MECHANICS OF BI-MATERIAL BEAMS AND ITS APPLICATION TO MIXED-MODE FRACTURE OF WOOD-FRP BONDED INTERFACES

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Cole S. Hamey

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MECHANICS OF BI-MATERIAL BEAMS AND ITS APPLICATION TO MIXED-MODE FRACTURE OF WOOD-FRP BONDED INTERFACES

Cole S. Hamey

Dissertation

Approved: ______________________________
Advisor
Dr. Wieslaw K. Binienda

Co-Advisor
Dr. Pizhong Qiao

Committee Member
Dr. Craig C. Menzemer

Committee Member
Dr. Xiaosheng Gao

Committee Member
Dr. Shing-Chung Wong

Accepted: ______________________________
Department Chair
Dr. Wieslaw K. Binienda

Dean of the College
Dr. George K. Haritos

Dean of the Graduate School
Dr. George R. Newkome

Date

__________________________

Dr. Kevin Kreider
ABSTRACT

In this study, mechanics models for bi-layer beams suitable for bi-material interface characterization are introduced, and their application to mixed mode fracture of wood-FRP bonded interface is studied.

First, an engineering approach for evaluating the mixed-mode (Mode-I/II) fracture toughness of wood/FRP composite bonded interfaces is presented. Two four-point bending specimens, i.e., four-point asymmetric end-notched flexure (4-AENF) and four-point asymmetric mixed-mode bending (4-AMMB), are proposed for the study of the mixed-mode fracture. With proper design, the rate of compliance change with respect to crack length can be determined to be independent of the crack length for the specimens. The proposed specimens can be used to determine the mixed-mode fracture toughness without the need of measuring the crack propagation length. Mode decompositions are evaluated for each specimen, and with previously determined Mode-I and Mode-II data, a number of failure criteria and failure envelopes are determined for the wood-FRP interface.

Second, an intuitive mechanics-based approach is presented to analyze layered beams using the split beam model, from which the layered beam is modeled as individual sub-beams and the stress acting throughout each sub-beam is used to determine the forces acting on the sub-beams. The split beam model is adapted to evaluate the compliance and ERR of three common groups of fracture specimens and loading conditions: Mode-I.
dominant (ADCB and DCB), Mode-II dominant (AENF and ENF), and Mixed-Mode (ASLB and SLB). It is demonstrated that the split beam model generates solutions consistent with those of existing specimens found in literature. Also, derived specimens are provided to allow for different ways of viewing the specimen configurations and possible ways of applying loads to achieve certain conditions. The compliance and ERR of two derived specimens are presented. The ERR for these two specimens is shown to be the same as the existing solutions, thus validating the accuracy of the approach.

The simplified bi-layer models and related application to fracture of bi-material interface introduced in this study improve the data reduction techniques in fracture characterization and facilitate analysis and design of bi-material fracture specimens.
DEDICATION

for my ancestors and my posterity
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CHAPTER I
INTRODUCTION AND LITERATURE REVIEW

1.1 Introduction

The usage of composites materials in all the engineering fields has grown in recent years, and will continue to grow into the foreseeable future. This growth in the use of composite materials has been spurred in part because composites exhibit a unique set of characteristics including being light weight, high strength and stiffness, and improved corrosive resistance. Wood bonded to fiber reinforced plastics (FRP) is a useful specimen in the study of interface bond. The wood-FRP specimen can be use to simulate numerous actual applications. Current and recent research trends have focused on the bonding of strips of FRP to wood members in order to increase the strength and stiffness of the specimen. The lamination process, whether performed in the manufacturing of large quantities of structural composites, the retrofitting of existing structures with composites to improve the overall performance of the system, or for some other means, can often result in an imperfect bond. This imperfect bond can lead to delamination and fracture growth along the interface bond between the laminated layers. The fracture mechanism is often classified as either Mode-I (opening mode), Mode-II (in-plane sliding mode), or mixed-mode (combination of Mode-I and Mode-II) crack propagation. Since much research and characterization of wood-FRP bond under Mode-I and Mode-II load
conditions has already been completed, a goal of this study is to experimentally characterize the wood-FRP interface bond under Mixed-Mode loading conditions.

The modeling of the bi-material interface fracture can be highly complex. Often models involve either the use of finite element method (FEM) or other numerical method to properly characterize the material properties and geometries of interface in the cracked and uncracked portions or the models make rough approximations of those geometric and material properties in order to analytically determine the energy release rate using the compliance. The use of FEM can require time consuming generation of meshes and often requires post-processing using methods like the Jacobian derivative method to determine the energy release rate. Analytical compliance methods use approximate beam theory to determine the beam compliances and the approximations can vary on a case by case basis and sometimes can be valid for symmetric bi-materials but then are invalid when an asymmetric sample is used. A goal of this study is to develop an analytical model that simplifies the modeling of the bi-layer interface fracture and the determination of the energy release rate.

1.2 Literature Review

Much study has been completed on interface fracture (e.g., wood-FRP bonded materials), particularly for the fracture behavior under Mode-I, Mode-II, and Mixed-Mode loading conditions. Some research has also been conducted on the modeling of the interface bond. In the following sections studies involving both the experimental characterization and the interface model of Mode-I, Mode-II, and Mixed-Mode loading conditions will be reviewed.
1.2.1 Mode I Fracture

Experimental determination of Mode-I fracture toughness has been studied extensively. Davalos et al. (1997) explored the Mode-I interface fracture between wood-wood and wood-FRP composite. Using the contoured double cantilever beam (CDCB) specimen, which was designed using the Rayleigh-Ritz method, the Mode-I fracture toughness was determined for the wood-FRP interface. The CDBC specimen was designed in such a way as to achieve a constant change of compliance with respect to crack length. This allows the experimental determination of the strain energy release rate without the need to know the crack length. It was determined from this study that it was feasible to use a CDBC specimen to measure Mode-I fracture toughness both experimentally and numerically. In a second study, Davalos et al. (1998a) further determine the Mode-I fracture toughness from the wood-FRP interface results. Tests were conducted on a set of CDCB specimens to determine the critical loads for crack initiation and crack arrest, from which the critical strain energy release rates \( G_{Ic} \) were determined. Experimental results were compared well with the results that had been predicted by the Jacobian Derivative Method (Barbero and Reddy 1990). In a third study, Davalos et al. (1998b) examined the compliance rate of change with respect to the crack length of the tapered double cantilever beam (TDCB) specimen, which differs from the CDCB specimen by approximating the contour as a linear relationship. The result of this study revealed that the linear slope of the TDCB specimen could provide confident results.

The interface bond under Mode-I loading conditions has also been examined. Nagaswara Rao and Acharya (1994) studied the Mode-I fracture toughness and Energy
Release Rate (ERR) of a slender Double Cantilever Beam (DCB) specimen. The beam analysis conducted showed good agreement with experimental results. Bazhenov (1994) conducted a study to examine the differences between intralaminar and interlaminar fractures in cross ply [0/90] S-glass/epoxy composites. The intralaminar fracture toughness was found to be roughly twice that of the interlaminar fracture toughness. This would result in the crack propagating through interfacial layer of the specimen. Wang (1997) studied Mode-I fracture properties in a sandwich composite beam under three-point loading conditions. Todo and Jar (1997) examined interlaminar fracture properties of fiber reinforced composite DCB specimens. Using FEM it was determined that that the crack geometry can have a significant role in the fracture growth. Ozdil and Carlsson (1999) examined the DCB specimen with angle ply laminates. A Winkler foundation model was used in this study to help in determining the fracture properties. Forte et al. (2000) employed Reissner's variational theorem to develop an analytical model for DCB specimens with adhesive reinforcement. It was determined that the amount of reinforcement in the interfacial adhesive layer had a significant effect on the fracture characteristics of the specimen. Morais et al. (2002) conducted a study on the Mode-I interlaminar fracture of carbon/epoxy cross-ply composites. The results of this study were that the ERR of an intralaminar fracture are noticeably smaller than that of an interlaminar fracture. Morais (2003) conducted a study on different types of DCB specimens (i.e. multidirectional, and unidirectional composites) using FEM. It was illustrated in this study that the multidirectional composites resulted in fracture toughness value greater than that of the unidirectional composites. The Winkler model was also used in the analysis of the Tapered DCB specimen used in the study of fracture toughness.
of bi-material interfaces by Qiao et al. (2003). Shahani and Forqani (2004) examined the shear effects on a DCB fracture specimen using both a static and dynamic approach. Using a Timoshenko beam on a Winkler foundation the ERR for both the static and dynamic beam were determined analytically. Pereira and Morais (2004) studied, experimentally, mode I interlaminar fracture of carbon/epoxy composites. Experimental results were verified using FEM and had good agreement with the FEM results. Jensen (2005) used a quasi-linear fracture model (obtained through use of the Winkler foundation model) to aid in the representation of the material properties of a wood DCB specimen. Comparisons of the analytical model with experimental results had good agreement. El-Hajjar and Haj-Ali (2005) conducted experimental and numerical analyses on a thick-section fiber reinforced polymeric composite using an eccentrically loaded, single-edge-notch tension specimen. This study was done using methods based on existing ASTM fracture standards and the validity of those standards for the given specimens was examined. Tsai and Chen (2005) examined the effects of stitching on Mode-I fracture toughness of graphite/epoxy composites. The results of this study showed that the stitch specimens had considerably higher fracture toughness values, than that of the non-stitched specimens.

1.2.2 Mode-II Fracture

Studies have also been conducted on Mode-II interface cracking. Schuecker and Davidson (2000) conducted a study to compare the Mode-II characteristics obtained from using a three-point bend end-notched flexure (ENF) specimen to those obtained from using a four-point bend end-notched flexure (4-ENF) specimen. The main difference between the two methods is that in a 4-ENF test the crack length and compliance do not
need to be monitored during the testing of the sample. They found that the large discrepancies between the ENF and 4-ENF, which had previously been attributed to frictional effects, did not exist if the measurements of the crack length and compliance were accurately obtained during the testing of the ENF specimens. Robinson (2001) examined the effect of friction in Mode-II testing. In Robinson’s study, a method for experimentally establishing the significance of friction in a 4-ENF specimen was presented. This includes details of a fixture designed to help apply Robinson’s method. Wang and Qiao (2003) conducted Mode-II testing on the wood-FRP interface similar to the Mode-I specimens used by Davalos et al. (1998a). In this study, a three-point bend tapered end-notched flexure (TENF) specimen was developed. This specimen was also designed to allow for a constant compliance rate of change with respect to the crack length. A tapered beam on elastic foundation (TBEF) model (Qiao et al. 2003) was also developed to aid in the design of the TENF specimen; so a comparison of the results could be made. The results of the experimental TENF specimen were sufficiently close to the results found using the TBEF model. Yoshihara (2004) examined the Mode-II R-curve of wood-wood interfaces using a 4-ENF testing configuration. In this study it was determined that the Mode-II R-curve could be accurately determined using the results from 4-ENF fracture test alone. This aids in validating the use of 4-ENF fracture tests for determination of Mode-II interface characteristics.

The Winkler model is used to help account for the interfacial deformation between the elastic layers of the laminated specimen. This model has been applied to the bi-layer beam in End-Notched Flexure (ENF) specimen by Corleto and Hogan (1995). Ding and Kortschot (1999) presented a simplified analysis of an ENF specimen. The ENF
specimen was modeled using elastic shear springs as the interface. Todo et al. (2000) examined crack initiation under Mode-II loading conditions. The effect of the pre-crack geometry was also examined. Compston et al. (2001) studied the effects of matrix toughness and loading rates in Mode-II fractures. Morais et al. (2002) presented a study of the interlaminar fracture of filament wound composites using ENF specimens. The results (fracture properties) of angle ply and unidirectional specimens were compared and had similar agreement with the Mode-I testing results. Pereira et al. (2004) conducted an experimental study on Mode-II interlaminar fracture of carbon/epoxy composites. ENF specimens were used to examine the effects of composite lay-up on the fracture properties. Morais (2004) provided an analysis of Mode-II interlaminar fractures in multidirectional composites. The results of this study were similar to the result of the Mode-I study. Yan et al. (2004) determined the effect of z-pins on Mode-II fractures in composite materials. Pereira and Morais (2004) studied interlaminar fractures in multidirectional and unidirectional composites. Similar to the result from Mode-I testing it was determined that the angle ply specimens had higher fracture toughness values than the unidirectional composites. Yoshihara (2005) examined the Mode-II crack initiation of wood-wood interfaces using a 3-ENF testing configuration. In this study it was determined that the Mode-II crack initiation could be accurately determined using the results from 3-ENF fracture test and the specimen compliance. Szekrenyes and Uj (2005) examined Mode-II fracture in E-glass/polyester composites with the inclusion of the Saint Venant effect at the clamped end of the specimens. Compliance was used to determine the ERR in this study. Davies et al. (2005) used a 4-ENF specimen to conduct a parametric study on the effects of the fiber volume fraction on the Mode-II fracture
toughness of composite materials. The results showed an inverse relationship between the fiber volume fraction and the ERR of the specimens.

1.2.3 Mixed-Mode Fracture

Mixed-mode interface fractures have been explored for sometime; however, most of the literature concerning the mixed-mode fractures only examines the total strain energy release rate and does not conduct mode decomposition. Klingbeil and Beuth (1996) presented the results for the energy release rate for both two-layer and three-layer deposited metal specimens under four-point bending. The experimental results were also compared to numerical results obtained using the finite element method. Using a similar four-point bending technique, Zhang and Lewandoski (1997) examined the bilayer interface of an aluminum-aluminum composite sample. This study allowed for the characterization of the interface of these materials. Hofinger et al. (1998) also used a four-point bending test specimen in order to obtain the interface fracture energy of a thin, brittle layered material. From experiments the energy release rate of ZrO$_2$-ceramic sprayed on alloyed steel was determined.

Some work has been done on mixed-mode fractures in which the mode decomposition was conducted and a failure envelope or failure criterion was developed for certain materials. These studies are of importance to the present study because similar techniques will be adapted in determination of a set of mixed-mode characteristics for the wood-FRP specimen. Jurf and Pipes (1982) examined the interlaminar fracture characteristics of composite materials. In the study, the opening mode, the shearing mode, and the mixed mode conditions were used. Both the critical stress intensity factor (SIF) and the critical energy release rate (ERR) were determined experimentally, and a
quadratic fit was used to show the mixed mode data from the experimental results. Donaldson (1987) built upon prior results found in literature to determine the effect of material toughness on the growth of delamination in a composite laminate. The results were then used to develop mixed-mode fracture criterion. Further it was determined that a linear criterion between the Mode I and Mode II cracks was only slightly conservative in regards to the critical load values, for the materials studied. Johnson and Mangalgiri (1987) conducted a study in which the influence of the resin on the interlaminar mixed-mode fracture was considered. The results demonstrated different fracture toughness values for different resin type. However, the failure criterion for each resin type could be approximated using a linear relationship. Garg (1988) reviewed the state of delamination growth and prediction. This review included numerous mixed mode failure criteria using the critical ERR. Suo and Hutchinson (1989) explored the relationship between the Mode I and Mode II SIF in sandwich test specimens. The results established a relationship between the stress field and the bi-material interface. Reeder and Crews (1990) developed a method and an apparatus for test mixed-mode bending specimens. In the study, the design of an apparatus capable of applying both an ENF type load and a DCB type load was presented. The experimental results using the apparatus were compared with results of finite element analysis, from which a close agreement was reached. In a subsequent study, Reeder and Crews (1991) redesigned the apparatus presented in the previous study in order to account for nonlinearities that were evident in the previous results. The redesigned apparatus was able to minimize the nonlinearities caused by the previous configurations. Reeder (1992) presented a comprehensive review to numerous different models of mixed mode fracture criteria. These models were then
applied to the previously obtained experimental results. This provided guidance for selecting the appropriate criterion for a given type of material. Hwu et al. (1995) examined the delamination fracture criteria for composites laminates. In the study the mixed-mode relationship was explored considering both the SIF and the ERR. The experimental results were then validated by comparison with finite element results. Ducept et al. (1997) conducted an experimental study to determine the mixed-mode failure criteria of glass/epoxy composites. In the experiment the influence of the specimen thickness was examined for Mode-I (DCB), Mode-II (ENF), and mixed-mode I/II loading. The analyses of the results of the mixed-mode test were performed using beam theory models and experimentally using displacement measurements. Davies et al. (1998) conducted a review of the current status of testing methods for delamination resistance in composite materials. In this review the Mixed-Mode Bending (MMB) specimen is discussed. The MMB mentioned in the review was based on beam theory. Parvatareddy and Dillard (1999) conducted a study to characterize the mixed-mode failure envelope of adhesive joints. Tests were completed to determine the Mode-I, Mode-II and mixed-mode I/II fracture toughness. To achieve this variety crack propagation modes, the DCB, ENF and mixed mode flexure (MMF) specimens were used. Parvatareddy and Dillard (1999) also considered aging effects on the joints in their study. In a second study with glass/epoxy composites, Ducept et al. (2000) compared their previous results to the ones obtained from testing a composite/composite joint. Using these results the fracture envelopes were developed to use in the prediction of failure loads on structural joints. Krueger and O’Brien (2001) studied the mixed-mode delamination in composite materials. Using finite element techniques, the large built-up
composite structures were modeled. The modeling employed the use of the shell/3D technique, which was also developed during the study. Chang et al. (2002) observed the mixed-mode failures of granite. Mode-I and Mode-II results were compared using curve fitting by means of linear, elliptical, and quadratic functions. The results demonstrated that the experimental failures could be fit to certain types of regression curves. Krueger et al. (2003) completed a brief overview of the implementation of fracture mechanics in design, including an examination of mixed-mode failure. Tay (2003) completed an overview of advances in the characterization and analysis of composite fractures. In this overview, examples of failure envelopes developed by curve fitting using linear functions were displayed. The results of the linear curve fitting displayed in Tay (2003) were similar to those found in Chang et al. (2002).

