{sgn}(\hat{x}_1 + \hat{x}_3) + u_y 
$$

(7.23)

The observer error dynamics becomes

$$
\dot{\hat{\xi}}_1 = \hat{\xi}_2 - l_1 \hat{\xi}_1 \\
\dot{\hat{\xi}}_2 = -w_x^2 \hat{\xi}_1 - d_x \hat{\xi}_2 - w_{xy} \hat{\xi}_3 - (d_y - 2\Omega_c) \hat{\xi}_4 - l_2 \hat{\xi}_1 - \rho_2 \text{sgn}(\hat{\xi}_1 + \hat{\xi}_3) - f_{m1} \\
\dot{\hat{\xi}}_3 = \hat{\xi}_4 - l_3 \hat{\xi}_1 \\
\dot{\hat{\xi}}_4 = -w_{xy} \hat{\xi}_1 - (d_y + 2\Omega_c) \hat{\xi}_2 - w_x^2 \hat{\xi}_3 - d_y \hat{\xi}_4 - l_4 \hat{\xi}_1 - \rho_2 \text{sgn}(\hat{\xi}_1 + \hat{\xi}_3) - f_{m2} 
$$

(7.24)

where $f_m = [f_{m1} \ f_{m2}]^T$.

7.3. Observer-Based Adaptive Sliding Mode Controller Design.

In this section, the effect of using the state estimates in the control law will be explored. Let us consider the problem where both the state and the angular velocity $\Omega_c$ and all gyroscope parameters are to be estimated in real time using an adaptive sliding mode controller with a sliding mode observer. We will apply the sliding mode control method and prove that the control performance is robust to estimation errors when used in conjunction with the sliding mode observer. The block diagram of the adaptive sliding mode control with sliding mode observer is shown in Fig. 7.1.
If not all states are measurable, define the tracking error between observed state and reference model state as

\[ \dot{e}(t) = \dot{\hat{X}}(t) - X_m(t). \]  

(7.25)

The sliding surface in term of \( \dot{e}(t) \) is defined as

\[ \dot{s}(t) = \lambda \dot{e} - \int_0^t \lambda (A_m + BK_e) \dot{e} \, d\tau. \]  

(7.26)

The adaptive controller with the estimated states is proposed as

\[ u(t) = K^T(t) \dot{X}(t) + K_e \dot{e} + (\lambda B)^{-1} v_c \]  

where \( K^T(t) = \begin{bmatrix} k_{11} & k_{21} & k_{31} & k_{41} \\ k_{12} & k_{22} & k_{32} & k_{42} \end{bmatrix} \) is an estimate of \( K^* \), the robust component of the control signal is defined through

\[ v_c = \begin{cases} -\rho \frac{\hat{s}}{\|\hat{s}\|} & \text{if } s \neq 0 \\ 0 & \text{otherwise.} \end{cases} \]  

(7.28)

![Block diagram of sliding mode observer based direct adaptive sliding mode control for a MEMS gyroscope](image)

Figure 7.1: Block diagram of sliding mode observer based direct adaptive sliding mode control for a MEMS gyroscope

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Define the controller estimation error as
\[ \tilde{K}(t) = K(t) - K^* . \] (7.29)

Substituting equation (7.29) and (7.27) into equation (7.19) yields
\[ \dot{\tilde{X}}(t) = A_m \tilde{X}(t) + BK^T(t) \dot{\tilde{X}}(t) + BK_c(\tilde{X} - X_m) + B(\lambda B)^{-1}v_c - LC\tilde{X} + Bv_0. \] (7.30)

The derivative of \( \hat{e}(t) \) is
\[ \dot{\hat{e}}(t) = (A_m + BK_c)\hat{e} + B\tilde{K}^T(t)\dot{\tilde{X}}(t) + B(\lambda B)^{-1}v_c - LC\tilde{X} + Bv_0. \] (7.31)

The derivative of sliding surface with the estimated states is
\[ \dot{s}(t) = \lambda B\tilde{K}^T(t) + v_c - \lambda LC\tilde{X} + \lambda Bv_0. \] (7.32)

Define a Lyapunov function as
\[ V = \frac{1}{2} \tilde{X}^T \dot{\tilde{X}} + \frac{1}{2} \text{tr}[\tilde{K}M^{-1}\tilde{K}^T] + \frac{1}{2} \tilde{X}^T P\tilde{X} \] (7.33)

where the positive definite matrix \( P = P^T > 0, M = M^T > 0, M = \text{diag}\{m_1, m_2\} \).

Differentiating \( V \) with respect to time yields
\[ \dot{V} = \frac{1}{2} \tilde{X}^T \dot{\tilde{X}} - \rho_2 \frac{\tilde{X}^T PBFC\tilde{X}}{||FC\tilde{X}||} - \frac{1}{2} \tilde{X}^T P \tilde{B} f_m + \frac{1}{2} [\tilde{X}^T (A - LC)^T P\tilde{X} + \tilde{X}^T P(A - LC)\tilde{X}] \]
\[ = -\rho_1 \|\hat{e}\|^2 - \tilde{s}^T \lambda LC\tilde{X} + \tilde{s}^T \lambda Bv_0 + \tilde{s}^T \lambda B\tilde{K}^T(t)\dot{\tilde{X}}(t) + \text{tr}[\tilde{K}M^{-1}\tilde{K}^T] - \frac{1}{2} \tilde{X}^T Q\tilde{X} \]
\[ - \rho_2 \frac{\tilde{X}^T PBFC\tilde{X}}{||FC\tilde{X}||} - \tilde{X}^T P \tilde{B} f_m. \] (7.34)

Using (7.21) and \( FC = B^TP \), we get
\[ \dot{V} = -\rho_1 \|\hat{e}\|^2 - \tilde{s}^T \lambda LC\tilde{X} + \tilde{s}^T \lambda Bv_0 - \frac{1}{2} \tilde{X}^T Q\tilde{X} - \rho_2 \frac{\tilde{X}^T C^T F^T f_m}{||FC\tilde{X}||} - \tilde{X}^T C^T F^T f_m \]
\[ + (\tilde{s}^T \lambda B\tilde{K}^T \dot{\tilde{X}} + \text{tr}[\tilde{K}M^{-1}\tilde{K}^T]) \] (7.35)
In order to make $\dot{V}$ negative definite, the updating law is chosen as

$$
\ddot{K}^T (t) = \dot{K}^T (t) = -MB^T \dot{X}^T (t)
$$

(7.36)

with $K(0)$ being arbitrary. This yields

$$
\dot{V} = -\rho_1 \|\dot{s}\| + \dot{s}^T \lambda LC \dot{X} + \dot{s}^T \lambda BV_x - \frac{1}{2} \dot{X}^T Q \dot{X} - \rho_2 \|FC \dot{X}\| - \dot{X}^T C^T F^T f_m
$$

$$
\leq -\rho_1 \|\dot{s}\| + \|s\| \lambda L \|C \dot{X}\| + \|s\| \lambda \|B\| \|\rho_2 - \frac{1}{2} \dot{X}^T Q \dot{X} - \rho_2 \|FC \dot{X}\| + \|FC \dot{X}\| \|f_m\|
$$

