NONSTANDARD HULLS OF GROUPS

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NONSTANDARD HULLS OF GROUPS

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ABSTRACT

The nonstandard hull construction, one of the most useful applications of Abraham Robinson’s nonstandard analysis, is generalized to an arbitrary group. Some important connections between the classical construction as applied to vector spaces and groups are noted, and several useful results regarding this generalized construction are developed. The theory developed provides particularly interesting results when applied to certain discrete groups.
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CHAPTER I
INTRODUCTION

1.1 Preliminaries

The framework of nonstandard analysis we will work with is outlined in [1], which is based on the ideas of Abraham Robinson, who first widely introduced the subject in [2]. For what follows we will have a fixed universe, $\mathcal{U}$, which contains all objects of interest. The key idea in nonstandard analysis is that given this universe $\mathcal{U}$ there exists an enlargement $\mathcal{U}^*$. Furthermore, given any set $X$ from the universe, there exists a corresponding enlarged set $X^*$ in $\mathcal{U}^*$ with all of the same first-order properties as $X$. To work with elements, we make the following natural identifications. For each $\alpha \in X$, we identify $\alpha$ with $\alpha^* \in X^*$. Thus we identify $X$ as a subset of $X^*$, and it follows from the properties of the construction of $X^*$ that we write $X = X^*$ if and only if $X$ is finite. When working with elements, we say that $z \in X^*$ is standard if $z = x^*$ for some $x \in X$. Consequently $z$ is said to be nonstandard if it has no preimage in $X$ under $\ast$. If $X = \mathbb{R}$, then when building the enlargement $X^*$ the standard elements of $X^*$ are the familiar real numbers, and it is shown in [1] that there are nonstandard elements of this set that possess some interesting properties. Specifically, there exists elements of $\mathbb{R}^*$ that are smaller than every standard nonzero
real number, and elements of $\mathbb{R}^*$ that are larger than every standard real number. This enlarged set $\mathbb{R}^*$ is often referred to as the set of hyperreal numbers. We now give formal names to these nonstandard elements and discuss their basic properties.

**Definition 1.** If $x \in \mathbb{R}^*$, then we have the following:

1. $x$ is said to be infinitesimal if $|x| < M$ for all standard, positive $M \in \mathbb{R}^*$;

2. $x$ is said to be limited if $|x| \leq M$ for some standard, positive $M \in \mathbb{R}^*$;

3. $x$ is said to be unlimited if $|x| > M$ for all standard, positive $M \in \mathbb{R}^*$.

We will write $x \approx y$ when the difference of $x$ and $y$ is infinitesimal, and say that $x$ and $y$ are infinitely close. We will write $x \ll \infty$ when $x$ is limited. We now state the following Theorem which gives the basic properties of arithmetic for hyperreal numbers.

**Theorem 2.** Let $x, y \in \mathbb{R}^*$. Then the following are true.

1. If $x \approx 0$ and $y \approx 0$, then $xy \approx 0$.

2. If $x \ll \infty$ and $y \approx 0$, then $xy \approx 0$.

3. If $x \ll \infty$ and $y \ll \infty$, then $xy \ll \infty$.

4. If $x$ is unlimited and $y \ll \infty$, then $xy$ is unlimited.

5. If $x \approx 0$, then $\frac{1}{x}$ is unlimited.

6. If $x$ is unlimited, then $\frac{1}{x} \approx 0$. 


The proof of the theorem is a direct application of Definition 1. Notice that given \( x \approx 0 \) and \( y \) unlimited, the product \( xy \) is indeterminate.

When working with limited real numbers, we have the following useful result.

**Theorem 3.** Let \( x, y \in \mathbb{R}^* \). If \( x \) is limited, then there exists a unique standard element, denoted \( \text{st}(x) \), such that \( \text{st}(x) \approx x \). Furthermore, if \( y \) is limited, then the following hold:

1. \( \text{st}(x \pm y) = \text{st}(x) \pm \text{st}(y) \);
2. \( \text{st}(xy) = \text{st}(x)\text{st}(y) \);
3. \( \text{st} \left( \frac{x}{y} \right) = \frac{\text{st}(x)}{\text{st}(y)} \), provided \( \text{st}(y) \neq 0 \);
4. If \( x \leq y \), then \( \text{st}(x) \leq \text{st}(y) \).

The interested reader can find a proof of this theorem in [1]. Observe that we require \( x \) to be limited; otherwise, there will be no standard point close enough to \( x \) for \( \text{st}(x) \) to make sense. Intuitively, one can observe that all limited \( x \in \mathbb{R}^* \) are of the form \( x + \delta \) where \( x \) is standard and \( \delta \approx 0 \); thus taking the standard part is merely choosing to set \( \delta = 0 \).

We can also extend the preceding notions in a natural way when considering objects other than the real line. For instance given a normed vector space \( V \) over \( \mathbb{R} \) with norm \( \| \cdot \| \), if \( v \in V \) we say \( v \) is \( \| \cdot \| \)-limited or simply limited if \( \| v \| \ll \infty \); similarly, one can call \( v \) \( \| \cdot \| \)-infinitesimal or infinitesimal whenever \( \| v \| \approx 0 \). If \( v = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and \( v \) is limited, we can define \( \text{st}(v) = (\text{st}(x_1) \ldots \text{st}(x_n)) \),
provided \( x_i \ll \infty \) for all \( i \). Given a matrix \( A \) over \( \mathbb{R}^* \) with limited entries, we can define \( \text{st}(A) \) to be the matrix over \( \mathbb{R} \) whose entries are the standard parts of each entry of \( A \). We now give a formal definition in this spirit.

**Definition 4.** Let \( V \) be a vector space over \( \mathbb{R} \). Suppose that \( v \in V^* \) can be written as \( \sum_i \alpha_i e_i \) where each \( e_i \) is a basis element, and \( \alpha_i \in \mathbb{R}^* \) is limited for all \( i \). Then we define the standard part of \( v \), denoted \( \text{st}(v) \), to be \( \sum_i \text{st}(\alpha_i)e_i \).

We now show that in the finite dimensional case that the standard part of a vector does not depend on the choice of basis.

**Lemma 5.** Let \( V \) be a finite dimensional vector space over \( \mathbb{R} \). Let \( v \in V^* \). Suppose that \( \beta = \{e_1, \ldots, e_n\} \) and \( \beta' = \{e'_1, \ldots, e'_n\} \) are bases for \( V \). Then \( \text{st}(v) \) is the same with respect to both \( \beta \) and \( \beta' \).

**Proof.** Let \( v \in V^* \), and write \( v = \sum_i \alpha_i e_i = \sum_i \alpha'_i e'_i \). Where \( \alpha_i \ll \infty \) for all \( i \). Let \( M \) be the change of basis matrix from \( \beta \) to \( \beta' \). Then observe

\[
\begin{pmatrix}
\alpha'_1 \\
\vdots \\
\alpha'_n
\end{pmatrix} = M
\begin{pmatrix}
\alpha_1 \\
\vdots \\
\alpha_n
\end{pmatrix}
\]

Note that since the standard part of the vector on the right is defined, and \( M \) is standard, we may write

\[
\begin{pmatrix}
\text{st}(\alpha'_1) \\
\vdots \\
\text{st}(\alpha'_n)
\end{pmatrix} = M
\begin{pmatrix}
\text{st}(\alpha_1) \\
\vdots \\
\text{st}(\alpha_n)
\end{pmatrix}
\]

This shows the result. \( \square \)
One of the most important notions of nonstandard analysis is the following principle.

1.2 The Transfer Principle

**Theorem 6.** Given any statement $m$ about our standard universe $\mathcal{U}$, there exists an analogous statement $m^*$ about $\mathcal{U}^*$. Such a statement $m$ is true in $\mathcal{U}$ if and only if $m^*$ is true in $\mathcal{U}^*$.

No trouble arises in using transfer when starting with a statement about the standard universe. However, in order to apply transfer when starting with a statement in $\mathcal{U}^*$, all parameters must be standard. If this requirement is not met, then the Transfer Principle will typically give false results. The reader is referred to [2] or [1] for proof of the transfer principle (as stated in [1] the proof is not particularly enlightening and its understanding is not requisite to effectively working with the transfer principle). We now give two examples which show the proper use of the transfer principle and a misapplication.

**Example 7.** Consider the following statement:

$$ (\exists! y \in \mathbb{R}) \ (\forall x \in \mathbb{R}) \ (xy = x). $$
Applying transfer we have

\[(\exists! \ y \in \mathbb{R}^*) \ (\forall x \in \mathbb{R}^*) \ (xy = x).\]

Thus we may assert that \(\mathbb{R}^*\) has a unique multiplicative identity. Furthermore, recalling that we can identify \(\mathbb{R}\) as a subset of \(\mathbb{R}^*\), we can say that this is 1.

Here transfer causes no problems since our formulas involved only first-order properties. Let us contrast this to our next example.

**Example 8.** Consider the true statement in the standard universe that \(\mathbb{Q}\) has no proper subfield. By transfer we would like to say that \(\mathbb{Q}^*\) has no proper subfield. But this is clearly false since \(\mathbb{Q} \subset \mathbb{Q}^*\). The misapplication of transfer is that our original statement did not involve only first-order properties.

The primary use of the transfer principle is that if one wishes to prove a theorem about the standard universe, it suffices to prove an analogous theorem with standard parameters in the enlarged universe. Many times the advantages provided by rigorously defined unlimited and infinitesimal elements allows for a much simpler proof.

1.3 Basic Topological Concepts From Nonstandard Analysis

We noted earlier that when considering a normed vector space \((V, \| \cdot \|)\) over \(\mathbb{R}\) that there is a natural way to define when a vector is infinitesimal. Thus an obvious way to define when two vectors \(v, w \in V\) are infinitely close is when \(\|v - w\| \simeq 0\), in which
case we write \( v \simeq w \). We now see how the concept of infinitely close can be defined in an arbitrary topological space.

**Definition 9.** Let \( X \) be a topological space. Two points \( x, y \in X \) are said to be infinitely close, written \( x \simeq y \), if for all standard open sets \( O \), where \( x \in O \), \( y \in O^* \).

Note that if \((X, d)\) is a metric space, we have \( x \simeq y \) if and only if \( d^*(x, y) \simeq 0 \), and thus the definition above complies with our definition of infinitely close for vectors. Using this definition, we can use nonstandard analysis to give an extremely useful characterization of compactness which we will employ in a future result.

**Definition 10.** A subset \( Y \) of a topological space \( X \) is compact if for all \( x \in Y^* \), there exists a point \( a \in Y \) such that \( x \simeq a \).

If the sets \( X \) and \( Y \) are standard, then this definition is equivalent to the classical definition of compactness. The interested reader can find a proof of this fact in [1].