In order to conduct thorough experimental and analytical analysis of mixed-mode fracture of wood-FRP bonded interface, proper theoretical models must be developed. Some of the models used in the present study were adopted from the aid of the following literature. Suo and Hutchinson (1990) examined the interface crack between two isotropic elastic layers. To solve this problem analytically, with the exception of a single real scalar independent of the loading, the superposition principle was used. The value that was unsolvable analytically was found using numerical models. Mode decomposition was conducted in this study using the stress intensity factors (SIF). Methods for such complex mode decomposition were also outlined in this study. Hutchinson and Suo (1992) completed a thorough review of analytical and testing procedures for mixed-mode cracks in layered materials, and they provided a comprehensive review including procedures for mode decomposition using both the
strain energy release rates and the SIF. Wang and Qiao (2005) recently added to the bilayer beam theory that had been previously developed by allowing the two shear deformable sub-beams to deform separately from one another. This resulted in two fictitious concentrated forces at the crack tip, rather than the three forces in Suo and Hutchinson (1992) and Schapery and Davidson (1990). In comparisons, this model showed better agreement with finite element modeling than the existing composite beam model. The major advancement of this model is that the shear deformation of the uncracked bilayer beam is included. In companion studies, Wang and Qiao (2004a, b) applied the shear deformable bi-layer beam developed in the previous study to the fracture problem. Using the method of superposition, the J-integral was derived along with the separate Mode-I and Mode-II energy release rates. This method is further examined in the next section of this study.

Mixed-mode interface fractures have been explored for sometime. Sheinman and Kardomateas (1996) determined stress intensity factors and ERR for laminated composites with delamination. Using a cylindrical bending model a method of mode decomposition was presented. In an experimental study, Benzeggagh and Kenane (1996) measured the mixed-mode fracture toughness of E-glass/epoxy composites. Buchholtz et al. (1997) presented a study in which numerical analysis was used to examine many different crack types including interlaminar and intralaminar. Agrawal and Ben Jar (2003) employed FEM to evaluate the interlaminar fracture toughness of thick composite materials. Stevanovic et al. (2003) experimentally determine Mode-I and Mode-II fracture properties for glass/polyester composites toughened with particulates. The particulate volume was varied and the fracture toughness was determined based on each
volume. Jimenez and Miravete (2004) also used FEM to predict the onset of delamination in ENF, DCB, and mixed-mode bending specimens. Using the FEM and experimental techniques a failure criterion was developed. Morais and Moura (2005) made use of FEM to evaluate different fracture initiation criterions. ENF and DCB specimens were evaluated to determine the effects of non-linearity in the criterions. Mathews and Swanson (2005) employed FEM to perform mode decomposition on mixed mode specimens. The numerical approaches used had good agreement with known specimen results.

One of the most common approaches used when developing an analytical model for laminate mechanics is a Winkler Foundation Model (Beam on Elastic Foundation Model) used in both the Single-Leg Bending (SLB) and End-Loaded Split (ELS) specimens by Szekrenyes and Uj (2004). Qiao and Wang (2005) and Wang and Qiao (2005) employed Reissner-Mindlin plate theory to explore the modeling of rigid, semi-rigid, and flexible crack joints. Kim and Kong (2002) developed an analytical model for free edge delamination using CLPT. Comparison between the analytical model and FEM using the virtual crack closure technique had good agreement. Kim et al. (2002) examined free edge delamination in layered composite materials. Using CLPT an analytical 3D model is developed, and results found with FEM were compared with analytical results with good agreement. In this study, CLPT will be used to aid in the development of the Split Beam Model.

1.3 Objectives and Scope

Composite materials, wood-FRP in particular, are increasingly used in civil structures. The use of composite laminates can lead to issues with the durability and
strength of the interface bond. In order to assure the longevity and safety of laminated structure it is imperative to properly determine the characteristic of the interface bond through both experimental and analytical studies.

The main goal of this study is to develop an experimental characterization of the wood-FRP interface bond under Mixed-Mode loading conditions, and to develop a simplified analytical model to determine fracture and structural properties such as the energy release rate, and compliance for numerous beam configurations. In particular, the objectives of this study are:

1. To employ appropriate analytical models to develop wood-FRP specimens and to experimentally measure the total critical Mixed-Mode energy release rate (ERR).
2. To use mode decomposition techniques to determine the Mode-I and Mode-II contributions to the total ERR, and use this data along with previously determined Mode-I and Mode-II results to develop failure criterion envelops for the wood-FRP interface bond.
3. To develop a simplified Split Beam Model that incorporates the composite joint and allows for the determination of the beam forces in each layer of the laminated beam.
4. To apply the simplified Split Beam Model to the bi-layer beam using the rigid joint model, and determine in a uniform manner the compliance and ERR for a number of common fracture specimens. Also, to show how these fracture specimens using the Simplified Split Beam Model can relate to one another as well as some more complex loading conditions.
5. To apply the simplified Split Beam Model to the bi-layer beam using the semi-rigid joint model, and determine the constants of integration found during the process in a uniform manner as it applies to the Split Beam Model.

1.4 Research Overview

This study combines both experimental and analytical determination of the Mixed-Mode fracture toughness of wood-FRP with the development of a simplified Split beam model for determining the energy release rate (ERR) of many common bi-layer beam structures.

In chapter II, analytical approaches for determine the total ERR along with methods for determining the Mode-I and Mode-II contributions will be reviewed. Using these analytical approaches two four point bending fracture specimens will be designed. These two four point bending specimens, i.e., the mixed-mode bending (4-MMB) and the asymmetric end notch flexure (4-AENF), have been chosen because the configuration allows for the measurement of the fracture toughness without measuring the fracture growth (i.e., crack length). Also in this chapter the method for data reduction for each of the two specimens will be discussed and outlined.

In chapter III, the materials used in the experimental characterization of the mixed-mode fracture toughness of the wood-FRP specimens will be presented. The description of the fabrication process will be given along with the testing procedure used in the determination of the critical crack initiation and arrest forces. Next the experimental results of each specimen tested are presented and average values and coefficients of variance are determined for the 4-MMB and 4-AENF configurations. Finally, a number
of failure criterions are examined, utilized, and compared, with the values of the constants determined for the best fit solutions provided.

In chapter IV, a generalized simplified Split Beam Model is presented. The Split Beam Model simplifies the determination for the forces acting in individual members of a laminate using a kinematic approach. With the Split Beam Model the bi-layer beam system is examined in greater depth in conjunction with the rigid joint model. A major advantage of the Split Beam Model is the case at which it can deal with both the symmetric and asymmetric bi-layer beam conditions. It will be illustrated how the Split Beam models solutions for asymmetric conditions will converge to the solutions symmetric case. Many common fracture testing configurations will be solved using the Split Beam Model and results will be compared to the one found in literature where available. Finally, load configurations referred to as Derived Specimens will be solved, and the ERR from these specimens will be shown to be equivalent to the specimens from which the derived specimens come from.

In chapter V, the Split Beam Model will be used in the semi-rigid joint model to solve problems similar to the ones in chapter IV. Using the semi-rigid joint the rotations of the sub-beams at the crack tip no longer need to be equal to one another, this allows for a more accurate determination of the ERR. To apply the semi-rigid joint, a differential equation must be solved. The constants of integration are determined in a uniform manner for all the beam configurations based on the Split Beam Model. Finally, a summary of both the rigid and semi-rigid joint results is given in this chapter to allow for a comparison of both methods.
In chapter VI, a summary of the present study is presented, highlighting the major points of the present study. Also, a brief description of some possible research topics to build on the present study will be given.
CHAPTER II
MIXED MODE ANALYSIS OF THE BI-MATERIAL INTERFACE

2.1 Introduction

Fiber-reinforced plastic (FRP) composites are being extensively used in structural applications. In particular, wood and cement structures are more often being externally reinforced with FRP composites. Although significant stiffness and strength increases can be achieved with this reinforcement technique, there is concern about the performance of the interface bond between the composite reinforcement and the primary material. The interface bond is often susceptible to delamination and crack growth. These inadequacies of the interface bond can cause changes in the stiffness and strength of the bond, which in turn may lead to a complete structural failure of the composite reinforced members. A need exists to develop an effective characterization of the bond strength of such a type of material to avoid unexpected failures.

In this section the analytical model used during the development and design of the specimens in the present study will be examined and explained. Also a thorough description of the design process and design considerations will be presented. Finally, a brief description of the data reduction process, including the testing parameters that must be obtained during experimentation, is given.
2.2 Analytical Model

As mentioned near the end of the literature review section in chapter I, in the present study, the bi-layer shear deformable beam theory developed in Wang and Qiao (2004a) is used in this study. This model will be briefly reviewed in this section and placed into the context of the actual specimens used in this study. Using superposition of crack tip elements (CTE), we can calculate the energy release rate (ERR) with greater ease. As shown in Figure 2.1, a bi-layered material with a crack and an unknown ERR (Figure 2.1 (a)) can be viewed as the summation of a composite beam without a crack (i.e., the ERR = 0) (Figure 2.1 (b)) and a CTE under self-equilibrate forces (i.e., the ERR $= f(M, N, Q)$) (Figure 2.1 (c)).

The terms $h_1$ and $h_2$ in Figure 2.1 denote the heights of the individual materials of the bi-layer beam system. Based on Reisser-Mindlin plate theory, the deformation of the sub-layer beams can be written as

$$U_i(x_i, z_i) = u_i(x_i) + z_i\phi_i(x_i)$$

$$W_i(x_i, z_i) = w_i(x_i)$$

where the subscript $i = 1, 2$ representing the sub-layers 1 and 2 respectively. The constitutive equations can be written as

$$\begin{bmatrix} N_i(x) \\ M_i(x) \end{bmatrix} = \begin{bmatrix} C_i & 0 \\ 0 & D_i \end{bmatrix} \begin{bmatrix} du_i(x) \\ dx \\ \frac{d\phi_i(x)}{dx} \end{bmatrix}$$

$$Q_i = B_i(\phi_i(x) + \frac{dw_i(x)}{dx})$$
where $N_i$, $Q_i$, and $M_i$ are respectively the resulting axial force, transverse shear force, and bending moment. $C_i$, $D_i$, and $B_i$ denote, respectively, the axial, bending, and transverse shear stiffness properties of the beams and can be calculated as follows

$$C_i = (E_{xx})_i b_i h_i$$

$$D_i = \frac{(E_{xx})_i b_i h_i^3}{12}$$

$$B_i = \frac{5}{6} (G_{xz})_i b_i h_i$$

Figure 2.1 Crack tip element of bi-layer beams

$$M^* = N(h_1 + h_2)/2$$

where $E_{xx}$ is the longitudinal Young modulus, $G_{xz}$ is the in-plane shear modulus, and $b_i$ is the width of the specimen. After the material properties have been determined, the composite beam forces (see Figure 2.1 (b)), are computed as follows

$$N_{iC} = A_m M_T + A_n N_T$$
\[ M_{ic} = \left( \eta \xi A_M - \frac{h_2}{2D_2 \xi} \right) M_T + \left( \eta \xi A_N - \frac{1}{C_2 \xi} \right) N_T \]

\[ Q_{ic} = \left( \eta \xi + \frac{h_1}{2} \right) A_M Q_T \]

where

\[ \xi = \frac{h_1}{2D_1} - \frac{h_2}{2D_2} \]

\[ \eta = \frac{1}{C_1} + \frac{1}{C_2} + \frac{(h_1 + h_2)h_2}{4D_2} \]

\[ A_M = \frac{\left( \frac{1}{D_1} + \frac{1}{D_2} \right) h_2 + \xi}{2D_2} + \frac{\xi h_1 + h_2}{2D_2} \]

\[ A_N = \frac{\left( \frac{1}{D_1} + \frac{1}{D_2} \right)}{C_2 \left( \frac{1}{D_1} + \frac{1}{D_2} \right) \eta + \frac{\xi (h_1 + h_2)}{2D_2}} \]

and the total forces are expressed in terms of the applied forces as

\[ M_T = M_{10} + M_{20} + \frac{h_1 + h_2}{2} N_{10} + Q_T x \]

\[ N_T = N_{10} + N_{20} \]

\[ Q_T = Q_{10} + Q_{20} \]

Then, it is possible to calculate the effective forces that are applied to the self-equilibrate CTE (see Figure 2.1 (c)) are expressed

\[ N = N_{10} - N_{1c} \bigg|_{x=0} \]
\[ M = M_{i0} - M_{ic} \bigg|_{x=0} \]
\[ Q = Q_{i0} - Q_{ic} \bigg|_{x=0} \]

The effective forces can now be used to calculate the ERR of the fractured beam. The ERR is given as (Wang and Qiao 2004a)

\[ G = \frac{1}{2} \left( C_N N^2 + C_M M^2 + C_Q Q^2 + C_{NM} N M + C_{NQ} N Q + C_{MQ} M Q \right) \]

where

\[ C_N = \frac{1}{C_1} + \frac{1}{C_2} + \frac{(h_1 + h_2)^2}{4D_2}, \quad C_M = \frac{1}{D_1} + \frac{1}{D_2}, \quad C_Q = \frac{1}{B_1} + \frac{1}{B_2}, \]
\[ C_{NM} = \frac{(h_1 + h_2)}{D_2}, \quad C_{NQ} = k \left( \frac{1}{B_1} + \frac{1}{B_2} \right) h_1, \quad C_{MQ} = 2k \left( \frac{1}{B_1} + \frac{1}{B_2} \right) \]

\[ k = \sqrt{\left( \frac{1}{D_1} + \frac{1}{D_2} \right) \eta + \xi (h_1 + h_2) - \frac{2D_2}{\xi (h_1 + h_2)}} \]

Based on a global mode decomposition approach (Wang and Qiao 2004a), the ERR can further be decomposed into pure Mode-I \( G_I \) and pure Mode-II \( G_{II} \) components, from which a phase angle \( \psi \) can be determined as well, and they are given as

\[ G_I = \frac{1}{2} \left( \frac{1}{B_1} + \frac{1}{B_2} \right) \left( Q + k \left( M + \frac{h_1 N}{2} \right) \right)^2 \]
\[ G_{II} = \frac{(\xi M - \eta N)^2}{(h_1 \xi + 2\eta)} \]

\[ \psi = \tan^{-1} \frac{G_{II}}{G_I} \]
This model will now be applied to the loading conditions used in the present study in order to design a set of specimens that will produce the mixed-mode failure envelope.

2.3 Specimen Design

In the present study two specimen load configurations will be used. The first is similar to the 4-ENF as described in the aforementioned literature review. The only difference between this sample and the 4-ENF is that the specimen is asymmetric. The asymmetry is what causes this specimen to be mixed-mode; due to the specimen configuration, this specimen is Mode-II dominant. The configuration is illustrated in Figure 2.2. This loading configuration will be referred to as four-point bend asymmetric end-notched flexure (4-AENF).

