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NEWTON’S METHOD AS A MEAN VALUE METHOD

A Thesis

Presented to

The Graduate Faculty of The University of Akron

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

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May, 2007
NEWTON’S METHOD AS A MEAN VALUE METHOD

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Thesis

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ABSTRACT

In this thesis, the relationships between fixed-point problems, relaxation methods and Newton’s method were investigated. It was proven that Newton’s method was a modified relaxation method, whose mean value parameter approached the optimal parameter for relaxation method. Also, under some conditions on the derivative of the function whose fixed point was sought, a convergent relaxation sequence that converged to a fixed point of the function was introduced.
ACKNOWLEDGEMENTS

Thank you Dr. Hajjafar for picking up a stray cat and for giving it the chance to grow and accomplish its goal. Without you, that cat might still be wondering around in the department looking for a thesis. Thanks to Dr. Krishna for taking the time out of his busy days to read the cat’s thesis. Thanks to Dr. Saliga for knowing the cat’s English grammar so well and for teaching the cat the “no-person” method. Thanks to Dr. Curtis Clemons for the weird catnip called LaTeX that rocks any cat’s mind at first try. Thanks to Dr. Kevin Kreider for handing out Matlab snack when the cat scratches his door. Thanks to Patty Shelton and the gang for brightening the cat’s days and helping it in countless ways. Thanks to Dave Thompson for giving the cat sweets of candy bars and cinnamon rolls on long, hungry days. Thanks to Katie Cerrone for baking heavenly muffins that leave the cat wanting more. Thanks to The Department of Theoretical and Applied Mathematics for being a place where the cat got to know interesting people and experienced a variety of unexpected situations. Lastly, thanks to mom, dad, and the siblings for supporting the cat’s education and lending their ears and shoulder to meow on.
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CHAPTER I
INTRODUCTION

"The Newton-Rhapson method is one of the most useful and best known algorithms that relies on the continuity of $f'(x)$ and $f''(x)$." [1]. This paper proposed a new relaxation method that related to Newton’s method. Given a function $f(x)$ on an interval $[a, b]$, the goal was to find values $s$ such that $f(s) = 0$. By constructing another function $g(x)$ such that $s = g(s)$ whenever $f(x) = 0$, the problem of finding a root for $f$ became a problem of finding a fixed-point of $g$. Given $x_0$, the fixed-point iteration given by $x_{n+1} = g(x_n)$ for $n \geq 0$ would converge to $s$ as stated in the following theorems:

**Theorem 1.1** Given an interval $I = [a, b]$, if for any $x \in I$, $g(x) \in I$ (denoted $g(I) \subseteq I$) and if $g(x)$ is continuous, then $g(x)$ has at least one fixed-point in $I$ [2][Pg. 152].

**Theorem 1.2** If $g(I) \subseteq I$ and $|g'(x)| \leq L < 1$ for all $x \in I$, then there exists exactly one $s \in I$ such that $g(s) = s$ [2][Pg. 152].

**Theorem 1.3** Let $g(I) \subseteq I = [a, b]$ and $|g'(x)| \leq L < 1$ for all $x \in I$. For any $x_0 \in I$, the sequence $x_{n+1} = g(x_n)$, $n = 1, 2, 3, \ldots$ converges to the fixed-point $s$ and the $n^{th}$ error, $e_n = x_n - s$ satisfies $|e_n| \leq \frac{L^n|x_1 - x_0|}{1 - L}$. [2][Pg. 153]
In general, if \(|g'(s)| < 1\), then the fixed-point iteration \(x_{n+1} = g(x_n)\) converges to \(s\) for an appropriate \(x_0\). It is shown below that if \(|g'(s)| > 1\), then the sequence does not converge to \(s\) for any choice of \(x_0\).

**Theorem 1.4** Let \(g(x)\) and \(g'(x)\) be continuous on an interval \([a, b]\) containing \(s\) where \(g(s) = s\) and \(|g'(s)| > 1\). Then the fixed-point iteration will not converge to \(s\) for any choice of \(x_0\) [3].

**Proof:** By continuity of \(g\) at \(s\), there exists an \(M > 1\) and a \(\delta > 0\) such that \(|g'(x)| \geq M\) for all \(x \in (s - \delta, s + \delta)\). Let \(x_0 \in (s - \delta, s + \delta)\). Then \(x_1 - s = g(x_0) - s = g(x_0) - g(s) = g'(k)(x_0 - s)\) for some \(k \in (s - \delta, s + \delta)\). This implies that \(|x_1 - s| > |s - x_0|\). Thus, for any \(\delta\), the distance between \(x_1\) and \(s\) is bigger than the distance between \(x_0\) and \(s\). Therefore, if \(|g'(s)| > 1\), the sequence will not converge.

The next chapter contains results of a restated problem so that a fixed-point iteration will converge to \(s\) for \(|g'(s)| > 1\).
CHAPTER II

RELAXATION METHOD

2.1 Restating The Problem

**Theorem 2.1** Let \( g(x) \) be a function and define \( h(x) = \omega x + (1 - \omega)g(x) \) for some constant \( \omega \neq 1 \). Then \( s = g(s) \) if and only if \( s = h(s) \) [3].

**Proof**

\[
(\Rightarrow) \text{ Let } s = g(s), \text{ Then, } h(s) = ws + (1 - \omega)g(s) \\
\quad = ws + g(s) - wg(s) \\
\quad = ws + s - ws \\
\quad = s
\]

\[
(\Leftarrow) \text{ Let } h(s) = s, \text{ Then, } h(s) = ws + (1 - \omega)g(s) \\
\quad = s(1 - \omega) = (1 - \omega)g(s) \\
\quad = s \text{ if } \omega \neq 1
\]

Therefore, any fixed-point of \( g(x) \) is also a fixed-point of \( h(x) \).