We make one final remark regarding notation. The symbol \( \prec \) will be used to denote when one group is a subgroup of another. That is if \( H \) is a subgroup of \( G \), we will write this as \( H \prec G \).

Applying these ideas, we will define and then investigate a new structure based on an already well known and useful structure. With these basic notions, we may proceed to define and consider the properties of the nonstandard hull of a group.
2.1 Definitions and Basic Examples

One of the most useful constructions in nonstandard analysis is the nonstandard hull construction. This idea was originally introduced in [3]. The definition of the nonstandard hull of a vector space $V$ we use is given in [4], which also gives a thorough listing of many results proved about Banach spaces through this construction. Let $V$ be a vector space over the real numbers with norm $\| \cdot \|$. By the Transfer Principle, $V^*$ is a vector space over $\mathbb{R}^*$, and $\| \cdot \|^*$ is a norm on $V^*$ into $\mathbb{R}^*$. With this one can define the sets:

\[
\mu(0) = \{ x \in V^* \mid \| x \|^* \simeq 0 \}
\]
\[
\text{fin}(V) = \{ x \in V^* \mid \| x \|^* \ll \infty \}
\]

Both are vector spaces over $\mathbb{R}$, and $\mu(0)$ is a subspace of $\text{fin}(V)$. Often $\mu(0)$ is called the monad about 0, and $\text{fin}(V)$ is called the Galaxy about 0.

**Definition 11.** The nonstandard hull of a vector space $V$ is the quotient vector space $\text{fin}(V)/\mu(0)$. 
We now wish to extend this definition to an arbitrary group. We first fix a positive hyperreal number \( \epsilon \approx 0 \) whose use will become evident in Remark 19. Note that in order to consider an analogous structure on a general group \( G \), we must find an appropriate way to define when elements of \( G^* \) are limited and infinitesimal. In order to consider a wide variety of groups we have the following definition.

**Definition 12.** Let \( G \) be a group and \( G^* \) its enlargement. Then any nonnegative function \( h : G^* \to \mathbb{R}^* \cup \{\infty\} \) is called a seminorm for \( G \) if the following conditions hold for \( x, y \in G^* \):

1. \( h(e) = 0 \), where \( e \) denotes the group identity.
2. \( h(x) \ll \infty \) if and only if \( h(x^{-1}) \ll \infty \).
3. \( h(x) \simeq 0 \) if and only if \( h(x^{-1}) \simeq 0 \).
4. If \( h(x), h(y) \ll \infty \), then \( h(xy) \ll \infty \).
5. If \( h(x), h(y) \simeq 0 \), then \( h(xy) \simeq 0 \).
6. If \( h(x) \ll \infty \) and \( h(y) \simeq 0 \), then \( h(xyx^{-1}) \simeq 0 \) (normality).

Notice that Condition 6 is automatically satisfied for all abelian groups as a result of Properties 1 and 5. Also, if \( h(x) = \infty \), then we will use the convention that \( h(x) \) is unlimited. On occasion we may have a function \( h \) on \( G \) such that \( h^* \) on \( G^* \) is a seminorm. In such a case, however, we will not make any distinction between \( h \) and \( h^* \) and only refer to \( h \). Furthermore, whenever we say \( h \) is a seminorm on \( G \), we implicitly mean that \( h \) satisfies the properties of a seminorm on the enlargement \( G^* \).
We now note some classes of seminorms that we will often come across and whose stronger properties give some interesting results.

**Definition 13.** A seminorm $h$ is said to be

1. **Subadditive** if $h(xy) \leq h(x) + h(y)$ whenever $h(x), h(y) \ll \infty$;

2. **Inverse-preserving** if for all $x \in G^*$, $h(x) = h(x^{-1})$;

3. **Controlled** if for all standard $x \in G^*$, $h(x) \simeq 0$.

We will assume that all seminorms considered are controlled unless otherwise specified. In order to avoid certain pathologies, we will also assume that any subadditive seminorm considered satisfies the condition: $h(x^n) \leq |n|h(x)$ for all $n \in \mathbb{N}^*$.

**Definition 14.** Let $G$ be a group and $\mathcal{F} = \{h_i\}_{i \in I}$ a family of seminorms for $G$, where $I$ is some index set. We will say $x \in G$ is $\mathcal{F}$-limited if $h_i(x) \ll \infty$ for all $i \in I$. Similarly, we will say $x$ is $\mathcal{F}$-infinitesimal if $h_i(x) \simeq 0$ for all $i \in I$.

**Definition 15.** Given a group $G$ and family of seminorms $\mathcal{F}$ for $G$, we define the following nonstandard sets,

1. $(G, \mathcal{F})_{\text{lim}} = \{x \in G^* \mid x \text{ is } \mathcal{F}\text{-limited}\}$;

2. $(G, \mathcal{F})_{\text{inf}} = \{x \in G^* \mid x \text{ is } \mathcal{F}\text{-infinitesimal}\}$.

Often we will simply write $G_{\text{inf}}$ and $G_{\text{lim}}$ if the choice of seminorm family is clear from context. Also, we may choose to use the notation $G_{\text{inf}}$ and $G_{\text{lim}}$ when we encounter the following situation.
Definition 16. Let \( \mathcal{F} \) and \( \mathcal{G} \) be two families of seminorms for a group \( G \). We will say \( \mathcal{F} \) and \( \mathcal{G} \) are equivalent seminorm families if \( (G, \mathcal{F})_{\text{lim}} = (G, \mathcal{G})_{\text{lim}} \) and \( (G, \mathcal{F})_{\text{inf}} = (G, \mathcal{G})_{\text{inf}} \).

One can easily verify that this defines an equivalence relation.

Using the properties of seminorms, we have the following result that allows us to define our construction.

Theorem 17. \( G_{\text{lim}}, G_{\text{inf}} \) are subgroups of \( G^* \). Furthermore \( G_{\text{inf}} \triangleleft G_{\text{lim}} \).

Proof. First observe that \( G_{\text{lim}} \) and \( G_{\text{inf}} \) are nonempty since for all \( h \in \mathcal{F} \), \( h(e) = 0 \), and thus both sets contain the identity element. Given \( x, y \in G_{\text{lim}} \), for all \( h \in \mathcal{F} \), by Condition 4 of seminorms, \( h(xy) \ll \infty \), and \( h(x^{-1}) \ll \infty \) by Condition 2. Similar reasoning may be employed with \( G_{\text{inf}} \). Normality follows by recalling Condition 6 of seminorms.

Definition 18. The quotient \( \text{NSH}(G, \mathcal{F}) = G_{\text{lim}}/G_{\text{inf}} \) will be called the nonstandard hull of \( G \).

Often it will be sufficient to simply choose our family of seminorms to be a single seminorm. In these cases we will use the notation \( \text{NSH}(G, h) \), and call elements \( h \)-limited and \( h \)-infinitesimal when appropriate. It is immediate that the nonstandard hull construction produces the same results when replacing a seminorm family \( \mathcal{F} \) with an equivalent seminorm family. It will later be shown that we can often exploit this fact and replace \( \mathcal{F} \) with a single seminorm \( h \).
We now wish to show that Definition 18 is compatible with Definition 11 and thus justify our claim that our nonstandard hull construction is an extension of the classical definition.

**Remark 19.** We fix a positive hyperreal number $\epsilon \simeq 0$, and whenever we use the symbol $\epsilon$ it will be this fixed number.

Note that this choice is arbitrary; thus any results we state using $\epsilon$ are independent of this fixed choice.

**Theorem 20.** Suppose that $(V, \| \cdot \|)$ is a normed vector space over $\mathbb{R}$. Then $NSH(V, \epsilon \| \cdot \|)^* \cong \text{fin}(V)/\mu(0)$ as vector spaces over $\mathbb{R}$ where $\text{fin}(V)$ and $\mu(0)$ are as in Definition 11.

**Proof.** First observe that $h = \epsilon \| \cdot \|^*$ is a subadditive, inverse-preserving, and controlled seminorm. Consider the map

$$\phi : V_{\lim} \to \text{fin}(V)/\mu(0)$$

given by $\phi(x) = \text{st}(\epsilon x) + \mu(0)$. This map is surjective since given $x + \mu(0) \in \text{fin}(V)/\mu(0)$ it is clear that $\frac{x}{\epsilon} \in V_{\lim}$, and $\phi \left( \frac{x}{\epsilon} \right) = x + \mu(0)$ . One can now see $\phi$ induces an isomorphism, via the first isomorphism theorem, between $NSH(V, h)$ and $\text{fin}(V)/\mu(0)$ by noting two facts. First, every element $x + \mu(0) \in \text{fin}(V)/\mu(0)$ is really just the equivalence class of all $y \in V^*$ such that $x \simeq y$ (with respect to $\| \cdot \|^*$). Furthermore, one can use the properties of subadditive, inverse preserving seminorms and verify a coset $x + V_{\inf}$ is nothing more than the equivalence class of all $y$ such
that \( h(y) \simeq h(x) \). That is, \( \phi(x) = 0 + \mu(0) \) if and only if \( \text{st}(\epsilon x) \in \mu(0) \) if and only if 
\[
\|\epsilon x\| = \epsilon \|x\| = h(x) \simeq 0 \text{ if and only if } x \in V_{\inf}. \quad \blacksquare
\]

Thus we have that our definition of nonstandard hull a group and the classical definition agree in the cases where we simply take a normed vector space \((V, \| \cdot \|)\) and choose our seminorm to be \( \epsilon \| \cdot \|^{*} \) rather that \( \| \cdot \|^{*} \). It should be noted that we are considering \( V \) as a group under the operation of addition.

Using our convention of considering only controlled seminorms, we have the following observation.

**Theorem 21.** The nonstandard hull of any finite group is the trivial group.

**Proof.** Since \( G \) is finite, \( G^{*} = G \). Therefore, every element of \( G^{*} \) is \( \mathcal{F} \)-infinitesimal, and \( G_{\text{lim}} = G_{\text{inf}} = G \). \( \blacksquare \)

One would hope that this construction provides a slightly more interesting result in the case of an infinite group. Consider the group \( \mathbb{Z} \) of integers. Note that in \( \mathbb{Z}^{*} \) there are no nonzero infinitesimal elements. This is because by the Transfer Principle, we can assert that there is no element of \( \mathbb{Z}^{*} \) between 0 and 1. Thus if we consider the seminorm \( h = | \cdot | \), it is clear that the traditional nonstandard hull construction will give the set of standard integers \( \mathbb{Z} \). But if we use \( h = \epsilon | \cdot | \), we see that there is a substantial difference in what the nonstandard hull construction yields.