![Delamination](image)

**Figure 2.2** Four-point bend asymmetric end-notched flexure (4-AENF) specimen

The second load configuration is also in four-point bend loading configuration. It is similar to the mixed-mode bending (MMB) described in literature, the main difference is that a single leg on one side sits on the support to allow for the Mode-I peeling component to be introduced to the bottom free end, as observed in Figure 2.3. This configuration will henceforth be referred to as four-point bend mixed-mode bending (4- MMB). The 4- MMB specimen used in the present study is Mode-I dominant.
Using these configurations in Figure 2.2 and Figure 2.3 has an advantage of allowing the ERR and the mode mixity to become independent of the crack length as long as the crack length is greater than the value of “d”, the distance from a support to the nearest load point. Another advantage of these configurations can be observed upon examination of the CTE’s. The CTE of the 4-AENF will be examined first, as shown in Figure 2.4. It is determined from the CTE that there is not a presence of shear or normal forces, and thus it greatly simplifies the calculation of the ERR.

\[
M_{10} = \frac{D_1}{D_1 + D_2} Pd \\
M_{20} = \frac{D_2}{D_1 + D_2} Pd \\
M_T = Pd
\]

Figure 2.4 CTE of 4-AENF bi-layer beam specimen

It is now possible to determine the effective forces acting on the CTE in terms of the applied loading conditions. These forces are then used to determine the total ERR, the Mode-I ERR, and the Mode-II ERR in terms of the applied loading. The effective forces and ERRs of the 4-AENF specimen can be expressed as follows.
\[ N = A_M Pd \]
\[ M = -\frac{D_1 Pd}{D_1 + D_2} + \left( \eta \frac{A_M}{\xi} - \frac{h_2}{2D_2\xi} \right) Pd \]
\[ Q = 0 \]
\[ G = \frac{1}{2} \left( C_N A_M^2 + C_M \left( \frac{-D_1}{D_1 + D_2} + \left( \frac{\eta A_M - \frac{h_2}{2D_2\xi}}{2} \right) \right)^2 \right) P^2 d^2 \]
\[ G_I = \frac{1}{2} \left( \frac{1}{B_1} + \frac{1}{B_2} \right) \left( k \left( \frac{-D_1}{D_1 + D_2} + \left( \frac{\eta A_M - \frac{h_2}{2D_2\xi}}{2} \right) \right) + \frac{h_1 A_M}{2} \right)^2 P^2 d^2 \]
\[ G_H = \frac{\left( \frac{\xi}{\eta} \left( \frac{-D_1}{D_1 + D_2} + \left( \frac{\eta A_M - \frac{h_2}{2D_2\xi}}{2} \right) \right) - \eta A_M \right)^2}{(h_1\xi + 2\eta)} P^2 d^2 \]

In a similar fashion, the CTE of the 4-MMB specimen is also examined. Once again it is determined from the CTE (Figure 2.5) that there is not a presence of shear or normal forces. Also in this specimen there is not a presence of a bending moment in the lower beam (sub-beam 2) before the crack tip. Thus, the value of \( M_{10} \) is the only term that needs to be determined in order to calculate the effective forces and the ERRs, in terms of the applied loading. The effective forces and the ERRs were determined as

\[ M_{10} = Pd \]

Figure 2.5 CTE of the 4-MMB bi-layer beam specimen

25
\[ N = A_M P d \]

\[ M = \left( \frac{\eta}{\xi} A_M - \frac{h_2}{2D_2^\xi} - 1 \right) P d \]

\[ Q = 0 \]

\[ G = \frac{1}{2} \left[ C_N A_M^2 + C_M \left( \frac{\eta}{\xi} A_M - \frac{h_2}{2D_2^\xi} - 1 \right)^2 + C_{NM} A_M \left( \frac{\eta}{\xi} A_M - \frac{h_2}{2D_2^\xi} - 1 \right) \right] P^2 d^2 \]

\[ G_1 = \frac{1}{2} \left( \frac{1}{B_1} + \frac{1}{B_2} \right) \left[ k \left( \left( \frac{\eta}{\xi} A_M - \frac{h_2}{2D_2^\xi} - 1 \right) + \frac{h_1 A_M}{2} \right) \right]^2 P^2 d^2 \]

\[ G_{II} = \frac{\left( \frac{\eta}{\xi} A_M - \frac{h_2}{2D_2^\xi} - 1 \right) + \eta A_M}{\left( h_1^\xi + 2\eta \right)} \left( \frac{P^2 d^2}{2} \right) \]

It is important to note that in the present study one sub-beam of the specimen described by the subscript 1 is actually a composite beam made up of FRP and wood; therefore the whole specimen is a wood/FRP composite beam bonded to a wood beam to form the bilayer beam sample. The values of \( C_1, D_1, \) and \( B_1 \) can no longer be simply determined. The appropriate stiffness values are determined using composite beam theory, and the method of transformed sections is used in the present study. Besides the stiffness values an effective height \( (h_1) \) of the composite beam is also needed. The distance from the plane of crack propagation to the centrioidal axis of the composite beam was assumed to be \( \frac{h_1}{2} \). The values of \( C_1, D_1, B_1, \) and \( h_1 \) can be determined as follows.
\[ C_1 = E_w b_w h_w + E_c b_c h_c \]

\[ D_1 = E_w \left( \frac{b_w h_w^3}{12} \right) + b_w h_w \left( \frac{h_1}{2} - \left( h_c + \frac{h_w}{2} \right) \right)^2 + n \left( \frac{b_c h_c^3}{12} + b_c h_c \left( \frac{h_1}{2} - \frac{h_c}{2} \right)^2 \right) \]

\[ B_1 = \frac{5}{6} \left( \mu_w b_w h_w + \mu_c b_c h_c \right) \]

\[ \frac{h_1}{2} = \frac{b_w h_w \left( h_c + \frac{h_w}{2} \right) + n \frac{b_c h_c^2}{2}}{b_w h_w + n b_c h_c} \]

\[ n = \frac{E_c}{E_w} \]

where \( E_w, E_c, \mu_w, \) and \( \mu_c \) are the Young’s and shear moduli of the wood and FRP, respectively, and \( h_w, b_w, h_c, \) and \( b_c \) are the heights and widths of the wood and FRP, respectively.

The wood used in both sub-beams one and two of the specimen in the present study is considered to have the same material properties. Thus, \( (E_w)_2 = E_w \) and \( (G_w)_2 = \mu_w \), these values are assumed to be known during the design process. Also, \( b_c = b_w = b_2 \) and \( h_c \) are known throughout the design. This leaves the height of the wood in the composite sub-beam and sub-beam 2, \( h_w \) and \( h_2 \), as the only remaining design variables. To achieve a variety of mixed-mode results the ratio of \( G_{II} \) to \( G_I \) was computed versus the ratio of \( h_2 \) to \( h_w \) for both the 4-AENF and 4-MMB loading configurations. The results are displayed in Figure 2.6 and Figure 2.7. The value of the ratio of \( h_2 \) to \( h_w \) was then
chosen to be approximately one for both the cases. With the knowledge of this ratio, the value of $h_2$ and $h_w$ can then be decided and the design of the specimens is concluded.

![Graph](image1)

**Figure 2.6 Height ratio determination for the 4-MMB specimen**

![Graph](image2)

**Figure 2.7 Height ratio determination for the 4-AENF specimen**

2.4 Data Reduction

In this section, a description of the data reduction methods employed in the present study for both the 4-AENF configuration and the 4-MMB configuration tests is presented. These methods are very similar to the methods used in the design of the specimen. In order to determine the ERRs from these equations, the material properties,
specimen geometry, and loading conditions \((P \text{ and } d)\) are needed. The material properties can be determined prior to testing and at this stage are known values. The specimen geometry was determined in the previous section during the design of the specimens. This leaves the loading conditions \((P \text{ and } d)\) as the values needed. The distance between the load and the supports \(d\) can be assigned any value the testing equipment is capable of producing as long as \(d\) is less than \(a\) the crack length. This means \(d\) is also a known value. From testing of the specimen the load \(P\) will be set equal to the critical load for crack propagation \(P_{cr}\). This in turn will result in the ERRs \((G, G_{I},\) and \(G_{II}\)) becoming the critical ERRs \((G_{c}, G_{Ic},\) and \(G_{IIc}\)). The critical ERRs for the 4-AENF configuration and 4-MMB configuration are given as follows.

\[
G_{c}^{AENF} = \frac{1}{2} \left( C_{N} A_{M}^2 + C_{M} \left( \frac{-D_{1}}{D_{1} + D_{2}} + \left( \frac{\eta}{\xi} A_{M} - \frac{h_{2}}{2D_{2}\xi} \right) \right)^2 \right) P_{cr}^2 d^2
+ C_{NM} A_{M} \left( \frac{-D_{1}}{D_{1} + D_{2}} + \left( \frac{\eta}{\xi} A_{M} - \frac{h_{2}}{2D_{2}\xi} \right) \right) P_{cr}^2 d^2
\]

\[
G_{lc}^{AENF} = \frac{1}{2} \left( \frac{1}{B_{1}} + \frac{1}{B_{2}} \right) k \left( \left( \frac{-D_{1}}{D_{1} + D_{2}} + \left( \frac{\eta}{\xi} A_{M} - \frac{h_{2}}{2D_{2}\xi} \right) \right) + \frac{h_{1} A_{M}}{2} \right) P_{cr}^2 d^2
\]

\[
G_{IIc}^{AENF} = \frac{\left( \frac{-D_{1}}{D_{1} + D_{2}} + \left( \frac{\eta}{\xi} A_{M} - \frac{h_{2}}{2D_{2}\xi} \right) \right) - \eta A_{M}}{(h_{1}\xi + 2\eta)} P_{cr}^2 d^2
\]

\[
G_{c}^{MMB} = \frac{1}{2} \left( C_{N} A_{M}^2 + C_{M} \left( \frac{\eta}{\xi} A_{M} - \frac{h_{2}}{2D_{2}\xi} - 1 \right) \right)^2 \left( C_{NM} A_{M} \left( \frac{\eta}{\xi} A_{M} - \frac{h_{2}}{2D_{2}\xi} - 1 \right) \right) P_{cr}^2 d^2
\]

\[
G_{lc}^{MMB} = \frac{1}{2} \left( \frac{1}{B_{1}} + \frac{1}{B_{2}} \right) k \left( \left( \frac{\eta}{\xi} A_{M} - \frac{h_{2}}{2D_{2}\xi} - 1 \right) + \frac{h_{1} A_{M}}{2} \right) P_{cr}^2 d^2
\]
\[
G_{MMB}^{\text{Mlc}} = \frac{\left( \xi \eta A_M - \frac{h_2}{2D_2} \xi - 1 \right) - \eta A_M}{(h_1 \xi + 2\eta)} \left( P_\text{cr}^2 d^2 \right)^2
\]

2.5 Conclusions

In this section the analytical model used during the development and design of the fracture specimens (i.e., 4-AENF and 4-AMMB) in the present study was examined and explained. Using superposition of crack tip elements (CTE), we were able to calculate the energy release rate (ERR) with greater ease. Also, a thorough description of the design process and design considerations was presented. Finally, a brief description of the data reduction process, including the testing parameters that must be obtained during experimentation, was given.
CHAPTER III
EXPERIMENTAL CHARACTERIZATION OF THE MIXED MODE FRACTURE TOUGHNESS OF WOOD-FRP INTERFACIAL BOND

3.1 Introduction

Wood-FRP reinforced materials are being increasingly used because of their high stiffness and strength combined with their cost effectiveness and ease of production. Current research on wood reinforcement has focused on the use of FRP strips or fiber fabrics bonded to wooden members. Fracture along the bonded interface of such a material is often due to preexisting cracks or flaws introduced during the fabrication process of the material. Opening tension loading associated with Mode-I crack extension (Davalos et al. 2007; Davalos et al. 2008a,b) and in-plane shear loading related to Mode-II crack extension (Wang and Qiao 2003; Qiao et al. 2003a) have been examined separately in previous works on various different samples. However, the combination of Mode-I and Mode-II in a mixed-mode sample for wood-FRP bonded structures has not yet been fully explored or characterized.

In this section the materials that are used in the present study will be explored. Also, the manufacturing process used to achieve the desired specimen geometry and specimen uniformity will be described. Finally, the specimen testing procedure will be explained.
3.2 Materials

Three separate materials were mainly used in the present study. These materials included wood in particular Red Maple, pultruded FRP, and adhesive. The materials will be discussed.

Red Maple was chosen as the substrate to be bonded to the pultruded FRP. The Red Maple specimens contained no visible defects and were cut in such a way (with the grain of the wood) as to insure that the crack would always propagate along the bond interface and not through the wood. The materials properties of the Red Maple were obtained from Qiao et al. (2000) and are displayed in Table 3.1.

Pultruded FRP laminate was chosen as substrate to be bonded to the Red Maple hardwood. The pultrusion process is ideal for the mass production of structural FRP members. The FRP used in the present study consisted of E-glass fiber roving embedded within a Phenolic resin matrix. The pultrusion process resulted in composite sheets of size 0.254 cm × 11.43 cm × 152.4 cm (0.1 in × 4.5 in × 60 in), which were later machined into strips of 0.254 cm × 3.556 cm × 76.2 cm (0.1 in × 1.4 in × 30 in) for use in the specimens. The FRP sheets were manufactured by Creative Pultrusions, Inc. Alum Bank, Pa. The material properties were also obtained from Qiao et al. (2000) and are displayed in Table 3.1.

Table 3.1 Material Properties of Red Maple and FRP composite

<table>
<thead>
<tr>
<th></th>
<th>E GPa (10^6 psi)</th>
<th>G GPa (10^6 psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Red Maple</td>
<td>13.714 (1.989)</td>
<td>1.241 (0.180)</td>
</tr>
<tr>
<td>Pultruded FRP Laminate</td>
<td>37.440 (5.430)</td>
<td>5.351 (0.776)</td>
</tr>
</tbody>
</table>
The third material used in the present study was adhesive. Penacolite adhesive G-1131 from Indspec Chemical Corporation, capable of curing at room temperature and providing strong, waterproof bonds of utmost durability, was used for bonding the wood to the FRP. Penacolite adhesive G-1131 consists of two parts, a liquid resin G-1131-A and a powder hardener G-1131-B. The liquid resin G-1131-A is an alcohol water solution of partially condensed resorcinol-formaldehyde resin. The hardener G-1131-B is a tan powder formulated from paraformaldehyde and suitable cellulosic filler. The adhesive is mixed in the following portion: 5 parts G-1131-A to 1 part G-1131-B by weight. G-1131-A is charged to the mixer and then G-1131-B is added slowly with agitation and mixing continuing for 5 minutes to make the setting agent thoroughly dispersed. Penacolite adhesive G-1131 is adaptable to accelerated curing time at intermediate temperatures, as well as room temperature.

The open/closed assembly times used in bonding were 5 minutes and 35 minutes, respectively, for all specimens. Sufficient pressure, 1.38 MPa (200 psi) used in the present study, was provided in the bonding to reduce the glue line to a thickness of 0.0508-0.1778 mm (0.002-0.007 in). The assembly time and pressure chosen in the present study were focused on the study of Davalos et al. (2000). The adhesive was allowed to cure for at least 10 hours before the pressure was removed, and all specimens were allowed to cure for at least 6 days at room temperature before testing was conducted.

3.3 Specimen Fabrication

In order to produce specimens for the present study, 24 pieces of Red Maple were saw-cut to the dimensions of 2.032 cm × 3.556 cm × 76.2 cm (0.1 in × 1.4 in × 30 in).
The FRP was into twelve 0.254 cm × 3.556 cm × 76.2 cm (0.1 in × 1.4 in × 30 in) strips. For each specimen two pieces of Red Maple and one Piece of FRP were used (Figure 3.1). A total of twelve specimens were created in six of each for the 4-AENF and 4-MMB configurations. This was done to insure that at least three test from each configuration would be viable.

![Figure 3.1 Schematic of specimen dimensions and design variables](image)

(a) 4-AENF

(b) 4-MMB

\[
\begin{align*}
L &= 76.2 \text{ cm (30 in)} \\
h_w &= h_2 = 2.032 \text{ cm (0.8 in)} \\
h_C &= 0.254 \text{ cm (0.1 in)} \\
b \text{(width)} &= 3.56 \text{ cm (1.4 in)}
\end{align*}
\]

Figure 3.1 Schematic of specimen dimensions and design variables

To bond the materials, adhesive was spread on the bonding surface using a wooden spatula. First the adhesive was spread completely on one side of the FRP and the top piece of Red Maple. Next, the adhesive was applied to the bottom Red maple piece and the opposite side of the FRP, being careful to avoid the initial crack location. The crack
location was initially defined by placing a piece of cellophane tape at the desired location, in this case 29.21 cm (11.5 in) from either support. Then the sample were clamped with the prescribed pressure and clamping time (Figure 3.2) and cured for the prescribed curing times. After the specimens had been sufficiently cured the 4-MMB samples had a small notch of wood removed from the bottom piece to achieve the desired final configuration. This piece was removed by cutting through the bottom piece of red maple with a handsaw at two separate locations. The final samples resemble the ones pictured in Figure 3.3.