(7.37)

$$
\leq -\|s\| (\rho_1 - \|L\| \|x_1 + \ddot{x}_1\| - \rho_2 \|B\|) - \frac{1}{2} \ddot{X}^T Q \dot{X} - \|FC \dot{X}\| (\rho_2 - \alpha_1 \|X\| - \alpha_{m_2})
$$

With the choices of $\rho_2 > \alpha_{m_1} \|X\| + \alpha_{m_2}$ and $\rho_1 > \|L\| \|x_1 + \ddot{x}_1\| + \rho_2 \|B\|$, $\dot{V}$ becomes negative semi-definite, i.e., $\dot{V} \leq 0$. This implies that $\dot{s}$, $\ddot{K}$, and $\dot{X}$ are all bounded. $\dot{V}$ is negative definite implies that $\dot{s}$, $\ddot{K}$ and $\dot{X}$ converge to zero. $\dot{V}$ is negative semi-definite ensures that $V$, $\dot{s}$, $\ddot{K}$ and $\dot{X}$ are all bounded. $\dot{V} = 0$ implies $\ddot{X}(t) = 0$ and $\dot{s} = 0$. It can be shown that $\dot{s}(t)$ will asymptotically converge to zero, $\lim_{t \to \infty} \dot{s}(t) = 0$, that is $\dot{s}(t)$ and $\ddot{e}(t)$ all converge to zero asymptotically.

LaSalle’s invariant set theorem can be used to prove that $\lim_{t \to \infty} \dot{s}(t) = 0$. $\dot{V} = 0$ implies that $\dot{s} = 0$ and there is no other solution but $\dot{s} = 0$. According to LaSalle’s invariant set theorem, defining $R = \{\dot{s} \in R^n \mid \dot{V}(x) = 0\}$, then if $R$ contains no other trajectories other than $\dot{s} = 0$, the origin 0 is asymptotically stable. Therefore the sliding surface $\dot{s} = 0$ is an invariant set which implies that any trajectory starting from an initial condition within the set remains in the set all the time, that is $\ddot{s}(t)$ will asymptotically converge to zero, $\lim_{t \to \infty} \ddot{s}(t) = 0$. Barbalat’s lemma can also be used to prove that $\lim_{t \to \infty} \dot{s}(t) = 0$. 

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To make conclusions about the parameter errors $\tilde{K}$ converge to zero other than the fact that they are bounded, we need to make the persistence of excitation argument. From the adaptive law $\hat{K}^T(t) = -MB^T \hat{X}^T \hat{s} \hat{X}^T(t)$, according to the persistence excitation theory [5], if $\hat{X}$ meets persistence of excitation condition, then $\tilde{K} \to 0$, $K(t)$ will converges to its true values. It can be shown that there exist some positive scalar constants $\alpha$ and $T$ such that for all $t > 0$, $\int_{t}^{t+T} \hat{X}^T \hat{X} d\tau \geq \alpha t$, where $\hat{X}^T = \begin{bmatrix} x_1^2 & x_1 x_1 & x_1 x_2 & x_1 \dot{x}_2 \\
 x_1 x_1 & x_1^2 & x_1 x_2 & x_1 \dot{x}_2 \\
 x_2 x_1 & x_2 x_1 & x_2^2 & x_2 \dot{x}_2 \\
 x_2 x_1 & x_2 x_1 & x_2 x_2 & x_2^2 \end{bmatrix}$

From (7.11) and (7.26) it can be shown that $\hat{X}^T \hat{X}$ has full rank if $w_i \neq w_z$, i.e. the excitation frequencies on $x$ and $y$ axes should be different; In other word, excitation of proof mass should be persistently exciting [22].

Since $\tilde{K} \to 0$, then the unknown angular velocity as well as all other unknown parameters can be determined from $A + BK^T = A_m$ and we obtain $\Omega_\zeta = 0.25(k_{22} - k_{41})$.

In summary if persistently exciting drive signals, $x_m = A_1 \sin(w_i t)$ and $y_m = A_2 \sin(w_z t)$ are used, then $\tilde{K}(t), \tilde{X}(t), \hat{s}(t)$ and $\hat{c}(t)$ all converge to zero asymptotically. Consequently the unknown angular velocity can be determined as $\lim_{t \to \infty} \hat{\Omega}_\zeta(t) = \Omega_\zeta$. However it is difficult to establish the convergence rate.

Remark 1. In order to eliminate the chattering, the discontinuous control component $(\lambda B)^{-1} v_c$ in (7.27) can be replaced by a smooth sliding mode component to yield

$$u(t) = K^T(t)\hat{X}(t) + K_c \hat{c} - (\lambda B)^{-1} \rho \frac{\hat{s}}{\|\hat{s}\| + \varepsilon}$$  \hspace{1cm} (7.38)
where $\epsilon > 0$ is small constant. This modification creates a small boundary layer around the switching surface in which the system trajectory will remain. Therefore, the chattering problem can be reduced significantly.

**Remark 2. Separation Principle**

Substituting (7.27) into $\dot{X}(t) = AX(t) + Bu + Bf_m$, yields

$$
\dot{X}(t) = AX(t) + Bu(t) + Bf_m
= AX(t) + B\left(K^T \dot{X}(t) + K_c \dot{\epsilon} + (\lambda B)^{-1} \nu_c\right) + Bf_m
= AX(t) + B(K^T (X + \tilde{X}) + K_c \dot{\epsilon} + (\lambda B)^{-1} \nu_c) + Bf_m
$$

(7.39)

So the closed-loop system including the estimation error in the sliding mode can be written as

$$
\begin{bmatrix}
\dot{X} \\
\dot{\tilde{X}}
\end{bmatrix} = \begin{bmatrix}
A + B(K^T + K_c) & B(K^T + K_c) \\
0 & A - LC
\end{bmatrix} \begin{bmatrix}
X \\
\tilde{X}
\end{bmatrix} + \begin{bmatrix}
BK_c \\
0
\end{bmatrix} X_m - \begin{bmatrix}
B(\lambda B)^{-1} \rho_s \\
0
\end{bmatrix} - \begin{bmatrix}
0 \\
B KFC \tilde{X}
\end{bmatrix} + \begin{bmatrix}
0 \\
-B
\end{bmatrix} f_m
$$

(7.40)