It is shown in the next section that if \( |g'(s)| > 1 \), there exist an \( \omega \) such that \( |h'(s)| < 1 \).

2.2 Choosing \( \omega \)

Let \( s \) be a fixed-point of \( g(x) \) and \( |g'(x)| > 1 \). Let \( h \) be defined by \( h(x) = \omega x + (1 - \omega)g(x), \omega \neq 1 \). To find an \( \omega \) such that \( |h'(s)| < 1 \), two cases were considered.
Case 1: ($g'(s)>1$) [3]

Given $h(x) = \omega x + (1-\omega)g(x)$, $\omega \neq 1$, then $h'(x) = \omega + (1-\omega)g'(x)$. The inequality $|h'(s)| < 1$ is equivalent to $|h'(s)| = |\omega + (1-\omega)g'(s)| < 1$ or, written in another way:

\[
-1 < \omega + (1-\omega)g'(s) \quad \text{and} \quad 1 > \omega + (1-\omega)g'(s)
\]

\[
\Leftrightarrow -1 < \omega + g'(s) - \omega g'(s) \quad \text{and} \quad 1 > \omega + g'(s) - \omega g'(s)
\]

\[
\Leftrightarrow \omega g'(s) - \omega < g'(s) + 1 \quad \text{and} \quad \Leftrightarrow g'(s) - 1 < -\omega + \omega g'(s)
\]

\[
\Leftrightarrow \omega(g'(s) - 1) < g'(s) + 1 \quad \text{and} \quad \Leftrightarrow g'(s) - 1 < \omega(g'(s) - 1)
\]

Since $g'(s) > 1$, $g'(s) - 1 > 0$

\[
\therefore \omega < \frac{g'(s) + 1}{g'(s) - 1}
\]

Thus, if $g'(s) > 1$, choose $\omega$ such that

\[
1 < \omega < \frac{g'(s) + 1}{g'(s) - 1}. \tag{2.1}
\]

But since $s$ is unknown, $g'(s)$ is unknown, and an upper bound for $\omega$ is still unknown. The following theorems provide appropriate $\omega$ that are in the interval given by equation 2.1.

**Theorem 2.2** Let $g(x)$ be differentiable and continuous on $[a, b]$ that contains a fixed point $s$ of $g(x)$. Let $g'(s) > 1$ and let $M$ be any upper bound of $g'(x)$ on $[a, b]$.

Letting $\omega = \frac{M}{M-1}$, the iteration given by

\[
x_{j+1} = \omega x_j + (1-\omega)g(x_j) \tag{2.2}
\]

will converge to $x = s$ for an appropriate $x_0$. 

4
Proof: To show that the iteration converges in this case, showing that \( \omega = \frac{M - 1}{M} \in \left( 1, \frac{g'(s) + 1}{g'(s) - 1} \right) \) is sufficient. Since \( M > g'(s) > 1, M - 1 > 0 \). So, since \( M - 1 < M, 1 < \frac{M}{M - 1} \). Now,

\[
\frac{M}{M - 1} < \frac{g'(s) + 1}{g'(s) - 1} \iff M(g'(s) - 1) < (g'(s) + 1)(M - 1) \\
\iff Mg'(s) - M < Mg'(s) + M - g'(s) - 1 \\
\iff g'(s) + 1 < 2M
\]

Notice, since \( 1 < g'(s) < M, g'(s) + M < M + M = 2M \). Since \( M > 1, g'(s) + 1 < g'(s) + M \). So \( g'(s) + 1 < g'(s) + M < 2M \). Therefore \( g'(s) + 1 < 2M \).

Thus the theorem was proved.

**Theorem 2.3** Let \( g(x) \) be differentiable and continuous on \([a, b] \) that contains a fixed point \( s \) of \( g(x) \). Let \( g'(s) > 1 \) and let \( M \) be any upper bound of \( g'(x) \) on \([a, b] \).

Letting \( \omega = \frac{M + 1}{M} \), the iteration given by equation 2.2 will converge to \( x = s \) for an appropriate \( x_0 \).

Proof: As before, \( \omega = \frac{M + 1}{M} \in \left( 1, \frac{g'(s) + 1}{g'(s) - 1} \right) \) is needed.

Since \( M < M + 1, 1 < \frac{M + 1}{M} \). Now,

\[
\frac{M + 1}{M} < \frac{g'(s) + 1}{g'(s) - 1} \iff (M + 1)(g'(s) - 1) < M(g'(s) + 1) \\
\iff Mg'(s) - M + g'(s) - 1 < Mg'(s) + M \\
\iff g'(s) - 1 < 2M
\]

Since \( M > g'(s), M > g'(s) - 1, \) and so \( 2M > g'(s) - 1 \). Thus the theorem was proved.
**Case 2:** \( (g'(s) < -1) \) [3]

As before, \( |h'(s)| < 1 \) is equivalent to \( |h'(s)| = |\omega + (1 - \omega)g'(s)| < 1 \) or, written in another way:

\[
\begin{align*}
-1 &< \omega + (1 - \omega)g'(s) & 1 &> \omega + (1 - \omega)g'(s) \\
\iff -1 &< \omega + g'(s) - \omega g'(s) & \iff 1 &> \omega + g'(s) - \omega g'(s) \\
\iff \omega g'(s) - \omega &< g'(s) + 1 & \iff \omega g'(s) - \omega &> g'(s) - 1 \\
\iff \omega(g'(s) - 1) &< g'(s) + 1 & \iff \omega(g'(s) - 1) &> g'(s) - 1
\end{align*}
\]