![Figure 3.2 Bonding of wood-FRP interface](image)

Figure 3.2 Bonding of wood-FRP interface
3.4 Fracture Test

Fracture test of the mixed-mode wood-FRP bonded interfaces were conducted in this study. The test procedures for both the 4-AENF specimens and the 4-MMB specimens are exactly the same, and the same type experimental data should be monitored and recorded. The experiments were preformed on an MTS servo hydraulic testing machine, and a four-point bending loading was applied using a steel-loading fixture. This loading fixture allowed for the clear span length between the supports to be 68.58 cm (27.0 in) and the distance between the loads and the supports to be 21.59 cm (8.5 in) as illustrated in Figure 3.4. In the experiment, the specimens were loaded to failure to measure the critical load for crack initiation and crack arrest. Since the displacement control will tend to result in more stable crack growth than the load control, the experiment was conducted using the displacement control with a loading rate of 3.81 mm/min. (0.15 in/min). During the testing, the MTS machine (Figure 3.5) was allowed to monitor and record the experiment time, displacement and force. The force is divided evenly between the two loading points, this means that the value $P_{cr}$ used in the data reduction equations is actually one-half the value of $P_{cr, MTS}$, which is the critical load value measured by the
MTS machine. From the critical load obtained the critical ERRs (fracture toughness) can be determined.

\[ d = 21.59 \text{ cm (8.5 in)} \]

![Testing configuration for 4-MMB and 4-AENF specimens](image)

Figure 3.4 Testing configuration for 4-MMB and 4-AENF specimens

![4-AENF wood-FRP specimen under loading on the MTS machine](image)

Figure 3.5 4-AENF wood-FRP specimen under loading on the MTS machine

3.5 Experimental Results

A number of experiments were conducted for each specimen configuration. These results are presented in the following sections.
3.5.1 Results of 4-AENF Tests

Six specimens were tested in the 4-AENF configuration (Figure 3.1 (a)), of which four were able to yield favorable results. Two tests using 4-AENF 4 and 4-AENF 5 specimens, had results that were highly inconsistent when compared to the other four tests. The results based on the tests of 4-AENF 4 and 4-AENF 5 are displayed but these values were not used for further data analysis. The loading values for crack initiation and crack arrest of all the six tests are given in Table 3.2. It was determined from these values that a 10.0% coefficient of variance (COV) and a mean value of 1,161.4 N existed for the crack initiation load values. The crack arrest loads had an 11.0% COV and a mean value of 1,081.0 N. The average brittleness index was 0.069, which indicated a moderately strong and stable crack growth. The stable crack growth can also be determined from the load versus displacement plots of the specimen. Saw-toothed shape plots which consist of crack initiations and crack arrests indicates stable crack growth. This is clearly evident from an examination of Figure 3.6 to Figure 3.11.
**Table 3.2 Crack initiation and arrest loads of the 4-AENF specimen**

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Crack Initiation $P_{cr}$ N (lb)</th>
<th>Crack Arrest $P_{cr}^a$ N (lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-AENF 1</td>
<td>1008 (227) 953 (214)</td>
<td>865 (194) 924 (208)</td>
</tr>
<tr>
<td>4-AENF 2</td>
<td>1210 (272) 1277 (287)</td>
<td>1194 (268) 1166 (262)</td>
</tr>
<tr>
<td>4-AENF 3</td>
<td>1190 (268) 1189</td>
<td>1142 (257) 1133 (255)</td>
</tr>
<tr>
<td>4-AENF 4*</td>
<td>664 (149) 672 (151) 704 (158) 719 (162)</td>
<td>625 (141) 624 (140) 665 (149) 599 (135)</td>
</tr>
<tr>
<td>4-AENF 5*</td>
<td>519 (117) 433 (97)</td>
<td>562 (126) 520 (117)</td>
</tr>
<tr>
<td>4-AENF 6</td>
<td>1240 (279) 1224 (275)</td>
<td>1122 (252) 1102 (248)</td>
</tr>
<tr>
<td>Mean*</td>
<td>1161.4 (260.2)</td>
<td>1081.0 (243.0)</td>
</tr>
<tr>
<td>(COV)</td>
<td>(10.0%)</td>
<td>(11.0%)</td>
</tr>
</tbody>
</table>

* Values of 4-AENF 4 and 4-AENF 5 were recorded, but were not used in the calculation of the ERRs, Mean Value, or COV.

**Figure 3.6 Load-displacement curve for specimen 4-AENF 1**
Figure 3.7 Load-displacement curve for specimen 4-AENF 2

Figure 3.8 Load-displacement curve for specimen 4-AENF 3

Figure 3.9 Load-displacement curve for specimen 4-AENF 4
3.5.2 Results of 4-MMB Tests

Five specimens were tested in the 4-MMB configuration (Figure 3.1 (b)), of which three were able to yield favorable results. Two tests, 4-MMB 2 and 4-MMB 4, had results that were highly inconsistent when compared to the other three tests. The results of tests 4-MMB 2 and 4-MMB 4 are displayed but these values were not used for further data analysis. The loading values for crack initiation and crack arrest of all the five tests are given in Table 3.3. It was determined from these values that a 10.5% COV and a
mean value of 843.7 N existed for the crack initiation load values. The crack arrest loads had an 11.4% COV and a mean value of 788.4 N. The average brittleness index was 0.065 which indicated a moderately strong and stable crack growth. Once again, the stable crack growth can also be determined from the load versus displacement plots of the specimen. A saw-toothed shape plot, prompting several crack initiations and crack arrests, indicates stable crack growth (Figure 3.12 through Figure 3.16).

Table 3.3 Crack initiation and arrest loads of the 4-MMB specimen

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Crack Initiation $P_{cr}$</th>
<th>Crack Arrest $P_{cr}^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>N (lb)</td>
<td>N (lb)</td>
</tr>
<tr>
<td>4-MMB 1</td>
<td>769 (173)</td>
<td>728 (164)</td>
</tr>
<tr>
<td></td>
<td>841 (189)</td>
<td>805 (181)</td>
</tr>
<tr>
<td></td>
<td>849 (191)</td>
<td>803 (181)</td>
</tr>
<tr>
<td></td>
<td>841 (189)</td>
<td>806 (181)</td>
</tr>
<tr>
<td>4-MMB 2*</td>
<td>468 (105)</td>
<td>437 (98)</td>
</tr>
<tr>
<td></td>
<td>463 (104)</td>
<td>443 (100)</td>
</tr>
<tr>
<td></td>
<td>702 (158)</td>
<td>667 (150)</td>
</tr>
<tr>
<td>4-MMB 3</td>
<td>734 (165)</td>
<td>655 (147)</td>
</tr>
<tr>
<td></td>
<td>805 (181)</td>
<td>729 (164)</td>
</tr>
<tr>
<td></td>
<td>785 (176)</td>
<td>729 (164)</td>
</tr>
<tr>
<td>4-MMB 4*</td>
<td>81 (18)</td>
<td>56 (13)</td>
</tr>
<tr>
<td></td>
<td>188 (42)</td>
<td>125 (28)</td>
</tr>
<tr>
<td>4-MMB 5</td>
<td>997 (224)</td>
<td>911 (205)</td>
</tr>
<tr>
<td></td>
<td>972 (219)</td>
<td>930 (209)</td>
</tr>
<tr>
<td>Mean*</td>
<td>843.67 (189.66)</td>
<td>788.44 (177.25)</td>
</tr>
<tr>
<td>(COV)</td>
<td>(10.5%)</td>
<td>(11.4%)</td>
</tr>
</tbody>
</table>

* Values of 4-MMB 2 and 4-MMB 4 were recorded, but were not used in the calculation of the ERRs, Mean Value, or COV.
Figure 3.12 Load-displacement curve for specimen 4-MMB 1

Figure 3.13 Load-displacement curve for specimen 4-MMB 2

Figure 3.14 Load-displacement curve for specimen 4-MMB 3
3.5.3 Testing Results Discussions

From the mean values for the crack initiation and the crack arrest for both the 4-AENF and 4-MMB specimens, the ERRs and mode mixity were able to be determined (Table 3.4). The values for only Mode-I and only Mode-II ERR of the wood-FRP were obtained from Qiao et al. (2000) and Wang and Qiao (2003), respectively. The $G_c$ values determined from the experiment of 4-AENF and 4-MMB correlate well with those values determined from the pure Mode-II results. The phase angle was determined in relation to
the mode mixity found from the specimens and can be observed in Figure 3.17. However, the present results varied widely from the pure Mode-I result found in Qiao et al. (2000). The samples for the 4-MMB tests demonstrated the fiber bridging phenomena as illustrated in Figure 3.18. The fiber bridging may account for inconsistencies between the pure Mode-I results and the present results. The effect of the fiber bridging may have caused the present results to be higher than the actual value. However, this effect is not accounted for in the present study.

Table 3.4 Mode decomposition results

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Crack Propagation</th>
<th>$G_{IIc}$ Experimental</th>
<th>$G_{IC}$ Experimental</th>
<th>$G_{IIIc}$ Experimental</th>
<th>Phase Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-AENF</td>
<td>Crack Initiation</td>
<td>568.9 (3.27)</td>
<td>47.1 (0.27)</td>
<td>521.9 (3.00)</td>
<td>73.3</td>
</tr>
<tr>
<td></td>
<td>Crack Arrest</td>
<td>496.2 (2.85)</td>
<td>41.0 (0.24)</td>
<td>455.1 (2.62)</td>
<td></td>
</tr>
<tr>
<td>4-MMB</td>
<td>Crack Initiation</td>
<td>538.0 (3.09)</td>
<td>380.1 (2.19)</td>
<td>157.9 (0.91)</td>
<td>32.8</td>
</tr>
<tr>
<td></td>
<td>Crack Arrest</td>
<td>469.9 (2.70)</td>
<td>331.9 (1.91)</td>
<td>137.9 (0.79)</td>
<td></td>
</tr>
<tr>
<td>Mode-I*</td>
<td>Crack Initiation</td>
<td>224.2 (1.29)</td>
<td>224.2 (1.29)</td>
<td>0.0 (0.00)</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>Crack Arrest</td>
<td>185.6 (1.07)</td>
<td>185.6 (1.07)</td>
<td>0.0 (0.00)</td>
<td></td>
</tr>
<tr>
<td>Mode-II**</td>
<td>Crack Initiation</td>
<td>569.0 (3.27)</td>
<td>0.0 (0.00)</td>
<td>569.0 (3.27)</td>
<td>90</td>
</tr>
<tr>
<td></td>
<td>Crack Arrest</td>
<td>536.0 (3.08)</td>
<td>0.0 (0.00)</td>
<td>536.0 (3.08)</td>
<td></td>
</tr>
</tbody>
</table>

* Values Obtained From Qiao et al. (2000)
** Values Obtained From Wang and Qiao (2003)

Figure 3.17 ERR-Phase angle curves
3.6 Mixed Mode Design Criterion

In this section, the idea of design criterion, which characterizes the wood (Red Maple)-FRP interface will be explored. Using methods similar to that found in Reeder (1992) the values of $G_{Ic}$ and $G_{IIc}$ are plotted, taking one as a dependent variable and the other as an independent variable. Six different mixed-mode failure criteria are fit to the experimental data recorded in the present study. The six criteria that will be considered include the Linear Criterion, the Power Law Criterion, the Exponential Hackle Criterion, the Exponential $K_I/K_{II}$ Criterion, the Linear Interaction Criterion, and the Bilinear Criterion. Each of the methods will be reviewed in detail. A coefficient of correlation ($R^2$) is also calculated for each criterion for both the actual experimental data for each individual specimen ($R^2$ Exp.) and also for the average experimental value calculated for each experimental configuration ($R^2$ Avg.). These coefficients of correlation are used to aid in the comparison all of the failure criteria.
3.6.1 The Linear Failure Criterion

The Linear Failure Criterion is the most simplistic of the six failure criterion to be examined. This criterion is easily expressed as

$$\left(\frac{G_{lc}^m}{G_{lc}}\right) + \left(\frac{G_{lhc}^m}{G_{lhc}}\right) = 1$$

where $G_{lc}$ and $G_{lhc}$ are the pure Mode-I and pure Mode-II fracture toughnesses and $G_{lc}^m$ and $G_{lhc}^m$ are the corresponding fracture toughness obtained from the mixed-mode fracture test. The above equation results in a linear relationship between Mode-I and Mode-II fracture toughness values (Figure 3.19).

![Figure 3.19 Linear Criterion](image)
3.6.2 The Power Law Failure Criterion

The Power Law Failure Criterion is a more generalized form of the Linear Criterion. This criterion is expressed as

\[
\left(\frac{G_{ik}^m}{G_{kc}}\right)^\alpha + \left(\frac{G_{ilc}^m}{G_{ilc}}\right)^\beta = 1
\]

where \( \alpha \) and \( \beta \) are the constants that can be used to fit a wide range of material responses. Values of \( \alpha \) and \( \beta \) can be determined, for a given material, by curve fitting with the experimental results (Figure 3.20).

![Power Law Criterion](image)

Figure 3.20 Power Law Failure Criterion
3.6.3 The Exponential Hackle Failure Criterion

The Exponential Hackle Failure Criterion is more complex than the prior two criteria, consequentially allowing for a greater variety of material responses to be modeled. The Exponential Hackle Failure Criterion is expressed as

\[ G_{lc}^m + G_{hc}^m = (G_{lc} - G_{hc}) e^{\gamma(1-N)} + G_{hc} \]

Where \( N = \sqrt{1 + \frac{G_{hc}}{G_{lc}} \sqrt{\frac{E_{11}}{E_{22}}}} \) and \( \gamma \) is a constant chosen to model the material response (Figure 3.21).

![Figure 3.21 Exponential Hackle Failure Criterion](image)

3.6.4 The Exponential K_I/K_II Failure Criterion

The Exponential K_I/K_II Failure Criterion, like the Exponential Hackle Failure Criterion, is more complex than both the Linear and Power Law criteria. This model
allows for a greater variety of material responses and is expressed similar to the Exponential Hackle Criterion as

\[ G_{IC}^m + G_{IIc}^m = (G_{IIc} - G_{IC}) e^{\eta \sqrt{G_{IIc}/G_{IC}}} + G_{IC} \]

where \( \eta \) is a constant chosen to model the material response (Figure 3.22).

![Exponential KI/KII Criterion](image)

**Figure 3.22 Exponential K\textsubscript{I}/K\textsubscript{II} Failure Criterion**

3.6.5 The Linear Interaction Failure Criterion

The Linear Interaction Failure Criterion is the most complex criterion employed in the present study. This criterion models the interaction between the Mode-I and Mode-II components. The Linear Interaction Failure Criterion is expressed as

\[ \left( \frac{G_{IC}^m}{G_{IC}} - 1 \right) \left( \frac{G_{IIc}^m}{G_{IIc}} - 1 \right) - \left[ \kappa + \varphi \left( \frac{G_{IC}^m}{G_{IC} + G_{IIc}^m} \right) \right] \left( \frac{G_{IC}^m}{G_{IIc}} \right) = 0 \]
where $\kappa$ is an arbitrary interaction parameter chosen to model the material response, and $\varphi$ is a parameter employed to allow for the linear interaction between the Mode-I and Mode-II components (Figure 3.23).

![Linear Interaction Criterion](image)

Figure 3.23 Linear Interaction Failure Criterion

3.6.6 The Bilinear Failure Criterion

The Bilinear Failure Criterion is employed when the failure mechanism resembles the failure mechanism of epoxy. This meaning that there appears to be two linear regions to the mixed-mode failure. In such a case it is feasible to assume a linear response for each independent region. This reasoning leads the Bilinear Failure Criterion to be expressed as

$$G^m_K = \xi G^m_{IIc} + G^m_{Ic}$$
\[ G^m_K = \xi G^m_{IIc} - \zeta G^m_{IIc} \]

where \( \xi \) and \( \zeta \) are the slopes of the two lines segments used in this criterion (Figure 3.24).

Figure 3.24 Bilinear Failure Criterion

3.6.7 Failure Criterion Results

A number of failure criteria were fitted with good agreement to the experimental data. The Exponential Hackle Failure Criterion, Exponential \( K_I/K_{II} \) Failure Criterion, Linear Interaction Failure Criterion, and Bilinear Failure Criterion all have the coefficient of correlation (\( R^2 \)) that close to 1. The Linear Failure Criterion and Power Law Failure Criterion do not fit the experimental data with good agreement. The fitting results for
each criterion with the constants used to fit the respective criterion are summarized in Table 3.5.

Table 3.5 Failure Criterion constants and R-squared values

<table>
<thead>
<tr>
<th>Criterion</th>
<th>Constants</th>
<th>$R^2$ Exp</th>
<th>$R^2$ Avg</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>$m = 0.3945$</td>
<td>0.660</td>
<td>0.628</td>
</tr>
<tr>
<td>Power Law</td>
<td>$\alpha = 0.644$</td>
<td>0.596</td>
<td>0.777</td>
</tr>
<tr>
<td></td>
<td>$\beta = 5.272$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exponential Hackle</td>
<td>$\gamma = 3.254$</td>
<td>0.603</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>$E_1/E_2 = 5$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exponential KI/KII</td>
<td>$\eta = -0.579$</td>
<td>0.629</td>
<td>0.999</td>
</tr>
<tr>
<td>Linear Interaction</td>
<td>$\kappa = 2.737$</td>
<td>0.891</td>
<td>0.971</td>
</tr>
<tr>
<td></td>
<td>$\omega = -5.33$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bilinear</td>
<td>$\xi = 0.989$</td>
<td>0.562</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>$\zeta = -0.928$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3.7 Conclusions

In this study the interface fracture characteristic of wood-FRP bonds under mixed-mode (I/II) was examined. Analytical bi-layer beam models were developed using formulations put forth in Wang and Qiao (2004a, b). These analytical models were further designed to be the 4-AENF specimen configuration and the 4-MMB specimen configuration. Using a relation between the ratio of $G_{II}$ to $G_{I}$ and the thickness ratio (e.g., $h_2$ to $h_1$), the specimen dimensions were designed. Experimental samples were tested in both the configurations. The data from the experiments was processed in order to conduct mode decomposition and obtain the $G_{Ic}$ and $G_{IIc}$ values for each specimen. These values were finally used to construct a set of figures that outlined the interface fracture characteristic of the wood-FRP bond.