The state equation of the compensator by including $u(t) = K^T \dot{X}(t) + K_c \dot{\epsilon} + (\lambda B)^{-1} \nu_c$ in the estimator equation is

$$
\dot{\hat{X}}(t) = A \dot{\hat{X}}(t) + B \left(K^T \dot{\hat{X}}(t) + K_c \dot{\epsilon} + (\lambda B)^{-1} \nu_c\right) - LC\tilde{X} + Bv_0
= (A + BK^T) \dot{\hat{X}}(t) + BK_c \dot{\epsilon} + B(\lambda B)^{-1} \nu_c - LC\tilde{X} + Bv_0
= (A + BK^T + BK_c) \dot{\hat{X}}(t) - BK_c X_m + B(\lambda B)^{-1} \nu_c - LC\tilde{X} + Bv_0
= (A + BK^T + BK_c - LC) \dot{\hat{X}}(t) - BK_c X_m + B(\lambda B)^{-1} \nu_c + LCX + Bv_0.
$$

(7.41)

If $\hat{s} = 0$ and $\tilde{X} = 0$ which mean $\nu_0 = 0$ and $\nu_c = 0$, therefore
\[
\begin{bmatrix}
\dot{X}
\end{bmatrix} = 
\begin{bmatrix}
A + B(K^T + K_e) & B(K^T + K_e) \\
0 & A - LC
\end{bmatrix}
\begin{bmatrix}
X
\end{bmatrix} + 
\begin{bmatrix}
B K_e \\
0
\end{bmatrix} X_m + 
\begin{bmatrix}
B
\end{bmatrix} f_m.
\] (7.42)

It is clearly shown from above equation that the eigenvalues of the closed loop system are determined by those of \( A + B(K^T + K_e) \) and those assigned for the sliding mode observer \( A - LC \). Therefore the design of controllers and observers can be conducted separately.

Remark 3. We will derive the combined observer-controller closed-loop system dynamics in sliding mode.

\[
\dot{s}(t) = \dot{\lambda} e_i - \lambda(A_m + BK_e) e_i
\]
\[
= \lambda(\dot{\hat{X}} - \hat{X}_m) - \lambda(A_m + BK_e)(\hat{X} - X_m)
\]
\[
= \lambda \left[ (A - LC)\hat{X} + Bu + LCX + Bv_0 - A_m X_m \right] - \lambda(A_m + BK_e)(\hat{X} - X_m)
\]
\[
= \lambda (A - LC)\hat{X} + \lambda Bu + \lambda LCX + \lambda Bv_0 - \lambda (A_m + BK_e)\hat{X} + \lambda BK_e X_m.
\] (7.43)

Setting \( \dot{s} = 0 \) to solve equivalent control \( u_{eq} \) gives

\[
u_{eq} = -(\lambda B)^{-1} \lambda (A - LC)\hat{X} - (\lambda B)^{-1} \lambda LCX - (\lambda B)^{-1} \lambda Bv_0
\]
\[
+ (\lambda B)^{-1} \lambda (A_m + BK_e)\hat{X} - K_e X_m.
\] (7.44)

Substituting \( u_{eq} \) into the observer equation yields

\[
\dot{\hat{X}} = (A - LC)\hat{X} + Bu_{eq} + LCX + Bv_0
\]
\[
= \left[ I - B(\lambda B)^{-1} \lambda \right] (A - LC)\hat{X} + LCX + Bv_0 + B(\lambda B)^{-1} \lambda A_m \hat{X} + BK_e (\hat{X} - X_m)
\]
\[
\dot{\hat{X}} = (A - LC)\hat{X} + Bv_0 - Bf_m
\] (7.45)

which describes the combined closed-loop system in sliding mode.
Remark 4. In the previous derivation, the proposed sliding mode observer is full order observer which all states including measurable position state are estimated. Since the position of system can be measurable, the measurable position state variables need not to be estimated, only the velocity state variable should be estimated, therefore reduced order sliding mode observer can be designed to estimate the velocity. It is important to note that if the measurement of output variable involves significant noises and is relatively inaccurate, the use of full order observer may result in better performance for estimation of angular velocity and cross stiffness and damping coefficients.

7.4. Simulation Studies

We evaluated the proposed adaptive sliding mode control on a lumped MEMS gyroscope model (2.18-2.19) using MATLAB/SIMULINK. The control objective is to design an adaptive state tracking sliding mode controller so that a consistent estimate of $\Omega_z$ can be obtained. The Matlab/Simulink block of adaptive sliding mode control with sliding mode observer of MEMS gyroscope and Matlab/Simulink sub-block of sliding mode observer, adaptive estimator, sliding mode controller and system dynamics of MEMS gyroscope, and are depicted in Figs. 7.2-7.6 respectively.

We allowed $\pm 2\%$ parameter variations for the spring and damping coefficients with respect to their nominal values. We further assumed $\pm 1\%$ magnitude changes in the coupling terms $d_{xy}$ and $\omega_{xy}$. Both matched and unmatched disturbance are assumed to be random variable with zero mean and unity variance.
The unknown angular velocity is assumed $\Omega_z = 5.0 \text{ rad/s}$ and the initial condition on $K$ matrix is $K(0) = 0.9K^*$. The desired motion trajectories are $x_m = \sin(\omega_1 t)$ and $y_m = 1.2 \sin(\omega_2 t)$, where $\omega_1 = 4.17kHz$ and $\omega_2 = 5.11kHz$. The sliding mode parameter $\lambda$ in (7.13) is chosen as $\lambda = \begin{bmatrix} 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}$, and matrix $K_{c}$ in (7.17) is designed as $K_{c} = \begin{bmatrix} -10 & -10 & 10 & 10 \\ -10 & -10 & -10 & -10 \end{bmatrix}$ to place the poles of the matrix $(A_m + BK_{c})$ at $\{-8.1036\pm11.4386i, -1.8964\pm1.3956i\}$. The sliding mode gain of (7.17) is $\rho_1 = 10000$. The adaptive gain of (7.36) is $m_1 = m_2 = 20$. The parameter of smooth sliding mode component $\tanh[\alpha(x_1 + \tilde{x}_3)]$ of (7.38) is $\alpha = 5$. The observer feedback gain matrix $L$ of (7.19) are designed as $L = \begin{bmatrix} 99.9 & -21482 & 1946 & -3385 \end{bmatrix}^T$ to place the poles of the matrix $A - LC$ at $\{-32\pm103i, -18\pm101i\}$. The initial condition of the sliding mode observer in (7.19) is $\hat{X}(0) = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \end{bmatrix}$ and the sliding gain of (7.19) is $\rho_2 = 4$. when $\varepsilon = 0.1e^{-11}$, $\frac{S}{\|s\|} = \frac{S}{\|s\| + \varepsilon}$, therefore for the convenience of numerical simulation the non-smooth sliding mode component $\frac{S}{\|s\|}$ in (7.30) can be approximately calculated by $\frac{S}{\|s\| + \varepsilon}$, where $\varepsilon = 0.1e^{-11}$. In the simulation of smooth sliding mode component in (7.41), $\varepsilon = 0.15$. The convergence behavior of sliding surface and tracking error are shown in Fig. 7.7 and Fig. 7.9. Fig. 7.8 depicts the convergence of observation error. It is shown that the estimated states of the sliding mode observer can asymptotically
approximate the unmeasured states of the system. Fig. 7.10 shows the adaptation of controller parameters. Fig. 7.11 and Fig. 7.12 compare the angular velocity estimation between the smooth sliding mode controller and non-smooth sliding mode controller. Both figures show that estimation of angular converges to its true values with persistent sinusoidal frequency \( w_1 \neq w_2 \). It is observed that the estimated angular velocity with smooth sliding mode controller has better convergence performance. Fig. 7.13 and Fig. 7.14 compare the sliding mode control force between the smooth sliding mode controller and non-smooth sliding mode controller. It is shown that the adaptive sliding mode system with the smooth sliding mode controller can reduce chattering significantly.
Adaptive Sliding Mode Control with Sliding Mode Observer for a MEMS Gyroscope Model