**Theorem 22.** \( NSH(\mathbb{Z}, h) = \mathbb{R} \).
Proof. Consider the map $\phi : \mathbb{Z}_{\text{lim}} \to \mathbb{R}$ defined by $x \mapsto \text{st}(\epsilon x)$ which is clearly a homomorphism. To see the map is surjective, consider any $y \in \mathbb{R}$, and let $n = \left\lfloor \frac{y}{\epsilon} \right\rfloor$, where $\lfloor \cdot \rfloor$ denotes the greatest integer function. Observe that

$$\frac{|y - \epsilon n|}{\epsilon} = \frac{|y - n|}{n} < 1.$$ 

This implies that $|y - \epsilon n| < \epsilon$ showing that $y \simeq \epsilon n$. Therefore $\phi(n) = y$, as desired.

Now note that $\text{Ker} \phi = \mathbb{Z}_{\text{inf}}$ since $\phi(n) = 0$ if and only if $\text{st}(\epsilon n) = 0$ if and only if $\epsilon n \simeq 0$ if and only if $n \in \mathbb{Z}_{\text{inf}}$. This gives the result $\text{NSH}(\mathbb{Z}, h) \cong \mathbb{R}$.

Thus we see that the result of applying this construction is not always trivial.

With Theorem 22, we can also easily determine for any infinite subgroup $G \subset \mathbb{R}$ that $\text{NSH}(G, h) = \mathbb{R}$. To see this we may employ the same exact mapping used in Theorem 22. The only changes in the proof will be in showing the map is surjective. If it is necessary, we replace the greatest integer function with a different function $\text{rnd}_G(x) : \mathbb{R}^* \to \mathbb{R}^*$ defined by

$$\text{rnd}_G(x) = \begin{cases} x, & \text{if } G^* \text{ is dense in } \mathbb{R}^*; \\ \max\{g \in G^* \mid g < x\}, & \text{otherwise.} \end{cases}$$

Then the remainder of the proof will follow word for word. This shows that two different infinite groups need not have distinct nonstandard hulls. Thus an obvious question of interest is what the conditions are for two different groups to have the same nonstandard hulls.
Again, note that since \( \mathbb{R}^* \) is a complete space with infinitesimal elements, we could have simply considered the sets

\[
\mathbb{R}_{\text{inf}} = \{ x \in \mathbb{R}^* \mid x \simeq 0 \}
\]

\[
\mathbb{R}_{\text{lim}} = \{ x \in \mathbb{R}^* \mid x \ll \infty \}
\]

and used the absolute value function as a seminorm. If one was only concerned with finding the nonstandard hull of \( \mathbb{R} \) we know this gives the same result by Theorem 20. However, one can now see that in the case of \( \mathbb{Z} \), that there is no natural concept of an infinitesimal element. But by using a controlled seminorm, namely \( h = \epsilon \cdot | \cdot | \), we effectively create a set of infinitesimal elements which allows the nonstandard hull construction to give results noted above.

2.2 Some Pathological Examples

Before proceeding further in investigating the nonstandard hull construction, we highlight a few examples of certain groups and seminorms where the nonstandard hull construction yields results that are not as clean as our first class of examples. First we consider an example where our group \( G \) is nonstandard. One can foresee that there should be some problems when considering the nonstandard hull of such a group since we forfeit any use of the transfer principle. We consider a simple case and observe that the nonstandard hull depends heavily on our fixed choice of \( \epsilon \) and choice of \( h \).
Example 23. Let $G = \mathbb{Z}_n$, where $n \in \mathbb{N}^*$ is unlimited. This group is not the image of any group under $\ast$, and is therefore nonstandard. We consider any natural choice of seminorm $h$ that sends the generator $1$ to a positive value less than or equal to $\epsilon$. There are three possible cases. If $n\epsilon \simeq 0$, then we will have that for any such $h$, $\text{NSH}(G; h)$ will be trivial. If $n\epsilon \ll \infty$, but $\epsilon n$ is not infinitesimal, then we will have that $G_{\text{inf}} \subset G_{\text{lim}}$, where the containment is proper. Thus $\text{NSH}(G, h)$ will be a nontrivial group. Finally if $\epsilon n$ is unlimited $\text{NSH}(G, h)$ will vary depending on the choice of $h$.

Since we wish to have the transfer principle at our disposal, and avoid situation as listed above, we will from now on assume all groups considered are standard. However, even when still working with standard groups, our next example shows our choice of seminorm can radically alter our results.

Example 24. Consider the seminorm $h$ for $\mathbb{Z}$ defined by

$$h(x) = \begin{cases} 
\epsilon |x|, & \text{if } \epsilon x \ll \infty; \\
\epsilon^2 |x|, & \text{otherwise}.
\end{cases}$$

By Theorem 22, we can immediately see that $\text{NSH}(\mathbb{Z}, h)$ is some group which contains $\mathbb{R}$ as a proper subgroup.

One would hope that there is, at least in some situations, a more systematic way for determining a seminorm to use on a given group for which the nonstandard
hull construction will yield more useful results. We now proceed to give a method for determining a family of seminorms on a wide class of groups that possess useful properties. These seminorms will depend not on arbitrary choice but rather the structure of the group under consideration.

2.3 Finitely Generated Groups and $\varepsilon$-Maximal Seminorms

We wish to find a way to replace a family of seminorms with a single equivalent seminorm. A natural way to approach this is to consider the supremum of a family of seminorms. However, there are some caveats to such a simple approach. Consider the following situation. Suppose that $A \subset \mathbb{R}^*$ is a set bounded above by $x \in \mathbb{R}^*$. One would like to say that by transfer that $A$ has a supremum. This, however, is a misapplication of the transfer principle just as in Example 8. For instance let $A = \mathbb{N}$ and let $x$ be unlimited. Then clearly $A$ is bounded above by $x$, but $A$ has no least upper bound. Even if our set $A$ is bounded above by a limited value, we cannot assert that $A$ has a supremum. For instance, let $A = \{\delta \in \mathbb{R}^* \mid \delta \simeq 0\}$. Concluding that this set has a least upper bound is equivalent to asserting the existence of a largest infinitesimal. One can use the definition of an infinitesimal element and see the existence of a largest infinitesimal in $\mathbb{R}^*$ is not true. Thus we must take a more careful approach when dealing with the problem of replacing a family of seminorms with a single seminorm.
Lemma 25. Let $A \subseteq \mathbb{R}^*$ be bounded above by $x \in \mathbb{R}^*$ where $x \ll \infty$. Then the set $\text{st}(A) := \{\text{st}(x) \mid x \in A\}$ has a supremum.

Proof. For all $z \in \text{st}(A)$, $z = \text{st}(y)$ for some $y \in A$, and by Theorem 3 it follows that $\text{st}(y) \leq \text{st}(x)$. Hence $\text{st}(A)$ is a set of standard real numbers that has a standard upper bound. Therefore $\text{st}(A)$ has a supremum. $\square$

Theorem 26. Let $\mathcal{F} = \{h_i \mid i \in I\}$ be a family of inverse-preserving, subadditive seminorms for a group $G$. Define $h(x) = \sup(\text{st}(h_i(x)))$ provided that $\text{st}(h_i(x))$ is defined for all $i \in I$; otherwise, define $h(x) = \infty$. Then $h$ is an inverse-preserving, subadditive seminorm for $G$.

Proof. Clearly for all $i$, $h_i(e) = 0$, so $h(e) = \sup 0 = 0$. Observe that for all $x, y$ and $i \in I$ that $h_i(xy) \leq h_i(x) + h_i(y)$. Hence $\text{st}(h_i(xy)) \leq h(x) + h(y)$. This shows $h(x) + h(y)$ is an upper bound on $\text{st}(h_i(xy))$; thus by definition the supremum $h(xy) \leq h(x) + h(y)$. Note that properties 4 and 5 of seminorms are automatically satisfied by $h$ since $h$ is subadditive. Also, $h(x) = h(x^{-1})$ follows since each $h_i$ is inverse-preserving. Finally, suppose that $h(x) \ll \infty$ and $h(y) = 0$. Then this implies that for all $i \in I$ that $h_i(x) \ll \infty$ and $h_i(y) \simeq 0$. Hence $h_i(xyx^{-1}) \simeq 0$ for all $i \in I$. Then it follows that $\text{st}(h_i(xyx^{-1})) = 0$ for all $i$, and taking the supremum over $i$ results in $h(xyx^{-1}) = 0$ which is infinitesimal. $\square$

Remark 27. The restriction to only considering inverse-preserving subadditive seminorms is necessary. For suppose that $x, y$ were $\mathcal{F}$-limited, and the set $\{h_i(xy) \mid i \in I\}$ is unbounded. Then in this situation, we would have that $h(x), h(y) \ll \infty$ but
\( h(xy) = \infty \); consequently, \( h \) is not a seminorm. Also, it could be possible that \( h(x) \ll \infty \) but the set \( \{ h_i(x^{-1}) \mid i \in I \} \) is unbounded. However if for each \( x \in (G, \mathcal{F})_{\text{lim}} \), the set \( \{ h_i(x) \mid i \in I \} \) is bounded above by a limited value, then it is clear that \( h \) as defined in Theorem 26 is a seminorm for \( G \). Thus even if \( \mathcal{F} \) does not consist of only inverse-preserving subadditive seminorms, it still may be possible that \( h \) is a seminorm. We keep this situation in mind in the definition that follows.

**Definition 28.** Let \( h \) be defined as in Theorem 26. If \( h \) is a seminorm, then we will call \( h \) the maximal seminorm with respect to \( \mathcal{F} \) for \( G \), or simply a maximal seminorm when the underlying family \( \mathcal{F} \) is understood.

We now make some observations about maximal seminorms, and notice that the notion of a maximal seminorm agrees with what we would want if we could simply consider the supremum of a family of seminorms. For if \( h \) is the maximal seminorm with respect to \( \mathcal{F} = \{ h_i \}_{i \in I} \) and \( h(x) = \infty \), then this implies either for all \( h_i \in \mathcal{F} \), \( h_i(x) \ll \infty \) and the set \( \{ h_i(x) \} \) is unbounded or there exists at least one \( i \) where \( h_i(x) \) is unlimited. Also, if \( h_i(x) \simeq 0 \) for all \( i \in I \), we have \( h(x) = 0 \) which is infinitesimal. And by Lemma 25, we see that if each \( h_i(x) \) is limited and the set \( \{ h_i(x) \} \) has a limited upper bound, then \( h(x) \) is a well defined limited value.