The result of this study showed that the two specimen configurations (4-AENF and 4-MMB) explored in this study could be used to develop an array of data points for mixed-mode bending of a sample. Also, the simplified equations for the mode decomposition of
such samples were presented in this study, thus enabling them for future on other material interfaces then wood-FRP.
CHAPTER IV

SPLIT BEAM THEORY AND ITS APPLICATION TO RIGID JOINT MODEL

4.1 Introduction

The increasing use of hybrid layered or laminated composite materials have promoted a need for methods to analyze and experimentally test such materials. Also, because the lamination process often creates flaws that may induce fractures and delamination between material layers, methods involving fracture mechanics and fracture analysis of laminated composite materials are also being developed. To address all of these fracture problems numerous methods and research has been generated. In this study, classical lamination plate theory (CLPT) will be used to aid in the development of the Split Beam Model.

4.2 Split Beam Model with Composite Joint

By making use of mechanics principles it is feasible to model a laminated beam system in a novel way. Partitioning the internal stress fields of the laminated beam allows for each beam in the laminated system to be examined individually as split beam. The internal stresses dictate the actions of the beam and can be used to determine the force fields carried by each split beam, from which the deformation fields can be obtained. A generalized laminated beam system is presented in Figure 4.4.1.
In Figure 4.1, $N_i(x)$, and $M_i(x)$, $h_i$ ($i = 1, 2, ..., n$) are the axial forces, bending moments and thickness in the $i$th layer; $b$ is the width of the beam; the shear forces ($Q_i(x)$) acting each sub-beam is assumed to be a function of the total shear force ($Q_T(x)$) multiplied by the ratio of the shear stiffness of each sub-beam to the total shear stiffness of the laminate as:

$$Q_i = Q_T \frac{B_i}{\sum_{i=1}^{n} B_i} = \beta Q_i$$

The sub-beams can be treated as individual beams once the forces in each beam are determined.

The forces acting on the individual layers are determined using CLPT. Using CLPT the strain fields can be determined as

$$
\begin{bmatrix}
\varepsilon_{x} \\
\varepsilon_{y} \\
\gamma_{xy}
\end{bmatrix} = 
\begin{bmatrix}
\varepsilon_{x}^0 \\
\varepsilon_{y}^0 \\
\gamma_{xy}^0
\end{bmatrix} + 
\begin{bmatrix}
\kappa_{x} \\
\kappa_{y} \\
\kappa_{xy}
\end{bmatrix}
$$
where $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ are the total strain fields, $\varepsilon_x^o, \varepsilon_y^o, \gamma_{xy}^o$ are the strains at the neutral axis caused by the global axial forces, and $\kappa_x, \kappa_y, \kappa_{xy}$ are the curvatures caused by the global bending moments. $\varepsilon_x^o, \varepsilon_y^o, \gamma_{xy}^o$ and $\kappa_x, \kappa_y, \kappa_{xy}$ can be determined for the global applied forces using the compliance equations.

$$
\begin{bmatrix}
\varepsilon_x^o \\
\varepsilon_y^o \\
\gamma_{xy}^o \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix} = 
\begin{bmatrix}
[a] & [b] \\
[b] & [\delta]
\end{bmatrix}
\begin{bmatrix}
N_x \\
N_y \\
N_{xy} \\
M_x \\
M_y \\
M_{xy}
\end{bmatrix}
$$

where $N_x, N_y, N_{xy}, M_x, M_y, M_{xy}$ are the global forces, and $\alpha, \beta, \delta$ are the compliance matrices.

These equations can be reduced to

$$
\varepsilon_x = \varepsilon_x^o + z \kappa_x
$$

for a one-dimensional beam. The total strain field can then be used to calculate the forces acting on the individual beams (see Figure 4.1).

$$
N_i(x) = \frac{1}{2} (\varepsilon_U + \varepsilon_L) E_i b h_i
$$

$$
M_i(x) = N_i(x) * \varepsilon
$$

where $h_i$ is the height of sub-beam $i$, $E_i$ is the Young’s modulus of sub-beam $i$, $\varepsilon_U$ and $\varepsilon_L$ are the strain at the upper and lower edges of the beam $i$, and $\varepsilon$ is the eccentricity of the calculated axial load of beam $i$. 

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Using the split beam models, the forces \( N_i(x) \) and \( M_i(x) \) as shown in Figure 4.1 can be determined based on the composite beam model. These forces combined with the transverse shear force enable the displacement fields to be determined based on Timoshenko’s governing equations of the individual layers.

4.3 Application of Split Beam Model: The Bi-Layer Beam

The conventional composite beam theory is used to analyze the bi-layer beam system with an interface fracture, in which the cross-sections of the two sub-beams are assumed to remain in the same plane after the deformation, i.e.,

\[
\phi_1(x) = \phi_2(x)
\]

Along the interface of the sub-beams the displacement continuity is given by

\[
u_2(x) - \frac{h_1}{2} \phi_1(x) = u_1(x) + \frac{h_2}{2} \phi_2(x)
\]

\[
w_1(x) = w_2(x)
\]

These continuity conditions will collectively be referred as the rigid joint model. An illustration of the composite beam model can be seen in Figure 4.2.

![Figure 4.2 Composite beam model](image)
By examining only one sub-beam, the laminated beam system (Figure 4.1) can be modeled as individual beams with a generalized loading as illustrated in Figure 4.3. In Figure 4.3 \(\sigma(x)\) and \(\tau(x)\) are the interface normal (peel) and shear stresses, respectively; \(m(x)\) is a distributed moment at the neutral axis of the sub-beam; and \(p(x)\) is a generalized external loading and \(a\) is the crack length in a fracture specimen.

![Sub-beam 1 using the split beam model](image)

Figure 4.3 Sub-beam 1 using the split beam model

For a bi-layer beam system the split beam model can be simplified. The total strain \(\varepsilon\) across a cross section can be determined as a summation of the strain at neutral axis \(\varepsilon^o\) caused by the global axial force \(N_G\), and the curvature \(\kappa_x\) caused by the global moment \(M_G\) multiplied by the distance from the neutral axis. In equation form \(\varepsilon\) can be given as

\[
\varepsilon = \varepsilon^o + z' \kappa_x
\]

where \(z'\) is the distance from the neutral axis to the location at which the strain is to be determined. The strain at neutral axis caused \(\varepsilon^o\) by the global axial force \(N_G\), and the curvature \(\kappa_x\) caused by the global moment \(M_G\) can be determine as;
\[ \varepsilon^o = \frac{N_g}{A_T} \]
\[ \kappa_x = \frac{M_g}{D_T} \]

where \(A_T\) and \(D_T\) are the total axial stiffness and total bending stiffness, respectively, of the uncracked beam. These values can be determined using the transformed sections method, and they are written as

\[ A_T = E_1 A \]
\[ D_T = E_1 I_T^z \]

where \( I_T^z \) and \( A \) are the total moment of inertia and area, respectively, of the uncracked beam, \( E_1 \) and \( E_2 \) are the longitudinal Young's modulus of sub-laminated beams 1 and 2, respectively. Using the transformed sections method, the total area is given as

\[ A = b(h_1 + nh_2) \]
\[ n = \frac{E_2}{E_1} \]

The force fields \( (M_1(x), N_1(x), Q_1(x), M_2(x), N_2(x), Q_2(x)) \) can be determined from the strain distribution across the whole bi-layer beam. A sample strain distribution is displayed in Figure 4.4. From this strain distribution diagram the forces acting on the sub-beam 1 are computed as

\[ N_1(x) = \frac{1}{2}(\varepsilon_1 + \varepsilon_2)E_1bh_1 \]
\[ M_1(x) = N_1(x) \ast e \]
\[ Q_1(x) = \frac{B_1}{B_1 + B_2}Q_T \]

where
\[
e = \frac{h_1^2}{6(h_1 - 2\bar{z})}
\]
\[
\bar{z} = \frac{h_1^2 - nh_2^2}{2(h_1 + nh_2)}
\]

\(\bar{z}\) is the distance from the interface to the neutral axis.

To illustrate more clearly the employment of the above in the split bi-layer beam analysis three common fracture specimens (i.e., the ADCB, AENF, and AELS specimens), which represent three respective type groups of loadings (i.e., Mode-I dominant, Mode-II dominant, and Mixed-Mode) are used. The DCB, ENF, and ELS specimens are analyzed from reduction of the asymmetric models.

The stress resultants and displacements relationships are expressed as:

\[
N_i = A_i \frac{du_i}{dx}, \quad M_i = D_i \frac{d\phi_i}{dx}, \quad Q_i = B_i \left( \phi_i + \frac{dw_i}{dx} \right)
\]
where \( A_i, B_i, \) and \( D_i \) \((i = 1, 2)\) are the axial, shear and bending stiffness coefficients of layer \( i \), respectively, and they are expressed as

\[
A_i = E_i bh_i, \quad B_i = \frac{5}{6} G_i bh_i, \quad D_i = E_i \frac{bh_i^3}{12}
\]

where \( E_i \) and \( G_i \) \((i = 1, 2)\) are the longitudinal Young’s modulus and transverse shear modulus of layer \( i \), respectively.

Based on the above composite beam model of bi-layer structures, the displacement fields of individual sub-beams can be determined, from which the compliance of the split bi-layer beam can be derived. The compliances of the split bi-layer beams can next be used to determine the energy release rate of the common fracture specimens using the compliance method.

4.4 The Fracture Specimens

In this study three classes of fracture specimens will be presented. Each class of fracture specimens contains either transverse force or moment loading. Also, all specimens will be first examined using asymmetric material and geometrical properties, and for verification it will be shown that the asymmetric models converge to known symmetric models for the applicable cases.

4.4.1 Fracture Specimen Description

The three classes of specimens will be referred as Mode-I dominant specimens, Mode-II dominant specimens, and Mixed-Mode specimens. When an asymmetric Mode-I dominant specimen is allowed to converge to the symmetric case it will become pure Mode-I, this is similarly true for the Mode-II dominant specimen in that it will converge to the pure Mode-II case. For the Mixed-Mode specimens there are no practical material
and geometrical conditions that will allow it to become pure Mode-I or pure Mode-II. In Table 4.1 there is an illustration for load configurations of each specimen that will be examined in this study, along with the specimen load configuration for symmetric material and geometrical properties. The derived specimens are also provided, and they can be seen as equal to the asymmetric specimen or symmetric specimen depending on material and geometrical properties, if you take a slice of the derived specimen around the crack tip. The derived specimens can also be used as a reference for ways of applying point loads in order to achieve the desired load configurations. It is important to note that the compliance and displacement fields determined for the asymmetric and symmetric specimens are not necessarily equal to those values for the derived specimens. However, those values can be used as a reference point to help determine the proper values for the derived sections. Also, the compliance and displacement fields for all of the derived specimens are not presented in the present study.

Upon further examination of the asymmetric and symmetric specimens in Table 4.1, it can be observed that the boundary and compatibility conditions are similar for all of the specimens. These boundary and compatibility conditions can be expressed as follows:

\[ w_i (L - a) = 0 \]
\[ \phi_i (L - a) = 0 \]
\[ [w_i (0)]^+ = [w_i (0)]^- \]
\[ [\phi_i (0)]^+ = [\phi_i (0)]^- \]
Table 4.1 Specimen load configurations

### Mode-I Dominant Specimens

<table>
<thead>
<tr>
<th>Mode</th>
<th>Specimen</th>
<th>Configuration</th>
</tr>
</thead>
<tbody>
<tr>
<td>ADCB</td>
<td>$P_1$</td>
<td>$A_1, B_1, D_1, h_1$</td>
</tr>
<tr>
<td></td>
<td>$P_2$</td>
<td>$A_2, B_2, D_2, h_2$</td>
</tr>
<tr>
<td>DCB</td>
<td>$P_1$</td>
<td>$A_1, B_1, D_1, h_1$</td>
</tr>
<tr>
<td></td>
<td>$P_2$</td>
<td>$A_1, B_1, D_1, h_1$</td>
</tr>
</tbody>
</table>

### Mode-II Dominant Specimens

<table>
<thead>
<tr>
<th>Mode</th>
<th>Specimen</th>
<th>Configuration</th>
</tr>
</thead>
<tbody>
<tr>
<td>AENF</td>
<td>$P_1$</td>
<td>$A_1, B_1, D_1, h_1$</td>
</tr>
<tr>
<td></td>
<td>$P_2$</td>
<td>$A_2, B_2, D_2, h_2$</td>
</tr>
<tr>
<td>ENF</td>
<td>$P_1$</td>
<td>$A_1, B_1, D_1, h_1$</td>
</tr>
<tr>
<td></td>
<td>$P_2$</td>
<td>$A_1, B_1, D_1, h_1$</td>
</tr>
</tbody>
</table>

### Mixed-Mode Specimens

<table>
<thead>
<tr>
<th>Mode</th>
<th>Specimen</th>
<th>Configuration</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASLB</td>
<td>$P_1$</td>
<td>$A_1, B_1, D_1, h_1$</td>
</tr>
<tr>
<td></td>
<td>$P_2$</td>
<td>$A_2, B_2, D_2, h_2$</td>
</tr>
<tr>
<td>SLB</td>
<td>$P_1$</td>
<td>$A_1, B_1, D_1, h_1$</td>
</tr>
<tr>
<td></td>
<td>$P_2$</td>
<td>$A_1, B_1, D_1, h_1$</td>
</tr>
</tbody>
</table>
4.4.2 Mode-I Dominant Specimens

One of the commonly used specimens in the study of fracture mechanics, particularly in the study of mode-I fracture toughness, is the Double Cantilever Beam (DCB) specimen. Similar in loading to the DCB specimen and capable of achieving mixed-mode fracture toughness, the Asymmetric Double Cantilever Beam (ADCB) specimen is also used in the study of fracture mechanics. The ADCB only differs from the DCB in that the sub-beams 1 and 2 are no longer symmetric; this causes the mixed-mode fracture. In this study the ADCB specimen will be examined in thorough detail when loaded with a transverse point load or a point moment load. It will also be shown that under symmetric material and geometric conditions the ADCB specimen reverts back to the DCB specimen.

4.4.2.1 Transversely Loaded ADCB Specimen

The transversely loaded ADCB specimen consists of an ADCB specimen with two transverse forces applied to the cracked end of the specimen in opposite directions (Figure 4.5). The ADCB with a transverse load condition will henceforth be referred to as TL-ADCB.

![Figure 4.5 TL-ADCB specimen](image)

To model this type of specimen, it is split into 2 parts, and then the force and displacement fields can be calculated for each part. Using the compatibility and
boundary conditions, the constants that come from integration in the calculation of the displacement field can then be found. The force and displacement fields will be found using mechanics approaches from the internal stress fields.

For $-a \leq x \leq 0$, the force fields and displacement fields are found using Timoshenko beam theory because at this point sub-beams 1 and 2 do not interact with one another

$$M_1 (x) = P(x + a)$$
$$N_1 (x) = 0$$
$$Q_1 (x) = P$$
$$\phi_1 (x) = \frac{P}{2D_1} x(x + 2a) + c_1$$
$$w_1 (x) = \frac{P}{B_1} x - \frac{P}{6D_1} x^2 (x + 3a) - c_1 x + c_2$$

For $0 \leq x \leq L - a$ the force fields are

$$N_1 (x) = 0$$
$$M_1 (x) = 0$$
$$Q_1 (x) = 0$$

and the displacement fields are for $0 \leq x \leq L - a$

$$\phi_1 (x) = c_3$$
$$w_1 (x) = -c_3 x + c_4$$

Applying the boundary conditions and compatibility equations the constants can be solved and substituted back into the displacement equations. Also, since the rigid model is being used and the loading is equal and opposite in the top sub-beam and bottom sub-beam the displacement for each beam for $x > 0$ is equal to zero which means that only the beam $-a < x < 0$ needs to be examined, with the boundary conditions being $w_1(0) = 0$ and $\phi_1(0) = 0$ (since $c_3 = c_4 = 0$). The constants can then be solved as $c_1 = c_2 = 0$. 