Since \( g'(s) < -1, \ g'(s) - 1 < 0 \) and \( g'(s) < -1, \ g'(s) - 1 < 0 \)

\[
\therefore \omega > \frac{g'(s) + 1}{g'(s) - 1}
\]

Thus, if \( g'(s) < -1 \), choose \( \omega \) such that

\[
\frac{g'(s) + 1}{g'(s) - 1} < \omega < 1.
\] (2.3)

Once again, since \( s \) is unknown, \( g'(s) \) is not known, and a lower bound for \( \omega \) is not known. This problem was solved by the following theorems.

**Theorem 2.4** Let \( g(x) \) be differentiable on \([c, d]\) that contains a fixed-point \( s \) of \( g(x) \). Let \( g'(s) < -1 \) and let \( m \) be any lower bound of \( g'(x) \) on \([c, d]\). Letting

\[
\omega = \frac{m}{m - 1},
\]

the iteration given by equation 2.2 will converge to the fixed-point for an appropriate \( x_0 \).

**Proof:** To show that the iteration converges in this case, it suffices to show that

\[
\omega = \frac{m}{m - 1} \in \left( \frac{g'(s) + 1}{g'(s) - 1}, 1 \right).
\]
Since $m < g'(s) < -1$, $m > m - 1$. So \( \frac{m}{m - 1} < 1 \). Now,

\[
\frac{g'(s) + 1}{g'(s) - 1} < \frac{m}{m - 1} \iff (g'(s) + 1)(m - 1) < m(g'(s) - 1)
\]

\[
\iff mg'(s) - g'(s) + m - 1 < mg'(s) - m
\]

\[
\iff 2m < g'(s) + 1
\]

Notice, since

\[
m < g'(s) < -1,
\]

\[
m - 1 < g'(s) - 1 < -2,
\]

\[
2(m - 1) < g'(s) - 1 < -2,
\]

\[
2m - 2 < g'(s) - 1 < -2,
\]

\[
2m < g'(s) + 1 < 0
\]

So $2m < g'(s) + 1$ and the theorem was proved.

**Theorem 2.5** Let $g(x)$ be differentiable on $[c, d]$ that contains a fixed-point $s$ of $g(x)$. Let $g'(s) < -1$ and let $m$ be any lower bound of $g'(x)$ on $[c, d]$. Letting $\omega = \frac{m + 1}{m}$, the iteration given by equation 2.2 will converge to the fixed-point for an appropriate $x_0$.

**Proof:** As before, $\omega = \frac{m + 1}{m} \in \left(\frac{g'(s) + 1}{g'(s) - 1}, 1\right)$ is needed.

Since $m < g'(s) < -1$, $0 > m + 1 > m$. Thus $\frac{m + 1}{m} < 1$. Now,

\[
\frac{g'(s) + 1}{g'(s) - 1} < \frac{m + 1}{m} \iff m(g'(s) + 1) < (m + 1)(g'(s) - 1)
\]

\[
\iff mg'(s) + m < mg'(s) - m + g'(s) - 1
\]

\[
\iff 2m < g'(s) - 1
\]
Notice, since $m < g'(s) < -1$, $2m = m + m < g'(s) + m$. Since $m < -1$ and $g'(s) < -1$, $g(s) - 1 > g'(s) + m$. So $2m < g'(s) + m < g'(s) - 1$. Thus the theorem was proved.

2.3 Example

Let $f(x) = -\frac{x^2}{4} - \frac{x}{4} + 3$. Since $f(-6) = -4.5$ and $f(-2) = 2.5$, there exists a root $s_1 \in (-6, -2)$ by the Intermediate Value theorem. Similarly, since $f(1) = 2.5$ and $f(5) = -4.5$, there exists a root $s_2 \in (1, 5)$. The equation $f(x) = 0$ is equivalent to $g(x) = x$ where $g(x) = -x^2 + 12$. The iteration given by

$$x_{j+1} = g(x_j)$$

will converge to a root for an appropriate $x_0$ if $|g'(s)| < 1$.

The function $g'(x)$ is decreasing since $g'(x) = -2x$ and $g''(x) = -2$. Further, $g'(-6) = 12$ and $g'(-2) = 4$. Since $g'(x)$ is decreasing, 12 is an upper bound for $g'(x)$ on the interval $[-6, -2]$. Also, since $g'(x) > 1$ for all $x \in [-6, -2]$, $g'(s) > 1$ and clearly the sequence generated by equation 2.4 will not converge to $s_1$. To create a sequence that may converge to $s_1$, let $h(x)$ be defined as follows:

$$h(x) = \omega x + (1 - \omega)g(x) = \omega x + (1 - \omega)(-x^2 + 12)$$

i) By theorem 2.2, with $\omega = \frac{M}{M - 1} = \frac{12}{12 - 1} = \frac{12}{11}$, the following sequence converges to $s_1$ for an appropriate $x_0$:

$$x_{j+1} = \frac{12x_j + (x_j^2 - 12)}{11}$$
When \( x_0 = -3 \), equation 2.6 generates the following sequence that converges to \( s_1 = -4 \).