**Lemma 29.** Let \( (G, \mathcal{F}) \) be a group and a family of subadditive, inverse-preserving seminorms that is bounded above by a function taking on limited values. Let \( h \) be the maximal seminorm with respect to \( \mathcal{F} \). Then \( \mathcal{F} \) and \( \{ h \} \) are equivalent seminorm families.
**Proof.** First we show that \((G, h)_{\text{lim}} \subseteq (G, F)_{\text{lim}}\). Suppose \(x \in (G, h)_{\text{lim}}\). Then \(h_i(x) \leq h(x) + 1 \ll \infty\) for all \(h_i \in F\), and thus we have \(x \in (G, F)_{\text{lim}}\) which gives the desired containment. Now suppose that \(x \in (G, h)_{\text{inf}}\). Then \(h(x) = 0\), which implies that \(\text{st}(h_i(x)) = 0\) for all \(h_i \in F\), or equivalently, \(h_i(x) \simeq 0\). This shows \(x \in (G, F)_{\text{inf}}\).

To see \((G, F)_{\text{lim}} \subseteq (G, h)_{\text{lim}}\), suppose \(x \in (G, F)_{\text{lim}}\). By assumption, the set \(\{h_i(x)\}\) is bounded above by some limited value. Hence it is clear that \(h(x)\) is limited. Finally, to see \((G, F)_{\text{inf}} \subseteq (G, h)_{\text{inf}}\), suppose that \(x \in (G, F)_{\text{lim}}\). Then it follows that \(\text{st}(h_i(x)) = 0\) for all \(h_i \in F\). Thus we have that \(h(x) = 0\), putting \(x \in (G, h)_{\text{inf}}\). \(\square\)

**Remark 30.** The conditions of Lemma 29 require that we can bound a given seminorm family \(F\) by a function taking on limited values. However, often there will be elements \(x \in G^*\) such that \(h_i(x)\) is unlimited for some \(h_i \in F\). However, such a point plays no role in considering the nonstandard hull of \(G\). Thus if there exists a function \(f\) taking on limited values that for each \(x \in (G, F)_{\text{lim}}\), and the set \(\{h_i(x)\}\) is bounded above by \(f(x)\), then ideas Lemma 29 show that \(F\) and \(\{h\}\) are equivalent seminorm families.

In order to use results regarding equivalence of seminorms, we will be required to limit the types of finitely generated groups that will will consider.

**Definition 31.** Let \(G = \langle S \rangle\) where \(S\) is a finite generating set for \(G\). Let \(N \in \mathbb{N}\). We will say \(G\) is of \(N\)-capacity if for every \(x \in G\), there exists \(w_1, w_2, \ldots, w_N \in S\) such that for every \(x \in G\), \(x = \prod_{i=1}^{N} w_i^{r_i}\) where \(r_i \in \mathbb{Z}\). Furthermore, any word representing \(x \in G\) of this form will be called an \(N\)-word.
Observe that this definition can be expressed in the formula

\[(\forall x \in G) \left( \exists (w_1, \ldots, w_n) \in S^N \right) \left( \exists (r_1, \ldots, r_N) \in \mathbb{Z}^N \right) \left( x = \prod_{i=1}^{N} w_i^{r_i} \right). \]

Applying transfer to this formula we have

\[(\forall x \in G^*) \left( \exists (w_1, \ldots, w_n) \in S^N \right) \left( \exists (r_1, \ldots, r_N) \in \mathbb{Z}^{*N} \right) \left( x = \prod_{i=1}^{N} w_i^{r_i} \right); \]

where we recall that \( S^* = S \) since \( S \) is finite. Now consider the function \( f : \mathbb{Z}^N \to G \) defined by

\[f(r_1, \ldots, r_N) = \prod_{i=1}^{N} w_i^{r_i},\]

and the function \( g : \mathbb{Z}^N \to \mathbb{N} \) defined by

\[g(r_1, \ldots, r_N) = \sum_{i=1}^{N} |r_i|.\]

Then we have the formula,

\[(\forall x \in G) \left( \exists n \in \mathbb{N} \right) \left[ (\exists \vec{r}) \left( f(\vec{r}) = x \land g(\vec{r}) = n \right) \land (\forall \vec{r}) \left( g(\vec{r}) < n \implies f(\vec{r}) \neq x \right) \right].\]

We call \( n \) the length of the minimal \( N \)-word for \( x \in G \). Observe by applying transfer we can assert the existence of a minimal \( N \)-word length for each \( x \in G^* \).

If we have a finitely generated group \( G \) with \( N \)-capacity together with a choice of finite generating set \( S \), then we can use Theorem 26 to pick a controlled seminorm \( h \). Let \( G \) be finitely generated; consider the set \( \mathcal{F} \) of all inverse-preserving, subadditive seminorms that send elements in \( S \) to values less than or equal to \( \epsilon \). Now observe that for any \( x \in G^* \), we can assign to \( x \) a value \( \ell(x) \) that is the length for the
minimal \( N \)-word in \( S \) that gives \( x \). Thus we may define a function \( f : G^* \rightarrow \mathbb{R}^* \) where 
\[ f(x) = \epsilon \cdot \ell(x) \]. Since \( f \) bounds \( \mathcal{F} \) from above, Theorem 26 gives a subadditive, inverse-preserving seminorm \( h \) on \( G \). Thus in this case we may think of the nonstandard hull of \( G \) as depending on the choice of generating set and \( \epsilon \) rather than the choice of seminorm. We now give the seminorm we have just constructed a formal name.

**Definition 32.** Let \( G \) be a finitely generated group with generating set \( S \). We will call a seminorm \( h \) on \( G \) \( \epsilon \)-maximal if \( h \) is maximal seminorm with respect to the family of seminorms \( \mathcal{F} \), where \( \mathcal{F} \) is the collection of all subadditive, inverse-preserving seminorms sending elements of \( S \) to values less than or equal to \( \epsilon \).

2.4 Arbitrary Products of Finitely Generated Groups

Now that we have a systematic way for determining an appropriate seminorm on a finitely generated group with \( N \)-capacity, we close the chapter by showing that we can use such seminorms to define a family of seminorms on products of finitely generated groups for which the nonstandard hull construction gives particularly useful results. As an immediate consequence we will be able to determine the nonstandard hull of all finitely generated groups. For the remainder of the section, we will assume that all finitely generated groups considered are of \( N \)-capacity for some \( N \).

We first need the following lemma in order to determine an appropriate family of seminorms when considering an arbitrary direct product of finitely generated groups.
Lemma 33. Let $G_1, \ldots, G_n$ be groups and suppose that $G = G_1 \times G_2 \times \cdots \times G_n$. For each $i$, let $h_i$ be a subadditive, inverse-preserving seminorm for $G_i$. Then the function $h$ on $G^*$ defined by $h((g_1, \ldots, g_n)) = \sum_{i=1}^n h_i$ is a subadditive, inverse-preserving seminorm for $G$.

Proof. We show that the lemma holds in the case $n = 2$; the general result follows by induction. Let $e = (e_1, e_2)$ where $e_i$ is the identity element of $G_i$. Observe that $h(e) = h((e_1, e_2)) = h_1(e_1) + h_2(e_2) = 0$. Let $a, b \in G$, where $(a_1, a_2) = a$, $(b_1, b_2) = b$. Then we have

$$h(ab) = h(a_1, a_2)(b_1, b_2) = h_1(a_1b_1) + h_2(a_2b_2)$$

$$\leq h_1(a_1) + h_1(b_1) + h_2(a_2) + h_2(b_2) = h((a_1, a_2)) + h((b_1, b_2)) = h(a) + h(b).$$

The normality condition holds as well. For suppose $h(a) \ll \infty$, and $h(b) \simeq 0$. Then clearly $h_1(a_1), h_2(a_2) \ll \infty$, and $h_1(b_1), h_2(b_2) \simeq 0$. Hence $h_1(a_1b_1a_1^{-1}) \simeq 0$ and $h_2(a_2b_2a_2^{-1}) \simeq 0$ which shows $h(aba^{-1}) \simeq 0$. Finally, we have

$$h(a^{-1}) = h_1(a_1^{-1}) + h_2(a_2^{-1}) = h_1(a_1) + h_2(a_2) = h(a).$$

We can now observe that there is a relationship between $\epsilon$-maximal seminorms and the type of seminorm described in Lemma 33.

Lemma 34. Let $G$ be a finitely generated group. Suppose that $G \cong G_1 \times G_2$. Let $S_1$ and $S_2$ be generating sets for $G_1$ and $G_2$. Let $S$ be the corresponding generating set for $S$. Let $h_i$ be the $\epsilon$-maximal seminorm for $G_i$ with respect to $S_i$. If $h$ is the
\( \epsilon \)-maximal seminorm for \( G \) with respect to \( S \), then \( h \) is equivalent to the seminorm given in Lemma 33.

**Proof.** We can think of \( S \) as the disjoint union, \( S_1 \cup S_2 \). Let \( x \in G^* \). Then \( x = g_1g_2 \) for some \( g_1 \in G_1^* \) and \( g_2 \in G_2^* \). Write

\[
S_1 = \{x_1, \ldots, x_n\}, \text{ and } S_2 = \{y_1, \ldots, y_m\}.
\]

Then \( g_1 = x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k} \), and \( g_2 = y_{i_1}^{\beta_1} \cdots y_{i_l}^{\beta_l} \) as minimal words in \( S_1 \) and \( S_2 \) respectively. Thus

\[
h(x) = h(g_1g_2) \leq \epsilon \ell(g_1g_2) = \epsilon (\ell(g_1) + \ell(g_2)).
\]

Observe that \( \ell(g_1) \) and \( \ell(g_2) \) are upper bounds for all seminorms for \( G_1 \) and \( G_2 \) sending elements in \( S_1 \) and \( S_2 \) respectively to values less than or equal to \( \epsilon \). Since \( h \) is the \( \epsilon \)-maximal seminorm for \( S \) and \( S_1 \cap S_2 \) is empty,

\[
h(g_1) + h(g_2) = h_1(g_1) + h_2(g_2).
\]

We can now consider how the nonstandard hull construction interacts with arbitrary products of finitely generated groups. Let \( I \) be some index set and suppose that

\[
G \cong \prod_{i \in I} G_i
\]

where each \( G_i \) is a finitely generated group. For each \( G_i \) choose a finite generating set. For \( J \subset I \) a finite set, define

\[
G_J = \prod_{j \in J} G_j.
\]
From our preceding lemmas, we have a natural choice of seminorm \( \hat{h}_J \) on each \( G^*_J \).

We now will extend each \( \hat{h}_J \) to be a seminorm \( h_J \) on all of \( G^* \). To do this, note that we have the natural projection map

\[
\pi_J : G \to G_J.
\]

Then we may define \( h_J = \hat{h}_J \circ \pi^*_J \). Let \( \Gamma \) be the set of all finite subsets of \( I \). Then we now have a family \( \{h_J\}_{J \in \Gamma} \) of seminorms for \( G \).