Figure 7.2: Matlab/Simulink block of adaptive sliding mode control with sliding observer for a MEMS gyroscope
Figure 7.3: Matlab/Simulink sub-block of sliding mode observer

Figure 7.4: Matlab/Simulink sub-block of adaptive estimator
Figure 7.5: Matlab/Simulink sub-block of sliding mode controller
Figure 7.6: Matlab/Simulink sub-block of system dynamics

Figure 7.7: Convergence of the sliding surface
Figure 7.8: Convergence of the observation error

Figure 7.9: Convergence of the tracking error
Figure 7.10. Adaptation of controller parameters
Figure 7.11: Adaptation of angular velocity with the smooth sliding mode controller

Figure 7.12: Adaptation of angular velocity with the non-smooth sliding mode controller
Figure 7.13: Smooth sliding mode control forces with the sliding mode observer

Figure 7.14: Non-smooth sliding mode control forces with the sliding mode observer
7.5 Concluding Remarks

It is shown that with the control law (7.27), the parameter adaptation laws (7.36), and the sliding mode observer (7.23), if the gyroscope is controlled to follow the mode-unmatched reference model with \( w_1 \neq w_2 \), all unknown gyroscope parameters including the angular velocity converge to their true values and tracking error is going to zero asymptotically.

This chapter proposed an adaptive sliding mode controller with a proportional and integral sliding surface for a MEMS gyroscope. A nonlinear sliding mode observer using only measured output information is incorporated into the adaptive sliding mode control to estimate the unmeasured states, the angular velocity and all gyroscope parameters. The combined observer-controller synthesis involves three steps. First a sliding mode controller is developed assuming the availability of the state vector, then a sliding mode observer of the state vector is designed to estimate the unmeasured states, finally the sliding mode observer is combined with the proposed sliding mode controller that utilizes the estimate instead of the true state vector. Smooth sliding mode compensators are used to reduce control chattering. Simulations demonstrate the efficiencies and robustness of the proposed adaptive sliding mode controller with the sliding mode observer. It is shown that the angular velocity and gyroscope parameters can be consistently estimated in the presence of model uncertainty, external disturbance and unmeasured states.
CHAPTER VIII

ADAPTIVE SLIDING MODE CONTROLLED TRIAXIAL GYROSCOPE

In this chapter, adaptive sliding mode control for two axes angular sensor will be extended to triaxial angular sensors and a novel concept for an adaptively controlled triaxial angular velocity sensor device that is able to detect rotation in three orthogonal axes, using a single vibrating mass is proposed. To enable all unknown parameter estimates to converge to their true value, the necessary model trajectory is shown to be a 3D Lissajous patterns. The proposed device includes a single suspended mass that can move in three axes, having actuation and sensing element in three orthogonal axes. The use of a single mass to sense triaxial rotation promises to reduce cost and energy consumption and avoid mechanical interference from three axes. These advantages meet the requirements of many emerging applications such as pedestrian navigation. The triaxial angular velocity sensor will be based on a surface micromachining technology capable of sensing angular motion about three orthogonal axes. It provides analog outputs for angular velocity and precision references about the X, Y, and Z-axes.
8.1 Dynamics of triaxial gyroscope system.

Assume that

(i) The gyroscope mounting table (substrate) is moving with a constant translational speed;

(ii) The gyroscope is rotating at constant angular velocities with respect to $x$, $y$ and $z$ axis;

(iii) The centrifugal forces are assumed negligible;

(iv) The gyroscope undergoes rotations about the $x$, $y$ and $z$ axis.

Referring to [58], the dynamics of triaxial gyroscope system is as follows

\[
\begin{align*}
mx + dx_{xx} \dot{x} + dx_{xy} \dot{y} + dx_{xz} \dot{z} + k_{xx} x + k_{xy} y + k_{xz} z &= u_x + 2m\Omega_z \dot{y} - 2m\Omega_y \dot{z} \\
my + dy_{xy} \dot{x} + dy_{yy} \dot{y} + dy_{yz} \dot{z} + k_{xy} x + k_{yy} y + k_{yz} z &= u_y - 2m\Omega_z \dot{x} + 2m\Omega_x \dot{z} \\
mz + dz_{xz} \dot{x} + dz_{yz} \dot{y} + dz_{zz} \dot{z} + k_{xz} x + k_{yz} y + k_{zz} z &= u_z + 2m\Omega_y \dot{x} - 2m\Omega_x \dot{y}.
\end{align*}
\]  

(8.1)

Referring to [22], dividing the equation by the reference mass and rewriting the gyroscope dynamics in vector forms result in

\[
\ddot{q} + \frac{D}{m} \dot{q} + \frac{K_s}{m} q = \frac{u}{m} - 2\Omega \dot{q}
\]

(8.2)

where

\[
q = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad u = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}, \quad \Omega = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}, \quad D = \begin{bmatrix} d_{xx} & d_{xy} & d_{xz} \\ d_{xy} & d_{yy} & d_{yz} \\ d_{xz} & d_{yz} & d_{zz} \end{bmatrix}
\]

\[
K_s = \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{xy} & k_{yy} & k_{yz} \\ k_{xz} & k_{yz} & k_{zz} \end{bmatrix}.
\]
Since the non-dimensional time is $t^* = w_0 t$, dividing both sides of equation by $w_0^2 q_0$ and reference length $q_0$ gives the final form of the non-dimensional equation of motion for the z-axis gyroscope.

$$\frac{\ddot{q}}{q_0} + \frac{D}{m w_0 q_0} \dot{q} + \frac{K_s}{m w_0^2 q_0} q = \frac{u}{m w_0^2 q_0} - 2 \frac{\Omega}{w_0} \dot{q}.$$  (8.3)

Define new parameters as follows:

$$q^* = \frac{q}{q_0}, \quad D^* = \frac{D}{m w_0}, \quad u^*_x = \frac{u_x}{m w_0^2 q_0}, \quad u^*_y = \frac{u_y}{m w_0^2 q_0}, \quad u^*_z = \frac{u_z}{m w_0^2 q_0}, \quad \Omega^* = \frac{\Omega}{w_0},$$

$$w_x = \sqrt{\frac{k_{xx}}{m w_0^2}}, \quad w_y = \sqrt{\frac{k_{yy}}{m w_0^2}}, \quad w_z = \sqrt{\frac{k_{zz}}{m w_0^2}}.$$  

Ignoring the superscript (*) , the nondimensional representation of (8.1) is

$$\ddot{q} + D \dot{q} + K_p q = u - 2\Omega \dot{q}$$  (8.4)

where $K_P = \begin{bmatrix} w_x^2 & w_{xy} & w_{xz} \\
                        w_{xy} & w_y^2 & w_{yz} \\
                        w_{xz} & w_{yz} & w_z^2 \end{bmatrix}$.