\[
\begin{align*}
    x_1 &= -3.54545 \\
    x_2 &= -3.81593 \\
    x_3 &= -3.92998 \\
    x_4 &= -3.97409 \\
    x_5 &= -3.99052 \\
    x_6 &= -3.99654 \\
    x_7 &= -3.99874 \\
    x_8 &= -3.99954 \\
    x_9 &= -3.99983 \\
    x_{10} &= -3.99994 \\
    x_{11} &= -3.99998 \\
    x_{12} &= -3.99999 \\
    x_{13} &= -4.00000 = s_1 \\
    x_{14} &= -3.99999 \\
    x_{15} &= -4.00000 = s_1
\end{align*}
\]

ii) By theorem 2.3, with \( \omega = \frac{M + 1}{M} = \frac{12 + 1}{12} = \frac{13}{12} \), the following sequence converges to \( s_1 \) for an appropriate \( x_0 \):

\[
x_{j+1} = \frac{13x_j + (x_j^2 - 12)}{12} \quad (2.7)
\]

When \( x_0 = -3 \), equation 2.7 generates the following sequence that converges to \( s_1 = -4 \).

\[
\begin{align*}
    x_1 &= -3.5 \\
    x_2 &= -3.77083 \\
    x_3 &= -3.90014 \\
    x_4 &= -3.95756 \\
    x_5 &= -3.98217 \\
    x_6 &= -3.99254 \\
    x_7 &= -3.99689 \\
    x_8 &= -3.99946 \\
    x_9 &= -3.99977 \\
    x_{10} &= -3.99994 \\
    x_{11} &= -3.99998 \\
    x_{12} &= -3.99999 \\
    x_{13} &= -4.00000 = s_1 \\
    x_{14} &= -3.99999 \\
    x_{15} &= -4.00000 = s_1
\end{align*}
\]

To approximate \( s_2 \), the fact that \( g'(x) \) is a decreasing function was used.

Since \( g'(1) = -2 \) and \( g'(5) = -10 \), -10 is a lower bound for \( g'(x) \) on the interval [1,5]. Also, since \( g'(x) < -1 \) for all \( x \in [1,5] \), \( g'(s) < -1 \) and clearly the sequence
generated by equation 2.4 will not converge to $s_2$. In order to generate a sequence that may converge to $s_2$, then $h(x)$ defined in equation 2.5 was used.

i) By theorem 2.4, with $\omega = \frac{m}{m-1} = \frac{-10}{-10-1} = \frac{10}{11}$, the following sequence converges to $s_2$ for an appropriate $x_0$:

$$x_{j+1} = \frac{10x_j + (12 - x_j^2)}{11} \quad (2.8)$$

When let $x_0 = 4$, equation 2.8 generates the following sequence that converges to $s_2 = 3$:

$$x_1 = 3.27272 \quad x_5 = 3.00429 \quad x_9 = 3.00008$$

$$x_2 = 3.09241 \quad x_6 = 3.00156 \quad x_{10} = 3.00003$$

$$x_3 = 3.03283 \quad x_7 = 3.00057 \quad x_{11} = 3.00001$$

$$x_4 = 3.01184 \quad x_8 = 3.00021 \quad x_{12} = 3.00000 = s_2$$

ii) By theorem 2.5, with $\omega = \frac{m+1}{m} = \frac{-10+1}{-10} = \frac{9}{10}$, the following sequence will converges to $s_2$ for an appropriate $x_0$:

$$x_{j+1} = \frac{9x_j + (12 - x_j^2)}{10} \quad (2.9)$$

When let $x_0 = 4$, equation 2.9 generates the following sequence that converges to $s_2 = 3$:

$$x_1 = 3.2 \quad x_5 = 3.00147 \quad x_9 = 3.00001$$

$$x_2 = 3.056 \quad x_6 = 3.00044 \quad x_{10} = 3.00000 = s_2$$

$$x_3 = 3.01649 \quad x_7 = 3.00013$$

$$x_4 = 3.00492 \quad x_8 = 3.00004$$
For these $\omega$, the sequences generated by equation 2.2 converged at a slow rate. An $\omega$ that helps speed up convergence was desired.

2.4 Another $\omega$

By the work of Linhart, the following theorems also provided appropriate $\omega$ that are in the interval given by 2.1 and 2.3.

**Theorem 2.6** [3] Let $g(x)$ be differentiable and continuous on $[a, b]$ that contains a fixed point $s$ of $g(x)$. Let $g'(s) > 1$ and let $M$ be any upper bound of $g'(x)$ on $[a, b]$. Letting $\omega = \frac{M + 1}{M - 1}$, the iteration given by equation 2.2 will converge to $x = s$ for an appropriate $x_0$.

**Proof:** To show that the iteration converges in this case, $\omega = \frac{M + 1}{M - 1} \in \left(1, \frac{g'(s) + 1}{g'(s) - 1}\right)$ is needed.

Obviously, $1 < \frac{M + 1}{M - 1}$ since $M - 1 < M + 1$. Now,

$$\frac{M + 1}{M - 1} < \frac{g'(s) + 1}{g'(s) - 1} \iff (M + 1)(g'(s) - 1) < (g'(s) + 1)(M - 1)$$

$$\iff Mg'(s) - M + g'(s) - 1 < Mg'(s) - g'(s) + M - 1$$

$$\iff g'(s) - M < M - g'(s)$$

$$\iff 2g'(s) < 2M$$

$$\iff g'(s) < M$$

Since $M$ is an upperbound, $g'(s)$ is less than $M$. Thus the theorem was proved.
Theorem 2.7 [3] Let \( g(x) \) be differentiable on \([c, d]\) that contains a fixed-point \( s \) of \( g(x) \). Let \( g'(s) < -1 \) and let \( m \) be any lower bound of \( g'(x) \) on \([c, d]\). Letting 
\[ \omega = \frac{m + 1}{m - 1}, \]
the iteration given by equation 2.2 will converge to the fixed-point for an appropriate \( x_0 \).