**Theorem 35.** Let \( G \cong \prod_{i \in I} G_i \) where each \( G_i \) is a finitely generated group. Then

\[
\text{NSH}(G, \{h_J\}_{J \in \Gamma}) \cong \prod_{i \in I} \text{NSH}(G_i, h_i),
\]

where \( h_J \) and \( h_i \) are as given above.

**Proof.** Consider the map

\[
\phi : (G, \{h_J\})_{\lim} \to \prod_{i \in I} G_i_{\lim}/G_i_{\inf}.
\]

Where \( \phi \) is the identity map on \( G^* \) restricted to \( (G, \{h_J\})_{\lim} \) composed with the natural quotient map. It is clear from this definition of \( \phi \) and each \( h_J \) that \( \phi \) is a surjective homomorphism. Now suppose that \( x \in \text{Ker}\ \phi \). Then each coordinate \( x_i \) of \( x \) is \( h_i \)-infinitesimal. Thus it is clear that each finite sum of the \( h_i(x_i) \) is infinitesimal as well. Hence \( h_J(x) \simeq 0 \) for all \( J \); this shows \( x \in (G, \{h_J\})_{\inf} \). Conversely, if \( x \in (G, \{h_J\})_{\inf} \), then any finite sum of the \( h_i(x_i) \) is infinitesimal. This shows that each individual \( h_i(x_i) \) must be infinitesimal since the \( h_i \) are nonnegative functions; hence we have \( x \in \text{Ker}\ \phi \). We now see the desired result via the first isomorphism theorem. \( \square \)
Notice, that this theorem shows the motivation for defining the nonstandard hull of a group with respect to a family of seminorms. If we were merely considering a direct sum, then we could have avoided the use of families. But in order to ensure that the nonstandard hull construction interacts with products as shown above, we must resort to using families. We now finish by determining the nonstandard hulls of a wide class of groups.

**Corollary 36.** The nonstandard hull of any finitely generated abelian group is isomorphic to \( \mathbb{R}^n \) for some \( n \geq 0 \).

*Proof.* Suppose \( G \) is a finitely generated abelian group. Then \( G \cong \mathbb{Z}^n \times A \) for some \( n \geq 1 \) and some finite group \( A \). Note that \( h = \epsilon \cdot | \cdot | \) is an \( \epsilon \)-maximal seminorm on the integers. Let \( h_0 \) be some \( \epsilon \)-maximal seminorm on \( A \). Then we know that we may form a family \( \{h_j\} \) on \( G \) that satisfies the conditions of Theorem 35. Thus we have

\[
\text{NSH}(G, \{h_j\}) \cong \text{NSH}(\mathbb{Z}, h) \times \cdots \times \text{NSH}(\mathbb{Z}, h) \times \text{NSH}(A, h_0) \cong \mathbb{R}^n
\]

by Theorem 21 and Theorem 22. \( \square \)

Now that we have found a wide class of examples where the nonstandard hull construction is well behaved. Having a nice class of examples, we now move on to proving some general statements regarding the results of applying this construction. Several of the results will not depend at all on the class of seminorms chosen. Because of this independence, these results will prove to be particularly interesting.
CHAPTER III

PROPERTIES OF THE NONSTANDARD HULL CONSTRUCTION

3.1 Some Topological and Categorical Results

Taking advantage of the extra structure provided by inverse-preserving, subadditive
seminorms, we can now proceed to make some interesting observations regarding the
result of the nonstandard hull construction. For the remainder of this section, we will
assume that all seminorms are inverse-preserving and subadditive.

**Theorem 37.** Let $G$ be a group with seminorm $h$. Then $\text{NSH}(G, h)$ is a metric space
where the metric is $d(xG_{\text{inf}}, yG_{\text{inf}}) = \text{st}(h(xy^{-1}))$.

**Proof.** Let $xG_{\text{inf}}, yG_{\text{inf}} \in \text{NSH}(G, h)$. One can verify from the properties of seminorms
that if $x_1$ and $x_2$ are different coset representatives for $xG_{\text{inf}}$, then $h(x_1) \sim h(x_2)$.
From this observation, it is now clear that $d(xG_{\text{inf}}, yG_{\text{inf}}) = \text{st}(h(xy^{-1}))$ has the same
value regardless of the chosen coset representatives. This same observation also shows
that $d(xG_{\text{inf}}, yG_{\text{inf}}) = 0$ if and only if $xG_{\text{inf}} = yG_{\text{inf}}$. Now observe that

$$d(xG_{\text{inf}}, yG_{\text{inf}}) = \text{st}(h(xy^{-1})) = \text{st}(h((yx^{-1})^{-1})) = \text{st}(h(yx^{-1})) = d(yG_{\text{inf}}, xG_{\text{inf}}).$$
Finally, for all \( zG_{\text{inf}} \in \text{NSH}(G, h) \)

\[
d(xG_{\text{inf}}, yG_{\text{inf}}) = \text{st}(h(xz^{-1}zy^{-1})) \leq \text{st}(h(xz^{-1}) + h(zy^{-1}))
\]

\[
= \text{st}(h(xz^{-1})) + \text{st}(h(zy^{-1})) = d(xG_{\text{inf}}, zG_{\text{inf}}) + d(zG_{\text{inf}}, yG_{\text{inf}}) \quad \square
\]

Note that we need not be concerned with the fact that \( h \) can take on the value \( \infty \). For if \( h(x) = \infty \), then it follows that \( x \) is not \( h \)-limited. Since our metric is defined on the nonstandard hull, it follows that such \( x \) will never be considered.

Also, note that above we only considered a group \( G \) with a single seminorm \( h \); however, we know that we have a natural way to derive a single seminorm from a family. Furthermore, we know that if we choose a bounded family of seminorms, this seminorm will be equivalent to the family.

We now make some categorical observations about the nonstandard hull construction. In order to do this, we consider the following type of mapping which preserves not only group structure but also preserves information between seminorm families.

\textbf{Definition 38.} Let \((G, \mathcal{F})\) be a group with seminorm family. Let \( \phi \) be a homomorphism from \( G \) into another group \( H \) with seminorm family \( \mathcal{G} \). Let \( h \) and \( \hat{h} \) be the maximal seminorms with respect to \( \mathcal{F} \) and \( \mathcal{G} \) respectively. The mapping \( \phi : G \to H \) will be called a rough-transformation with respect to \( \mathcal{F} \) and \( \mathcal{G} \) if for all \( x \in G^* \) there exists a limited constant \( C \) such that \( C \cdot h(x) \leq \hat{h}(\phi(x)) \) whenever \( h(x) \ll \infty \).
Observe that given any isomorphism $\phi : G \to H$ we can always define a seminorm $\hat{h}$ on $H$ that makes $\phi$ into an rough-transformation by setting $\hat{h}(x) = h(\phi^{-1}(x))$. Also, it follows that given any rough-transformation $\phi$, $h(x) \ll \infty$ if and only if $\hat{h}(\phi(x)) \ll \infty$, and $h(x) \simeq 0$ if and only if $\hat{h}(\phi(x)) \simeq 0$.

**Lemma 39.** Let $(G, F)$ and $(H, G)$ be groups with seminorm families. Suppose that $\phi : (G, F) \to (H, G)$ is a rough-isometry. Then there exists a continuous homomorphism $\hat{\phi} : \text{NSH}(G, F) \to \text{NSH}(H, G)$ induced by $\phi$.

**Proof.** Define $\hat{\phi}$ by $\hat{\phi}(x_{G_{\text{inf}}}) = \phi^*(x)H_{\text{inf}}$. This is clearly a homomorphism. To see that this is a continuous map, let $\eta > 0$ and set $\delta = \frac{\eta}{\text{st}(C)}$. Suppose that $d(x_{G_{\text{inf}}}, y_{G_{\text{inf}}}) < \delta$, then $d'(\phi^*(x)H_{\text{inf}}, \phi^*(y)H_{\text{inf}}) \leq \text{st}(C) \cdot d(x_{G_{\text{inf}}}, y_{G_{\text{inf}}}) < \eta$.

We will denote the category of groups with seminorm family by $\text{SNgrp}$, and the category of metric space groups by $\text{MSgrp}$. Note that the arrows of the respective categories are rough-transformations and continuous homomorphisms. Noting this, we can now make our categorical observation.

**Theorem 40.** $\text{NSH}$ is a functor from $\text{SNgrp}$ into $\text{MSgrp}$.

**Proof.** Let $(G, \{h_i\})$ be an object from $\text{SNgrp}$. By Theorem 37 we know there is a corresponding metric space group $(\text{NSH}(G, \{h_i\}), d)$. By Lemma 39, we know that given any rough-transformation $\phi$ there is a corresponding continuous homomorphism $\text{NSH}(\phi)$. One can easily verify that $\text{NSH}(\text{id}_G) = \text{id}_{\text{NSH}(G)}$. Also given any composable
arrows (rough-transformations) $g, f$, it is immediate from the definitions that $\text{NSH}(g \circ f) = \text{NSH}(g) \circ \text{NSH}(f)$. □

3.2 Subgroups, Normality, Quotients, and Semi-Direct Products

The previous section provided us with an idea of what the nonstandard hull construction yields when restricted to inverse-preserving, subadditive seminorms. Moving in a more general direction, we now prove three theorems that explicate relationship between the subgroups of a given group $G$ and the subgroups of $\text{NSH}(G, F)$. These results are particularly interesting since we need not make any assumptions on the nature of the seminorms considered.

\textbf{Theorem 41.} If $H \prec G$, then $\text{NSH}(H, F)$ has a natural identification with a subgroup of $\text{NSH}(G, F)$.

\textit{Proof.} Consider the map $\phi : H_{\text{lim}} \to \text{NSH}(G, F)$, defined by $\phi(x) = xG_{\text{inf}}$. The map is clearly a homomorphism. It is evident that the kernel is $\text{Ker } \phi = \{x \in H_{\text{lim}} \mid x \in G_{\text{inf}}\}$, but this is exactly the set $\{x \in H_{\text{lim}} \mid x \in H^* \text{ and for } h \in F, h(x) \simeq 0\} = H_{\text{inf}}$. Thus by the first isomorphism theorem,

$$H_{\text{lim}}/H_{\text{inf}} \cong \phi(H_{\text{lim}})/G_{\text{inf}} \prec \text{NSH}(G, F)$$

Theorem 41 shows that we have a natural identification between the set $K = \{xG_{\text{inf}} \in \text{NSH}(G, h) \mid x \in H_{\text{lim}}\} \prec \text{NSH}(G, F)$ and $\text{NSH}(H, F)$. From now on, we will simply use this identification and write $\text{NSH}(H, F) \prec \text{NSH}(G, F)$. 