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The displacement fields for sub-beam 2 can be found in a similar method. With the displacement fields solved for each sub-beam, the total displacement can be determined. The total displacement field can then be used to determine the compliance.

\[
C = \frac{w(-a)}{P}
\]

\[
C = \frac{a^3}{3} \left( \frac{1}{D_1} + \frac{1}{D_2} \right) + a \left( \frac{1}{B_1} + \frac{1}{B_2} \right)
\]

Under symmetric material and geometric properties the TL-ADCB specimen should become equivalent to the TL-DCB specimen. Under symmetric conditions, \(D_1 = D_2, B_1 = B_2\). The compliance of the TL-DCB specimen becomes:

\[
C = \frac{2a^3}{3D_1} + \frac{2a}{B_1}
\]

The energy release rate (ERR), \((G_C)\) can also be determined. The ERRs for both the TL-ADCB and the TL-DCB specimens and are as follows:

\[
G_C = \frac{P^2}{2b} \frac{dC}{da}
\]

\[
G_C^{ADCB} = \frac{P^2}{2b} \left( a^2 \left( \frac{1}{D_1} + \frac{1}{D_2} \right) + \left( \frac{1}{B_1} + \frac{1}{B_2} \right) \right)
\]

\[
G_C^{DCB} = \frac{P^2}{2b} \left( \frac{2a^2}{D_1} + \frac{2}{B_1} \right)
\]

The compliance and ERR are equal to those values found in literature for the symmetric specimen (Qiao and Wang 2005).

4.4.2.2 Moment Loaded ADCB Specimen

The moment loaded ADCB specimen consists of an ADCB specimen with two moments applied to the cracked end of the specimen in the opposite directions (Figure
The ADCB with a moment load condition will henceforth be referred to as ML-ADCB.

Similarly, the force and displacement fields will be found using mechanics approaches from the internal stress fields. For \(-a \leq x \leq 0\) the force fields and displacement fields are found using beam theory because at this point sub-beams 1 and 2 do not interact with one another

\[
\begin{align*}
M_1 (x) &= M \\
N_1 (x) &= 0 \\
Q_1 (x) &= 0 \\
\phi_1 (x) &= \frac{M}{D_1} x + c_1 \\
w_1 (x) &= -\frac{M}{2D_1} x^2 - c_1 x + c_2
\end{align*}
\]

For \(0 \leq x \leq L - a\) the force fields are

\[
\begin{align*}
N_1 (x) &= 0 \\
M_1 (x) &= 0 \\
Q_1 (x) &= 0
\end{align*}
\]

and the displacement fields are for \(0 \leq x \leq L - a\)

\[
\begin{align*}
\phi_1 (x) &= c_3 \\
w_1 (x) &= -c_3 x + c_4
\end{align*}
\]
Applying the boundary conditions and compatibility equations the constants can be solved and substituted back into the displacement equations. The rotational compliance of the beam at the location of the load \((x = -a)\) is calculated as:

\[
C_\phi = \frac{\phi_l(-a)}{M}
\]

\[
C_\phi = a \left( \frac{1}{D_1} + \frac{1}{D_2} \right)
\]

Under symmetric material properties the ML-ADC specimen should become equivalent to the ML-DCB specimen. Thus, the compliance of the ML-DCB specimen was determined to be:

\[
C_\phi = \frac{2a}{D_1}
\]

The energy release rate (ERR), \((G_\phi)\) can also be determined. The ERR for both the ML-ADC and the ML-DCB specimens and is as follows:

\[
G_\phi = \frac{M^2}{2b} \frac{dC}{da}
\]

\[
G_{\phi}^{ADC} = \frac{M^2}{2b} \left( \frac{1}{D_1} + \frac{1}{D_2} \right)
\]

\[
G_{\phi}^{DCB} = \frac{M^2}{2b} \left( \frac{2}{D_1} \right)
\]

The compliance and ERR are similar to the values found for the TL-ADC specimen and are equivalent for the ERR if \(M = Pa\).

4.4.3 Mode-II Dominant Specimens

Another commonly used specimen in the study of fracture mechanics, particularly in the study of mode-II fracture toughness, is the End-Notched Flexure (ENF) specimen. Similar in loading to the ENF specimen, and capable of achieving mixed-mode fracture
toughness, the Asymmetric End-Notched Flexure (AENF) specimen is also used in the study of fracture mechanics. The AENF only differs from the ENF in that the sub-beams 1 and 2 are no longer symmetric; this causes the mixed-mode fracture. In this study the AENF specimen will be examined in thorough detail when loaded with a transverse point load and a point moment load. It will also be shown that under symmetric material conditions the AENF specimen reverts back to the ENF specimen.

4.4.3.1 Transversely Loaded AENF Specimen

The transversely loaded AENF specimen consists of an AENF specimen with a transverse force applied to the cracked end of the specimen (Figure 4.7). The AENF with a transverse load condition will henceforth be referred to as TL- AENF.

![Figure 4.7 TL-AENF specimen](image)

To model this beam it is split into 2 parts and then the force and displacement fields can be calculated for each part. Using the compatibility and boundary conditions the constants that come from integration in the calculation of the displacement field can then be found. The force and displacement fields will be found using mechanics approaches from the internal stress fields.

For \(-a \leq x \leq 0\) the force and displacement fields are found using beam theory because at this point sub-beams 1 and 2 do not interact with one another

\[ M_1(x) = \alpha P(x + a) \]
\[ N_1(x) = 0 \]
\[ Q_1(x) = \alpha P \]
\[ \phi_1(x) = \frac{\alpha P}{2D_1} x(x + 2a) + c_1 \]
\[ w_1(x) = \frac{\alpha P}{B_1} x - \frac{\alpha P}{6D_1} x^2 (x + 3a) - c_1 x + c_2 \]

For \( 0 \leq x \leq L - a \) the force fields are

\[ N_1(x) = \frac{1}{2} \left( h_1 - 2\psi \right) B h_1 M_T = \chi M_T \]
\[ M_1(x) = \chi \psi M_T = \psi M_T \]
\[ Q_1(x) = \frac{B_1}{B_1 + B_2} Q_T = \beta Q_T \]

where

\[ M_T = P(a + x) \]
\[ Q_T = P \]

and the displacement fields are for \( 0 \leq x \leq L - a \)

\[ \phi_1(x) = \frac{\psi P x(x + 2a)}{2D_1} + c_3 \]
\[ w_1(x) = \frac{\beta P x}{B_1} - \frac{\psi P x^2}{6D_1} (x + 3a) - c_3 x + c_4 \]

Applying the boundary conditions and compatibility equations the constants can be solved and substituted back into the displacement equations. The constants can be determined as

\[ c_1 = c_3 = -\frac{\psi P}{2D_1} (L + a)(L - a) \]
\[ c_2 = c_4 = -\frac{\beta P}{B_1} (L - a) - \frac{\psi P}{6D_1} (L - a)^2 (2L + a) \]

and can be substituted into the appropriate displacement equations above to determine the displacement at any location along the sub-beam 1.
The compliance of the beam at the location of the load \((x = -a)\) is calculated as:

\[
C = \frac{a^3(\alpha - \psi)}{3D_1} + \frac{L^3\psi}{3D_1} + \frac{a(\alpha - \beta)}{B_1} + \frac{L\beta}{B_1}
\]

Under symmetric material properties the TL-AENF specimen should become equivalent to the TL-ENF specimen. This is a true statement because under symmetric conditions \(\psi = \frac{1}{8}, \alpha = \frac{1}{2}, \text{and} \beta = \frac{1}{2}\). The compliance of the TL-ENF specimen was determined to be:

\[
C = \frac{a^3}{8D_1} + \frac{L^3}{24D_1} + \frac{L}{2B_1}
\]

The ERR, \(G_C\), can also be determined. The ERR for both the TL-AENF and the TL-ENF specimens and is as follows:

\[
G_{C_{AENF}} = \frac{P^2}{2b} \left( \frac{a^2(\alpha - \psi)}{D_1} + \frac{(\alpha - \beta)}{B_1} \right)
\]

\[
G_{C_{ENF}} = \frac{P^2}{2b} \frac{3a^2}{8D_1}
\]

The compliance and ERR are the same as those formulas found in literature for the symmetric specimen (Qiao and Wang 2005).

4.4.3.2 Moment Loaded AENF Specimen

The moment loaded AENF specimen consists of an AENF specimen with a moment applied to the cracked end of the specimen (Figure 4.8). The AENF with a moment load condition will henceforth be referred to as ML-AENF.
Similarly, the force and displacement fields will be found using mechanics approaches from the internal stress fields. For \(-a \leq x \leq 0\) the force and displacement fields are found using beam theory because at this point sub-beams 1 and 2 do not interact with one another

\[
M_1 (x) = \alpha M \\
N_1 (x) = 0 \\
Q_1 (x) = 0 \\
\phi_1 (x) = \frac{\alpha M}{D_1} x + c_1 \\
w_1 (x) = -\frac{\alpha M}{2D_1} x^2 - c_1 x + c_2
\]

For \(0 \leq x \leq L - a\) the force fields are

\[
N_1 (x) = \frac{1}{2} \left( \frac{h_1 - 2\bar{z}}{I_1^z} \right) M_\tau = \chi M_\tau \\
M_1 (x) = c \chi M_\tau = \psi M_\tau \\
Q_1 (x) = \frac{B_1}{B_1 + B_2} Q_\tau = \beta Q_\tau
\]

where

\[
M_\tau = M \\
Q_\tau = 0
\]

and the displacement fields are for \(0 \leq x \leq L - a\)
\[
\phi_1(x) = \frac{\psi M x}{D_1} + c_3
\]
\[
w_1(x) = \frac{\psi M x^2}{2D_1} - c_3 x + c_4
\]

Applying the boundary conditions and compatibility equations the constants can be solved and substituted back into the displacement equations. The constants can be determined as

\[
c_1 = c_3 = -\frac{\psi M}{D_1} (L - a)
\]
\[
c_2 = c_4 = -\frac{\psi M}{2D_1} (L - a)^2
\]

and can be substituted into the appropriate displacement equations above to determine the displacement at any location along the sub-beam 1. The compliance of the beam at the location of the load \((x = -a)\) is calculated as:

\[
C_\phi = \frac{a(\alpha - \psi)}{D_1} + \frac{L \psi}{D_1}
\]

Under symmetric material properties the ML-AENF specimen should become equivalent to the ML-ENF specimen. This is a true statement because under symmetric conditions \(\psi = \frac{1}{8}\) and \(\alpha = \frac{1}{2}\) in the mechanics model. The compliance of the ML-ENF specimen is determined as:

\[
C_\phi = \frac{3a}{8D_1} + \frac{L}{8D_1}
\]

The ERR, \(G_\phi\), can also be determined. The ERR for both the ML-AENF and the ML-ENF specimens and is as follows:

\[
G_\phi^{\text{AENF}} = \frac{M^2}{2b} \left( \frac{(\alpha - \psi)}{D_1} \right)
\]

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\[ G_c^{\text{ENF}} = \frac{M^2}{2b} \frac{3}{8D_1} \]

The compliance and ERR are similar to the values found for the TL-ADCB specimens and are equivalent for the ERR if \( M = Pa \).

4.4.4 Mixed-Mode Specimens

A third set of specimens used in the study of fracture mechanics are Mixed-Mode Bending or Single-Leg Bending (SLB) specimens. The SLB specimen is capable of achieving mixed-mode fracture toughnesses. The Asymmetric Single-Leg Bending (ASLB) specimen is also used in the study of fracture mechanics. The ASLB only differs from the SLB in that the sub-beams 1 and 2 are no longer symmetric. Both the SLB and ASLB cause the mixed-mode fracture due to asymmetric nature of the specimens. In this study the ASLB specimen will be examined in thorough detail when loaded with a transverse point load and a point moment load. It will also be shown that under symmetric material conditions the ASLB specimen reverts back to the SLB specimen.

4.4.4.1 Transversely Loaded ASLB Specimen

The transversely loaded ASLB specimen consists of an ASLB specimen with a transverse force applied to the cracked end of the specimen (Figure 4.9) on a single leg of the beam. The ASLB with a transverse load condition will henceforth be referred to as TL-ASLB.
The force and displacement fields will be found using mechanics approaches from the internal stress fields. For \(-a \leq x \leq 0\) the force and displacement fields are found using beam theory because at this point sub-beams 1 and 2 do not interact with one another.

\[
M_1(x) = P(x + a) \\
N_1(x) = 0 \\
Q_1(x) = P \\
\phi_1(x) = \frac{P}{2D_1}x(x + 2a) + c_1 \\
w_1(x) = \frac{P}{B_1}x - \frac{P}{6D_1}x^2(x + 3a) - c_1x + c_2
\]

For \(0 \leq x \leq L - a\) the force fields are

\[
N_1(x) = \frac{1}{2} \left( \frac{h_i - 2\bar{z}}{I_i} \right)h_i M_T = \chi M_T \\
M_1(x) = e M_T = \psi M_T \\
Q_1(x) = \frac{B_1}{B_1 + B_2} Q_T = \beta Q_T
\]

where

\[
M_T = P(a + x) \\
Q_T = P
\]

and the displacement fields are for \(0 \leq x \leq L - a\)

\[
\phi_1(x) = \frac{\psi P x(x + 2a)}{2D_1} + c_3
\]
Applying the boundary conditions and compatibility equations the constants can be solved and substituted back into the displacement equations. The constants can be determined as

\[ c_1 = c_3 = -\frac{\psi P}{2D_1} (L + a)(L - a) \]
\[ c_2 = c_4 = -\frac{\beta P}{B_1} (L - a) - \frac{\psi P}{6D_1} (L - a)^2 (2L + a) \]

and can be substituted into the appropriate displacement equations above to determine the displacement at any location along the sub-beam 1. The compliance of the beam at the location of the load \((x = -a)\) was calculated to be:

\[ C = \frac{a^3 (1 - \psi)}{3D_1} + \frac{L^3 \psi}{3D_1} + \frac{a(1 - \beta)}{B_1} + \frac{L\beta}{B_1} \]

Under symmetric material properties the TL-ASLB specimen should become equivalent to the TL-SLB specimen. This is a true statement because under symmetric conditions \(\psi = \frac{1}{8}\) and \(\beta = \frac{1}{2}\) in the mechanics model. The compliance of the TL-SLB specimen was determined to be:

\[ C = \frac{7a^3}{24D_1} + \frac{L^3}{24D_1} + \frac{a + L}{2B_1} \]

The ERR, \(G_C\), can also be determined. The ERR for both the TL-ASLB and the TL-SLB specimens and is as follows:

\[ G_C^{ASLB} = \frac{P^2}{2b} \left( \frac{a^2 (1 - \psi)}{D_1} + \frac{(1 - \beta)}{B_1} \right) \]
The expressions for the compliance and ERR of ASLB and SLB could not be found in literature.

4.4.4.2 Moment Loaded ASLB Specimen

The moment loaded ASLB specimen consists of an ASLB specimen with a moment applied to the cracked end of the specimen (Figure 4.10). The ASLB with a moment load condition will henceforth be referred to as ML-ASLB.

![Figure 4.10 ML-ASLB specimen](image)

To model this beam it is split into 2 parts and then the force and displacement fields can be calculated for each part. Using the compatibility and boundary conditions the constants that come from integration in the calculation of the displacement field can then be found. The force fields and displacement fields will be found using mechanics approaches from the internal stress fields.

For \(-a \leq x \leq 0\) the force fields and displacement fields are found using beam theory because at this point sub-beams 1 and 2 do not interact with one another

\[
M_1(x) = M \\
N_1(x) = 0 \\
Q_1(x) = 0
\]
\[ \phi_1 (x) = \frac{M}{D_1} x + c_1 \]

\[ w_1 (x) = -\frac{M}{2D_1} x^2 - c_1 x + c_2 \]

For \( 0 \leq x \leq L - a \) the force fields are

\[ N_1 (x) = \frac{1}{2} \left( \frac{h_1 - 2z}{h_1} \right) h_1 M_T = \chi M_T \]

\[ M_1 (x) = c \chi M_T = \psi M_T \]

\[ Q_1 (x) = \frac{B_1}{B_1 + B_2} Q_T = \beta Q_T \]

where

\[ M_T = M \]

\[ Q_T = 0 \]

and the displacement fields are for \( 0 \leq x \leq L - a \)

\[ \phi_1 (x) = \frac{\psi M x}{D_1} + c_3 \]

\[ w_1 (x) = \frac{\psi M x^2}{2D_1} - c_3 x + c_4 \]

Applying the boundary conditions and compatibility equations the constants can be solved and substituted back into the displacement equations. The constants can be determined as

\[ c_1 = c_3 = \frac{\psi M}{D_1} (L - a) \]

\[ c_2 = c_4 = -\frac{\psi M}{2D_1} (L - a)^2 \]

and can be substituted into the appropriate displacement equations above to determine the displacement at any location along the sub-beam 1. The compliance of the beam at the location of the load \( x = -a \) was calculated to be:
\[ C_\psi = \frac{a(1-\psi)}{D_1} + \frac{L\psi}{D_1} \]

Under symmetric material properties the ML-ASLB specimen should become equivalent to the ML-SLB specimen. This is a true statement because under symmetric conditions \( \psi = \frac{1}{8} \) in the mechanics model. The compliance of the ML-SLB specimen was determined to be:

\[ C_\psi = \frac{7a}{8D_1} + \frac{L}{8D_1} \]

The ERR, \( G_\phi \), can also be determined. The ERR for both the ML-ASLB and the ML-SLB specimens and is as follows:

\[
G_{\phi}^{\text{ASLB}} = \frac{M^2}{2b} \left( \frac{1-\psi}{D_1} \right) \\
G_{\phi}^{\text{SLB}} = \frac{M^2}{2b} \frac{7}{8D_1}
\]

The compliance and ERR rate are similar to the values found for the TL-ASLB specimens and are equivalent for the ERR if \( M = Pa \).