Proof: To show that the iteration converges in this case, \( \omega = \frac{m + 1}{m - 1} \in \left( \frac{g'(s) + 1}{g'(s) - 1}, 1 \right) \)
is needed.

Since \( m < g'(s) < -1 \), \( \frac{m + 1}{m - 1} < 1 \iff m + 1 > m - 1 \iff 0 > -2 \), which is always true. Similarly,
\[
\frac{g'(s) + 1}{g'(s) - 1} < \frac{m + 1}{m - 1} \iff (g'(s) + 1)(m - 1) < (m + 1)(g'(s) - 1)
\]
\[
\iff mg'(s) - g'(s) + m - 1 < mg'(s) - m + g'(s) - 1
\]
\[
\iff m - g'(s) < g'(s) - m
\]
\[
\iff 2m < 2g'(s)
\]
\[
\iff m < g'(s)
\]

Since \( m \) is a lowerbound, \( g'(s) \) is greater than \( m \). Therefore, the theorem was proved.

2.5 Example

The previous example, solving for \( f(x) = 0 \) when \( f(x) = \frac{-x^2}{4} - \frac{x}{4} + 3 \) was solved using theorem 2.6 and 2.7. As before, \( g(x) = -x^2 + 12 \) and \( g'(x) = -2x \).

Given the interval \([-6, -2]\), an upper bound is \( M = 12 \).
Given the interval $[1, 5]$, a lower bound is $m = -10$.

By theorem 2.6, with $\omega = \frac{M + 1}{M - 1} = \frac{12 + 1}{12 - 1} = \frac{13}{11}$, the following sequence converges to $s_1$ for an appropriate $x_0$:

$$x_{j+1} = \frac{13x_j + 2(x_j^2 - 12)}{11} \quad (2.10)$$

Let $x_0 = -3$. Then equation 2.10 generates the following sequence that converges to $s_1 = -4$:

$$x_1 = -4.09091 \quad x_5 = -4.00053 \quad x_9 = -4.00000$$

$$x_2 = -3.97370 \quad x_6 = -3.99986 \quad x_{10} = -4.00000 = s_1$$

$$x_3 = -4.00705 \quad x_7 = -4.00004$$

$$x_4 = -3.99807 \quad x_8 = -3.99999$$

Now, by theorem 2.7, with $\omega = \frac{m + 1}{m - 1} = \frac{-10 + 1}{-10 - 1} = \frac{9}{11}$, the following sequence converges to $s_2$ for an appropriate $x_0$:

$$x_{j+1} = \frac{9x_j + 2(12 - x_j^2)}{11} \quad (2.11)$$

Let $x_0 = 4$. Then equation 2.11 generates the following sequence that converges to $s_2 = 3$:

$$x_1 = 2.54545 \quad x_4 = 3.00668 \quad x_7 = 2.99986 \quad x_{10} = 3.00000$$

$$x_2 = 3.08640 \quad x_5 = 2.99817 \quad x_8 = 3.00004 \quad x_{11} = 3.00000 = s_2$$

$$x_3 = 2.97508 \quad x_6 = 3.00050 \quad x_9 = 2.99999$$

The sequences generated by equation 2.2 using these $\omega$ converged a little faster but still at a slow rate. The next chapter presents results on how to find an $\omega$ that gave faster convergence.
3.1 Finding the optimal $\omega$

In order to find an optimal $\omega$, the rate of convergence of the traditional fixed-point algorithm needed to be examined. Let $s$ be a fixed-point of $g(x)$ with $g'(x)$ continuous on an open interval $I$ containing $s$ with $|g'(s)| < 1$. Suppose $x_0 \in I$ and let $e_k = x_k - s$, for all $k$. Also, suppose that the $(k + 1)^{st}$ derivative of $g(x)$ is continuous on $I$. By expanding $g(x)$ in a Taylor’s series about $s$ the error of this algorithm was analyzed. The error was found to be: [2][Pg. 156]

$$e_{n+1} = g(x_n) - g(s)$$

$$= g'(s)e_n + \frac{g''(s)e_n^2}{2!} + \frac{g^{(k)}(s)e_n^k}{k!} + \frac{g^{(k+1)}(b_n)e_n^{k+1}}{(k + 1)!}$$

(3.1)

where $b_n$ lies between $x_n$ and $s$. Assume $g'(x) \neq 0$ for all $x \in I$. When $k = 0$, $e_{n+1} = g'(b_n)e_n$ or

$$\frac{e_{n+1}}{e_n} = g'(b_n) \quad [2][Pg.156]$$

(3.2)

But as $n \to \infty$, $x_n \to s$ which implies $\lim_{n \to \infty} \left( \frac{e_{n+1}}{e_n} \right) = g'(s)$. This assumption suggested that for large $n$, $e_{n+1} \approx g'(s)e_n$. This rate of convergence was called first-
order or linear convergence. But if \( g'(s) = 0 \), and \( g'(x) \neq 0 \) for all \( x \in I \), then

\[
\lim_{n \to \infty} \left( \frac{e_{n+1}}{e_n^2} \right) = \frac{g''(s)}{2!} \quad [2][Pg.156] \tag{3.3}
\]

This rate of convergence was called second-order or quadratic convergence. A quadratic convergence was much faster than a linear convergence. By forcing quadratic convergence in the traditional fixed-point problem, one arrived at Newton’s method. Similarly, the restriction of \( h'(s) = 0 \) was made to the modified fixed-point algorithm to speed up convergence.