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This gives a partial, though not completely satisfactory, answer to when two different groups have the same nonstandard hull. Suppose that for some reason one knew $\text{NSH}(H, \mathcal{F}) = \text{NSH}(G, \mathcal{F})$ where $H \preceq G$. Then for any group $K$ such that $H \preceq K \preceq G$, it follows that $\text{NSH}(K, \mathcal{F}) = \text{NSH}(G, \mathcal{F})$ by Theorem 41.

**Lemma 42.** If $H \triangleleft G$, then $H_{\text{lim}} \triangleleft G_{\text{lim}}$ and $H_{\text{inf}} \triangleleft G_{\text{lim}}$.

**Proof.** Note by the Transfer Principle that since $H \triangleleft G$, then $H^* \triangleleft G^*$. Consider $x \in H_{\text{lim}}$ and $y \in G_{\text{lim}}$. First we show that $yxy^{-1}$ is $\mathcal{F}$-limited. This is because for any $h \in \mathcal{F}$, $h(x), h(y) \ll \infty$ implies $h(xy) \ll \infty$, and $h(x) \ll \infty$ implies $h(x^{-1}) \ll \infty$; therefore, $h(xy) \ll \infty$ and $h(y^{-1}) \ll \infty$ implies that $h(xy^{-1}) \ll \infty$. Now suppose that $x \in H_{\text{inf}}$ instead. Then it is a direct consequence of the properties of $h$ that $h(xy^{-1}) \simeq 0$. 

**Theorem 43.** If $H \triangleleft G$, then $\text{NSH}(H, \mathcal{F}) \triangleleft \text{NSH}(G, \mathcal{F})$.

**Proof.** Let $xG_{\text{inf}} \in \text{NSH}(H, \mathcal{F})$. Then

$$(yG_{\text{inf}})(xG_{\text{inf}})(y^{-1}G_{\text{inf}}) = (yxy^{-1})G_{\text{inf}}.$$ 

By Lemma 42, $yxy^{-1} \in H_{\text{lim}}$; this gives the result.

The next natural question is what form the group $\text{NSH}(G, \mathcal{F})/\text{NSH}(H, \mathcal{F})$ has when $H \triangleleft G$. In order to answer this it is first necessary to find an appropriate seminorm family on $G/H$ when $H \triangleleft G$. Consider $(G, \mathcal{F})$ and and suppose $H \triangleleft G$. We use $h \in \mathcal{F}$ to induce a seminorm $\tilde{h}$ on $G/H$ that will preserve a relationship between
when \( h(x) \simeq 0 \) and \( \bar{h}(xH) \simeq 0 \). To achieve this, note that \( h \) is bounded below by 0. Thus we may consider the infimum of \( h(x) \) as \( x \) ranges over any particular subset of \( G \). We now use this observation and define \( \bar{h} \).

**Lemma 44.** Let \( H \triangleleft G \). Let \( h \) be a seminorm on \( G \). Then \( \bar{h} \) defined by

\[
\bar{h}(xH) = \inf \{ h(x_i) \mid x_iH = xH \}
\]  

(3.1)

is a seminorm on \( G/H \). Furthermore, if \( h \) is subadditive then \( \bar{h} \) is subadditive, and if \( h \) is inverse-preserving, then \( \bar{h} \) is as well.

**Proof.** Let \( xH \) be any coset of \( H \) in \( G \). It is clear that if \( xH \) is the trivial coset then \( \bar{h}(xH) = 0 \) as desired. Suppose that \( \bar{h}(xH) \ll \infty \) and \( \bar{h}(yH) \ll \infty \). Then clearly, there exists \( x_0 \in xH, y_0 \in yH \) such that \( h(x_0), h(y_0) \ll \infty \). By the properties of \( h \) and \( \bar{h} \), we have \( \bar{h}(xyH) \leq h(x_0y_0) \ll \infty \). A similar argument holds for the infinitesimal case. Now suppose that \( \bar{h}(xH) \ll \infty \) and \( \bar{h}(yH) \simeq 0 \). This implies that there is some \( x_0 \in xH \) such that \( h(x_0) \ll \infty \) and some \( y_0 \in yH \) such that \( h(y_0) \simeq 0 \). Then it is clear that \( \bar{h}(xyx^{-1}H) \leq h(x_0y_0x_0^{-1}) \simeq 0 \). Conditions 2 and 3 follow by similar
arguments. To see that the subadditivity of $h$ implies the subaddivity of $\tilde{h}$, consider

$$
\tilde{h}(xyH) = \inf \{ h(bc) \mid bcH = xyH \}
$$

$$
= \inf \{ h(bc) \mid x^{-1}bH = yc^{-1}H \}
$$

$$
\leq \inf \{ h(bc) \mid x^{-1}bH = H = yc^{-1}H = H \}
$$

$$
\leq \inf \{ h(b) + h(c) \mid bH = xH \text{ and } yH = cH \}
$$

$$
\leq \inf \{ h(b) \mid bH = xH \} + \inf \{ h(c) \mid cH = yH \}
$$

$$
= \tilde{h}(xH) + \tilde{h}(yH)
$$

which gives the result. The fact that $\tilde{h}(xH) = \tilde{h}(x^{-1}H)$ when $h$ is inverse-preserving is clear.

Using the seminorm defined by (3.1), it is now clear that given a family of seminorms $\mathcal{F}$ on $G$, we can define a family $\tilde{\mathcal{F}}$ on $G/H$.

**Lemma 45.** If $H \triangleleft G$ and $\mathcal{F}$ is as above, then $xH$ is $\tilde{\mathcal{F}}$-infinitesimal if and only there is a $y \in G_{\inf}$ such that $xy^{-1} \in H^*$, or equivalently, $x \in G_{\inf} H^*$. Similarly, $\tilde{\mathcal{F}}$-limited if and only there is a $y \in G_{\lim}$ such that $xy^{-1} \in H^*$, or equivalently, $x \in G_{\lim} H^*$.

**Proof.** This is obvious since if there is a $\mathcal{F}$-infinitesimal element that is in the coset $xH$, then for all $\tilde{h} \in \tilde{\mathcal{F}}$, $\tilde{h}(xH)$ must be infinitesimal, and thus there is obviously some $\mathcal{F}$-infinitesimal element in the coset. Conversely, suppose that $x \in G_{\inf} H^*$. Then $x = ab$ where $a \in G_{\inf}$ and $b \in H^*$. Then clearly $aH = xH$, and $h(x) \simeq 0$ for all $h \in \mathcal{F}$. This implies that $\tilde{h}(xH) \simeq 0$ for all $\tilde{h} \in \tilde{\mathcal{F}}$. The second half of the theorem follows similarly. \qed

33
With this we can now consider what form the quotient $\text{NSH}(G, \mathcal{F})/\text{NSH}(H, \mathcal{F})$ has when $H \vartriangleleft G$.

**Theorem 46.** If $H \vartriangleleft G$, then there is a natural isomorphism

$$f : \text{NSH}(G, \mathcal{F})/\text{NSH}(H, \mathcal{F}) \to \text{NSH}(G/H, \mathcal{F})$$

where $\mathcal{F}$ is the seminorm family defined above.

**Proof.** By Lemma 45, we have that

$$\text{NSH}(G/H, \mathcal{F}) = \frac{(G/H)_{\lim}}{(G/H)_{\inf}} = \frac{G_{\lim}H^*}{G_{\inf}H^*}. \quad (3.2)$$

By the third isomorphism theorem the right-most group in (3.2) is isomorphic to

$$\frac{G_{\lim}H^*/H^*}{G_{\inf}H^*/H^*}. \quad (3.3)$$

By the second isomorphism theorem, (3.3) is isomorphic to

$$\frac{G_{\lim}/(G_{\lim} \cap H^*)}{G_{\inf}/(G_{\inf} \cap H^*)} = \frac{G_{\lim}/H_{\lim}}{G_{\inf}/H_{\inf}}. \quad (3.4)$$

Now observe that we have a natural choice of surjective map

$$\phi : G_{\lim} \to \frac{G_{\lim}/H_{\lim}}{G_{\inf}/H_{\inf}}$$

which has kernel $H_{\lim}G_{\inf}$. Then this induces a surjective mapping

$$\Phi : G_{\lim}/G_{\inf} \to \frac{G_{\lim}/H_{\lim}}{G_{\inf}/H_{\inf}}$$

with kernel $H_{\lim}G_{\inf}/G_{\inf} \cong H_{\lim}/(H_{\lim} \cap G_{\inf}) = H_{\lim}/H_{\inf} = \text{NSH}(H, \mathcal{F})$. Naturality follows from the naturality of all the maps we have used. \qed
Note that by (3.4) we can now work with the more convenient forms \(G_{\text{lim}}/H_{\text{lim}}\) and \(G_{\text{inf}}/H_{\text{inf}}\) instead of \((G/H)_{\text{lim}}\) and \((G/H)_{\text{inf}}\) respectively.

We now consider how the nonstandard hull construction interacts with semidirect products.

**Lemma 47.** If \(A\) is a subgroup of \(G\), then \(G = A \ltimes N\) if and only if there exists a surjective homomorphism \(\phi : G \rightarrow A\) that is the identity on \(A\) and has kernel \(N\).

**Theorem 48.** If \(G \cong A \ltimes N\) where the corresponding map \(\Phi : (G, \mathcal{F}) \rightarrow (A, \mathcal{F})\) is a rough-transformation, then \(\text{NSH}(G, \mathcal{F}) \cong \text{NSH}(A, \mathcal{F}) \ltimes \text{NSH}(N, \mathcal{F})\).

**Proof.** Suppose \(G \cong A \ltimes N\) with seminorm family \(\mathcal{F}\). Since \(G \cong A \ltimes N\), by assumption, there is a surjective rough-transformation

\[\Phi : G \rightarrow A\]

where \(\Phi\) acts as the identity on \(A\), and \(\text{Ker} \Phi = N\). We may then use \(*\) to extend \(\Phi\) to a map \(\Phi^*\), and note that \(\Phi^*(a) = a\) for all \(a \in A^*\) and \(\text{Ker} \Phi^* = N^*\) by the transfer principle. Now recall by Theorem 41 we have the natural identifications

\[\text{NSH}(A, \mathcal{F}) \equiv \{aG_{\text{inf}} \mid a \in A_{\text{lim}}\}\]

\[\text{NSH}(N, \mathcal{F}) \equiv \{nG_{\text{inf}} \mid n \in N_{\text{lim}}\}\]

Define the homomorphism

\[\phi : \text{NSH}(G, \mathcal{F}) \rightarrow \text{NSH}(A, \mathcal{F})\]
by \( \phi(xG_{inf}) = \Phi^*(x)G_{inf} \). By the definition of \( \Phi^* \), if \( aG_{inf} \in NSH(A, \mathcal{F}) \), then

\[
\phi(aG_{inf}) = \Phi^*(a)G_{inf} = aG_{inf}.
\]

We now show that \( \text{Ker } \phi = NSH(N, \mathcal{F}) \). Suppose that \( nG_{inf} \in NSH(N, \mathcal{F}) \). Then

\[
\phi(nG_{inf}) = \Phi^*(n)G_{inf} = eG_{inf} \in \text{Ker } \phi.
\]

To see the other inclusion, suppose that \( xG_{inf} \in \text{Ker } \phi \). Since \( G^* = A^* \times N^* \), we may write \( x = an \) for some \( a \in A^* \) and \( n \in N^* \). But since \( \Phi \) is a rough-transformation, we have that \( x \in G_{lim} \) implies that \( a \in A_{lim} \) and \( n \in N_{lim} \). Thus \( a = \Phi^*(x) \in G_{inf} \).