The control target for MEMS gyroscope is to maintain the proof mass to oscillate in the x y and z direction at given frequency and amplitude as $x_m = A_1 \sin(w_1 t), \ y_m = A_2 \sin(w_2 t)$ and $z_m = A_3 \sin(w_3 t)$.

Rewriting the gyroscope model in state space equation:
which is \( \dot{X} = AX + Bu \), where \( X = [x \quad \dot{x} \quad y \quad \dot{y} \quad z \quad \dot{z}]^T \).

Rewriting the reference model in state space equation:

\[
\dot{X}_m = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-w_x^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -w_y^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -w_z^2 & 0
\end{bmatrix} X_m
\]

which is \( \dot{X}_m = A_m X_m \), where \( X_m = [x_m \quad \dot{x}_m \quad y_m \quad \dot{y}_m \quad z_m \quad \dot{z}_m]^T \).

Figure 8.1: The adaptive sliding mode block diagram for the triaxial MEMS gyroscope.
8.2 Adaptive Sliding Mode Control for Triaxial Angular Velocity Sensor

The adaptive sliding mode block diagram for the triaxial MEMS gyroscope is shown in Fig. 8.1. Consider the system with parametric uncertainties and external disturbance:

\[
\dot{X}(t) = (A + \Delta A)X(t) + Bu + f(t)
\]  

(8.7)

where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \) and \( A \in \mathbb{R}^{n \times n} \) is unknown matrix, \( \Delta A \) is the unknown parameter uncertainties of the matrix \( A \), \( f(t) \) is an uncertain extraneous disturbance or unknown nonlinearity of the system.

We make the same assumptions as in section 6.1.

1. \( \Delta A \) and \( f(t) \) have matched and unmatched terms. There exists unknown matrices of appropriate dimension \( D \), \( G \) such that \( \Delta A(t) = BD(t) + \Delta \tilde{A}(t) \) and \( f(t) = BG(t) + \tilde{f}(t) \),

where \( BD(t) \) is matched uncertainty and \( \Delta \tilde{A}(t) \) is unmatched uncertainty, \( BG(t) \) is matched disturbance and \( \tilde{f}(t) \) is unmatched disturbance.

From this assumptions, (8.7) can be rewritten as

\[
\dot{X}(t) = AX(t) + Bu(t) + \Delta AX(t) + f(t) \\
= AX(t) + Bu(t) + BDX(t) + \Delta \tilde{A}X(t) + BG + \tilde{f}(t) \\
= AX(t) + Bu(t) + Bf_m + f_u 
\]

(8.8)

where \( Bf_m(t, X, u) \) represents the lumped matched uncertainty and disturbance which is given by

\[
f_m(t, X) = DX(t) + G
\]

(8.9)
The term \( f_u(t, X) \) represents the lumped unmatched uncertainty and disturbance which is given by

\[
f_u(t, X) = \Delta \tilde{X}(t) + \tilde{f}(t)
\]

(8.10)

2. The matched and unmatched lumped uncertainty and external disturbance \( f_m \) and \( f_u \) are bounded such as \( \|f_m(t, X)\| \leq \alpha_{m1} \|X\| + \alpha_{m2} \) and \( \|f_u(t, X)\| \leq \alpha_{u1} \|X\| + \alpha_{u2} \), where \( \alpha_{m1}, \alpha_{m2}, \alpha_{u1}, \alpha_{u2} \) are known positive constants.

3. There exists a constant matrix \( K^* \) such that the following matching condition \( A + BK^* = A_m \) can always be satisfied.

Remark 1. From \( A + BK^* = A_m \), we get \( K^* = (B^T B)^{-1} B^T (A_m - A) \),

\[
A_m - A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-w_1^2 + w_x^2 & d_{xx} & w_{xy} & d_{xy} - 2\Omega_z & w_{xz} & d_{xy} + 2\Omega_y \\
0 & 0 & 0 & 0 & 0 & 0 \\
w_{xy} & d_{xy} + 2\Omega_z & -w_2^2 + w_y^2 & d_{xy} & w_{yz} & d_{yz} - 2\Omega_x \\
0 & 0 & 0 & 0 & 0 & 0 \\
w_{xz} & d_{xy} - 2\Omega_y & w_{yz} & d_{yz} + 2\Omega_x & -w_3^2 + w_z^2 & d_{zz}
\end{bmatrix}
\]

Thus from \( K^* = (B^T B)^{-1} B^T (A_m - A) \), i.e.

\[
\begin{bmatrix}
k_{11} & k_{21} & k_{31} & k_{41} & k_{51} & k_{61} \\
k_{12} & k_{22} & k_{32} & k_{42} & k_{52} & k_{62} \\
k_{13} & k_{23} & k_{33} & k_{43} & k_{53} & k_{63}
\end{bmatrix} = \begin{bmatrix}
-w_1^2 + w_x^2 & d_{xx} & w_{xy} & d_{xy} - 2\Omega_z & w_{xz} & d_{xy} + 2\Omega_y \\
w_{xy} & d_{xy} + 2\Omega_z & -w_2^2 + w_y^2 & d_{xy} & w_{yz} & d_{yz} - 2\Omega_x \\
w_{xz} & d_{xy} - 2\Omega_y & w_{yz} & d_{yz} + 2\Omega_x & -w_3^2 + w_z^2 & d_{zz}
\end{bmatrix}
\]
We can obtain the parameters of MEMS gyroscope as follows

\[ d_{xx} = k_{21}, \quad d_{yy} = k_{42}, \quad d_{zz} = k_{63}, \quad w_{x}^2 = k_{11} + w_{1}^2, \quad w_{y}^2 = k_{32} + w_{2}^2, \quad w_{z}^2 = k_{53} + w_{3}^2, \]

\[ w_{xy} = k_{31}, \quad w_{yz} = k_{52}, \quad w_{xz} = k_{51} = k_{13}, \]