Assume that \( g(x) \) is continuous on an interval \( I \) that contains a fixed-point \( s \in I \). Also, suppose \( g'(x) \) is continuous on \( I \) and \( |g'(s)| > 1 \). Let \( h(x) = \omega x + (1 - \omega)g(x) \).

To find an \( \omega \) such that \( h(x) \) converges quadratically, two cases were considered.

**Case 1:** \[[3] \ (g'(s) > 1)\] For quadratic convergence, \( h'(s) = 0 \) is needed. Thus,

\[
h'(s) = \omega + (1 - \omega)g'(s) = 0
\]

\[
\Rightarrow \omega + g'(s) - \omega g'(s) = 0
\]

\[
\Rightarrow \omega(1 - g'(s)) = -g'(s)
\]

\[
\Rightarrow \omega = \frac{-g'(s)}{1 - g'(s)}
\]

\[
\Rightarrow \omega = \frac{g'(s)}{g'(s) - 1} \tag{3.4}
\]

This \( \omega \) gives quadratic convergence. To see that this \( \omega \) satisfies equation 2.1, \( \omega \) needs to satisfy both \( 1 < \omega \) and \( \omega < \frac{g'(s) + 1}{g'(s) - 1} \).

For the first part, since \( g'(s) > 1 \), \( g'(s) - 1 > 0 \), and \( g'(s) > g'(s) - 1 \). This implies that \( \frac{g'(s)}{g'(s) - 1} > 1 \), and \( \omega > 1 \). Now, since \( g'(s) > 1 \), it follows that
\[ g'(s)[g'(s) - 1] < [g'(s) + 1][g'(s) - 1], \text{ and that } \frac{g'(s)}{g'(s) - 1} < \frac{g'(s) + 1}{g'(s) - 1}. \] Thus \( \omega < \frac{g'(s) + 1}{g'(s) - 1} \). Therefore, an \( \omega \) chosen using equation 3.4 not only provided quadratic convergence but also satisfied equation 2.1.

**Case 2:** (3) \( (g'(s) < -1) \) For quadratic convergence, \( h'(s) = 0 \) is needed. But when \( h'(s) = 0 \) is solved for \( \omega \), equation 3.4 was the result. To see that this \( \omega \) satisfies equation 2.3, \( \omega \) needs to satisfy both \( \frac{g'(s) + 1}{g'(s) - 1} < \omega \) and \( \omega < 1 \).

For the first part, since \( g'(s) < -1 \), it follows that \( g'(s)[g'(s) - 1] > [g'(s) + 1][g'(s) - 1] \). This implies that \( \frac{g'(s)}{g'(s) - 1} > \frac{g'(s) + 1}{g'(s) - 1} \). Thus \( \omega > \frac{g'(s) + 1}{g'(s) - 1} \).

Now, \( g'(s) < -1 \) implies that \( \frac{g'(s)}{g'(s) - 1} < 1 \). Thus \( \omega < 1 \). Hence, an \( \omega \) chosen using equation 3.4 satisfied equation 2.3.

Therefore, in both cases, for \( \omega = \frac{g'(s)}{g'(s) - 1} \), the iteration \( x_{n+1} = h(x_n) \) converges quadratically.

The same result was obtained when \( g'(s) \neq 1 \). To do that, looked back at equation \( h(x) = \omega x + (1 - \omega)g(x) \) and the fact that \( h(s) = g(s) = s \). Suppose \( x \) is close to \( s \) but not \( s \), written \( x = s + \delta s \) where \( \delta s \) is very small. So,

\[
x_1 \approx h(x_1) = \omega(s + \delta s) + (1 - \omega)g(s + \delta s)
= \omega s + \omega \delta s + (1 - \omega) \left[ g(s) + \delta g'(s) + \frac{(\delta s)^2}{2} g''(s) + \ldots \right]
= \omega s + \omega \delta s + (1 - \omega) \left[ g(s) + \delta g'(s) + O(\delta s^2) \right]
= \omega s + \omega \delta s + (1 - \omega)g(s) + (1 - \omega)\delta g'(s) + O(\delta s^2)
= \omega s + \omega \delta s + (1 - \omega)s + (1 - \omega)\delta g'(s) + O(\delta s^2)
\]

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So, \( x_1 \approx \omega s + \omega \delta s + s - \omega s + (1 - \omega) \delta s g'(s) \)

\[ x_1 - s \approx \delta s [\omega + (1 - \omega) g'(s)] \]

Choosing \( \omega + (1 - \omega) g'(s) \approx 0 \) so that \( x_1 \approx s \), gives

\[
\begin{align*}
\omega + g'(s) - \omega g'(s) & \approx 0 \\
\omega (1 - g'(s)) & \approx -g'(s) \\
\omega & \approx \frac{g'(s)}{g'(s) - 1}
\end{align*}
\]

which is exactly the same result. This method was known as Lagrange Multiplier [4].