Therefore, \( a \in A^* \cap G_{inf} = A_{inf} \), showing \( xG_{inf} = nG_{inf} \in \text{Ker } \phi \). This gives that \( \text{Ker } \phi = NSH(N, \mathcal{F}) \) which completes the result.

**Example 5**: Using Theorem 48 we can quickly apply the nonstandard hull construction to a nonabelian group. We consider the infinite dihedral group, \( D_\infty \cong \mathbb{Z} \times \mathbb{Z}_2 \), where we denote an element of this group by \((x, y)\) where \( x \) is an integer and \( y \) is either 0 or 1 depending on which coset the element is from. If we let \( h(x, 0) = \epsilon |x| + \epsilon \cdot 0 \) and \( h(x, 1) = \epsilon |x| + \epsilon \cdot 1 \), then we may observe that the nonstandard hull of the infinite dihedral group is isomorphic to \( NSH(\mathbb{Z}, h) \times NSH(\mathbb{Z}_2, h) \cong \mathbb{R} \times \{\epsilon\} \cong \mathbb{R} \).

**Theorem 49.** Let \( G \) be a topological group. Let \( h \) be a continuous seminorm for \( G \), that is if \( x \in G^* \) and \( x \simeq y \) (with respect to the topology on \( G \)), then \( h(x) \simeq h(y) \).

Let \( \Gamma \lt G \) be a subgroup such that \( G = \Gamma K \) for some compact subset \( K \) of \( G \), Then \( NSH(\Gamma, h) \cong NSH(G, h) \).

**Proof.** Let \( h, \Gamma, \) and \( G \) be as in the statement of the theorem. By Theorem 41 we know \( NSH(\Gamma, h) \lt NSH(G, h) \). Thus it suffices to show \( NSH(G, h) \subset NSH(\Gamma, h) \). Let
$xG_{\inf} \in \text{NSH}(G, h)$. Then $x = \gamma k$ for some $\gamma \in \Gamma^*$ and $k \in K^*$. Thus we have

$$x\gamma^{-1} = k$$

$$h(x\gamma^{-1}) = h(k).$$

By the nonstandard definition of compactness, there exists some $k_0 \in K$ where $k_0 \simeq k$. Hence $h(k) \simeq h(k_0) \simeq 0$. Therefore, $xG_{\inf} = \gamma G_{\inf}$. Recall by Theorem 41 that $\gamma G_{\inf}$ can be identified as an element of $\text{NSH}(\Gamma, h)$. This together with the observation that $xG_{\inf} = \gamma G_{\inf}$ shows $\text{NSH}(G, h) \subset \text{NSH}(\Gamma, h)$ which completes the result. 

This is a significant generalization to the observation we made after Theorem 22. In the next chapter, we will see another class of groups where this result can be employed in a similar manner.
CHAPTER IV

THE HEISENBERG GROUP

4.1 Bilinear Forms and the Heisenberg Group

To this point, with the exception of Example 5, we have only applied the nonstandard
hull construction to abelian groups. Furthermore, even in the case of the infinite
dihedral group, the nonstandard hull construction yielded an abelian group. We now
will show this is not always the case by investigating a generalized version of the
Heisenberg group. Often the Heisenberg group is described as

\[
H^3(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right| x, y, z \in \mathbb{R} \right\}
\]

where the operation is matrix multiplication. However, we can give a much more
general definition for which the group listed above is a special case. First we require
the following definition.

**Definition 50.** Let \( V \) be a vector space over the field \( F \). A function \( B : V \times V \to F \)
is called a **bilinear form** if

1. The map \( v \mapsto B(v, w) \) is linear for each \( w \in V \), and

2. The map \( v \mapsto B(w, v) \) is linear for each \( w \in V \).
Furthermore, $B$ is said to be skew-symmetric if $B(v, w) = -B(w, v)$ for all $v, w \in V$, and $B$ is said to be nondegenerate if $B(v, w) = 0$ for all $w \in V$ implies $v = 0$.

Given a vector space $V$ over a field $F$ and a nondegenerate, skew-symmetric bilinear form $B$, we define the Heisenberg group $H(V, B)$ to be the set $V \times F$ with the group law
\[(v, z) \cdot (v', z') = (v + v', z + z' + B(v, v')).\]

We now show that if $V$ is a finite dimensional vector space over $\mathbb{R}$ and $B$ is a bilinear form for $V$, then the $\mathbb{R}^*$ bilinear form $B^*$ for $V^*$ possesses some nice properties.

**Lemma 51.** Let $V$ be a finite dimensional vector space over $\mathbb{R}$ with basis $\beta$. If $B$ is a bilinear form for $V$, then the bilinear form $B^*$ for $V^*$ is determined by its behavior on $\beta$ and $B^*(e_i, e_j)$ is standard for all elements $e_i, e_j \in \beta^*$.

**Proof.** First we observe that a bilinear form $B$ is completely determined by its behavior on $\beta$. That is if we let $b_{ij} = B(e_i, e_j)$, for each $e_i, e_j \in \beta$, then given $v = \sum_i \alpha_i e_i$ and $v' = \sum_j \gamma_j e_j$, we have
\[B(v, v') = \sum_i \sum_j \alpha_i \gamma_j b_{ij}.\]

Since $\beta$ is finite, $\beta^* = \beta$; thus the $\mathbb{R}^*$-bilinear form $B^*$ on $V^*$ is also completely determined by $\beta$, and it is clear that given $e_i, e_j \in \beta^*$, that $B^*(e_i, e_j) = b_{ij}$ is standard. 

Now suppose that $(V, \| \cdot \|)$ is a normed vector space over $\mathbb{R}$ (not necessarily finite dimensional), and $B$ is a nondegenerate, skew-symmetric bilinear form for $V$. 39
We introduce the seminorm we will use on $H(V,B)$. In order to make a very general statement later, we must impose several conditions on $B$ and $\| \cdot \|$.

**Hypothesis 52.** Let $(V, \| \cdot \|)$ be a vector space over $\mathbb{R}$ with basis $\beta$ and a skew-symmetric, nondegenerate bilinear form $B$. Let $v \in V^*$ and write $v = \sum i \alpha_i e_i$ where the $\alpha_i \in \mathbb{R}^*$ and $e_i \in \beta$. We make the following assumptions on $V$, $\| \cdot \|$, and $B$:

1. $\|v\| \ll \infty$ implies $\sum_i |\alpha_i| \ll \infty$;

2. $\|v\| \simeq 0$ implies $\sum_i |\alpha_i| \simeq 0$;

3. There exists an $M \ll \infty$ such that for all $e_i, e_j \in \beta^*$, $|B^*(e_i, e_j)| \leq M$.

If $V$ is a normed vector space over $\mathbb{R}$ with even dimension, then we can always find a norm and appropriate bilinear form so that Hypothesis 52 is satisfied.

**Lemma 53.** Let $V$ be a normed vector space over $\mathbb{R}$ with basis $\beta$ and bilinear form $B$ satisfying Hypothesis 52. Then $h(v, z) = \epsilon \|v\| + \epsilon^2 |z|$ defines a seminorm for $H(V,B)$.

**Proof.** First note that $h$ is inverse-preserving since $(v, z)^{-1} = (-v, -z)$, and it is clear $h$ takes the identity element to zero. Now suppose $(v, z), (v', z') \in H(V,B)^*$ and $h(v, z), h(v', z') \ll \infty$. Then

\[
h((v, z) \cdot (v', z')) = h(v + v', z + z' + B^*(v, v'))
\]

\[
= \epsilon \|v + v'\| + \epsilon^2 |z + z' + B^*(v, v')|
\]

\[
\leq \epsilon \|v\| + \epsilon \|v'\| + \epsilon^2 |z| + \epsilon^2 |z'| + \epsilon^2 |B^*(v, v')|.
\]
It is obvious each term in the last sum is limited with the possible exception of the term 
\( \epsilon^2 |B^* (v, v')| = |B^* (\epsilon v, \epsilon v')| \). But if we let 
\( v = \sum_i \alpha_i e_i \) and \( v' = \sum_j \gamma_j e_j \), then

\[
|B^* (\epsilon v, \epsilon v')| \leq \sum_i \sum_j |\epsilon \alpha_i \epsilon \gamma_j B^* (e_i, e_j)| \leq M \left( \sum_i |\epsilon \alpha_i| \right) \left( \sum_j |\epsilon \gamma_j| \right). 
\]

(4.1)

By Hypothesis 52, we must have that (4.1) is limited. A similar argument holds for the case where \( h(v, z), h(v', z') \simeq 0 \). Now suppose \( h(v, z) \ll \infty \) and \( h(v', z') \simeq 0 \). Then

\[
h((v, z) \cdot (v', z') \cdot (-v, -z)) = \epsilon \|v'\| + \epsilon^2 |z'| + 2B^* (v, v').
\]

(4.2)

Thus it suffices to see that \( \epsilon^2 |B^* (v, v')| = |B^* (\epsilon v, \epsilon v')| \simeq 0 \). Again, if we let 
\( v = \sum_i \alpha_i e_i \) and \( v' = \sum_j \gamma_j e_j \), then (4.1) again gives the desired result. \( \square \)

Notice that \( h \) is a controlled, inverse-preserving seminorm that is not subadditive.

4.2 The Nonstandard Hull of the Heisenberg Group

We now wish to give a significant generalization of Theorem 22, for the real line is simply the Heisenberg group where \( V \) is the vector space of dimension zero. Before doing this we have the following rather unsurprising result. However, when combined with Theorem 49, we will have determined the nonstandard hull for a plethora of discrete groups. Recall that when considering a vector \( v \in V^* \), where \( v = \sum_i \alpha_i e_i \) with respect to a standard basis \( \beta^* \), we define \( \text{st}(v) = \sum_i \text{st}(\alpha_i) e_i \) where \( \text{st}(\alpha) \) is the usual standard part of \( \alpha \in \mathbb{R}^* \). As was shown in Chapter 1, \( \text{st}(v) \) is independent of the choice of basis in the finite dimensional case.
Theorem 54. Let $V$ be a normed vector space over $\mathbb{R}$. Using the seminorm $h$ from Lemma 53, $\text{NSH}(H(V, B), h) \cong H(\hat{V}, \hat{B})$ where $\hat{V}$ is the nonstandard hull of $V$ as given in Definition 11, and $\hat{B}$ is the $\mathbb{R}^*$-bilinear form for $\hat{V}$ induced by $B^*$.