\( (8.11) \)

\[ \Omega_x = 0.25(k_{43} - k_{62}), \quad \Omega_y = 0.25(k_{61} - k_{23}), \quad \Omega_z = 0.25(k_{22} - k_{41}), \]

\[ d_{xy} = 0.5(k_{41} + k_{22}), \quad d_{yz} = 0.5(k_{43} + k_{62}), \quad d_{xz} = 0.5(k_{61} + k_{23}). \]

The tracking error is defined as \( e(t) = X(t) - X_m(t), \) and the derivative of tracking error is

\[ \dot{e} = A_m e + (A - A_m)X + Bu + Bf_m + f_u. \]  

\( (8.12) \)

The proportional-integral sliding surface is defined as

\[ s(t) = \lambda e - \int_0^t \lambda (A_m + BK_e) e d\tau, \]

where \( \lambda = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24} & \lambda_{25} & \lambda_{26} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & \lambda_{34} & \lambda_{35} & \lambda_{36} \end{bmatrix} \)

which satisfies the condition that \( \lambda B \) is nonsingular, \( s \) is a column vector as \( s = [s_1 \quad s_2 \quad s_3]^T, K_e \) is a constant matrix which satisfies the condition that \( (A_m + BK_e) \) is Hurwitz.

The derivative of the sliding surface is

\[ \dot{s} = \lambda (A - A_m)X + \lambda Bu + \lambda Bf_m + \lambda f_u - \lambda BK_e e. \]  

\( (8.13) \)

Setting \( \dot{s} = 0 \) to solve equivalent control \( u_{eq} \) gives

\[ u_{eq} = -(\lambda B)^{-1} \lambda (A - A_m)X + K_e e - f_m - (\lambda B)^{-1} \lambda f_u \]

\[ = K_e^T X(t) + K_e e - f_m - (\lambda B)^{-1} \lambda f_u. \]  

\( (8.14) \)

Substituting \( u_{eq} \) into equation (8.11) yields

\[ \dot{e}(t) = (A_m + BK_e) e(t) + (I - B(\lambda B)^{-1}) f_u. \]

\( (8.15) \)
which is the closed-loop error dynamic equation after system enters the sliding surface.

The adaptive control signal \( u \) is proposed

\[
u(t) = K^T(t)X(t) + K_e e(t) - \rho(\lambda B)^{-1} \frac{s}{\|s\|}
\]  

(8.15)

where, \( \rho \) is constant, \( \frac{s}{\|s\|} \) is the sliding mode unit control signal, \( K(t) \) is estimate of \( K^* \),

\[
K^T = \begin{bmatrix} k_{11} & k_{21} & k_{31} & k_{41} & k_{51} & k_{61} \\
 k_{12} & k_{22} & k_{32} & k_{42} & k_{52} & k_{62} \\
 k_{13} & k_{23} & k_{33} & k_{43} & k_{53} & k_{63} \end{bmatrix}.
\]

Define the estimation error as

\[
\tilde{K}(t) = K(t) - K^*.
\]  

(8.17)

Substituting (8.17) into (8.16) yields

\[
\dot{X}(t) = AX(t) + BK^T(t)X(t) + BK_e e + Bf_m + f_u - B\rho(\lambda B)^{-1} \frac{s}{\|s\|}
\]  

(8.18)

\[
= (A_m - BK^*e^T)X(t) + BK^T(t)X(t) + BK_e e + Bf_m + f_u - B\rho(\lambda B)^{-1} \frac{s}{\|s\|}
\]

\[
= A_m X(t) + BK^T(t)X(t) + BK_e e + Bf_m + f_u - B\rho(\lambda B)^{-1} \frac{s}{\|s\|},
\]

Then, we have the tracking error equation

\[
\dot{e}(t) = A_m e + BK^T(t)X(t) + BK_e e + Bf_m + f_u - B\rho(\lambda B)^{-1} \frac{s}{\|s\|}
\]  

(8.19)

\[
= (A_m + BK_e) e + BK^T(t)X(t) + Bf_m + f_u - B\rho(\lambda B)^{-1} \frac{s}{\|s\|},
\]

The sliding surface dynamics becomes
\[ \dot{s}(t) = \lambda \dot{e}(t) - \lambda (A_m + BK_x)e(t) = \lambda (A_m + BK_x)e(t) + \lambda B\tilde{K}^T(t)X(t) + \dot{\lambda}f_m + \dot{\lambda}f_u - \lambda B\rho(\lambda B)^{-1}\frac{s}{\|s\|} - \lambda (A_m + BK_x)e(t) \tag{8.20} \]

\[ = \lambda B\tilde{K}^T(t)X(t) + \lambda Bf_m + \dot{\lambda}f_u - \rho \frac{s}{\|s\|.} \]

Define a Lyapunov function
\[ V = \frac{1}{2} s^T s + \frac{1}{2} \text{tr}[\tilde{K}M^{-1}\tilde{K}^T] \tag{8.21} \]

where \( M = M^T > 0 \), \( M \) is positive definite matrix, \( \text{tr}[M] \) denoting the trace of a square matrix \( M \).

Differentiating \( V \) with respect to time yields
\[ \dot{V} = s^T \dot{s} + \frac{1}{2} \text{tr}[\tilde{K}M^{-1}\dot{\tilde{K}}^T] = s^T \left[ \lambda B\tilde{K}^T(t)X(t) + \lambda Bf_m + \dot{\lambda}f_u - \rho \frac{s}{\|s\|} \right] + \text{tr}[\tilde{K}M^{-1}\dot{\tilde{K}}^T] \tag{8.22} \]

\[ = -s^T \rho \frac{s}{\|s\|} + s^T \lambda Bf_m + s^T \dot{\lambda}f_u + \left( s^T \lambda B\tilde{K}^T(t)X(t) + \text{tr}[\tilde{K}M^{-1}\dot{\tilde{K}}^T] \right). \]

To make \( \dot{V} \leq 0 \), we choose the adaptive laws as
\[ \dot{\tilde{K}}^T(t) = \dot{\tilde{K}}^T(t) = -MB^T \lambda^T sX^T(t) \tag{8.23} \]
\[ \dot{\tilde{K}}(t) = \dot{\tilde{K}}(t) = -X(t)s^T \lambda BM \tag{8.24} \]

with \( \kappa(0) \) being arbitrary. This adaptive law yields
\[ \dot{V} = -\rho \frac{s}{\|s\|} + s^T \lambda Bf_m + s^T \dot{\lambda}f_u \leq -\rho \frac{s}{\|s\|} + s^T \lambda B\|f_m\| + s^T \|\lambda f_u\| \leq -\rho \frac{s}{\|s\|} + s^T \lambda B\left(\alpha_{m1}\|X\| + \alpha_{m2}\right) + s^T \|\lambda\left(\alpha_{u1}\|X\| + \alpha_{u2}\right) \leq 0 \tag{8.25} \]
With $\rho \geq \|\mathbf{A}\| (\alpha_{m_1} \|X\| + \alpha_{m_2}) + \|\mathbf{A}\| (\alpha_{s_1} \|X\| + \alpha_{s_2}) + \eta$, where $\eta$ is a positive constant, $\dot{V}$ becomes negative semi-definite, i.e., $\dot{V} \leq -\eta \|s\|$. This implies that the trajectory reaches the sliding surface in finite time and remains on the sliding surface. $\dot{V}$ is negative definite implies that $s$ and $\tilde{K}$ converge to zero. $\dot{V}$ is negative semi-definite ensures that $V, s$ and $\tilde{K}$ are all bounded. It can be concluded from (8.20) that $\dot{s}$ is also bounded.