Unfortunately, to calculate the true optimum \( \omega \), the exact fixed point \( s \) was needed. However, this parameter \( \omega \) was a Mean Value parameter where \( x_1 - s = O(\delta s) \) and therefore an approximation to \( \omega \) was available. \( \omega \) was replaced in \( s \approx \omega x + (1 - \omega) g(x) \) by \( \frac{g'(x_1)}{g'(x_1) - 1} \) without causing an error exceeding \( O(\delta s^2) \). Consequently, the variational estimate

\[ s \approx x_2 = \frac{g'(x_1)}{g'(x_1) - 1} x_1 + \left(1 - \frac{g'(x_1)}{g'(x_1) - 1}\right) g(x_1) \]

was obtained and the process was continued with \( x_2 \) instead of \( x_1 \). Continuing provided

\[
\begin{align*}
x_{n+1} = \frac{g'(x_n)}{g'(x_n) - 1} x_n + \left(1 - \frac{g'(x_n)}{g'(x_n) - 1}\right) g(x_n)
\end{align*}
\]

(3.5)

The convergence of this iteration procedure needed to be determined.
Rewriting the process as follows:

\[
x_{n+1} = \frac{g'(x_n)}{g'(x_n) - 1}x_n + \left(1 - \frac{g'(x_n)}{g'(x_n) - 1}\right)g(x_n)
\]

\[
= \frac{g'(x_n)}{g'(x_n) - 1}x_n + \left(-1\right)\frac{g'(x_n) - 1}{g'(x_n) - 1}g(x_n)
\]

\[
= \frac{g'(x_n) + 1}{g'(x_n) - 1}x_n + \left(-1\right)\frac{1}{g'(x_n) - 1}g(x_n)
\]

\[
= x_n + \frac{1}{g'(x_n) - 1}\left(x_n - g(x_n)\right)
\]

\[
= x_n - \frac{x_n - g(x_n)}{1 - g'(x_n)}
\]

showed that this new iteration process was actually Newton’s method for the function \( f(x) = x - g(x) \). Therefore the sequence converges to a zero of \( f(x) \), i.e. when \( f(s) = 0, s = g(s) \).

The next step was to show that any Newton’s method is a modified relaxation method. Let \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \). For \( f(x) + x - x = 0 \), define \( g(x) = f(x) + x \). Then \( f(x) = 0 \) is equivalent to \( g(x) = x \).

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]

\[
= x_n (g'(x_n) - 1) - (g(x_n) - x_n)
\]

\[
= \frac{g'(x_n)}{g'(x_n) - 1}x_n + \frac{g'(x_n) - 1 - g'(x_n)}{g'(x_n) - 1}g(x_n)
\]

\[
= \frac{g'(x_n)}{g'(x_n) - 1}x_n + \left(1 - \frac{g'(x_n)}{g'(x_n) - 1}\right)g(x_n)
\]

\[
= \omega_n x_n + (1 - \omega_n)g(x_n)
\]
3.2 Examples

The previous example solving \( f(x) = 0 \) when \( f(x) = \frac{-x^2}{4} - \frac{x}{4} + 3 \) was solved using the new modified algorithm. As before, let \( g(x) = -x^2 + 12 \) and \( g'(x) = -2x \). Then

\[
\omega_n = \frac{g'(x_n)}{g'(x_n) - 1} = \frac{-2x_n}{-2x_n - 1}. \quad \text{Using sequence in 3.5,}
\]

\[
x_{n+1} = \frac{-2x_n}{-2x_n - 1} x_n + \left( 1 - \frac{-2x_n}{-2x_n - 1} \right) (-x_n^2 + 12) = \frac{2x_n^2 + 2x_n + 1 - 2x_n}{2x_n + 1} (-x_n^2 + 12) = \frac{2x_n^2 - x_n^2 + 12}{2x_n + 1} = \frac{x_n^2 + 12}{2x_n + 1}
\]

The same \( x_0 \)s were used as before so the rates of convergence could be compared.

By letting \( x_0 = -3 \),

\[
\begin{align*}
x_1 &= -4.2 \\
x_2 &= -4.00540 \\
x_3 &= -4.00000
\end{align*}
\]

By letting \( x_0 = 4 \),

\[
\begin{align*}
x_1 &= 3.11111 \\
x_2 &= 3.00171 \\
x_3 &= 3.00000
\end{align*}
\]

Clearly, the convergence rate for the modified algorithm is faster.
3.3 Using $\omega_0$

In the modified algorithm $x_{n+1} = \omega_n x_n + (1 - \omega_n) g(x_n)$, since as $n \to \infty$, $\omega_n$ converges to $\frac{g'(s)}{g'(s) - 1}$, after a few calculations $\omega_n$ must fall in the range of acceptable $\omega$ for the algorithm $x_{n+1} = \omega x_n + (1 - \omega) g(x_n)$. So after a few steps, say $n_0$, the same $\omega_{n_0}$ can be used for the iteration, saving computer time and even truncation error. Actually, when the minimum of $g'(x)$ on $[a, b]$ is greater than 1 or when the maximum of $g'(x)$ on $[a, b]$ is less than 1, $\omega = \omega_0$ can be used. In the following theorem, $x_0$ was chosen so that the corresponding Newton’s method converges.

**Theorem 2.7** Let $m = \text{minimum of } g'(x)$ on $[a, b]$ and $M = \text{maximum of } g'(x)$ on $[a, b]$. Suppose either $m > 1$ or $M < 1$ and let $\omega = \frac{g'(x_0)}{g'(x_0) - 1}$, then

$$x_{n+1} = \omega x_n + (1 - \omega) g(x_n)$$

converges to $s$, a fixed point of $g(x)$.