4.5 Derived Specimens

To illustrate the use of the derived specimens two of the derived specimens are further examined. Using the three-point bending specimens from both the Mode-II dominant and the Mixed-Mode specimen groups, comparisons between the AENF/ENF, ASLB/SLB and the corresponding derived specimens can be developed.

4.5.1 The Three-Point Bending AENF/ENF Specimen

The three-point bending AENF/ENF (3AENF or 3ENF) specimen (Figure 4.11) is the derived specimen from the TL-AENF/ENF specimen. As previously stated the derived
specimens can be seen as equal to the asymmetric specimen or symmetric specimen depending on material and geometrical properties, if you take a slice of the derived specimen around the crack tip. The derived specimens can also be used as a reference for ways of applying point loads in order to achieve the desired load configurations.

![Figure 4.11 Three-point bending AENF specimen](image)

The compliance for the 3AENF specimen can be found in similar fashion to the other compliances determined in the present study, and is found to be:

\[
C = \frac{a^3(\alpha - \psi)}{12D_1} + \frac{L^3\psi}{6D_1} + \frac{a(\alpha - \beta)}{4B_1} + \frac{L\beta}{2B_1}
\]

This compliance is nearly equivalent to the compliance determined above for the TL-AENF specimen except that the terms which are a function of \(a\) have been multiplied by \(\left(\frac{1}{2}\right)^2\) and the terms which are not functions of \(a\) have been multiplied by \(\frac{1}{2}\). Similarly the 3ENF specimen has a compliance determined as:

\[
C = \frac{a^3}{32D_1} + \frac{L^3}{48D_1} + \frac{L}{4B_1}
\]

The ERR, \(G_C\), can also be determined. The ERR for both the 3AENF and the 3ENF specimens and is as follows:

\[
G_C^{AENF} = \frac{P^2}{8b} \left( \frac{a^2(\alpha - \psi)}{D_1} + \frac{(\alpha - \beta)}{B_1} \right)
\]
The ERR for these specimens appears to be nearly equivalent. However, a close examination of the forces acting on sub-beam 1 in the specimens for the 3AENF and 3ENF and the TL-AENF and TL-ENF the ERRs are found to be equivalent (Figure 4.7 and Figure 4.11). The forces acting around the crack tip in the sub-beams of the 3AENF and 3ENF specimens are actually equal to half of the forces acting in the TL-AENF and TL-ENF specimens. When squared in the calculation of the energy release rate this accounts for the difference of the two by a factor of \((\frac{1}{2})^2\).

4.5.2 The Three-Point Bending ASLB/SLB Specimen

Similarly, the three-point bending ASLB/SLB (3ASLB or 3SLB) specimen (Figure 4.12) is the derived specimen for the TL-ASLB/SLB specimen.

![Figure 4.12 Three-point bending ASLB specimen](image)

The compliance for the 3ASLB specimen can be found in similar fashion to the other compliances determined in the present study, and is found to be:

\[
C = \frac{a^3(1-\psi)}{12D_1} + \frac{L^3\psi}{6D_1} + \frac{a(1-\beta)}{4B_1} + \frac{L\beta}{2B_1}
\]

This compliance is nearly equivalent to the compliance determined above for the TL-ASLB specimen except that the terms that are a function of \(a\) have been multiplied by
and the terms that are not functions of $a$ have been multiplied by $\frac{1}{2}$. Similarly the 3SLB specimen has a compliance determined as:

$$C = \frac{7a^3}{96D_1} + \frac{L^3}{48D_1} + \frac{L}{4B_1} + \frac{a}{8B_1}$$

The ERR, $G_C$, can also be determined. The ERR for both the 3ASLB and the 3SLB specimens and is as follows:

$$G_C^{\text{ENF}} = \frac{P^2}{8b} \left( \frac{a^2(1-\psi)}{D_1} + \frac{(1-\beta)}{B_1} \right)$$

$$G_C^{\text{ENF}} = \frac{P^2}{16b} \left( \frac{7a^2}{4D_1} + \frac{1}{B_1} \right)$$

The ERR for these specimens appears to be nearly equivalent. However, a close examination of the forces acting on sub-beam 1 in the specimens for the 3ASLB and 3SLB and the TL-ASLB and TL-SLB the ERRs are found to be equivalent (Figure 4.9 and Figure 4.12). The forces acting around the crack tip in the sub-beams of the 3ASLB and 3SLB specimens are actually equal to half of the forces acting in the TL-ASLB and TL-SLB specimens. When squared in the calculation of the energy release rate this accounts for the difference of the two by a factor of $\left( \frac{1}{2} \right)^2$.

### 4.6 Conclusions

In this study, an explicit mechanics-based approach was presented for analyzing layered beams using the split beam model. Whereby the layered beam was modeled as individual sub-beams and the stress acting throughout each sub-beam is used to determine the forces acting on the sub-beam. The forces along with the material and geometric properties of the sub-beam could then be used with the aid of Timoshenko’s
method to determine the displacement fields, compliance and the overall energy release rate of the beam.

The split beam model was used to evaluate the compliance of many common (Mode-I dominant, Mode-II dominant, and Mixed-Mode) fracture specimens and loading conditions. It was demonstrated that the split beam model generated solutions to these common specimens that were equivalent to those found in literature. Specimens presented that could not be found in previous literature had good agreement with similar specimens

The derived specimens were provided to allow for different ways of viewing the specimen configurations and possible ways of applying loads to achieve certain conditions. Two derived specimens were presented. The ERR for these two specimens was shown to be equivalent to the common specimens that the derived specimens could be considered to represent.
CHAPTER V
APPLICATION OF SPLIT BEAM THEORY TO SEMI-RIGID JOINT MODEL

5.1 Introduction

In this chapter, the split beam model as introduced in the previous chapter will be applied to the semi-rigid joint model (Wang and Qiao 2004a), by modifying the compatibility condition. Also, the constants of integration from the semi-rigid joint model will be determined using a uniform procedure for a number of common bi-layer beam configurations. Finally, expressions for the ERR of numerous bi-layer beams will be provided.

5.2 The Semi-Rigid Joint Model

The semi-rigid joint model differs from the rigid joint model in that each sub-layer is allowed to rotate freely from one another (i.e. $\varphi_1 \neq \varphi_2$). This allows for there to be no addition moment to be attaching along the interfacial layer (Figure 5.1). This effectively means that the summation of the moment at the interfacial layer in a single sub-beam should equal zero. This also results in a higher order differential equation when solving for the displacement fields, compliances, and ERRs of fracture specimens. To determine these higher order differential equations it is first necessary to examine a differential element of the beam system and determine the equilibrium equations there. This has
been done in Figure 5.2. The derivation of the moment acting in sub-beam 1 then follows.

![Figure 5.1 Sub-beam 1 using the semi-rigid joint model](image)

\[ Q_1 + \Delta Q_1 + \Delta N_1 + M_1 + \Delta M_1 = Q_2 + \Delta Q_2 + \Delta N_2 + M_2 + \Delta M_2 \]

![Figure 5.2 Free body diagram and force equilibriums of sub-layers](image)

\[ \frac{dN_1(x)}{dx} = b \tau(x) \quad \frac{dN_2(x)}{dx} = -b \tau(x) \]

\[ \frac{dQ_1(x)}{dx} = b \sigma(x), \quad \frac{dQ_2(x)}{dx} = -b \sigma(x) \]

\[ \frac{dM_1(x)}{dx} = Q_1(x) - \frac{h_1}{2} b \tau(x), \quad \frac{dM_2(x)}{dx} = Q_2(x) - \frac{h_2}{2} b \tau(x) \]

Similar to the rigid joint model the compliance equations for the semi-rigid joint model can be determined as
\[ N_i = C_i \frac{du_i(x)}{dx}, \quad M_i = D_i \frac{d\phi_i(x)}{dx}, \quad Q_i = B_i \left( \phi_i + \frac{dw_i(x)}{dx} \right) \]

where \( i = 1, 2 \) represent sub-beams 1 and 2. The global equilibrium equations for the semi-rigid joint model can also be determined:

\[
N_1 + N_2 = N_T \\
Q_1 + Q_2 = Q_T \\
M_1 + M_2 + \frac{h_1 + h_2}{2} N_i = M_T
\]

Finally, the compatibility equations for the semi-rigid joint can be determined. These equations differ from the compatibility equations for the rigid joint model only in that the restriction on the equal rotations of sub-beam 1 and sub-beam 2 is no longer required. The compatibility equations for the semi-rigid joint are:

\[
w_1 = w_2 \\
u_1 - \frac{h_1}{2} \varphi_1 = u_2 + \frac{h_2}{2} \varphi_2 \quad \varphi_1 \neq \varphi_2
\]

Differentiating the compatibility equations and substituting the compliance equations into them yields:

\[
\frac{N_1}{C_1} - \frac{h_1 M_1}{2D_1} = \frac{N_2}{C_2} + \frac{h_2 M_2}{2D_2} \\
\frac{dQ_1}{B_1 dx} - \frac{M_1}{D_1} = \frac{dQ_2}{B_2 dx} - \frac{M_2}{D_2}
\]

The global equilibrium equations can be substituted into the above equations resulting in the following two expressions:

\[
\frac{N_1}{C_1} - \frac{h_1 M_1}{2D_1} = \frac{N_T - N_1}{C_2} + \frac{h_2}{2D_2} \left( M_T - M_1 - N_1 \cdot \frac{h_1 + h_2}{2} \right) \\
\left( \frac{1}{B_1} + \frac{1}{B_2} \right) \frac{dQ_1}{dx} = \frac{M_1}{D_1} - \frac{1}{D_2} \left( M_T - M_1 - N_1 \cdot \frac{h_1 + h_2}{2} \right)
\]
With \( Q_x = \frac{dM_1}{dx} + \frac{h_1}{2} \frac{dN_1}{dx} \), the above expressions can be combined to form a second order differential equation in terms of \( M_1 \).

\[
\frac{d^2 M_1}{dx^2} - k^2 M_1 = F(x)
\]

where

\[
k^2 = \left( \frac{1}{D_1} + \frac{1}{D_2} \right) + \frac{\xi}{\eta} \left( \frac{h_1 + h_2}{2D_2} \right)
\]

\[
F(x) = \left( \frac{h_1 + h_2}{2\eta D_2 C_2} \right) N_T + \left( \frac{(h_1 + h_2)h_2}{4\eta D_2^2} - \frac{1}{D_1} \right) M_R
\]

and

\[
\eta = \frac{1}{C_1} + \frac{1}{C_2} + \frac{(h_1 + h_2)h_2}{4D_2}
\]

\[
\xi = \frac{h_1}{2D_1} - \frac{h_2}{2D_2}
\]

Solving the differential equation results in:

\[ M_1 = c_1 e^{\xi x} + c_2 e^{-\xi x} + M_R \]

where \( c_1 \) and \( c_2 \) are constants that can be determined from moment equilibrium at the crack tip and at a location far away from the crack tip, and \( M_R \) can be seen as the moment from the rigid joint model.

Equilibrium far away from the crack tip means \( \lim_{x \to \infty} (M_1) = M_R \) this can only happen if \( c_1 = 0 \). The moment \( (M_1) \) can now be described by the rigid part \( (M_R) \) and the semi-rigid part \( (M_s = ce^{-\xi x}) \), where \( c \) is determined from the moment equilibrium at the crack tip and will be done on a per specimen basis later on, as:
\[ M_1 = M_s + M_R \]

The shear force \( Q_1 \) and normal force \( N_1 \) can be determined from \( M_1 \).

\[
N_1 = \frac{\varepsilon}{\eta} M_s + N_R = N_s + N_R
\]

\[
Q_1 = \left( 1 + \frac{h_1 \varepsilon}{2\eta} \right) M_s + Q_R = Q_s + Q_R
\]

It is now possible to determine the displacement fields, compliances, ERRs, and constant \( c \) for each specimen type. The most important part of the calculation of the compliance, ERR, and force fields is the determination of the constant \( c \). Form the free body diagram (Figure 5.3), the summation of the moments around the crack tip can be calculated and must be equal to zero. The summation of the moment about \( x = 0, z = 0 \) must equal 0 for the crack tip element at zero

\[
\sum M = 0 \Rightarrow M_{10}(0) + Q_{10}(0) \frac{dx}{2} + \sigma(0) \frac{dx^2}{2} + N_{10}(0) \frac{h_1}{2} = M_1(0) + Q_1(0) \frac{dx}{2} + N_1(0) \frac{h_1}{2}
\]

![Figure 5.3 Free body diagram for a crack tip element of sub-beam 1](image)

Figure 5.3 Free body diagram for a crack tip element of sub-beam 1

Taking the limit of the above equation as \( dx \to 0 \) and substituting in the physical values it is possible to determine the constant as
5.3 The TL-ADCB Specimen

The transversely loaded ADCB specimen consists of an ADCB specimen with two transverse forces applied to the cracked end of the specimen in opposite directions (Figure 5.4). The ADCB with a transverse load condition will henceforth be referred to as TL-ADCB.

Figure 5.4 TL-ADCB specimen

The forces at \( x = 0 \) are found using Timoshenko beam theory and are determined to be:

\[
M_{10}(0) = Pa \\
N_{10}(0) = 0 \\
N_R(x) = 0 \\
M_R(x) = 0
\]

Substituting these values into the equation for determining the constant allows the constant \( c \) for the ADCB specimen to be determined as:

\[
c = \frac{Pa}{1 - \frac{\bar{h}_1}{\eta^2}}
\]

and for the symmetric DCB specimen:
\( c = Pa \)

The compliance can now be determined. Similar to the forces the compliances can be split into two parts, the part contributed by the rigid joint which was previously determine and the part contributed by the semi-rigid joint:

\[ C = C_R + C_S \]

The compliance from the semi-rigid joint for the ADCB and the DCB specimens are determined to be:

\[
C_S = \frac{a^2}{\left(1-\frac{\frac{\eta}{h_1}}{k}\right)^2}\left(\frac{1}{D_1} + \frac{1}{D_2}\right) + \frac{a}{\left(1-\frac{\frac{\eta}{h_1}}{k}\right)}\left[-\left(1 + \frac{\eta}{h_1}\right)\left(\frac{1}{B_1} + \frac{1}{B_2}\right) + \frac{1}{k^2}\left(\frac{1}{D_1} + \frac{1}{D_2}\right)\right]
\]

\[
C_S^{sym} = \frac{2a^2}{\sqrt{B_1D_1}}
\]

Similar to the compliance the ERR can also be divided into a semi-rigid part and a rigid part \( (G = G_S + G_R) \). The semi-rigid part of the ERR was determined as:

\[
G_S = \frac{P^2}{2b} \frac{dC_S}{da} = \frac{P^2}{2b}\left\{\frac{2a}{\left(1-\frac{\frac{\eta}{h_1}}{k}\right)^2}\left(\frac{1}{D_1} + \frac{1}{D_2}\right) + \frac{1}{\left(1-\frac{\frac{\eta}{h_1}}{k}\right)}\left[-\left(1 + \frac{\eta}{h_1}\right)\left(\frac{1}{B_1} + \frac{1}{B_2}\right) + \frac{1}{k^2}\left(\frac{1}{D_1} + \frac{1}{D_2}\right)\right]\right\}
\]

\[
G_S^{sym} = \frac{P^2}{2b} \frac{dC_S^{sym}}{da} = \frac{P^2}{2b} \frac{4a}{\sqrt{B_1D_1}}
\]

The total compliance and ERR can be determined by adding the rigid part and the semi-rigid part together and is determined to be:

\[
C_{ADCB}^{sym} = \frac{a^3}{3}\left(\frac{1}{D_1} + \frac{1}{D_2}\right) + a\left(\frac{1}{B_1} + \frac{1}{B_2}\right) + \frac{a^2}{\left(1-\frac{\eta}{h_1}\right)^2}\left(\frac{1}{D_1} + \frac{1}{D_2}\right) + \frac{a}{\left(1-\frac{\eta}{h_1}\right)}\left[-\left(1 + \frac{\eta}{h_1}\right)\left(\frac{1}{B_1} + \frac{1}{B_2}\right) + \frac{1}{k^2}\left(\frac{1}{D_1} + \frac{1}{D_2}\right)\right]
\]

\[
C_{DCB}^{sym} = \frac{2a^3}{3D_1} + \frac{2a^2}{B_1} + \frac{2a^2}{\sqrt{B_1D_1}}
\]

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\[ G^{\text{ADC}} = \frac{P^2}{2b} \left( a^2 \left( \frac{1}{D_1} + \frac{1}{D_2} \right) + \frac{1}{B_1 + B_2} + \frac{2a}{1 - \frac{\eta}{\eta^2}} \left( \frac{1}{D_1} + \frac{1}{D_2} \right) + \frac{1}{1 - \frac{\eta}{\eta^2}} \left( \frac{1}{D_1} + \frac{1}{D_2} \right) \right) \]

\[ G^{\text{DCB}} = \frac{P^2}{2b} \left( \frac{2a^2}{D_1} + \frac{2}{B_1 + \sqrt{B_1 D_1}} + \frac{4a}{\sqrt{B_1 D_1}} \right) \]

The ERR expression for the symmetric DCB is the same as the one given in Qiao and Wang (2005).