LaSalle’s invariant set theorem can be used to prove that $\lim_{t \to \infty} s(t) = 0$. Barbalat’s lemma can also be used to prove that $\lim_{t \to \infty} s(t) = 0$. The inequality $\dot{V} \leq -\eta \|s\|$ implies that $s$ is integrable as $\int_0^\infty \|s\| dt \leq \frac{1}{\eta} [V(0) - V(t)]$. Since $V(0)$ is bounded and $V(t)$ is nonincreasing and bounded, it can be concluded that $\lim_{t \to \infty} \int_0^\infty \|s\| dt$ is bounded. Since $\lim_{t \to \infty} \int_0^\infty \|s\| dt$ is bounded and $\dot{s}$ is also bounded, according to Barbalat’s lemma, $s(t)$ will asymptotically converge to zero, that is $s(t)$ and $e(t)$ all converge to zero asymptotically.

To make conclusions about the parameter errors $\tilde{K} = 0$, we need to make the persistence of excitation argument. From the adaptive law $\dot{\tilde{K}}^T(t) = -MB^T \tilde{\lambda}^T sX^T$, according to the persistence excitation theory [5], if $X$ is persistent excitation signal, then $\dot{\tilde{K}}^T(t) = -MB^T \tilde{\lambda}^T sX^T$ guarantees that $\tilde{K} \to 0$, $K$ will converges to its true values. It can be shown that there exist some positive scalar constants $\alpha$ and $T$ such that for all $t > 0$, $\int_t^{t+T} XX^T d\tau \geq \alpha I$. where
It can be shown that $XX^T$ has full rank if $w_1 \neq w_2 \neq w_3$, i.e. the excitation frequencies on $x$ and $y$ axes should be different. In other word, excitation of proof mass should be persistently exciting [22]. Since $\tilde{K} \to 0$, then the unknown angular velocity as well as all other unknown parameters can be determined from $A + BK^T = A_m$.

In summary, if persistently exciting drive signals, $x_m = A_1 \sin(w_1t), y_m = A_2 \sin(w_2t)$ and $z_m = A_3 \sin(w_3t)$ are used, then $\tilde{K}$, $s(t)$ and $e(t)$ all converge to zero asymptotically. Consequently the unknown angular velocity can be determined as $\lim_{t \to \infty} \hat{\Omega}_z(t) = \Omega_z$, $\lim_{t \to \infty} \hat{\Omega}_x(t) = \Omega_x$ and $\lim_{t \to \infty} \hat{\Omega}_y(t) = \Omega_y$. However it is difficult to establish the convergence rate.
8.3 Simulation Studies

We evaluated the proposed adaptive sliding mode control on a lumped MEMS gyroscope model [58] using MATLAB/SIMULINK. The control objective is to design an adaptive state tracking sliding mode controller so that a consistent estimate of $\Omega_x$, $\Omega_y$, and $\Omega_z$ can be obtained. The Matlab/Simulink block of adaptive sliding mode control for MEMS triaxial gyroscope and Matlab/Simulink sub-block of system dynamics, adaptive estimator, PI sliding surface and sliding mode controller of MEMS gyroscope are depicted in Figs. 8.3-8.7, respectively.

In the simulation, we allowed $\pm$ 5% parameter variations for the spring and damping coefficients and further assumed $\pm$ 1% magnitude changes in the coupling terms. The external disturbance is a random variable with zero mean and unit variance. Referring to [58], parameters of the MEMS gyroscope are as follows:

$m = 0.57e-8$ kg, $w_0 = 3kH\Omega$, $q_0 = 10^{-6} m$.

$d_{xx} = 0.429e-6$ N s/m, $d_{yy} = 0.687e-6$ N s/m, $d_{zz} = 0.895e-6$ N s/m, $k_{xz} = 7$ N/m,

$d_{xy} = 0.0429e-6$ N s/m, $d_{xz} = 0.0687e-6$ N s/m, $d_{yz} = 0.0895e-6$ N s/m,

$k_{xx} = 80.98$ N/m, $k_{xy} = 5$ N/m, $k_{yx} = 71.62$ N/m, $k_{zz} = 60.97$ N/m, $k_{xz} = 6$ N/m,

The unknown angular velocities are assumed $\Omega_z = 5.0$ rad/s, $\Omega_x = 3.0$ rad/s and $\Omega_y = 2.0$ rad/s. The initial condition on $K$ matrix is $K(0) = 0.9K^{-\top}$. The desired motion trajectories are $x_m = \sin(w_1 t)$, $y_m = 1.2\sin(w_2 t)$ and $z_m = 1.5\sin(w_3 t)$ where $w_1 = 4.17kHz$, $w_2 = 5.11kHz$ and $w_3 = 6.71kHz$. Fig. 8.2 depicts the reference model three-dimensional Lssajous pattern trajectory when $t$ is from 0s-10s. The sliding gain
\[ \rho = \text{diag}[10000 \ 10000 \ 10000], \text{ the sliding mode parameter } \lambda = \begin{bmatrix} 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix}, \]

the adaptive gain is \( M = \text{diag}\{20 \ 20 \ 20\}, \)

\[ K_c = \begin{bmatrix} -10000 & -10000 & -10000 & -10000 & 0 & 10000 \\ -10000 & -10000 & -10000 & 10000 & 10000 & 10000 \end{bmatrix}, \text{ therefore the eigenvalues of matrix } (A_m + BK_c) \text{ are } \]

\[ \{-13371+11738i \ -13371-11738i \ -1624+1911i \ -1624-1911i \ -1 \ -5\}. \]

The convergence behavior of sliding surface and tracking error are shown in Fig. 8.7 and Fig. 8.8. Figs 8.9-11 show the adaptation of controller parameters. Fig. 8.12 and Fig. 8.13 compare the angular velocity estimate between the smooth sliding mode controller and non-smooth sliding mode controller. Both figures show that the estimates of angular velocities converge to their true values. It is observed that the estimate of angular velocity with smooth sliding mode controller has better convergence performance. Fig. 8.14 and Fig. 8.15 compare the sliding mode control force between the smooth sliding mode controller and non-smooth sliding mode controller. It is shown that the adaptive sliding mode system with the smooth sliding mode controller can reduce chattering significantly.
Figure 8.2: Reference model Lsajous trajectory