**Proof:** Suppose $m > 1$. Obviously $M > 1$. So $0 < m - 1 \leq g'(x_0) - 1 \leq M - 1$.

So

$$\frac{1}{m - 1} \geq \frac{1}{g'(x_0) - 1} \geq \frac{1}{M - 1},$$

$$\frac{x_n - g(x_n)}{m - 1} \text{ and } \frac{x_n - g(x_n)}{M - 1}.$$  

It follows that

$$x_n + \frac{x_n - g(x_n)}{g'(x_0) - 1} = \frac{g'(x_0) x_n - g(x_n)}{g'(x_0) - 1} = \frac{g'(x_0)}{g'(x_0) - 1} x_n + \left(1 - \frac{g'(x_0)}{g'(x_0) - 1}\right) g(x_n)$$

is between

$$x_n + \frac{x_n - g(x_n)}{M - 1} = \frac{M x_n - g(x_n)}{M - 1} = \frac{M}{M - 1} x_n + \left(1 - \frac{M}{M - 1}\right) g(x_n)$$

and

$$x_n + \frac{x_n - g(x_n)}{m - 1} = \frac{m x_n - g(x_n)}{m - 1} = \frac{m}{m - 1} x_n + \left(1 - \frac{m}{m - 1}\right) g(x_n)$$
By the squeeze theorem and theorems 2.2 and 2.4, since
\[\frac{M}{M-1}x_n + \left(1 - \frac{M}{M-1}\right)g(x_n)\] and \[\frac{m}{m-1}x_n + \left(1 - \frac{m}{m-1}\right)g(x_n)\] both converge to \(s\), a fixed point of \(g(x)\), for \(\omega = \frac{g'(x_0)}{g'(x_0) - 1}\), \(x_{n+1} = \omega x_n + (1 - \omega)g(x_n)\) converges to \(s\).

Now, suppose \(M < 1\). Obviously \(m < 1\). So \(\frac{m}{m-1}g(x_0) - 1 < 0\).

So \(\frac{1}{m-1} \geq \frac{1}{g'(x_0) - 1} \geq \frac{1}{M-1}\). With the same proof as before,
\[\frac{g'(x_0)}{g'(x_0) - 1}x_n + \left(1 - \frac{g'(x_0)}{g'(x_0) - 1}\right)g(x_n)\] is between
\[\frac{M}{M-1}x_n + \left(1 - \frac{M}{M-1}\right)g(x_n)\] and \[\frac{m}{m-1}x_n + \left(1 - \frac{m}{m-1}\right)g(x_n)\].

So again, by the squeeze theorem and theorems 2.2 and 2.4, for \(\omega = \frac{g'(x_0)}{g'(x_0) - 1}\), \(x_{n+1} = \omega x_n + (1 - \omega)g(x_n)\) converges to \(s\).

3.4 Example

Given the function \(f(x) = x^3 - 6x^2 - x + 30\), define \(g(x) = x^3 - 6x^2 + 30\), so that \(g'(x) = 3x^2 - 12x\). This problem was solved by letting \(\omega = \omega_0\) and \(\omega = \omega_1\).

\(i)\) \((\omega = \omega_0)\) On the interval \([-3, -1]\), the minimum of \(g'(x) > 1\), so by the theorem,
the iteration \(x_{n+1} = \omega_0 x_n + (1 - \omega_0)g(x_n)\) will converge. By letting \(x_0 = -1.5\),
\(\omega_0 = 1.04210526316\), giving
\[x_{n+1} = 1.04210526316x_n + (1 - 1.04210526316)(x_n^3 - 6x_n^2 + 30)\].

This iteration generates the following sequence that converges to \(-2\).
\[ x_1 = -2.11579 \quad x_7 = -2.00140 \quad x_{13} = -2.00002 \]
\[ x_2 = -1.93831 \quad x_8 = -1.99934 \quad x_{14} = -1.99999 \]
\[ x_3 = -2.02731 \quad x_9 = -2.00031 \quad x_{15} = -2.00000 \]
\[ x_4 = -1.98669 \quad x_{10} = -1.99985 \quad x_{16} = -2.00000 \]
\[ x_5 = -2.00622 \quad x_{11} = -2.00007 \]
\[ x_6 = -1.99704 \quad x_{12} = -1.99997 \]

ii) \( \omega = \omega_1 \) As above, for \( \omega = \frac{g'(x_0)}{g'(x_0) - 1} \), when \( x_0 = -1.5 \), \( \omega_0 = 1.04210526316 \) and \( x_1 = -2.11578947368 \) and \( \omega_1 = 1.02644161749 \).

Fixing \( \omega_1 = 1.02644161749 \), the iteration becomes

\[ x_{n+1} = 1.02644161749x_n + (1 - 1.02644161749)(x_n^3 - 6x_n^2 + 30) \]

generating the following sequence that converges to -2.

\[ x_1 = -1.88671 \quad x_5 = -1.99999 \]
\[ x_2 = -1.98752 \quad x_6 = -2.00000 \]
\[ x_3 = -1.99902 \quad x_7 = -2.00000 \]
\[ x_4 = -1.99993 \]

Using \( \omega = \omega_1 \), the iteration converged faster than when \( \omega = \omega_0 \) was used.
CHAPTER IV
CONCLUSION

The traditional $x_{n+1} = g(x_n)$ fixed-point method diverged when the absolute value of the derivative at the fixed point was bigger than one. This method was modified so that a convergent sequence was generated. This relaxation modified method depends on a parameter $\omega$. The optimal $\omega$ forced the relaxation method to converge quadratically. However, this optimal $\omega$ depends on the fixed point which was not at hand. A modified relaxation method was introduced and its relationship with Newton’s method examined. Also, it was shown that under some special cases, this modified relaxation method became a convergent relaxation method.