Proof. Write $\text{fin}(V)/\mu(0) = \hat{V}$, and write $\mathbb{R} = (\mathbb{R}_{\text{lim}}, \epsilon^2|\cdot|)/\mathbb{R}_{\text{inf}}$. Note the last equality is justified since the seminorms $\epsilon^2|\cdot|$ and $\epsilon|\cdot|$ for $\mathbb{R}$ are equivalent. Consider the map

$$\phi : H(V, B)_{\text{lim}} \rightarrow H(\hat{V}, \hat{B})$$

defined by $\phi(v, z) = (\text{st}(\epsilon v) + \mu(0), \text{st}(\epsilon^2 z) + \mathbb{R}_{\text{inf}})$. To see the map is surjective first recall Theorem 20 and Theorem 22. Thus given $(v + \mu(0), z + \mathbb{R}_{\text{inf}}) \in H(\hat{V}, \hat{B})$, we see that $\phi \left( \begin{pmatrix} v \\ \epsilon \end{pmatrix}, z \epsilon^2 \right) = (v + \mu(0), z + \mathbb{R}_{\text{inf}})$. Also, $\phi(v, z) = (0 + \mu(0), 0 + \mathbb{R}_{\text{inf}})$ if and only if $\epsilon\|v\|, \epsilon^2|z| \simeq 0$ if and only if $h(v, z) \simeq 0$ if and only if $(v, z) \in H(V, B)_{\text{inf}}$. The result then follows by the first isomorphism theorem. $\Box$

Now consider the metric on $H(V, B)$ given by

$$d((v, z), (v', z')) = \|v - v'\| + |z - z'|.$$ 

Note that this induces the usual topology on $H(V, B)$ which is equivalent topologically to $V \times \mathbb{R}$. We now show that the hypotheses of Theorem 49 are satisfied. Suppose that $(v, z) \simeq (v', z')$. Then $d((v, z), (v', z')) = \|v - v'\| + |z - z'| \simeq 0$. Hence $\|v - v'\| \simeq 0$ and $|z - z'| \simeq 0$. Now notice that

$$\|v\| = \|v - v' + v'\| \leq \|v - v'\| + \|v'\|$$
which implies that $\text{st}(\|v\|) \leq \text{st}(\|v'\|)$. Also,

$$\|v'\| = \|v' - v + v\| \leq \|v - v'\| + \|v\|$$

which implies $\text{st}(\|v'\|) \leq \text{st}(\|v\|)$. Thus $\text{st}(\|v\|) = \text{st}(\|v'\|)$, or equivalently, $\|v\| \simeq \|v'\|$. It then follows that $\epsilon\|v\| \simeq \epsilon\|v'\|$. A similar argument shows $\epsilon^2|z| \simeq \epsilon^2|z'|$. Hence

$$h(v, z) = \epsilon\|v\| + \epsilon^2|z| \simeq \epsilon\|v'\| + \epsilon^2|z| \simeq \epsilon\|v'\| + \epsilon^2|z'| = h(v', z').$$

This shows that we can employ Theorem 49 and observe that for any $\Gamma \vartriangleleft H(V, B)$ such that $G = \Gamma K$ where $K$ is compact, we have $\text{NSH}(\Gamma, h) \cong \text{NSH}((V, B), h)$.

**Example 55.** Note that in the finite-dimensional case, one particular realization of the Heisenberg group is the set of matrices

$$
\begin{pmatrix}
1 & x & c \\
0 & I_n & y \\
0 & 0 & 1
\end{pmatrix}
$$

where $x$ is a row vector of length $n$, $y$ is a column vector of length $n$, $I_n$ is the $n \times n$ identity matrix, and $c \in \mathbb{R}$. We can form a subgroup by letting $c$ the entries of $x$ and $y$ be integers. The above work shows that the nonstandard hull of this matrix group with integer entries is the corresponding group of matrices with real entries.
4.3 An Infinite-Dimensional Example

We conclude by showing a specific infinite dimensional vector space and bilinear form such that we can use the seminorm described in Lemma 53 and the result of Theorem 54. Recall that

\[
\ell^1(\mathbb{N}) = \left\{ (a_i)_{i \in \mathbb{N}} \mid \forall i, \ a_i \in \mathbb{R}, \ \sum_i |a_i| < \infty \right\}.
\]

Then by transfer, it follows that

\[
\ell^1(\mathbb{N})^* = \left\{ (a_i)_{i \in \mathbb{N}^*} \mid \forall i, \ a_i \in \mathbb{R}^*, \ \sum_i |a_i| < \infty \right\}.
\]

Let \( V = \ell^1(\mathbb{N}) \). Consider the basis \( \beta = \{ e_i \mid i \in \mathbb{N} \} \) for \( V \) where \( e_j \) is the sequence that is one at position \( j \) and zero elsewhere. Then clearly \( \beta^* = \{ e_i \mid i \in \mathbb{N}^* \} \) is a basis for \( V^* \). Now we define \( B \) on \( \beta \) as follows:

\[
B(e_i, e_j) = \begin{cases} 
1 & \text{if } i < j, \\
0 & \text{if } i = j, \\
-1 & \text{if } i > j.
\end{cases}
\]

It is now shown that \( B \) maps any pair of vectors from \( V \) to a real number. Let \( v = \sum_i \alpha_i e_i \) and \( w = \sum_j \beta_j e_j \) be elements of \( V \). Then

\[
B(v, w) = \sum_i \sum_j \alpha_i \beta_j B(e_i, e_j) \leq \sum_i \sum_j |\alpha_i||\beta_j| \leq \left( \sum_i |\alpha_i| \right) \cdot \left( \sum_j |\beta_j| \right) < \infty.
\]

It is clear from the definition of \( B \) that the set \( \{ B^*(e_i, e_j) \} \) is bounded above by \( M = 1 \). Using the usual norm for \( V \), we now can see that \( H(V, B) \) satisfies Hypothesis 52. Hence \( NSH(H(V, B), h) \cong H(\hat{V}, \hat{B}) \). An interesting aspect of the infinite dimensional
case is that \( \hat{V} \) is not isomorphic to \( V \). In fact, the structure of \( \hat{V} \) is much more complicated; the interested reader can find the details in [4]. Finally note, that we could employ Theorem 49 and consider the discrete subgroup

\[
L = \{(a_i) \in V \mid \forall i, i \cdot a_i \in \mathbb{Z}\},
\]

and consequently we would have \( \text{NSH}(H(L, B), h) \cong H(\hat{V}, \hat{B}) \).
CHAPTER V

OPEN QUESTIONS

There are many questions open to further investigation regarding the nonstandard hull construction. For instance, the version of nonstandard analysis in [1] is only required so that we may form the limited and infinitesimal sets legally. However, there are axiomatic approaches to nonstandard analysis such as a formulation known as internal set theory or IST presented in [5] that avoid many of the logical complexities necessary in approaches based on Robinson’s nonstandard analysis. From [6] the traditional nonstandard hull construction can be carried out within IST. It may very well be possible to employ the ideas used in this article to reformulate the nonstandard hull of a group construction in the much simpler setting of IST.

One may have noticed that conditions 2 and 3 of Definition 12 were never used. However, if we were to consider the group $G$ of $n \times n$ upper triangular matrices with ones on the diagonal and entries from $\mathbb{R}$, these conditions are necessary to use a natural choice of seminorm

$$h(A) = \sum_{i>j} e^{|a_{ij}|}$$

where $a_{ij}$ is in the $ij$ entry of $A$. It should be possible to prove that the nonstandard hull of this group and the corresponding group with integer entries is $G$ using the
methods developed in Chapter 3. Furthermore, it may be possible to obtain this result as a special case of a much wider class of Lie groups.

There seem to be many connections between the nonstandard hull construction and the ideas of coarse geometry. For instance, in coarse geometry finite sets and compact sets are coarsely equivalent to single points. Also, the integers are coarsely equivalent to the real line, and the discrete Heisenberg group is coarsely equivalent to the real Heisenberg group. All of the results guaranteeing that the nonstandard hull construction yields a metric space relied on having subadditive seminorms, and it may very well be that using the ideas of coarse equivalence that more general statements can be made in the same vein without requiring the condition that the seminorms must be subadditive. In fact, the examples of the Heisenberg group show that we can still recover a metric space without having a subadditive seminorm. It may very well be that Theorem 49 could be reformulated such that the seminorm considered do not need to be compatible with a group’s topology, but rather compatible in some way with the group’s coarse structure.

It was previously conjectured that the nonstandard hull of a group is a complete metric space provided a subadditive, inverse-preserving seminorm is used. However, if one were to consider the group of rational numbers $\mathbb{Q}$ and the seminorm $h_0 = | \cdot |$, then $\text{NSH}(\mathbb{Q}, h_0) \cong \mathbb{Q}$. Of course we know that if we instead chose $h = \epsilon | \cdot |$, then the nonstandard hull would be a complete metric space. The obvious difference
is that \( h \) is controlled but \( h_0 \) is not. Obviously, other similar examples could be constructed where using a controlled seminorm results in a complete space but using a seminorm that is not controlled gives a space that is not complete. It would be of great interest to explore in more detail the properties of controlled seminorms and formally state what effects they have on the nonstandard hull construction. Clearly, the definition of a seminorm is very weak, so determining in general what effect different restrictions placed on the class of seminorms considered has on the nonstandard hull construction would be particularly useful.

The conditions set in Hypothesis 52 were obviously chosen so that we could consider the nonstandard hull of the Heisenberg group when the underlying vector space was \( \ell^1(\mathbb{N}) \). A next natural question would be if there are other infinite-dimensional vector spaces that satisfy these hypotheses, or if not perhaps Hypothesis 52 could be modified to consider other \( \ell^p \) spaces. If these assumptions can be properly modified, then obviously an application of Theorem 54 and Theorem 49 would allow us to deal with finding the nonstandard hull of many discrete subgroups of \( H(\ell^p(\mathbb{N}, B)) \) quite easily.

A recurring theme was considering a discrete subgroup of a group and showing that the nonstandard hull construction recaptures the entire group. Thus an obvious application would be to see what statements can be made about a more complex group by studying a simpler discrete subgroup and its nonstandard hull.
BIBLIOGRAPHY