Figure 8.3: Matlab/Simulink block of adaptive sliding mode control for a MEMS triaxial gyroscope
Figure 8.4: Matlab/Simulink sub-block of system dynamics
Figure 8.5: Matlab/Simulink sub-block of adaptive estimator

Figure 8.6: Matlab/Simulink sub-block of PI sliding surface
Figure 8.7: Matlab/Simulink sub-block of sliding mode controller
Figure 8.8: Convergence of the sliding surface

Figure 8.9: Convergence of the tracking error
Figure 8.10: The adaptation of controller parameters k11-k61

Figure 8.11: The adaptation of controller parameters k12-k62
Figure 8.12: Adaptation of controller parameters k13-k63

Figure 8.13: Adaptation of angular velocity with smooth sliding mode controller
Figure 8.14: Adaptation of angular velocity with non-smooth sliding mode controller

Figure 8.15: Smooth sliding mode control force of the adaptive sliding mode controller
Figure 8.16: Non-smooth sliding mode control force of the adaptive sliding mode controller

8.4 Concluding Remarks.

1. The persistent excitation condition can always be met for the single z axis angular velocity sensor when the reference model’s x and y resonant frequencies are unmatched. This can be extended to triaxial angular velocity sensor by the triaxial persistent excitation $w_1 \neq w_2 \neq w_3$. This creates a 3D Lissajous trajectory for the reference model.

2. Detailed exploration of the convergence rate and transient behavior are not discussed in this dissertation. It should be noted that they depend on the adaptation gains and ratios between the three mismatched natural frequencies.

3. The proposed triaxial adaptive sliding mode scheme will require minimal energy during operation since there is only single mass to be driven.
4. A novel single mass adaptively sliding mode controlled triaxial angular velocity sensor has been presented. The proposed adaptive sliding mode control with PI sliding surface in the previous chapter in this dissertation has been extended to a triaxial adaptive sliding mode controller with necessary modifications.

5. A triaxial sensor substrate is adapted for use in measuring the acceleration and angular velocity of a moving body along three orthogonal axes.

Conclusion: This chapter presents a new sliding mode adaptive controller for MEMS triaxial angular sensors device that is able to detect rotation in three orthogonal axes, using a single vibrating mass. An adaptive sliding mode controller with proportional and integral sliding surface is developed and the stability of the closed-loop system can be established. The proposed adaptive sliding mode controller updates estimates of stiffness error, damping and input angular velocity parameters in real time, removing the need for any offline calibration stages. To enable all unknown parameter estimates to converge to their true values, the necessary model trajectory is shown to be a three-dimensional Lissajous pattern. The numerical simulation for MEMS triaxial angular sensor is presented to verify the effectiveness of the proposed adaptive sliding mode controller.
CHAPTER IX
CONCLUSION AND FUTURE RESEARCH

9.1 Summary and Conclusion

A sliding mode control algorithm is incorporated into an adaptive control system and the feasibility of adaptive sliding mode control in the presence of the model uncertainty and external disturbance is investigated. The technique has been applied to a MEMS gyroscope and the angular velocity and all gyroscope parameters are consistently determined.

Indirect adaptive control scheme without integral sliding surface and direct adaptive control scheme with proportional-integral sliding surface are investigated. The proposed control scheme differs from the previous sliding mode techniques in the sense that the sliding surface is based on the proportional integral sliding mode control. The additional integral term provides one more degree of freedom than the conventional sliding surface.

If we compare indirect adaptive sliding mode controller and direct adaptive sliding mode controller with integral sliding mode surface, in both case, $B$ matrix has to be assumed to be known. The advantage of direct adaptive sliding mode controller with integral sliding mode surface is that it can provide more flexibility in determining the sliding surface. The limitation of direct adaptive sliding mode controller is that controller
structures have only eight components which can only correspond to at most eight system parameters. The gyroscope system parameters are not determined by themselves directly but from controller parameters indirectly. In order to obtain the unique gyroscope system parameter solution, some restrictions have to be imposed on the controller structure.

This dissertation develops adaptive sliding mode control strategies for a MEMS z-axis gyroscope. A direct adaptive sliding mode controller with proportional and integral sliding surface for a general linear system with single and multiple inputs are also developed. The proposed adaptive sliding mode controllers for MEMS z-axis gyroscope make real-time estimates of the angular velocity as well as all gyroscope parameters including coupling stiffness and damping parameters. Therefore, fabrication imperfection and time varying noise and disturbance can be compensated. The reference model trajectory is designed to satisfy the persistent excitation to enable that the estimations of parameters to converge to their true values.

In the presence of the unmeasured states, this dissertation proposes an adaptive sliding mode controller with a sliding mode observer. A nonlinear sliding mode observer using only measured output information is incorporated into the adaptive sliding mode control to estimate the unmeasured states, the angular velocity and gyroscope parameters. Moreover, adaptive sliding mode control for two axes angular sensor is extended to triaxial angular sensor and a novel concept for an adaptively controlled triaxial angular velocity sensor device that is able to detect rotation in three orthogonal axes, using a single vibrating mass is proposed.
The numerical simulations of MEMS gyroscope are investigated to show the effectiveness of the proposed adaptive sliding mode control schemes. It is shown that the proposed adaptive sliding mode controllers offer several advantages such as consistent estimates of gyroscope parameters including angular velocity and large robustness to parameter variations and external disturbance.

9.2. Future Research

This dissertation proposed an adaptive sliding mode controller with a proportional and integral sliding surface for a MEMS gyroscope in the presence of unmeasured states. Real time experiment of adaptive sliding mode controller for the MEMS gyroscope should be performed to evaluate the effectiveness of the proposed adaptive sliding mode control algorithm in the near future.

Detailed exploration of the convergence rate and transient behavior should be discussed in the future. It should be noted that they depend on the adaptation gains and ratios between the two mismatched natural frequencies.

The value of sliding surface is converging to zero as time goes on. The smooth sliding mode control’s boundary layer is fixed in the simulation. It is necessary to adaptively adjusting the smooth sliding mode control parameter or boundary layer’s width based on the changing of sliding surface. There are some relations between boundary layer and sliding surface.
Some advanced control algorithm should be investigated to control the MEMS gyroscope, for example, adaptive sliding mode control with neural network and fuzzy logic controller for the MEMS gyroscope, adaptive backstepping control with sliding mode controller for the MEMS gyroscope. In the case of time varying angular velocity of MEMS gyroscope, polynomial approximation and other mathematical approximation methods can be combined with new adaptive control algorithm for time varying system.

The real time implementation would require a high performance digital controller. DSpace real time system or FPGA digital implementation can be investigated to control the MEMS Gyroscope. For the nano-position system and high speed implementation requirement, FPGA seems have great advantages over classic real time control algorithm because of its high speed ability. The high speed integrated circuit hardware descriptive language (VHDL) of the digital implementation will be designed to evaluate the adaptive control algorithm.
BIBLIOGRAPHY