5.4 The TL-AENF Specimen

The transversely loaded AENF specimen consists of an AENF specimen with a transverse force applied to the cracked end of the specimen (Figure 5.5). The AENF with a transverse load condition will henceforth be referred to as TL-AENF.

![Figure 5.5 TL-AENF specimen](image)

The forces at \( x = 0 \) are found using Timoshenko beam theory and are determined to be:

\[ M_{10}(0) = \alpha Pa \]
\[ N_{10}(0) = 0 \]
\[ N_{\beta}(x) = \beta Pa \]
\[ M_{\beta}(x) = \gamma Pa \]

Substituting these values into the equation for determining the constant allows the constant \( c \) for the ADCB specimen to be determined as:

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\[ c = \frac{\left( \alpha - \psi - \chi \frac{h_1}{2} \right) p a}{1 + \frac{\varphi h_1}{\eta^2}} \]

and for the symmetric DCB specimen:

\[ c = 0 \]

Similar to the ADCB specimen, the compliance can now be determined. The compliance from the semi-rigid joint for the AENF and the ENF specimens are determined to be:

\[ C_s = \left( \alpha - \psi - \chi \frac{h_1}{2} \right) a^2 + \left( \alpha - \psi - \chi \frac{h_1}{2} \right) a \left[ \left( 1 + \frac{\varphi h_1}{\eta^2} \right) \frac{1}{D_{ik}} + \frac{1}{D_{ik}} k^2 \right] \]

\[ C_{Sym} = 0 \]

The semi-rigid part of the ERR was determined as:

\[ G_s = \frac{P^2}{2b} \left[ \frac{2 \left( \alpha - \psi - \chi \frac{h_1}{2} \right) a}{1 + \frac{\varphi h_1}{\eta^2} D_{ik}} + \frac{\left( \alpha - \psi - \chi \frac{h_1}{2} \right) a}{\left( 1 + \frac{\varphi h_1}{\eta^2} \right) D_{ik}} \left[ \left( 1 + \frac{\varphi h_1}{\eta^2} \right) \frac{1}{D_{ik}} + \frac{1}{D_{ik}} k^2 \right] \right] \]

\[ G_{Sym} = 0 \]

The total compliance and ERR can be determined by adding the rigid part and the semi-rigid part together and is determined to be:

\[ C_{c, AENF} = \frac{a^3 (\alpha - \psi)}{3D_i} + \frac{L \psi}{3D_i} + \frac{a (\alpha - \beta)}{B_i} + \frac{L \beta}{B_i} + \frac{\left( \alpha - \psi - \chi \frac{h_1}{2} \right) a^2}{1 + \frac{\varphi h_1}{\eta^2} D_{ik}} + \frac{\left( \alpha - \psi - \chi \frac{h_1}{2} \right) a}{\left( 1 + \frac{\varphi h_1}{\eta^2} \right) D_{ik}} \left[ \left( 1 + \frac{\varphi h_1}{\eta^2} \right) \frac{1}{D_{ik}} + \frac{1}{D_{ik}} k^2 \right] \]

\[ C_{c, ENF} = \frac{a^3}{8D_i} + \frac{L^3}{24D_i} + \frac{L}{2B_i} \]
\[ G_c^{AENF} = \frac{p^2}{2b} \left( \frac{a^2(\alpha - \psi)}{D_1} + \frac{(\alpha - \beta)}{B_1} + \frac{2}{\eta x} \left( \frac{\alpha - \psi - \frac{h_1}{2}}{D_1} \right) + \frac{1}{\eta x} \left( \frac{1 + \frac{\eta h_1}{\eta x}}{B_1} \right) + \frac{1}{D_1 k^2} \right) \]

\[ G_c^{ENF} = \frac{p^2}{2b} \frac{3a^2}{8D_1} \]

In a similar fashion the solution for the TL-ASLB can be found. The specimens under moment loading can also be determined. The solutions to both the rigid joint and semi-rigid joint models for ADCB, AENF, and ASLB specimens under both transverse loading and moment loading conditions are given in Table 5.1-Table 5.6.
Table 5.1 Transversely loaded ADCB specimen

<table>
<thead>
<tr>
<th>Mode-I Dominant Specimens</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
</tr>
<tr>
<td>$z$</td>
</tr>
<tr>
<td>$a$</td>
</tr>
<tr>
<td>$L$</td>
</tr>
</tbody>
</table>

Sub-Beam 1

Sub-Beam 2

Rigid Joint Model

$$C_{R}^{ADCB} = \frac{a^3}{3} \left( \frac{1}{D_1} + \frac{1}{D_2} \right) + a \left( \frac{1}{B_1} + \frac{1}{B_2} \right)$$

$$G_{R}^{ADCB} = \frac{P^2}{2b} \left( \frac{a^2}{D_1} + \frac{1}{B_1} + \frac{1}{B_2} \right)$$

$$C_{R}^{DCB} = \frac{2a^3}{3D_1} + \frac{2a}{B_1}$$

$$G_{R}^{DCB} = \frac{P^2}{2b} \left( \frac{2a^2}{D_1} + \frac{2}{B_1} \right)$$

Semi-rigid Joint Model

$$C_{C}^{ADCB} = C_{R}^{ADCB} + \frac{a^2}{\left( 1 - \frac{\varphi h}{\eta^2} \right)} \left( \frac{1}{D_1} + \frac{1}{D_2} \right) + \frac{a}{\left( 1 - \frac{\varphi h}{\eta^2} \right)} \left( \frac{1}{B_1} + \frac{1}{B_2} \right) + \frac{1}{k} \left( \frac{1}{D_1} + \frac{1}{D_2} \right)$$

$$G_{C}^{ADCB} = G_{R}^{ADCB} + \frac{P^2}{2b} \left( \frac{2a}{\left( 1 - \frac{\varphi h}{\eta^2} \right)} \left( \frac{1}{D_1} + \frac{1}{D_2} \right) + \frac{1}{\left( 1 - \frac{\varphi h}{\eta^2} \right)} \left( \frac{1}{B_1} + \frac{1}{B_2} \right) + \frac{1}{k} \left( \frac{1}{D_1} + \frac{1}{D_2} \right) \right)$$

$$C_{C}^{DCB} = C_{R}^{DCB} + \frac{2a^2}{\sqrt{B_1D_1}}$$

$$G_{C}^{DCB} = G_{R}^{DCB} + \frac{P^2}{2b} \left( \frac{4a}{\sqrt{B_1D_1}} \right)$$
Table 5.2 Moment loaded ADCB specimen

### Mode-I Dominant Specimens

![Diagram of Mode-I Dominant Specimens]

<table>
<thead>
<tr>
<th>Rigid Joint Model</th>
<th>Semi-rigid Joint Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_{\phi-R}^{ADCB} = a \left( \frac{1}{D_1} + \frac{1}{D_2} \right) )</td>
<td>( C_{\phi-R}^{ADCB} = \frac{M^2}{2b} \left( \frac{1}{D_1} + \frac{1}{D_2} \right) )</td>
</tr>
<tr>
<td>( C_{\phi-R}^{DCB} = \frac{2a}{D_1} )</td>
<td>( G_{\phi-R}^{DCB} = \frac{M^2}{2b} \left( \frac{2}{D_1} \right) )</td>
</tr>
</tbody>
</table>

\[
C_{\phi}^{ADCB} = C_{\phi-R}^{ADCB} + \frac{1}{k} \left( \frac{1}{D_1} + \frac{1}{D_2} \right) \left( 1 - \frac{\bar{g} h_1}{\eta^2} \right)
\]

\[
G_{\phi}^{ADCB} = G_{\phi-R}^{ADCB}
\]

\[
C_{\phi}^{DCB} = C_{\phi-R}^{DCB} + \frac{2}{\sqrt{B_1 D_1}}
\]

\[
G_{\phi}^{DCB} = G_{\phi-R}^{DCB}
\]
Table 5.3 Transversely loaded AENF specimen

**Mode-II Dominant Specimens**

![Diagram of Mode-II Dominant Specimens](image)

<table>
<thead>
<tr>
<th>Rigid Joint Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ C_{R}^{\text{AENF}} = \frac{a^3}{3D_1} \left( 3 - \frac{L}{B_1} \right) + \frac{a(a-\beta)}{B_1} + \frac{L\beta}{B_1} ]</td>
</tr>
<tr>
<td>[ C_{R}^{\text{ENF}} = \frac{a^3}{8D_1} + \frac{L^3}{24D_1} + \frac{L}{2B_1} ]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Semi-rigid Joint Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ C_{C}^{\text{AENF}} = C_{R}^{\text{AENF}} \left( 1 + \frac{\varphi_h}{\eta^2} \right) D_1 k + \left( 1 + \frac{\varphi_h}{\eta^2} \right) \frac{a^2}{D_1 k^2} ]</td>
</tr>
<tr>
<td>[ G_{C}^{\text{AENF}} = G_{R}^{\text{AENF}} \left( 1 + \frac{\varphi_h}{\eta^2} \right) D_1 k + \left( 1 + \frac{\varphi_h}{\eta^2} \right) \frac{a^2}{D_1 k^2} ]</td>
</tr>
<tr>
<td>[ C_{C}^{\text{ENF}} = C_{R}^{\text{ENF}} ]</td>
</tr>
<tr>
<td>[ G_{C}^{\text{ENF}} = G_{R}^{\text{ENF}} ]</td>
</tr>
</tbody>
</table>
Table 5.4 Moment loaded ENF specimen

### Mode-II Dominant Specimens

![Diagram of ENF specimen with sub-beams and loading](image)

#### Rigid Joint Model

\[
\begin{align*}
    C_{\phi-R}^{ADCB} &= \frac{a(\alpha - \psi)}{D_1} + \frac{L\psi}{D_1} \\
    C_{\phi-R}^{ADCB} &= \frac{3a}{8D_1} + \frac{L}{8D_1} \\
    G_{\phi-R}^{ADCB} &= \frac{M^2}{2b} \left( \frac{\alpha - \psi}{D_1} \right) \\
    G_{\phi-R}^{ADCB} &= \frac{M^2}{2b} \cdot \frac{3}{8D_1}
\end{align*}
\]

#### Semi-rigid Joint Model

\[
\begin{align*}
    C_{\phi}^{ENF} &= C_{\phi-R}^{ENF} + \left( \frac{\alpha - \psi - \frac{h}{2}}{1 + \frac{\zeta h}{\eta^2}} \right) D_k \\
    C_{\phi}^{ENF} &= C_{\phi-R}^{ENF} \\
    G_{\phi}^{ADCB} &= G_{\phi-R}^{ADCB} \\
    G_{\phi}^{DCB} &= G_{\phi-R}^{DCB}
\end{align*}
\]
### Mixed Mode Specimens

![Mixed Mode Specimens Diagram](image)

#### Rigid Joint Model

\[
C_{R}^{ASLB} = \frac{a^3 (1-\psi)}{3D_1} + \frac{L \psi}{3D_1} + \frac{a(1-\beta)}{B_1} + \frac{L \beta}{B_1}
\]

\[
C_{R}^{SLB} = \frac{7a^3}{24D_1} + \frac{L^2}{24D_1} + \frac{a + L}{2B_1}
\]

\[
G_{R}^{ASLB} = \frac{P^2}{2b} \left( \frac{a^2 (1-\psi)}{D_1} + \frac{1-\beta}{B_1} \right)
\]

\[
G_{R}^{SLB} = \frac{P^2}{2b} \left( \frac{7a^2}{8D_1} + \frac{1}{2B_1} \right)
\]

#### Semi-rigid Joint Model

\[
C_{C}^{ASLB} = C_{R}^{ASLB} + \frac{\left(1-\psi - \frac{h_1}{2} \right) a^2}{\left(1+\frac{\phi h_1}{\eta^2} \right) D_1 k} + \frac{\left(1-\psi - \frac{h_1}{2} \right) a}{\left(1+\frac{\phi h_1}{\eta^2} \right) D_1 k} + \frac{1}{D_1 k^2}
\]

\[
G_{C}^{ASLB} = G_{R}^{ASLB} + \frac{P^2}{2b} \left[ \frac{2 \left(1-\psi - \frac{h_1}{2} \right) a}{\left(1+\frac{\phi h_1}{\eta^2} \right) D_1 k} + \frac{1-\psi - \frac{h_1}{2}}{\left(1+\frac{\phi h_1}{\eta^2} \right)} \left(1 + \frac{\phi h_1}{\eta^2} \left( \frac{1}{B_1} + \frac{1}{D_1 k^2} \right) \right) \right]
\]

\[
C_{C}^{SLB} = C_{C}^{ASLB} + \frac{a^2}{2D_1} \frac{B_1}{B_1}
\]

\[
G_{C}^{SLB} = G_{R}^{SLB} + \frac{P^2}{2b} \left( \frac{a}{\sqrt{B_1 D_1}} \right)
\]
5.5 Conclusions

The Split Beam Model was used with the semi-rigid joint model to solve problems similar to the ones in chapter IV. In the semi-rigid joint model, the rotations of the sub-beams at the crack tip no longer need to be equal to one another. This allows for a more accurate determination of the ERR. To apply the semi-rigid joint, a differential equation was solved. The constants of integration were determined in a uniform manner for all the beam configurations based on the Split Beam Model. Finally, summaries of both the rigid and semi-rigid joint results were given. From these summaries, it can be noted that
while the compliance of each specimen varies from rigid joint model to the semi rigid joint model, the ERR of the moment loaded specimens are equivalent for the rigid joint model and the semi-rigid joint model.
CHAPTER VI

CONCLUSIONS

Methods used in the determination of the Mixed-Mode fracture toughness of wood-FRP and the development of a simplified Split beam model to fracture specimens are presented. By using the Split beam model, it was possible to determine the fracture toughness of many common bi-layer beam structures.

Analytical approaches for determining the total ERR along with methods for determining the Mode-I and Mode-II contributions were reviewed. Using these analytical approaches, two four-point bending specimens were designed. These two four-point bending specimens, the mixed-mode bending (4-MMB), and the asymmetric end notched flexure (4-AENF) were chosen because the configuration allowed for the measurement of the fracture toughness without measuring the fracture growth. Also, the methods for data reduction for each of the two specimens were discussed and outlined.

The materials used in the experimental characterization of the mixed-mode fracture toughness of the wood-FRP specimens were presented. The description of the fabrication process was given along with the testing procedure used in the determination of the critical crack initiation forces. Experimental results of each specimen tested were presented and average values and coefficients of variance were determined for the 4-
MMB and 4-AENF configurations. Finally, a number of failure criterions were provided and compared with the values of the constants determined for the best fit solutions.

A generalized, simplified Split Beam Model was presented. The Split Beam Model simplifies the determination of the forces acting in individual sublayers of a laminate. Using a kinematic approach, with the Split Beam Model, the bi-layer beam system was examined in greater depth in conjunction with the rigid joint model. A major advantage of the Split Beam Model was shown to be the ease at which it can deal with both the symmetric and asymmetric bi-layer beam conditions. As illustrated, the Split Beam models solutions for asymmetric conditions converged to the solutions symmetric case. The energy release rates (ERR) of many common fracture testing configurations were solved using the Split Beam Model and the compliance method. Results were compared to results and found to be consistent with the ones available in literature. Load configurations, referred to as Derived Specimens, were solved, and the ERR from these specimens were shown to be equivalent to the generic specimens.

Finally, the Split Beam Model was used in conjunction with the semi-rigid joint model to solve problems similar to the ones found in the rigid joint model. Using the semi-rigid joint, the rotations of the sub-beams at the crack tip no longer need to be equal to one another. This allowed for a more accurate determination of the ERR. In applying the semi-rigid joint, a differential equation was solved. The constants of integration were determined in a uniform manner for all the beam configurations based on the Split Beam Model. In conclusion, a summary of both the rigid and semi-rigid joint results was given to allow for a comparison of both the methods.
REFERENCES


